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Decay of geometry for unimodal maps: An elementary proof

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Abstract

We prove that a nonrenormalizable smooth unimodal interval map with critical order between 1 and 2 displays decay of geometry, by an elementary and purely "real" argument. This completes a "real" approach to Milnor's attractor problem for smooth unimodal maps with critical order not greater than 2.

1. Introduction

The dynamical properties of unimodal interval maps have been extensively studied recently. A major breakthrough is a complete solution of Milnor's attractor problem for smooth unimodal maps with quadratic critical points.

Let f be a unimodal map. Following [19], let us define a (*minimal*) *measure-theoretical attractor* to be an invariant compact set A such that $\{x : \omega(x) \subset A\}$ has positive Lebesgue measure, but no invariant compact proper subset of A has this property. Similarly, we define a *topological attractor* by replacing "has positive Lebesgue measure" with "is a residual set". By a *wild attractor* we mean a measure-theoretical attractor which fails to be a topological one. In [19], Milnor asked if wild attractors can exist.

For smooth unimodal maps with nonflat critical points, this problem was reduced to the case that f is a nonrenormalizable map with a nonperiodic recurrent critical point, by a purely real argument. Furthermore, in [8], [12], it was shown that such a map f does not have a wild attractor if it displays decay of geometry.

A smooth unimodal map f with critical order ℓ sufficiently large may have a wild attractor. See [2]. But in the case $\ell \leq 2$, it was expected that f would have the decay of geometry property and thus have no wild attractor; this has been verified in the case $\ell = 2$ so far. In fact, in [8], [12], it was proved that for S-unimodal maps with critical order $\ell \leq 2$, the decay of geometry property follows from a "*starting condition*". Kozlovski [11] allowed one to get rid of the negative Schwarzian condition in this argument. The verification of the starting condition is more complicated, and it has only been done in the case $\ell = 2$. The first proof was given by Lyubich [12] with a gap fulfilled in [14]. (The argument in [12] is complete for quadratic maps, and more generally, for real analytic maps in the "Epstein class". The gap only appears in the passage to the smooth case.) More recently, Graczyk-Sands-Świątek [3], [4] gave an alternative proof of this result, using the method of "*asymptotically conformal extension*" which goes back to Dennis Sullivan and was discussed earlier in Section 3.1 of [9] (under the name of "tangent extension") and in Section 12.2 of [13]. We note that these proofs of the starting condition make elaborate use of "complex" methods and do not seem to work for the case $\ell < 2$.

In this paper, we shall prove the decay of geometry property for all critical order $\ell \leq 2$, which includes a new proof for the case $\ell = 2$. The proof is very elementary, where no complex analysis is involved. We shall only use the standard cross-ratio technique and the real Koebe principle. This completes a "real" attempt for the attractor problem for unimodal interval maps with critical order $1 < \ell \leq 2$.

Let us state the result more precisely. By a unimodal map, we mean a C^1 map $f : [-1, 1] \to [-1, 1]$ with a unique critical point 0, such that $f(-1) = f(1) = -1$. We shall assume that f is C^3 except at 0, and there are C^3 local diffeomorphisms ϕ, ψ such that $f(x) = \psi(|\phi(x)|^\ell)$ for x close to 0, where $\ell > 1$ is a constant, called the *critical order*. We shall refer to such a map as a C^3 *unimodal map with critical order* ℓ. Recall that f is *renormalizable* if there exist an interval I which contains the critical point 0 in its interior, and a positive integer $s > 1$, such that the intervals $I, f(I), \cdots, f^{s-1}(I)$ have pairwise disjoint interiors, $f^s(I) \subset I$, and $f^s(\partial I) \subset \partial I$.

Main Theorem. *Let $f : [-1, 1] \to [-1, 1]$ be a nonrenormalizable C^3 unimodal map with critical order $\ell \in (1, 2]$. Assume that f has a nonperiodic recurrent critical point. Then f displays decay of geometry.*

Corollary 1.1. *A C^3 unimodal map with critical order $\ell \in (1, 2]$ does not have a wild attractor.*

To explain the meaning of decay of geometry, we follow the notation according to Lyubich [12]. Let q denote the unique orientation-reversing fixed point of f, and let $\hat{q}$ be the other preimage of q. The *principal nest* is the sequence of nested neighborhoods of the critical point

$$I^0 \supset I^1 \supset I^2 \supset I^3 \supset \cdots,$$

where $I^0 = (q, \hat{q})$, and I^{n+1} is the critical return domain to I^n for all $n \geq 0$. Let $m(1) < m(2) < \cdots$ be all the *noncentral return moments*, that is, these are all the positive integers such that the first return of the critical point to $I^{m(k)-1}$ is not contained in $I^{m(k)}$.

Definition. We say that f displays *decay of geometry* if there are constants $C > 0$ and $\lambda > 1$ such that

$$\frac{|I^{m(k)}|}{|I^{m(k)+1}|} \geq C\lambda^k.$$

According to [8], [12], for any $1 < \ell \leq 2$, there is a constant $\epsilon = \epsilon(\ell) > 0$, such that f displays decay of geometry if

$$\liminf_n \frac{|I^{n+1}|}{|I^n|} \leq \epsilon.$$

The last inequality is called the *starting condition*.

Prior to this work, real methods were known to work for some special examples. The so-called "essentially unbounded" combinatorics admits a rather simple argument ([8], [12]). The more difficult cases, namely the Fibonacci combinatorics and the so-called "rotation-like" combinatorics, are also resolved in [10] and [5] respectively. Those arguments are again complicated and seem difficult to generalize to cover all combinatorics.

Let us say a few words on our method. As in [10], we shall look at the closest critical return times $s_1 < s_2 < \cdots$, and find a geometric parameter for each n which monotonically increases exponentially fast. The parameters used here are, however, very different from those therein: we consider the location of the closest critical returns in the principal nest. For each closest return time s_n (with n sufficiently large), let k be such that $f^{s_n}(0) \in I^{m(k)} - I^{m(k+1)}$. Note that $f^{s_n}(0)$ must be contained in $I^{m(k+1)-1} - I^{m(k+1)}$. Set

$$A_n = \frac{|f(b)| - |f(f^{s_{n+1}}(0))|}{|f(b)| - |f(f^{s_n}(0))|}, \quad B_n = \left(\frac{|f^{s_n}(0)|}{|f^{s_{n+1}}(0)|}\right)^{\ell/2},$$

where b is an endpoint of $I^{m(k+1)-1}$. It is not difficult to show that the Main Theorem follows from the following:

MAIN LEMMA. *There exists a universal constant $\sigma > 0$ such that for all n sufficiently large*,

$$|(f^{s_{n+1}})'(f(0))|\frac{B_n}{A_n} \geq (1+\sigma)|(f^{s_n})'(f(0))|\frac{B_{n-1}}{A_{n-1}}.$$

To prove the Main Lemma, we use the standard cross-ratio distortion estimate. For any two intervals $J \Subset T$, define as usual the *cross-ratio*

$$C(T, J) = \frac{|T||J|}{|L||R|},$$

where L, R are the components of $T - J$. We shall apply the following fundamental fact: if $T \subset (-1, 1)$ and $n \in \mathbb{N}$ are such that $f^n|T$ is a diffeomorphism, and if $f^n(T)$ is contained in a small neighborhood of the critical point, then for any interval $J \Subset T$, $C(f^n(T), f^n(J))/C(T, J)$ is bounded from below by a constant close to 1. (See §2.4.) In particular, for any $x \in T$, this gives us a lower bound on $|(f^n)'(x)|$ in terms of the length of the intervals $T - \{x\}$ and their images under f^n.

We shall choose an appropriate neighborhood T_n of $f^{s_n}(0)$, such that $f^{s_{n+1}-s_n}|T_n$ is a diffeomorphism. Using the argument described above, we obtain lower bounds on $|(f^{s_{n+1}-s_n-1})'(f^{s_n+1}(0))|$, as desired. We should note that we do not choose T_n to be the maximal interval on which $f^{s_{n+1}-s_n}$ is monotone, but require $f^{s_{n+1}-s_n}(T_n)$ not to exceed $I^{m(k)-1}$.

Our proof can be modified to deal with a nonrenormalizable C^3 unimodal map with critical order $2+\epsilon$, with $\epsilon > 0$ sufficiently small. In general, the decay of geometry property does not hold, but we can show that $\liminf |I^{m(k)}|/|I^{m(k)+1}|$ is bounded from below by a universal constant $C(\epsilon)$, and $C(\epsilon) \to \infty$ as $\epsilon \to 0$. The argument in [12] is still valid to show that such a map does not have a wild attractor as well. It is also possible to weaken the smoothness condition to be C^2. These (minor) issues will not be discussed further in this paper.

In Section 2, we shall give the necessary definitions and recall some known facts which will be used in our argument. These facts include Martens' real bounds ([16]) and Kozlovski's result on cross-ratio distortion ([11]). We shall deduce the Main Theorem from the Main Lemma. In Section 3, we shall define the intervals T_n and investigate the location of the boundary points of T_n and $f^{s_{n+1}-s_n}(T_n)$ in the principal nest. In Section 4, we shall prove the Main Lemma by means of cross-ratio, and complete our argument. As we shall see, the argument is particularly simple if there is no central low return in the principal nest in which case all the closest return s_n are of type I (defined in Section 3).

Throughout this paper, f is a unimodal map as in the Main Theorem. Note that by means of a C^3 coordinate change, we may assume that

- f is an even function,

- $f(x) = -|x|^\ell + f(0)$ on a neighborhood of 0,

and we shall do so from now on. We use (a, b) to denote the open interval with endpoints a, b, not necessarily with $a < b$.

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2. Preliminaries

2.1. *Pull back, nice intervals.* Given an open interval $I \subset [-1, 1]$, and an orbit x, $f(x)$, $\cdots$, $f^n(x)$ with $f^n(x) \in I$, by *pulling back* I along $\{f^i(x)\}_{i=0}^n$, we get a sequence of intervals $I_i \ni f^i(x)$ such that $I_n = I$, and I_i is a component of $f^{-1}(I_{i+1})$ for each $0 \le i \le n-1$. The interval I_0 is *produced* by this pull back procedure, and will be denoted by $I(n; x)$. The pull back is *monotone* if none of these intervals I_i, $0 \le i \le n-1$ contains the critical point, and it is *unimodal* if I_i, $1 \le i \le n-1$, does not contain the critical point but I_0 does.

Following [16], an open interval $I \subset [-1, 1]$ is called *nice* if $f^n(\partial I) \cap I = \emptyset$ for all $n \in \mathbb{N}$. Given a nice interval, let

$$D_I = \{x \in [-1, 1] : \text{there exists } k \in \mathbb{N} \text{ such that } f^k(x) \in I\}.$$

A component J of D_I is an *entry domain* to I. If $J \subset I$, then we shall also call it a *return domain* to I. For any $x \in D_I$, the minimal positive integer $k = k(x)$ with $f^k(x) \in I$ is the *entry time* of x to I. This integer will also be called the *return time* of x to I if $x \in I$. Note that $k(x)$ is constant on any entry domain. The *first entry map* to I is the map $R_I : D_I \to I$ defined by $x \mapsto f^{k(x)}(x)$. The *first return map* to I is the restriction of R_I on $D_I \cap I$.

For any given $x \in D_I$, the pull back of I along the orbit $x, f(x), \ldots, f^{k(x)}(x)$ is either unimodal or monotone, according to whether $I(k(x); x) \ni 0$ or not. This follows from the basic property of a nice interval that any two intervals obtained by pulling back this interval are either disjoint, or nested, i.e., one contains the other.

2.2. *The principal nest.* Let q denote the orientation-reversing fixed point of f. Let $I^0 = (-q, q)$, and for all $n \ge 1$, let I^n be the return domain to I^{n-1} which contains the critical point. All these intervals I^n are nice. The sequence

$$I^0 \supset I^1 \supset I^2 \supset \cdots,$$

is called *the principal nest*. Let g_n denote the first return map to I^n. Let $m(0) = 0$, and let $m(1) < m(2) < \cdots$ be all the *noncentral return moments*; i.e., these are positive integers such that

$$g_{m(k)-1}(0) \notin I^{m(k)}.$$

Note that $\bigcap_n I^n = \{c\}$ since we are assuming that f is nonrenormalizable and since f does not have a wandering interval ([17]).

Lemma 2.1. *For any $z \in (I^{m(k)} - I^{m(k+1)}) \cap D_{I^{m(k)}}$, if $|g_{m(k)}(z)| \le |z|$, then $z \in I^{m(k)} - I^{m(k)+1}$.*

Proof. If $z \in I^{m(k)+i} - I^{m(k)+i+1}$ for some $1 \le i \le m(k+1) - m(k) - 1$, then $g_{m(k)}(z) \in I^{m(k)+i-1} - I^{m(k)+i}$, and hence $|g_{m(k)}(z)| > |z|$, which contradicts the hypothesis of this lemma. $\square$

Lemma 2.2. *Let $J \subset I^{m(k)-1} - I^{m(k)}$ be a return domain to $I^{m(k)-1}$ with return time s. Then there is an interval J' with $J \subset J' \subset I^{m(k)-1} - I^{m(k)}$ such that $f^s : J' \to I^{m(k-1)}$ is a diffeomorphism.*

Proof. Assume $g_{m(k)-1}|J = g^p_{m(k-1)}|J$. Then

$$g^{p-1}_{m(k-1)}(J) \subset I^{m(k-1)} - I^{m(k-1)+1}$$

by Lemma 2.1. For any $0 \le i \le p-1$, let $0 \le j_i \le m(k) - m(k-1) - 1$ be such that $g^i_{m(k-1)}(J) \subset I^{m(k-1)+j_i} - I^{m(k-1)+j_i+1}$ and let P_i be the component of $I^{m(k-1)+j_i} - I^{m(k-1)+j_i+1}$ which contains $g^i_{m(k-1)}(J)$. Then it is easy to see that for any $0 \le i \le p-2$, $g_{m(k-1)}$ maps a neighborhood of $g^i_{m(k-1)}(J)$ in P_i onto P_{i+1}, diffeomorphically. Since there is a neighborhood of $g^{p-1}_{m(k-1)}(J)$ in P_{p-1} which is mapped onto $I^{m(k-1)}$ by $g_{m(k-1)}$ diffeomorphically as well, the lemma follows. □

Corollary 2.3. *Let s be the return time of 0 to $I^{m(k)}$. Then there is an interval $J \ni f(0)$ with $f^{-1}(J) \subset I^{m(k)}$, such that $f^{s-1} : J \to I^{m(k-1)}$ is a diffeomorphism.*

Proof. Let s' be the return time of 0 to $I^{m(k-1)}$. We pull back the nice interval $I^{m(k-1)}$ along $\{f^i(0)\}_{i=s'}^{s}$ and denote by $P \ni f^{s'}(0)$ the interval produced. By the previous lemma, this pull back is monotone and P is contained in $I^{m(k)-1}$. The pull back of P along $\{f^i(0)\}_{i=0}^{s'}$ is certainly unimodal, and the interval produced is contained in $D_{I^{m(k)-1}}$, and hence in $I^{m(k)}$. The corollary follows. □

2.3. *Martens' real bounds.* The following result was proved by Martens [16] in the case that f has negative Schwarzian, and extended to general smooth unimodal maps in [20], [11].

Lemma 2.4. *There exists a constant $\rho > 1$ which depends only on the critical order of f, such that for all k sufficiently large,*

$$|I^{m(k)}| \ge \rho |I^{m(k)+1}|. \tag{2.1}$$

Moreover, if $g_{m(k)}(I^{m(k)+1}) \not\ni 0$, then

$$|I^{m(k+1)-1}| \ge \rho |I^{m(k+1)}|.$$

2.4. *Cross ratio distortion.* For any two intervals $J \Subset T$, we define the cross-ratio

$$C(T, J) = \frac{|T||J|}{|L||R|},$$

where L, R are the components of $T - J$. If $h : T \to \mathbb{R}$ is a homeomorphism onto its image, we write

$$\mathbf{C}(h; T, J) = \frac{C(h(T), h(J))}{C(T, J)}.$$

A diffeomorphism with negative Schwarzian always expands the cross-ratio. In general, a smooth map does not expand the cross-ratio, but in small scales, cross-ratios are still "almost expanded" by the dynamics of f.

LEMMA 2.5 (Theorem C, [11]). *For each k sufficiently large, there is a positive number $\mathcal{O}_k$, with $\mathcal{O}_k \to 1$ as $k \to \infty$ and with the following property. Let $T \subset [-1, 1]$ be an interval and let n be a positive integer. Assume that $f^n|T$ is monotone and $f^n(T) \subset I^{m(k-1)}$. Then for any interval $J \Subset T$,*

$$\mathbf{C}(f^n; T, J) \geq \mathcal{O}_k.$$

Note that even when $J = \{z\}$ consists of one point, the left-hand side of the above inequality makes sense. In fact, it gives

$$\frac{|f^n(T)|}{|T|} |(f^n)'(z)| \geq \mathcal{O}_k \frac{|f^n(T^+)|}{|T^+|} \frac{|f^n(T^-)|}{|T^-|},$$

where T^+, T^- are the components of $T - \{z\}$. To see this, we just apply the lemma to $J_\epsilon = (z - \epsilon, z + \epsilon)$, and let ϵ go to 0.

The estimate on cross-ratio distortion enables us to apply the following lemma, called the real Koebe principle. This lemma is well-known, and a proof can be found, for example, in [18].

LEMMA 2.6. *Let $\tau > 0$ and $0 < C \leq 1$ be constants. Let I be an interval, and let $h : I \to h(I) = (-\tau, 1 + \tau)$ be a diffeomorphism. Assume that for any intervals $J \Subset T \subset I$, there exists $\mathbf{C}(h; T, J) \geq C$. Then for any $x, y \in h^{-1}([0, 1])$,*

$$\frac{h'(x)}{h'(y)} \leq \frac{1}{C^6} \frac{(1 + \tau)^2}{\tau^2}.$$

2.5. *Closest returns and proof of main theorem.*

Definition. For any $k \geq 0$, denote $c_k = f^k(0)$. A *closest* (*critical*) *return time* is a positive integer s such that $c_k \notin (c_s, -c_s)$ for all $1 \leq k \leq s$. The point $f^s(c)$ will be called a *closest* (*critical*) *return.*

Let us now deduce the Main Theorem from the Main Lemma.

Proof of Main Theorem. By Main Lemma, there exist $C \in (0, 1)$ and $\lambda > 1$ such that

$$|(f^{s_n})'(c_1)| B_{n-1} \geq |(f^{s_n})'(c_1)| B_{n-1} / A_{n-1} \geq C \lambda^n,$$

where we use the fact $A_{n-1} > 1$.

For any $k \geq 0$, consider the first return of the critical point to $I^{m(k)}$, which is a closest return, denoted by $f^{s_{n_k}}(0)$. Obviously, $n_k \geq k$, and thus

$$|(f^{s_{n_k}})'(c_1)|B_{n_k-1} \geq C\lambda^k.$$

We claim that there are constants $C' > 0$ and $\lambda' > 1$ such that

$$\frac{|I^{m(k-1)}|}{|I^{m(k+1)}|} \geq C'\lambda'^k.$$

By Corollary 2.3, there is an interval $J \ni f(0)$ with $f^{-1}(J) \subset I^{m(k)}$ and such that $f^{s_{n_k}-1} : J \to I^{m(k-1)}$ is a diffeomorphism. By Lemma 2.4, $f^{s_{n_k}}(I^{m(k)+1}) \subset I^{m(k)}$ is well inside $I^{m(k-1)}$, and so by Lemmas 2.5 and 2.6, the map $f^{s_{n_k}-1}|f(I^{m(k)+1})$ has uniformly bounded distortion. In particular, there is a universal constant K such that

$$|(f^{s_{n_k}-1})'(c_1)| \leq K\frac{|I^{m(k)}|}{|f(I^{m(k)+1})|} \leq K\frac{|I^{m(k)}|}{|f(I^{m(k+1)})|},$$

and hence for k sufficiently large,

$$|(f^{s_{n_k}})'(c_1)| \leq K\frac{|I^{m(k)}|}{|f(I^{m(k+1)})|}\ell|c_{s_{n_k}}|^{\ell-1}.$$

Since $c_{s_{n_k}} \in I^{m(k)}$, this implies

$$|(f^{s_{n_k}})'(c_1)| \leq K\ell\left(\frac{|I^{m(k)}|}{|I^{m(k+1)}|}\right)^{\ell}. \tag{2.2}$$

On the other hand, $c_{s_{n_{k-1}}} \in I^{m(k-1)}$, and $c_{s_{n_k}} \notin I^{m(k+1)}$, and thus

$$B_{n_k-1} \leq \left(\frac{|I^{m(k-1)}|}{|I^{m(k+1)}|}\right)^{\ell/2}. \tag{2.3}$$

These inequalities (2.2) and (2.3) imply the claim.

Let us consider again the map $f^{s_{n_k}-1}|J$ as above. Applying Lemma 2.5, we have

$$C(J, f(I^{m(k)+1}))^{-1} \geq \mathcal{O}_k C(I^{m(k-1)}, f^{s_{n_k}}(I^{m(k)+1}))^{-1} \geq \mathcal{O}_k C(I^{m(k-1)}, I^{m(k)})^{-1},$$

which implies that

$$\frac{|I^{m(k)}|}{|I^{m(k)+1}|} \geq C''\left(\frac{|I^{m(k-1)}|}{|I^{m(k)}|}\right)^{1/\ell}.$$

This inequality, together with the claim above, implies that $|I^{m(k)}|/|I^{m(k)+1}|$ grows exponentially fast. The proof of the Main Theorem is completed. □

2.6. *Two elementary lemmas.* We shall need the following two elementary lemmas to deal with the case $\ell < 2$.

Lemma 2.7. *For any $\alpha \in (0,1)$, the function*

$$\ell \mapsto \phi(\alpha, \ell) = \alpha^{1-\frac{\ell}{2}} \int_\alpha^1 t^{\ell-1} dt$$

is a monotone increasing function on $(0, \infty)$.

Proof. Direct computation shows:

$$\begin{aligned}
\frac{\partial \phi_(\alpha, \ell)}{\partial \ell} &= \alpha^{1-\frac{\ell}{2}} \int_\alpha^1 t^{\ell-1}(\log t - \log \sqrt{\alpha}) dt \\
&= \alpha^{\ell/2} \int_{\log \sqrt{\alpha}}^{-\log \sqrt{\alpha}} e^{\ell t} t dt \\
&= \alpha^{\ell/2} \int_0^{-\log \sqrt{\alpha}} t(e^{\ell t} - e^{-\ell t}) dt \\
&> 0.
\end{aligned}$$

□

Lemma 2.8. *For any $1 > a > b$, and any $1 \le \ell \le 2$,*

$$\frac{1-b^\ell}{1-a^\ell} \ge \frac{1-b^2}{1-a^2}.$$

Proof. By a continuity argument, it suffices to prove the lemma when ℓ is rational. Let $\ell = m/n$, with $m, n \in \mathbb{N}$, and let $x = b^{1/n}$, $y = a^{1/n}$. Then

$$1 > \frac{x}{y} \ge \left(\frac{x}{y}\right)^2 \ge \cdots \ge \left(\frac{x}{y}\right)^{2n-1},$$

which implies that

$$\frac{1 + x + x^2 + \cdots + x^{m-1}}{1 + y + y^2 + \cdots + y^{m-1}} \ge \frac{1 + x + x^2 + \cdots + x^{2n-1}}{1 + y + y^2 + \cdots + y^{2n-1}}.$$

Multiplying by $(1-x)/(1-y)$ on both sides, we obtain the desired inequality.

□

3. The closest critical returns

Let $s_1 < s_2 < \cdots$ be all the closest return times. Let n_0 be such that s_{n_0} is the return time of 0 to $I^{m(1)}$. For any $n \ge n_0$, let $k = k(n)$ be so that $c_{s_n} \in I^{m(k)} - I^{m(k+1)}$. Note that we have $c_{s_n} \in I^{m(k+1)-1} - I^{m(k+1)}$, because the first return of 0 to $I^{m(k)}$ lies in $I^{m(k+1)-1} - I^{m(k+1)}$ and it is a closest return. Let $T_n \ni c_{s_n}$ be the maximal open interval such that the following two conditions are satisfied:

- $f^{s_{n+1}-s_n}|T_n$ is monotone,
- $f^{s_{n+1}-s_n}(T_n) \subset I^{m(k)-1}$.

We shall use the cross-ratio estimate to get a lower bound for

$$|(f^{s_{n+1}-s_n-1})'(f(c_{s_n}))|.$$

To do this, it will be necessary to know the location of the boundary points of T_n and their images under $f^{s_{n+1}-s_n}$.

Note. Let u_n be the endpoint of T_n which is closer to the critical point 0, and v_n the other one. Also, let $L_n = (u_n, c_{s_n})$, and $R_n = (v_n, c_{s_n})$. Let x_n, y_n denote the endpoints of $f^{s_{n+1}-s_n}(T_n)$, so organized that $|x_n| \leq |y_n|$.

LEMMA 3.1. *$T_n \subset I^{m(k+1)-1}$.*

Proof. Arguing by contradiction, assume $T_n \not\subset I^{m(k+1)-1}$. Then there exists $z \in T_n \cap \partial I^{m(k+1)-1}$. Clearly, $g^i_{m(k)}(z) \in \partial I^{m(k+1)-i-1}$ for all $0 \leq i \leq m(k+1) - m(k) - 1$. In particular,

$$w = g_{m(k)}^{m(k+1)-m(k)-1}(z) \in \partial I^{m(k)}.$$

Now let $\nu \in \mathbb{N}$ be such that $g^\nu_{m(k)} = f^{s_{n+1}-s_n}$ near c_{s_n}. Since $g^i_{m(k)}(c_{s_n}) \in I^{m(k+1)-i-1} - I^{m(k+1)-i}$ for all $0 \leq i \leq m(k+1) - m(k) - 1$, and $g^\nu_{m(k)}(c_{s_n}) \in (c_{s_n}, -c_{s_n})$, we have $\nu \geq m(k+1) - m(k)$. So the forward orbit of w intersects $I^{m(k)-1}$, i.e., $w \in D_{I^{m(k)-1}}$. But this is absurd since $I^{m(k)}$ is a return domain to the nice interval $I^{m(k)-1}$. $\square$

Definition. We say that s_n is of *type* I if $f^{s_{n+1}-s_n}(T_n) \supset (c_{s_{n-1}}, -c_{s_{n-1}})$. Otherwise, we say that s_n is of *type* II.

The following lemma contains the combinatorial information which we are going to use.

LEMMA 3.2. *Let $n \geq n_0$, and let $k \in \mathbb{N}$ be such that $c_{s_n} \in I^{m(k)} - I^{m(k+1)}$. Let $p = m(k+1) - m(k)$. Then $y_n \in \partial I^{m(k)-1}$, $x_n \notin I^{m(k)}$, and $(x_n, y_n) \ni 0$.*

Moreover, if s_n is of type II, *then*

- *$p \geq 2$ and $g_{m(k)}(I^{m(k)+1}) \not\ni 0$,*
- *c_{s_n} is the first return of 0 to $I^{m(k)}$,*
- *If $q \in \mathbb{N}$ is minimal such that $g^q_{m(k)-1}(0) \in I^{m(k)}$, then there exist $1 \leq q' < q$, $1 \leq p' \leq p-1$ such that $x_n = g^{q'}_{m(k)-1}(g^{p'}_{m(k)}(0))$, and $c_{s_{n-1}} = g^{q'}_{m(k)-1}(0)$.*

Remark 3.1. Let us see what happens if f has the so called Fibonacci combinatorics, i.e., the closest critical return times exactly form the Fibonacci sequence: $s_1 = 1$, $s_2 = 2$, and $s_{n+1} = s_n + s_{n-1}$ for all $n \geq 2$. In this case, c_{s_n} is the first return of the critical point to I^{n-3}, and

$$c_{s_{n+1}} = g_{n-2}(c) = g_{n-3}^2(c) = g_{n-3}(c_{s_n}),$$

for all $n \geq 3$. Thus for all $n \geq 4$, T_n is the component of $I^{n-3} - \{c\}$ which contains c_{s_n}, and $f^{s_{n-1}} = f^{s_{n+1}-s_n}$ maps T_n diffeomorphically onto the interval bounded by $c_{s_{n-1}}$ and an endpoint of I^{n-4}. In particular, all s_n are of type I.

To prove this lemma, let us first do some preparation. Let $z \in D_{I^{m(k)}} \cap (I^{m(k)} - I^{m(k)+1})$, and let r be the return time of z to $I^{m(k)}$. Consider the pull back of $I^{m(k)-1}$ along the orbit $\{f^i(z)\}_{i=0}^r$, and denote by $J_i \ni f^i(z)$ the intervals obtained. Then this pull back is unimodal or monotone according to $J = J_0 \ni 0$ or not. We say that z is *good* if $0 \notin J$, and *bad* otherwise.

LEMMA 3.3. *Let z, r, J, and J_1 be as above. Then the following hold.*

1) *If z is good, then $J \cap I^{m(k)+1} = \emptyset$;*

2) *If z is bad, then f^r is monotone on $(0, z)$, and $f^i(0) \notin I^{m(k)}$ for all $1 \leq i \leq r$.*

Moreover, in either case, there is a closest return $c_s \in I^{m(k)-1} - I^{m(k)}$ such that $(c_s, -c_s) \subset f^r(J)$.

Proof. Since both J and $I^{m(k)+1}$ are produced by pull back of the nice interval $I^{m(k)-1}$, either $J \cap I^{m(k)+1} = \emptyset$, or $J \supset I^{m(k)+1}$. (Note that $J \subset I^{m(k)+1}$ cannot happen.) If $J \not\ni 0$, then $J \cap I^{m(k)+1} = \emptyset$. This proves 1). In this case, we take c_s to be the first return of the critical point to $I^{m(k)-1}$, which is necessarily not in $I^{m(k)}$, to verify the last statement of the lemma.

Assume now that z is bad. As f^r is monotone on each component of $J - \{0\}$ and $(0, z) \subset J$, $f^r|(0, z)$ is monotone. As we noted above, $f^i(J_1)$ is disjoint from $I^{m(k)}$ for all $0 \leq i \leq r-2$, which proves that $f^i(0) \notin I^{m(k)}$ for all $1 \leq i \leq r-1$. The statement $f^r(0) \notin I^{m(k)}$ is obvious. We proved 2). To verify the last statement of the lemma in this case, we just take c_s to be the point in $\{f^i(0)\}_{i=1}^r$ which is closest to the critical point. $\square$

Proof of Lemma 3.2. Let $\nu \in \mathbb{N}$ be such that $f^{s_{n+1}-s_n} = g_{m(k)}^{\nu}$ on a neighborhood of c_{s_n}. By Lemma 2.1, $z := g_{m(k)}^{\nu-1}(c_{s_n}) \in I^{m(k)} - I^{m(k)+1}$. Let r be the return time of z to $I^{m(k)}$. As above, let $J \ni z$ denote the interval obtained by pulling back $I^{m(k)-1}$ along $\{f^i(z)\}_{i=0}^r$, and let J' be the component of $J - \{0\}$ which contains z. Then $J' \subset I^{m(k)}$, $f^r|J'$ is monotone, and $f^r(J') \supset (c_{s_{n-1}}, -c_{s_{n-1}}) \cup I^{m(k)}$. Moreover, $f^r(J')$ contains a component of $I^{m(k)-1} - I^{m(k)}$. If z is good, then $J' \subset I^{m(k)} - I^{m(k)+1}$, and if z is bad, then J' contains 0 in its closure.

For each $1 \le i \le \nu - 1$, let γ_i be the maximal interval which contains c_{s_n} such that

- $g^i_{m(k)}|\gamma_i$ is well-defined and monotone,
- $g^i_{m(k)}(\gamma_i) \subset I^{m(k)} - \{0\}$.

Moreover, let $0 \le j_i \le p-1$ be such that $g^i_{m(k)}(c_{s_n}) \in I^{m(k)+j_i} - I^{m(k)+j_i+1}$. Note that T_n is equal to $(g^{\nu-1}_{m(k)}|\gamma_{\nu-1})^{-1}(J')$ and that if $g^{\nu-1}_{m(k)}(\gamma_{\nu-1}) \supset J'$, then

$$f^{s_{n+1}-s_n}(T_n) = f^r \circ g^{\nu-1}_{m(k)}(T_n) = f^r(J') \supset (c_{s_{n-1}}, -c_{s_{n-1}}),$$

and s_n is of type I. In particular, this is the case if $g^{\nu-1}_{m(k)}(\gamma_{\nu-1})$ is a component of $I^{m(k)} - \{0\}$.

If $p = 1$ or $g_{m(k)}(I^{m(k)+1}) \ni 0$, then for each $0 \le i \le \nu - 2$, $g_{m(k)}$ maps a neighborhood $K_i (\subset I^{m(k)})$ of $g^i_{m(k)}(c_{s_n})$ diffeomorphically onto a component of $I^{m(k)} - \{0\}$, and thus $g^i_{m(k)}(\gamma_i)$ is a component of $I^{m(k)} - \{0\}$ for all $1 \le i \le \nu - 1$. By the above remark, s_n is of type I. The statements about the positions of x_n and y_n are also clear.

Now we assume that $p \ge 2$ and $g_{m(k)}(I^{m(k)+1}) \not\ni 0$. We claim that for each $1 \le i \le \nu - 1$, there is $0 \le p_i \le p - 1$ with $g^{p_i}_{m(k)}(0) \in I^{m(k)+j_i+1}$, such that $g^i_{m(k)}(\gamma_i)$ is the component of $I^{m(k)} - \{g^{p_i}_{m(k)}(0)\}$ which does not contain the critical point.

Let us prove this claim by induction on i. For $i = 1$, the claim is true with $p_1 = 1$ because γ_1 is a component of $I^{m(k)+1} - \{0\}$ and $j_1 = m(k+1) - m(k) - 2$. Now assume that the claim holds for $i \le \nu - 2$ and let us prove it for $i + 1$. To this end, we distinguish two cases. If $j_i > 0$, then $g^i_{m(k)}(c_{s_n}) \in I^{m(k)+1}$ which implies $\gamma_{i+1} = (g^i_{m(k)}|\gamma_i)^{-1} I^{m(k)+1}$; so the claim is true with $p_{i+1} = p_i + 1$. If $j_i = 0$, then $g^i_{m(k)}(\gamma_i)$ contains a component $I^{m(k)} - I^{m(k)+1}$, and thus it contains the return domain to $I^{m(k)}$ which contains $g^i_{m(k)}(c_{s_n})$. Therefore, $g^{i+1}_{m(k)}(\gamma_{i+1})$ is a component of $I^{m(k)} - \{0\}$, and the claim is true with $p_{i+1} = 0$. This completes the induction.

The statements about the positions of x_n and y_n follow immediately from this claim. Let us assume that s_n is of type II, and prove the other terms. It is clear that c_{s_n} is the first return of 0 to $I^{m(k)}$ since $f^{s_{n+1}-s_n}(T_n) \supset I^{m(k)}$. To prove the last term of the lemma, we first notice that $p' := p_{\nu-1} \ne 0$ and that $z = g^{\nu-1}_{m(k)}(c_{s_n})$ is bad, for otherwise, $g^{\nu-1}_{m(k)}(\gamma_{\nu-1}) \supset J'$ which implies that s_n is of type I by our previous remark. Let $q' \in \mathbb{N}$ be such that $g^{q'}_{m(k)-1} = f^r$ near z. As $T_n = (g^{\nu-1}_{m(k)}|\gamma_{\nu-1})^{-1}(J')$, we have $x_n = g^{q'}_{m(k)-1}(g^{p'}_{m(k)}(0))$. By Lemma 3.3, $g^i_{m(k)-1}(0) \notin I^{m(k)}$ for all $1 \le i \le q'$. In particular, $q' < q$. Since $c_{s_{n-1}}$ is exactly the point in $\{g^i_{m(k)-1}(0), 1 \le i \le q-1\}$ which is closest to the critical point and since the intervals $g^i_{m(k)-1}(0, g^{p'}_{m(k)}(0))$, $1 \le i \le q$, are pairwise disjoint, we have $c_{s_{n-1}} = g^{q'}_{m(k)-1}(0)$. $\square$

Remark 3.2. In the case that $p = 1$, we see from the above proof that $f^{s_{n+1}-s_n}(T_n) = f^r(J')(= f^r(J))$. In particular, if z is good, then y_n, x_n are the endpoints of $I^{m(k)-1}$; and if z is bad, then $x_n = f^r(0)$. We shall make use of this fact in the proof of Lemma 4.1.

4. Proof of the Main Lemma

For any $n \ge n_0$, let k be such that $c_{s_n} \in I^{m(k)} - I^{m(k+1)}$, and let b_n be an endpoint of $I^{m(k+1)-1}$. Recall that

$$A_n = \frac{|b_n|^\ell - |c_{s_{n+1}}|^\ell}{|b_n|^\ell - |c_{s_n}|^\ell},\ B_n = \left(\frac{|c_{s_n}|}{|c_{s_{n+1}}|}\right)^{\ell/2}.$$

The goal of this section is to prove the following:

Main Lemma. *There exists a universal constant $\sigma > 0$ such that for all n sufficiently large,*

$$|(f^{s_{n+1}-s_n})'(f(c_{s_n}))| \ge (1+\sigma)\frac{A_n}{A_{n-1}}\frac{B_{n-1}}{B_n}.$$

The proof is organized as follows. First of all, by means of cross-ratio, we prove

$$|(f^{s_{n+1}-s_n})'(f(c_{s_n}))|\frac{A_{n-1}B_n}{A_nB_{n-1}} \ge \mathcal{O}_k A_{n-1}V_nW_n, \tag{4.1}$$

where

$$V_n = \frac{2|x_n|(|y_n|+|c_{s_n}|)}{(|y_n|+|x_n|)(|x_n|+|c_{s_n}|)} = 1 + \frac{|y_n|-|x_n|}{|y_n|+|x_n|}\frac{|x_n|-|c_{s_n}|}{|x_n|+|c_{s_n}|},$$

and

$$W_n = \left(\frac{|x_n|}{|c_{s_{n-1}}|}\right)^{\ell/2}.$$

Then we distinguish three cases to check that the left-hand side of (4.1) is bounded from below by a constant greater than 1. Note that $A_{n-1} > 1$ and $V_n \ge 1$ for all n, and that $W_n \ge 1$ if and only if s_n is of type I.

Case 1. s_n is of type I, and $|I^{m(k)-1}|/|I^{m(k)}|$ is bounded from below by a constant greater than 1. In this case, we prove that $|x_n|/|c_{s_n}|$ is strictly bigger than 1, and then the desired estimate follows from easy observations.

Case 2. *s_n is of type* I, *and* $|I^{m(k)-1}|/|I^{m(k)}|$ *is close to* 1. In this case, by Martens' real bounds, we have $m(k-1)-m(k)>1$ and that $g_{m(k-1)}$ displays a high return, i.e., $g_{m(k-1)}(I^{m(k-1)+1}) \ni 0$. According to the relative position of c_{s_n} with the orientation-preserving fixed point of $g_{m(k-1)}|I^{m(k-1)+1}$, two subcases will be considered.

Case 3. *s_n is of type* II. In this case, W_n is smaller than 1. Using the combinatorial information given by Lemma 3.2, we shall show that this loss can be compensated by the gain from $A_{n-1}V_n$.

Remark 4.1. It has been noticed by Martens [16], using the distortion control of the first return maps, that if $|I^{m(k)-1}|/|I^{m(k)}|$ is very close to 1, then $g_{m(k)-1}|I^{m(k)}$ has a high return and $|I^{m(k)}|/|I^{m(k)+1}|$ is very big. As we mentioned in the introduction, decay of geometry follows from the starting condition. Therefore arguing by contradiction the Main Theorem follows if we can prove the Main Lemma under the assumption that f does not satisfy the starting condition. From this point of view, the second case above is not necessary. We include an argument for this case as well so that we can prove the decay of geometry property without reference to the starting condition.

4.1. *Proof of* (4.1). Applying Lemma 2.5 to the map $f^{s_{n+1}-s_n-1}: f(T_n) \to (x_n, y_n)$, we obtain

$$\mathbf{C}(f^{s_{n+1}-s_n-1}; f(T_n), \{f(c_{s_n})\}) \geq \mathcal{O}_k,$$

which means

$$|(f^{s_{n+1}-s_n-1})'(f(c_{s_n}))| \geq \mathcal{O}_k \frac{|y_n - c_{s_{n+1}}||x_n - c_{s_{n+1}}|}{|y_n - x_n|} \frac{|f(L_n)| + |f(R_n)|}{|f(L_n)||f(R_n)|}. \tag{4.2}$$

By Lemma 3.1, T_n is contained in a component of $I^{m(k+1)-1} - \{0\}$. So for all n sufficiently large, we have

$$|f(L_n)| \leq |c_{s_n}|^\ell,\ |f(R_n)| \leq |b_n|^\ell - |c_{s_n}|^\ell,$$

which implies

$$\frac{|f(L_n)| + |f(R_n)|}{|f(L_n)||f(R_n)|} \geq \frac{|b_n|^\ell}{(|b_n|^\ell - |c_{s_n}|^\ell)|c_{s_n}|^\ell} \geq A_n \frac{|b_n|^\ell}{(|b_n|^\ell - |c_{s_{n+1}}|^\ell)|c_{s_n}|^\ell}. \tag{4.3}$$

Since $|x_n| \geq |b_n|$, this implies

$$\frac{|f(L_n)| + |f(R_n)|}{|f(L_n)||f(R_n)|} \geq A_n \frac{|x_n|^\ell}{(|x_n|^\ell - |c_{s_{n+1}}|^\ell)|c_{s_n}|^\ell}. \tag{4.4}$$

Since $|y_n| \geq |x_n|$, we have

$$|y_n - c_{s_{n+1}}||x_n - c_{s_{n+1}}| \geq (|y_n| + |c_{s_{n+1}}|)(|x_n| - |c_{s_{n+1}}|). \tag{4.5}$$

Recall that $\phi(\alpha,\ell) = \alpha^{1-\ell/2}\int_\alpha^1 t^{\ell-1}dt$ is a monotone increasing function with respect to ℓ (Lemma 2.7). Now,

$$\begin{aligned}
&|(f^{s_{n+1}-s_n})'(f(c_{s_n}))| \\
&\quad = |(f^{s_{n+1}-s_n-1})'(f(c_{s_n}))|\ell|c_{s_{n+1}}|^{\ell-1} \\
&\quad \geq \mathcal{O}_k \frac{(|y_n|+|c_{s_{n+1}}|)(|x_n|-|c_{s_{n+1}}|)}{|y_n|+|x_n|} A_n \frac{|x_n|^\ell}{(|x_n|^\ell-|c_{s_{n+1}}|^\ell)|c_{s_n}|^\ell}\ell|c_{s_{n+1}}|^{\ell-1} \\
&\quad = \mathcal{O}_k A_n \left(\frac{|x_n||c_{s_{n+1}}|}{|c_{s_n}|^2}\right)^{\ell/2} \frac{(|y_n|+|c_{s_{n+1}}|)(|x_n|-|c_{s_{n+1}}|)}{(|y_n|+|x_n|)|x_n|} \frac{1}{\phi(|c_{s_{n+1}}/x_n|,\ell)} \\
&\quad \geq \mathcal{O}_k A_n \left(\frac{|x_n||c_{s_{n+1}}|}{|c_{s_n}|^2}\right)^{\ell/2} \frac{(|y_n|+|c_{s_{n+1}}|)(|x_n|-|c_{s_{n+1}}|)}{(|y_n|+|x_n|)|x_n|} \frac{1}{\phi(|c_{s_{n+1}}/x_n|,2)} \\
&\quad = \mathcal{O}_k A_n \left(\frac{|x_n||c_{s_{n+1}}|}{|c_{s_n}|^2}\right)^{\ell/2} \frac{2|x_n|(|y_n|+|c_{s_{n+1}}|)}{(|y_n|+|x_n|)(|x_n|+|c_{s_{n+1}}|)} \\
&\quad \geq \mathcal{O}_k A_n \left(\frac{|x_n||c_{s_{n+1}}|}{|c_{s_n}|^2}\right)^{\ell/2} \frac{2|x_n|(|y_n|+|c_{s_n}|)}{(|y_n|+|x_n|)(|x_n|+|c_{s_n}|)} \\
&\quad = \mathcal{O}_k A_n \left(\frac{|x_n|}{|c_{s_{n-1}}|}\right)^{\ell/2} \frac{B_{n-1}}{B_n} V_n,
\end{aligned}$$

which implies (4.1)

4.2. *Case* 1. In this case, we assume that s_n is of type I, and that $|I^{m(k)-1}|/|I^{m(k)}|$ is bounded from below by some constant $\rho_1 > 1$.

LEMMA 4.1. *There exists a constant $\rho_2 > 1$, such that $|x_n|/|c_{s_n}| > \rho_2$.*

Proof. First notice that

$$\frac{|y_n|}{|c_{s_n}|} \geq \frac{|I^{m(k)-1}|}{|I^{m(k)}|} \geq \rho_1,$$

and thus if $|y_n|/|x_n| \leq \sqrt{\rho_1}$, then $|x_n|/|c_{s_n}| \geq \sqrt{\rho_1}$, and we are done. So assume that $|y_n|/|x_n| \geq \sqrt{\rho_1}$. In particular, $|x_n|$ is strictly smaller than $|y_n|$ so that $x_n \in I^{m(k)-1}$.

As before, let $\nu \in \mathbb{N}$ be such that $c_{s_{n+1}} = g_{m(k)}^\nu(c_{s_n})$, and let $z = g_{m(k)}^{\nu-1}(c_{s_n})$. Then $|z| \geq |c_{s_n}|$. If $m(k+1) > m(k)+1$, then $c_{s_n} \in I^{m(k)+1}$, and thus

$$\frac{|x_n|}{|c_{s_n}|} \geq \frac{|I^{m(k)}|}{|I^{m(k)+1}|} \geq \rho$$

by Lemma 2.4. So we may assume that $m(k+1) = m(k)+1$. Let r be the return time of z to $I^{m(k)}$. By Lemma 2.2, there is an interval $J_1 \ni f(z)$, with $f^{-1}(J_1) \subset I^{m(k)}$, such that $f^{r-1} : J_1 \to I^{m(k)-1}$ is a diffeomorphism. From the fact $x_n \in I^{m(k)-1}$, by Remark 3.2, it follows that z is bad (so $J_1 \ni c_1$), and that $x_n = f^{r-1}(c_1)$. Since $f^{r-1}(f(0,z)) = (x_n, c_{s_{n+1}})$ is well inside $I^{m(k)-1}$, it follows from Lemma 2.5 that $|f(I^{m(k)})|/|f(0,z)|$ is uniformly bounded away from 1. In particular, so is $|x_n|/|c_{s_n}|(\geq |I^{m(k)}|/2|z|)$. □

Let us prove the right-hand side of (4.1) is strictly greater than 1 (for large n). It suffices to show that $\max(A_{n-1}, V_n, W_n)$ is uniformly bounded from 1, since each of these three terms is not less than 1. Here we use the assumption that s_n is of type I.

Assume that V_n is close to 1. Then either $|y_n|/|x_n|$ or $|x_n|/|c_{s_n}|$ is close to 1. Lemma 4.1 shows that we are in the former case. If W_n is also close to 1, then so is $|y_n|/|c_{s_{n-1}}|$. As $|I^{m(k)-1}|/|I^{m(k)}| \geq \rho_1$, it follows that $c_{s_{n-1}} \in I^{m(k)-1} - I^{m(k)}$ and A_{n-1} is uniformly bigger than 1.

4.3. *Case* 2. In this case, we assume that s_n is of type I and that $|I^{m(k)-1}|/|I^{m(k)}| < \rho_1$, where $\rho_1 > 1$ is a constant close to 1. In particular, we assume that ρ_1 is less than the constant ρ in Lemma 2.4. Then we have

$$m(k-1) - m(k) \geq 2, \text{ and } g_{m(k-1)}(I^{m(k-1)+1}) \ni 0.$$

Let ζ denote the orientation-preserving fixed point of $g_{m(k-1)}|I^{m(k-1)+1}$ which is farthest from the critical point. (In fact, there is only one orientation-preserving fixed point of the map.) Note that $\zeta \in I^{m(k)} - I^{m(k)+1}$.

LEMMA 4.2. $|g'_{m(k-1)}(\zeta)| > 1$ *is uniformly bounded from above, and uniformly bounded away from* 1.

Proof. Let s be the return time of 0 to $I^{m(k)-1}$. Let M be the component of $I^{m(k-1)+1} - \{0\}$ containing ζ. By Lemma 2.5, $\mathbf{C}(f^s, M, \{\zeta\}) \geq \mathcal{O}_k$. Observe that for each component J of $M - \{\zeta\}$, $|g_{m(k-1)}(J)|/|J|$ is uniformly bounded below from 1. It follows that $|g'_{m(k-1)}(\zeta)|$ is bounded below from 1. Since the pull back of $I^{m(k-1)}$ along the orbit $\{f^i(0)\}_{i=1}^s$ is monotone, $f^{s-1}|f(I^{m(k)})$ has uniformly bounded distortion. Thus, there exists a universal constant $C > 1$ such that

$$\begin{aligned}|(f^s)'(\zeta)| &= |(f^{s-1})')(f(\zeta))|\ell|\zeta|^{\ell-1}\\ &\leq C\frac{|I^{m(k)-1}|}{|f(I^{m(k)})|}\ell|I^{m(k)}|^{\ell-1} \leq C\ell\left(\frac{|I^{m(k)-1}|}{|I^{m(k)})|}\right)^{\ell},\end{aligned}$$

and hence the derivative is uniformly bounded. □

Case 2.1. $c_{s_n} \notin (\zeta, -\zeta)$. Since $c_{s_n} \in I^{m(k+1)-1} - I^{m(k+1)}$, we have $m(k+1) = m(k) + 1$. We claim that there is an interval T'_n with $I^{m(k)} \supset T'_n \ni c_{s_n}$ and such that $f^{s_{n+1}-s_n} : T'_n \to I^{m(k-1)}$ is a diffeomorphism. Once this is proved, we can then repeat the above argument by using T'_n instead of T_n to conclude that the left side of (4.1) is uniformly bounded from 1. To prove the claim, let $\nu' \in \mathbb{N}$ be such that $g_{m(k)-1}^{\nu'}(c_{s_n}) = c_{s_{n+1}}$. Since $g_{m(k)-1}$ pushes points in $I^{m(k)} - (-\zeta, \zeta)$ farther away from 0, $g_{m(k)-1}^{\nu'-1}(c_{s_n})$ is contained in a component P of $I^{m(k)-1} - I^{m(k)}$. Note that $g_{m(k)-1}^{\nu'-1}$ maps a neighborhood of c_{s_n} (in $I^{m(k)}$) diffeomorphically onto P. Thus the existence of T'_n is guaranteed by Lemma 2.2. The claim is proved.

Case 2.2. $c_{s_n} \in (\zeta, -\zeta)$. In this case, our strategy is to assume that $\max(A_{n-1}, V_n, |x_n|/|c_{s_{n-1}}|)$ is close to 1, and prove that the left-hand side of (4.1) is bounded from below by a constant greater than 1.

Let b be the endpoint of $I^{m(k)}$ which is on the same side of 0 as ζ. By Lemma 2.5 and Lemma 2.6, it is easy to see that $g_{m(k-1)}$ has uniformly bounded distortion on $[\zeta, b]$, and thus by Lemma 4.2, $|g'_{m(k-1)}|$ is uniformly bounded from above on $[\zeta, b]$. Consequently, $(|b| - |\zeta|)/(|y_n| - |b|)$ is bounded from zero. Since A_{n-1} is close to 1 and $|c_{s_n}| < |\zeta|$, it follows that $c_{s_{n-1}} \in I^{m(k)}$ and $(|c_{s_{n-1}}| - |c_{s_n}|)/(|b| - |c_{s_{n-1}}|)$ is very small. Since we are also assuming that $|x_n|/|c_{s_{n-1}}|$ is close to 1, it follows that $|b|/|c_{s_n}|$ is very close to 1. This implies that $g_{m(k-1)}(c_{s_n})$ is closer to the critical point than c_{s_n}, and hence it is $c_{s_{n+1}}$. Let $\xi \in (0, \zeta)$ be such that $g_{m(k)-1}(\xi) = c_{s_n}$; then $|\zeta| - |\xi| \asymp |c_{s_n}| - |c_{s_{n+1}}|$. Note that $|c_{s_{n-1}}| \geq |\xi|$, and hence $|c_{s_{n-1}}| - |c_{s_n}|$ is not much smaller than $|c_{s_n}| - |c_{s_{n+1}}|$. Therefore, A_n and B_n/B_{n-1} are both close to 1 as well. Moreover, $|(f^{s_{n+1}-s_n})'(f(c_{s_n}))|$ is almost equal to $|g'_{m(k-1)}(\zeta)|$, and hence uniformly bounded away from 1. All these imply the left-hand side of (4.1) is bounded from below by a constant greater than 1.

4.4. *Case* 3. We assume now that s_n is of type II. Let p, p', q, q' be as in Lemma 3.2, and let $w_n = g_{m(k)}^p(0)$. As $c_{s_n} \in I^{m(k+1)-1} - I^{m(k+1)}$ and $g_{m(k)}$ displays a low return, $g_{m(k)}(I^{m(k)+1}) \not\ni 0$, the following hold:

- $|c_{s_n}| < |w_n| < |x_n| < |y_n|$;
- the points $g_{m(k)}^i(0)$, $1 \leq i \leq p$, lie on the same side of the critical point.

We no longer have $W_n \geq 1$, but instead, Lemma 3.2 provides more combinatorial information to apply. We claim

$$\left(\frac{|x_n|}{|c_{s_{n-1}}|}\right)^{\ell/2} \geq \frac{|x_n|}{|c_{s_{n-1}}|} \geq \mathcal{O}_k \frac{|y_n| - |w_n| + 2|w_n|^2/|c_{s_{n-1}}|}{|y_n| + |w_n|}. \tag{4.6}$$

To see this, let J be the entry domain to $I^{m(k)}$ which contains $c_{s_{n-1}} = g_{m(k)-1}^{q'}(0)$. Then $g_{m(k)-1}^{q-q'}|J : J \to I^{m(k)}$ is a diffeomorphism and

$$I^{m(k)-1} - I^{m(k)} \supset J \supset g_{m(k)-1}^{q'}(I^{m(k)+1}) \ni g_{m(k)-1}^{q'}(g_{m(k)}^{p'}(0)) = x_n.$$

Since $g^{q-q'}_{m(k)-1}((x_n, c_{s_{n-1}}))$ is contained in (c_{s_n}, w_n) which does not contain 0, and since $|w_n| < |x_n|$, we have by Lemma 2.5 that

$$\begin{aligned}\frac{|x_n| - |w_n|}{|c_{s_{n-1}}| - |x_n|} &\geq C(J, (x_n, c_{s_{n-1}}))^{-1} \geq \mathcal{O}_k C(g^{q-q'}_{m(k)-1}(J), g^{q-q'}_{m(k)-1}((x_n, c_{s_{n-1}})))^{-1}\\ &\geq \mathcal{O}_k \frac{(|y_n| - |w_n|)(|y_n| + |c_{s_n}|)}{(|w_n| - |c_{s_n}|)2|y_n|}\\ &\geq \mathcal{O}_k \frac{|y_n| - |w_n|}{2|w_n|},\end{aligned}$$

which implies (4.6) by rearranging.

Note. Set

$$U_n = A_{n-1}W_n = \left(\frac{x_n}{c_{s_{n-1}}}\right)^{\ell/2} \frac{|y_n|^\ell - |c_{s_n}|^\ell}{|y_n|^\ell - |c_{s_{n-1}}|^\ell},$$

and

$$\lambda = \frac{|y_n|}{|c_{s_{n-1}}|}, \quad \mu = \frac{|c_{s_{n-1}}|}{|w_n|}.$$

Then, (4.1) becomes

$$|(f^{s_{n+1}-s_n})'(f(c_{s_n}))| \frac{A_{n-1}}{A_n} \frac{B_n}{B_{n-1}} \geq \mathcal{O}_k U_n V_n,$$

where $V_n = 2|x_n|(|y_n| + |c_{s_n}|)/\{(|y_n| + |x_n|)(|x_n| + |c_{s_n}|)\}$ is as before.

By Lemma 2.8,

$$\frac{|y_n|^\ell - |c_{s_n}|^\ell}{|y_n|^\ell - |c_{s_{n-1}}|^\ell} \geq \frac{|y_n|^2 - |c_{s_n}|^2}{|y_n|^2 - |c_{s_{n-1}}|^2} \geq \frac{|y_n|^2 - |w_n|^2}{|y_n|^2 - |c_{s_{n-1}}|^2},$$

and by (4.6), this implies

$$U_n \geq \frac{|y_n|^2 - |w_n|^2}{|y_n|^2 - |c_{s_{n-1}}|^2} \frac{|y_n| - |w_n| + 2|w_n|^2/|c_{s_{n-1}}|}{|y_n| + |w_n|} = \frac{(\lambda\mu - 1)(\lambda\mu^2 - \mu + 2)}{\mu^3(\lambda^2 - 1)}.$$

Let us first prove the Main Lemma in the case that V_n is close to 1. In fact, by Lemma 2.4, $|x_n|/|c_{s_n}| \geq |I^{m(k)}|/|I^{m(k)+1}|$ is bounded away from 1, and thus $|y_n|/|x_n|$ is close to 1. This implies that $W_n = (|x_n|/|c_{s_{n-1}}|)^{\ell/2}$ is close to 1, and $A_{n-1} = (|y_n|^\ell - |c_{s_n}|^\ell)/(|y_n|^\ell - |c_{s_{n-1}}|^\ell)$ is very big. The Main Lemma follows.

Let us assume that $V_n > 1$ is uniformly bounded away from 1. To show that U_nV_n is bounded from below by a constant greater than 1, we may assume that $U_n < 1$. Write $P = (\lambda\mu - 1)(\lambda\mu^2 - \mu + 2)$ and $Q = \mu^3(\lambda^2 - 1)$. Then

$$Q - P = \mu^3(\lambda^2 - 1) - (\lambda\mu - 1)(\lambda\mu^2 - \mu + 2) = (\mu - 1)(2\lambda\mu - \mu^2 - \mu - 2) \geq 0.$$

Since $\mu > 1$, this implies

$$\lambda \geq \frac{\mu^2 + \mu + 2}{2\mu} \geq \frac{2\sqrt{2} + 1}{2} > 1.9. \tag{4.7}$$

Moreover,

$$\frac{1}{1-U_n} \geq \frac{Q}{Q-P} \geq \frac{\mu^3(\lambda^2-1)}{(\mu-1)(2\lambda\mu-\mu^2-\mu-2)}. \tag{4.8}$$

Let $\theta = 2\lambda\mu - (\mu^2+\mu+2)$. Then $\lambda = (\mu^2+\mu+2+\theta)/2\mu$. Substituting this equality to (4.8), we obtain

$$\frac{1}{1-U_n} \geq \frac{\theta^2+2(\mu^2+\mu+2)\theta+(\mu^2+3\mu+2)(\mu^2-\mu+2)}{\theta}\frac{\mu}{4(\mu-1)}. \tag{4.9}$$

LEMMA 4.3. *Let* $\alpha = |x_n|/|c_{s_n}|$. *Then*

$$\alpha^2 \geq \frac{\lambda\mu-1}{2}\frac{\alpha}{\alpha-1}\mathcal{O}_k + 1. \tag{4.10}$$

Proof. By Lemma 3.2, c_{s_n} is the first return of 0 to $I^{m(k)}$. So there is an interval $J \ni c_1$ such that $f^{s_n-1} : J \to I^{m(k)-1}$ is a diffeomorphism, and $f^{-1}(J) \subset I^{m(k)}$. As $f^{s_n}((0, c_{s_n})) \subset (c_{s_n}, w_n)$, and $x_n \notin I^{m(k)}$, applying Lemma 2.5, we obtain

$$\begin{aligned}
\frac{|x_n|^\ell}{|c_{s_n}|^\ell} &= 1 + \frac{|x_n|^\ell - |c_{s_n}|^\ell}{|c_{s_n}|^\ell} \geq 1 + C(J, (f(0), f(c_{s_n})))^{-1} \\
&\geq 1 + C(I^{m(k)-1}, f^{s_n}((0, c_{s_n})))^{-1}\mathcal{O}_k \\
&\geq 1 + C(I^{m(k)-1}, (c_{s_n}, w_n))^{-1}\mathcal{O}_k \\
&= 1 + \frac{|y_n| - |w_n|}{|w_n| - |c_{s_n}|}\frac{|y_n| + |c_{s_n}|}{2|y_n|}\mathcal{O}_k \\
&\geq 1 + \frac{|y_n| - |w_n|}{2|w_n|}\frac{|w_n|}{|w_n| - |c_{s_n}|}\mathcal{O}_k \\
&\geq 1 + \frac{\lambda\mu-1}{2}\frac{|x_n|}{|x_n| - |c_{s_n}|}\mathcal{O}_k \\
&\geq 1 + \frac{\lambda\mu-1}{2}\frac{\alpha}{\alpha-1}\mathcal{O}_k.
\end{aligned}$$

Since the left-hand side of this inequality is less than or equal to α^2, the lemma follows. □

Completion of the Main Lemma in the type II *case.* In the following, we distinguish two cases. In each case, we estimate U_n and V_n to check that U_nV_n is greater than 1 by a definite amount.

Case 3.1. $\theta \le 1$. By (4.9),

$$\begin{aligned}
\frac{1}{1-U_n} &\ge (1+2(\mu^2+\mu+2)+(\mu^2+3\mu+2)(\mu^2-\mu+2))\frac{\mu}{4(\mu-1)}\\
&= \frac{(\mu-1)^5+7(\mu-1)^4+21(\mu-1)^3+37(\mu-1)^2+43(\mu-1)+21}{4(\mu-1)}\\
&\ge \frac{37(\mu-1)^2+43(\mu-1)+21}{4(\mu-1)}\\
&\ge \frac{2\sqrt{37\times 21}+43}{4}\\
&\ge 24,
\end{aligned}$$

which implies $U_n \ge 23/24$.

Since $\lambda\mu = (\mu^2+\mu+2+\theta)/2 \ge 2$, we have $\alpha^2 \ge 1 + \mathcal{O}_k 0.5\alpha/(\alpha-1)$, which implies $\alpha > 3/2$ (provided that k is sufficiently large). Thus,

$$V_n - 1 = \frac{|y_n|-|x_n|}{|y_n|+|x_n|}\frac{|x_n|-|c_{s_n}|}{|x_n|+|c_{s_n}|} \ge \frac{\lambda-1}{\lambda+1}\frac{\alpha-1}{\alpha+1} \ge \frac{0.9}{2.9}\cdot\frac{0.5}{2.5} > \frac{1}{20},$$

and consequently,

$$U_nV_n > \frac{23}{24}\cdot\frac{21}{20} > 1.$$

Case 3.2. $\theta > 1$. By (4.9),

$$\begin{aligned}
\frac{1}{1-U_n} &\ge \frac{\theta^2+2(\mu^2+\mu+2)\theta+(\mu^2+3\mu+2)(\mu^2-\mu+2)}{\theta}\frac{\mu}{4(\mu-1)}\\
&\ge \frac{\mu}{2(\mu-1)}(\sqrt{(\mu^2+3\mu+2)(\mu^2-\mu+2)}+\mu^2+\mu+2)\\
&\ge \frac{\mu}{2(\mu-1)}\{\mu^2+(1+\sqrt{5+4\sqrt{2}})\mu+2\}\\
&\ge \frac{\mu(\mu^2+4.2\mu+2)}{2(\mu-1)}\\
&\ge \frac{7.2(\mu-1)^2+13.4(\mu-1)+7.2}{2(\mu-1)}\\
&\ge 13.
\end{aligned}$$

Thus, $U_n \ge 12/13$. Moreover,

$$\begin{aligned}
\lambda &= \frac{\mu^2+\mu+2+\theta}{2\mu}\\
&\ge \frac{\mu^2+\mu+3}{2\mu} \ge \sqrt{3}+\frac{1}{2} > 2.2.\\
\lambda\mu &= \frac{\mu^2+\mu+2+\theta}{2} \ge 2.5.
\end{aligned}$$

By Lemma 4.3, $\alpha^2 \geq 1 + 0.75\alpha/(\alpha - 1)\mathcal{O}_k$, and hence $\alpha > 1.6$. Therefore,

$$V_n - 1 \geq \frac{1.2}{3.2} \cdot \frac{0.6}{2.6} > \frac{1}{12},$$

which implies that $U_n V_n$ is bounded from below by a constant greater than 1. $\square$

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