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The Parisi formula

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Dedicated to Francesco Guerra

Abstract

Using Guerra's interpolation scheme, we compute the free energy of the Sherrington-Kirkpatrick model for spin glasses at any temperature, confirming a celebrated prediction of G. Parisi.

1. Introduction

The Hamiltonian of the Sherrington-Kirkpatrick (SK) model for spin glasses [10] is given at inverse temperature β by

$$H_N(\boldsymbol{\sigma}) = -\frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij}\sigma_i\sigma_j. \tag{1.1}$$

Here $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N) \in \Sigma_N = \{-1, 1\}^N$, and $(g_{ij})_{i<j}$ are independent and identically distributed (i.i.d.) standard Gaussian random variable (r.v.). It is unexpected that the simple, basic formula (1.1) should give rise to a very intricate structure. This was discovered over 20 years ago by G. Parisi [8]. The predictions of Parisi became the starting point of a whole theory, the breadth and the ambitions of which can be measured in the books [6] and [9]. Literally hundreds of papers of theoretical physics have been inspired by these ideas.

The SK model is a purely mathematical object, but the methods by which it has been studied by Parisi and followers are not likely to be recognized as legitimate by most mathematicians. The present paper will correct this discrepancy and will make one of the central predictions of Parisi, the computation of the "free energy" of the SK model appear as a consequence of a general mathematical principle. This general principle will also apply for even p to the "p-spin" generalization of (1.1), where the Hamiltonian is given at inverse

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temperature β by

$$H_N(\boldsymbol{\sigma}) = -\beta\Big(\frac{p!}{2N^{p-1}}\Big)^{1/2} \sum_{i_1<\ldots<i_p} g_{i_1,\ldots,i_p}\sigma_{i_1}\cdots\sigma_{i_p}. \tag{1.2}$$

We consider for each N a Gaussian Hamiltonian H_N on Σ_N, that is a jointly Gaussian family of r.v. indexed by Σ_N. (Here, as everywhere in the paper, by Gaussian r.v., we mean that the variable is centered.) We assume that for a certain sequence $c(N) \to 0$ and a certain function $\xi : \mathbb{R} \to \mathbb{R}$, we have

$$\forall \boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2 \in \Sigma_N \ , \ \Big|\frac{1}{N}\mathsf{E}H_N(\boldsymbol{\sigma}^1)H_N(\boldsymbol{\sigma}^2) - \xi(R_{1,2})\Big| \leq c(N), \tag{1.3}$$

where

$$R_{1,2} = R_{1,2}(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \frac{1}{N}\sum_{i\leq N} \sigma_i^1\sigma_i^2 \tag{1.4}$$

is called the overlap of the configurations $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$. A simple computation shows that for the Hamiltonian (1.2), we have (1.3) for $\xi(x) = \beta^2 x^p/2$ and $c(N) \leq K(p)/N$, where $K(p)$ depends on p only.

When ξ is three times continuously differentiable, and satisfies

$$\xi(0) = 0,\ \xi(x) = \xi(-x),\ \xi''(x) > 0 \ \text{ if } x > 0\ , \tag{1.5}$$

we will compute the asymptotic free energy of Hamiltonians satisfying (1.3).

We fix once and for all a number h (that represents the strength of an "external field").

Consider an integer $k \geq 1$ and numbers

$$0 = m_0 \leq m_1 \leq \cdots \leq m_{k-1} \leq m_k = 1 \tag{1.6}$$

and

$$0 = q_0 \leq q_1 \leq \cdots \leq q_{k+1} = 1. \tag{1.7}$$

It helps to think of m_ℓ as being a parameter attached to the interval $[q_\ell, q_{\ell+1}[$. To lighten notation, we write

$$\boldsymbol{m} = (m_0, \ldots, m_{k-1}, m_k)\ ; \quad \boldsymbol{q} = (q_0, \ldots, q_k, q_{k+1}). \tag{1.8}$$

Consider independent Gaussian r.v. $(z_p)_{0\leq p\leq k}$ with

$$\mathsf{E}z_p^2 = \xi'(q_{p+1}) - \xi'(q_p). \tag{1.9}$$

We define the r.v.

$$X_{k+1} = \log\operatorname{ch}\Big(h + \sum_{0\leq p\leq k} z_p\Big)$$

and recursively, for $\ell \geq 0$

$$X_\ell = \frac{1}{m_\ell}\log \mathsf{E}_\ell \exp m_\ell X_{\ell+1}, \tag{1.10}$$

where E_ℓ denotes expectation in the r.v. $z_p, p \geq \ell$. When $m_\ell = 0$ this means $X_\ell = \mathsf{E}_\ell X_{\ell+1}$. Thus $X_0 = \mathsf{E}_0 X_1$ is a number. We set

$$\mathcal{P}_k(\boldsymbol{m}, \boldsymbol{q}) = \log 2 + X_0 - \frac{1}{2} \sum_{1 \leq \ell \leq k} m_\ell \big(\theta(q_{\ell+1}) - \theta(q_\ell)\big) \tag{1.11}$$

where

$$\theta(q) = q\xi'(q) - \xi(q). \tag{1.12}$$

We define

$$\mathcal{P}(\xi, h) = \inf \mathcal{P}_k(\boldsymbol{m}, \boldsymbol{q}), \tag{1.13}$$

where the infimum is over all choices of k and all choices of the sequences $\boldsymbol{m}$ and $\boldsymbol{q}$ as above.

One might notice that giving sequences $\boldsymbol{m}$ and $\boldsymbol{q}$ as in (1.8) is the same as giving a probability measure μ on $[0, 1]$ that charges at most k points (the points q_ℓ for $1 \leq \ell \leq k$, the mass of q_ℓ being $m_\ell - m_{\ell-1}$). One can then write $\mathcal{P}(\mu)$ rather than $\mathcal{P}_k(\boldsymbol{m}, \boldsymbol{q})$. Moreover Guerra [3] proves that this definition can be extended by a continuity argument to any probability measure μ on $[0, 1]$, and the distribution function of such a probability is the "functional order parameter" of the theoretical physicists. We do not adopt this point of view since an essential ingredient of our approach is that we need only consider discrete objects rather than continuous ones. We refer the reader to [18] for further results in this direction.

THEOREM 1.1 (The Parisi formula). *We have*

$$\lim_{N\to\infty} \frac{1}{N} \mathsf{E} \log \sum_{\boldsymbol{\sigma}} \exp\Big(H_N(\boldsymbol{\sigma}) + h \sum_{i \leq N} \sigma_i\Big) = \mathcal{P}(\xi, h). \tag{1.14}$$

The summation is of course over all values of $\boldsymbol{\sigma} \in \Sigma_N$. To lighten the exposition, we do not follow the convention of physics to put a minus sign in front of the Hamiltonian.

We learned the present formulation in Guerra's work [3], to which we refer for further discussion of its connections with Parisi's original formulation. In this truly remarkable paper Guerra proves that the left-hand side of (1.14) is bounded by the right-hand side, using an interpolation scheme that is the backbone of the present work. Guerra and Toninelli [5] had previously established the existence of the limit in (1.14).

Even in concrete cases, the computation of the quantity $\mathcal{P}(\xi, h)$ is certainly a nontrivial issue. In fact, it is possibly a difficult problem. This problem however is of a different nature, and we will not investigate it. It should be pointed out that one of the reasons that make our proof of Theorem 1.1 possible is that we have succeeded in separating the proof of this theorem from the issue of computing $\mathcal{P}(\xi, h)$.

When the infimum in (1.13) is a minimum, and if $k \geq 1$ is the smallest integer for which $\mathcal{P}(\xi, h) = \mathcal{P}_k(\boldsymbol{m}, \boldsymbol{q})$ for a certain choice of $\boldsymbol{m}$ and $\boldsymbol{q}$, one says in physics that the system exhibits "$k-1$ steps of replica symmetry breaking". Only the case $k = 1$ ("high temperature behavior") and $k = 2$ (as in the p-spin interaction model for $p \geq 3$ at suitable temperatures) have been described in the physics literature but it is possible (elaborating on the ideas of [14]) to show that suitable choices of ξ can produce situations where k is any integer. The most interesting situation is however when the infimum is not attained in (1.13), which is expected to be the case for the SK model (where $\xi(x) = \beta x^2/2$) when β is large enough.

The Parisi formula can be seen as a theorem of mathematical analysis. The proof we present is self-contained, and requires no knowledge whatsoever of physics. It could however be of some interest to briefly discuss some of the results and of the ideas that led to this proof. This discussion, that occupies the rest of the present paragraph, assumes that the reader is somewhat familiar with the area and its recent history, and understands it is in no way a prerequisite to read the rest of the paper. We will discuss only the history of the SK model (where $\xi(x) = \beta x^2/2$). In that case, at given h, for β small enough, the infimum in (1.13) is obtained for $k = 1$, and the corresponding value is known as the "replica-symmetric solution". The region of parameters β, h where this occurs is known as the "high-temperature region". For sufficiently small β, (say, $\beta \leq 1/10$), and any value of h, the author [21] first proved in 1996 the validity of (1.14) using the so-called "cavity method" (which is developed at length in his book [16]). Soon after, and independently, M. Shcherbina [11] produced a proof using somewhat different ideas, valid in a larger region of parameters and, in particular, for all h and all $\beta \leq 1$. It became soon apparent however that the cavity method is powerless to obtain (1.14) in the entire high-temperature region.

One of the key ideas of our approach is the observation (to be detailed later) that, in order to prove *lower* bounds for the left-hand side of (1.14), it is sufficient to prove *upper* bounds on similar quantities that involve two copies of the system (what is called real replicas in physics). The author observed this in 1998 while writing the paper [13]. This observation was not very useful at that time, since there was no method to prove upper bounds. In 2000, F. Guerra [2] invented an interpolation method (which he later improved in his marvelous paper [3] that plays an essential role in our approach) to prove such upper bounds, and soon after the author [15] attempted to combine Guerra's method of proving upper bounds with his method to turn upper bounds into lower bounds to try to prove (1.14) in the entire high temperature region.

The main difficulty is that when one tries to use Guerra's method for two replicas, some terms due to the interaction between these replicas have the wrong sign. The device used by the author [15] in an attempt to overcome this difficulty unfortunately runs into intractable technical problems. The paper [15] inspired in turn a work by Guerra and Toninelli [4], with a more straightforward approach, but that also fails to reach the entire high-temperature region. The author then improved in [16, Th. 2.9.10], the result of Guerra and Toninelli [4], and it was at this time that he made the simple, yet critical, observation that the difficulties occurring when one attempts to use Guerra's scheme of [2] for two replicas largely disappear when, rather than considering the system consisting of two replicas, one considers instead the subsystem of the set of pairs of configurations with a given overlap. The region reached by this theorem still seems smaller than the high-temperature region. The author obtained somewhat later, in spring 2003, the proof of (1.14) in the entire high-temperature region, and presented it in [16, Th. 2.11.16]. Even though our proof of Theorem 1.1 is self-contained, to penetrate the underlying ideas, the reader might find it useful to look first at this simpler use of our main techniques.

The basic mechanism of the proof extracts crucial information from the fact that one cannot improve the bound obtained for $k = 1$ when one uses instead $k = 2$. This mechanism is simpler to describe in the case of the control of the high-temperature region than in the general case, which involves more details. It should be stressed however that the conventional wisdom, that asserted that the proof of (1.14) would be much easier in the high-temperature region than in general, turned out to be completely wrong. Rather surprisingly, the main ideas of our proof of the Parisi formula seem already required to prove it in the entire high temperature region. A crucial difficulty in the control of this region is that in some sense low temperature behavior seems to occur earlier when one considers two replicas rather than one. Even to control the high temperature region, our proof uses one idea of the type "symmetry breaking" (as inspired by Guerra [3]). Thus, unexpectedly, while it took many years to prove the Parisi formula in the entire high-temperature region, it took only a few weeks more to prove it for all values of the parameters.

Interestingly, and despite Theorem 1.1, it is still not known exactly what is the high temperature region of the SK model. This is due to the difficulty of computing $\mathcal{P}(\xi, h)$. F. Guerra proved that for any values of β and h, if the r.v. z is standard Gaussian, the equation $q = \mathsf{E}\mathrm{th}^2(\beta z\sqrt{q} + h)$ has a unique solution, and F. Toninelli [22] deduced from Guerra's upper bound of [3] that, if q is this unique solution, in the high temperature region one has

$$\beta^2 \,\mathsf{E}\frac{1}{\mathrm{ch}^4(\beta z\sqrt{q} + h)} \le 1. \tag{1.15}$$

It seems possible that the region where Condition (1.15) holds is exactly the high temperature region, but this has not been proved yet. (This question boils down to a nasty calculus problem, see [16, p. 154].)

It seems of interest to mention some of the developments that occurred during the rather lengthy interval that separated the submission of this work from its revision. The author [19] extended Theorem 1.1 to the case of the spherical model and obtained some information on the physical meaning of the parameters occurring in $\mathcal{P}(\xi, h)$ [18], [21]. Moreover, D. Panchenko [7] extended Theorem 1.1 to the case where the spins can take more general values than -1 and 1 .

The Parisi conjecture (1.14) was probably the most widely known open problem about "spin glasses", and it is certainly nice to have been able to prove it. The author would like however to stress that, when seen as part of the global area of spin glass models, this is a rather limited progress. It is not more than a very first step in a very rich area. Many of the most fundamental and fascinating predictions of the Parisi theory remain conjectures, even in the case of the SK model. This is in particular the case of ultrametricity and of the so-called chaos problem. These problems apparently cannot be solved using only the techniques of the present paper, or simple modifications of these. It is even conceivable that they will turn out to be very difficult. In fact, very little is presently known about the structure of the Gibbs measure. Moreover, the techniques of the present paper rely on rather specific arguments, namely using the convexity of ξ, to ensure that certain remainder terms are nonnegative. It is not known at this time how to use a similar approach for *any* of the important spin glass models other than the class described here (and variations of it). A detailed description in mathematical terms of some of the most blatant open problems on spin glasses can be found in [20].

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2. Methodology

To lighten notation, we will not indicate the dependence in N, so that our basic Hamiltonian is denoted by H. Central to our approach is the interpolation scheme recently discovered by F. Guerra [3]. Consider an integer k and sequences $\boldsymbol{m}, \boldsymbol{q}$ as above. Consider independent copies $(z_{i,p})_{0\le p\le k}$ of the sequence $(z_p)_{0\le p\le k}$ of (1.9), that are independent of the randomness of H. We denote by E_ℓ expectation in the r.v. $(z_{i,p})_{i\le N, p\ge \ell}$. We consider the Hamiltonian

$$H_t(\boldsymbol{\sigma}) = \sqrt{t}H(\boldsymbol{\sigma}) + \sum_{i\le N} \sigma_i \Big(h + \sqrt{1-t} \sum_{0\le p\le k} z_{i,p}\Big). \tag{2.1}$$

We define

$$F_{k+1,t} = \log \sum_{\boldsymbol{\sigma}} \exp H_t(\boldsymbol{\sigma}), \tag{2.2}$$

and, for $\ell \geq 1$, we define recursively

$$F_{\ell,t} = \frac{1}{m_\ell} \log \mathsf{E}_\ell \exp m_\ell F_{\ell+1,t}. \tag{2.3}$$

When $m_\ell = 0$ this means that $F_{\ell,t} = \mathsf{E}_\ell F_{\ell+1,t}$. We set

$$\varphi(t) = \frac{1}{N} \mathsf{E} F_{1,t}. \tag{2.4}$$

The expectation here is in both the randomness of H and the r.v. $(z_{i,0})_{i \leq N}$.

We write, for $1 \leq \ell \leq k$,

$$W_\ell = \exp m_\ell (F_{\ell+1,t} - F_{\ell,t}). \tag{2.5}$$

(To lighten notation, the dependence in t is kept implicit.) We denote by Ξ_ℓ the σ-algebra generated by H and the variables $(z_{i,p})_{i \leq N, p < \ell}$ so that $F_{\ell,t}$ is Ξ_ℓ-measurable, and

$$W_\ell \text{ is } \Xi_{\ell+1}\text{-measurable.} \tag{2.6}$$

Since $\mathsf{E}_\ell(\cdot) = \mathsf{E}(\cdot | \Xi_\ell)$, it follows from (2.3) that

$$\mathsf{E}_\ell(W_\ell) = 1. \tag{2.7}$$

Using (2.6), and since $\mathsf{E}_\ell = \mathsf{E}_\ell \mathsf{E}_{\ell+1}$, we see inductively from (2.7) that

$$\mathsf{E}_\ell(W_\ell \cdots W_k) = \mathsf{E}_\ell(W_\ell) \mathsf{E}_{\ell+1}(W_{\ell+1} \cdots W_k) = 1. \tag{2.8}$$

Let us denote by $\langle f \rangle_t$ the average of a function f for the Gibbs measure with Hamiltonian H_t, i.e.

$$\langle f \rangle_t \exp F_{k+1,t} = \sum_{\boldsymbol{\sigma}} f(\boldsymbol{\sigma}) \exp H_t(\boldsymbol{\sigma}).$$

We then see from (2.8) that the functional

$$f \mapsto \mathsf{E}_\ell\big(W_\ell \cdots W_k \langle f \rangle_t\big)$$

is a probability γ_ℓ on Σ_N. We denote by $\gamma_\ell^{\otimes 2}$ its product on Σ_N^2, and for a function $f : \Sigma_N^2 \to \mathbb{R}$ we set

$$\mu_\ell(f) = \mathsf{E}\big(W_1 \cdots W_{\ell-1} \gamma_\ell^{\otimes 2}(f)\big). \tag{2.9}$$

THEOREM 2.1 (Guerra's identity [3]). *For $0 < t < 1$ we have*

$$\begin{aligned} \varphi'(t) = &-\frac{1}{2} \sum_{1 \leq \ell \leq k} m_\ell(\theta(q_{\ell+1}) - \theta(q_\ell)) \\ &-\frac{1}{2} \sum_{1 \leq \ell \leq k} (m_\ell - m_{\ell-1}) \mu_\ell\big(\xi(R_{1,2}) - R_{1,2}\xi'(q_\ell) + \theta(q_\ell)\big) + \mathcal{R} \end{aligned} \tag{2.10}$$

where $|\mathcal{R}| \leq c(N)$.

The convexity of ξ implies that

$$\forall x\,,\ \xi(x) - x\xi'(q) + \theta(q) \geq 0 \tag{2.11}$$

so by (2.10) we have

$$\varphi(1) \leq \varphi(0) - \frac{1}{2} \sum_{1\leq \ell \leq k} m_\ell\big(\theta(q_{\ell+1}) - \theta(q_\ell)\big) + c(N). \tag{2.12}$$

One basic idea of (2.1) is that for $t = 0$, there is no interaction between the sites, so that $\varphi(0)$ is easy to compute. In fact, if we denote by $X_{i,\ell}$ the r.v. defined as in (1.10) but starting with the sequence $(z_{i,p})_{0\leq p\leq k}$ rather than with the sequence $(z_p)_{0\leq p\leq k}$, we see immediately by decreasing induction over ℓ that

$$F_{\ell,0} = N\log 2 + \sum_{i\leq N} X_{i,\ell} \tag{2.13}$$

so that

$$\varphi(0) = \log 2 + X_0 \tag{2.14}$$

and (2.12) implies

$$\frac{1}{N}\mathsf{E}\log\sum_{\boldsymbol{\sigma}} \exp\Big(H_N(\boldsymbol{\sigma}) + h\sum_{i\leq N}\sigma_i\Big) \leq \mathcal{P}_k(\boldsymbol{m},\boldsymbol{q}) + c(N), \tag{2.15}$$

which proves "half" of Theorem 1.1, the main result of [3].

Soon after the present work was submitted for publication, Aizenman, Sims and Starr [1] produced a generalization of Guerra's interpolation scheme (nontrivial arguments are required to show that this scheme actually contains Guerra's scheme). The main purpose of this scheme seems to have been to try to improve on Guerra's bound (2.15). As Theorem 1.1 shows, this is not possible. However the scheme of [1] is still of interest, and is more transparent than Guerra's scheme. It was used in particular by the author [17] to prove that Guerra's bound (2.15) still holds if one relaxes condition (1.5) into assuming that ξ is convex on $\mathbb{R}^+$ rather than on $\mathbb{R}$ as is assumed in [3]. It would be nice to be able to prove Theorem 1.1 under these weaker conditions on ξ. This would in particular cover the case of the p-spin interaction model for odd p.

We will deduce the other half of Theorem 1.1 from the following, where we recall that φ depends implicitly on $k, \boldsymbol{m}$ and $\boldsymbol{q}$.

Theorem 2.2. *Given $t_0 < 1$, there exists a number $\varepsilon > 0$, depending only on t_0, ξ and h, with the following property. Assume that for some number k and for some sequences $\boldsymbol{m}$ and $\boldsymbol{q}$ as in* (1.8), *we have*

$$\mathcal{P}_k(\boldsymbol{m},\boldsymbol{q}) \leq \mathcal{P}(\xi, h) + \varepsilon, \tag{2.16}$$

(2.17) *$\mathcal{P}_k(\boldsymbol{m},\boldsymbol{q})$ realizes the minimum over all choices of $\boldsymbol{m}$ and $\boldsymbol{q}$.*

Then, for $t \le t_0$, we have

$$\lim_{N\to\infty} \varphi(t) = \psi(t) := \varphi(0) - \frac{t}{2} \sum_{1\le \ell \le k} m_\ell\big(\theta(q_{\ell+1}) - \theta(q_\ell)\big). \tag{2.18}$$

The existence of $\boldsymbol{m}$ and $\boldsymbol{q}$ satisfying (2.17) is obvious by a compactness argument. It is to permit this compactness argument that equality is allowed in (1.6) and (1.7). However, when $\boldsymbol{m}$ and $\boldsymbol{q}$ are as in (2.17), without loss of generality, we can assume (decreasing k if necessary) that

(2.19)
$$0 = q_0 < q_1 < \cdots < q_k < q_{k+1} = 1 \ , \ 0 = m_0 < m_1 < \cdots < m_{k-1} < m_k = 1.$$

This is because if $q_\ell = q_{\ell+1}$ then $z_\ell = 0$, so that we can remove $q_{\ell+1}$ from the list $\boldsymbol{q}$ and m_ℓ from the list $\boldsymbol{m}$ without changing anything. If $m_\ell = m_{\ell+1}$ we can "merge the intervals $[q_\ell, q_{\ell+1}[$ and $[q_{\ell+1}, q_{\ell+2}[$" and remove $q_{\ell+1}$ from $\boldsymbol{q}$ and m_ℓ from $\boldsymbol{m}$.

The central point of Theorem 2.2 is the fact that $t_0 < 1$ can be as close to 1 as one wishes. The expert about the cavity method should have already guessed that if instead of (2.17) we fix $\boldsymbol{m}$ and we assume that $\mathcal{P}_k(\boldsymbol{m},\boldsymbol{q})$ realizes the minimum over all choices of $\boldsymbol{q}$, then the conclusion of Theorem 2.2 holds for some $t_0 > 0$ (a result that is in the spirit of the fact that "the replica-symmetric solution is true at high enough temperature"). The key mechanism of the proof extracts information from the fact that $\mathcal{P}_k(\boldsymbol{m},\boldsymbol{q})$ is also minimal over all choices of $\boldsymbol{m}$ to reach any value $t_0 < 1$ (a result that is in the spirit of "the control of the entire high-temperature region").

It might be useful to stress the considerable simplification that is brought by Theorem 2.2. One only has to consider structures with a "finite level on complexity" independent of N. It is of course much easier to bring out these structures in a large system than it would be to bring out the whole Parisi structure with "an infinite level of complexity". One can surely expect that this idea of reducing to a "finite level of complexity" through interpolation to be useful in the study of other spin glass systems.

When $\xi''(0) > 0$, one can actually take ε of order $(1-t_0)^6$ in Theorem 2.2. We see no reason why this rate would be optimal.

To prove Theorem 1.1, we see from Guerra's identity that $|\varphi'(t)| \le L + c(N)$, where, as everywhere in this paper, L denotes a number depending on ξ and h only, that need not be the same at each occurrence. Since $\psi(1) = \mathcal{P}_k(\boldsymbol{m},\boldsymbol{q})$, we see from (2.18) that

$$\limsup_{N\to\infty} |\varphi(1) - \mathcal{P}_k(\boldsymbol{m},\boldsymbol{q})| \le L(1-t_0)$$

so that

$$\liminf_{N\to\infty} \varphi(1) \geq \mathcal{P}(\xi, h) - L(1-t_0),$$

and this implies Theorem 1.1 since $t_0 < 1$ is arbitrary.

We will deduce Theorem 2.2 from the following, where, for simplicity, we write $\mu_r(A)$ rather than $\mu_r(\mathbf{1}_A)$ for a subset A of Σ_N^2.

PROPOSITION 2.3. *Given $t_0 < 1$, there exists $\varepsilon > 0$, depending only on t_0, ξ and h, with the following properties. Assume that $k, \boldsymbol{m}, \boldsymbol{q}$ are as in* (2.16), (2.17) *and* (2.19). *Then for any $\varepsilon_1 > 0$, and any $1 \leq r \leq k$, for N large enough, we have for all $t \leq t_0$ that*

$$\mu_r\big(\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2); (R_{1,2} - q_r)^2 \geq K(\psi(t) - \varphi(t)) + \varepsilon_1\}\big) \leq \varepsilon_1. \tag{2.20}$$

Here, as well as in the rest of the paper, K denotes a number depending on $\xi, t_0, h, \boldsymbol{q}$ and $\boldsymbol{m}$ only, and that need not be the same at each occurrence. (Thus here K does not depend on N, t or ε_1.)

Proof of Theorem 2.2. Since ξ is twice continuously differentiable, we have

$$|\xi(R_{1,2}) - R_{1,2}\xi'(q_r) + \theta(q_r)| \leq L(R_{1,2} - q_r)^2 \tag{2.21}$$

and thus (2.20) implies (since $|R_{1,2} - q_r| \leq 2$) that for $t \leq t_0$, we have

$$\mu_r\big(\xi(R_{1,2}) - R_{1,2}\xi'(q_r) + \theta(q_r)\big) \leq K(\psi(t) - \varphi(t)) + L\varepsilon_1$$

and (2.10) implies that

$$\big(\psi(t) - \varphi(t)\big)' \leq K\big(\psi(t) - \varphi(t)\big) + L\varepsilon_1 + c(N). \tag{2.22}$$

Since $\varphi(0) = \psi(0)$, (2.18) follows by integration. $\square$

The essential ingredient in the proof of (2.20) is an *a priori* bound of the same nature as (2.15), but for two copies of the system coupled in a special way. This construction will make the functionals μ_ℓ of (2.9) appear as very natural objects. We fix $1 \leq r \leq k$ and sequences $\boldsymbol{m}$ and $\boldsymbol{q}$ as in (2.16), (2.17), and (2.19) once and for all. (Thus $m_1 > 0$.) We consider a sequence of pairs of Gaussian r.v. (z_p^1, z_p^2), for $0 \leq p \leq k$. Each pair is independent of the others. For $j = 1$ or $j = 2$ the sequence (z_p^j) is as in (1.9); but

$$z_p^1 = z_p^2 \text{ if } p < r; \quad z_p^1 \text{ and } z_p^2 \text{ are independent if } \ p \geq r. \tag{2.23}$$

We consider the Hamiltonian

$$H_t(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \sqrt{t}\big(H(\boldsymbol{\sigma}^1) + H(\boldsymbol{\sigma}^2)\big) + \sum_{j=1,2}\sum_{i\leq N} \sigma_i^j\Big(h + \sqrt{1-t}\sum_{0\leq p\leq k} z_{i,p}^j\Big), \tag{2.24}$$

where $(z_{i,p}^1, z_{i,p}^2)_{0\le p\le k}$ are independent copies of the sequence $(z_p^1, z_p^2)_{0\le p\le k}$, that are also independent of the randomness in H. We define

$$n_\ell = \frac{m_\ell}{2} \ \text{ if } 0 \le \ell < r; \quad n_\ell = m_\ell \ \text{ if } r \le \ell \le k, \tag{2.25}$$

and

$$J_{k+1,t,u} = \log \sum_{R_{1,2}=u} \exp H_t(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2). \tag{2.26}$$

Thus, the sum is taken only over all pairs $(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)$ for which $R_{1,2} = u$. (We always assume that u is taken such that such pairs exist.) For $\ell \ge 0$, we define recursively

$$J_{\ell,t,u} = \frac{1}{n_\ell} \log \mathsf{E}_\ell \exp n_\ell J_{\ell+1,t,u} \tag{2.27}$$

where E_ℓ denotes expectation in the r.v. $z_{i,p}^j$ for $p \ge \ell$, and we set

$$\Psi(t,u) = \frac{1}{N} \mathsf{E} J_{1,t,u}, \tag{2.28}$$

where the expectation is in the randomness of H and the r.v. $z_{i,0}^j$. The *a priori* estimate on which the paper relies is the following.

THEOREM 2.4. *If* $t_0 < 1$, *there is a number* $\varepsilon > 0$, *depending only on* t_0, ξ *and* h *such that whenever* (2.16), (2.17) *and* (2.19) *hold*, *for all* $t \le t_0$ *we have*

$$\Psi(t,u) \le 2\psi(t) - \frac{(u-q_r)^2}{K} + 2c(N), \tag{2.29}$$

where K *does not depend on* t *or* N.

It is very likely that with a further effort, one could get an explicit dependence of K in t_0, probably $K = L/(1-t_0)^2$, thereby obtaining a rate of convergence in Theorem 1.1. This line of investigation is better left for further research.

To obtain Proposition 2.3, we will combine (2.29) with the following.

PROPOSITION 2.5. *Assume that for some* $\varepsilon_2 > 0$ *we have*

$$\Psi(t,u) \le 2\varphi(t) - \varepsilon_2. \tag{2.30}$$

Then we have

$$\mu_r(\{R_{1,2} = u\}) \le K \exp\Big(-\frac{N}{K}\Big), \tag{2.31}$$

where K *does not depend on* N *or* t.

Proof of Proposition 2.3. Consider $t_0 < 1$ and let $\varepsilon > 0$ be as in Theorem 2.4. Let K_0 be the constant of (2.29). Consider $\varepsilon_1 > 0$. Then if

$$(u - q_r)^2 \geq 2K_0(\psi(t) - \varphi(t)) + \varepsilon_1,$$

by (2.29) we have $\Psi(t,u) \leq 2\varphi(t) - \varepsilon_1/K_0 + 2c(N)$, so that (2.30) holds for N large with $\varepsilon_2 = \varepsilon_1/2K_0$. Since there are at most $2N+1$ values of u to consider (because $NR_{1,2} \in \mathbb{Z}$), it follows from (2.31) that

$$\mu_r\Big(\big\{(R_{1,2} - q_r)^2 \geq 2K_0(\psi(t) - \varphi(t)) + \varepsilon_1\big\}\Big) \leq (2N+1)K \exp\Big(-\frac{N}{K}\Big),$$

and for N large enough the right-hand side is $\leq \varepsilon_1$ for all $t \leq t_0$. □

The proof of Proposition 2.5 has two parts. The first part relies on a rather general principle, but the second will shed some light on the conditions (2.23) and (2.25).

Keeping the dependence in t implicit, we define

$$J_{k+1} = \log \sum_{\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2} \exp H_t(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2), \tag{2.32}$$

where the sum is now over all pairs of configurations, and we define recursively J_ℓ as in (2.27). We set

$$V_\ell = \exp n_\ell (J_{\ell+1} - J_\ell) \tag{2.33}$$

and denote by $\langle \cdot \rangle$ an average for Gibbs' measure with Hamiltonian (2.24). To lighten notation we write $J_{\ell,u}$ rather than $J_{\ell,t,u}$.

Lemma 2.6. *If we have* $\mathsf{E}(J_{1,u}) \leq \mathsf{E}(J_1) - \varepsilon_2 N$, *then for some number* K' *not depending on* N *or* t *we have*

$$\mathsf{E}\big(V_1 \cdots V_k \langle \mathbf{1}_{\{R_{1,2}=u\}} \rangle\big) \leq K' \exp\Big(-\frac{N}{K'}\Big).$$

Proof. Let $U = \langle \mathbf{1}_{\{R_{1,2}=u\}} \rangle$, so that $U \leq 1$ and

$$J_{k+1,u} = J_{k+1} + \log U. \tag{2.34}$$

Arguing as in (2.8), we see that

$$\forall \ell \geq 0\ ,\ \mathsf{E}_\ell\big(V_\ell \cdots V_k U\big) \leq 1. \tag{2.35}$$

We prove by decreasing induction over ℓ that

$$J_{\ell+1,u} \geq J_{\ell+1} + \frac{1}{n_{\ell+1}} \log \mathsf{E}_{\ell+1}\big(V_{\ell+1} \cdots V_k U\big). \tag{2.36}$$

For $\ell = k$, this is (2.34). For the induction from $\ell+1$ to ℓ, using (2.35) for $\ell+1$ and that $n_\ell \leq n_{\ell+1}$, we see first that

$$J_{\ell+1,u} \geq J_{\ell+1} + \frac{1}{n_\ell} \log \mathsf{E}_{\ell+1}\big(V_{\ell+1} \cdots V_k U\big) \tag{2.37}$$

and thus, using the definition of V_ℓ in the second line,

$$\begin{aligned}\exp n_\ell J_{\ell+1,u} &\geq \mathsf{E}_{\ell+1}(V_{\ell+1}\cdots V_k U)\exp n_\ell J_{\ell+1}\\ &= V_\ell \mathsf{E}_{\ell+1}(V_{\ell+1}\cdots V_k U)\exp n_\ell J_\ell\\ &= \mathsf{E}_{\ell+1}(V_\ell\cdots V_k U)\exp n_\ell J_\ell.\end{aligned}$$

Since J_ℓ does not depend on the r.v. $(z^j_{i,p})$ for $p \geq \ell$, and since $\mathsf{E}_\ell = \mathsf{E}_\ell\mathsf{E}_{\ell+1}$ we have

$$\mathsf{E}_\ell \exp n_\ell J_{\ell+1,u} \geq \exp n_\ell J_\ell \mathsf{E}_\ell(V_\ell\cdots V_k U),$$

and taking logarithms completes the induction. Thus, using (2.36) for $\ell = 0$ we have

$$\log \mathsf{E}_1(V_1\cdots V_k U) \leq n_1(J_{1,u} - J_1)$$

and hence, taking expectation,

$$\mathsf{E}\log \mathsf{E}_1(V_1\cdots V_k U) \leq -\varepsilon_2 n_1 N.$$

Moreover since $m_1 > 0$ we have $n_1 > 0$. It then follows from concentration of measure (as detailed in this setting e.g. in [16, §2.2]) that $\log \mathsf{E}_1(V_1\cdots V_k U) \geq -\varepsilon_2 n_1 N/2$ with a probability at most $K_1\exp(-N/K_1)$, where K_1 does not depend on N or t. Thus $\mathsf{E}_1(V_1\cdots V_k U) \geq \exp(-\varepsilon_2 n_1 N/2)$ with the same probability and the conclusion using (2.35) for $\ell = 1$. □

Lemma 2.7. *We have*

$$\frac{1}{N}\mathsf{E}J_1 = 2\varphi(t), \tag{2.38}$$

and for any function f on $\Sigma_N^{\otimes 2}$, we have

$$\mathsf{E}(V_1\cdots V_k\langle f\rangle) = \mu_r(f). \tag{2.39}$$

Combining this with Lemma 2.6, we prove Proposition 2.5.

Proof. The ideas underlying this proof are very simple, but will play a fundamental role in the sequel. Therefore, we try choose clarity over formality. Writing $\boldsymbol{z}_p = (z_{i,p})_{i\leq N}$, we see that the quantities $F_\ell = F_{\ell,t}$ of (2.3) depend on the randomness of H and the r.v. $(\boldsymbol{z}_p)$ for $p < \ell$, so we can write them as $F_\ell(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_{\ell-1})$. For $j = 1, 2$, we write, with obvious notation

$$F^j_\ell = F_\ell(\boldsymbol{z}^j_1,\ldots,\boldsymbol{z}^j_{\ell-1}).$$

We claim that for $\ell \geq 1$ we have

$$J_\ell = F^1_\ell + F^2_\ell. \tag{2.40}$$

This is obvious for $\ell = k+1$. If $\ell \geq r$, since $\boldsymbol{z}^1_\ell$ and $\boldsymbol{z}^2_\ell$ are independent,

$$\mathsf{E}_\ell\exp m_\ell(F^1_{\ell+1} + F^2_{\ell+1}) = \mathsf{E}_\ell\exp m_\ell F^1_{\ell+1}\mathsf{E}_\ell\exp m_\ell F^2_{\ell+1} = \exp m_\ell(F^1_\ell + F^2_\ell)$$

and this performs the induction step from $\ell+1$ to ℓ in (2.40). If $\ell < r$, since $F^j_{\ell+1}$ depends only on $(\boldsymbol{z}^j_1, \dots, \boldsymbol{z}^j_{r-1})$, we have by (2.23) that $F^1_{\ell+1} = F^2_{\ell+1}$, so, since $n_\ell = m_\ell/2$,

$$\mathsf{E}_\ell \exp n_\ell (F^1_{\ell+1} + F^2_{\ell+1}) = \mathsf{E}_\ell \exp m_\ell F^1_{\ell+1} = \mathsf{E}_\ell \exp m_\ell F^1_\ell = \exp n_\ell (F^1_\ell + F^2_\ell)$$

and this completes the proof of (2.40). Taking $\ell = 1$ and expectation implies (2.38).

Since W_ℓ depends only on $\boldsymbol{z}_1, \dots, \boldsymbol{z}_\ell$, it follows with obvious notation that

$$V_\ell = W^1_\ell = W^2_\ell \text{ if } \ell < r; \quad V_\ell = W^1_\ell W^2_\ell \text{ if } \ell \geq r,$$

from which it is straightforward to check (2.39). $\square$

3. Guerra's bound and its extension

We will first prove Theorem 2.1. Our approach to the computations is slightly simpler than Guerra's [3]. This simplification will be quite helpful when we will consider the more complicated situation of Theorem 3.1.

The main tool of the proof is integration by parts. Consider a jointly Gaussian family of r.v. $\boldsymbol{h} = (h_j)_{j \in J}$, J finite. Then for a function $F : \mathbb{R}^J \to \mathbb{R}$, of moderate growth, we have

$$\mathsf{E} h_i F(\boldsymbol{h}) = \sum_{j \in J} \mathsf{E}(h_i h_j) \mathsf{E} \frac{\partial F}{\partial x_j}(\boldsymbol{h}). \tag{3.1}$$

Since $\exp m_\ell F_{\ell,t} = \mathsf{E}_\ell \exp m_\ell F_{\ell+1,t}$, by (2.3) we get

$$\frac{\partial F_{\ell,t}}{\partial t} \exp m_\ell F_{\ell,t} = \mathsf{E}_\ell \frac{\partial F_{\ell+1,t}}{\partial t} \exp m_\ell F_{\ell+1,t},$$

and since $F_{\ell,t}$ is Ξ_ℓ-measurable, we get

$$\frac{\partial F_{\ell,t}}{\partial t} = \mathsf{E}_\ell W_\ell \frac{\partial F_{\ell+1,t}}{\partial t}$$

where W_ℓ is given by (2.5). By iteration (and arguing as in the proof of (2.8)), we get

$$\varphi'(t) = \mathsf{E}\Big(W_1 \cdots W_k \frac{\partial F_{k+1,t}}{\partial t} \Big). \tag{3.2}$$

Since $m_0 = 0$ and $m_k = 1$, for any numbers $c_1, \dots, c_{k+1}$, we have

$$\sum_{1 \leq \ell \leq k} m_\ell (c_{\ell+1} - c_\ell) = c_{k+1} + \sum_{1 \leq \ell \leq k} c_\ell (m_{\ell-1} - m_\ell). \tag{3.3}$$

Using this for $c_\ell = F_{\ell,t}$, we get

$$W_1 \cdots W_k = T \exp F_{k+1,t} \tag{3.4}$$

where $T = T_1 \cdots T_k$ and

$$T_\ell = \exp F_{\ell,t}(m_{\ell-1} - m_\ell), \tag{3.5}$$

so that

$$\varphi'(t) = \frac{1}{N}\mathsf{E}\Big(T\frac{\partial}{\partial t}\exp F_{k+1,t}\Big) = \mathrm{I} + \sum_{0\le p\le k}\mathrm{II}(p), \tag{3.6}$$

where

$$\mathrm{I} = \frac{1}{2N\sqrt{t}}\mathsf{E}\Big(T\sum_{\boldsymbol{\sigma}} H(\boldsymbol{\sigma})\exp H_t(\boldsymbol{\sigma})\Big), \tag{3.7}$$

$$\mathrm{II}(p) = -\frac{1}{2N\sqrt{1-t}}\mathsf{E}\Big(T\sum_{\boldsymbol{\sigma},i}\sigma_i z_{i,p}\exp H_t(\boldsymbol{\sigma})\Big). \tag{3.8}$$

To compute I, we use (3.1) for the family $(H(\boldsymbol{\sigma}))_{\boldsymbol{\sigma}\in\Sigma_N}$. We write

$$\zeta(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2) = \frac{1}{N}\mathsf{E}\big(H(\boldsymbol{\sigma}^1)H(\boldsymbol{\sigma}^2)\big) \tag{3.9}$$

so that by (1.3) we have

$$|\zeta(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2) - \xi(R_{1,2})| \le c(N). \tag{3.10}$$

We think of the quantities $H(\boldsymbol{\sigma})$ as independent variables, and, with a slight abuse of notation, we have from (3.5) that

$$\frac{\partial T_\ell}{\partial H(\boldsymbol{\rho})} = (m_{\ell-1} - m_\ell)\frac{\partial F_{\ell,t}}{\partial H(\boldsymbol{\rho})}T_\ell$$

so that $\mathrm{I} = \mathrm{III} + \sum_{1\le\ell\le k}\mathrm{I}(\ell)$ where

$$\mathrm{III} = \frac{1}{2\sqrt{t}}\mathsf{E}\Big(T\sum_{\boldsymbol{\sigma},\boldsymbol{\rho}}\zeta(\boldsymbol{\sigma},\boldsymbol{\rho})\frac{\partial}{\partial H(\boldsymbol{\rho})}\exp H_t(\boldsymbol{\sigma})\Big) \tag{3.11}$$

$$\mathrm{I}(\ell) = \frac{m_{\ell-1}-m_\ell}{2\sqrt{t}}\mathsf{E}\Big(T\sum_{\boldsymbol{\sigma},\boldsymbol{\rho}}\zeta(\boldsymbol{\sigma},\boldsymbol{\rho})\exp H_t(\boldsymbol{\sigma})\frac{\partial F_{\ell,t}}{\partial H(\boldsymbol{\rho})}\Big). \tag{3.12}$$

Now

$$\begin{aligned}\frac{\partial}{\partial H(\boldsymbol{\rho})}\exp H_t(\boldsymbol{\sigma}) &= \sqrt{t}\ \mathbf{1}_{\{\boldsymbol{\rho}=\boldsymbol{\sigma}\}}\exp H_t(\boldsymbol{\sigma})\\ &= \sqrt{t}\ \mathbf{1}_{\{\boldsymbol{\rho}=\boldsymbol{\sigma}\}}\langle\mathbf{1}_{\{\boldsymbol{\sigma}\}}\rangle_t\exp F_{k+1,t}.\end{aligned}$$

Here $\mathbf{1}_{\{\boldsymbol{\rho}=\boldsymbol{\sigma}\}}$ is 1 if $\boldsymbol{\rho}=\boldsymbol{\sigma}$ and is 0 otherwise. The function $\mathbf{1}_{\{\boldsymbol{\sigma}\}}$ is such that $\mathbf{1}_{\{\boldsymbol{\sigma}\}}(\boldsymbol{\tau}) = \mathbf{1}_{\{\boldsymbol{\rho}=\boldsymbol{\tau}\}}$ so that $\langle\mathbf{1}_{\{\boldsymbol{\sigma}\}}\rangle_t$ is the mass at $\boldsymbol{\sigma}$ of the Gibbs measure. Thus,

$$\begin{aligned}\sum_{\boldsymbol{\sigma},\boldsymbol{\rho}}\zeta(\boldsymbol{\sigma},\boldsymbol{\rho})\frac{\partial}{\partial H(\boldsymbol{\rho})}\exp H_t(\boldsymbol{\sigma}) &= \sqrt{t}\sum_{\boldsymbol{\sigma}}\zeta(\boldsymbol{\sigma},\boldsymbol{\sigma})\langle\mathbf{1}_{\{\boldsymbol{\sigma}\}}\rangle_t\exp F_{k+1,t}\\ &= \sqrt{t}\langle\zeta(\boldsymbol{\sigma},\boldsymbol{\sigma})\rangle_t\exp F_{k+1,t},\end{aligned}$$

and using (3.10), (3.4) and (2.8) for $\ell = 1$, we get

$$\mathrm{III} = \frac{1}{2}\xi(1) + \mathcal{R}$$

where $|\mathcal{R}| \le c(N)/2$. We have

$$\frac{\partial}{\partial H(\boldsymbol{\rho})} F_{k+1,t} = \sqrt{t}\langle \mathbf{1}_{\{\boldsymbol{\rho}\}}\rangle_t$$

so that, proceeding as in (3.2), we have

$$\frac{\partial}{\partial H(\boldsymbol{\rho})} F_{\ell,t} = \sqrt{t}\mathsf{E}_\ell\big(W_\ell \cdots W_k \langle \mathbf{1}_{\{\boldsymbol{\rho}\}}\rangle_t\big) = \sqrt{t}\ \gamma_\ell(\mathbf{1}_{\{\boldsymbol{\rho}\}}). \tag{3.13}$$

Since $\exp H_t(\boldsymbol{\sigma}) = \langle \mathbf{1}_{\{\boldsymbol{\sigma}\}}\rangle_t \exp F_{k+1,t}$ we get from (3.4) that

$$\mathrm{I}(\ell) = \frac{m_{\ell-1} - m_\ell}{2} \sum_{\boldsymbol{\sigma},\boldsymbol{\rho}} \zeta(\boldsymbol{\sigma},\boldsymbol{\rho}) \mathsf{E}\big(W_1 \cdots W_k \langle \mathbf{1}_{\{\boldsymbol{\sigma}\}}\rangle_t \gamma_\ell(\mathbf{1}_{\{\boldsymbol{\rho}\}})\big). \tag{3.14}$$

Since $\mathsf{E} = \mathsf{E}\mathsf{E}_\ell$ and $W_1, \dots, W_{\ell-1}, \gamma_\ell$ are Ξ_ℓ-measurable, we get that

$$\begin{aligned}
\mathsf{E}\big(W_1 \cdots W_k \langle \mathbf{1}_{\{\boldsymbol{\sigma}\}}\rangle_t \gamma_\ell(\mathbf{1}_{\{\boldsymbol{\rho}\}})\big) &= \mathsf{E}\big(W_1 \cdots W_{\ell-1}\gamma_\ell(\mathbf{1}_{\{\boldsymbol{\rho}\}})\mathsf{E}_\ell(W_\ell \cdots W_k \langle \mathbf{1}_{\{\boldsymbol{\sigma}\}}\rangle_t)\big)\\
&= \mathsf{E}\big(W_1 \cdots W_{\ell-1}\gamma_\ell(\mathbf{1}_{\{\boldsymbol{\rho}\}})\gamma_\ell(\mathbf{1}_{\{\boldsymbol{\sigma}\}})\big)\\
&= \mathsf{E}\big(W_1 \cdots W_{\ell-1}\gamma_\ell^{\otimes 2}(\mathbf{1}_{\{(\boldsymbol{\sigma},\boldsymbol{\rho})\}})\big)\\
&= \mu_\ell(\mathbf{1}_{\{(\boldsymbol{\sigma},\boldsymbol{\rho})\}})
\end{aligned}$$

and thus

$$\mathrm{I}(\ell) = \frac{m_{\ell-1} - m_\ell}{2}\mu_\ell(\zeta(\boldsymbol{\sigma},\boldsymbol{\rho})).$$

Again using (3.10), we get

$$\mathrm{I} = \frac{1}{2}\Big(\xi(1) + \sum_{1\le \ell\le k}(m_{\ell-1} - m_\ell)\mu_\ell(\xi(R_{1,2}))\Big) + \mathcal{R} \tag{3.15}$$

where $|\mathcal{R}| \le c(N)$.

Since $F_{\ell,t}$ does not depend on $z_{i,p}$ for $\ell \le p$, a similar (but easier) computation yields

$$\mathrm{II}(p) = -\frac{1}{2}\big(\xi'(q_{p+1}) - \xi'(q_p)\big)\Big(1 + \sum_{p<\ell\le k}(m_{\ell-1} - m_\ell)\mu_\ell(R_{1,2})\Big). \tag{3.16}$$

Since $\xi'(q_0) = \xi'(0) = 0$, summation of these formulas for $0 \le p \le k$ yields

$$\sum_{0\le p\le k}\mathrm{II}(p) = -\frac{1}{2}\Big(\xi'(1) + \sum_{1\le \ell\le k}(m_{\ell-1} - m_\ell)\xi'(q_\ell)\mu_\ell(R_{1,2})\Big)$$

so that

(3.17)
$$2\varphi'(t) = \xi(1) - \xi'(1) + \sum_{1\le\ell\le k} (m_{\ell-1} - m_\ell)\mu_\ell\big(\xi(R_{1,2}) - R_{1,2}\xi'(q_\ell)\big) + 2\mathcal{R}$$
$$= -\theta(1) - \sum_{1\le\ell\le k} (m_{\ell-1} - m_\ell)\theta(q_\ell)$$
$$+ \sum_{1\le\ell\le k} (m_{\ell-1} - m_\ell)\mu_\ell\big(\xi(R_{1,2}) - R_{1,2}\xi'(q_\ell) + \theta(q_\ell)\big) + 2\mathcal{R}$$

and the result follows using (3.3) for $c_\ell = \theta(q_\ell)$. □

We now turn to the principle on which the paper relies. We consider integers κ, τ, with $\tau \le \kappa$, a number $\eta = \pm 1$, a sequence $n_0 = 0 \le n_1 \le \cdots \le n_\kappa = 1$, and a sequence $\rho_0 = 0 \le \rho_1 \le \cdots \le \rho_{\kappa+1} = 1$. We consider independent pairs of random variables $(Z_p^1, Z_p^2)_{0\le p\le\kappa}$. We construct independent pairs of Gaussian random variables $(y_p^1, y_p^2)_{0\le p\le\kappa}$ with the following properties:

$$y_p^1 = \eta y_p^2 \text{ if } p < \tau, \tag{3.18}$$

$$y_p^1 \text{ and } y_p^2 \text{ are independent if } p \ge \tau, \tag{3.19}$$

$$\mathsf{E}(y_p^j)^2 = t\big(\xi'(\rho_{p+1}) - \xi'(\rho_p)\big). \tag{3.20}$$

We consider independent copies $(Z_{i,p}^1, Z_{i,p}^2)_{0\le p\le\kappa}$ of the sequence $(Z_p^1, Z_p^2)_{0\le p\le\kappa}$, and independent copies $(y_{i,p}^1, y_{i,p}^2)_{0\le p\le\kappa}$ of the sequence $(y_p^1, y_p^2)_{0\le p\le\kappa}$. We assume that these are independent of each other and of the randomness of H. For $0 \le v \le 1$, we define

(3.21)
$$H_v(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \sqrt{vt}H(\boldsymbol{\sigma}^1) + \sqrt{vt}H(\boldsymbol{\sigma}^2) + \sum_{j=1,2}\sum_{i\le N} \sigma_i^j \Big(h + \sum_{0\le p\le\kappa} (Z_{i,p}^j + \sqrt{1-v}\,y_{i,p}^j)\Big).$$

We think of t as fixed, so the dependence in t is not indicated. To lighten notation we set

$$u = \eta\rho_\tau. \tag{3.22}$$

We define

$$F_{\kappa+1,v} = \log \sum_{R_{1,2}=u} \exp H_v(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2), \tag{3.23}$$

that is, the sum is taken only over the pairs $(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)$ of configurations such that $R_{1,2} = u$. We denote by E_ℓ expectation in the variables $Z_{i,p}^j$ and $y_{i,p}^j$ for $p \ge \ell$, and define recursively

$$F_{\ell,v} = \frac{1}{n_\ell} \log \mathsf{E}_\ell \exp n_\ell F_{\ell+1,v}.$$

(If $n_\ell = 0$, this means that $F_{\ell,v} = \mathsf{E}_\ell F_{\ell+1,v}$.) We define

$$\eta(v) = \frac{1}{N}\mathsf{E}F_{1,v}. \tag{3.24}$$

THEOREM 3.1. *For $0 < v < 1$ we have*

$$\eta'(v) \le -t\Big(2\sum_{\ell<\tau} n_\ell\big(\theta(\rho_{\ell+1}) - \theta(\rho_\ell)\big) + \sum_{\ell\ge\tau} n_\ell\big(\theta(\rho_{\ell+1}) - \theta(\rho_\ell)\big)\Big) + 4c(N) \tag{3.25}$$

and, consequently,

$$\eta(1) \le \eta(0) - t\Big(2\sum_{\ell<\tau} n_\ell\big(\theta(\rho_{\ell+1}) - \theta(\rho_\ell)\big) + \sum_{\ell\ge\tau} n_\ell\big(\theta(\rho_{\ell+1}) - \theta(\rho_\ell)\big)\Big) + 4c(N). \tag{3.26}$$

The underlying idea is that, as in Theorem 2.1, for $v = 0$, there is no coupling between the sites, so that we will be able to estimate $\eta(0)$, and thus to bound $\eta(1)$ with (3.26).

Proof. This relies on the same principles as the proof of Theorem 2.1. The main new feature is that new terms are created by the interaction between the two copies of the system we consider now. These terms tend to have the wrong sign to make the argument of Theorem 2.1 work, but the device of restricting the summation to $R_{1,2} = u$ in (3.23) makes these terms much easier to handle. We write

$$V_\ell = \exp n_\ell(F_{\ell+1,v} - F_{\ell,v}); \quad T_\ell = \exp F_{\ell,v}(n_{\ell-1} - n_\ell),$$

so that if $T = T_1 \cdots T_\kappa$ we have $V_1 \cdots V_\kappa = T \exp F_{\kappa+1,v}$. We consider the set

$$S_u = \{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \in \Sigma_N^2; \quad R_{1,2} = u\}$$

and, for a function f on S_u, define $\langle f\rangle_v$ by

$$\langle f\rangle_v \exp F_{\kappa+1,v} = \sum_{(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\in S_u} f(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \exp H_v(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2).$$

We define a probability γ_ℓ on S_u by

$$\gamma_\ell(f) = \mathsf{E}_\ell(V_\ell \cdots V_\kappa \langle f\rangle_v)$$

and for a function f on S_u^2, we write

$$\mu_\ell(f) = \mathsf{E}\big(V_1 \cdots V_{\ell-1}\gamma_\ell^{\otimes 2}(f)\big).$$

As in the case of Theorem 2.1, we obtain

$$\eta'(v) = \mathrm{I} + \sum_{0\le p\le\kappa} \mathrm{II}(p), \tag{3.27}$$

where

$$\mathrm{I} = \frac{\sqrt{t}}{2N\sqrt{v}}\mathsf{E}\Big(T\sum_{R_{1,2}=u}(H(\boldsymbol{\sigma}^1)+H(\boldsymbol{\sigma}^2))\exp H_v(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\Big), \tag{3.28}$$

$$\mathrm{II}(p) = -\frac{\sqrt{t}}{2N\sqrt{1-v}}\mathsf{E}\Big(T\sum_{R_{1,2}=u}\sum_{i\le N, j=1,2}\sigma_i^j y_{i,p}^j \exp H_v(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\Big). \tag{3.29}$$

We have

(3.30)
$$\begin{aligned}\frac{\partial}{\partial H(\boldsymbol{\sigma})}\exp H_v(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2) &= \sqrt{vt}(\mathbf{1}_{\{\boldsymbol{\sigma}^1=\boldsymbol{\sigma}\}}+\mathbf{1}_{\{\boldsymbol{\sigma}^2=\boldsymbol{\sigma}\}})\exp H_v(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\\ &= \sqrt{vt}(\mathbf{1}_{\{\boldsymbol{\sigma}^1=\boldsymbol{\sigma}\}}+\mathbf{1}_{\{\boldsymbol{\sigma}^2=\boldsymbol{\sigma}\}})\langle\mathbf{1}_{\{(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\}}\rangle_v\exp F_{\kappa+1,v}\end{aligned}$$

so that

$$\frac{\partial}{\partial H(\boldsymbol{\sigma})}F_{\kappa+1,v} = \sqrt{vt}\sum_{(\boldsymbol{\tau}^1,\boldsymbol{\tau}^2)\in S_u}(\mathbf{1}_{\{\boldsymbol{\tau}^1=\boldsymbol{\sigma}\}}+\mathbf{1}_{\{\boldsymbol{\tau}^2=\boldsymbol{\sigma}\}})\langle\mathbf{1}_{\{(\boldsymbol{\tau}^1,\boldsymbol{\tau}^2)\}}\rangle_v.$$

Thus, integrating by parts in (3.28), as in the case of Theorem 2.1, we get

$$\mathrm{I} = \mathrm{III} + \sum_{0\le\ell\le\kappa}\mathrm{I}(\ell),$$

where

$$\mathrm{III} = \frac{t}{2}\mathsf{E}\Big(V_1\cdots V_\kappa\sum_{R_{1,2}=u}D_1(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\Big) \tag{3.31}$$

$$\mathrm{I}(\ell) = \frac{t}{2}(n_{\ell-1}-n_\ell)\mathsf{E}\Big(V_1\cdots V_\kappa\sum_{R_{1,2}=u}D_2(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\Big), \tag{3.32}$$

for

(3.33)
$$\begin{aligned}D_1(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2) &= \sum_{\boldsymbol{\sigma}}(\mathbf{1}_{\{\boldsymbol{\sigma}^1=\boldsymbol{\sigma}\}}+\mathbf{1}_{\{\boldsymbol{\sigma}^2=\boldsymbol{\sigma}\}})\big(\zeta(\boldsymbol{\sigma}^1,\boldsymbol{\sigma})+\zeta(\boldsymbol{\sigma}^2,\boldsymbol{\sigma})\big)\langle\mathbf{1}_{\{(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\}}\rangle_v\\ &= \big(\zeta(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^1)+\zeta(\boldsymbol{\sigma}^2,\boldsymbol{\sigma}^2)+2\zeta(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\big)\langle\mathbf{1}_{\{(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\}}\rangle_v\end{aligned}$$

and

$$\begin{aligned}D_2(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2) &= \sum_{\boldsymbol{\sigma}}\big(\zeta(\boldsymbol{\sigma}^1,\boldsymbol{\sigma})+\zeta(\boldsymbol{\sigma}^2,\boldsymbol{\sigma})\big)\langle\mathbf{1}_{\{(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\}}\rangle_v\\ &\qquad\times\sum_{(\boldsymbol{\tau}^1,\boldsymbol{\tau}^2)\in S_u}(\mathbf{1}_{\{\boldsymbol{\tau}^1=\boldsymbol{\sigma}\}}+\mathbf{1}_{\{\boldsymbol{\tau}^2=\boldsymbol{\sigma}\}})\gamma_\ell\big(\mathbf{1}_{\{(\boldsymbol{\tau}^1,\boldsymbol{\tau}^2)\}}\big)\\ &= \langle\mathbf{1}_{\{(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\}}\rangle_v\sum_{(\boldsymbol{\tau}^1,\boldsymbol{\tau}^2)\in S_u}\gamma_\ell\big(\mathbf{1}_{\{(\boldsymbol{\tau}^1,\boldsymbol{\tau}^2)\}}\big)\\ &\qquad\times\big(\zeta(\boldsymbol{\sigma}^1,\boldsymbol{\tau}^1)+\zeta(\boldsymbol{\sigma}^1,\boldsymbol{\tau}^2)+\zeta(\boldsymbol{\sigma}^2,\boldsymbol{\tau}^1)+\zeta(\boldsymbol{\sigma}^2,\boldsymbol{\tau}^2)\big).\end{aligned} \tag{3.34}$$

Using, as in the case of Theorem 2.1, the fact that

$$\mathsf{E}\big(V_1\cdots V_\kappa\langle f_1\rangle_v\gamma_\ell(f_2)\big)=\mathsf{E}\big(V_1\cdots V_{\ell-1}\gamma_\ell(f_1)\gamma_\ell(f_2)\big)$$

we get

$$\mathrm{I}(\ell)=\frac{t}{2}(n_{\ell-1}-n_\ell)\mu_\ell\big(\zeta(\boldsymbol{\sigma}^1,\boldsymbol{\tau}^1)+\zeta(\boldsymbol{\sigma}^1,\boldsymbol{\tau}^2)+\zeta(\boldsymbol{\sigma}^2,\boldsymbol{\tau}^1)+\zeta(\boldsymbol{\sigma}^2,\boldsymbol{\tau}^2)\big),$$

where the four quantities $\zeta(\cdot,\cdot)$ are seen as functions of $((\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2),(\boldsymbol{\tau}^1,\boldsymbol{\tau}^2))\in S_u^2$.

Using (3.10), and since in (3.31) the summation is only over $R_{1,2}=u$, we have

$$\begin{aligned}\mathrm{I}=\frac{t}{2}\bigg(&2\xi(1)+2\xi(u)+\sum_{1\le\ell\le\kappa}(n_{\ell-1}-n_\ell)\mu_\ell\Big(\xi\big(R(\boldsymbol{\sigma}^1,\boldsymbol{\tau}^1)\big)+\xi\big(R(\boldsymbol{\sigma}^1,\boldsymbol{\tau}^2)\big)\\&+\xi\big(R(\boldsymbol{\sigma}^2,\boldsymbol{\tau}^1)\big)+\xi\big(R(\boldsymbol{\sigma}^2,\boldsymbol{\tau}^2)\big)\Big)\bigg)+\mathcal{R},\end{aligned}$$

where $|\mathcal{R}|\le 4c(N)$.

To compute the term II, we have to keep in mind (3.18) and (3.19). When $p\ge\tau$ we find by a similar computation

$$\begin{aligned}\mathrm{II}(p)=C_p:=&-\frac{t}{2}\big(\xi'(\rho_{p+1})-\xi'(\rho_p)\big)\\&\times\Big(2+\sum_{p<\ell\le\kappa}(n_{\ell-1}-n_\ell)\mu_\ell\big(R(\boldsymbol{\sigma}^1,\boldsymbol{\tau}^1)+R(\boldsymbol{\sigma}^2,\boldsymbol{\tau}^2)\big)\Big)\end{aligned}$$

and when $p<\tau$, we find

$$\begin{aligned}\mathrm{II}(p)=C_p&-\frac{\eta t}{2}\big(\xi'(\rho_{p+1})-\xi'(\rho_p)\big)\\&\times\Big(2u+\sum_{p<\ell\le\kappa}(n_{\ell-1}-n_\ell)\mu_\ell\big(R(\boldsymbol{\sigma}^1,\boldsymbol{\tau}^2)+R(\boldsymbol{\sigma}^2,\boldsymbol{\tau}^1)\big)\Big).\end{aligned}$$

By summation of these formulas, we get

$$\begin{aligned}\sum_{0\le p\le\kappa}\mathrm{II}(p)=-\frac{t}{2}\Big(&2\xi'(1)+2\eta u\xi'(\rho_\tau)\\&+\sum_{1\le\ell\le\kappa}\xi'(\rho_\ell)(n_{\ell-1}-n_\ell)\mu_\ell\big(R(\boldsymbol{\sigma}^1,\boldsymbol{\tau}^1)+R(\boldsymbol{\sigma}^2,\boldsymbol{\tau}^2)\big)\\&+\eta\sum_{1\le\ell\le\kappa}\xi'(\rho_{\min(\ell,\tau)})(n_{\ell-1}-n_\ell)\mu_\ell\big(R(\boldsymbol{\sigma}^1,\boldsymbol{\tau}^2)+R(\boldsymbol{\sigma}^2,\boldsymbol{\tau}^1)\big)\Big).\end{aligned}$$

We note that, since we assume that $\xi(x)=\xi(-x)$, besides (2.11) we also have

$$\xi(x)-\eta x\xi'(q)+\theta(q)=\xi(\eta x)-\eta x\xi'(q)+\theta(q)\ge 0.$$

Finally, writing

$$S^j(\rho)=\xi(R(\boldsymbol{\sigma}^j,\boldsymbol{\tau}^j))-\xi'(\rho)R(\boldsymbol{\sigma}^j,\boldsymbol{\tau}^j)+\theta(\rho)\ge 0,$$

$$T^{j,j'}(\rho) = \xi(R(\boldsymbol{\sigma}^j, \boldsymbol{\tau}^{j'})) - \eta\xi'(\rho)R(\boldsymbol{\sigma}^j, \boldsymbol{\tau}^{j'}) + \theta(\rho) \geq 0$$

we get

$$\begin{aligned}\eta'(v) \leq \frac{t}{2}\Big(&2\big(\xi(1) - \xi'(1) + \xi(u) - \eta u\xi'(\rho_\tau)\big)\\ &+ \sum_{1\leq\ell\leq\kappa}(n_{\ell-1} - n_\ell)\mu_\ell\big(S^1(\rho_\ell) + S^2(\rho_\ell)\big)\\ &+ \sum_{1\leq\ell\leq\kappa}(n_{\ell-1} - n_\ell)\mu_\ell\big(T^{1,2}(\rho_{\min(\ell,\tau)}) + T^{2,1}(\rho_{\min(\ell,\tau)})\big)\\ &-2\sum_{1\leq\ell\leq\kappa}(n_{\ell-1} - n_\ell)\theta(\rho_\ell) - 2\sum_{1\leq\ell\leq\kappa}(n_{\ell-1} - n_\ell)\theta(\rho_{\min(\ell,\tau)})\Big) + 4c(N).\end{aligned}$$

Now, since $\xi(\eta x) = \xi(x)$ and $\rho_\tau = \eta u$, we have $\xi(u) - \eta u\xi'(\rho_\tau) = \xi(\rho_\tau) - \rho_\tau\xi'(\rho_\tau) = -\theta(\rho_\tau)$, so using (3.3) twice, and since $S^j(\rho), T^{j,j'}(\rho) \geq 0$ we get

$$\begin{aligned}\eta'(v) &\leq -t\Big(\sum_{1\leq\ell\leq\kappa} n_\ell\big(\theta(\rho_{\ell+1}) - \theta(\rho_\ell)\big)\\ &\qquad + \sum_{1\leq\ell\leq\kappa} n_\ell\big(\theta(\rho_{\min(\ell+1,\tau)}) - \theta(\rho_{\min(\ell,\tau)})\big)\Big) + 4c(N)\\ &= -t\Big(\sum_{1\leq\ell\leq\kappa} n_\ell\big(\theta(\rho_{\ell+1}) - \theta(\rho_\ell)\big)\\ &\qquad + \sum_{1\leq\ell<\tau} n_\ell\big(\theta(\rho_{\ell+1}) - \theta(\rho_\ell)\big)\Big) + 4c(N).\end{aligned}$$

This proves (3.25). □

4. The basic operators.

In this section, we perform some basic calculations, and then learn how to use conditions (2.16), (2.17) and (2.19).

We consider a standard Gaussian r.v. g, and an infinitely differentiable function A such that $\mathsf{E}\exp A(x + g\sqrt{v}) < \infty$ for each x and each $v \geq 0$. For $0 < m \leq 1$, we define

$$B(x, v, m) = \frac{1}{m}\log \mathsf{E}\exp mA\big(x + g\sqrt{v}\big), \tag{4.1}$$

and $B(x, v, 0) = \mathsf{E}A\big(x + g\sqrt{v}\big)$. Since the case $m = 0$ is essentially trivial, it will never be considered in the proofs below. To lighten notation, we write B' for $\partial B/\partial x$, B'' for $\partial^2 B/\partial x^2$, etc. and omit the arguments x, v and m in the next lemma and its proof.

Lemma 4.1. *We have*

$$\exp B(x, v, m) \leq \mathsf{E}\exp A\big(x + g\sqrt{v}\big). \tag{4.2}$$

$$\text{(4.3)} \qquad \textit{If } A \textit{ is strictly convex, so is } x \mapsto B(x,v,m),$$

$$\frac{\partial B}{\partial v} = \frac{1}{2}B'' + \frac{m}{2}B'^2. \tag{4.4}$$

Proof. By Hölder's inequality, we have

$$\mathsf{E}\exp mA(x+g\sqrt{v}) \le \big(\mathsf{E}\exp A(x+g\sqrt{v})\big)^m.$$

This proves (4.2). To lighten notation, we write $Y = x + g\sqrt{v}$ and

$$Q = \exp m\big(A(Y) - B(x,v,m)\big) \tag{4.5}$$

so that $\mathsf{E}(Q) = 1$ and

$$B' = \mathsf{E}(A'(Y)Q), \tag{4.6}$$

$$\begin{aligned} B'' &= \mathsf{E}(A''(Y)Q) + m\mathsf{E}\big(A'(Y)^2Q\big) - mB'\mathsf{E}(A'(Y)Q) \\ &= \mathsf{E}(A''(Y)Q) + m\mathsf{E}\big(A'(Y)^2Q\big) - mB'^2 \end{aligned} \tag{4.7}$$

by (4.6). Since $\mathsf{E}Q = 1$, the Cauchy-Schwarz inequality shows that

$$B' = \mathsf{E}(A'(Y)Q) \le \mathsf{E}\big(A'(Y)^2Q\big)^{1/2},$$

so (4.7) implies that $B'' \ge \mathsf{E}(A''(Y)Q)$ and this proves (4.3). Using integration by parts, we have

$$\frac{\partial B}{\partial v} = \frac{1}{2\sqrt{v}}\mathsf{E}(gA'(Y)Q) = \frac{1}{2}\mathsf{E}(A''(Y)Q) + \frac{m}{2}\mathsf{E}\big(A'(Y)^2Q\big) \tag{4.8}$$

and together with (4.7) this proves (4.4). □

We consider another standard Gaussian r.v. g', independent of g. We consider $a > 0$ and $0 \le m' \le 1$. We think of these quantities as fixed, so they remain implicit in the notation. We consider $0 \le v \le a$ and write $Z = x + g'\sqrt{a-v}$ and

$$\begin{aligned} C(x,v,m) &= \frac{1}{m'}\log\mathsf{E}\exp m'B(x+g'\sqrt{a-v},v,m) \\ &= \frac{1}{m'}\log\mathsf{E}\exp m'B(Z,v,m), \end{aligned} \tag{4.9}$$

where B is as given in (4.1). We write

$$R = \exp m'\big(B(Z,v,m) - C(x,v,m)\big). \tag{4.10}$$

Lemma 4.2. *We have*

$$\frac{\partial C}{\partial v}(x,v,m) = \frac{1}{2}(m-m')\mathsf{E}\big(B'^2(Z,v,m)R\big). \tag{4.11}$$

Proof. From (4.9), we have $\partial C/\partial v = \mathrm{I} + \mathrm{II}$, where

$$\begin{aligned}\mathrm{I} &= \mathsf{E}\Big(\frac{\partial B}{\partial v}(Z,v,m)R\Big),\\ \mathrm{II} &= -\frac{1}{2\sqrt{a-v}}\mathsf{E}\big(g'B'(Z,v,m)R\big)\\ &= -\frac{1}{2}\mathsf{E}\big((B''(Z,v,m)+m'B'^2(Z,v,m))R\big)\end{aligned}$$

after integration by parts, and we use (4.4). □

We write

$$\Delta(x,v) = \frac{\partial}{\partial m}C(x,v,m)\Big|_{m=m'}. \tag{4.12}$$

To lighten notation, (and since we think of m' as fixed) we write $B(x,v)$ rather than $B(x,v,m')$, and similarly for B', B'', etc.

LEMMA 4.3. *Writing $Y = x + g\sqrt{v}$ and $Z = x + g'\sqrt{a-v}$, we have*

$$\Delta(x,v) = \mathsf{E}(D(Z,v)R) \tag{4.13}$$

where

$$D(x,y) = -\frac{1}{m'}B(x,v) + \frac{1}{m'}\mathsf{E}\big(A(Y)\exp m'(A(Y)-B(x,v))\big), \tag{4.14}$$

$$\frac{\partial\Delta}{\partial v}(x,v) = \frac{1}{2}\mathsf{E}\big(B'^2(Z,v)R\big), \tag{4.15}$$

$$\frac{\partial}{\partial v}\mathsf{E}\big(B'^2(Z,v)R\big) = -\mathsf{E}\big(B''^2(Z,v)R\big). \tag{4.16}$$

Proof. It is straightforward to see that

$$D(x,v) = \frac{\partial}{\partial m}B(x,v,m)\Big|_{m=m'} \tag{4.17}$$

and using (4.9) this yields (4.13). Next, we observe that $C(x) := C(x,v,m')$ is independent of v, because

$$C(x,v,m') = \frac{1}{m'}\log\mathsf{E}\exp m'A\big(x+g\sqrt{a}\big) \tag{4.18}$$

since $x + g\sqrt{v} + g'\sqrt{a-v}$ has the same distribution as $x + g\sqrt{a}$. Also, if we denote by $V(x,v)$ the last term of (4.14), then,

$$\mathsf{E}(V(Z,v)R) = \frac{1}{m'}\mathsf{E}\Big(A(x+g\sqrt{a})\exp m'\big(A(x+g\sqrt{a})-C(x)\big)\Big)$$

is also independent of v, so that

$$\frac{\partial\Delta}{\partial v}(x,v) = -\frac{1}{m'}\frac{\partial}{\partial v}\mathsf{E}(B(Z,v)R). \tag{4.19}$$

For simplicity, we write $B = B(Z,v), B' = B'(Z,v)$ etc. and $C = C(x)$, so that $R = \exp m'(B-C)$. We have

$$\frac{\partial}{\partial v}\mathsf{E}(BR) = \frac{\partial}{\partial v}\mathsf{E}(B\exp m'(B-C)) = \mathrm{III} + \mathrm{IV}$$

where, using (4.4),

$$\mathrm{III} = \mathsf{E}\Big(\frac{\partial B}{\partial v}(1+m'B)R\Big) = \frac{1}{2}\mathsf{E}\big((B''+m'B'^2)(1+m'B)R\big),$$

$$\begin{aligned}\mathrm{IV} &= -\frac{1}{2\sqrt{v-a}}\mathsf{E}\big(g'B'(1+m'B)\exp m'(B-C)\big)\\ &= -\frac{1}{2}\mathsf{E}\Big(\big((B''+m'B')(1+m'B)+m'B'^2\big)R\Big),\end{aligned}$$

and thus

$$\frac{\partial}{\partial v}\mathsf{E}(BR) = -\frac{m'}{2}\mathsf{E}(B'^2R). \tag{4.20}$$

Combining this with (4.19) proves (4.15). In the same manner,

$$\frac{\partial}{\partial v}\mathsf{E}\big(B'^2\exp m'(B-C)\big) = \mathrm{V} + \mathrm{VI},$$

where, by (4.2)

$$\begin{aligned}\mathrm{V} &= \mathsf{E}\big((2\frac{\partial B'}{\partial v}B' + m'B'^2\frac{\partial B}{\partial v})R\big)\\ &= \mathsf{E}\big((B^{(3)}B' + 2m'B'^2B'' + \frac{1}{2}m'B'^2B'' + \frac{1}{2}m'^2B'^4)R\big).\end{aligned}$$

Integration by parts gives

$$\begin{aligned}\mathrm{VI} &= -\frac{1}{2\sqrt{v-a}}\mathsf{E}\big(g'(2B'B''+m'B'^3)\exp m'(B-C)\big)\\ &= -\frac{1}{2}\mathsf{E}\big((2B''^2 + 2B'B^{(3)} + 3m'B'^2B'' + 2m'B'^2B'' + m'^2B'^4)R\big),\end{aligned}$$

which yields (4.6). $\square$

LEMMA 4.4. *For a* r.v. Y *and* $0 < m \le 1$, *we have, for a certain number* L,

$$\Big|\frac{d}{dm}\Big(\frac{1}{m}\log\mathsf{E}\exp mY\Big)\Big| \le L\mathsf{E}\exp L|Y|, \tag{4.21}$$

$$\Big|\frac{d^2}{dm^2}\Big(\frac{1}{m}\log\mathsf{E}\exp mY\Big)\Big| \le L\mathsf{E}\exp L|Y|. \tag{4.22}$$

Proof. Setting $M = m^{-1}\log\mathsf{E}\exp mY$ and

$$U = \frac{\exp mY}{\mathsf{E}\exp mY} = \exp m(Y-M),$$

we have

$$\begin{aligned}\frac{d}{dm}\Big(\frac{1}{m}\log \mathsf{E}\exp mY\Big) &= -\frac{1}{m^2}\log \mathsf{E}\exp mY + \frac{1}{m}\frac{\mathsf{E}Y\exp mY}{\mathsf{E}\exp mY}\\ &= \frac{1}{m^2}\mathsf{E}U\log U.\end{aligned}$$

Now

$$U-1 \le U\log U \le U-1+L(U-1)^2,$$

so that, since $\mathsf{E}U=1$,

$$0 \le \frac{1}{m^2}\mathsf{E}U\log U \le \frac{L}{m^2}\mathsf{E}(U-1)^2 \le L\mathsf{E}(Y-M)^2\exp 2|Y-M|, \tag{4.23}$$

by the fact that $|e^x-1| \le |x|e^{|x|}$. We have $\exp M \le \mathsf{E}\exp Y$ by Hölder's inequality, and $M \ge \mathsf{E}Y$ by Jensen's inequality, and the proof of (4.21) is easily concluded.

Simple algebra shows that

$$\frac{d^2}{dm^2}\Big(\frac{1}{m}\log \mathsf{E}\exp mY\Big) = \frac{1}{m^3}\big(-2\mathsf{E}U\log U + \mathsf{E}U\log^2 U - (\mathsf{E}U\log U)^2\big).$$

The last term is taken care of by (4.23). We note that

$$\big|-2U\log U + U\log^2 U + 2(U-1)\big| \le L|U-1|^3$$

so that

$$\big|-2\mathsf{E}U\log U + \mathsf{E}U\log^2 U\big| \le L\mathsf{E}|U-1|^3$$

and we conclude as before. □

After these preliminaries, we turn to the main goal of this section, learning how to use conditions (2.16) and (2.17). We remind the reader that we have fixed an integer k and sequences $\boldsymbol{m}$ and $\boldsymbol{q}$, that satisfy these conditions and (2.19), and Gaussian r.v. $(z_\ell)_{0\le\ell\le k}$ as in (1.9).

We consider the function

$$A_{k+1}(x) = \log \mathrm{ch} x \tag{4.24}$$

and we define recursively, for $\ell \ge 1$, the function

$$A_\ell(x) = \frac{1}{m_\ell}\log \mathsf{E}\exp m_\ell A_{\ell+1}(x+z_\ell), \tag{4.25}$$

and $A_0(x) = \mathsf{E}A_1(x+z_0)$. Then the r.v. X_ℓ of (1.10) is given by $X_\ell = A_\ell\big(h + \sum_{0\le p<\ell} z_p\big)$ so that $X_0 = A_0(h)$ and (2.14) becomes

$$\varphi(0) = \log 2 + A_0(h). \tag{4.26}$$

We recall that we think of m_ℓ as being a parameter attached to the interval $[q_\ell, q_{\ell+1}[$. A basic procedure is to consider $q_{r-1} \le u \le q_r$ and to split the interval $[q_{r-1}, q_r[$ into the subintervals $[q_{r-1}, u[$ and $[u, q_r[$, to which one attaches the parameters m_{r-1} and m respectively, where $m_{r-1} \leq m \leq m_r$. Accordingly, we consider the sequences

$$\boldsymbol{q}(u) = (q_0, \dots, q_{r-1}, u, q_r, \dots, q_{k+1}), \tag{4.27}$$

$$\boldsymbol{m}(m) = (m_0, \dots, m_{r-1}, m, m_r, \dots, m_k). \tag{4.28}$$

We write

$$\Phi(m, u) = \mathcal{P}_{k+1}(\boldsymbol{m}(m), \boldsymbol{q}(u)). \tag{4.29}$$

It should be obvious that $\Phi(m_{r-1}, u) = \mathcal{P}_k(\boldsymbol{m}, \boldsymbol{q})$, so that by (2.16) we have

$$\Phi(m, u) \geq \Phi(m_{r-1}, u) - \varepsilon. \tag{4.30}$$

It is also useful to note that for $m = m_r$, one can merge the intervals $[u, q_r[$ and $[q_r, q_{r+1}[$, so that by (2.17), we have

$$\Phi(m_r, u) \geq \mathcal{P}_k(\boldsymbol{m}, \boldsymbol{q}) = \Phi(m_{r-1}, u). \tag{4.31}$$

Let us now examine how this splitting procedure affects the construction (4.25). We set $A = A_r$, and consider the quantities (4.1) and (4.9), where we take

$$a = \xi'(q_r) - \xi'(q_{r-1}); \quad m' = m_{r-1}. \tag{4.32}$$

We set

$$C_r(x, v, m) = B(x, v, m); \quad C_{r-1}(x, v, m) = C(x, v, m), \tag{4.33}$$

and for $1 \leq \ell \leq r - 2$, we define recursively

$$C_\ell(x, v, m) = \frac{1}{m_\ell} \log \mathsf{E} \exp m_\ell C_{\ell+1}(x + z_\ell, v, m). \tag{4.34}$$

We set

$$T(v, m) = \mathsf{E} C_1(h + z_0, v, m). \tag{4.35}$$

From (4.18) we see that $C(x, v, m_{r-1}) = A_{r-1}(x)$. Then we see inductively that $C_\ell(x, v, m_{r-1}) = A_\ell(x)$ so that we have

$$T(v, m_{r-1}) = A_0(h). \tag{4.36}$$

Taking into account that we have replaced m_{r-1} by m on the interval $[u, q_r[$, we get, using (4.29) and (4.11),

$$\begin{aligned}\Phi(m, u) = {} & \log 2 + T(\xi'(q_r) - \xi'(u), m) \\ & - \frac{1}{2} \sum_{1 \leq \ell \leq k} m_\ell \big(\theta(q_{\ell+1}) - \theta(q_\ell)\big) + \frac{1}{2}(m_{r-1} - m)\big(\theta(q_r) - \theta(u)\big).\end{aligned} \tag{4.37}$$

We define

$$w_\ell = h + \sum_{p<\ell} z_p \text{ if } \ell \le r-1 \ ; \ w_r = w_{r-1} + g'\sqrt{a-v}. \tag{4.38}$$

For $\ell \le r-1$, we set

$$R_\ell(v,m) = \exp m_\ell\big(C_{\ell+1}(w_{\ell+1},v,m) - C_\ell(w_\ell,v,m)\big) \tag{4.39}$$

and, to lighten notation,

$$S_{r-2}(v,m) = R_1(v,m)\cdots R_{r-2}(v,m). \tag{4.40}$$

From (4.35), and proceeding as in (3.2), we get the formula

$$\frac{\partial T}{\partial m}(v,m) = \mathsf{E}\Big(S_{r-2}(v,m)\frac{\partial C}{\partial m}(w_{r-1},v,m)\Big). \tag{4.41}$$

When $m = m_{r-1}$, this formula is particularly well adapted to differentiation in v because then the quantity $S_{r-2}(v,m_{r-1})$ does not depend on v (see (4.18)). We then write it S_{r-2}.

We define

$$U(v) = 2\frac{\partial T}{\partial m}(v,m)\Big|_{m=m_{r-1}}. \tag{4.42}$$

Using (4.12) and (4.41), we have

$$U(v) = 2\mathsf{E}(S_{r-2}\Delta(w_{r-1},v)). \tag{4.43}$$

Since S_{r-2} does not depend on v, to differentiate in v we can use (4.15) and (4.16). Writing

$$S_{r-1}(v) = S_{r-2}R_{r-1}(v,m_{r-1}),$$

this yields the following (fundamental) relations

$$U'(v) = \mathsf{E}\big(S_{r-1}(v)B'^2(w_r,v)\big) \tag{4.44}$$

$$U''(v) = -\mathsf{E}\big(S_{r-1}(v)B''^2(w_r,v)\big) \le 0. \tag{4.45}$$

Let us note the following relation following from (4.37) and (4.42),

$$f(u) := \frac{\partial \Phi}{\partial m}(m,u)\Big|_{m=m_{r-1}} = \frac{1}{2}U(\xi'(q_r)-\xi'(u)) - \frac{1}{2}\big(\theta(q_r)-\theta(u)\big). \tag{4.46}$$

Lemma 4.5. *We have*

$$\Big|\frac{\partial \Phi}{\partial m}(m,u)\Big| \le L,\ \ \Big|\frac{\partial^2 \Phi}{\partial m^2}(m,u)\Big| \le L.$$

The point here is that these derivatives are bounded by a quantity depending on ξ and h but not on k. This is essential for our approach.

Proof. It suffices to prove this boundedness for $\partial T/\partial m$ and $\partial^2 T/\partial m^2$. Using (4.41) and computing $\partial C/\partial m$ from (4.9), we obtain

$$\frac{\partial T}{\partial m}(v,m) = \mathsf{E}\Big(S_{r-1}(v,m)\frac{\partial B}{\partial m}(w_r,v,m)\Big) \tag{4.47}$$

where $S_{r-1}(v,m) = S_{r-2}(v,m)R_{r-1}(v,m)$. To lighten notation, we will not indicate the dependence in v (so we write $S_{r-1}(m)$ rather than $S_{r-1}(v,m)$, etc.). We have

$$\begin{aligned}\sum_{0\le \ell\le r-1} & m_\ell\big(C_{\ell+1}(w_{\ell+1},m) - C_\ell(w_\ell,m)\big)\\ &= m_{r-1}C_r(w_r,m) + \sum_{0\le\ell\le r-1} C_\ell(w_\ell,m)(m_{\ell-1}-m_\ell)\end{aligned} \tag{4.48}$$

and thus, recalling (4.39) and (4.40),

$$\frac{\partial}{\partial m}S_{r-1}(m) = \Big(m_{r-1}\frac{\partial C_r}{\partial m}(w_r,m) + \sum_{0\le\ell\le r-1}(m_{\ell-1}-m_\ell)\frac{\partial C_\ell}{\partial m}(w_\ell,m)\Big)S_{r-1}(m)$$

and hence, from (4.47)

$$\begin{aligned}\frac{\partial^2 T}{\partial m^2}(w,m) = {} & \mathsf{E}\Big(S_{r-1}(m)\frac{\partial^2 B}{\partial m^2}(w_r,m)\Big)\\ & + m_{r-1}\mathsf{E}\Big(S_{r-1}(m)\frac{\partial C_r}{\partial m}(w_r,m)\frac{\partial B}{\partial m}(w_r,m)\Big)\\ & + \sum_{0\le\ell\le r-1}(m_{\ell-1}-m_\ell)\mathsf{E}\Big(S_{r-1}(m)\frac{\partial C_\ell}{\partial m}(w_\ell,m)\frac{\partial B}{\partial m}(w_r,m)\Big).\end{aligned}$$

By Hölder's inequality, it suffices to show that

$$\mathsf{E}S^4_{r-1}(m) \le L, \tag{4.49}$$

$$\mathsf{E}\Big(\frac{\partial B}{\partial m}(w_r,m)\Big)^4 \le L;\quad \mathsf{E}\Big(\frac{\partial^2 B}{\partial^2 m}(w_r,m)\Big)^4 \le L, \tag{4.50}$$

$$\mathsf{E}\Big(\frac{\partial C_\ell}{\partial m}(w_\ell,m)\Big)^4 \le L. \tag{4.51}$$

If L_0 is the constant of Lemma 4.4, iteration of (4.2) and use of Hölder's inequality show that

$$\mathsf{E}\exp 4L_0 B(w_r,m) \le \mathsf{E}\mathrm{ch}^{4L_0}\Big(w_r + g\sqrt{v} + \sum_{\ell\ge r} z_\ell\Big) \le L$$

so that (since $B\ge 0$), Lemma 4.4 implies (4.50) by Hölder's inequality. Use of (4.48), of (4.2) and Hölder's inequality yields (4.49).

To prove (4.51), one computes $\partial C_\ell/\partial m$ in the same manner as we computed $\partial C_1/\partial m$ and proceeds similarly. □

There is a general principle at work in Lemma 4.5, namely that the "change of density" S_{r-1} does not affect the boundedness of derivatives. This will be used again.

Proposition 4.6. *The function* $f(u)$ *of* (4.46) *satisfies*

$$f(u) \geq -L\sqrt{\varepsilon}. \tag{4.52}$$

Proof. Since there is nothing to prove if $f(u) \geq 0$, we assume $f(u) < 0$. It follows from Lemma 4.5 that

$$\Phi(m,u) \leq \Phi(m_{r-1},u) + (m - m_{r-1})f(u) + L_1(m - m_{r-1})^2. \tag{4.53}$$

By (4.31), we have

$$(m_r - m_{r-1})f(u) + L_1(m_r - m_{r-1})^2 \geq 0$$

so that

$$m_r \geq m_{r-1} - \frac{f(u)}{L_1}.$$

Thus

$$m_{r-1} \leq m := m_{r-1} - \frac{f(u)}{2L_1} \leq m_r$$

and by (4.30), (4.53) for this value of m yields $-\varepsilon \leq -f(u)^2/4L_1$. □

Proposition 4.7. *We have*

$$q_r = U'(0). \tag{4.54}$$

Proof. Computing $\partial T/\partial v$ from (4.35) by proceeding as in (3.2) and using (4.11) with $m' = m_{r-1}$, we get

$$\frac{\partial T}{\partial v}(v,m) = \frac{1}{2}(m - m_{r-1})\mathsf{E}\big(S_{r-1}(v,m)B'^2(w_r,v,m)\big), \tag{4.55}$$

where $S_{r-1}(v,m) = S_{r-2}(v,m)R_{r-1}(v,m)$. (It is interesting to observe that differentiating this relation in m at $m = m_{r-1}$ yields again the relation (4.44).)

The relation $v = \xi'(q_r) - \xi'(u)$ defines u as a function $u(v)$ and $u'(v)\xi''(u(v)) = -1$. Since $\theta'(u) = u\xi''(u)$, we have $d\theta(u(v))/dv = -u(v)$ and, by (4.37)

$$\frac{\partial \Phi}{\partial v}(m,u(v)) = \frac{1}{2}(m - m_{r-1})\big(\mathsf{E}(S_{r-1}(v,m)B'^2(w_r,v,m)) - u\big).$$

Thus, if $m \neq m_{r-1}$ and $q_{r-1} < u < q_r$, if we cannot decrease $\Phi(m,u)$ by a small variation of u, we must have

$$u = \mathsf{E}\big(S_{r-1}(v,m)B'^2(w_r,v,m)\big). \tag{4.56}$$

When $r = 1$ and $u = 0$, one still obtains this relation by expressing that a small increase of u does not decrease Φ. (This case is the reason why we consider u

as a function of v; if we differentiate (4.37) with respect to u, we get an extra factor $\xi''(u)$, and we might well have $\xi''(0) = 0$.)

Thus (4.56) must hold if we cannot decrease $\Phi(m, u) = \mathcal{P}_{k+1}(\boldsymbol{m}(m), \boldsymbol{q}(u))$ by a small variation of u. If we write $\boldsymbol{q}(u) = (q'_\ell)_{\ell \le k+1}$, we see that $u = q'_r$ is simply one of the terms of this sequence. Changing k into $k-1$, and using the fact that $m_{r-1} \neq m_r$ by (2.19), we see that (4.54) must hold, for otherwise, we could decrease $\mathcal{P}_k(\boldsymbol{m}, \boldsymbol{q})$ by a small variation of q_r. □

PROPOSITION 4.8. *We have*

$$f(q_r) = f'(q_r) = 0; \quad f(q_{r-1}) \ge 0.$$

Proof. We have $f(q_r) = 0$ simply because when $u = q_r, \Phi(m, u)$ does not depend on m. By (2.19) we have $m_{r-1} < 1$. Thus we have $f(q_{r-1}) \ge 0$, for otherwise we could decrease $\mathcal{P}_k(\boldsymbol{m}, \boldsymbol{q})$ by a small increase of m_{r-1}. (In fact, if $r \ge 2$, we have $m_{r-1} > 0$ so that we even have $f(q_{r-1}) = 0$.) Using (4.46), we get

$$f'(q_r) = \frac{1}{2}\xi''(q_r)\big(q_r - U'(0)\big) = 0$$

by (4.54). □

LEMMA 4.9. *We have* $|f^{(3)}(u)| \le L$.

Proof. This should be obvious by the method of Lemma 4.5. □

PROPOSITION 4.10. *We have*

$$-f''(q_r) = \frac{1}{2}\xi''(q_r)(-\xi''(q_r)U''(0) - 1) \le L\varepsilon^{1/6}. \tag{4.57}$$

Proof. Since $f'(q_r) = f(q_r) = 0$, by Propositions 4.6 and 4.8, we have

$$-L\sqrt{\varepsilon} \le f(u) \le \frac{1}{2}(u - q_r)^2 f''(q_r) + L|u - q_r|^3 \tag{4.58}$$

so that

$$f''(q_r) \ge -\frac{L\sqrt{\varepsilon}}{(u - q_r)^2} - L|u - q_r|. \tag{4.59}$$

Also, since $f(q_{r-1}) \ge 0$, we see from (4.58) that

$$f''(q_r) \ge -2L(q_r - q_{r-1}). \tag{4.60}$$

If $q_r - q_{r-1} \le \varepsilon^{1/6}$, this implies $f''(q_r) \ge -2L\varepsilon^{1/6}$. Otherwise, taking $u = q_r - \varepsilon^{1/6} \ge q_{r-1}$ we get again $f''(q_r) \ge -L\varepsilon^{1/6}$ from (4.59). □

Our basic construction splits the interval $[q_{r-1}, q_r[$ into the intervals $[q_{r-1}, u[$ and $[u, q_r[$. A "dual" construction splits instead the interval $[q_r, q_{r+1}[$ into $[q_r, u[$ and $[u, q_{r+1}[$, to which we now attach parameters $m_{r-1} \le m \le m_r$ and m_r respectively. This dual construction is studied by very similar methods, so we do not detail it.

5. The main estimate

The goal of this section is to prove Theorem 2.4. In the following statements, $L_1, L_2, \dots$, denote specific quantities depending only upon ξ and h.

PROPOSITION 5.1. *If* $L_1\varepsilon^{1/6} \le 1 - t_0$, *then*

(5.1)
$$q_{r-1} \le u \le q_r,\ L_1(q_r - u) \le 1 - t_0 \Rightarrow \Psi(t,u) \le 2\psi(t) - \frac{(1-t_0)^2}{L_1}(u - q_r)^2.$$

PROPOSITION 5.2. *If* $L_1\varepsilon^{1/6} \le 1 - t_0$, *then*

(5.2)
$$q_r \le u \le q_{r+1},\ L_1(u - q_r) \le 1 - t_0 \Rightarrow \Psi(t,u) \le 2\psi(t) - \frac{(1-t_0)^2}{L_1}(u - q_r)^2.$$

PROPOSITION 5.3. *If* $L_2\varepsilon^{1/6} \le 1 - t_0, r = 1$ *and* $0 \le u < q_1$, *then*

$$\Psi(t,u) < 2\psi(t). \tag{5.3}$$

PROPOSITION 5.4. *If* $L_2\varepsilon^{1/6} \le 1 - t_0, r = 1$ *and* $-q_1 \le u < 0$, *then* (5.3) *holds.*

We consider the function

$$\gamma(c) = \inf\bigl\{|\xi(y) - \xi(x) + (x - y)\xi'(y)|; 0 \le x, y \le 1, |x - y| \ge c\bigr\}. \tag{5.4}$$

Then, since we assume $\xi''(x) > 0$ for $x > 0$, we have $\gamma(c) > 0$ for $c > 0$.

PROPOSITION 5.5. *If* $L_3\varepsilon^{1/2} \le (1 - t_0)\gamma((1 - t_0)/L_1)$, *then*

$$q_{r-1} \le u \le q_r,\ L_1(q_r - u) \ge 1 - t_0 \Rightarrow \Psi(t,u) < 2\psi(t).$$

PROPOSITION 5.6. *If* $L_3\varepsilon^{1/2} \le (1 - t_0)\gamma((1 - t_0)/L_1)$, *then*

$$q_r \le u \le q_{r+1},\ L_1(u - q_r) \ge 1 - t_0 \Rightarrow \Psi(t,u) < 2\psi(t).$$

PROPOSITION 5.7. *If either* $u < q_{r-1}$ *or* $u > q_{r+1}$, *then* (5.3) *holds.*

Proof of Theorem 2.4. Combining Propositions 5.6 to 5.7 we see that if $t \le t_0$, if $L_1|u - q_r| \ge 1 - t_0$ and if $L_3\varepsilon^{1/2} \le (1 - t_0)\gamma((1 - t_0)/L_1)$, we have $\Psi(t,u) < 2\psi(t)$. By compactness, there exists K such that

$$t \le t_0\ ;\ L_1|u - q_r| \ge 1 - t_0 \Longrightarrow \Psi(t,u) \le 2\psi(t) - \frac{(u - q_r)^2}{K}$$

and (5.1) and (5.2) finish the proof. □

Propositions 5.3 and 5.4 are not used as such in the proof of Theroem 2.4, but are intermediate results used in the proof of Proposition 5.7.

Proposition 5.1, 5.3 and 5.5 rely on a common scheme of proof. We consider

(5.5)
$$\rho_0 = 0, \rho_1 = q_1, \dots, \rho_{r-1} = q_{r-1}, \rho_r = u, \rho_{r+1} = q_r, \dots, \rho_{k+1} = q_k, \rho_{k+2} = 1,$$

$$\begin{aligned} n_0 = 0, n_1 = \frac{m_1}{2}, \dots, n_{r-1} = \frac{m_{r-1}}{2}, \\ n_r = m, n_{r+1} = m_r, \dots, n_k = m_{k-1}, n_{k+1} = 1, \end{aligned} \tag{5.6}$$

where m is a number satisfying $m_{r-1}/2 \leq m \leq m_r$. Thus (when $r \geq 2$, and hence $m_{r-1} > 0$) the number m can be either larger or smaller than m_{r-1}, a key feature of the construction.

We recall the r.v. (z_p^j) of (2.23), for $0 \leq p \leq k$, and we set

$$Z_p^j = \sqrt{1-t} z_p^j \ \text{ if } p < r; \quad Z_r^j = 0; \quad Z_p^j = \sqrt{1-t} z_{p-1}^j \ \text{ if } \ p > r. \tag{5.7}$$

We consider the r.v. y_p^j of (3.18) for $\eta = 1$. We will use Theorem 3.1 with $\kappa = k+1$ and $\tau = r$. (Note that $u = \rho_\tau$.)

It should be obvious from (5.7) (since "nothing happens for n_r" because $Z_r^j = 0$) that, with the notation (3.24), we have

$$\Psi(t, u) = \eta(1). \tag{5.8}$$

We have

$$\begin{aligned} \sum_{\ell \leq r-2} n_\ell \big(\theta(\rho_{\ell+1}) - \theta(\rho_\ell)\big) &= \frac{1}{2} \sum_{\ell \leq r-2} m_\ell \big(\theta(q_{\ell+1}) - \theta(q_\ell)\big), \\ n_{r-1}\big(\theta(\rho_r) - \theta(\rho_{r-1})\big) &= \frac{m_{r-1}}{2}\big(\theta(u) - \theta(q_{r-1})\big), \\ n_r\big(\theta(\rho_{r+1}) - \theta(\rho_r)\big) &= m\big(\theta(q_r) - \theta(u)\big), \\ \sum_{\ell \geq r+1} n_\ell \big(\theta(\rho_{\ell+1}) - \theta(\rho_\ell)\big) &= \sum_{\ell \geq r} m_\ell \big(\theta(q_{\ell+1}) - \theta(q_\ell)\big), \end{aligned}$$

so that, collecting the terms and using (5.8) we get from (3.26) that

(5.9)
$$\Psi(t, u) \leq \eta(0) - t \sum_{\ell \leq k} m_\ell \big(\theta(q_{\ell+1}) - \theta(q_\ell)\big) - t(m - m_{r-1})\big(\theta(q_r) - \theta(u)\big).$$

To bound $\eta(0)$, we define an auxiliary function $V(\lambda, m, v)$ as follows. Consider independent pairs (g_p^1, g_p^2) of Gaussian r.v., for $0 \leq p \leq k+1$, such that, if $a = \xi'(q_r) - \xi'(q_{r-1})$, and if $0 \leq v \leq a$, we have

(5.10)
For $p > r, g_p^1$ and g_p^2 are independent ; $\mathsf{E}(g_p^1)^2 = \mathsf{E}(g_p^2)^2 = \xi'(\rho_{p+1}) - \xi'(\rho_p)$.

$$g_r^1 \text{ and } g_r^2 \text{ are independent } ; \ \mathsf{E}(g_r^1)^2 = \mathsf{E}(g_r^2)^2 = v. \tag{5.11}$$

$$g_{r-1}^1 = g_{r-1}^2; \ \mathsf{E}(g_{r-1}^1)^2 = \mathsf{E}(g_{r-1}^2)^2 = a - v. \tag{5.12}$$

$$\text{For } p < r-1, \ g_p^1 = g_p^2 \ , \ \mathsf{E}(g_p^1)^2 = \mathsf{E}(g_p^2)^2 = \xi'(\rho_{p+1}) - \xi'(\rho_p). \tag{5.13}$$

Keeping the dependence in v and λ implicit, we define the functions

$$A_{k+2}^*(x_1, x_2) = \log\big(\mathrm{ch}x_1\mathrm{ch}x_2\mathrm{ch}\lambda + \mathrm{sh}x_1\mathrm{sh}x_2\mathrm{sh}\lambda\big), \tag{5.14}$$

and for $1 \le \ell \le k+1$ we define recursively the functions

$$A_\ell^*(x_1, x_2) = \frac{1}{n_\ell} \log \mathsf{E}_\ell \exp n_\ell A_\ell^*(x_1 + g_\ell^1, x_2 + g_\ell^2),$$

defining as usual $A_\ell^*(x_1, x_2) = \mathsf{E}_\ell A_\ell^*(x_1 + g_\ell^1, x_2 + g_\ell^2)$ when $n_\ell = 0$. We set

$$V(\lambda, m, v) = A_0^*(h, h). \tag{5.15}$$

To relate this function to $\eta(0)$, we note that

$$\sum_{R_{1,2}=u} \exp H_0(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \le \exp(-\lambda N u) \sum_{\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2} \exp\Big(H_0(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) + \sum_{i \le N} \lambda \sigma_i^1 \sigma_i^2\Big),$$

$$\sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \exp(a_1\varepsilon_1 + a_2\varepsilon_2 + \lambda\varepsilon_1\varepsilon_2) = 4(\mathrm{ch}a_1\mathrm{ch}a_2\mathrm{ch}\lambda + \mathrm{sh}a_1\mathrm{sh}a_2\mathrm{sh}\lambda)$$

and that the r.v. $g_p^j = Z_p^j + y_p^j$ satisfy (5.10) to (5.13) with

$$v = v(u) = t\big(\xi'(q_r) - \xi'(u)\big). \tag{5.16}$$

It should then be obvious that

$$\eta(0) \le 2\log 2 + V(\lambda, m, v(u)) - \lambda u. \tag{5.17}$$

We note that

$$V(0, m, v) = 2T(v, m), \tag{5.18}$$

where $T(v, m)$ is defined as in (4.35). This is seen by the same argument as that used in (2.39). In particular from (4.36) we have

$$V(0, m_{r-1}, v(u)) = 2A_0(h). \tag{5.19}$$

Combining (5.9) and (5.17), we get

$$\begin{aligned} \Psi(t,u) \le {} & 2\log 2 + V(\lambda, m, v(u)) - \lambda u \\ & - t\sum_{\ell \le k} m_\ell\big(\theta(q_{\ell+1}) - \theta(q_\ell)\big) - t(m - m_{r-1})\big(\theta(q_r) - \theta(u)\big). \end{aligned} \tag{5.20}$$

By (5.19), for $\lambda = 0$ and $m = m_{r-1}$, the right-hand side of (5.20) is $2\psi(t)$. Thus, to prove that $\Psi(t, u) < 2\psi(t)$, it suffices to show that the partial derivative of this right-hand side at $\lambda = 0, m = m_{r-1}$ with respect to λ is not zero, or, if $r \geq 2$, and hence $m_{r-1} > 0$, to show that the partial derivative of this right-hand side with respect to m is not zero (or to obtain a quantitative control of these derivatives to prove (5.1)). It is because $m_0 = 0$ that special arguments are required when $r = 1$.

The nice fact is that the partial derivatives of the right-hand side of (5.20) relate well to quantities studied in Section 4. This is obvious through (5.18) concerning the partial derivative with respect to m. Less obvious is the following.

LEMMA 5.8. *We have*

$$\frac{\partial}{\partial \lambda} V(\lambda, m_{r-1}, v)\Big|_{\lambda=0} = U'(v) \tag{5.21}$$

where $U'(v)$ is as given by (4.44).

Proof. Let us write $\zeta_\ell^j = h + \sum_{0 \leq p < \ell} g_p^j$ and

$$R_\ell^* = \exp n_\ell \big(A_{\ell+1}^*(\zeta_{\ell+1}^1, \zeta_{\ell+1}^2) - A_\ell^*(\zeta_\ell^1, \zeta_\ell^2) \big)$$

so that from (5.15) and proceeding as in (3.2) we have

$$\frac{\partial V}{\partial \lambda}(\lambda, m_{r-1}, v)\Big|_{\lambda=0} = \mathsf{E}\big(R_1^* \cdots R_{k+1}^* \mathrm{th}(\zeta_{k+2}^1) \mathrm{th}(\zeta_{k+2}^2) \big). \tag{5.22}$$

We recall the sequence (A_ℓ) of functions of (4.25), and the quantity $B(x, v) = B(x, v, m_{r-1})$ of (4.1). Then, recalling (5.14), proceeding as in the proof of (2.38), and using (5.10) to (5.13), we see that when $\lambda = 0$ and $m = m_{r-1}$ we have

$$\ell \geq r + 1 \Longrightarrow A_\ell^*(x_1, x_2) = A_{\ell-1}(x_1) + A_{\ell-1}(x_2), \tag{5.23}$$

$$A_r^*(x_1, x_2) = B(x_1, v) + B(x_2, v), \tag{5.24}$$

$$\ell \leq r - 1 \Longrightarrow A_\ell^*(x, x) = 2A_\ell(x). \tag{5.25}$$

Thus we have

$$\ell \leq r - 2 \Longrightarrow R_\ell^* = \exp m_\ell \big(A_{\ell+1}(\zeta_{\ell+1}^1) - A_\ell(\zeta_\ell^1) \big), \tag{5.26}$$

$$R_{r-1}^* = \exp m_{r-1} \big(B(\zeta_r^1, v) - A_{r-1}(\zeta_{r-1}^1) \big), \tag{5.27}$$

$$\ell \geq r \Longrightarrow R_\ell^* = Q_\ell^1 Q_\ell^2, \tag{5.28}$$

where

$$Q_r^j = \exp m_{r-1} \big(A_r(\zeta_{r+1}^j) - B(\zeta_r^j, v) \big),$$

$$\ell > r \Rightarrow Q_\ell^j = \exp m_{\ell-1} \big(A_r(\zeta_{\ell+1}^j) - A_{r-1}(\zeta_\ell^j) \big).$$

If we denote by E_r expectation in the r.v. g_p^j for $p \geq r$, the right-hand side of (5.22) is then (using independence as provided by (5.10) and (5.11))

$$\begin{aligned}&\mathsf{E}\big(R_1^* \cdots R_{r-1}^* \mathsf{E}_r\big(R_r^* \cdots R_{k+1}^* \operatorname{th}(\zeta_{k+2}^1)\operatorname{th}(\zeta_{k+2}^2)\big)\big)\\&= \mathsf{E}\big(R_1^* \cdots R_{r-1}^* \mathsf{E}_r\big(Q_r^1 \cdots Q_{k+1}^1 \operatorname{th}(\zeta_{k+2}^1)\big)^2\big).\end{aligned}$$

From (4.25), (5.10), (5.11) and (4.1), and proceeding as in (3.2) we have

$$B'(\zeta_r^1) = \mathsf{E}_r\big(Q_r^1 \cdots Q_{k+1}^1 \operatorname{th}(\zeta_{k+2}^1)\big),$$

and the result from (4.44), (5.26) and (5.27), since the sequences $(\zeta_\ell^1)_{\ell \leq r}$ and $(w_\ell)_{\ell \leq r}$ of (4.38) have the same distribution. $\square$

LEMMA 5.9. *We have*

$$\left|\frac{\partial^2}{\partial \lambda^2} V(\lambda, m_{r-1}, v)\right| \leq L. \tag{5.29}$$

Proof. Similar to Lemma 4.5. $\square$

Proof of Proposition 5.1. We recall the function $v(u)$ of (5.16) and consider the function

$$h(u) = U'(v(u)) - u, \tag{5.30}$$

so that, by (4.54), we have $h(q_r) = 0$. We have

$$h'(q_r) = -t\xi''(q_r)U''(0) - 1.$$

We claim that

$$L\varepsilon^{1/6} \leq 1 - t_0 \Rightarrow h'(q_r) \leq -\frac{1 - t_0}{2}. \tag{5.31}$$

Indeed, if $-\xi''(q_r)U''(0) \leq 1/2$, we have $h'(q_r) \leq -1/2$. Otherwise, since $0 \leq -U''(0) \leq L$ we have $\xi''(q_r) \geq 1/L$, so that by (4.57) we have $-\xi''(q_r)U''(0) \leq 1 + L\varepsilon^{1/6}$, and hence

$$-t\xi''(q_r)U''(0) - 1 \leq t_0 + L\varepsilon^{1/6} - 1,$$

from which (5.31) follows.

Now (as in Lemmas 4.5 and 5.9), $|h''(u)| \leq L$, so that, since $h(q_r) = 0$,

$$h(u) \geq (u - q_r)h'(q_r) - L(u - q_r)^2 \geq \frac{1}{4}(q_r - u)(1 - t_0) \tag{5.32}$$

when $h'(q_r) \leq -(1 - t_0)/2$ and $u - q_r \leq (1 - t_0)/4L$.

By Lemma 5.9, the function $\alpha(\lambda) = V(\lambda, m_{r-1}, v(u)) - \lambda u$ satisfies $|\alpha''(\lambda)| \leq L$, so that

$$\inf_\lambda \alpha(\lambda) \leq \alpha(0) - \frac{\alpha'(0)^2}{L} \tag{5.33}$$

and since by (5.21) we have $\alpha'(0) = h(u)$, combining with (5.32) and (5.20) finishes the proof. $\square$

Proof of Proposition 5.3. In the case $r = 1$, we have $B(x, v) = \mathsf{E}A(x + g\sqrt{v})$, so that $B''(x, v) = \mathsf{E}A''(x + g\sqrt{v})$, and one sees from (4.45) that

$$-U''(v) = \mathsf{E}\big(A''(g\sqrt{a-v} + g_1\sqrt{v})A''(g\sqrt{a-v} + g_2\sqrt{v})\big)$$

where $a = \xi'(q_1)$ and g, g_1, g_2 are standard Gaussian, so that, using the Cauchy-Schwarz inequality,

$$-U''(v) \le \mathsf{E}\big(A''^2(g\sqrt{a-v} + g_1\sqrt{v})\big) = \mathsf{E}\big(A''^2(g\sqrt{a})\big) = -U''(0).$$

Thus, the function $h(u)$ of (5.30) satisfies

$$h'(u) = -t\xi''(u)U''(v(u)) - 1 \le -t\xi''(q_1)U''(0) - 1 = h'(q_1).$$

Using (5.31), we see that if $L\varepsilon^{1/6} \le 1 - t_0$ we have $h'(u) < 0$ for $u \le q_1$ so that since $h(q_1) = 0$, $h(u) > 0$ for $u < q_1$ and we are done by (5.33) as in Proposition 5.1. $\square$

Proof of Proposition 5.5. We can assume $U'(v(u)) = u$, for otherwise we are done by (5.33) as in Proposition 5.1. Let us denote by D the partial derivative in m of the right-hand side of (5.20) at $m = m_{r-1}$ and $\lambda = 0$. By (5.18) and (4.42),

$$D = U(v(u)) - t\big(\theta(q_r) - \theta(u)\big).$$

The function

$$x \mapsto \beta(x) = U\big(x(\xi'(q_r) - \xi'(u))\big) - x\big(\theta(q_r) - \theta(u)\big)$$

is concave because its derivative

$$\beta'(x) = \big(\xi'(q_r) - \xi'(u)\big)U'\big(x(\xi'(q_r) - \xi'(u))\big) - \theta(q_r) + \theta(u)$$

is decreasing by (4.45). Thus

$$\beta(1) \le \beta(t) + (1-t)\beta'(t). \tag{5.34}$$

Since we assume that $U'(v(u)) = u$, we have

$$\beta'(t) = \big(\xi'(q_r) - \xi'(u)\big)u - \theta(q_r) + \theta(u) = \xi(q_r) - \xi(u) + (u - q_r)\xi'(q_r)$$

after replacing θ by its value (1.12). Thus, by definition of the function γ of (5.4) we have $\beta'(t) \le -\gamma(q_r - u)$ and (5.34) yields

$$D = \beta(t) \ge (1 - t_0)\gamma(q_r - u) + \beta(1).$$

By (4.46) and (4.52), we have $\beta(1) \ge -L\sqrt{\varepsilon}$. Thus if $q_r - u \ge (1 - t_0)/L_1$ and $L\sqrt{\varepsilon} \le (1 - t_0)\gamma((1 - t_0)/L_1)$, we have $D > 0$. $\square$

Proof of Proposition 5.4. We modify the previous scheme of proof by taking now the r.v. y_p^j as in (3.18) to (3.20) for $\eta = -1$. We recall that $\rho_r = |u|$. Proceeding as in Lemma 5.8 we get

$$W(v) := \frac{\partial}{\partial \lambda} V(\lambda, 0, v)\Big|_{\lambda=0} = \mathsf{E}\big(A'(g\sqrt{a-v} + g_1\sqrt{v})A'(-g\sqrt{a-v} + g_2\sqrt{v})\big)$$

where $A = A_1, a = \xi'(q_1)$ and g, g_1, g_2 are independent standard Gaussian. We show that for $v = v(u) = \xi'(q_1) - t\xi'(|u|) = \xi'(q_1) + t\xi'(u)$ we have $W(v(u)) - u > 0$. First we observe that $W(v(0)) \geq 0$ and then show as in Proposition 5.3 that the derivative in u is < 0. $\square$

The proofs of Propositions 5.2 and 5.6 are very similar to the proofs of Propositions 5.1 and 5.5 respectively, using now the "dual construction" that we described at the end of Section 4, and we turn to the proof of Proposition 5.7. This proof is comparatively easier than the previous ones since it does not require the work of Section 4. However, a new construction is required. We consider s with $q_{s-1} \leq |u| \leq q_s$, and the two sequences

$$(m'_\ell) = \Big(0, \frac{m_1}{2}, \dots, \frac{m_{r-1}}{2}, m_r, \dots, m_{k-1}, m_k\Big)$$

$$(m''_\ell) = \Big(0, \frac{m_1}{2}, \dots, \frac{m_{s-1}}{2}, m_{s-1}, m_s, \dots, m_{k-1}, m_k\Big).$$

We consider a sequence $n_0 \leq n_1 \leq \cdots \leq n_{2k+2}$ and two disjoint subsets I and J of $\{0, \dots, 2k+2\}$ with $\mathrm{card} I = k+1$ and $\mathrm{card} J = k+2$ such that the numbers $(n_\ell)_{\ell \in I}$ are the numbers $(m'_\ell)_{0 \leq \ell \leq k}$, while the numbers $(n_\ell)_{\ell \in J}$ are the numbers $(m''_\ell)_{0 \leq \ell \leq k+1}$. The purpose of the sets I and J is that we keep track whether a number n_ℓ occurs from the list (m'_ℓ) or the list (m''_ℓ). This is particularly useful in the case where, accidentally, for some ℓ, ℓ' we have $m_\ell = m_{\ell'}/2$. We consider the sequence

$$(q'_\ell)_{0 \leq \ell \leq k+2} = (q_0, q_1, q_2, \dots, q_{s-1}, |u|, q_s, \dots, q_k, q_{k+1}), \tag{5.35}$$

so that $|u| = q'_s$. We define the sequence $(\rho_\ell)_{0 \leq \ell \leq 2k+3}$ by $\rho_\ell = q'_p$, where p is the smallest integer for which $n_\ell \leq m''_p$. We define τ as the unique $\tau \in J$ such that $\rho_\tau = |u|$. It should be clear with this construction that

$$\begin{aligned}\textstyle\sum_{1 \leq \ell < \tau} 2n_\ell\big(\theta(\rho_{\ell+1}) - \theta(\rho_\ell)\big) &+ \sum_{\tau \leq \ell \leq 2k+2} n_\ell\big(\theta(\rho_{\ell+1}) - \theta(\rho_\ell)\big) \\ &= \sum_{1 \leq \ell \leq s-1} 2m''_\ell\big(\theta(q'_{\ell+1}) - \theta(q'_\ell)\big) + \sum_{s \leq \ell \leq k} m''_\ell\big(\theta(q'_{\ell+1}) - \theta(q'_\ell)\big) \\ &= \sum_{1 \leq \ell \leq k} m_\ell\big(\theta(q_{\ell+1}) - \theta(q_\ell)\big).\end{aligned} \tag{5.36}$$

We define $Z_\ell^j = \sqrt{1-t}\,z_p^j$ if $\ell \in I$ and $n_\ell = m'_p$, and $Z_\ell^j = 0$ if $\ell \in J$. We consider the sequence (y_p^j) as defined in (3.18) to (3.20) where $\eta = 1$ if $u \geq 0$

and $\eta = -1$ if $u < 0$. It should be obvious that, with the notation of (3.26), we have $\Psi(t,u) = \eta(1)$, so that to prove that $\Psi(t,u) < 2\psi(t)$ it suffices to prove that $\eta(0) < 2\varphi(0) = 2\log 2 + 2A_0(h)$.

We set $g_\ell^j = Z_\ell^j + y_\ell^j$. We observe that if $\ell \in I$ we have $g_\ell^j = Z_\ell^j$, while if $\ell \in J$ we have $g_\ell^j = y_\ell^j$. Thus for all ℓ, either $g_\ell^1 = \pm g_\ell^2$ or else g_ℓ^1 and g_ℓ^2 are independent. The first case occurs exactly when $\ell \in I$ and $n_\ell = m'_p = m_p/2$ for $p \le r-1$ or $\ell \in J$ and $n_\ell = m''_p = m_p/2$ for $p \le s-1$. The second case occurs exactly when $\ell \in I$ and $n_\ell = m'_p = m_p$ for $p \ge r$ or $\ell \in J$ and $n_\ell = m''_{p+1} = m_p$ for $p \ge s-1$.

We define the sequence (n_ℓ^*) by $n_\ell^* = 2n_\ell$ if $g_\ell^1 = \pm g_\ell^2$ and $n_\ell^* = n_\ell$ if g_ℓ^1 and g_ℓ^2 are independent. We observe that n^* is one of the numbers $m_0, \dots, m_{k+1}$.

We define

$$D_{2k+3}(x_1,x_2) = \log \mathrm{ch} x_1 + \log \mathrm{ch} x_2$$

and, recursively, for $0 \le \ell \le 2k+2$,

$$D_\ell(x_1,x_2) = \frac{1}{n_\ell}\log \mathsf{E}\exp n_\ell D_{\ell+1}(x_1+g_\ell^1, x_2+g_\ell^2).$$

As usual, $\eta(0) \le 2\log 2 + D_0(h,h)$, so that all we have to prove is that

$$D_0(h,h) < 2A_0(h). \tag{5.37}$$

We define

$$D^*_{2k+3}(x) = \log \mathrm{ch} x$$

and, recursively, for $\ell \ge 0$,

$$D^*_\ell(x) = \frac{1}{n^*_\ell}\log \mathsf{E}\exp n^*_\ell D^*_{\ell+1}(x+g_\ell^1).$$

Lemma 5.10. *Consider $n \ge 0$, two independent Gaussian* r.v. g^1 *and* g^2, *with $\mathsf{E}(g^1)^2 = \mathsf{E}(g^2)^2 \ne 0$, and functions $G(x)$ and $D(x_1,x_2)$ such that*

$$\forall x_1, x_2,\ D(x_1,x_2) \le G(x_1) + G(x_2). \tag{5.38}$$

Define

$$\begin{aligned}
D_{(1)}(x_1,x_2) &= \frac{1}{n}\log \mathsf{E}\exp nD(x_1+g^1, x_2+g^2),\\
D_{(2)}(x_1,x_2) &= \frac{1}{n}\log \mathsf{E}\exp nD(x_1+g^1, x_2+g^1),\\
D_{(3)}(x_1,x_2) &= \frac{1}{n}\log \mathsf{E}\exp nD(x_1+g^1, x_2-g^1),\\
G'(x) &= \frac{1}{n}\log \mathsf{E}\exp nG(x+g^1),\\
G''(x) &= \frac{1}{2n}\log \mathsf{E}\exp 2nG(x+g^1).
\end{aligned}$$

Then,

(a) *We have*

$$D_{(1)}(x_1, x_2) \leq G'(x_1) + G'(x_2). \tag{5.39}$$

If equality occurs in (5.38) *only for* $x_1 = \pm x_2$, *we then have*

$$\forall x_1, x_2,\ D_{(1)}(x_1, x_2) < G'(x_1) + G'(x_2). \tag{5.40}$$

(b) *We have*

$$\forall x_1, x_2,\ D_{(2)}(x_1, x_2) \leq G''(x_1) + G''(x_2). \tag{5.41}$$

If $n > 0$ *and* G *is strictly convex, we can have equality only if* $x_1 = x_2$. *If there is never equality in* (5.38), *there is never equality in* (5.41). *If* $n = 0$ *and equality occurs in* (5.38) *only for* $x_1 = x_2$, *equality occurs in* (5.41) *only for* $x_1 = x_2$.

(c) *We have*

$$\forall x_1, x_2,\ D_{(3)}(x_1, x_2) \leq G''(x_1) + G''(x_2). \tag{5.42}$$

If $n > 0$, G *is strictly convex and* $G(x) = G(-x)$, *we can have equality in* (5.42) *only for* $x_1 = -x_2$. *If there is never equality in* (5.38), *there is never equality in* (5.42). *If* $n = 0$ *and equality occurs in* (5.38) *only for* $x_1 = -x_2$, *equality occurs in* (5.42) *only for* $x_1 = -x_2$.

Proof. The first part of (a) follows by independence. The second part is obvious.

The first part of (b) follows from the Cauchy-Schwarz inequality. If $n > 0$, there can be equality only if the two functions $y \mapsto \exp nG(x_1 + y)$ and $y \mapsto \exp nG(x_2 + y)$ are proportional, i.e. if $y \mapsto G(x_1 + y) - G(x_2 + y)$ is constant. Since G is strictly convex we must have $x_1 = x_2$. The rest of (b) is obvious.

The first part of (c) follows from the Cauchy-Schwarz inequality. There can be equality only if the two functions $y \mapsto \exp nG(x_1 + y)$ and $y \mapsto \exp nG(x_2 - y) = \exp nG(y - x_2)$ are proportional. If G is strictly convex, this implies as before that $x_1 = -x_2$. The rest is obvious. □

PROPOSITION 5.11. *We have*

$$D_0(h, h) \leq 2D_0^*(h). \tag{5.43}$$

Equality can occur only when

$$\forall b, c \leq 2k + 2,\ (g_b^1 \neq 0,\ g_c^1 \neq 0,\ n_b^* = n_b,\ n_c^* = 2n_c > 0) \Rightarrow c < b. \tag{5.44}$$

Moreover,

$$u < -q_1 \ \Rightarrow\ D_0(h, h) < 2D_0^*(h). \tag{5.45}$$

Proof. (5.43) follows from recursive use of (5.39) and (5.41). Suppose that (5.44) fails, so that we can find $b < c$ with $n_b^* = n_b, n_c^* = 2n_c > 0$. We can assume that $c - b$ is as small as possible under these conditions. Then either $b = c - 1$, or else $n_\ell = 0$ for $\ell < c$. Also, $c \geq 2$ since $n_c > 0$ and the first two terms of the sequence (n_ℓ) are zero. We observe that by Lemma 4.1, the functions D_ℓ^* are strictly convex. Thus by Lemma 5.10(b), we have

$$D_{c-1}(x_1, x_2) < D_{c-1}^*(x_1) + D_{c-1}^*(x_2) \quad \text{if } x_1 \neq x_2.$$

If $b = c - 1$, by (5.40), we have $D_\ell(x_1, x_2) < D_\ell^*(x_1) + D_\ell^*(x_2)$ for $\ell = b - 1$, and hence by Lemma 5.10, (b), for all $0 \leq \ell < b$. If instead $n_\ell = 0$ for $\ell < c$, we have recursively

$$\forall\ b \leq \ell < c - 1\ ,\ x_1 \neq x_2 \Rightarrow D_\ell(x_1, x_2) < D_\ell^*(x_1) + D_\ell^*(x_2)$$

and we conclude as before.

To prove (5.45) we observe that if $u < -q_1$ there exists b in J with $n_b = m_1/2$ and $g_b^1 = -g_b^2$, and that $n_c = 0$ for $c \in J$ and $c < b$. By (5.42), and, since $D_\ell^*(x) = D_\ell^*(-x)$, we have

$$D_{b-1}(x_1, x_2) < D_{b-1}^*(x_1) + D_{b-1}^*(x_2) \quad \text{if } x_1 \neq -x_2,$$

and we conclude as before. □

Since we have $q_1 < q_2 < \cdots < q_k$, the only possibility to have $g_\ell^1 = 0$ is when $\ell \in J, n_\ell = m_{s-1}/2$ and $|u| = q_{s-1}$ or $n_\ell = m_{s-1}$ and $|u| = q_s$.

Given a function Q and numbers $a \geq 0$ and $m > 0$, we write

$$T_{m,a}(Q)(x) = \frac{1}{m} \log \mathsf{E} \exp mQ\big(x + g\sqrt{a}\big),$$

where g is standard Gaussian.

Lemma 5.12. *If $a, a' \geq 0$ and $m \geq m'$, for each x,*

$$T_{m,a} \circ T_{m',a'}(Q)(x) \leq T_{m',a'} \circ T_{m,a}(Q)(x). \tag{5.46}$$

(5.47) *If $a, a' > 0$ and $m > m'$, we can have equality only if the function Q is constant.*

Proof. Consider independent standard normal r.v. g and g', and denote by E and E' expectation in g and g' respectively. Then

$$T_{m,a} \circ T_{m',a'}(Q)(x) = \frac{1}{m} \log \mathsf{E} \exp \frac{m}{m'} \log \mathsf{E}' \exp m'Q\big(x + g\sqrt{a} + g'\sqrt{a'}\big),$$

$$T_{m',a'} \circ T_{m,a}(Q)(x) = \frac{1}{m'} \log \mathsf{E}' \exp \frac{m'}{m} \log \mathsf{E} \exp mQ\big(x + g\sqrt{a} + g'\sqrt{a'}\big).$$

Setting $\alpha = m/m' \geq 1$ and $X = \exp m'Q\big(x + g\sqrt{a} + g'\sqrt{a'}\big)$, the required inequality is

$$(\mathsf{E}(\mathsf{E}'X)^{\alpha})^{1/\alpha} \leq \mathsf{E}'(\mathsf{E}X^{\alpha})^{1/\alpha}$$

or

$$||\mathsf{E}'X||_{\alpha} \leq \mathsf{E}'||X||_{\alpha} \tag{5.48}$$

if one sets $||Y||_{\alpha} = (\mathsf{E}Y^{\alpha})^{1/\alpha}$. This inequality holds true by convexity. Moreover, when $\alpha > 1$, the norm $||\cdot||_{\alpha}$ is strictly convex, so there can be equality in (5.48) only if, seen as a function of g, X does not depend on g', i.e. Q is constant. □

Proof of Proposition 5.7. For $0 \leq \ell \leq 2k+2$, let

$$a_{\ell} = \mathsf{E}(g_{\ell}^1)^2 = \mathsf{E}(g_{\ell}^2)^2.$$

Each number n_{ℓ}^* is one of the numbers $m_0, \ldots, m_k$, and, moreover,

(5.49) the sum of the numbers a_{ℓ} such that $n_{\ell}^* = m_p$ is $\xi'(q_{p+1}) - \xi'(q_p)$.

This is seen by inspection. For example, there are three values of ℓ for which $n_{\ell}^* = m_{s-1}$. There is one value in I, for which $a_{\ell} = (1-t)(\xi'(q_s) - \xi'(q_{s-1}))$. There is one value in J where $n_{\ell}^* = n_{\ell}$ and $a_{\ell} = t(\xi'(q_s) - \xi'(|u|))$, and one value in J where $n_{\ell}^* = 2n_{\ell}$ and $a_{\ell} = t(\xi'(|u|) - \xi'(q_{s-1}))$.

Consider the operators $W_{\ell} = T_{n_{\ell}^*, a_{\ell}}$. The function D_0^* is constructed by starting with the function $D_{2k+2}^*(x) = \log \mathrm{ch} x$ and applying successively the operators $W_{2k+2}, W_{2k+1}, \ldots, W_1, W_0$.

Consider a permutation π of $\{0, \ldots, 2k+2\}$ such that the sequence $(n_{\pi(\ell)}^*)$ is nondecreasing. If we apply successively the operators $W_{\pi(2k+2)}, W_{\pi(2k+1)}, \ldots$ $\ldots, W_{\pi(1)}, W_{\pi(0)}$ to the function $D_{2k+2}^*(x)$, we obtain the function A_0 (where $A_0(x) = \mathsf{E}A_1(x + z_1)$. This follows from (5.49) and the fact that

$$T_{m,a_1} \circ T_{m,a_2} = T_{m,a_1+a_2}.$$

Lemma 5.12 then shows that

$$D_0^*(h) \leq A_0(h). \tag{5.50}$$

Moreover, (5.47) shows that we can have equality only if

$$c < b \Rightarrow n_c^* \leq n_b^*. \tag{5.51}$$

We have to show that equality cannot simultaneously occur in (5.43) and (5.50). We will consider only the case where $q_{s-1} < u < q_s$, leaving the (easy) equality cases to the reader. We then have $a_{\ell} \neq 0$ for each ℓ.

By (5.45) we have already proved (5.37) if $u < -q_1$, so we can assume that $u \geq -q_1$. Then we have either $s = 1$ or $u \geq 0$. If $s = 1$, by Propositions 5.3 and 5.4, there is nothing to prove if $r = 1$, so we can assume $r \geq 2$ and hence

$s \le r-1$. If $u \ge 0$, since we assume that $u \notin [q_{r-1}, q_{r+1}]$ we can assume that either $s \le r-1$ or $r+1 \le s-1$. Thus, in any case, we can assume that either $s \le r-1$ or $r+1 \le s-1$.

Assume first that $s \le r-1$. Consider $c \in I$ and $b \in J$ with $n_c = m_{r-1}/2$ and $n_b = m_{s-1}$, so that $n_c^* = m_{r-1} = 2n_c > 0$ and $n_b^* = n_b$. We cannot have equality in (5.43) unless $c < b$. But then

$$n_c^* = m_{r-1} \ge m_s > m_{s-1} = n_b = n_b^*,$$

and by (5.51) we cannot have inequality in (5.50).

Assume now that $r+1 \le s-1$. Consider $c \in J$ and $b \in I$ with $n_c = m_{s-1}/2$ and $n_b = m_r$. Thus $n_c^* = m_{s-1} = 2n_c \ge 0$ and $n_b^* = n_b$. If we have equality in (5.43), then we must have $c < b$ by (5.44). Then

$$n_c^* = m_{s-1} \ge m_{r+1} > m_r = n_b = n_b^*,$$

and by (5.51) we cannot have equality in (5.50). □

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