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Gromov-Witten theory, Hurwitz theory, and completed cycles

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Contents

0. Introduction

0.1. *Overview.*

0.1.1. There are two enumerative theories of maps from curves to curves. Our goal here is to study their relationship. All curves in the paper will be projective over $\mathbb{C}$.

The first theory, introduced in the 19^{th} century by Hurwitz, concerns the enumeration of degree d covers,

$$\pi : C \to X,$$

of nonsingular curves X with specified ramification data. In 1902, Hurwitz published a closed formula for the number of covers,

$$\pi : \mathbf{P}^1 \to \mathbf{P}^1,$$

with specified simple ramification over $\mathbf{A}^1 \subset \mathbf{P}^1$ and arbitrary ramification over ∞ (see [17] and also [10], [36]).

Cover enumeration is easily expressed in the class algebra of the symmetric group $S(d)$. The formulas involve the characters of $S(d)$. Though great strides have been taken in the past century, the characters of $S(d)$ remain objects of substantial combinatorial complexity. While any particular Hurwitz number may be calculated, very few explicit formulas are available.

The second theory, the Gromov-Witten theory of target curves X, is modern. It is defined via intersection in the moduli space $\overline{M}_{g,n}(X,d)$ of degree d stable maps,

$$\pi : C \to X,$$

from genus g, n-pointed curves. A sequence of descendents,

$$\tau_0(\gamma), \tau_1(\gamma), \tau_2(\gamma), \ldots,$$

is determined by each cohomology class $\gamma \in H^*(X,\mathbb{Q})$. The descendents $\tau_k(\gamma)$ correspond to classes in the cohomology of $\overline{M}_{g,n}(X,d)$. Full definitions are given in Section 0.2 below. The Gromov-Witten invariants of X are defined as integrals of products of descendent classes against the virtual fundamental class of $\overline{M}_{g,n}(X,d)$.

Let $\omega \in H^2(X,\mathbb{Q})$ denote the (Poincaré dual) class of a point. We define the *stationary sector* of the Gromov-Witten theory X to be the integrals involving only the descendents of ω. The stationary sector is the most basic and fundamental part of the Gromov-Witten theory of X.

Since Gromov-Witten theory and Hurwitz theory are both enumerative theories of maps, we may ask whether there is any precise relationship between the two. We prove the stationary sector of Gromov-Witten is in fact *equivalent* to Hurwitz theory.

0.1.2. Let X be a nonsingular target curve. The main result of the paper is a correspondence, termed here the GW/H *correspondence*, between the stationary sector of Gromov-Witten theory and Hurwitz theory.

Each descendent $\tau_k(\omega)$ corresponds to an explicit linear combination of ramification conditions in Hurwitz theory. A stationary Gromov-Witten invariant of X is equal to the sum of the Hurwitz numbers obtained by replacing $\tau_k(\omega)$ by the associated ramification conditions. The ramification conditions associated to $\tau_k(\omega)$ are universal — independent of all factors including the target X.

0.1.3. The GW/H correspondence may be alternatively expressed as associating to each descendent $\tau_k(\omega)$ an explicit element of the class algebra of the symmetric group. The associated elements, the completed cycles, have been considered previously in Hurwitz theory — the term *completed cycle* first appears in [12] following unnamed appearances of the associated elements in [1], [11]. In fact, completed cycles, implicitly, are ubiquitous in the theory of shifted symmetric functions.

The completed k-cycle is the ordinary k-cycle corrected by a nonnegative linear combination of permutations with smaller support (except, possibly, for the constant term corresponding to the empty permutation, which may be of either sign). The corrections are viewed as completing the cycle. In [12], the corrections to the ordinary k-cycle were understood as counting degenerations of Hurwitz coverings with appropriate combinatorial weights. Similarly, in Gromov-Witten theory, the correction terms will be seen to arise from the boundary strata of $\overline{M}_{g,n}(X,d)$.

0.1.4. The GW/H correspondence is important from several points of view. From the geometric perspective, the correspondence provides a combinatorial approach to the stationary Gromov-Witten invariants of X, leading to very concrete and efficient formulas. From the perspective of symmetric functions, a geometrization of the theory of completed cycles is obtained.

Hurwitz theory with completed cycles is combinatorially much more accessible than standard Hurwitz theory — a major motivation for the introduction of completed cycles. Completed cycles calculations may be naturally evaluated in the operator formalism of the infinite wedge representation, $\Lambda^{\frac{\infty}{2}}V$. In particular, closed formulas for the completed cycle correction terms are obtained. If the target X is either genus 0 or 1, closed form evaluations of all corresponding generating functions may be found; see Sections 3 and 5. In fact, the completed cycle corrections appear in the theory with target genus 0.

Hurwitz theory, while elementary to define, leads to substantial combinatorial difficulties. Gromov-Witten theory, with much more sophisticated foundations, provides a simplifying completion of Hurwitz theory.

0.1.5. The present paper is the first of a series devoted to the Gromov-Witten theory of target curves X. In subsequent papers, we will consider the equivariant theory for $\mathbf{P}^1$, the descendents of the other cohomology classes of X, and the connections to integrable hierarchies. The equivariant Gromov-Witten theory of $\mathbf{P}^1$ and the associated 2-Toda hierarchy will be the subject of [32].

The introduction is organized as follows. We review the definitions of Gromov-Witten and Hurwitz theory in Sections 0.2 and 0.3. Shifted symmetric functions and completed cycles are discussed in Section 0.4. The basic GW/H correspondence is stated in Section 0.5.

0.2. *Gromov-Witten theory.* The Gromov-Witten theory of a nonsingular target X concerns integration over the moduli space $\overline{M}_{g,n}(X,d)$ of stable degree d maps from genus g, n-pointed curves to X. Two types of cohomology classes are integrated. The *primary classes* are:

$$\mathrm{ev}_i^*(\gamma) \in H^2(\overline{M}_{g,n}(X,d),\mathbb{Q}),$$

where ev_i is the morphism defined by evaluation at the i^{th} marked point,

$$\mathrm{ev}_i : \overline{M}_{g,n}(X) \to X,$$

and $\gamma \in H^*(X,\mathbb{Q})$. The *descendent classes* are:

$$\psi_i^k \mathrm{ev}_i^*(\gamma),$$

where $\psi_i \in H^2(\overline{M}_{g,n}(X,d),\mathbb{Q})$ is the first Chern class of the cotangent line bundle L_i on the moduli space of maps.

Let $\omega \in H^2(X,\mathbb{Q})$ denote the Poincaré dual of the point class. We will be interested here exclusively in the integrals of the descendent classes of ω:

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\omega) \right\rangle_{g,d}^{\circ X} = \int_{[\overline{M}_{g,n}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^n \psi_i^{k_i}\, \mathrm{ev}_i^*(\omega). \tag{0.1}$$

The theory is defined for all $d \geq 0$.

Let $g(X)$ denote the genus of the target. The integral (0.1) is defined to vanish unless the dimension constraint,

$$2g - 2 + d(2 - 2g(X)) = \sum_{i=1}^n k_i, \tag{0.2}$$

is satisfied. If the subscript g is omitted in the bracket notation $\langle \prod_i \tau_{k_i}(\omega)\rangle_d^X$, the genus is specified by the dimension constraint from the remaining data. If the resulting genus is not an integer, the integral is defined as vanishing. Unless emphasis is required, the genus subscript will be omitted.

The integrals (0.1) constitute the *stationary sector* of the Gromov-Witten theory of X since the images in X of the marked points are pinned by the integrand. The total Gromov-Witten theory involves also the descendants of the identity and odd classes of $H^*(X,\mathbb{Q})$.

The moduli space $\overline{M}_{g,n}(X,d)$ parametrizes stable maps with connected domain curves. However, Gromov-Witten theory may also be defined with disconnected domains. If $C = \bigcup_{i=1}^{l} C_i$ is a disconnected curve with connected components C_i, the arithmetic genus of C is defined by:

$$g(C) = \sum_i g(C_i) - l + 1,$$

where $g(C_i)$ is the arithmetic genus of C_i. In the disconnected theory, the genus may be negative. Let $\overline{M}^{\bullet}_{g,n}(X,d)$ denote the moduli space of stable maps with possibly disconnected domains.

We will use the brackets $\langle\ \rangle^{\circ}$ as above in (0.1) for integration in connected Gromov-Witten theory. The brackets $\langle\ \rangle^{\bullet}$ will be used for the disconnected theory obtained by integration against $[\overline{M}^{\bullet}_{g,d}(X,d)]^{\mathrm{vir}}$. The brackets $\langle\ \rangle$ will be used when it is not necessary to distinguish between the connected and disconnected theories.

0.3. *Hurwitz theory.*

0.3.1. The Hurwitz theory of a nonsingular curve X concerns the enumeration of covers of X with specified ramification. The ramifications are determined by the profile of the cover over the branch points.

For Hurwitz theory, we will only consider covers,

$$\pi : C \to X,$$

where C is nonsingular and π is dominant on each component of C. Let $d > 0$ be the degree of π. The *profile* of π over a point $q \in X$ is the partition η of d obtained from multiplicities of $\pi^{-1}(q)$.

By definition, a partition η of d is a sequence of integers,

$$\eta = (\eta_1 \geq \eta_2 \geq \cdots \geq 0),$$

where $|\eta| = \sum \eta_i = d$. Let $\ell(\eta)$ denote the length of the partition η, and let $m_i(\eta)$ denote the multiplicity of the part i. The profile of π over q is the partition (1^d) if and only if π is unramified over q.

Let $d > 0$, and let $\eta^1, \ldots, \eta^n$ be partitions of d assigned to n distinct points $q_1, \ldots, q_n$ of X. A Hurwitz cover of X of genus g, degree d, and monodromy η^i at q_i is a morphism

$$\pi : C \to X \tag{0.3}$$

satisfying:

(i) C is a nonsingular curve of genus g,

(ii) π has profile η^i over q_i,

(iii) π is unramified over $X \setminus \{q_1, \dots, q_n\}$.

Hurwitz covers may exist with connected or disconnected domains. The Riemann-Hurwitz formula,

$$2g(C) - 2 + d(2 - 2g(X)) = \sum_{i=1}^{n} (d - \ell(\eta^i)) \,, \tag{0.4}$$

is valid for both connected and disconnected Hurwitz covers. In disconnected theory, the domain genus may be negative. Since $g(C)$ is uniquely determined by the remaining data, the domain genus will be omitted in the notation below.

Two covers $\pi : C \to X$, $\pi' : C' \to X$ are isomorphic if there exists an isomorphism of curves $\phi : C \to C'$ satisfying $\pi' \circ \phi = \pi$. Up to isomorphism, there are only finitely many Hurwitz covers of X of genus g, degree d, and monodromy η^i at q_i. Each cover π has a finite group of automorphisms $\mathrm{Aut}(\pi)$.

The Hurwitz number,

$$\mathsf{H}_d^X(\eta^1, \dots, \eta^n),$$

is defined to be the weighted count of the distinct, *possibly disconnected* Hurwitz covers π with the prescribed data. Each such cover is weighted by $1/|\mathrm{Aut}(\pi)|$.

The GW/H correspondence is most naturally expressed as a relationship between the disconnected theories, hence the disconnected theories will be of primary interest to us.

0.3.2. We will require an extended definition of Hurwitz numbers valid in the degree 0 case and in case the ramification conditions η satisfy $|\eta| \neq d$. The Hurwitz numbers H_d^X are defined for all degrees $d \geq 0$ and all partitions η^i by the following rules:

(i) $\mathsf{H}_0^X(\emptyset, \dots, \emptyset) = 1$, where $\emptyset$ denotes the empty partition.

(ii) If $|\eta^i| > d$ for some i then the Hurwitz number vanishes.

(iii) If $|\eta^i| \leq d$ for all i then

$$\mathsf{H}_d^X(\eta^1, \dots, \eta^n) = \prod_{i=1}^{n} \binom{m_1(\boldsymbol{\eta}^i)}{m_1(\eta^i)} \cdot \mathsf{H}_d^X(\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^n) \,, \tag{0.5}$$

where $\boldsymbol{\eta}^i$ is the partition of size d obtained from η^i by adding $d - |\eta^i|$ parts of size 1.

In other words, the monodromy condition η at $q \in X$ with $|\eta| < d$ corresponds to counting Hurwitz covers with monodromy $\boldsymbol{\eta}$ at q together with the data of a subdivisor of $\pi^{-1}(q)$ of profile η.

0.3.3. The enumeration of Hurwitz covers of $\mathbf{P}^1$ is classically known to be equivalent to multiplication in the class algebra of the symmetric group. We review the theory here.

Let $S(d)$ be the symmetric group. Let $\mathbb{Q}S(d)$ be the group algebra. The class algebra,

$$\mathcal{Z}(d) \subset \mathbb{Q}S(d),$$

is the center of the group algebra.

Hurwitz covers with profile η^i over $q_i \in \mathbf{P}^1$ canonically yield n-tuples of permutations $(s_1, \ldots, s_n)$ defined up to conjugation satisfying:

(i) s_i has cycle type η_i,

(ii) $s_1 s_2 \cdots s_n = 1$.

The elements s_i are determined by the monodromies of π around the points q_i.

Therefore, $\mathsf{H}_d^{\mathbf{P}^1}(\eta^1, \ldots, \eta^n)$ equals the number of n-tuples satisfying conditions (ii) and (ii) divided by $|S(d)|$. The factor $|S(d)|$ accounts for over counting and automorphisms.

Let $C_\eta \in \mathcal{Z}(d)$ be the conjugacy class corresponding to η. We have shown:

$$\begin{aligned} \mathsf{H}_d^{\mathbf{P}^1}(\eta^1, \ldots, \eta^n) &= \frac{1}{d!} \left[C_{(1^d)}\right] \prod C_{\eta^i} \\ &= \frac{1}{(d!)^2} \operatorname{tr}_{\mathbb{Q}S(d)} \prod C_{\eta^i} \end{aligned} \tag{0.6}$$

where $\left[C_{(1^d)}\right]$ stands for the coefficient of the identity class and $\operatorname{tr}_{\mathbb{Q}S(d)}$ denotes the trace in the adjoint representation.

Let λ be an irreducible representation λ of $S(d)$ of dimension $\dim \lambda$. The conjugacy class C_η acts as a scalar operator with eigenvalue

$$\mathbf{f}_\eta(\lambda) = |C_\eta| \frac{\chi_\eta^\lambda}{\dim \lambda}, \quad |\lambda| = |\eta|, \tag{0.7}$$

where χ_η^λ is the character of any element of C_η in the representation λ. The trace in equation (0.6) may be evaluated to yield the basic character formula for Hurwitz numbers:

$$\mathsf{H}_d^{\mathbf{P}^1}(\eta^1, \ldots, \eta^n) = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^2 \prod_{i=1}^n \mathbf{f}_{\eta^i}(\lambda). \tag{0.8}$$

The character formula is easily generalized to include the extended Hurwitz numbers (of Section 0.3.2) of target curves X of arbitrary genus g. The character formula can be traced to Burnside (exercise 7 in §238 of [2]); see also [4], [19].

Define $\mathbf{f}_\eta(\lambda)$ for arbitrary partitions η and irreducible representations λ of $S(d)$ by:

$$\mathbf{f}_\eta(\lambda) = \binom{|\lambda|}{|\eta|} |C_\eta| \frac{\chi^\lambda_\eta}{\dim \lambda} . \tag{0.9}$$

If $\eta = \emptyset$, the formula is interpreted as:

$$\mathbf{f}_\emptyset(\lambda) = 1 .$$

For $|\eta| < |\lambda|$, the function χ^λ_η is defined via the natural inclusion of symmetric groups $S(|\eta|) \subset S(d)$. If $|\eta| > |\lambda|$, the binomial in (0.9) vanishes.

The character formula for extended Hurwitz numbers of genus g targets X is:

$$\mathsf{H}^X_d(\eta^1, \ldots, \eta^n) = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^{2-2g(X)} \prod_{i=1}^n \mathbf{f}_{\eta^i}(\lambda) . \tag{0.10}$$

0.4. *Completed cycles.*

0.4.1. Let $\mathcal{P}(d)$ denote the set of partitions of d indexing the irreducible representations of $S(d)$. The Fourier transform,

$$\mathcal{Z}(d) \ni C_\mu \mapsto \mathbf{f}_\mu \in \mathbb{Q}^{\mathcal{P}(d)} , \quad |\mu| = d , \tag{0.11}$$

determines an isomorphism between $\mathcal{Z}(d)$ and the algebra of functions on $\mathcal{P}(d)$. Formula (0.8) may be alternatively derived as a consequence of the Fourier transform isomorphism.

Let $\mathcal{P}$ denote the set of all partitions (including the empty partition $\emptyset$). We may extend the Fourier transform (0.11) to define a map,

$$\phi : \bigoplus_{d=0}^{\infty} \mathcal{Z}(d) \ni C_\mu \mapsto \mathbf{f}_\mu \in \mathbb{Q}^{\mathcal{P}} , \tag{0.12}$$

via definition (0.9). The extended Fourier transform ϕ is no longer an isomorphism of algebras. However, ϕ is linear and injective.

We will see the image of ϕ in $\mathbb{Q}^{\mathcal{P}}$ is the algebra of *shifted symmetric functions* defined below (see [23] and also [31]).

0.4.2. The shifted action of the symmetric group $S(n)$ on the algebra $\mathbb{Q}[\lambda_1, \ldots, \lambda_n]$ is defined by permutation of the variables $\lambda_i - i$. Let

$$\mathbb{Q}[\lambda_1, \ldots, \lambda_n]^{*S(n)}$$

denote the invariants of the shifted action. The algebra $\mathbb{Q}[\lambda_1, \ldots, \lambda_n]^{*S(n)}$ has a natural filtration by degree.

Define the algebra of shifted symmetric functions Λ^* in an infinite number of variables by

$$\Lambda^* = \varprojlim \mathbb{Q}[\lambda_1, \dots, \lambda_n]^{*S(n)}\,, \tag{0.13}$$

where the projective limit is taken in the category of filtered algebras with respect to the homomorphisms which send the last variable λ_n to 0.

Concretely, an element $\mathbf{f} \in \Lambda^*$ is a sequence (usually presented as a series),

$$\mathbf{f} = \left\{\mathbf{f}^{(n)}\right\}\,, \quad \mathbf{f}^{(n)} \in \mathbb{Q}[\lambda_1, \dots, \lambda_n]^{*S(n)}\,,$$

satisfying:

(i) the polynomials $\mathbf{f}^{(n)}$ are of uniformly bounded degree,

(ii) the polynomials $\mathbf{f}^{(n)}$ are stable under restriction,

$$\mathbf{f}^{(n+1)}\Big|_{\lambda_{n+1}=0} = \mathbf{f}^{(n)}\,.$$

The elements of Λ^* will be denoted by boldface letters.

The algebra Λ^* is filtered by degree. The associated graded algebra $\operatorname{gr}\Lambda^*$ is canonically isomorphic to the usual algebra Λ of symmetric functions as defined, for example, in [27].

A point $(x_1, x_2, x_3, \dots) \in \mathbb{Q}^\infty$ is *finite* if all but finitely many coordinates vanish. By construction, any element $\mathbf{f} \in \Lambda^*$ has a well-defined evaluation at any finite point. In particular, $\mathbf{f}$ can be evaluated at any point

$$\lambda = (\lambda_1, \lambda_2, \dots, 0, 0, \dots)\,,$$

corresponding to a partition λ. An elementary argument shows functions $\mathbf{f} \in \Lambda^*$ are uniquely determined by their values $\mathbf{f}(\lambda)$. Hence, Λ^* is canonically a subalgebra of $\mathbb{Q}^{\mathcal{P}}$.

0.4.3. The shifted symmetric power sum $\mathbf{p}_k$ will play a central role in our study. Define $\mathbf{p}_k \in \Lambda^*$ by:

$$\mathbf{p}_k(\lambda) = \sum_{i=1}^{\infty} \left[(\lambda_i - i + \tfrac{1}{2})^k - (-i + \tfrac{1}{2})^k\right] + (1 - 2^{-k})\zeta(-k)\,. \tag{0.14}$$

The shifted symmetric polynomials,

$$\sum_{i=1}^{n} \left[(\lambda_i - i + \tfrac{1}{2})^k - (-i + \tfrac{1}{2})^k\right] + (1 - 2^{-k})\zeta(-k)\,, \quad n = 1, 2, 3, \dots\,,$$

are of degree k and are stable under restriction. Hence, $\mathbf{p}_k$ is well-defined.

The shifts by $\frac{1}{2}$ in the definition of $\mathbf{p}_k$ appear arbitrary — their significance will be clear later. The peculiar ζ-function constant term in $\mathbf{p}_k$ will be explained below.

The image of $\mathbf{p}_k$ in $\operatorname{gr}\Lambda^* \cong \Lambda$ is the usual k^{th} power-sum functions. Since the power-sums are well known to be free commutative generators of Λ, we conclude that

$$\Lambda^* = \mathbb{Q}[\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots]\,.$$

The explanation of the constant term in (0.14) is the following. Ideally, we would like to define $\mathbf{p}_k$ by

$$\mathbf{p}_k \text{ ``=''} \sum_{i=1}^{\infty} (\lambda_i - i + \tfrac{1}{2})^k\,. \tag{0.15}$$

However, the above formula violates stability and diverges when evaluated at any partition λ. In particular, evaluation at the empty partition $\emptyset$ yields:

$$\mathbf{p}_k(\emptyset) \text{ ``=''} \sum_{i=1}^{\infty} (-i + \tfrac{1}{2})^k\,. \tag{0.16}$$

Definition (0.15) can be repaired by subtracting the infinite constant (0.16) inside the sum in (0.14) and compensating by adding the ζ-regularized value outside the sum.

The same regularization can be obtained in a more elementary fashion by summing the following generating series:

$$\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \frac{(-i+\frac{1}{2})^k z^k}{k!} = \sum_{i=1}^{\infty} e^{z(-i+\frac{1}{2})} = \frac{1}{z\,\mathcal{S}(z)}\,,$$

where, by definition,

$$\mathcal{S}(z) = \frac{\sinh(z/2)}{z/2} = \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}\,(2k+1)!}\,.$$

The coefficients c_i in the expansion,

$$\frac{1}{\mathcal{S}(z)} = \sum_{i=0}^{\infty} c_i z^i\,, \tag{0.17}$$

are essentially Bernoulli numbers. Since

$$(1-2^{-k})\,\zeta(-k) = k!\,c_{k+1}\,,$$

the two above regularizations are equivalent. The constants c_k will play an important role.

It is convenient to arrange the polynomials $\mathbf{p}_k$ into a generating function:

$$\mathbf{p}_k(\lambda) = k!\,[z^k]\,\mathbf{e}(\lambda, z)\,, \quad \mathbf{e}(\lambda, z) = \sum_{i=0}^{\infty} e^{z(\lambda_i - i + \frac{1}{2})}\,, \tag{0.18}$$

where $[z^k]$ denotes the coefficient of z^k in the expansion of the meromorphic function $\mathbf{e}(\lambda, z)$ in Laurent series about $z = 0$.

0.4.4. The function $\mathbf{f}_\mu(\lambda)$, arising in the character formulas for Hurwitz numbers, is shifted symmetric,

$$\mathbf{f}_\mu \in \Lambda^*,$$

a nontrivial result due to Kerov and Olshanski (see [23] and also [31], [33]). Moreover, the Fourier transform (0.12) is a linear isomorphism,

$$\phi : \bigoplus_{d=0}^{\infty} \mathcal{Z}(d) \ni C_\mu \mapsto \mathbf{f}_\mu \in \Lambda^* . \tag{0.19}$$

The identification of the highest degree term of $\mathbf{f}_\mu$ by Vershik and Kerov ([39], [23]) yields:

$$\mathbf{f}_\mu = \frac{1}{\prod \mu_i} \mathbf{p}_\mu + \dots , \tag{0.20}$$

where $\mathbf{p}_\mu = \prod \mathbf{p}_{\mu_i}$ and the dots stand for terms of degree lower than $|\mu|$.

The combinatorial interplay between the two mutually triangular linear bases $\{\mathbf{p}_\mu\}$ and $\{\mathbf{f}_\mu\}$ of Λ^* is a fundamental aspect of the algebra Λ^*. In fact, these two bases will define the GW/H correspondence.

Following [12], we define the *completed conjugacy classes* by

$$\overline{C}_\mu = \frac{1}{\prod_i \mu_i} \phi^{-1}(\mathbf{p}_\mu) \in \bigoplus_{d=0}^{|\mu|} \mathcal{Z}(d) .$$

Since the basis $\{\mathbf{p}_\mu\}$ is multiplicative, a special role is played by the classes

$$\overline{(k)} = \overline{C}_{(k)} , \quad k = 1, 2, \dots ,$$

which we call the *completed cycles*. The formulas for the first few completed cycles are:

$$\begin{aligned}
\overline{(1)} =& (1) - \frac{1}{24} \cdot () , \\
\overline{(2)} =& (2) , \\
\overline{(3)} =& (3) + (1,1) + \frac{1}{12} \cdot (1) + \frac{7}{2880} \cdot () , \\
\overline{(4)} =& (4) + 2 \cdot (2,1) + \frac{5}{4} \cdot (2) ,
\end{aligned}$$

where, for example,

$$(1,1) = C_{(1,1)} \in \mathcal{Z}(2) ,$$

is our shorthand notation for conjugacy classes.

Since $\mathbf{f}_\mu(\emptyset) = 0$ for any $\mu \neq \emptyset$, the coefficient of the empty partition,

$$() = C_\emptyset,$$

in $\overline{(k)}$ equals the constant term of $\frac{1}{k}\mathbf{p}_k$.

The *completion coefficients* $\rho_{k,\mu}$ determine the expansions of the completed cycles,

$$\overline{(k)} = \sum_{\mu} \rho_{k,\mu} \cdot (\mu) \,. \tag{0.21}$$

Formula (0.17) determining the constants,

$$\rho_{k,\emptyset} = (k-1)!\, c_{k+1} \,,$$

admits a generalization determining all the completion coefficients,

$$\rho_{k,\mu} = (k-1)! \frac{\prod \mu_i}{|\mu|!} \left[z^{k+1-|\mu|-\ell(\mu)}\right] \mathcal{S}(z)^{|\mu|-1} \prod \mathcal{S}(\mu_i z) \,, \tag{0.22}$$

where, as before, $[z^i]$ stands for the coefficient of z^i. Formula (0.22) will be derived in Section 3.2.4

The term *completed cycle* is appropriate as $\overline{(k)}$ is obtained from (k) by adding nonnegative multiples of conjugacy classes of strictly smaller size (with the possible exception of the constant term, which may be of either sign). The nonnegativity of $\rho_{k,\mu}$ for $\mu \neq \emptyset$ is clear from formula (0.22). Also, the coefficient $\rho_{k,\mu}$ vanishes unless the integer $k+1-|\mu|-\ell(\mu)$ is even and nonnegative.

We note the transposition (2) is the unique cycle with no corrections required for completion.

0.4.5. The term *completed cycle* was suggested in [12] when the functions $\mathbf{p}_k$ in [1], [11] were understood to count degenerations of Hurwitz coverings. The GW/H correspondence explains the geometric meaning of the completed cycles and, in particular, identifies the degenerate terms as contributions from the boundary of the moduli space of stable maps.

In fact, completed cycles implicitly penetrate much of the theory of shifted symmetric functions. While the algebra Λ^* has a very natural analog of the Schur functions (namely, the *shifted Schur functions*, studied in [31] and many subsequent papers), there are several competing candidates for the analog of the power-sum symmetric functions. The bases $\{\mathbf{f}_\mu\}$ and $\{\mathbf{p}_\mu\}$ are arguably the two finalists in this contest. The relationship between these two linear bases can be studied using various techniques; in particular, the methods of [31], [33], [24] can be applied.

0.5. *The* GW/H *correspondence.*

0.5.1. The GW/H correspondence may be stated symbolically as:

$$\boxed{\tau_k(\omega) = \frac{1}{k!}\,\overline{(k+1)}}\,. \tag{0.23}$$

That is, descendents of ω are equivalent to completed cycles.

Let X be a nonsingular target curve. The GW/H correspondence is the following relation between the disconnected Gromov-Witten and disconnected Hurwitz theories:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle_d^{\bullet X} = \frac{1}{\prod k_i!} \mathsf{H}_d^X \left(\overline{(k_1+1)}, \dots, \overline{(k_n+1)} \right) , \tag{0.24}$$

where the right-hand side is defined by linearity via the expansion of the completed cycles in ordinary conjugacy classes.

The GW/H correspondence, the completed cycle definition, and formula (0.10) together yield:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle_d^{\bullet X} = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^{2-2g(X)} \prod_{i=1}^{n} \frac{\mathbf{p}_{k_i+1}(\lambda)}{(k_i+1)!} . \tag{0.25}$$

For $g(X) = 0$ and 1, the right side can be expressed in the operator formalism of the infinite wedge $\Lambda^{\frac{\infty}{2}} V$ and explicitly evaluated, see Sections 3 and 5.

The GW/H correspondence naturally extends to relative Gromov-Witten theory; see Theorem 1. In the relative context, the GW/H correspondence provides an invertible rule for exchanging descendent insertions $\tau_k(\omega)$ for ramification conditions.

The coefficients $\rho_{k,\mu}$ are identified as connected 1-point Gromov-Witten invariants of $\mathbf{P}^1$ relative to $0 \in \mathbf{P}^1$. The explicit formula (0.22) for the coefficients is a particular case of the formula for 1-point connected GW invariants of $\mathbf{P}^1$ relative to $0, \infty \in \mathbf{P}^1$; see Theorem 2.

0.5.2. Let us illustrate the GW/H correspondence in the special case of maps of degree 0. In particular, we will see the role played by the constant terms in the definition of $\mathbf{p}_k$.

In the degree 0 case, the only partition λ in the sum (0.25) is the empty partition $\lambda = \emptyset$. Since, by definition,

$$\mathbf{p}_k(\emptyset) = k!\, c_{k+1} ,$$

the formula (0.25) yields

$$\left\langle \prod \tau_{k_i}(\omega) \right\rangle_0^{\bullet X} = \prod c_{k_i+2} .$$

The result is equivalent to the (geometrically obvious) vanishing of all multi-point connected invariants,

$$\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \rangle_0^{\circ X} = 0 , \quad n > 1 ,$$

together with the following evaluation of the connected degree 0, 1-point function,

$$1+\sum_{g=1}^{\infty} \langle \tau_{2g-2}(\omega)\rangle^{\circ X}_{g,0} \; z^{2g} = \frac{1}{\mathcal{S}(z)}\,. \tag{0.26}$$

And, indeed, the result is correct; see [13], [34] .

0.5.3. A useful convention is to formally set the contribution $\langle \tau_{-2}(\omega)\rangle^{\bullet X}_{0,0}$ of the unstable moduli space $\overline{M}_{0,1}(X,0)$ to equal 1,

$$\langle \tau_{-2}(\omega)\rangle^{\bullet X}_{0,0} = 1\,. \tag{0.27}$$

This convention simplifies the form of the generating function (0.26) and several others functions in the paper. In the disconnected theory, the unstable contribution (0.27) is allowed to appear in any degree and genus. Hence, in the disconnected theory, the convention is equivalent to setting

$$\tau_{-2}(\omega) = 1\,. \tag{0.28}$$

The parallel convention for the completed cycles

$$\mathbf{p}_0 = 0\,, \quad \tfrac{1}{(-1)!}\,\mathbf{p}_{-1} = 1$$

fits well with the formula (0.18) .

0.6. *Plan of the paper.*

0.6.1. A geometric study of descendent integrals concluding with a proof of the GW/H correspondence in the context of relative Gromov-Witten theory is presented in Section 1. The GW/H correspondence is Theorem 1. A special case of GW/H correspondence is assumed in the proof. The special case, the GW/H correspondence for the absolute Gromov-Witten theory of $\mathbf{P}^1$, will be established by equivariant computations in [32].

Relative Gromov-Witten theory is discussed in Section 1.2. The completion coefficients (0.21) are identified in Section 1.7 as 1-point Gromov-Witten invariants of $\mathbf{P}^1$ relative to $0 \in \mathbf{P}^1$.

0.6.2. The remainder of the paper deals with applications of the GW/H correspondence. In particular, generating functions for the stationary Gromov-Witten invariants of targets of genus 0 and 1 are evaluated. These computations are most naturally executed in the infinite wedge formalism. We review the infinite representation $\Lambda^{\frac{\infty}{2}}V$ in Section 2. The formalism also provides a convenient and powerful approach to the study of integrable hierarchies; see for example [20], [28], [35].

The stationary GW theory of $\mathbf{P}^1$ relative to $0, \infty \in \mathbf{P}^1$ is considered in Section 3. We obtain a closed formula for the corresponding 1-point function in Theorem 2. The formula (0.22) for the completion coefficients is obtained as a special case. A generalization of Theorem 2 for the n-point function is given in Theorem 3.

0.6.3. The 2-Toda hierarchy governing the Gromov-Witten theory of $\mathbf{P}^1$ relative to $\{0, \infty\} \subset \mathbf{P}^1$ is discussed in Section 4. The main result is Theorem 4 which states that the natural generating function for relative GW-invariants is a τ-function of the 2-Toda hierarchy of Ueno and Takasaki [38]. Theorem 4 generalizes a result of [30].

The flows of the Toda hierarchy are associated to the ramification conditions μ and ν imposed at $\{0, \infty\}$. The equations of the Toda hierarchy are equivalent to certain recurrence relations for relative Gromov-Witten invariants, the simplest of which is made explicit in Proposition 4.3.

0.6.4. The Gromov-Witten theory of $\mathbf{P}^1$ was conjectured to be governed by the Toda equation by Eguchi and Yang [8], and also by Dubrovin [5]. The Toda conjecture was further studied in in [6], [7], [16], [30], [34].

The Toda conjecture naturally extends to the $\mathbb{C}^\times$-equivariant Gromov-Witten theory of $\mathbf{P}^1$. We will prove in [32] that the equivariant theory of $\mathbf{P}^1$ is governed by an integrable hierarchy which can also be identified with the 2-Toda of [38]. The flows in the equivariant 2-Toda correspond to the insertions of $\tau_k([0])$ and $\tau_k([\infty])$, where

$$[0], [\infty] \in H^*_{\mathbb{C}^\times}(\mathbf{P}^1, Q)\,,$$

are the classes of the torus fixed points.

The equivariant 2-Toda hierarchy is *different* from the relative 2-Toda studied here. However, the lowest equations of both hierarchies agree on their common domain of applicability.

0.6.5. In Section 5, we discuss the stationary Gromov-Witten theory of an elliptic curve E. The GW/H correspondence identifies the n-point function of Gromov-Witten invariants of E with the character of the infinite wedge representation of $\mathfrak{gl}(\infty)$. This character has been previously computed in [1], see also [29], [11]. We quote the results of [1] here and briefly discuss some of their implications, in particular, the appearance of quasimodular forms.

While the GW/H correspondence is valid for all nonsingular target curves X, we do not know closed form evaluations for targets of genus $g(X) \geq 2$. The targets $\mathbf{P}^1$ and E yield very beautiful theories. Perhaps the study of the Gromov-Witten theory of higher genus targets will lead to the discovery of new structures.

0.7. *Acknowledgments.* An important impulse for this work came from the results of [12] and, more generally, from the line of research pursued in

[11], [12]. Our interaction with S. Bloch, A. Eskin, and A. Zorich played a very significant role in the development of the ideas presented here.

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1. The geometry of descendents

1.1. *Motivation*: *nondegenerate maps.* We begin by examining the relation between Gromov-Witten and Hurwitz theory in the context of nondegenerate maps with nonsingular domains.

Let $M^\bullet_{g,n}(X,d) \subset \overline{M}^\bullet_{g,n}(X,d)$ be the open locus of maps,

$$\pi : (C, p_1, \ldots, p_n) \to X,$$

where each connected component $C_i \subset C$ is nonsingular and dominates X.

Let $q_1, \ldots, q_n \in X$ be distinct points. Define the closed substack V by:

$$V = \mathrm{ev}_1^{-1}(q_1) \cap \cdots \cap \mathrm{ev}_n^{-1}(q_n) \subset M^\bullet_{g,n}(X,d)\,.$$

The stacks $M^\bullet_{g,n}(X,d)$ and V are nonsingular Deligne-Mumford stacks of the expected dimensions — see [14] for proofs.

The Hurwitz number $\mathsf{H}^X_d((k_1+1), \ldots, (k_n+1))$ may be defined by the enumeration of pointed Hurwitz covers

$$\pi : (C, p_1, \ldots, p_n) \to (X, q_1, \ldots, q_n)\,,$$

where

(i) $\pi(p_i) = q_i$,

(ii) π has ramification order k_i at p_i.

Here, π has *ramification order* k at p if π takes the local form $z \to z^{k+1}$ at p_i. The count of pointed Hurwitz covers is weighted by $1/|\mathrm{Aut}(\pi)|$ where $\mathrm{Aut}(\pi)$ is the automorphism group of the *pointed* cover.

The above enumeration of pointed covers coincides with the definition of $\mathsf{H}^X_d((k_1+1), \ldots, (k_n+1))$ given in Section 0.3.

Proposition 1.1. *Let $d > 0$. The algebraic cycle class*,

$$\Big(\prod_{i=1}^n k_i!\, c_1(L_i)^{k_i}\, \mathrm{ev}_i^*(\omega)\Big) \cap [M^\bullet_{g,n}(X,d)] \in A_0(M^\bullet_{g,n}(X,d)),$$

is represented by the locus of covers enumerated by $\mathsf{H}^X_d((k_1+1), \ldots, (k_n+1))$.

Proof. Since V represents $\prod_{i=1}^n \mathrm{ev}_i^*(\omega)$ in the Chow theory of $M^\bullet_{g,n}(X,d)$, we may prove that the locus of Hurwitz covers represents

$$\prod_{i=1}^n k_i!\, c_1(L_i)^{k_i} \ \cap [V]$$

in the Chow theory of V.

First, consider the marked point p_1. There exists a canonical section $s \in H^0(V, L_1)$ obtained from π by the following construction. Let π^* denote the pull-back map on functions:

$$\pi^* : m_{q_1}/m_{q_1}^2 \to m_{p_1}/m_{p_1}^2, \tag{1.1}$$

where m_{q_1}, m_{p_1} are the maximal ideals of the points $q_1 \in X$ and $p_1 \in C$ respectively. Via the canonical isomorphisms,

$$m_{q_1}/m_{q_1}^2 \stackrel{\sim}{=} T^*_{q_1}(X), \quad m_{p_1}/m_{p_1}^2 \stackrel{\sim}{=} T^*_{p_1}(C),$$

the map (1.1) is the dual of the differential of π. Since q_1 is fixed, the identification $m_{q_1}/m_{q_1}^2 \stackrel{\sim}{=} \mathbb{C}$ yields a section s of L_1 by (1.1).

The scheme theoretic zero locus $Z(s) \subset V$ is easily seen to be the (reduced) substack of maps where p_1 has ramification order at least 1 over q_1. The cycle $Z(s)$ represents $c_1(L_1) \cap [V]$ in the Chow theory of V.

When restricted to $Z(s)$, the pull-back of functions yields a map:

$$\pi^* : m_{q_1}/m_{q_1}^2 \to m_{p_1}^2/m_{p_1}^3.$$

Hence, via the isomorphisms,

$$m_{q_1}/m_{q_1}^2 \stackrel{\sim}{=} \mathbb{C}, \quad m_{p_1}^2/m_{p_1}^3 \stackrel{\sim}{=} L_1^{\otimes 2},$$

a canonical section $s' \in H^0(Z(s), L_1^{\otimes 2})$ is obtained. A direct scheme theoretic verification shows that $Z(s') \subset Z(s)$ is the (reduced) substack where p_1 has ramification order at least 2 over q. Hence the cycle $Z(s')$ represents the cycle class $2c_1(L_1)^2$.

After iterating the above construction, we find that $k_1!\, c_1(L_1)^{k_1}$ is represented by the substack where p_1 has ramification order at least k_1. At each stage, the reducedness of the zero locus is obtained by a check in the versal deformation space of the ramified map (the issue of reducedness is local).

Since the cycles determined by ramification conditions at distinct markings p_i are transverse, we conclude that $\prod_{i=1}^n k_i!\, c_1(L_i)^{k_i} \cap [V]$ is represented by the locus of Hurwitz covers enumerated by $\mathsf{H}^X_d((k_1+1), \ldots, (k_n+1))$. □

Proposition 1.1 shows a connection between descendent classes and Hurwitz covers for the *open* moduli space $M^\bullet_{g,n}(X,d)$. We therefore expect a geometric formula:

$$\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \rangle^{\bullet X}_d = \frac{\mathsf{H}^X_d((k_1+1), \ldots, (k_n+1))}{\prod k_i!} + \Delta, \tag{1.2}$$

where Δ is a correction term obtained from the boundary,

$$\overline{M}^{\bullet}_{g,n}(X,d) \setminus M^{\bullet}_{g,n}(X,d).$$

The GW/H correspondence gives a description of this correction term Δ.

For example, consider the case where $k_i = 1$ for all i. Then, since 2-cycles are already complete (see Section 0.4), the basic GW/H correspondence (0.24) yields an exact equality,

$$\langle \tau_1(\omega) \cdots \tau_1(\omega) \rangle^{\bullet X}_{d} = \mathsf{H}^X_d((2), \ldots, (2)), \tag{1.3}$$

which appears in [34]. However, the correction term Δ will not vanish in general.

We note that Proposition 1.1 holds for the connected moduli of maps and connected Hurwitz numbers by the same proof. Since the disconnected case will be more natural for the study of the correction equation (1.2), the results have been stated in the disconnected case.

1.2. *Relative Gromov-Witten theory.* We will study the GW/H correspondence in the richer context of the Gromov-Witten theory of X relative to a finite set of distinct points $q_1, \ldots, q_m \in X$. Let $\eta^1, \ldots, \eta^m$ be partitions of d. The moduli space

$$\overline{M}_{g,n}(X, \eta^1, \ldots, \eta^m)$$

parametrizes genus g, n-pointed relative stable maps with monodromy η^i at q_i. Foundational developments of relative Gromov-Witten theory in symplectic and algebraic geometry can be found in [9], [18], [25], [26]. The stationary sector of the relative Gromov-Witten theory is:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega), \eta^1, \ldots, \eta^m \right\rangle^{\circ X}_{g,d} = \int_{[\overline{M}_{g,n}(X,\eta^1,\ldots,\eta^m)]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{k_i} \, \mathrm{ev}_i^*(\omega), \tag{1.4}$$

the integrals of descendents of ω relative to $q_1, \ldots, q_m \in X$.

The genus *and* the degree may be omitted in the notation (1.4) as long as $m > 0$. Again, the corresponding disconnected theory is denoted by the brackets $\langle\ \rangle^{\bullet}$.

The stationary theory relative to $q_1, \ldots, q_m$ specializes to the stationary theory relative to $q_1, \ldots, q_{m-1}$ when η^m is the trivial partition (1^d). In particular, when all the partitions η^i are trivial, the standard stationary theory of X is recovered. A proof of this specialization property is obtained from the degeneration formula discussed in Section 1.3 below.

The stationary Gromov-Witten theory of $\mathbf{P}^1$ relative to $0, \infty \in \mathbf{P}^1$ will play a special role. Let μ, ν be partitions of d prescribing the profiles over

$0, \infty \in \mathbf{P}^1$ respectively. We will use the notation,

$$\left\langle \mu, \prod \tau_{k_i}(\omega), \nu \right\rangle^{\mathbf{P}^1}, \tag{1.5}$$

to denote integrals in the stationary theory of $\mathbf{P}^1$ relative to $0, \infty \in \mathbf{P}^1$.

1.3. *Degeneration.* The degeneration formula for relative Gromov-Witten theory provides a formal approach to the descendent integrals

$$\left\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega), \eta^1, \ldots, \eta^m \right\rangle_d^{\bullet X}.$$

Let $x_1, \ldots, x_n \in X$ be distinct fixed points. Consider a family of curves with n sections over the affine line,

$$\pi : (\mathcal{X}, s_1, \ldots, s_n) \to \mathbb{A}^1,$$

defined by the following properties:

(i) $(\mathcal{X}_t, s_1(t), \ldots, s_n(t))$ is isomorphic to the fixed data $(X, x_1, \ldots, x_n)$ for all $t \neq 0$.

(ii) $(\mathcal{X}_0, s_1(0), \ldots, s_n(0))$ is a comb consisting of $n+1$ components (1 backbone isomorphic to X and n teeth isomorphic to $\mathbf{P}^1$). The teeth are attached to the points $x_1, \ldots, x_n$ of the backbone. The section $s_i(0)$ lies on the i^{th} tooth.

The degeneration π can be easily constructed by blowing-up the n points $(x_i, 0)$ of the trivial family $X \times \mathbb{A}^1$.

The following result is obtained by viewing the family π as a degeneration of the target in relative Gromov-Witten theory.

PROPOSITION 1.2 ([9], [18], [25], [26]). *A degeneration formula holds for relative Gromov-Witten invariants*:

$$\begin{aligned} &\left\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega), \eta^1, \ldots, \eta^m \right\rangle_d^{\bullet X} \\ &\quad = \sum_{|\mu^1|, \ldots, |\mu^n| = d} \mathsf{H}_d^X(\mu^1, \ldots, \mu^n, \eta^1, \ldots, \eta^m) \prod_{i=1}^n \mathfrak{z}(\mu^i) \left\langle \mu^i, \tau_{k_i}(\omega) \right\rangle^{\bullet \mathbf{P}^1}, \end{aligned} \tag{1.6}$$

where the sum is over all n-tuples $\mu^1, \ldots, \mu^n$ of partitions of d.

Here, the factor $\mathfrak{z}(\mu)$ is defined by:

$$\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)| \prod_{i=1}^{\ell(\mu)} \mu_i$$

where $\mathrm{Aut}(\mu) \cong \prod_{i \geq 1} S(m_i(\mu))$ is the symmetry group permuting equal parts of μ. The factor $\mathfrak{z}(\mu)$ will occur often.

The right side of the degeneration formula (1.6) involves the Hurwitz numbers and 1-point stationary Gromov-Witten invariants of $\mathbf{P}^1$ relative to $0 \in \mathbf{P}^1$. The degeneration formula together with the definition of the Hurwitz numbers implies the specialization property of relative Gromov-Witten invariants when $\eta^m = (1^d)$.

There exists an elementary analog of this degeneration formula in Hurwitz theory which yields:

$$\begin{aligned}(1.7)\quad & \mathsf{H}_d^X\left(\overline{(k_1)}, \ldots, \overline{(k_n)}, \eta^1, \ldots, \eta^m\right) \\ & = \sum_{|\mu^1|, \ldots, |\mu^n| = d} \mathsf{H}_d^X\left(\mu^1, \ldots, \mu^n, \eta^1, \ldots, \eta^m\right) \prod_{i=1}^{n} \mathfrak{z}(\mu^i)\, \mathsf{H}_d^{\mathbf{P}^1}\left(\mu^i, \overline{(k_i)}\right),\end{aligned}$$

where the sum is again over partitions μ^i of d.

1.4. *The abstract* GW/H *correspondence.* Formula (1.6) can be restated as a substitution rule valid in degree d:

$$\tau_k(\omega) = \sum_{|\mu|=d} \left(\mathfrak{z}(\mu)\ \langle \mu, \tau_k(\omega)\rangle^{\bullet \mathbf{P}^1}\right) \cdot (\mu)\,. \tag{1.8}$$

The substitution rule replaces the descendents $\tau_k(\omega)$ by ramification conditions in Hurwitz theory:

$$\left\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega), \eta^1, \ldots, \eta^m \right\rangle_d^{\bullet X} = \mathsf{H}_d^X\left(-, \ldots, -, \eta^1, \ldots, \eta^m\right).$$

Hurwitz numbers on the right side are defined by inserting the respective ramification conditions (1.8) and expanding multilinearly. The substitution rule, however, is degree dependent by definition.

A degree independent substitution rule is obtained by studying the *connected* relative invariants. Disconnected invariants may be expressed as sums of products of connected invariants obtained by all possible decompositions of the domain and distributions of the integrand. As the invariant $\langle \mu, \tau_k(\omega)\rangle^{\bullet \mathbf{P}^1}$ has a single term in the integrand and

$$\langle \nu \rangle^{\circ \mathbf{P}^1} = \frac{\delta_{\nu, 1^{|\nu|}}}{|\nu|!}$$

it follows that

$$\langle \mu, \tau_k(\omega)\rangle^{\bullet \mathbf{P}^1} = \sum_{i=0}^{m_1(\mu)} \frac{1}{i!} \left\langle \mu - 1^i, \tau_k(\omega)\right\rangle^{\circ \mathbf{P}^1}, \tag{1.9}$$

where $\mu - 1^i$ denotes the partition μ with i parts equal to 1 removed. Since

$$\frac{\mathfrak{z}(\mu)}{i!} = \binom{m_1(\mu)}{i} \mathfrak{z}(\mu - 1^i)\,,$$

we may rewrite (1.9) as:

$$\mathfrak{z}(\mu)\,\langle \mu, \tau_k(\omega)\rangle^{\bullet \mathbf{P}^1} = \sum_{i=0}^{m_1(\mu)} \binom{m_1(\mu)}{i} \mathfrak{z}(\mu - 1^i)\,\left\langle \mu - 1^i, \tau_k(\omega)\right\rangle^{\circ \mathbf{P}^1}.$$

The following result is then obtained from the definition of the extended Hurwitz numbers (0.5).

Proposition 1.3. *A substitution rule for converting descendents to ramification conditions holds*:

$$\tau_k(\omega) = \sum_\nu \left(\mathfrak{z}(\nu)\ \langle \nu, \tau_k(\omega)\rangle^{\circ \mathbf{P}^1}\right) \cdot (\nu)\,, \tag{1.10}$$

where the summation is over all *partitions* ν*.*

Proposition 1.3 is a degree independent, abstract form of the GW/H correspondence. Clearly, only partitions ν of size at most d contribute to the degree d invariants. What remains is the explicit identification of the coefficients in (1.10).

1.5. *The leading term.* Equating the dimension of the integrand in $\langle \nu, \tau_k(\omega)\rangle^{\circ \mathbf{P}^1}$ with the virtual dimension of the moduli space, we obtain

$$k+1 = 2g - 1 + |\nu| + \ell(\nu)\,.$$

Since $g \geq 0$ and $\ell(\nu) \geq 1$, we find

$$|\nu| \leq k+1\,.$$

Moreover, $\nu = (k+1)$ is the only partition of size $k+1$ which actually appears in (1.10). All other partitions ν appearing in (1.10) have a strictly smaller size.

We will now determine the coefficient of $\nu = (k+1)$ in (1.10) by the method of Proposition 1.1. The corresponding relative invariant is computed in the following lemma.

Lemma 1.4. *For* $d > 0$,

$$\langle (d), \tau_{d-1}(\omega)\rangle^{\mathbf{P}^1} = \frac{1}{d!}\,.$$

Proof. We first note that the connected and disconnected invariants coincide,

$$\langle (d), \tau_{d-1}(\omega)\rangle^{\circ \mathbf{P}^1} = \langle (d), \tau_{d-1}(\omega)\rangle^{\bullet \mathbf{P}^1}\,,$$

since the imposed monodromy is transitive. The genus of the domain is 0 by the dimension constraint.

Let $[\pi] \in \overline{M}_{0,1}(\mathbf{P}^1, (d))$ be a stable map relative to $0 \in \mathbf{P}^1$,

$$\pi : (C, p_1) \to T \to \mathbf{P}^1,$$

where T is a destabilization of $\mathbf{P}^1$ at 0 and $\pi(p_1) = \infty \in \mathbf{P}^1$. If p_1 lies on a π-contracted component $C_1 \subset C$ then,

(i) C_1 must meet $\overline{C \setminus C_1}$ in at least two points by stability,

(ii) $\overline{C \setminus C_1}$ must be connected by the imposed monodromy at 0.

Since conditions (i) and (ii) violate the genus constraint $g(C) = 0$, the marked point p_1 is *not* allowed to lie on a π-contracted component of C.

The moduli space $\overline{M}_{0,1}(\mathbf{P}^1, (d))$ is of expected dimension d. By Proposition 1.1 pursued for relative maps, the cycle

$$(d-1)!\, c_1(L_1)^{d-1} \operatorname{ev}_1^*(\omega) \cap [M_{0,1}(\mathbf{P}^1, (d))] \in A_0(M_{0,1}(\mathbf{P}^1, (d)))$$

is represented by the locus of covers enumerated by $\mathsf{H}_{0,d}((d),(d))$.

In fact, since p_1 does not lie on a π-contracted component of the domain for any moduli point $[\pi] \in \operatorname{ev}_1^{-1}(\infty) \subset \overline{M}_{0,1}(\mathbf{P}^1, (d))$, the proof of Proposition 1.1 is valid for the compact moduli space. The cycle

$$(d-1)!\, c_1(L_1)^{d-1} \operatorname{ev}_1^*(\omega) \cap [\overline{M}_{0,1}(\mathbf{P}^1, (d))] \in A_0(\overline{M}_{0,1}(\mathbf{P}^1, (d)))$$

is represented by the locus of covers enumerated by $\mathsf{H}_{0,d}((d),(d))$.

There is a unique cover $[\zeta]$ enumerated by $\mathsf{H}_{0,d}((d),(d))$. We may now complete the calculation:

$$\begin{aligned} \langle (d), \tau_{d-1}(\omega) \rangle^{\bullet \mathbf{P}^1} &= \int_{[\overline{M}_{0,1}(\mathbf{P}^1,(d))]} c_1(L_1)^{d-1} \operatorname{ev}_1^*(\omega) \\ &= \frac{1}{(d-1)!} \int_{[\zeta]} 1 \\ &= \frac{1}{d!}, \end{aligned}$$

since $[\zeta]$ is a cyclic Galois cover with automorphism group of order d. $\square$

Lemma 1.4 provides an identification of the leading term in the abstract GW/H correspondence (1.10).

Corollary 1.5.

$$\tau_k(\omega) = \frac{1}{k!}\,(k+1) + \dots\,, \tag{1.11}$$

where the dots stand for conjugacy classes (ν) *with* $|\nu| < k+1$.

1.6. *The full* GW/H *correspondence.* Let X be a nonsingular curve. The main result of the paper is a substitution rule for the relative Gromov-Witten theory of X.

THEOREM 1. *A substitution rule for converting descendents to ramification conditions holds*:

$$\tau_k(\omega) = \frac{1}{k!}\,\overline{(k+1)}\,. \tag{1.12}$$

The full correspondence for the relative theory yields:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega), \eta^1, \dots, \eta^m \right\rangle_d^{\bullet X} = \frac{1}{\prod k_i!}\,\mathsf{H}_d^X\left(\overline{(k_1+1)}, \dots, \overline{(k_n+1)}, \eta^1, \dots, \eta^m\right).$$

Our proof of Theorem 1 will rely upon a special case — the case of the absolute Gromov-Witten theory of $\mathbf{P}^1$. The formula,

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle_d^{\bullet \mathbf{P}^1} = \frac{1}{\prod k_i!}\,\mathsf{H}_d^{\mathbf{P}^1}\left(\overline{(k_1+1)}, \dots, \overline{(k_n+1)}\right), \tag{1.13}$$

will be proven in [32] as a result of equivariant computations. We will now deduce the general statement (1.12) from (1.13).

Proof. Let $\frac{1}{k!}\widetilde{(k+1)}$ denote the right side of the equality (1.10),

$$\frac{1}{k!}\widetilde{(k+1)} = \sum_\nu \left(\mathfrak{z}(\nu)\ \langle \nu, \tau_k(\omega)\rangle^{\circ \mathbf{P}^1}\right) \cdot (\nu)\,.$$

Define $\widetilde{\mathbf{p}}_k$ by the the Fourier transform (0.19),

$$\phi\left(\widetilde{(k)}\right) = \frac{1}{k}\widetilde{\mathbf{p}}_k.$$

The equality (1.12) is equivalent to the equality

$$\widetilde{\mathbf{p}}_k \stackrel{?}{=} \mathbf{p}_k\,. \tag{1.14}$$

As a result of (1.11), we find:

$$\widetilde{\mathbf{p}}_\mu = \mathbf{p}_\mu + \dots\,, \tag{1.15}$$

where $\widetilde{\mathbf{p}}_\mu = \prod \widetilde{\mathbf{p}}_{\mu_i}$ and the dots stand for lower degree terms. In other words, the transition matrix between the bases $\{\widetilde{\mathbf{p}}_\mu\}$ and $\{\mathbf{p}_\mu\}$ is unitriangular.

Let l be the following linear form on the algebra Λ^*:

$$l(\mathbf{f}) = \sum_\lambda \left(\frac{\dim \lambda}{|\lambda|!}\right)^2 \mathbf{f}(\lambda)\,.$$

This series obviously converges for any polynomial $\mathbf{f}$. For example, $l(1) = e$.

The associated quadratic form,

$$(\mathbf{f},\mathbf{g}) \mapsto l(\mathbf{fg})\,, \tag{1.16}$$

is, clearly, positive definite.

Formula (0.8), formula (1.13), and the definitions of the functions $\widetilde{\mathbf{p}}_\mu, \mathbf{p}_\mu$ yield the equality,

$$l(\widetilde{\mathbf{p}}_\mu) = l(\mathbf{p}_\mu)\,,$$

for all μ. In particular, we find

$$l(\widetilde{\mathbf{p}}_\mu \cdot \widetilde{\mathbf{p}}_\nu) = l(\mathbf{p}_\mu \cdot \mathbf{p}_\nu)\,,$$

for all μ and ν. The transition matrix between the bases $\{\widetilde{\mathbf{p}}_\mu\}$ and $\{\mathbf{p}_\mu\}$ is therefore orthogonal with respect to the positive definite quadratic form (1.16). By (1.15), the transition matrix is also unitriangular. Hence, the transition is the identity and equality is established in (1.14). □

1.7. *Completion coefficients.* Theorem 1 together with a comparison of the formulas (1.10) and (0.21) yields the following result.

PROPOSITION 1.6. *The completion coefficients satisfy*:

$$\frac{\rho_{k+1,\mu}}{k!} = \mathfrak{z}(\mu)\ \left\langle \mu, \tau_k(\omega) \right\rangle^{\circ \mathbf{P}^1}\ . \tag{1.17}$$

In other words, the coefficients $\rho_{k,\mu}$ are determined by connected relative 1*-point Gromov-Witten invariants of* $\mathbf{P}^1$ *relative to* $0 \in \mathbf{P}^1$.

We will perform the actual computation of these completion coefficients in Section 3, using the operator formalism reviewed in Section 2. An explicit formula for the completion coefficients will be given in Proposition 3.2.

2. The operator formalism

The fermionic Fock space formalism reviewed here is a convenient tool for manipulating the sums (0.25). The operator calculus of the formalism is basic to the rest of the paper. In Sections 3 and 5, the formalism is applied to the Gromov-Witten theory of targets of genus 0 and 1 respectively. The formalism underlies the study of the Toda hierarchy in Section 4.

2.1. *The infinite wedge.*

2.1.1. Let V be a linear space with basis $\{\underline{k}\}$ indexed by the half-integers:

$$V = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\,\underline{k}.$$

For each subset $S = \{s_1 > s_2 > s_3 > \dots\} \subset \mathbb{Z} + \frac{1}{2}$ satisfying:

(i) $S_+ = S \setminus \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right)$ is finite,

(ii) $S_- = \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \setminus S$ is finite,

we denote by v_S the following infinite wedge product:

$$v_S = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \dots \,. \tag{2.1}$$

By definition,

$$\Lambda^{\frac{\infty}{2}} V = \bigoplus \mathbb{C}\, v_S$$

is the linear space with basis $\{v_S\}$. Let $(\,\cdot\,,\,\cdot\,)$ be the inner product on $\Lambda^{\frac{\infty}{2}} V$ for which $\{v_S\}$ is an orthonormal basis.

2.1.2. The fermionic operator ψ_k on $\Lambda^{\frac{\infty}{2}} V$ is defined by wedge product with the vector $\underline{k}$,

$$\psi_k \cdot v = \underline{k} \wedge v\,.$$

The operator ψ_k^* is defined as the adjoint of ψ_k with respect to the inner product $(\,\cdot\,,\,\cdot\,)$.

These operators satisfy the canonical anti-commutation relations:

$$\psi_i \psi_j^* + \psi_i^* \psi_j = \delta_{ij}\,, \tag{2.2}$$

$$\psi_i \psi_j + \psi_j \psi_1 = \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0. \tag{2.3}$$

The *normally ordered* products are defined by:

$$:\psi_i\, \psi_j^*: = \begin{cases} \psi_i\, \psi_j^*\,, & j > 0\,, \\ -\psi_j^*\, \psi_i\,, & j < 0\,. \end{cases} \tag{2.4}$$

2.1.3. Let E_{ij}, for $i, j \in \mathbb{Z} + \frac{1}{2}$, be the standard basis of matrix units of $\mathfrak{gl}(\infty)$. The assignment

$$E_{ij} \mapsto :\psi_i\, \psi_j^*: \;\;,$$

defines a projective representation of the Lie algebra $\mathfrak{gl}(\infty) = \mathfrak{gl}(V)$ on $\Lambda^{\frac{\infty}{2}} V$.

Normal ordering is introduced to avoid the infinite constants which appear in the naive definition of the $\mathfrak{gl}(\infty)$-action on $\Lambda^{\frac{\infty}{2}} V$. The ordering and divergence issues here are closely related to the discussion in Section 0.4.3.

For example, the action on $\Lambda^{\frac{\infty}{2}} V$ of the identity matrix in $\mathfrak{gl}(\infty)$ is well-defined only after normal ordering. Indeed, the operator,

$$C = \sum_{k \in \mathbb{Z} + \frac{1}{2}} E_{kk},$$

corresponding to the identity matrix, acts on the basis v_S by:

$$C\, v_S = (|S_+| - |S_-|) v_S\,.$$

The operator C is known as the *charge* operator.[1] The kernel of C, the zero charge subspace, is spanned by the vectors

$$v_\lambda = \underline{\lambda_1 - \tfrac{1}{2}} \wedge \underline{\lambda_2 - \tfrac{3}{2}} \wedge \underline{\lambda_3 - \tfrac{5}{2}} \wedge \dots$$

indexed by all partitions λ. We will denote the kernel by $\Lambda_0^{\frac{\infty}{2}} V$.

The operator

$$H = \sum_{k \in \mathbb{Z}+\frac{1}{2}} k\, E_{kk}$$

is called the *energy operator*. The eigenvalues of H on $\Lambda_0^{\frac{\infty}{2}} V$ are easily identified:

$$H\, v_\lambda = |\lambda|\, v_\lambda\,.$$

The vacuum vector

$$v_\emptyset = \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \dots$$

is the unique vector with the minimal (zero) eigenvalue of H.

2.1.4. Define the translation operator T by:

$$T\, \underline{k_1} \wedge \underline{k_2} \wedge \underline{k_3} \wedge \dots = \underline{k_1+1} \wedge \underline{k_2+1} \wedge \underline{k_3+1} \wedge \dots\,. \tag{2.5}$$

We see,

$$T\, \psi_k\, T^{-1} = \psi_{k+1}\,, \quad T\, \psi_k^*\, T^{-1} = \psi_{k+1}^*\,.$$

We also find,

$$T^{-1}\, C\, T = C + 1\,.$$

Hence, T increases the charge by 1.

2.2. *Operators $\mathcal{E}$.*

2.2.1. The operator $\mathcal{E}_0(z)$ on $\Lambda^{\frac{\infty}{2}} V$ is defined by:

$$\mathcal{E}_0(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{zk}\, E_{kk} + \frac{1}{e^{z/2} - e^{-z/2}}\,, \tag{2.6}$$

where the second term is a scalar operator on $\Lambda^{\frac{\infty}{2}} V$. In fact, the scalar term in (2.6) and the the constant term in (0.14) have the same origin.

Ideally, we would like $\mathcal{E}_0(z)$ to be the naive action on $\Lambda^{\frac{\infty}{2}} V$ of the following diagonal operator in $\mathfrak{gl}(\infty)$:

$$\underline{k} \mapsto e^{zk}\, \underline{k}\,.$$

[1]The infinite wedge space is the mathematical formalization of Dirac's idea of a sea of fermions filling all but finitely many negative energy levels. The operator C measures the difference between the number $|S_+|$ of occupied positive energy levels (particles) and the number $|S_-|$ of vacant negative energy levels (holes), whence the name.

In other words, we would like to set

$$\mathcal{E}_0(z) \text{ “=” } \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk}\,\psi_k\,\psi_k^*\,, \tag{2.7}$$

without normal ordering. However, applied to the vacuum, definition (2.7) yields:

$$\sum_{k=-\frac{1}{2},-\frac{3}{2},\dots} e^{zk} = \frac{1}{e^{z/2}-e^{-z/2}}\,, \quad \Re z > 0\,, \tag{2.8}$$

which may or may not make sense depending on z. We therefore define $\mathcal{E}_0(z)$ using the normal ordering and then compensate by adding the scalar (2.8) by hand.

In particular, we observe

$$\frac{1}{e^{z/2}-e^{-z/2}} = \mathbf{e}(\emptyset, z)\,,$$

where the function

$$\mathbf{e}(\lambda, z) = \sum_{i=0}^{\infty} e^{z(\lambda_i - i + \frac{1}{2})}$$

is as defined in (0.18). More generally, we find

$$\mathcal{E}_0(z)\,v_\lambda = \mathbf{e}(\lambda, z)\,v_\lambda\,. \tag{2.9}$$

In other words, the functions $\mathbf{e}(\lambda, z)$ are the eigenvalues of the operator $\mathcal{E}_0(z)$.

2.2.2. Define the operators $\mathcal{P}_k$ for $k > 0$ by:

$$\mathcal{P}_k = k!\,[z^k]\,\mathcal{E}_0(z)\,, \tag{2.10}$$

where $[z^k]$ stands for the coefficient of z^k. From (0.18) and (2.9) we conclude:

$$\mathcal{P}_k\,v_\lambda = \mathbf{p}_k(\lambda)\,v_\lambda\,. \tag{2.11}$$

In particular, we find

$$\mathcal{P}_1 = H - \tfrac{1}{24}\,.$$

The definition of the operators $\mathcal{P}_k$ is naturally extended as follows:

$$\mathcal{P}_0 = C\,, \quad \tfrac{1}{(-1)!}\,\mathcal{P}_{-1} = 1\,. \tag{2.12}$$

The extension is related to convention (0.28).

2.2.3. The translation operator T acts on the operator $\mathcal{E}_0(z)$ by

$$T^{-1}\,\mathcal{E}_0(z)\,T = e^z\,\mathcal{E}_0(z)\,. \tag{2.13}$$

We find

$$T^{-1}\,\frac{\mathcal{P}_k}{k!}\,T=\sum_{m=0}^{k+1}\frac{1}{m!}\,\frac{\mathcal{P}_{k-m}}{(k-m)!}\,, \tag{2.14}$$

using convention (2.12) .

2.2.4. For any $r\in\mathbb{Z}$, we define

$$\mathcal{E}_r(z)=\sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{r}{2})}\,E_{k-r,k}+\frac{\delta_{r,0}}{\varsigma(z)}\,, \tag{2.15}$$

where the function $\varsigma(z)$ is defined by

$$\varsigma(z)=e^{z/2}-e^{-z/2}\,. \tag{2.16}$$

For $r\neq 0$, the normal ordering is not an issue and no constant term is required. The exponent in (2.15) is set to satisfy:

$$\mathcal{E}_r(z)^*=\mathcal{E}_{-r}(z)^*\,,$$

where the adjoint is with respect to the standard inner product on $\Lambda^{\frac{\infty}{2}}V$.

The operators $\mathcal{E}$ satisfy the following fundamental commutation relation:

$$[\mathcal{E}_a(z),\mathcal{E}_b(w)]=\varsigma\left(\det\left[\begin{smallmatrix} a & z\\ b & w\end{smallmatrix}\right]\right)\,\mathcal{E}_{a+b}(z+w)\,. \tag{2.17}$$

Equation (2.17) automatically incorporates the central extension of the $\mathfrak{gl}(\infty)$-action, which appears as the constant term in $\mathcal{E}_0$ when $r=-s$.

2.2.5. The operators $\mathcal{E}$ specialize to the standard bosonic operators on $\Lambda^{\frac{\infty}{2}}V$:

$$\alpha_k=\mathcal{E}_k(0)\,,\quad k\neq 0\,.$$

The commutation relation (2.17) specializes to the following equation:

$$[\alpha_k,\mathcal{E}_r(z)]=\varsigma(kz)\,\mathcal{E}_{k+r}(z)\,. \tag{2.18}$$

When $k+r=0$, equation (2.18) has the following constant term:

$$\frac{\varsigma(kz)}{\varsigma(z)}=\frac{e^{kz/2}-e^{-kz/2}}{e^{z/2}-e^{-z/2}}\,.$$

Letting $z\to 0$, we recover the standard relation:

$$[\alpha_k,\alpha_r]=k\,\delta_{k+r}\,. \tag{2.19}$$

2.2.6. The operators $\mathcal{E}$ form a projective representation of the (completed) Lie algebra of differential operators on $\mathbb{C}^\times$; see for example [21], [1].

Let x be the coordinate on $\mathbb{C}^\times$. Identify V with $x^{1/2}\mathbb{C}[x^{\pm 1}]$ via the assignment

$$\underline{k}\mapsto x^k\,.$$

We then find the following correspondences:

$$\mathcal{P}_k \leftrightarrow \left(x \frac{d}{dx} \right)^k , \quad \alpha_k \leftrightarrow x^k ,$$

where x^k is considered as the operator of multiplication by x. The correspondence is only a Lie algebra representation, and not a representation of an associative algebra.

The operator $\mathcal{E}_0(z)$ corresponds to the following differential operator of infinite order

$$T_z = \sum_k \frac{z^k}{k!} \left(x \frac{d}{dx} \right)^k ,$$

which acts on functions by rescaling their arguments:

$$T_z \cdot f(x) = f(e^z x) .$$

3. The Gromov-Witten theory of $\mathbf{P}^1$

3.1. *The operator formula.*

3.1.1. The operator formalism will be used here to study the stationary Gromov-Witten invariants of $\mathbf{P}^1$ relative to $0, \infty \in \mathbf{P}^1$,

$$\left\langle \mu, \prod \tau_{k_i}(\omega), \nu \right\rangle^{\bullet \mathbf{P}^1} ,$$

and the corresponding connected invariants.

The GW/H correspondence (1.12) together with (0.8) results in the following formula:

$$\left\langle \mu, \prod_{i=1}^n \tau_{k_i}(\omega), \nu \right\rangle^{\bullet \mathbf{P}^1} = \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \sum_{|\lambda|=|\mu|} \chi^\lambda_\mu \, \chi^\lambda_\nu \prod_{i=1}^n \frac{\mathbf{p}_{k_i+1}(\lambda)}{(k_i+1)!} , \tag{3.1}$$

the derivation of which uses the equality

$$|C_\mu| = |\mu|!/\mathfrak{z}(\mu) .$$

3.1.2. We first consider the following generating function,

$$F^\bullet_{\mu,\nu}(z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n=-2}^{\infty} \left\langle \mu, \prod_{i=1}^n \tau_{k_i}(\omega), \nu \right\rangle^{\bullet \mathbf{P}^1} \prod_{i=1}^n z_i^{k_i+1} , \tag{3.2}$$

where convention (0.28) is used for the $\tau_{-2}(\omega)$ insertions. Invariants in (3.2) with $\tau_{-1}(\omega)$ insertions are defined to vanish. Then, formula (3.1) may be rewritten as

$$F^\bullet_{\mu,\nu}(z_1,\dots,z_n) = \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \sum_{|\lambda|=|\mu|} \chi^\lambda_\mu \, \chi^\lambda_\nu \prod_{i=1}^n \mathbf{e}(\lambda, z_i)\,. \tag{3.3}$$

Our next goal is to recast the formula in terms of operators on $\Lambda^{\frac{\infty}{2}}V$.

3.1.3. The following formula in $\Lambda^{\frac{\infty}{2}}V$ is well-known (see e.g. [28]):

$$\prod_{i=1}^{\ell(\nu)} \alpha_{-\nu_i}\, v_\emptyset = \sum_{|\lambda|=d} \chi^\lambda_\nu \, v_\lambda\,, \quad |\nu| = d\,.$$

It is equivalent, for example, to the Murnaghan-Nakayama rule for characters of a symmetric group. Therefore, using (2.9), we can express the sum on the right side of (3.3) as:

$$\left(\prod \mathcal{E}_0(z_i) \prod \alpha_{-\nu_i}\, v_\emptyset, \prod \alpha_{-\mu_i}\, v_\emptyset\right)\,.$$

For any operator A, we denote the diagonal matrix element of A with respect to the vacuum vector $v_\emptyset$ by angle brackets:

$$\langle A \rangle = (A v_\emptyset, v_\emptyset)\,.$$

The above vacuum matrix element is the *vacuum expectation.* Since, clearly,

$$\alpha_k^* = \alpha_{-k}\,,$$

formula (3.3) can be recast in the following operator form.

PROPOSITION 3.1. *There exists*

$$F^\bullet_{\mu,\nu}(z_1,\dots,z_n) = \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \left\langle \prod_{i=1}^{\ell(\mu)} \alpha_{\mu_i} \prod_{i=1}^{n} \mathcal{E}_0(z_i) \prod_{i=1}^{\ell(\nu)} \alpha_{-\nu_i} \right\rangle\,. \tag{3.4}$$

3.2. *The* 1*-point series.*

3.2.1. We start by examining how the formalism works in the (geometrically trivial) case of the 0-point series. Formula (3.4) specializes to the following expression:

$$F^\bullet_{\mu,\nu}() = \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \left\langle \prod \alpha_{\mu_i} \prod \alpha_{-\nu_i} \right\rangle\,. \tag{3.5}$$

Observe, for positive k, that the operator α_k annihilates the vacuum

$$\alpha_k\, v_\emptyset = 0\,, \quad k > 0\,.$$

We can use the commutation relation (2.19) repeatedly to move the operators α_{μ_i} all the way to the right, after which the vacuum expectation vanishes.

Moving the operator $\alpha_{-\nu_i}$ all the way to the left has the same effect. Thus, a nonzero result is obtained only in case all operators α_{μ_i} and $\alpha_{-\nu_i}$ annihilate in pairs via the commutation relation

$$[\alpha_k, \alpha_{-k}] = k\,. \tag{3.6}$$

This leads to the expected result

$$F^\bullet_{\mu,\nu}() = \frac{\delta_{\mu,\nu}}{\mathfrak{z}(\mu)}\,.$$

From the geometric point of view, the commutation relation (3.6), or the equivalent relation

$$\langle \alpha_k \, \alpha_{-k} \rangle = k,$$

is responsible for a k-fold covering of $\mathbf{P}^1$ totally ramified over 0 and ∞.

3.2.2. Now we want to compute the 1-point series

$$F^\bullet_{\mu,\nu}(z) = \sum_{k=-2}^{\infty} \langle \mu, \tau_k(\omega), \nu \rangle^{\bullet \mathbf{P}^1} \, z^{k+1}\,, \tag{3.7}$$

or, rather, the associated connected series

$$F^\circ_{\mu,\nu}(z) = \sum_{k=-2}^{\infty} \langle \mu, \tau_k(\omega), \nu \rangle^{\circ \mathbf{P}^1} \, z^{k+1}\,. \tag{3.8}$$

We have

$$F^\bullet_{\mu,\nu}(z) = \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \left\langle \prod \alpha_{\mu_i} \, \mathcal{E}_0(z) \, \prod \alpha_{-\nu_i} \right\rangle\,. \tag{3.9}$$

We apply here the same strategy used to evaluate (3.5): we move the operators α_{μ_i} to the right and move the operators $\alpha_{-\nu_i}$ to the left.

We saw in the evaluation of the 0-point series that the commutation relation (3.6) accounts for a connected component without marked points. The commutators (3.6) make no contribution to the connected series (3.8).

All the action, therefore, happens as we commute the α's through the operator $\mathcal{E}_0(z)$. The commutation is given by (2.18). Applying formula (2.18) a total of $\ell(\mu) + \ell(\nu)$ times and using the obvious relation

$$|\mu| = |\nu|\,,$$

we obtain the following result

$$\begin{aligned} F^\circ_{\mu,\nu}(z) &= \frac{\prod \varsigma(\mu_i z) \prod \varsigma(\nu_i z)}{\mathfrak{z}(\mu)\,\mathfrak{z}(\nu)} \, \langle \mathcal{E}_0(z) \rangle \\ &= \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \, \frac{\prod \varsigma(\mu_i z) \prod \varsigma(\nu_i z)}{\varsigma(z)}\,. \end{aligned} \tag{3.10}$$

Using the function

$$\mathcal{S}(z) = \frac{\varsigma(z)}{z} = \frac{\sinh z/2}{z/2}\,,$$

we may state formula (3.10) as follows.

THEOREM 2. *For any two partitions μ and ν of the same size,*

$$(3.11)\quad \sum_{g=0}^{\infty} z^{2g} \left\langle \mu, \tau_{2g-2+\ell(\mu)+\ell(\nu)}(\omega), \nu \right\rangle^{\circ \mathbf{P}^1} = \frac{1}{|\operatorname{Aut}(\mu)|\,|\operatorname{Aut}(\nu)|} \frac{\prod \mathcal{S}(\mu_i z) \prod \mathcal{S}(\nu_i z)}{\mathcal{S}(z)}\,.$$

Formula (0.26) is recovered as the degree 0 case of Theorem 2. More generally, for $\mu = \nu = (1^d)$, we obtain

$$(3.12)\quad \sum_{g=0}^{\infty} z^{2g} \left\langle \tau_{2g-2+2d}(\omega) \right\rangle^{\circ \mathbf{P}^1} = \frac{1}{(d!)^2}\, \mathcal{S}(z)^{2d-1}\,,$$

which is the formula predicted in [34] from the (then) conjectural Toda equation. The Toda equation will be discussed in Section 4. In particular, formula (3.12) can also be deduced from Proposition 4.3.

3.2.3. The product of $\mathcal{S}$-functions in (3.11) satisfies an important property: the product is symmetric in the combined set of variables $\{\mu_i\} \cup \{\nu_i\}$. This *crossing symmetry* is very restrictive; see [12]. In particular, the symmetry implies that the full formula (3.11) may be obtained from the very special and degenerate case in which $\mu = (d)$. In fact, the property is almost equivalent to the GW/H correspondence: the symmetry alone forces $\tau_k(\omega)$ to correspond to a *linear* combination of the $\mathbf{p}_i$'s.

We also observe that since $\mathcal{S}(0) = 1$, the coefficient of z^{2g} in the product of $\mathcal{S}$-functions in (3.11) is well defined as a symmetric functions of degree $2g$ in infinitely many variables. In other words, we have the following stability: setting any variable to zero gives the analogous function in fewer variables, see the discussion in Section 0.4.2.

3.2.4. From (3.11) and Proposition 1.6 we obtain the following result determining the completion coefficients.

PROPOSITION 3.2. *The completion coefficients* (0.21) *are given by*

$$(3.13)\quad \rho_{k,\mu} = (k-1)!\,\frac{\prod \mu_i}{d!}\,[z^{2g}]\, \mathcal{S}(z)^{d-1} \prod \mathcal{S}(\mu_i z)\,,$$

where $[z^{2g}]$ stands for the coefficient of z^{2g} and the numbers g and d are defined by

$$d = |\mu|\,, \quad k+1 = |\mu| + \ell(\mu) + 2g\,.$$

In particular, the terms for which $|\mu| + \ell(\mu)$ reaches the maximal value $k+1$ may be viewed together as principal terms of the completed cycle $\overline{(k)}$. For such terms, the genus g vanishes and the coefficient $\rho_{k,\mu}$ simply becomes

$$\rho_{k,\mu} = (k-1)! \frac{\prod \mu_i}{|\mu|!}\,, \quad |\mu| + \ell(\mu) = k+1\,.$$

The geometric interpretation of the coefficients $\rho_{k,\mu}$ given in Proposition 1.6 was *not* essential for the derivation of formula (3.13).

3.3. *The n-point series.*

3.3.1. The same strategy works for the evaluation of the general n-point series (3.2), or, rather, the associated connected series $F^{\circ}_{\mu,\nu}(z_1,\dots,z_n)$. The result, however, is somewhat more complicated to state. In particular, we require the following auxiliary function

$$G\begin{pmatrix} a_1 & \dots & a_n \\ z_1 & \dots & z_n \end{pmatrix} = \left\langle \mathcal{E}_{a_1}(z_1) \, \dots \, \mathcal{E}_{a_n}(z_n) \right\rangle^{\circ} \,, \tag{3.14}$$

where the superscripted circle indicates the connected part of the vacuum expectation; that is,

$$\left\langle \mathcal{E}_{a_1}(z_1)\, \mathcal{E}_{a_2}(z_2) \right\rangle^{\circ} = \left\langle \mathcal{E}_{a_1}(z_1)\, \mathcal{E}_{a_2}(z_2) \right\rangle - \left\langle \mathcal{E}_{a_1}(z_1) \right\rangle \, \left\langle \mathcal{E}_{a_2}(z_2) \right\rangle \,,$$

et cetera. The function (3.14) clearly vanishes unless the condition

$$a_1 + \dots + a_n = 0$$

is satisfied. Also, the equation

$$G\begin{pmatrix} 0 \\ z \end{pmatrix} = \frac{1}{\varsigma(z)}$$

is clear.

3.3.2. For $n > 1$, the function (3.14) can be computed recursively as follows. First, if $a_1 \le 0$ then (3.14) vanishes:

$$G\begin{pmatrix} a_1 & \dots & a_n \\ z_1 & \dots & z_n \end{pmatrix} = 0\,, \quad a_1 \le 0\,.$$

If $a_1 > 0$, then by commuting the operator $\mathcal{E}_{a_1}(z_1)$ all the way to the right using the commutation relation (2.17), we obtain

$$\begin{aligned} &G\begin{pmatrix} a_1 & \dots & a_n \\ z_1 & \dots & z_n \end{pmatrix} \\ &\qquad = \sum_{i=2}^{n} \varsigma\left(\det \begin{bmatrix} a_1 & a_i \\ z_1 & z_i \end{bmatrix} \right) G\begin{pmatrix} a_2 & \dots & a_i + a_1 & \dots & a_n \\ z_2 & \dots & z_i + z_1 & \dots & z_n \end{pmatrix}\,, \quad a_1 > 0\,. \end{aligned}$$

The above rules can be easily converted into a nonrecursive form. For example, for $n = 2$

$$G\begin{pmatrix} a & -a \\ z_1 & z_2 \end{pmatrix} = \begin{cases} \dfrac{\varsigma(a(z_1+z_2))}{\varsigma(z_1+z_2)}\,, & a > 0\,, \\ 0 & a \le 0\,. \end{cases}$$

3.3.3. The strategy of Section 3.2 applies to evaluation of (3.4) with minor modification. Each of the α's now has a *choice* of operator $\mathcal{E}$ with which to interact (that is, with which to commute). This choice can be conveniently formalized in terms of a function

$$f : \{\mu_i\} \cup \{-\nu_i\} \to \{1, \dots, n\}\,,$$

where $\{\mu_i\} \cup \{-\nu_i\}$ is considered as a multiset, that is, a set with possible repetitions. The evaluation $f(\mu_i) = j$ indicates that the commutator of α_{μ_i} with $\mathcal{E}_0(z_j)$ is taken.

THEOREM 3. *Let M denote the multiset $\{\mu_i\} \cup \{-\nu_i\}$. Then*

$$\begin{aligned} &F^{\circ}_{\mu,\nu}(z_1, \dots, z_n) \\ &\quad = \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \sum_f \left(\prod_{m \in M} \varsigma\left(|m|\, z_{f(m)}\right) \right) G\begin{pmatrix} \dots & \sum_{f^{-1}(i)} m & \dots \\ \dots & z_i & \dots \end{pmatrix}, \end{aligned} \tag{3.15}$$

where the sum is over all functions $f : M \to \{1, \dots, n\}$.

Since the summation over f in (3.15) involves $n^{\ell(\mu)+\ell(\nu)}$ terms, formula (3.15) is only effective if the partitions μ and ν have few parts. For partitions of large length, especially for the case $\mu = \nu = (1^d)$, a more effective answer is given by the Toda equations; see in particular Proposition 4.3.

4. The Toda equation

We study here the Toda equations for the relative Gromov-Witten theory of $\mathbf{P}^1$. The Toda equations are equivalent to certain recurrence relations for the relative invariants.

4.1. *The τ-function.* The τ-function is a generating function (of the relative invariants of $\mathbf{P}^1$) which is convenient from the point of view of integrable hierarchies.

4.1.1. Let $t_1, t_2, \dots$ be a sequence of indeterminates. Consider the following *vertex operators*:

$$\Gamma_{\pm}(t) = \exp\left(\sum_{k>0} t_k \, \frac{\alpha_{\pm k}}{k} \right).$$

We easily obtain:

$$\Gamma_-(t) = \Gamma_+(t)^*,$$
$$\Gamma_+(t) = \sum_{\mu} \frac{t_\mu}{\mathfrak{z}(\mu)} \prod \alpha_{\mu_i}\,,$$

where $t_\mu = \prod t_{\mu_i}$. The above sum is over all partitions μ.

4.1.2. Define the τ-function for Gromov-Witten theory of $\mathbf{P}^1$ relative to $0, \infty \in \mathbf{P}^1$ by:

$$\tau_{\mathbf{P}^1}(x,t,s) = \sum_{|\mu|=|\nu|} t_\mu\, s_\nu \left\langle \mu, \exp\left(\sum_{i=0}^{\infty} x_i\, \tau_i(\omega)\right), \nu \right\rangle^{\bullet \mathbf{P}^1}, \tag{4.1}$$

where $x_0, x_1, \ldots$ is a new set of variables. The following conventions will hold for the degree 0 constant terms:

$$\langle \emptyset, \emptyset \rangle_0^{\bullet} = 1\,, \quad \langle \emptyset, \emptyset \rangle_0^{\circ} = 0\,.$$

The τ-function is often called the *partition function.*

Formula (3.4) and the definition (2.10) of the operators $\mathcal{P}_k$ together yield an operator formula for $\tau_{\mathbf{P}^1}$.

PROPOSITION 4.1. *We have*

$$\tau_{\mathbf{P}^1}(x,t,s) = \left\langle \Gamma_+(t)\, \exp\left(\sum_{k=0}^{\infty} \frac{x_k}{(k+1)!}\, \mathcal{P}_{k+1}\right) \Gamma_-(s) \right\rangle.$$

4.1.3. By the usual relation between the connected and disconnected theories, the logarithm of τ generates the connected invariants:

$$\begin{aligned} \mathcal{F}_{\mathbf{P}^1}(x,t,s) &= \sum_{|\mu|=|\nu|} t_\mu\, s_\nu \left\langle \mu, \exp\left(\sum_{i=0}^{\infty} x_i\, \tau_i(\omega)\right), \nu \right\rangle^{\circ \mathbf{P}^1} \\ &= \ln \tau_{\mathbf{P}^1}(x,t,s)\,, \end{aligned} \tag{4.2}$$

where the first equality is the definition of the function $\mathcal{F}_{\mathbf{P}^1}$. The function $\mathcal{F}_{\mathbf{P}^1}$ is known as the *free energy.*

4.2. *The string equation.*

4.2.1. As a slight extension of stationary Gromov-Witten theory, we allow the appearance of

$$\tau_0(1)\,,$$

a marked point with no imposed conditions. The $\tau_0(1)$-insertions are known as *punctures.* In both the connected and disconnected theory, the insertions of

$\tau_0(1)$ can be removed using the *string equation*:

$$\left\langle \tau_0(1) \prod_i \tau_{k_i}(\omega) \right\rangle = \sum_j \left\langle \prod_i \tau_{k_i-\delta_{i,j}}(\omega) \right\rangle . \tag{4.3}$$

By the same principle, one removes any number of punctures, which can be expressed as follows:

$$\left\langle e^{y\tau_0(1)} \prod_i \tau_{k_i}(\omega) \right\rangle = \left\langle \prod_i \left(\sum_{m\geq 0} \frac{y^m}{m!} \, \tau_{k_i-m}(\omega) \right) \right\rangle . \tag{4.4}$$

4.2.2. In the standard interpretation of the string equation, all the negative descendants are set to zero. Also, there is the following unique exception to the string equation in the connected theory:

$$\left\langle \tau_0(1)^2 \, \tau_0(\omega) \right\rangle_{0,0} = 1 \, .$$

In the disconnected theory, of course, the exception propagates in all degrees and genera.

An equivalent way of managing the exceptional case is to declare the string equation always valid, while simultaneously changing the interpretation of the output. Recall our conventions for the disconnected stationary theory:

$$\tau_k(\omega) = \begin{cases} 1 \, , & k = -2 \, , \\ 0 \, , & k \neq 2, \, k < 0 \, . \end{cases} \tag{4.5}$$

We now observe that the following interpretation of the string equation is equivalent to the standard one:

(i) We first apply the string equation in the form

$$\left\langle \tau_0(1) \prod \tau_{k_i}(\omega) \right\rangle = \sum_i \left\langle \tau_{k_1}(\omega) \dots \tau_{k_i-1}(\omega) \dots \right\rangle \, ,$$

that is, without exceptions and without setting $\tau_{-1}(\omega)$ and $\tau_{-2}(\omega)$ to zero, repeatedly to remove all $\tau_0(1)$-insertions,

(ii) After this we apply the rules (4.5) to the resulting stationary Gromov-Witten invariant.

4.2.3. The form of the string equation is unchanged in relative Gromov-Witten theory. Let us add an additional string variable y_0 to the generating function (4.1):

$$\tau_{\mathbf{P}^1}(x,t,s,y_0) = \sum_{|\mu|=|\nu|} t_\mu \, s_\nu \left\langle \mu, \exp\left(y_0 \, \tau_0(1) + \sum_{i=0}^{\infty} x_i \, \tau_i(\omega) \right), \nu \right\rangle^{\bullet \mathbf{P}^1} . \tag{4.6}$$

Similarly, the function

$$\mathcal{F}_{\mathbf{P}^1}(x,t,s,y_0) = \ln \tau_{\mathbf{P}^1}(x,t,s,y_0)$$

is the generating function for connected invariants in the presence of punctures.

Equations (4.4) and (2.14) together with the commutation of operator T with vertex operators $\Gamma_\pm$ results in the following generalization of Proposition 4.1:

PROPOSITION 4.2. *For $n \in \mathbb{Z}$,*

$$\tau_{\mathbf{P}^1}(x,t,s,n) = \left\langle T^{-n}\, \Gamma_+(t)\, \exp\left(\sum_{k=0}^{\infty} \frac{x_k}{(k+1)!}\, \mathcal{P}_{k+1}\right) \Gamma_-(s)\, T^n \right\rangle .$$

4.3. *The Toda hierarchy.*

4.3.1. By a standard argument (which can be found, for example, in [15], [37] and will be explained in more detail in [32]), Proposition 4.2 yields the following result.

THEOREM 4. *The sequence*

$$\{\tau_{\mathbf{P}^1}(x,t,s,n)\}\,, \quad n \in \mathbb{Z}\,,$$

is a τ-function of the 2-Toda hierarchy of Ueno and Takasaki [38] *in the variables t and s. In particular, the lowest equation of this hierarchy is*:

$$\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau(n) = \frac{\tau(n+1)\,\tau(n-1)}{\tau(n)^2}\,, \tag{4.7}$$

where $\tau(n) = \tau_{\mathbf{P}^1}(x,t,s,n)$.

The two sequences of flows in this hierarchy are connected with two ramification conditions μ and ν in the relative Gromov-Witten theory, and *not* with the descendent insertions $\tau_k(\omega)$.

In particular, since

$$\tau_1(\omega) = \overline{(2)} = (2)\,,$$

the function $\tau_{\mathbf{P}^1}$ specializes under the restriction

$$x_2 = x_3 = \cdots = 0$$

to the τ-function of [30] enumerating Hurwitz covers with arbitrary branching over $0, \infty \in \mathbf{P}^1$ and simple ramifications elsewhere. Thus, Theorem 4 generalizes the results of [30].

4.3.2. A 2-Toda hierarchy of a different kind arises in in the equivariant GW theory of $\mathbf{P}^1$; see [32]. The flows of the equivariant 2-Toda hierarchy are associated to the insertions of $\tau_k([0])$ and $\tau_k([\infty])$, where

$$[0], [\infty] \in H^*_{\mathbb{C}^\times}(\mathbf{P}^1)$$

are the classes of the $\mathbb{C}^\times$-fixed points in the equivariant cohomology of $\mathbf{P}^1$.

In the nonequivariant limit, both $[0]$ and $[\infty]$ yield the point class ω. In the nonequivariant specialization, the 2-Toda becomes a 1-Toda hierarchy for the absolute stationary Gromov-Witten theory of $\mathbf{P}^1$ described by the function

$$\tau^{\text{abs}}_{\mathbf{P}^1}(x,q) = \tau_{\mathbf{P}^1}(x, t_1, 0, 0, \dots, s_1, 0, 0, \dots),$$

where the variable $q = t_1 s_1$ keeps track of degree. On this absolute stationary submanifold, the lowest equations of the two different Toda hierarchies coincide; see Section 4.3.5.

Getzler in [16] and Zhang [40] have constructed an extension of the 1-Toda hierarchy, the extra flows of which correspond to the descendents $\tau_k(1)$. In other words, this extended Toda hierarchy describes the full absolute nonequivariant Gromov-Witten theory of $\mathbf{P}^1$. Getzler has proven the extended hierarchy is essentially equivalent to the union of the stationary 1-Toda hierarchy and the Virasoro constraints [16]. Further clarification of the structure of the extended Toda hierarchy was obtained by Carlet, Dubrovin, and Zhang in [3].

4.3.3. In terms of the free energy (4.2), equation (4.7) reads

$$\frac{\partial^2}{\partial t_1 \partial s_1} \mathcal{F}_{\mathbf{P}^1}(x,t,s,y_0) = \exp\left(\Delta\, \mathcal{F}_{\mathbf{P}^1}(x,t,s,y_0)\right), \tag{4.8}$$

where Δ is the following divided difference operator in the string variable y_0

$$\Delta f(y_0) = f(y_0+1) - 2f(y_0) + f(y_0-1).$$

Equivalently, the operator Δ can be interpreted as the insertion of

$$e^{\tau_0(1)} - 2 + e^{-\tau_0(1)} = \varsigma\left(\tau_0(1)\right)^2 .$$

The exponential on the right side of (4.8) can be interpreted as a generating function for disconnected invariants, modified by the action of the operator Δ.

Observe that, by the definition of $\mathcal{F}_{\mathbf{P}^1}$, the coefficient of $t_\mu s_\nu$ in the expansion of $\frac{\partial^2}{\partial t_1 \partial s_1}\mathcal{F}_{\mathbf{P}^1}(x,t,s)$ is equal to:

$$(m_1(\mu)+1)(m_1(\nu)+1)\left\langle \mu+1, \exp\left(\sum_{i=0}^{\infty} x_i\, \tau_i(\omega)\right), \nu+1 \right\rangle^{\circ \mathbf{P}^1},$$

where $\mu+1$ denotes the partition $\mu \cup \{1\}$.

4.3.4. By the string equation (4.4), the effect of the operator Δ on an n-point function is the following:

$$\sum_{k_i} \left\langle \mu, \varsigma\, (\tau_0(1))^2 \prod \tau_{k_i}(\omega), \nu \right\rangle \prod z_i^{k_i+1} = \varsigma \left(\sum z_i\right)^2 \sum_{k_i} \left\langle \mu, \prod \tau_{k_i}(\omega), \nu \right\rangle \prod z_i^{k_i+1} . \tag{4.9}$$

In particular, the result vanishes when $n = 0$. Hence, the 0-point functions do not appear in the right-hand side of (4.8).

We may now translate equation (4.8) to the following relation for n-point functions.

Proposition 4.3. *The Toda equations* (4.7), (4.8) *are equivalent to the following recurrence relation for n-point functions. For any μ and ν of the same size,*

$$F^{\circ}_{\mu+1,\nu+1}(z_1, \ldots, z_n) = \frac{1}{(m_1(\mu)+1)(m_1(\nu)+1)} \sum_{\{(S_i,\mu^i,\nu^i)\}} \prod_i \varsigma\, (\Sigma_{S_i})^2\, F^{\circ}_{\mu^i,\nu^i}\left(z_{S_i}\right) , \tag{4.10}$$

where the summation is over all sets of triples

$$\{(S_i, \mu^i, \nu^i)\} ,$$

such that $\{S_i\}$ is a partition of the set $\{1, \ldots, n\}$ into nonempty disjoint subsets:

$$\{1, \ldots, n\} = \bigsqcup S_i , \quad S_i \neq \emptyset ;$$

similarly, $\{\mu^i\}$ and $\{\nu^i\}$ satisfy

$$\mu = \bigcup \mu^i , \quad \nu = \bigcup \nu^i , \quad |\mu_i| = |\nu_i| ,$$

where, by definition, $z_S = \{z_i\}_{i \in S}$ and $\Sigma_S = \sum_{i \in S} z_i$.

It is instructive to notice the consistency of this result with the result of Theorem 2.

4.3.5. We will now consider the absolute stationary Gromov-Witten theory of $\mathbf{P}^1$. From the generating function $\tau_{\mathbf{P}^1}$, the absolute specialization $\tau^{\mathrm{abs}}_{\mathbf{P}^1}$ is obtained by setting

$$t_2 = t_3 = \cdots = s_2 = s_3 = \cdots = 0 .$$

The restricted function $\tau^{\mathrm{abs}}_{\mathbf{P}^1}$ depends on t_1 and s_1 only through the weight $(t_1 s_1)^d$ multiplying terms of degree d. Similarly, its dependence on the variable x_0 is exclusively through the weight $e^{x_0(d-\frac{1}{24})}$ multiplying terms of degree d in

$\tau^{\mathrm{abs}}_{\mathbf{P}^1}$. The constant term $-\frac{1}{24}$ can be transformed into an overall factor of $e^{-x_0/24}$. Since

$$\frac{\partial^2}{\partial x_0^2} \log e^{-x_0/24} = 0\,,$$

we see

$$t_1 s_1 \frac{\partial^2}{\partial t_1 \partial s_1} \log \tau^{\mathrm{abs}}_{\mathbf{P}^1} = \frac{\partial^2}{\partial x_0^2} \log \tau^{\mathrm{abs}}_{\mathbf{P}^1}\,.$$

We now replace the $\frac{\partial^2}{\partial t_1 \partial s_1}$ derivative in (4.8) by derivatives with respect to x_0. Then, we we set $t_1 s_1 = q$ and obtain the following result.

Proposition 4.4. *The generating function*

$$\mathcal{F}^{\mathrm{abs}}_{\mathbf{P}^1}(x, y_0, q) = \sum_d q^d \left\langle \exp\left(y_0\, \tau_0(1) + \sum_{i=0}^{\infty} x_i\, \tau_i(\omega) \right) \right\rangle^{\circ \mathbf{P}^1}_d$$

for the absolute invariants of $\mathbf{P}^1$ *satisfies the following version of the Toda equation* (4.8)

$$\frac{\partial^2}{\partial x_0^2} \mathcal{F}^{\mathrm{abs}}_{\mathbf{P}^1}(x, y_0, q) = q \exp\left(\Delta\, \mathcal{F}^{\mathrm{abs}}_{\mathbf{P}^1}(x, y_0, q) \right)\,. \tag{4.11}$$

In contrast to (4.7), (4.8), the differentiation in (4.11) is with respect to the variable coupled to the insertion of $\tau_0(\omega)$. Equation (4.11) is the lowest equation in another Toda hierarchy, mentioned in Section 4.3.2, the flows of which are associated to descendent insertions in the absolute Gromov-Witten theory of $\mathbf{P}^1$.

5. The Gromov-Witten theory of an elliptic curve

Gromov-Witten invariants are deformation invariant and, therefore, are the same for all elliptic curves E. Since a nonsingular cubic can be degenerated to a nodal rational curve, the degeneration principle explained in Section 1.3 yields the following expression for the Gromov-Witten invariants of an elliptic curve E in terms of relative invariants of $\mathbf{P}^1$:

$$\left\langle \prod \tau_{k_i}(\omega) \right\rangle^{\bullet E}_d = \sum_{|\mu|=d} \mathfrak{z}(\mu) \left\langle \mu, \prod \tau_{k_i}(\omega), \mu \right\rangle^{\bullet \mathbf{P}^1}\,. \tag{5.1}$$

Here the sum is taken over all partitions μ of d.

Consider the following n-point generating function

$$F_E(z_1, \ldots, z_n; q) = \sum_{d \geq 0} q^d \sum_{k_1, \ldots, k_n} \left\langle \prod_{i=1}^n \tau_{k_i}(\omega) \right\rangle^{\bullet E}_d \prod_{i=1}^n z_i^{k_i+1}\,,$$

which includes contributions of all degrees. From the degeneration formula (5.1) and the operator formula (3.4), we conclude that

$$F_E(z_1,\dots,z_n;q) = \sum_\mu \frac{q^{|\mu|}}{\mathfrak{z}(\mu)} \left\langle \prod \alpha_{\mu_i} \prod \mathcal{E}_0(z_i) \prod \alpha_{-\mu_i} \right\rangle = \operatorname{tr}_0 q^H \prod \mathcal{E}_0(z_i)\,, \tag{5.2}$$

where tr_0 denotes the trace in the charge zero subspace $\Lambda_0^{\frac{\infty}{2}} V \subset \Lambda^{\frac{\infty}{2}} V$, spanned by the vectors v_λ or, equivalently, by the vectors

$$\prod \alpha_{-\mu_i}\, v_\emptyset\,,$$

as λ or μ range over all partitions. The vectors $\prod \alpha_{-\mu_i}\, v_\emptyset$ are orthogonal with norm squared equal to $\mathfrak{z}(\mu)$; see Section 3.2.1. Also, the energy operator H in (5.2) was defined in Section 2.1.3.

The trace (5.2) has been previously computed in [1], see also [29], [11]. The result is as follows. Introduce the product

$$(q)_\infty = \prod_{n=1}^{\infty} (1-q^n)\,,$$

and the genus 1 theta function

$$\vartheta(z) = \vartheta_{\frac{1}{2},\frac{1}{2}}(z;q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{(n+\frac{1}{2})^2}{2}} e^{(n+\frac{1}{2})z}\,.$$

Up to normalization, $\vartheta(z)$ is the only odd genus 1 theta function — the normalization is immaterial as the formula will be homogeneous in ϑ.

Theorem 5 ([1]).

$$F_E(z_1,\dots,z_n;q) = \frac{1}{(q)_\infty} \sum_{\substack{\text{all } n! \text{ permutations} \\ \text{of } z_1,\dots,z_n}} \frac{\det \left[\dfrac{\vartheta^{(j-i+1)}(z_1+\dots+z_{n-j})}{(j-i+1)!} \right]_{i,j=1}^{n}}{\vartheta(z_1)\,\vartheta(z_1+z_2)\cdots\vartheta(z_1+\dots+z_n)}\,, \tag{5.3}$$

where in the $n!$ summands the z_i's are permuted in all possible ways.

Here, $\vartheta^{(k)}$ denotes the k-th derivative of ϑ. If $k<0$, the standard convention $1/k!=0$ is followed. Hence, negative derivatives do not appear in formula (5.3).

A qualitative conclusion which may be drawn is that the z-coefficients of (5.3) are quasimodular forms in the degree variable q. Concretely, for any collection of the k_i's,

$$(q)_\infty \sum_{d=0}^{\infty} q^d \left\langle \prod \tau_{k_i}(\omega) \right\rangle_d^{\bullet E} \in \mathbb{Q}[E_2, E_4, E_6]_{\sum(k_i+2)}\,, \tag{5.4}$$

where $\mathbb{Q}[E_2, E_4, E_6]$ denotes the ring (freely) generated by the Eisenstein series

$$E_k(q) = \frac{\zeta(1-k)}{2} + \sum_n \left(\sum_{d|n} d^{k-1} \right) q^n$$

of weight $k = 2, 4, 6$, and the lower index specifies the homogeneous component of weight $\sum(k_i + 2)$. This quasimodularity condition is both very useful and very restrictive. The modular transformation relates the $q \to 1$ behavior of the series (5.4) with its $q \to 0$ behavior, thus connecting large degree invariants with low degree invariants.

Since the 2-cycle is complete,

$$\tau_1(\omega) = \overline{(2)} = (2)\,.$$

the quasimodularity (5.4) generalizes the quasimodularity of generating functions for simply branched coverings of the torus studied in [4], [22].

Further discussion of the properties of the function (5.3) can be found in [1], [11]. In particular, [11] contains the asymptotic analysis of this function as $q \to 1$, which corresponds to the $d \to \infty$ asymptotics of the GW-invariants.

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