Annals of Mathematics, **163** (2006), 165–219

Invariant measures and arithmetic quantum unique ergodicity

By Elon Lindenstrauss*
Appendix with D. Rudolph

Abstract

We classify measures on the locally homogeneous space $\Gamma\backslash\,\mathrm{SL}(2,\mathbb{R})\times L$ which are invariant and have positive entropy under the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$ and recurrent under L. This classification can be used to show arithmetic quantum unique ergodicity for compact arithmetic surfaces, and a similar but slightly weaker result for the finite volume case. Other applications are also presented.

In the appendix, joint with D. Rudolph, we present a maximal ergodic theorem, related to a theorem of Hurewicz, which is used in *thepr*oof of the main result.

1. Introduction

We recall that the group L is S-algebraic if it is a finite product of algebraic groups over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}_p$, where S stands for the set of fields that appear in this product. An S-algebraic homogeneous space is the quotient of an S-algebraic group by a compact subgroup.

Let L be an S-algebraic group, K a compact subgroup of L, $G = \mathrm{SL}(2,\mathbb{R}) \times L$ and Γ a discrete subgroup of G (for example, Γ can be a lattice of G), and consider the quotient $X = \Gamma\backslash G/K$.

The diagonal subgroup

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\} \subset \mathrm{SL}(2,\mathbb{R})$$

acts on X by right translation. In this paper we wish to study probablilty measures μ on X invariant under this action.

Without further restrictions, one does not expect any meaningful classification of such measures. For example, one may take $L = \mathrm{SL}(2,\mathbb{Q}_p)$, $K =$

*The author acknowledges support of NSF grant DMS-0196124.

$\mathrm{SL}(2,\mathbb{Z}_p)$ and Γ the diagonal embedding of $\mathrm{SL}(2,\mathbb{Z}[\frac{1}{p}])$ in G. As is well-known,

$$\Gamma\backslash G/K \cong \mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R}). \tag{1.1}$$

Any A-invariant measure μ on $\Gamma\backslash G/K$ is identified with an A-invariant measure $\tilde{\mu}$ on $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})$. The A-action on $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})$ is very well understood, and in particular such measures $\tilde{\mu}$ are in finite-to-one correspondence with shift invariant measures on a specific shift of finite type [Ser85] — and there are plenty of these.

Another illustrative example is if L is $\mathrm{SL}(2,\mathbb{R})$ and $K = \{e\}$. In this case we assume that the projection of Γ to each $\mathrm{SL}(2,\mathbb{R})$ factor is injective (for example, Γ an irreducible lattice of G). No nice description of A-invariant measures on X is known in this case, but at least in the case that Γ is a lattice (the most interesting case) one can still show there are many such measures (for example, there are A-invariant measures supported on sets of fractal dimension).

An example of a very meaningful classification of invariant measures with far-reaching implications in dynamics, number theory and other subjects is M. Ratner's seminal work [Ra91], [Ra90b], [Ra90a] on the classification of measures on $\Gamma\backslash G$ invariant under groups $H < G$ generated by one-parameter unipotent subgroups. There it is shown that any such measure is a linear combination of algebraic measures: i.e. N invariant measures on a closed N-orbit for some $H < N < G$. This theorem was originally proved for G a real Lie group, but has been extended independently by Ratner and G. A. Margulis and G. Tomanov also to the S-algebraic context [MT94], [Ra95], [Ra98].

In order to get a similar classification of invariant measures, one needs to impose an additional assumption relating μ to the foliation of X by leaves isomorphic to L/K. The condition we consider is that of recurrence: that is that for every $B \subset X$ with $\mu(B) > 0$, for almost every $x \in X$ with $x \in B$ there are elements x' arbitrarily far (with respect to the leaf metric) in the L/K leaf of x with $x' \in B$; for a formal definition see Definition 2.3. For example, in our second example of $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, $K = \{e\}$ this recurrence condition is satisfied if μ in addition to being invariant under A is also invariant under the diagonal subgroup of the second copy of $\mathrm{SL}(2,\mathbb{R})$.

Though it is natural to conjecture that this recurrence condition is sufficient in order to classify invariant measures, for our proof we will need one additional assumption, namely that the entropy of μ under A is positive.

Our main theorem is the following:

THEOREM 1.1. *Let $G = \mathrm{SL}(2,\mathbb{R}) \times L$, where L is an S-algebraic group, $H < G$ is the $\mathrm{SL}(2,\mathbb{R})$ factor of G and K is a compact subgroup of L. Take Γ to be a discrete subgroup of G (not necessarily a lattice) such that $\Gamma \cap L$ is finite. Suppose μ is a probability measure on $X = \Gamma\backslash G/K$, invariant un-*

der multiplication from the right by elements of the diagonal group $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. *Assume that*

(1) *All ergodic components of* μ *with respect to the* A*-action have positive entropy.*

(2) μ *is* L/K*-recurrent.*

Then μ *is a linear combination of algebraic measures invariant under* H*.*

We give three applications of this theorem, the first of which is to a seemingly unrelated question: arithmetic quantum unique ergodicity. In [RS94], Z. Rudnick and P. Sarnak conjectured the following:

CONJECTURE 1.2. *Let* M *be a compact Riemannian manifold of negative sectional curvature. Let* ϕ_i *be a complete orthonormal sequence of eigenfunctions of the Laplacian on* M*. Then the probability measures* $d\tilde{\mu}_i = |\phi_i(x)|^2 \, d\,\mathrm{vol}$ *tend in the weak star topology to the uniform measure* $d\,\mathrm{vol}$ *on* M*.*

A. I. Šnirel′man, Y. Colin de Verdière and S. Zelditch have shown in great generality (specifically, for any manifold on which the geodesic flow is ergodic) that if one omits a subsequence of density 0 the remaining $\tilde{\mu}_i$ do indeed converge to $d\,\mathrm{vol}$ [Šni74], [CdV85], [Zel87]. An important component of their proof is the *microlocal lift* of any weak star limit $\tilde{\mu}$ of a subsequence of the $\tilde{\mu}_i$. The microlocal lift of $\tilde{\mu}$ is a measure μ on the unit tangent bundle SM of M whose projection on M is $\tilde{\mu}$, and most importantly it is always invariant under the geodesic flow on SM. We shall call any measure μ on SM arising as a microlocal lift of a weak star limit of $\tilde{\mu}_i$ a *quantum limit*. Thus a slightly stronger form of Conjecture 1.2 is the following conjecture, also due to Rudnick and Sarnak:

CONJECTURE 1.3 (Quantum Unique Ergodicity Conjecture). *For any compact negatively curved Riemannian manifold* M *the only quantum limit is the uniform measure* $d\,\mathrm{vol}_{SM}$ *on* SM*.*

Consider now a surface of constant curvature $M = \Gamma\backslash\mathbb{H}$. Then $SM \cong \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$, and under this isomorphism the geodesic flow on SM is conjugate to the action of the diagonal subgroup A on $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$, and as we have seen in (1.1) for certain $\Gamma < \mathrm{PSL}(2,\mathbb{R})$, we can view $X = \Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ as a double quotient $\tilde{\Gamma}\backslash G/K$ with $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{Q}_p)$. We will consider explicitly two kinds of lattices $\Gamma < \mathrm{SL}(2,\mathbb{R})$ with this property: congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ and of lattices derived from Eichler orders in an $\mathbb{R}$-split quaternion algebra over $\mathbb{Q}$; strictly speaking, the former does not fall in the framework of Conjecture 1.3 since Γ is not a uniform lattice. For simplicity, we will collectively call both types of lattices *congruence lattices over* $\mathbb{Q}$.

Any quantum limit μ on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ for Γ a congruence lattices over $\mathbb{Q}$ can thus be identified with an A-invariant measure on $\tilde{\Gamma}\backslash G/K$, so in order to deduce that μ is the natural volume on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ one needs only to verify that μ satisfies both conditions of Theorem 1.1.

Closely related to (1.1), which for general lattices over $\mathbb{Q}$ holds for all primes outside a finite exceptional set, are the Hecke operators which are self-adjoint operators on $L^2(M)$ which commute with each other and with the Laplacian on M. We now restricted ourselves to *arithmetic quantum limits*: quantum limits on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ for Γ a congruence lattice over $\mathbb{Q}$ that arises from a sequence of joint eigenfunctions of the Laplacian and all Hecke operators. It is expected that except for some harmless obvious multiplicities the spectrum of the Laplacian on M is simple, so presumably this is a rather mild assumption.

Jointly with J. Bourgain [BL03], [BL04], we have shown that arithmetic quantum limits have positive entropy: indeed, that all A-ergodic components of such measures have entropy $\geq 2/9$ (according to this normalization, the entropy of the volume measure is 2). Unlike the proof of Theorem 1.1 this proof is effective and gives explicit (in the compact case) uniform upper bounds on the measure of small tubes. The argument is based on a simple idea from [RS94], which was further refined in [Lin01a]; also worth mentioning in this context is a paper by Wolpert [Wol01]. That arithmetic quantum limits are $\mathrm{SL}(2,\mathbb{Q}_p)/\mathrm{SL}(2,\mathbb{Z}_p)$-recurrent is easier and follows directly from the argument in [Lin01a]; we provide a self-contained treatment of this in Section 8.

This establishes the following theorem:

THEOREM 1.4. *Let $M=\Gamma\backslash\mathbb{H}$ with Γ a congruence lattice over $\mathbb{Q}$. Then for compact M the only arithmetic quantum limit is the* (*normalized*) *volume* $d\,\mathrm{vol}_{SM}$. *For M not compact any arithmetic quantum limit is of the form* $c\,d\,\mathrm{vol}_{SM}$ *with* $0\leq c\leq 1$.

We remark that T. Watson [Wat01] proved this assuming the Generalized Riemann Hypothesis (GRH). Indeed, by assuming GRH Watson gets an optimal rate of convergence, and can show that even in the noncompact case any arithmetic quantum limit is the normalized volume (or in other words, that no mass escapes to infinity). We note that the techniques of [BL03] are not limited only to quantum limits; a sample of what can be proved using these techniques and Theorem 1.1 is the following theorem (for which we do not provide details, which will appear in [Lin04]) where no assumptions on entropy are needed (for the number theoretical background, see [Wei67]):

THEOREM 1.5. *Let $\mathbb{A}$ denote the ring of adeles over $\mathbb{Q}$. Let $A(\mathbb{A})$ denote the diagonal subgroup of* $\mathrm{SL}(2,\mathbb{A})$, *and let μ be an $A(\mathbb{A})$-invariant probability measure on* $X=\mathrm{SL}(2,\mathbb{Q})\backslash \mathrm{SL}(2,\mathbb{A})$. *Then μ is the* $\mathrm{SL}(2,\mathbb{A})$*-invariant measure on X.*

Theorem 1.1 also implies the following theorem:[1]

THEOREM 1.6. *Let* $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, *and* $H \subset G$ *as above. Take* Γ *to be a discrete subgroup of* G *such that the kernel of its projection to each* $\mathrm{SL}(2,\mathbb{R})$ *factor is finite* (*note that this is slightly more restrictive than in Theorem* 1.1). *Suppose* μ *is a probability measure on* $\Gamma\backslash G$ *which is invariant and ergodic under the two-parameter group* $B = \left(\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right)$. *Then either*

(1) μ *is an algebraic measure*, *or*

(2) *the entropy of* μ *with respect to* every *one-parameter subgroup of* B *is zero.*

This strengthens a previous, more general, result by A. Katok and R. Spatzier [KS96], which is of the same general form. However, Katok and Spatzier need an additional ergodicity assumption which is somewhat technical to state but is satisfied if, for example, every one-parameter subgroup of B acts ergodically on μ. While this ergodicity assumption is quite natural, it is very hard to establish it in most important applications. In a recent breakthrough, M. Einsiedler and A. Katok [EK03] have been able to prove without any ergodicity assumptions a similar specification of measures invariant under the full Cartan group on $\Gamma\backslash G$ for G an $\mathbb{R}$-split connected Lie group of rank ≥ 2. It should be noted that their proof does not work in a product situation as in Theorem 1.6; furthermore, Einsiedler and Katok need to assume that *all* one-parameter subgroups of the Cartan group act with positive entropy. In Section 6 of this paper we reproduce a key idea from [EK03] which is essential for proving Theorem 1.1 (if one is only interested in Theorem 1.6 this idea is not needed).

The proofs of both theorems uses heavily ideas introduced by M. Ratner in her study of horocycle flows and in her proof of Raghunathan's conjectures, particularly [Ra82], [Ra83]; see also [Mor05], particularly §1.4. Previous works on this subject have applied Ratner's work to classify invariant measures after some invariance under unipotent subgroups has been established; we use Ratner's ideas to establish this invariance in the first place. In order to apply Ratner's ideas one needs a generalized maximal inequality along the action of the horocyclic group which does not preserve the measure; a similar inequality was

[1]Indeed, let A be as above and A' be the group of diagonal matrices in the second $\mathrm{SL}(2,\mathbb{R})$ factor, so that $B = AA'$. By a result of H. Hu [Hu93], if there is some one-parameter subgroup of B with respect to which μ has positive entropy, μ has positive entropy with respect to either A or A' (note that in this case for any one-parameter subgroup of B all ergodic components have the same entropy). Without loss of generality, μ (and hence all its ergodic components) have positive entropy with respect to A; invariance under A' is used to verify the recurrence condition in Theorem 1.1.

discovered by W. Hurewicz a long time ago, but we present what we need (and a bit more) in the appendix, joint with D. Rudolph. We mention that a somewhat similar approach was used by Rudolph [Rud82] for a completely different problem (namely, establishing Bernoullicity of Patterson-Sullivan measures on certain infinite volume quotients of $\mathrm{SL}(2,\mathbb{R})$).

Both Theorem 1.1 and Theorem 1.6 have been motivated by results of several authors regarding invariant measures on $\mathbb{R}/\mathbb{Z}$. We give below only a brief discussion; for more details see [Lin03].

It has been conjectured by Furstenberg that the only nonatomic probability measure μ on $\mathbb{R}/\mathbb{Z}$ invariant under the multiplicative semigroup $\{a^n b^m\}$ with $a, b \in \mathbb{N} \setminus \{1\}$ multiplicative independent (i.e. $\log a/\log b \notin \mathbb{Q}$) is the Lebesgue measure. D. Rudolph [Rud90b] and A. Johnson [Joh92] have shown that any such μ which has positive entropy with respect to one element of the acting semigroup is indeed the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ (a special case of this has been proven earlier by R. Lyons [Lyo88]). It is explicitly pointed out in [Rud90b] that the proof simplifies considerably if one adds an ergodicity assumption. This theorem is in clear analogy with Theorem 1.6, though we note that in that case if one element of the acting semigroup has positive entropy it is quite easy to show that all elements of the acting semigroup have positive entropy.

B. Host [Hos95] has given an alternative proof of Rudolph's theorem. The basic ingredient of his proof is the following theorem: if μ is a invariant and is recurrent under the action of the additive group $\mathbb{Z}[\frac{1}{b}]/\mathbb{Z}$ for a, b relatively prime then μ is Lebesgue measure (a similar theorem for the multidimensional case is given in [Hos00]).

Jointly with K. Schmidt [LS04] we have proved that if $a \in M_n(\mathbb{Z})$ is a nonhyperbolic toral automorphism whose action on the n-dimensional torus is totally irreducible then any a-invariant measure which is recurrent with respect to the central foliation for the a action on the torus is Lebesgue measure. Like Host's results, this is a fairly good (but not perfect) analog to Theorem 1.1.

The scope of the methods developed in this paper is substantially wider than what I discuss here. In particular, in a forthcoming paper with M. Einsiedler and A. Katok [EKL06] we show how using the methods developed in this paper in conjunction with the methods of [EK03] one can substantially sharpen the results of the latter paper. These stronger results imply in particular that the set of exceptions to Littlewood's conjecture, i.e. those $(\alpha, \beta) \in \mathbb{R}^2$ for which $\underline{\lim}_{n\to\infty} n \|n\alpha\| \|n\beta\| > 0$, has Hausdorff dimension 0.

Acknowledgments. There are very many people I would like to thank for their help. I have had the good fortune to collaborate with several people about topics related to this work: in rough chronological order, with David Meiri and Yuval Peres [LMP99], Barak Weiss [LW01], Jean Bourgain [BL03], François

Ledrappier [LL03], Klaus Schmidt [LS04] and Manfred Einsiedler [EL03], and have learned a lot from each of these collaborations. I also talked about these questions quite a bit with Dan Rudolph; one ingredient of the proof, presented in the appendix, is due to these discussions.

It has been Peter Sarnak's suggestion to try to find a connection between quantum unique ergodicity and measure rigidity, and his consistent encouragement and help are very much appreciated.

I would like to thank David Fisher, Alex Gamburd, Boris Kalinin, Anatole Katok, Michael Larsen, Gregory Margulis, Shahar Moses, Nimish Shah, Ralf Spatzier, Marina Ratner, Benjamin Weiss and many others for helpful discussions. I particularly want to thank Dave Witte for patiently explaining to me some of the ideas behind Ratner's proof of Raghunathan's conjecture. Thanks are also due to Manfred Einsiedler, Shahar Mozes, Lior Silberman and Marina Ratner for careful reading and many corrections for an earlier version of this manuscript. I would like to thank the Newton Institute, the University of Indiana at Bloomington, and ETH Zurich for their hospitality.

This fairly long paper has been typeset in its entirety by voice. This would have not been possible without the help of Scotland Leman and of the Stanford University mathematics department which has made Scotland's help available to me. In dictating this paper I have used tools written by David Fox which are available on his website.[2]

Last but not least, this paper would have not been written without the help and support of my family, and in particular of my wife Abigail. This paper is dedicated with love to my parents, Joram and Naomi.

2. (G, T)-spaces

Let X be a locally compact separable metric space. We will denote the metric on all relevant metric spaces by $d(\cdot, \cdot)$; where this may cause confusion, we will give the metric space as a subscript, e.g. $d_X(\cdot, \cdot)$ etc. Similarly, $B_r(p)$ denotes the open ball of radius r in the metric space p belongs to; where needed, the space we work in will be given as a superscript, e.g. $B_r^X(x)$. We will assume implicitly that for any $x \in X$ (as well as any other locally compact metric space we will consider) and $r > 0$ the ball $B_r^X(x)$ is relatively compact.

We define the notion of a (G, T)-foliated space, or a (G, T)-space for short, for a locally compact separable metric space T with a distinguished point $e \in T$ and a locally compact second countable group G which acts transitively and

[2] *URL*: http://cfa-www.harvard.edu/ dcfox/dragon/natlatex.html. Since then Scotland with my help has written an improved version of these tools which I have used since and which I intend to post online when it is ready.

continuously on T (i.e. the orbit of e under G is T). This generalizes the notion of a G-space for (locally compact, metric) group G, i.e. a space with a continuous G action (see Example 2.2), as well as the notion of a (G,T)-manifold ([Thu97, §3.3]).

Definition 2.1. A *locally compact separable metric space* X is said to be a (G,T)-space if there is some open cover $\mathfrak{T}$ of X by relatively compact sets, and for every $U \in \mathfrak{T}$ a continuous map $t_U : U \times T \to X$ with the following properties:

(A-1) For every $x \in U \in \mathfrak{T}$, we have that $t_U(x,e) = x$.

(A-2) For any $x \in U \in \mathfrak{T}$, for any $y \in t_U(x,T)$ and $V \in \mathfrak{T}$ containing y, there is a $\theta \in G$ so that

$$t_V(y,\cdot) \circ \theta = t_U(x,\cdot). \tag{2.1}$$

In particular, For any $x \in U \in \mathfrak{T}$, and any $y \in t_U(x,T), V \in \mathfrak{T}(y)$ we have that $t_U(x,T) = t_V(y,T)$.

(A-1) There is some $r_U > 0$ so that for any $x \in U$ the map $t_U(x,\cdot)$ is injective on $\overline{B^T_{r_U}(e)}$.

X is T-space if it is an $(\mathrm{Isom}(T),T)$-space, where $\mathrm{Isom}(T)$ is the isometry group of T.

Note that if X is a (G,T)-space, and if the action of G on T extends to $H > G$ then X is automatically also an (H,T)-space. The most interesting case is when G acts on T by isometries. If the stabilizer in G of the point $e \in T$ is compact then it is always possible to find a metric on T so that G acts by isometries.

Example 2.2. Suppose that G is a locally compact metric group, acting continuously (say from the right) on a locally compact metric space X. Suppose that this action is locally free, i.e. there is some open neighborhood of the identity $B^G_r(e) \subset G$ so that for every $x \in X$

$$g \mapsto xg$$

is injective on $B^G_r(e)$. Then X is a (G,G)-space with $t_U(x,g) = xg$ for every $U \in \mathfrak{T}$ (if X is compact, we may take $\mathfrak{T} = \{X\}$ though in general a more refined open cover may be needed). We can identify G (more precisely, the action of G on itself from the left) as a subgroup of $\mathrm{Isom}(G)$ if we take d_G to be left invariant (i.e. $d_G(h_1,h_2) = d_G(gh_1,gh_2)$ for any $g,h_1,h_2 \in G$).

When G is a group we shall reserve the term G-space to denote this special case of the more general notion introduced in Definition 2.1.

For $x \in X$ we set

$$\mathfrak{T}(x) = \{U \in \mathfrak{T} : x \in U\}.$$

Notice that by property A-2, $y \in t_U(x,T)$ (which does not depend on U as long as $U \in \mathfrak{T}(x)$) is an equivalence relation which we will denote by $x \overset{T}{\sim} y$. For any x we will call its equivalence class under $\overset{T}{\sim}$ the T-orbit or T-leaf of x. This partition into equivalence classes gives us a foliation of X into leaves which are locally isometric to T. We say that a T-leaf is an embedded leaf if for any x in this leaf and $U \in \mathfrak{T}(x)$ the map $t_U(x,\cdot)$ is injective (note that if this is true for one choice of x in the leaf and $U \in \mathfrak{T}(x)$, it will also hold for any other choice).

Definition 2.3. We say that a Radon measure μ on a (G,T)-space X is *recurrent* if for every measurable $B \subset X$ with $\mu(B) > 0$, for almost every $x \in B$ and for every compact $K \subset T$ and $U \in \mathfrak{T}(x)$ there is a $t \in T \setminus K$ so that $t_U(x,t) \in B$.

Example 2.4. Suppose that G acts freely and continuously on X preserving a measure μ. Then by Poincaré recurrence, μ is G-recurrent if, and only if, G is not compact.

In the context of nonsingular $\mathbb{Z}$ or $\mathbb{R}$-actions (i.e. actions of these groups which preserve the measure class), what we have called the recurrent measures are known as conservative and play an important role; for example, see §1.1 in [Aar97]. This definition seems to be just what is needed in order to have nontrivial dynamics. For probability measures, there is an alternative interpretation of this condition in terms of conditional measures which we present later.

3. Restricted measures on leaves

Throughout this section, X is a (G,T)-space as in Definition 2.1 with $G \subset \mathrm{Isom}(T)$. For simplicity, we make the further assumption:

(3.1) The T -leaf of μ-almost every $x \in X$ is embedded.

Since X is second countable, it is also clearly permissible to assume without loss of generality that $\mathfrak{T}$ is countable. Let $\mathcal{M}_\infty(T)$ denote the space of all Radon (in particular, locally finite) measures on T, equipped with the smallest topology so that the map $\nu \mapsto \int f d\nu$ is continuous for every continuous compactly supported $f \in C_c(T)$. Note that since T is a locally compact separable metric space, $\mathcal{M}_\infty(T)$ is separable and metrizable (though in general not locally compact).

The purpose of this section is to show how the measure μ on X induces a locally finite measure on almost every T-orbit which is well defined up to a normalizing constant. More formally, if $U \in \mathfrak{T}(x)$ we define a measurable map $x \mapsto \mu_{x,T}^U \in \mathcal{M}_\infty(T)$ with the properties described below in Theorem 3.6; in particular, $x \mapsto \mu_{x,T}^U$ satisfies that there is a set of full measure so that for any two points x, y which are in this set and on the same T leaf, and if $\theta \in G$ is the isometry determined by (2.1) then

$$\theta_* \mu_{x,T}^U \propto \mu_{y,T}^V, \qquad \forall U \in \mathfrak{T}(x), V \in \mathfrak{T}(y),$$

i.e. the left-hand side is equal to a nonzero positive scalar times the right-hand side. Note that even if μ is a probability measure, in general $\mu_{x,T}^U$ will not be finite measures.

Sometimes, we will omit the upper index and write $\mu_{x,T} = \mu_{x,T}^U$. Usually this will not cause any real confusion since $t_U(x,\cdot)_* \mu_{x,T}^U$ does not depend on U. It is, however, somewhat more comfortable to think of $\mu_{x,T}$ as a measure on T since $t_U(x,\cdot)_* \mu_{x,T}^U$ is in general not a Radon measure.

Let $\mathcal{S}$ be the collection of Borel subsets of X. We recall that a sigma ring is a collection of sets $\mathcal{A}$ which is closed under countable unions and under set differences (i.e., if $A, B \in \mathcal{A}$ then so is $A \setminus B$). Unless specified otherwise, all sigma rings we consider will be countably generated sigma rings of Borel sets, and in particular have a maximal element.

Definition 3.1. Let $\mathcal{A} \subset \mathcal{S}$ be a countably generated sigma ring, and let $\mathcal{C} \subset \mathcal{A}$ be a countable ring of sets which generates $\mathcal{A}$. The *atom* $[x]_\mathcal{A}$ of a point $x \in X$ in $\mathcal{A}$ is defined as

$$[x]_\mathcal{A} = \bigcap_{C \in \mathcal{C} : x \in C} C = \bigcap_{A \in \mathcal{A} : x \in A} A.$$

Two countably generated sigma rings $\mathcal{A}, \mathcal{B} \subset \mathcal{S}$ with the same maximal element are *equivalent* (in symbols: $\mathcal{A} \sim \mathcal{B}$) if, for every $x \in X$, the atoms $[x]_\mathcal{A}$ and $[x]_\mathcal{B}$ are countable unions of atoms $[y]_{\mathcal{A} \vee \mathcal{B}}$ of the sigma ring $\mathcal{A} \vee \mathcal{B}$ generated by $\mathcal{A}$ and $\mathcal{B}$.

Let $\mathcal{A} \subset \mathcal{S}$ be a countably generated sigma ring, μ a Radon measure, and assume that the μ-measure of the maximal element of $\mathcal{A}$ is finite. Then we can consider the decomposition of μ with respect to the sigma ring $\mathcal{A}$, i.e. a set of probability measures $\{\mu_x^\mathcal{A} : x \in X\}$ on X with the following properties:

(1) For all $x, x' \in X$ with $[x]_\mathcal{A} = [x']_\mathcal{A}$,

$$\mu_x^\mathcal{A} = \mu_{x'}^\mathcal{A} \text{ and } i\mu_x^\mathcal{A}([x]_\mathcal{A}) = 1. \tag{3.2}$$

(2) For every $B \in \mathcal{S}$, the map $x \mapsto \mu_x^\mathcal{A}(B)$ is $\mathcal{A}$-measurable.

(3) For every $A \in \mathcal{A}$ and $B \in \mathcal{S}$,

$$\mu(A \cap B) = \int_A \mu_x^{\mathcal{A}}(B)\, d\mu(x). \tag{3.3}$$

We recall that if $\mathcal{A} \sim \mathcal{B}$ then there is a Borel set of full measure on which

$$\frac{\mu_x^{\mathcal{A}}|_{[x]_{\mathcal{A}\vee\mathcal{B}}}}{\mu_x^{\mathcal{A}}([x]_{\mathcal{A}\vee\mathcal{B}})} = \frac{\mu_x^{\mathcal{B}}|_{[x]_{\mathcal{A}\vee\mathcal{B}}}}{\mu_x^{\mathcal{B}}([x]_{\mathcal{A}\vee\mathcal{B}})}. \tag{3.4}$$

If $\mathcal{A}$ is a sigma ring with maximal element U, and $D \subset U$ we define $\mathcal{A}|_D = \{A \cap D : A \in \mathcal{A}\}$. Note that for any $x \in D$, $[x]_{\mathcal{A}|_D} = [x]_{\mathcal{A}} \cap D$. Similarly to (3.4), one has that on a Borel subset of D of full measure

$$\mu_x^{\mathcal{A}|_D} = \frac{\mu_x^{\mathcal{A}}|_{[x]_{\mathcal{A}} \cap D}}{\mu_x^{\mathcal{A}}([x]_{\mathcal{A}} \cap D)}. \tag{3.5}$$

Let $B_r^T = B_r^T(e)$ denote the ball of radius r around the distinguished point $e \in T$. Note that if $x \in U \in \mathfrak{T}$, then $t_U(x, B_r^T)$ does not depend on U; slightly abusing notation, we define for $x \in X$,

$$B_r^T(x) = t_U(x, B_r^T), \qquad U \in \mathfrak{T}(x);$$

we set $\overline{B_r^T}(x) = t_U(x, \overline{B_r^T})$. In this notation, the T-leaf of x is $B_\infty^T(x)$.

Lemma 3.2. *Let $x \in X$ and $r > 0$ be arbitrary. Fix $V \in \mathfrak{T}(x)$ and assume $t_V(x, \cdot)$ is injective on $\overline{B_{20r}^T}$. Then there is an $\epsilon > 0$ so that the set $U = t_V(B_\epsilon(x), B_r^T)$ satisfies*

(1) *any $y, z \in U$ with $y \in B_{10r}^T(z)$ actually satisfy $y \in B_{4r}^T(z)$.*

(2) *U is a relatively compact (i.e. $\overline{U}$ is compact) open subset of X.*

Proof. By our assumptions on x and r, we know that $x \notin t_V(x, \overline{B_{20r}^T} \setminus B_r^T(x))$. By continuity of t_V, and local compactness of T, we have that there is a $\epsilon > 0$ so that for every $x' \in B_\epsilon(x)$

$$B_{20r}^T(x') \cap B_\epsilon(x) \subset B_{2r}^T(x'). \tag{3.6}$$

In order to see that (1) holds, suppose $y_1, y_2 \in U$ with $y_1 \in B_{10r}^T(y_2)$. Then there are $x_1, x_2 \in B_\epsilon(x)$ so that $y_i \in B_r^T(x_i)$ for $i = 1, 2$. By the triangle inequality, $x_1 \in B_{12r}^T(x_2)$, and so by (3.6) $x_1 \in B_{2r}^T(x_2)$. This implies that indeed $y_1 \in B_{4r}^T(y_2)$.

Since clearly $\overline{U} \subset t_V(\overline{B_\epsilon}(x), \overline{B_r^T})$, and the latter is compact since it is the image by a continuous map of a compact set, the only thing which still needs explanation at this point is why U is open.

Suppose $z = t_V(y, q)$ with $y \in B_\epsilon(x)$ and $q \in B_r^T$. Take $V' \in \mathfrak{T}(z)$. By Definition 2.1 there is some $q' \in B_r^T$ with $y = t_{V'}(z, q')$. If z' is very close to z, we have that $y' = t_{V'}(z', q')$ is very close to y – close enough that $y' \in B_\epsilon(x)$ and then $z' \in B_r^T(y') \subset U$. □

Definition 3.3. A set $A \subset X$ is an *open T-plaque* if for any $x \in A$: (i) $A \subset B_r^T(x)$ for some $r > 0$ (ii) $t_V(x, \cdot)^{-1} A$ is open in T for some (equivalently for any) $V \in \mathfrak{T}(x)$.

Definition 3.4. A pair $(\mathcal{A}, U)$ with $\mathcal{A} \subset \mathcal{S}$ a countably generated sigma ring and $U \subset X$ its maximal element is called an *r, T-flower* with *center* $B \subset X$ if

(♣-1) $B \subset U$ and U is relatively compact.

(♣-2) For every $y \in U$

$$[y]_{\mathcal{A}} = U \cap B_{4r}^T(y)$$

(in particular, the atom $[y]_{\mathcal{A}}$ is an open T-plaque).

(♣-3) If $y \in B$ then $[y]_{\mathcal{A}} \supset B_r^T(y)$.

Corollary 3.5. *Under the assumptions of Lemma* 3.2, *and with $U \ni x$ as in that lemma, there is a countably generated sigma ring $\mathcal{A}$ so that $(\mathcal{A}, U)$ is a r, T-flower with center $B_\epsilon(x)$.*

Proof. Let $\mathcal{U}$ be the collection of all open subsets A of U so that if $y \in A$ then $B_{4r}^T(y) \cap U \subset A$.

We first show:

(∗) For every $y, y' \in U$ with $y \notin B_{4r}^T(y')$ one can find disjoint open subsets $A \ni y, A' \ni y'$ with $A, A' \in \mathcal{U}$.

By Lemma 3.2,

$$\overline{B_{4r}^T}(y) \cap \overline{B_{4r}^T}(y') = \emptyset;$$

since both sets are compact, there is an $\epsilon' > 0$ so that for all $z \in B(y, \epsilon'), z' \in B(y', \epsilon')$

$$\overline{B_{4r}^T}(z) \cap \overline{B_{4r}^T}(z') = \emptyset.$$

Suppose $y \in V \in \mathfrak{T}$, and that $B(y, \epsilon') \subset V$, and similarly for y' (and a corresponding $V' \in \mathfrak{T}$). Clearly,

$$A = t_V(B(y, \epsilon'), B_{4r}^T),$$
$$A' = t_{V'}(B(y', \epsilon'), B_{4r}^T)$$

have the desired properties.

Consider the sigma ring $\mathcal{A}$ generated by the collection $\mathcal{U}$. Clearly, $(\mathcal{A}, U)$ satisfies ♣-1.

Define a relation $y \smile y'$ on $U \times U$ if $y \in B_{4r}^T(y')$. This is clearly an equivalence relation. It is in fact a closed equivalence relation, since if $y_i \smile y_i'$

and $y_i \to y, y_i' \to y'$ with $y, y' \in U$ then $y \in \overline{B_{4r}^T}(y')$, and in view of definition of U this implies $y \in B_{4r}^T(y')$. By $(*)$ the quotient space $U/\smile$ is Hausdorff; since U is sigma compact so is $U/\smile$. By definition, the open sets on $U/\smile$ are precisely the images of sets in $\mathcal{U}$, and $\mathcal{A}$ can be identified with the Borel algebra on $U/\smile$, and so in particular is countably generated.

Furthermore, for any $y \in U$, if $y \in A \in \mathcal{U}$ then by definition $B_{4r}^T(y) \subset A$; if $y \notin A \in \mathcal{U}$ then $B_{4r}^T(y) \cap A = \emptyset$, so that

$$[y]_{\mathcal{A}} = \bigcap_{A \in \mathcal{U}:\, y \in A} A \cap \bigcap_{A \in \mathcal{U}:\, y \notin A} A^{\complement} \supset B_{4r}^T(y) \cap U. \tag{3.7}$$

On the other hand, by $(*)$, for every $y' \in U \setminus B_{4r}^T(y)$ there is an $A \in \mathcal{U}$ with $y' \notin A \ni y$, so in fact equality holds in (3.7), establishing ♣-2.

Since by Lemma 3.2 for any $y \in B$ we have that $B_r^T(y) \subset U$, ♣-2 implies ♣-3. □

The following theorem is the main result of this section:

Theorem 3.6. *Let X be a (G, T)-space, and μ a Radon measure on X so that μ-a.e. point has an embedded T-leaf. Then there are Borel measurable maps $\mu_{x,T}^V : V \mapsto \mathcal{M}_\infty(T)$ for $V \in \mathfrak{T}$ which are uniquely determined (up to μ-measure 0) by the following two conditions:*

1. *For almost every $x \in V$, $\mu_{x,T}^V(B_1^T) = 1$.*
2. *For any countably generated sigma ring $\mathcal{A} \subset \mathcal{S}$ with maximal element E, if for every $x \in E$ the atom $[x]_{\mathcal{A}}$ is an open T-plaque, then for μ-almost every $x \in E$, for all $V \in \mathfrak{T}$ containing x,*

$$t_V(x,\cdot)^{-1}{}_*\mu_x^{\mathcal{A}} \propto \mu_{x,T}^V|_{t_V(x,\cdot)^{-1}[x]_{\mathcal{A}}}.$$

In addition, $\mu_{x,T}^V$ satisfies the following:

3. *There is a set $X_0 \subset X$ of full μ-measure so that for every $x, y \in X_0$ with $x \overset{T}{\sim} y$, for any $U, V \in \mathfrak{T}$ with $x \in U, y \in V$ and for any isometry θ satisfying*

$$t_V(y,\cdot) \circ \theta = t_U(x,\cdot) \tag{3.8}$$

as in Definition 2.1

$$\theta_* \mu_{x,T}^U \propto \mu_{y,T}^V.$$

Proof. Define

$$X' = \{x : t_V(x,\cdot) \text{is injective for some (hence all)} V \in \mathfrak{T}(x)\}.$$

By our assumption (3.1), $\mu(X \setminus X') = 0$.

Since X is second countable, for any $V \in \mathfrak{T}$ and k we can cover $X' \cap V$ by countably many balls $B_{i,k}^V \subset V$ which are centers of $10^k, T$-flowers $(\mathcal{A}_{i,k}^V, U_{i,k}^V)$. Note that these flowers can be chosen independently of μ.

Now take $\mathcal{P}_k^V = \left\{P_{i,k}^V\right\}$ to be a partition of $V \cap X'$ into Borel sets with each $P_{i,k}^V \subset B_{i,k}^V$. Using this partition, we can define an approximation $\mu_{x,T}^{V,k,*}$: $V \cap X' \to \mathcal{M}_\infty(T)$ to the system of conditional measures on the T-leaves $\mu_{x,T}^V$ as follows:

$$\mu_{x,T}^{V,k,*} = t_V(x,\cdot)^{-1}{}_*(\mu_x^{\mathcal{A}_{i,k}})|_{B_{10^k}^T} \qquad \text{if } x \in P_{i,k}^V.$$

It would be convenient to normalize in a consistent way the $\mu_{x,T}^{V,k,*}$ for different k. For this we need the following easy lemma:

Lemma 3.7. *For every $V \in \mathfrak{T}$ and i,k, for μ-almost every $x \in U_{i,k}^V$ and for all $\rho > 0$*

$$\mu_x^{\mathcal{A}_{i,k}^V}(B_\rho^T(x)) > 0. \tag{3.9}$$

Proof. Set

$$Y = \left\{x \in U_{i,k}^V : \exists \rho > 0 \quad \mu_x^{\mathcal{A}_{i,k}^V}(B_\rho^T(x)) = 0\right\}.$$

By (3.3) and (3.2), we have that

$$\mu(Y) = \int_{U_{i,k}^V} \mu_x^{\mathcal{A}_{i,k}^V}(Y \cap [x]_{\mathcal{A}_{i,k}^V}) d\mu(x). \tag{3.10}$$

Let $x \in U_{i,k}^V \cap X'$ and $V' \in \mathfrak{T}(x)$. Set

$$\tilde{Y} = t_{V'}(x,\cdot)^{-1}\left(Y \cap [x]_{\mathcal{A}_{i,k}^V}\right).$$

Let $\tilde{y} \in \tilde{Y}$, and set $y = t_V(x,\tilde{y})$ (so in particular, $y \in [x]_{\mathcal{A}_{i,k}^V} \cap Y$). By definition of Y, for every such y there is a ρ_y so that

$$0 = \mu_y^{\mathcal{A}_{i,k}^V}(B_{\rho_y}^T(y)) = \mu_x^{\mathcal{A}_{i,k}^V}(t_{V'}(x, B_{\rho_y}^T(\tilde{y}))).$$

Since T is second countable, a countable number of such open neighborhoods $B_{\rho_y}^T(\tilde{y})$ suffice to cover $\tilde{Y}$, so that

$$\mu_x^{\mathcal{A}_{i,k}^V}(t_{V'}(x', \tilde{Y})) = \mu_{\tilde{x}}^{\mathcal{A}_{i,k}^V}(Y) = 0.$$

After we integrate, (3.10) implies that $\mu(Y) = 0$. □

We now proceed with the proof of Theorem 3.6. Suppose $(\mathcal{A}^{(i)}, U^{(i)})$ for $i = 1,2$ are r_i, T-flowers with centers $B^{(i)}$ respectively, with $1 < r = r_1 \leq r_2$ from the countable collection of flowers

$$\left\{(\mathcal{A}_{i,k}^V, U_{i,k}^V) : V \in \mathfrak{T},\, i,k \in \mathbb{N}\right\}. \tag{3.11}$$

Set $U^{(1,2)} = U^{(1)} \cap U^{(2)}$ and $\mathcal{A}^{(1,2)} = \mathcal{A}^{(1)}|_{U^{(1,2)}} \vee \mathcal{A}^{(2)}|_{U^{(1,2)}}$.

By (3.4) and (3.5) for μ almost every $x \in U^{(1,2)}$

$$\mu_x^{\mathcal{A}^{(1)}}|_{[x]_{\mathcal{A}^{(1,2)}}} \propto \mu_x^{\mathcal{A}^{(2)}}|_{[x]_{\mathcal{A}^{(1,2)}}} \tag{3.12}$$

so that for almost every $x \in B^{(1)} \cap B^{(2)}$

$$\frac{\mu_x^{\mathcal{A}^{(1)}}|_{B_r^T(x)}}{\mu_x^{\mathcal{A}^{(1)}}(B_r^T(x))} = \frac{\mu_x^{\mathcal{A}^{(2)}}|_{B_r^T(x)}}{\mu_x^{\mathcal{A}^{(1)}}(B_r^T(x))}. \tag{3.13}$$

Define X_0 to be the set of $x \in X'$ where

(1) Equation (3.9) holds for all flowers $(\mathcal{A}_{i,k}^V, U_{i,k}^V)$ with $x \in U_{i,k}^V$.

(2) For any two flowers as in (1), (3.12) holds.

Define for any $x \in X_0$ and $k \geq 1$

$$\mu_{x,T}^{V,k} = \frac{\mu_{x,T}^{V,k,*}}{\mu_{x,T}^{V,k,*}(B_1^T)};$$

by (3.13) we see that for every $k < k'$ and $x \in X_0$

$$\mu_{x,T}^{V,k} = \mu_{x,T}^{V,k'}|_{B_{10^k}^T}.$$

Define

$$\mu_{x,T}^V = \begin{cases} \lim_{k\to\infty} \mu_{x,T}^{V,k} & \text{for } x \in V \cap X_0 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that Theorem 3.6.(1) holds; we verify (2) and (3).

Suppose $\mathcal{A} \subset \mathcal{S}$ is a countably generated sigma ring with maximal element E, and that for every $x \in E$, $[x]_{\mathcal{A}}$ is an open T-plaque. Without loss of generality we may assume that there is some k_0 so that for every x,

$$[x]_{\mathcal{A}} \subset B_{10^{k_0}}^T(x), \tag{3.14}$$

since otherwise we may replace E by $\tilde{E} = \{x \in E : (3.14)\text{holds }\}$ for k_0 sufficiently large, and $\mathcal{A}$ by $\mathcal{A}|_{\tilde{E}}$. Note that by (3.14), for any i, V,

$$[x]_{\mathcal{A}} \subset [x]_{\mathcal{A}_{i,k_0}^V} \qquad \text{for every } x \in B_{i,k_0}^V \cap E.$$

To show (2), it is sufficient to note that by (3.4) and (3.5), for every i, V, for almost every $x \in E \cap B_{i,k_0}^V$

$$\mu_x^{\mathcal{A}} \propto \mu_x^{\mathcal{A}_{i,k_0}^V}|_{[x]_{\mathcal{A}}} \tag{3.15}$$

since by definition for almost every $x \in V$ there is some i for which

$$\mu_x^{\mathcal{A}_{i,k_0}^V}|_{B_{10^{k_0}}^T} \propto [t_V(x,\cdot)]_*(\mu_{x,T}^V)|_{B_{10^{k_0}}^T}.$$

We are left with showing (3). Suppose that $x, y \in X_0$ with $x \overset{T}{\sim} y$, and let U, V, θ be as in (3.8). Let $r > 0$ be arbitrary, and fix r_0 satisfying $x \in B^T_{r_0}(y)$. Choose k such that $10^k > r_0 + r$, and define i, j by

$$x \in P^U_{i,k}\ , \qquad y \in P^V_{j,k}.$$

We wish to show that

$$(\theta_* \mu^U_{x,T})|_{B^T_r} \propto \mu^V_{y,T}|_{B^T_r}. \tag{3.16}$$

Set $\mathcal{A}^{(1)} = \mathcal{A}^U_{i,k}, \mathcal{A}^{(2)} = \mathcal{A}^V_{j,k}$, and let $\mathcal{A}^{(1,2)}$ be a mutual refinement as above. By definition, the right-hand side is equal to $([t_V(y,\cdot)^{-1}]_*(\mu^{\mathcal{A}^{(2)}}_y))|_{B^T_r}$. For the left-hand side,

$$\begin{aligned}(\theta_* \mu^U_{x,T})|_{B^T_r} &= \left([\theta \circ t_U(x,\cdot)^{-1}]_*(\mu^{\mathcal{A}^{(1)}}_x)\right)|_{B^T_r} \\ &= [t_V(y,\cdot)^{-1}]_* \left(\mu^U_{x,T}|_{B^T_r(y)}\right).\end{aligned}$$

Since k was chosen sufficiently large so that $[x]_{\mathcal{A}^{(1,2)}} = [y]_{\mathcal{A}^{(1,2)}}$, by (3.12)

$$\mu^U_{x,T}|_{B^T_r(y)} \propto \mu^V_{x,T}|_{B^T_r(y)}$$

and (3.16) is established. □

We note the following easy consequence of the construction of the conditional measures; we leave the proof to the reader.

PROPOSITION 3.8. *Let $A \subset X$ be a measurable set with $\mu(A) > 0$. Then for μ-almost every $x \in A$ and $U \in \mathfrak{T}(x)$,*

$$(\mu|_A)^U_{x,T} \propto \mu^U_{x,T}|_{t_U(x,\cdot)^{-1}A}.$$

4. Recurrent measures and conditional measures on T-leaves

Throughout this section, X is a T-space as in Definition 2.1. In Definition 2.3 we have defined the notion of a T-recurrent measure. Here we give an alternative criterion when μ is a probability measure. As in the previous section, we assume for simplicity that μ-almost every T-leaf is embedded. For the case of a $\mathbb{Z}$-action which preserves the measure class of μ this is the Halmos Recurrence Theorem (see §1.1 in [Aar97]).

PROPOSITION 4.1. *A probability measure μ is T-recurrent if, and only if, for μ-almost every x and $U \in \mathfrak{T}(x)$,*

$$\mu^U_{x,T}(T) = \infty. \tag{4.1}$$

Remark. Consider the following very simple example of a T-structure where $X = T = G$, a noncompact locally compact metric group, with the

T-structure corresponding to the action of G on itself by multiplication from the right, and μ the Haar measure on G. This measure is clearly not recurrent. However for almost every x we have that $\mu^U_{x,T}$ is simply a Haar measure on G, in particular infinite.

Proof that (4.1) *holds a.s.* $\Longrightarrow$ μ *is recurrent.* Assume the contrary holds. Then there is an r_0 and a set B_1 with positive measure so that

$$B_1 \cap t_U(x, T \setminus B^T_{r_0}) = \emptyset, \qquad \forall x \in B_1, \quad x \in U \in \mathfrak{T}. \tag{4.2}$$

To simplify the analysis, we assume without loss of generality that there is some $U \in \mathfrak{T}$ with $B_1 \subset U$.

By (4.1), there is an $r_1 > r_0$ and a subset $U_1 \subset U$ with measure $\mu(U_1) > \mu(U) - \mu(B_1)/2$ so that for any $x \in U_1$

$$\mu^U_{x,T}(B^T_{r_1}) > 100\mu(B_1)^{-1}\mu^U_{x,T}(B^T_{r_0}). \tag{4.3}$$

We now take B to be $B_1 \cap U_1$; clearly $\mu(B) > \mu(B_1)/2$.

We will need the following:

Lemma 4.2. *There is r_1, T-flower $(\mathcal{A}, E)$ with base $B' \subset B$ satisfying $\mu(B') > \mu(B)/2$.*

Proof. By replacing B with a compact subset of measure only slightly less than $\mu(B)$ we may assume without loss of generality that B is compact. By our standing assumption (3.1), we can also assume that $t_U(x, \cdot)$ is injective on $\overline{B^T_{20r}}$ for every $x \in B$. We now take E to be the sigma compact set

$$E = t_U(B, B^T_{r_1}(y)).$$

Observe that for any $y_1, y_2 \in E$, if

$$y_1 \in B^T_\infty(y_2) \tag{4.4}$$

then in fact $y_1 \in B^T_{3r_1}(y_2)$. Indeed, since $y_i \in E$ there are $z_i \in B$ so that $y_i \in \overline{B^T_{r_1}}(z_i)$ (again for $i = 1, 2$). By (4.2), either

$$z_1 \in B^T_{r_0}(z_2), \qquad \text{or} \tag{4.5}$$

$$B^T_\infty(z_1) \cap B^T_\infty(z_2) = \emptyset. \tag{4.6}$$

Equation (4.4) is not consistent with (4.6), so (4.5) holds; hence by the triangle inequality $y_1 \in T_{2r_1+r_0}(y_2)$.

In the same way that Corollary 3.5 was deduced from Lemma 3.2, Lemma 4.2 can be deduced from the above observation: in particular, we define $\mathcal{A}$ as the sigma ring generated by the relatively open subsets A of E with the property that if $y \in A$ then $B^T_{3r_1}(y) \subset A$. $\square$

We now return to the proof of Proposition 4.1. Decompose the measure $\mu|_E$ according to the sigma ring $\mathcal{A}$ constructed in the above lemma. By Theorem 3.6, for almost every $x \in E$, and in particular for almost every $x \in B$

$$\mu_x^{\mathcal{A}} = c_{x,\mathcal{A}} t_U(x,\cdot)_* \left(\mu_{x,T}^U|_{t_U(x,\cdot)^{-1}([x]_{\mathcal{A}})}\right). \tag{4.7}$$

By (4.2) and (4.3), and by ♣-3 applied to the flower $(\mathcal{A}, E)$, for any x satisfying (4.7),

$$\begin{aligned} \mu_x^{\mathcal{A}}(B') &\le \mu_x^{\mathcal{A}}(B_{r_0}^T(x)) \\ &< \frac{\mu(B_1)}{100}\mu_x^{\mathcal{A}}(B_{r_1}^T) \\ &\le \frac{\mu(B_1)\mu_x^{\mathcal{A}}(E)}{100}. \end{aligned} \tag{4.8}$$

For almost every $y \in E$ with $\mu_y^{\mathcal{A}}(B') > 0$, (4.8) holds for at least one $x \in [y]_{\mathcal{A}} \cap B'$, and so

$$\begin{aligned} \mu(B') &= \int_E \mu_y^{\mathcal{A}}(B') d\mu(y) \\ &\le \frac{\mu(B_1)}{100} \int_E \mu_y^{\mathcal{A}}(E) d\mu(y) \\ &= \frac{\mu(B_1)\mu(E)}{100} \le \frac{\mu(B_1)}{100}. \end{aligned}$$

Since $\mu(B') \ge \mu(B)/2 \ge \mu(B_1)/4$ we have a contradiction. □

Proof that μ is recurrent $\Longrightarrow$ (4.1) *holds a.s.* Assume (4.1) does not hold on a set of positive μ measure. Then there is a set B of positive measure and $r_0 > 0$ so that for every $x \in B$

$$\mu_{x,T}^U(T) < \infty \quad \text{and} \quad \mu_{x,T}^U(B_{r_0}^T) > 0.9\mu_{x,T}^U(T) \tag{4.9}$$

(as usual, the above expression is independent of U as long as $x \in U \in \mathfrak{T}$). Without loss of generality, we can take this set B to be a subset of X_0, with X_0 as in Theorem 3.6 item (3).

Suppose now that $x \in B$ and $y = t_U(x,t) \in B$ with $t \in T$, $x \in U \in \mathfrak{T}$ and $y \in V \in \mathfrak{T}$. Then as in Theorem 3.6,

$$(\theta_{U,V}(x,y))_* \mu_{x,T}^U = c_{x,y}\mu_{y,T}^V;$$

hence

$$\frac{\mu_{y,T}^V(B_{r_0}^T)}{\mu_{y,T}^V(T)} = \frac{\mu_{x,T}^U(B_{r_0}^T(t))}{\mu_{x,T}^U(T)}$$

and so by (4.9) we have that

$$B_{r_0}^T \cap B_{r_0}^T(t) \ne \emptyset$$

and $t \in B_{2r_0}^T$. In other words, for any $x \in B$ we have that $t_U(x,T) \cap B \subset B_{2r_0}^T(x)$ and we are done. □

Proposition 4.3. *Let G be a locally compact metric group, and X a G-space as in Example 2.2. Let μ be a probability measure on X, and as usual assume that the G orbit of almost every x is embedded; i.e. the action is free on a co-null set. Then μ is G-invariant if, and only if, for μ-almost every x the conditional measure $\mu_{x,G}$ is a right invariant Haar measure on G.*

(Note that since in the case of G-spaces arising from a G-action the maps t_U are independant of $U \in \mathfrak{T}$, we can omit the elements of the atlas used in all notation.)

Proof that if $\mu_{x,G}$ is Haar measure almost surely then μ is G-invariant. Let $\mathcal{H}_G$ denote a right invariant Haar measure on G. We will show that for almost every $x \in X$ and $r > 0$ there is an $\epsilon > 0$ so that if $f \in L^\infty(\mu)$ with $\operatorname{supp} f \subset B_\epsilon(x)$ then

$$\int f(y) d\mu(y) = \int f(yg) d\mu(y) \qquad \forall g \in B_r^G. \tag{4.10}$$

Indeed, take x to be a point for which $g \mapsto xg \equiv t(x,g)$ is injective on B_{20r}^G, and $(\mathcal{A}, U)$ be an r, G-flower with center $B_\epsilon(x)$ (see Corollary 3.5).

Suppose $\operatorname{supp} f \subset B_\epsilon(x)$. Then

$$\int f(y) d\mu(y) = \int_U \int f(y') d\mu_y^{\mathcal{A}}(y') d\mu(y).$$

By Theorem 3.6.(3), and our assumption on $\mu_{y,G}$, for almost every y

$$\mu_y^{\mathcal{A}} \propto [t(y, \cdot)]_* \mathcal{H}_G|_{[y]_{\mathcal{A}}};$$

since $\operatorname{supp} f \subset B_\epsilon(x)$ we know by Corollary 3.5 that for any $y' \in [y]_{\mathcal{A}}$ for which $f(y') \neq 0, \quad y'g \in [y]_{\mathcal{A}}$ for $g \in B_r^G$. Hence for all $y \in U$

$$\int f(y') d\mu_y^{\mathcal{A}} = \int f(y'g) d\mu_y^{\mathcal{A}}.$$

Integrating, we get (4.10) for f satisfying $\operatorname{supp} f \subset B_\epsilon(x)$.

In order to obtain (4.10) for general bounded compactly supported measurable functions we proceed as follows: let f be such a function, and set $\tilde{f}(y) = f(yg)$. Let $\delta > 0$ be arbitrary. Find a compact set $K \subset X$ so that

$$\|f - f \cdot 1_K\|_{1,\mu}, \left\|\tilde{f} - \tilde{f} \cdot 1_{Kg^{-1}}\right\|_{1,\mu} < \delta .$$

We may further assume that the G-orbit of every $x \in K$ is an embedded orbit. Then we can write $f \cdot 1_K = f_1 + \cdots + f_k$ with each f_i as in the previous paragraph, and then (4.10) implies the same for $f \cdot 1_K$, and

$$\begin{aligned}\left|\int f d\mu - \int \tilde{f} d\mu\right| &\leq \left|\int f \cdot 1_K d\mu - \int \tilde{f} \cdot 1_{Kg^{-1}} d\mu\right| \\ &\quad + \|f - f \cdot 1_K\|_{1,\mu} + \left\|\tilde{f} - \tilde{f} \cdot 1_{Kg^{-1}}\right\|_{1,\mu} \leq 2\delta\end{aligned}$$

and we are done.

For the converse direction we need the following easy fact:

LEMMA 4.4. *Let ν be a Radon measure on a locally compact second countable group G. Let $V \subset G$ be an open neighborhood of the identity $e \in G$, and M a countable dense subset of G. Assume that for every open $A \subset V$ and for every $g \in M$,*

$$\nu(A) = \nu(Ag).$$

Then $\nu|_V \propto \mathcal{H}_G|_V$, with $\mathcal{H}_G$ a right invariant Haar measure on G.

This follows, for example, quite readily from the construction of Haar measure (§58, Theorem B of [Hal50]); alternatively, it is also an easy consequence of the existence and uniqueness of Haar measure. We omit the details. Note that if $V = B_r^G$ then since we have chosen d_G to be left invariant we see that $V^{-1} = V$ and $V^{-1}V \subset B_{2r}^G$.

Proof that if μ is G-invariant then $\mu_{x,G}$ is Haar measure almost surely. As in the converse direction, it is enough to show that for every $3r, G$-flower for μ-almost every every y in the center B of this flower

$$\mu_{y,G}|_{B_{r/2}^G} \propto \mathcal{H}_G|_{B_{r/2}^G}.$$

Suppose $A_1, A_2, \dots$ is a countable base for the topology of $\tilde{U} = t(B, B_r^G)$. By the definition of a $3r, G$-flower, for every i and $g \in B_{2r}^G$ we have that $A_i g \subset U$ and so by G-invariance of μ

$$\int_U \mu_y^{\mathcal{A}}(A_i) d\mu(y) = \mu(A_i) = \mu(A_i g) = \int_U \mu_y^{\mathcal{A}}(A_i g) d\mu(y).$$

By Theorem 3.6.(2) this gives that for every $g \in B_{2r}^G$ and μ-almost every $x \in B$

$$\mu_{x,G}((t(x,\cdot)^{-1}(A_i \cap [x]_{\mathcal{A}})) = \mu_{x,G}((t(x,\cdot)^{-1}(A_i \cap [x]_{\mathcal{A}})g)\ . \tag{4.11}$$

Note that since the A_i form a basis for the topology of U, any open subset of B_r^G is a countable union of sets from the collection

$$\left\{(t(x,\cdot)^{-1}(A_1 \cap [x]_{\mathcal{A}}), \dots\right\}.$$

Let M be a dense countable subset of B_{2r}^G. Then for almost every $x \in B$ equation (4.11) holds for every $g \in M$ and i. For such x the measure $\mu_{x,G}$ satisfies all the conditions of Lemma 4.4, and we are done. □

5. Expanding and contracting foliations

Definition 5.1. Let X be a (G, T)-space, and $\alpha : X \to X$ a homeomorphism of X. Let $H > G$ be a subgroup of the group of homeomorphisms $\mathrm{Hom}(T)$ of T. Then α *preserves the* (H, T)*-structure of* X if for any $U, V \in \mathfrak{T}$, for any $x \in U \cap \alpha^{-1}V$, there is a homeomorphism $\theta = \theta_{\alpha,x}^{U,V} \in H$ fixing e (i.e. $\theta(e) = e$) so that

$$\alpha \circ t_U(x, \cdot) = t_V(\alpha x, \cdot) \circ \theta. \tag{5.1}$$

Note that if $t_U(x, \cdot)$ is injective (which we assume holds for almost every x), then θ is uniquely determined.

We point out the following special important cases (as always, we assume that $G < \mathrm{Isom}(T)$):

(1) *α preserves the T-leaves* if it preserves the $(\mathrm{Hom}(T), T)$-structure.

(2) *α acts isometrically on the T-leaves* if it preserves the $(\mathrm{Isom}(T), T)$-structure.

(3) *α uniformly expands* (*contracts*) *the T-leaves* if it preserves the T-leaves and there is some $c > 1$ so that θ as in (5.1) can be chosen to satisfy $d(\theta x, \theta y) > cd(x, y)$ $(d(\theta x, \theta y) < c^{-1}d(x, y))$ respectively.

We remark that the notion as above can be extended to any group action (so Definition 5.1 treats the case of the $\mathbb{Z}$-action generated by α), with the exception of (3) above for which one needs at least an order on the acting group. Explicitly, we shall say that an $\mathbb{R}$-action $\alpha.$ uniformly expands T if for every $s > 0$ the homeomorphism α_s is uniformly expanding. Though for simplicity we state the results of this section for a $\mathbb{Z}$-action, all statements and their proofs remain equally valid for $\mathbb{R}$-actions.

An almost immediate corollary of the construction of the systems of conditional measures $\mu_{x,T}^U$ is the following:

PROPOSITION 5.2. *Let X be a T-space. Assume that $\alpha : X \to X$ is a homeomorphism that acts isometrically on T-leaves and preserves the measure μ. Then for μ almost every $x \in X$,*

$$\mu_{\alpha x,T}^V = [\theta_{\alpha,x}^{U,V}]_* \mu_{x,T}^U, \qquad U \in \mathfrak{T}(x), V \in \mathfrak{T}(\alpha x). \tag{5.2}$$

Proof. By the properties of conditional measures listed on p. 174, if $\mathcal{A}$ is a countably generated sigma ring of Borel subsets of a Borel set $E \subset X$, for μ almost every $x \in E$

$$\alpha_* \mu_x^{\mathcal{A}} = \mu_{\alpha x}^{\alpha \mathcal{A}}. \tag{5.3}$$

However, in view of Lemma 3.2, Corollary 3.5, and Theorem 3.6 item (2), the equation (5.3) implies the proposition. $\square$

Let μ be a probability measure on the space X, and α a homeomorphism of X preserving μ. The ergodic decomposition can be constructed in several ways, one of which is the following. Consider the sigma algebra $\mathcal{E}$ of Borel subsets of X which are (strictly) α-invariant (in the case of $\mathbb{R}$-action, $\mathcal{E}$ will be the collection of Borel subsets of X which are α_s-invariant for all s). This sigma algebra is usually *not* countably generated, and so has no well-defined atoms. However, since (X, μ) is a Lebesgue space, the conditional measures $\mu_x^{\mathcal{E}}$ are well-defined. It is fairly easy to see from the definition that almost surely the measures $\mu_x^{\mathcal{E}}$ are α-invariant. A slightly deeper fact is that they are also α-ergodic. The standard decomposition $\mu = \int \mu_x^{\mathcal{E}} d\mu(x)$ for this sigma algebra $\mathcal{E}$ is called the ergodic decomposition, and each $\mu_x^{\mathcal{E}}$ is called (in a somewhat loose sense) an ergodic component (see for example §3.5 of [Rud90a]).

We recall the following well known property of contracting foliations, which dates back at least to E. Hopf (cf. e.g. [KH95, §5.4]).

Proposition 5.3. *Let X be a T-space and $\alpha : X \to X$ a homeomorphism that uniformly expands the T-leaves. Let μ be an α-invariant probability measure on X, and $E \subset X$ an α-invariant Borel set. Then there is a Borel set $E' \subset X$ with $\mu(E \triangle E') = 0$ consisting of complete T-leaves, i.e. such that for every $x \in E'$ it holds that $B_\infty^T(x) \subset E'$.*

Proof. We first find, for every $\delta > 0$ a Borel E_δ consisting of complete T-leaves with $\mu(E \triangle E_\delta) < \delta$. By measurability, find $C \subset E \subset U$ with C compact, U open, and $\mu(U \setminus C) < \delta/2$. Let $f : X \to [0, 1]$ be a continuous function such that $f|_C = 1$ and $f|_{U^\complement} = 0$.

Set

$$E_\delta = \left\{ x : \underline{\lim}_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} f(T^{-n}x) > \tfrac{1}{2} \right\}.$$

Since f is continuous and α contracts T, the set E_δ is a union of complete T-leaves. Furthermore

$$E_\delta \setminus E \subset \left\{ x : \overline{\lim}_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} f(T^{-n}x) - 1_E(T^{-n}x) \geq \tfrac{1}{2} \right\},$$

$$E \setminus E_\delta \subset \left\{ x : \overline{\lim}_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} 1_E(T^{-n}x) - f(T^{-n}x) \geq \tfrac{1}{2} \right\},$$

so by the (usual) maximal inequality applied to α

$$\begin{aligned} \mu(E \triangle E_\delta) &\leq \mu \left\{ x : \overline{\lim}_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} \left| f(T^{-n}x) - 1_E(T^{-n}x) \right| \geq \tfrac{1}{2} \right\} \\ &\leq 2 \left\| f - 1_E \right\|_{1,\mu} \leq \delta. \end{aligned}$$

Once we have shown how to construct the sets E_δ, we can take

$$E' = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} E_{2^j}$$

which is easily seen to satisfy all the conditions of the proposition. □

Corollary 5.4. *Let X be a T-space, $\alpha : X \to X$ and μ be as in Proposition* 5.3. *Let $\mathcal{E}$ be the sigma algebra of α-invariant Borel sets. Then*:

(1) *For μ-almost every x and $\mu_x^{\mathcal{E}}$ almost every y*

$$(\mu_x^{\mathcal{E}})_{y,T} = \mu_{y,T}.$$

(2) *For every $E \in \mathcal{E}$ with positive μ measure*, *for μ*-a.e. *$x \in E$*

$$(\mu|_E)_{x,T} = \mu_{x,T}.$$

Proof. We first prove (1). By Proposition 5.3, without loss of generality E consists of full T-leaves. It follows that for every r, T-flower $(\mathcal{A}, U)$, the set $E \cap U$ is an element of $\mathcal{A}$.

It follows from the properties of conditional measures that for a.e. $x \in E \cap U$

$$(\mu|_E)_x^{\mathcal{A}} = \mu_x^{\mathcal{A}};$$

hence in view of the way the conditional measures $\mu_{x,T}$ have been constructed in the proof of Theorem 3.6 using a countable number of flowers $(\mu|_E)_{x,T} = \mu_{x,T}$ for a.e. $x \in E$ as claimed.

We proceed to prove (2). Again it is enough to show that for every r, T-flower $(\mathcal{A}, U)$, for μ-almost every $x \in U$ and $\mu_x^{\mathcal{E}}$ almost every y,

$$(\mu_x^{\mathcal{E}})_y^{\mathcal{A}} = \mu_y^{\mathcal{A}}. \tag{5.4}$$

Let $\mathcal{E}' = \{E \cap U : E \in \mathcal{E}\}$, $\tilde{\mathcal{E}} < \tilde{\mathcal{E}}$ a countably generated sub-sigma algebra equivalent to $\mathcal{E}$ modulo μ-null sets, and $\tilde{\mathcal{E}}' = \tilde{\mathcal{E}} \vee \left\{U, U^{\complement}\right\}$. Then for almost every $x \in U$,

$$\mu_x^{\mathcal{E}'} = \mu_x^{\tilde{\mathcal{E}}'} \propto \mu_x^{\tilde{\mathcal{E}}}|_U = \mu_x^{\mathcal{E}}|_U. \tag{5.5}$$

As we have already seen, it follows from Proposition 5.3 that up to sets of measures zero $\mathcal{E}'$ is contained in $\mathcal{A}$: i.e. that for every $E \in \mathcal{E}$ there is an $A \in \mathcal{A}$ so that $\mu((E \cap U) \triangle A) = 0$. Thus $(\mu_x^{\mathcal{E}'})_y^{\mathcal{A}} = \mu_y^{\mathcal{A}}$ for a.e $x \in U$ and $\mu_x^{\mathcal{E}}$ almost every y, and so by (5.5), equation (5.4), and hence this corollary, follow. □

6. A lemma of Einsiedler-Katok and its generalization

A key point in [EK03] is the following important observation. While the statement given in [EK03] is given in a somewhat less general context, their proof extends without any substantial difficulties to the framework of T-spaces. The heart of the arguments is a variation on Hopf's argument.

Definition 6.1. Let X be a T-space, and $\alpha : X \to X$ act isometrically on T-leaves. We shall say that $x' \in X$ is asymptotically in the T-leaf of $x \in X$ if there is some $x'' \overset{T}{\sim} x$ so that for any sequence n_i for which $\{\alpha^{n_i} x\}$ (hence $\{\alpha^{n_i} x''\}$) is relatively compact, $d(\alpha^{n_i} x'', \alpha^{n_i} x') \to 0$ as $i \to \infty$.

Note that in general there seems to be no reason why this should be a symmetric relation.

Lemma 6.2. *Let X be a T-space and $\alpha : X \mapsto X$ a homeomorphism that acts isometrically on T-leaves. Suppose that μ is an α-invariant probability measure on X (as usual, also assume that for μ almost every x, each T-leaf is embedded.)*

Then there is a co-null set X_0 such that for every $x, x' \in X_0$ so that x' is asymptotically in the T-leaf of x,

$$\mu^{U'}_{x',T} \propto \Phi_* \mu^U_{x,T}, \qquad U \in \mathfrak{T}(x), U' \in \mathfrak{T}(x'), \tag{6.1}$$

for some $\Phi \in \mathrm{Isom}(T)$.

Remark. It will transpire in the proof of Lemma 6.2 that this Φ can be chosen so that for some sequence n_i

$$\lim \alpha^{n_i} t_U(x,t) = \lim \alpha^{n_i} t_{U'}(x', \Phi(t)) \tag{6.2}$$

(in particular, both limits exist). Thus, if there is some Φ' which satisfies that whenever $\{\alpha^{n_i} t_U(x,t)\}$ is relatively compact,

$$d_X(\alpha^{n_i} t_U(x,t), \alpha^{n_i} t_{U'}(x', \Phi'(t))) \to 0$$

then $\Phi = \Phi'$, a fact that will be useful to us when we actually try to identify this element Φ in certain cases. Note that it is easy to calculate explicitly the constant of proportionality by comparing the measure of the set B^T_1.

Proof. We show that for every $\epsilon > 0$ there is a set X_ϵ on which (6.1) holds with $\mu(X_\epsilon) \geq 1 - \epsilon$. Since the maps $x \mapsto \mu^U_{x,T}$ are Borel, hence μ-measurable, for every $\epsilon > 0$ there is a compact set X'_ϵ of measure $\geq 1 - \epsilon^2/100$ on which this map is continuous. By the maximal ergodic theorem, there is a compact subset $X_\epsilon \subset X'_\epsilon$ so that:

(P-1) For every $x \in X_\epsilon$,

$$\underline{\lim}\, \tfrac{1}{n} \sum_{i=0}^{n} 1_{X'_\epsilon}(\alpha^n x) \geq 1 - \epsilon.$$

(P-2) For every $x \in X_\epsilon$ equation (5.2) holds.

(P-3) $\mu(X_\epsilon) > 1 - \epsilon$.

(P-4) X_ϵ is a subset of X_0 of Theorem 3.6.(3).

Suppose now that $x, x' \in X_\epsilon$ with x' asymptotically on the T-leaf of x. Let $x'' \stackrel{T}{\sim} x$ with $d(\alpha^n x'', \alpha^n x') \to 0$, and $U \in \mathfrak{T}(x), U' \in \mathfrak{T}(x')$. By P-1, there is an infinite sequence of n_i so that both $\alpha^{n_i} x$ and $\alpha^{n_i} x' \in X'_\epsilon$. Since X'_ϵ is compact, by passing to a subsequence if necessary we may assume that

$$\alpha^{n_i} x \to z, \qquad \alpha^{n_i} x' \to z', \qquad \big(z, z' \in X_\epsilon\big)\,.$$

Note that this implies in particular that $\alpha^{n_i} x'' \to z'$, and so from $x \stackrel{T}{\sim} x''$ it follows that $z \stackrel{T}{\sim} z'$.

Let $V \in \mathfrak{T}(z), V' \in \mathfrak{T}(z')$. For i large enough, $\alpha^{n_i} x \in V$ and $\alpha^{n_i} x' \in V'$. Let

$$\theta_{n_i} = \theta^{U,V}_{\alpha^{n_i},x}, \qquad \theta'_{n_i} = \theta^{U',V'}_{\alpha^{n_i},x'}$$

as in Definition 5.1.

Without loss of generality, by passing to a subsequence if necessary, we can assume that there is a limit $\theta = \lim_{i\to\infty} \theta_{n_i}$ and $\theta' = \lim_{i\to\infty} \theta'_{n_i}$. Let $\theta_{z,z'}$ be an isometry as in Definition 2.1 so that

$$t_{V'}(z', \cdot) \circ \theta_{z,z'} = t_V(z, \cdot).$$

Set $\Phi = [\theta']^{-1} \circ \theta_{z,z'} \circ \theta$. Then since $y \mapsto \mu^V_{y,T}$ is continuous and since for all i large enough $\alpha^{n_i} x \in V, \alpha^{n_i} x' \in V'$,

$$\mu^V_{z,T} = \lim \mu^V_{\alpha^{n_i} x,T} = \lim [\theta_{n_i}]_* \mu^U_{x,T} = \theta_* \mu^U_{x,T}, \tag{6.3}$$

$$\mu^{V'}_{z',T} = \lim \mu^{V'}_{\alpha^{n_i} x',T} = \lim [\theta'_{n_i}]_* \mu^{U'}_{x',T} = \theta'_* \mu^{U'}_{x',T}. \tag{6.4}$$

By Theorem 3.6,

$$\mu^{V'}_{z',T} \propto [\theta_{z,z'}]_* \mu^V_{z,T}; \tag{6.5}$$

together, equations (6.3) – (6.5) give (6.1).

Furthermore,

$$\begin{aligned}
\alpha^{n_i} t_U(x,t) &= t_V(\alpha^{n_i} x, \theta_{n_i}(t)) \to t_V(z, \theta(t)) = t_{V'}(z', \theta_{z,z'} \circ \theta(t)),\\
\alpha^{n_i} t_{U'}(x', \Phi(t)) &= t_{V'}(\alpha^{n_i} x', \theta'_{n_i} \circ \Phi(t)) \to t_{V'}(z', \theta' \circ \Phi(t))\\
&= t_{V'}(z', \theta_{z,z'} \circ \theta(t)).
\end{aligned}$$

establishing (6.2). ∎

Suppose that $H = H_1 \times H_2$ acts nicely from the right on X as in Example 2.2; this gives X an H_1-structure and an H_2-structure in the obvious way. We wish to extend this notion to more general circumstances. Since we will have to deal simultaneously with several different structures, where necessary we shall add the structure we are dealing with to the notation, e.g. $t_{U;S}$ etc. If S, T are metric spaces, we shall take $d_{S\times T} = \max(d_S, d_T)$. We will also assume that the components of the marked element $e \in S \times T$ are the marked elements (again denoted by the same symbol e) of S and T.

We shall say that an $S \times T$-structure of X is a product structure if it is an $(\mathrm{Isom}(S) \times \mathrm{Isom}(T), S \times T)$-structure. Note that it is immediate that if the $S \times T$-structure of an $S \times T$-space X is a product structure then it induces an S-structure on X and a T-structure on X with the same atlas

Let $\mathfrak{T}$ be as before by taking for any $x \in U \in \mathfrak{T}$, $s \in S$ and $t \in T$,

$$t_{U;T}(x, t) = t_{U;S\times T}(x, (e, t)), \qquad t_{U;S}(x, s) = t_{U;S\times T}(x, (s, e)).$$

Lemma 6.3. *Let X be an $(\mathrm{Isom}(S) \times \mathrm{Isom}(T), S \times T)$-space. Suppose that $x \in X$ is such that the map $t_{V,S\times T}(x, \cdot)$ is injective on $\overline{B^{S\times T}_{20r}}$ for some (hence all) $V \in \mathfrak{T}(x)$. Then there is an open set $U \ni x$ (not necessarily in $\mathfrak{T}$), and countably generated sigma rings $\mathcal{A} = \mathcal{A}_{;S\times T}$ and $\mathcal{A}_{;S}, \mathcal{A}_{;T} \supset \mathcal{A}$ of Borel subsets of U, and $\epsilon > 0$ so that*

(C-1) *$(U, \mathcal{A}_{;R})$ is an r, R-flower with base $B_\epsilon(x)$ for $R = S, T, S \times T$.*

(C-2) *for every $y \in V$,*

$$[y]_{\mathcal{A}_{;R}} = [y]_{\mathcal{A}} \cap B^R_{4r}(y), \qquad R = S, T.$$

Proof. Let U and ϵ be as in Lemma 3.2 applied for the $S \times T$-structure of X. Note that automatically, U and ϵ also satisfy (1) and (2) of Lemma 3.2, also for the T structure of X.

We can now apply Corollary 3.5 three times, once for the $S \times T$-structure, once for the S-structure and once for the T-structure of X, to obtain three countably generated sigma rings $\mathcal{A} = \mathcal{A}_{;S\times T}$, $\mathcal{A}_{;S}$ and $\mathcal{A}_{;T}$ of Borel subsets of V which satisfy C-1.

C-2 follows immediately from the way these sigma rings are constructed in Corollary 3.5. □

Proposition 6.4 (Einsiedler-Katok Lemma). *Suppose that X is an $(\mathrm{Isom}(S) \times \mathrm{Isom}(T), S \times T)$-space. Let $\alpha : X \mapsto X$ be a homeomorphism preserving the S, T, and $S \times T$ structures of X. Suppose that α acts isometrically on the S-leaves and uniformly contracts the T-leaves. Let μ be an α-invariant measure on X so that for almost every x its $S \times T$-leaf is an embedded leaf. Then for μ almost every x and all $U \in \mathfrak{T}(x)$*

$$\mu^U_{x,S\times T} = \mu^U_{x,S} \times \mu^U_{x,T}.$$

Proof. Let X_0 be a co-null set contained in both the co-null set of Lemma 6.2 applied to the S-structure of X, and the co-null set of Theorem 3.6.(3) applied to the three structures of X as an S-space, a T-space and an $S \times T$-space.

Let $r > 1$ be arbitrary, and $x_0 \in X$ any point whose $S\times T$-leaf is embedded.

Step 1. We show that there is some $\epsilon > 0$ so that for μ-almost every $x \in B_\epsilon^X(x_0)$ and any $V \in \mathfrak{T}(x)$ there is a measure $\nu_{x,r}$ on B_r^T so that

$$\mu_{x,S\times T}^V|_{B_r^{S\times T}} = \mu_{x,S}^V|_{B_r^S} \times \nu_{x,r}.$$

We now apply Lemma 6.3 on x_0 and r to get an $\epsilon > 0$, an open set U_0 and three sigma rings of subsets of U_0 with the properties cited above.

Fix $x \in X_0 \cap B(x_0, \epsilon)$ and $U \in \mathfrak{T}(x)$. Set

$$t_{(x)} = t_{U;S\times T}(x, \cdot), \qquad x_{s,t} = t_{(x)}(s,t).$$

Since the $S \times T$-structure of X is a product structure, we have for every $(s,t) \in S \times T$ and $V \in \mathfrak{T}(x_{s,t})$ isometries $\beta_{s,t}^V \in \mathrm{Isom}(S)$ and $\gamma_{s,t}^V \in \mathrm{Isom}(T)$ so that for all $s, s' \in S, t, t' \in T$

$$t_{U;S\times T}(x, (s', t')) = t_{V;S\times T}(x_{s,t}, (\beta_{s,t}^V(s'), \gamma_{s,t}^V(t'))). \tag{6.6}$$

Since α contracts the T-leaves, it follows that if $\{\alpha^{n_i} x\}$ is relatively compact (and so $\{\alpha^{n_i} x_{s,t}\}$ is relatively compact for all s, t) then

$$d_X(\alpha^{n_i} x_{s,t}, \alpha^{n_i} x_{s,t'}) \to 0 \qquad \forall t, t' \in T. \tag{6.7}$$

In particular, for every (s,t), we have that $x_{s,t}$ is asymptotically on the S-leaf of x and vice versa. By Lemma 6.2, we know that for every s, t for which $x_{s,t} \in X_0$ and $V \in \mathfrak{T}(x_{s,t})$, there is some Φ so that $\mu_{x,S}^U \propto \Phi_* \mu_{x_{s,t},S}^V$, and that this Φ satisfies (6.2) for x and $x_{s,t}$. By (6.6) and (6.7) we have that if $\{\alpha^{n_i} x\}$ is relatively compact

$$d_X(\alpha^{n_i} t_{U;S}(x, s'), \alpha^{n_i} t_{V,S}(x_{s,t}, \beta_{s,t}^V(s'))) \to 0;$$

so by the remark following Lemma 6.2 we have that $\Phi = \beta_{s,t}^V$, i.e.

$$\mu_{x_{s,t},S}^V \propto [\beta_{s,t}^V]_* \mu_{x,S}^{\cdot} U. \tag{6.8}$$

Let $\zeta_t : S \mapsto S \times T$ be the map $s \mapsto (s,t)$, and let $\pi_S : S \times T \to S$ and $\pi_T : S \times T \to T$ be the natural projections; in particular $\pi_S \circ \zeta_t$ is the identity transformation $S \to S$. Assume that $x_{s,t} \in X_0$, that $(s,t) \in B_r^{S\times T} = B_r^S \times B_r^T$ and $V \in \mathfrak{T}(x_{s,t})$. By (6.8) and (6.6) we know that for any bounded $K \subset S$

$$\begin{aligned}[t_{(x)}]_* \left(\mu_{x_{s,t},S}^V|_K\right) &\propto [t_{(x)}]_* \left(([\beta_{s,t}^V]_* \mu_{x,S}^U)|_K\right) \\ &= [t_{(x)} \circ \zeta_t]_* \left(\mu_{x,S}^U|_{\beta_{s,t}^V{}^{-1}(K)}\right).\end{aligned}$$

We now use Theorem 3.6 and the above to show

$$\mu^{\mathcal{A}_{;S}}_{x_{s,t}}|_{B^S_r(x_{e,t})} \propto [t_{V;S}(x_{s,t},\cdot)]_* \left(\mu^V_{x_{s,t},S}|_{B^S_r(\beta^V_{s,t}(e))}\right) \propto [t_{(x)} \circ \zeta_t]_* \left(\mu^U_{x,S}|_{B^S_r}\right). \tag{6.9}$$

We evaluate the implicit constant by evaluating the measure given to $B^S_1(x_{e,t})$ in both sides of (6.9). Applied to this set the right-hand side can be explicitly calculated:

$$\begin{aligned}\left([t_{(x)} \circ \zeta_t]_* \left(\mu^U_{x,S}|_{B^S_r}\right)\right)(B^S_1(x_{e,t})) &= \mu^U_{x,S}\left(B^S_r \cap \pi_S \circ t_{(x)}{}^{-1}(B^S_1(x_{e,t}))\right) \\ &= \mu^U_{x,S}(B^S_1) = 1;\end{aligned}$$

hence

$$\frac{1}{\mu^{\mathcal{A}_{;S}}_{x_{s,t}}(B^S_1(x_{e,t}))}\mu^{\mathcal{A}_{;S}}_{x_{s,t}}|_{B^S_r(x_{e,t})} = [t_{(x)} \circ \zeta_t]_* \left(\mu^U_{x,S}|_{B^S_r}\right). \tag{6.10}$$

Note that as long as $x_{s,t} \in [x]_{\mathcal{A}}$ the normalizing factor depends only on t (see (1) following Definition 3.1)

Since $\mathcal{A} = \mathcal{A}_{;S\times T} \subset \mathcal{A}_{;S}$ we know that for μ-almost every x

$$\mu^{\mathcal{A}}_x = \int \mu^{\mathcal{A}_{;S}}_y d\mu^{\mathcal{A}}_x(y). \tag{6.11}$$

We rewrite the above equation using (6.10)

$$\begin{aligned}\mu^{\mathcal{A}}_x|_{B^{S\times T}_r(x)} &= \int \mu^{\mathcal{A}_{;S}}_y|_{B^{S\times T}_r(x)\cap[y]_{\mathcal{A}_{;S}}} d\mu^{\mathcal{A}}_x(y) \\ &\propto [t_{(x)}]_* \left(\int_{\pi_T^{-1}(B^T_r)\cap t_{(x)}{}^{-1}([x]_{\mathcal{A}})} d\mu^U_{x,S\times T}(s,t)\, c(t)[\zeta_t]_*\mu^U_{x,S}|_{B^S_r}\right) \\ &= [t_{(x)}]_* \left(\nu^U_{x,r;T} \times \mu^U_{x,S}|_{B^S_r}\right)\end{aligned} \tag{6.12}$$

with

$$c(t) = \mu^{\mathcal{A}_{;S}}_{x_{s,t}}(B^S_1(x_{e,t})) = \mu^{\mathcal{A}_{;S}}_{x_{s,t}}(B^{S\times T}_1(x))$$

and ν a measure supported on $B^T_r \subset T$ defined by

$$\nu^U_{x,r;T}(A) = \int_{\pi_T^{-1}(A)\cap t_{(x)}{}^{-1}([x]_{\mathcal{A}})} c(t)\, d\mu^U_{x,S\times T}(s,t).$$

Step 2. We now show that for any $\delta > 0$ there is a set $B \subset B_\epsilon(x_0) \cap X_0$ of measure $\geq (1-\delta)\mu(B_\epsilon(x_0))$ so that

$$\nu^U_{x,r;T} \propto \mu^U_{x,T}|_{B^T_r} \qquad \forall x \in B. \tag{6.13}$$

Assume for the moment (6.13) is established. By taking $\delta \to 0$ we deduce that for almost every $x \in B_\epsilon(x_0)$ we have that

$$\mu^U_{x,S\times T}|_{B^{S\times T}_r} \propto \mu^U_{x,S}|_{B^S_r} \times \mu^U_{x,T}|_{B^S_t},$$

and from the way we have normalized the conditional measures it is immediate that in fact equality holds (i.e. the implicit constant above is one). By taking a countable sequence $r_i \to \infty$, and for every r_i a countable subcover of the collection of balls of the type $B_\epsilon^X(x_0)$ which covers all points of X whose $S \times T$-leaf is embedded, we establish the proposition (note that ϵ implicitly depends both on x_0 and on r_i).

It remains to establish (6.13). Similarly to (6.11), since $\mathcal{A} \subset \mathcal{A}_{;T}$, we can write

$$\begin{aligned} \mu_x^{\mathcal{A}}|_{B_r^{S\times T}(x)} &= \int \mu_y^{\mathcal{A}_{;T}}|_{B_r^{S\times T}(x)\cap[y]_{\mathcal{A}_{;T}}} d\mu_x^{\mathcal{A}}(y) \\ &\propto \int_{B_r^S} \mu_{x_{s,e}}^{\mathcal{A}_{;T}}|_{B_r^T(x_{s,e})}\, d[\pi_S]_*[\mu_{x,S\times T}^U|_{(t_{(x)})^{-1}[x]_{\mathcal{A}}}](s). \end{aligned} \tag{6.14}$$

Let $\tilde{\zeta}_s : T \mapsto S\times T$ be given by $\tilde{\zeta}_s : t \mapsto (s,t)$. Equation (6.12) can be rewritten as

$$\mu_x^{\mathcal{A}}|_{B_r^{S\times T}(x)} \propto \int_{B_r^S} [t_{(x)} \circ \tilde{\zeta}_s]_* \nu_{x,r;T}^U d\mu_{x,S}^U(s). \tag{6.15}$$

Comparing (6.14) and (6.15) we see that $\mu_{x,S}^U$ and $[\pi_S]_*[\mu_{x,S\times T}^U|_{(t_{(x)})^{-1}[x]_{\mathcal{A}}}]$ are in the same measure class and that for $\mu_{x,S}^U$ almost surely

$$\mu_{x_{s,e}}^{\mathcal{A}_{;T}}|_{B_r^T(x_{s,e})} \propto [t_{(x)} \circ \tilde{\zeta}_s]_* \nu_{x,r;T}^U. \tag{6.16}$$

Equation (6.16) is almost what we are seeking; however, we still need to show that for almost every x this equation holds at the specific value of $s = e$. This we achieve in the following way: Let $\tilde{B} \subset B_\epsilon(x_0) \cap X_0$ be a compact set with

$$\mu(B) \geq (1-\epsilon)\mu(B_\epsilon(x_0)),$$

on which

$$y \mapsto \mu_y^{\mathcal{A}_{;T}}|_{B_r^T(y)}$$

is continuous (with respect to the weak star topology of probability measures on X). By (6.16) there is a subset B of full measure of $x \in \tilde{B}$ for which there is some sequence $s_i \to e$ where (6.16) holds. We also require that Theorem 3.6.(2) holds for $x \in B$. Then, since

$$\mu_{x_{s_i,e}}^{\mathcal{A}_{;T}}|_{B_r^T(x_{s_i,e})} \to \mu_x^{\mathcal{A}_{;T}}|_{B_r^T(x)}, \qquad [t_{(x)} \circ \tilde{\zeta}_{s_i}]_* \nu_{x,r;T}^U \to [t_{(x)}]_* \nu_{x,r;T}^U,$$

by (6.16)

$$\mu_x^{\mathcal{A}_{;T}}|_{B_r^T(x)} \propto [t_{(x)}]_* \nu_{x,r;T}^U,$$

or, by Theorem 3.6.(2)

$$\mu_{x,T}^U|_{B_r^T(x)} \propto [t_{(x)}]_* \nu_{x,r;T}^U,$$

and we are done. □

Corollary 6.5. *Let X be an $S \times T$-space and $\alpha : X \to X$ as in Proposition* 6.4. *Then there is a set X_0 of full measure so that for every $x \overset{S\times T}{\sim} x'$ with $x, x' \in X_0, U \in \mathfrak{T}(x), U' \in \mathfrak{T}(x')$*

$$\mu^{U'}_{x';T} \propto \gamma_* \mu^{U}_{x;T},$$

where $\gamma \in \mathrm{Isom}(T)$ is defined by

$$t_{U';S\times T}(x', \cdot) \circ (\beta, \gamma) = t_{U;S\times T}(x, \cdot) \qquad \textit{for some } \beta \in \mathrm{Isom}(S).$$

7. Invariant structures and measure rigidity

We recall our main theorem: let $H = \mathrm{SL}(2, \mathbb{R})$, be equipped with some left invariant Riemannian metric d_H, L be an S-algebraic group, and $K < L$ be a compact subgroup. Set $T = L/K$ and let d_T be an L-invariant metric on T.

Let Γ be a discrete subgroup of $H \times L$, and take $X = \Gamma \backslash H \times T$. Note that we do *not* assume that Γ is a lattice. We take

$$d_{H\times T}((h,t),(h',t')) = \max(d_H(h,h'), d_T(t,t')).$$

Since the action of Γ preserves this metric, there is a unique metric d_X on X so that the projection $\pi : H \times T \to X$ is locally an isometry. For the sequel, we will need to assume that Γ is "irreducible" in the following (rather weak) sense that

$$\Gamma \cap L = \{e\} \tag{7.1}$$

(note that in the above equation L is identified with its image in $H \times L$).

The group H acts on X from the right, and in addition X has the structure of an (L, T)-space. Together this gives X the structure of an $(H \times L, H \times T)$-space; in particular this structure is a product structure. Let $\mathfrak{T}$ be a common atlas for the T and H-structures of X; since the H-structure of X comes from a group action, the local maps $t_{U;H}(x, h) = xh$ are independent of $U \in \mathfrak{T}$.

Let

$$a(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}, \qquad A = \{a(t) : t \in \mathbb{R}\},$$

$$n^+(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad N^+ = \{n^+(t) : t \in \mathbb{R}\},$$

$$n^-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \qquad N^- = \{n^-(t) : t \in \mathbb{R}\}.$$

Theorem 7.1. *Let $X = \Gamma \backslash H \times T$ be as above, and μ be an A-invariant and T-recurrent probability measure on X. Assume that all A-ergodic components of μ have positive entropy. Then μ is N^+-invariant.*

Proof of Theorem 1.1 *assuming Theorem* 7.1. By assumption, μ is A-invariant. Using the involution $i : g \mapsto (g^t)^{-1}$ on H (which we also consider as an involution on $H \times T$ fixing the second coordinate) we obtain a new measure μ' on $X' = \Gamma' \backslash H \times T$ with $\Gamma' = i(\Gamma)$ by first lifting μ to the product $H \times T$, applying the involution i and then projecting back to X'. The hypotheses in Theorem 7.1 remain satisfied for X' and μ', hence μ' is N^+-invariant, which shows that μ is N^- invariant.

It follows that the measure μ on $X = \Gamma \backslash H \times \Gamma$ is H-invariant, and Theorem 1.1 can now be deduced from the S-algebraic versions of Ratner's theorem [MT94], [Ra95]. Alternatively, one can use the elementary observation that a measure $\tilde{\mu}$ on $H \times T$ is left invariant under H if and only if $\tilde{\mu}$ is right invariant under H, hence by lifting μ to a measure $\tilde{\mu}$ on $H \times T$ the classification problem at hand reduces to classifying measures on $H \times T$ invariant from the left by $H\Gamma$, which is easy. □

LEMMA 7.2. *Let X be as in Theorem* 7.1, *and μ be a T-recurrent, A-invariant probability measure on X. Then for every sufficiently small $\epsilon > 0$, for every set $B \subset X$ with $\mu(B) > 0$ for almost every $x \in B$, there is a point $y \overset{T}{\sim} x$ with*

$$y \in B \cap (B_\epsilon(x) \setminus B_1^{N^+ \times T}(x)). \tag{7.2}$$

Proof. We first claim that for μ-almost every $x \in X$, it holds that the $N^+ \times T$-leaf of x is embedded. Indeed, the irreducibility condition on Γ implies that every T-leaf, without exception, is embedded. So if the $N^+ \times T$-leaf of x is not embedded, there are some $s \neq 0$ so that $x \overset{T}{\sim} xn^+(s)$, say $xn^+(s) = t_{U;T}(x,t)$ for $t \neq e$.

Consider the orbit of x under the semigroup $\{a(-t) : t \geq 0\}$. Almost surely, $xa(-t)$ would return infinitely often to some compact set K. Suppose $t_1 < t_2 < \dots$ is a sequence of such times with $t_i \to \infty$, and without loss of generality we may assume that $xa(-t_i) \to x_0$. Then $xn^+(s)a(-t_i) = xa(-t_i)n^+(e^{-2t_i}s) \to x_0$, and there is some $t' \neq e$ and U' so that $x_0 = t_{U';T}(x_0, t')$: a contradiction, which implies that almost surely the $N \times T$-leaf of xn is embedded.

Now let $\epsilon > 0$ be arbitrary. Cover X by countably many balls B_i of radius $\epsilon/2$, and throw away those whose intersection with B has measure 0. By T-recurrence, for μ-almost every $x \in B_i \cap B$ there is a $t \in T \setminus B_1^T$ and $U \in \mathfrak{T}(x)$ such that $y = t_{U;T}(x,t) \in B_i \cap B$. Note that $B_i \subset B_\epsilon(x)$. We also know that for μ-almost every $x \in B_i \cap B$, the $N^+ \times T$-leaf of x is embedded so that $y \notin B_1^{N^+ \times T}(x)$. Together this gives (7.2). □

Let $+_a : \mathbb{R} \to \mathbb{R}$ be the map $x \mapsto x + a$, and $\times_a : \mathbb{R} \to \mathbb{R}$ be the map $x \mapsto ax$.

Lemma 7.3. *Let μ be an A-invariant measure on X. Then the following sets*

$$\begin{aligned} Z &= \left\{x : \mu_{x,N^+} = \mathcal{H}_{N^+}\right\} \\ &= \left\{x \in X : \forall a \in \mathbb{R} \quad \mu_{x,N^+} = (+_a)_*\mu_{x,N^+}\right\}, \\ Y &= \left\{x \in X : \exists a \text{ such that } \mu_{x,N^+} \propto (+_a)_*\mu_{x,N^+}\right\} \end{aligned}$$

satisfy $\mu(Y \setminus Z) = 0$.

Proof. Set for $y \in Y$

$$\begin{aligned} \mathcal{R}_y &= \left\{a > 0 : \mu_{y,N^+} \propto (+_a)_*\mu_{x,N^+}\right\} \\ r(y) &= \inf \mathcal{R}_y. \end{aligned}$$

Since $r(\cdot)$ satisfies that $r(ya(-t)) = e^{-2t}r(y)$, by A-invariance of μ Poincaré recurrence implies that $r(y) = 0$ for μ-almost every $y \in Y$.

Choose some arbitrary nonnegative compactly supported test function $\phi \in C_c(\mathbb{R})$ which is nonzero in a neighborhood of 0. Then almost surely $\int \phi(t)d(+_a)_*\mu_{y,N^+} > 0$ for any $a \in \mathcal{R}_y$, and so we may define $\kappa_y : \mathcal{R}_y \to \mathbb{R}$ by

$$\exp(\kappa_y(a)) := \frac{d(+_a)_*\mu_{y,N^+}}{\mu_{y,N^+}} = \frac{\int \phi(t+a)d\mu_{y,N^+}}{\int \phi(t)d\mu_{y,N^+}}.$$

Since the map $a \mapsto \int \phi(t+a)d\mu_{y,N^+}$ is continuous, so is $\kappa_y(a)$; and if $r(y) = 0$ (which we recall happens a.s. for $y \in Y$) we now see that in fact $\mathcal{R}_y = \mathbb{R}^+$ and $\kappa_y(a) = \kappa_y(1) \cdot a$. In view of this last expression, we set $\kappa_y = \kappa_y(1)$.

We now again use the fact that μ is invariant under the A-action, which implies that

$$[\times_{e^{2t}}]_*\mu_{y,N^+} \propto \mu_{ya(t),N^+}.$$

Hence $\kappa_y = \kappa_{ya(t)}(e^{2t})$ or

$$\kappa_y = e^{2t}\kappa_{ya(t)}.$$

Again Poincaré recurrence implies that $\kappa_y = 0$ for almost every $y \in Y$; in other words almost every $y \in Y$ is in Z. □

A crucial ingredient in the proof is Ratner's H-property for the horocycle flow on $\mathrm{SL}(2,\mathbb{R})$ ([Ra82, Lemma 2.1] and [Ra83, Def. 1]). This property is related but distinct from Ratner's R-property which is used in the proof Raghunathan's conjecture (see [Ra92, p. 22] for the special case of $G = \mathrm{SL}(2,\mathbb{R})$ and [Ra90b] for the general case). We present below a form of the H-property that is convenient for our purposes. At its heart, is the following elementary calculation:

Lemma 7.4. *There is some universal constant $C > 0$ so that for any $\delta, t \in \mathbb{R}$ with $\frac{1}{\delta} > t > 1 > \delta$,*

$$n^-(\delta)n^+(t) \in n^+\left(\frac{t}{1+\delta t}\right) B^H_{Ct\delta}.$$

Proof. Indeed, this is simply an exercise in matrix multiplication:

$$\begin{aligned} n^-(\delta)n^+(t) &= \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ \delta & 1+t\delta \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{t}{1+\delta t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1+t\delta} & 0 \\ \delta & 1+t\delta \end{pmatrix} \\ &\in \begin{pmatrix} 1 & \frac{t}{1+t\delta} \\ 0 & 1 \end{pmatrix} B^H_{Ct\delta}. \end{aligned}$$

□

Lemma 7.5. *For any compact subset $X' \subset X$ and $\rho \in (0,1)$, there are C and $\eta_0 > 0$ so that for any $\epsilon < \eta_0$ and $x, x' \in X'$ with*

$$x' \in B_\epsilon(x) \setminus B_1^{N^+ \times T}(x)$$

there is some a so that for any τ with $\rho a < |\tau| < a$

$$x' n_+(\tau) \in B_{C\epsilon^{1/2}}(x n_+(\tau'))$$

with $C^{-1} < |\tau - \tau'| < C$.

In addition to our use of the H-property, our strategy of proof is similar to that used by Ratner, particularly in [Ra82], [Ra83].

7.1. *A simplified proof of Theorem* 7.1. Initially, we state the proof of Theorem 7.1 given an additional technical assumption, which allows us to avoid a complication in the proof, clarifying the ideas involved.

Additional assumptions. The additional assumption is that the conditional measures μ_{x,N^+} satisfy the doubling condition, i.e. there is a constant $\rho \in (0,1)$ so that for μ-almost every $x \in X$ and all $r > 1$

$$\mu_{x,N^+}(B_r^{N^+}) > 2\mu_{x,N^+}(B_{\rho r}^{N^+}). \tag{7.3}$$

Let Z and Y be as in Lemma 7.3. By Proposition 4.3, Theorem 7.1 is equivalent to $\mu(X \setminus Z) = 0$. Assume by contradiction that this is false. Let $\mu' = \mu|_{X \setminus Z}$. It is immediate from the definition of recurrent measures that the restriction of a recurrent measure is recurrent; so μ' is T-recurrent. Clearly Z is A-invariant (up to a set of μ-measure 0), and so μ' is A-invariant.

Since Z is A-invariant, it follows from Corollary 5.4(1) that for almost every $x \notin Z$

$$\mu'_{x,N^+} = \mu_{x,N^+}.$$

Replacing μ by μ' if necessary, it is enough to show that $\mu(Z) = 0$ (or equivalently that $\mu(Y) = 0$) leads to a contradiction.

Let $\epsilon > 0$ be arbitrary. For any such ϵ we can find a compact subset X_1 of X with measure $\geq 1 - \epsilon$ with the following properties:

(X-1) X_1 is disjoint from Y.

(X-2) The map $x \mapsto \mu_{x,N^+}$ is continuous on X_1 (with respect to the topology on $\mathcal{M}_\infty(N^+)$ given in §3).

(X-3) X_1 is a subset of the set of full measure in Corollary 6.5 applied to the $N^+ \times T$ structure of X.

(X-4) X_1 is a subset of the set of full measure in Theorem 3.6.(3) for the N^+, T, and $N^+ \times T$ structures of X.

We remark that we can find X_1 satisfying X-2 by Lusin's theorem [Fed69, p. 76], since $x \mapsto \mu_{x,N^+}$ is a Borel measurable map from X to the separable metric space $\mathcal{M}_\infty(N^+)$.

We now apply a version of the maximal ergodic theorem for not necessarily invariant measures which will be proved in the appendix (Theorem A.1). According to the theorem, there is a set X_2 (which we may as well assume is a compact subset of X_1) of measure $\geq 1 - C_1\epsilon^{1/2}$ (with C_1 some universal constant) so that:

(X-5) For every $x \in X_2$ and any $r > 0$

$$\int_{B_r^{N^+}} 1_{X_1}(xn^+(s))d\mu_{x,N^+}(s) \geq (1 - \epsilon^{1/2})\mu_{x,N^+}(B_r^{N^+}). \tag{7.4}$$

Let $\delta > 0$ be very small (depending on ϵ) to be determined later. Since μ is T-recurrent, by (7.2) it follows that for almost every $x \in X_2$ there is an $x' \overset{T}{\sim} x$ so that

$$x' \in X_2 \cap B_\delta(x) \setminus B_1^{N^+ \times T}(x).$$

As long as δ is small enough, this implies that x and x' satisfy the assumptions of Lemma 7.5. Let a be as in that corollary with ρ as in (7.3). Clearly, if δ is small enough a will be much bigger than 1.

Let

$$\begin{aligned} G_1 &= \{s \in \mathbb{R} : xn_+(s) \in X_1\}, \\ G_2 &= \left\{s \in \mathbb{R} : x'n_+(s) \in X_1\right\}. \end{aligned} \tag{7.5}$$

Since $x, x' \in X_1$ and $x \overset{T}{\sim} x'$ we have that $\mu_{x,N^+} = \mu_{x',N^+}$. Furthermore, since $x, x' \in X_2$ and $a > 1$,

$$\begin{aligned} \mu_{x,N^+}(\{s : \rho a < |s| < a\} \setminus G_i) &\le \epsilon^{1/2} \mu_{x,N^+}(B_a^{N^+}) \\ &\le 2\epsilon^{1/2} \mu_{x,N^+}(\{s : \rho a < |s| < a\}), \qquad i = 1, 2, \end{aligned} \tag{7.6}$$

where we have used (7.3) to pass from the first to the second line. By X-4, for all $x \in X_1$,

$$\mu_{x,N^+}(B_a^{N^+}) > 0.$$

Thus if $\epsilon < 0.01$

$$\mu_{x,N^+}(\{s : \rho a < |s| < a\} \cap G_1 \cap G_2) > 0$$

and in particular there is a $s_0 \in \{s : \rho a < |s| < a\} \cap G_1 \cap G_2$. Consider now the pair of points $y = xn_+(s_0), y' = x'n_+(s_0) \in X_1$. By Lemma 7.5, we know that

$$y' \in B_{C(\rho)\delta^{1/2}}(yn_+(\tau))$$

for some τ so that $|\tau|$ is in a fixed interval $I \subset \mathbb{R}^+$ which does not contain 0. Note that since $\mu_{x,N^+} = \mu_{x',N^+}$, and since x, x', y, y' are all in X_1,

$$\begin{aligned} \mu_{y,N^+} &\propto (+_{-s_0})_* \mu_{x,N^+} \\ &= (+_{-s_0})_* \mu_{x',N^+} \\ &\propto \mu_{y',N^+}. \end{aligned} \tag{7.7}$$

By comparing the measure of $B_1^{N^+}$ one sees that in fact

$$\mu_{y,N^+} = \mu_{y',N^+}. \tag{7.8}$$

Applying this with a sequence $\delta_i \to 0$ we get a sequence $y_i, y_i' \in X_1$; since X_1 is compact we may as well assume that $y_i \to y, y_i' \to y'$ and necessarily

$$\begin{aligned} y' = yn_+(\tau), &\qquad \tau \in I \cup -I, \\ y, y' \in X_1&. \end{aligned}$$

Furthermore, since on X_1 the map $x \mapsto \mu_{N^+,x}$ is continuous, and since for all i by (7.8)

$$\mu_{N^+,y_i} = \mu_{N^+,y_i'}$$

we get that

$$\mu_{N^+,y} = \mu_{N^+,y'} = \mu_{N^+,yn(\tau)}. \tag{7.9}$$

Once again using the fact that $y, y' \in X_1$ we also know that

$$\mu_{N^+,yn(t)} \propto (+_{-\tau})_* \mu_{N^+,y}. \tag{7.10}$$

Hence either y or y' is in Y, contrary to the fact that Y is disjoint from X_1.

7.2. *A complete proof of Theorem* 7.1. In the proof just given in §7.1, substantial use has been made of the doubling condition (7.3). The key to overcoming this difficulty is the observation that for a given constant $\rho < 1$ the set

$$\mathcal{R}_\rho(x) = \left\{ r : \mu_{x,N^+}(B_r^{N^+}) > 2\mu_{x,N^+}(B_{\rho r}^{N^+}) \right\}, \tag{7.11}$$

which is the set where a doubling condition holds, has a very different behavior, when we replace x by $xa(t)$, from the set of all r that satisfy the conclusion of Lemma 7.5, i.e. the set

$$D_{\rho,C,\gamma}(x,x') = \left\{ \begin{array}{c} r : \forall s, \rho r < |s| < r : x'n_+(s) \in B_\gamma(xn_+(s')) \\ \text{with } C^{-1} < \left|s - s'\right| < C \end{array} \right\},$$

for, e.g. $\gamma = C\epsilon^{1/2}$ (for technical reasons, we will actually need to use the slightly bigger γ). This gives us hope that by flowing along the flow associated with the subgroup A we might be able to arrange to have the doubling condition precisely where we need it.

Before we actually carry out the proof, we need the following standard fact in a nonstandard terminology:

THEOREM 7.6. *Let μ be an A-invariant probability measure on X. Then μ is N^+-recurrent if, and only if the entropy with respect to the action of $a(1)$ by right multiplication of almost every $a(1)$ ergodic component $\mu_\xi^{\mathcal{E}}$ is positive.*

We could have just as well considered ergodic components of the full A-action: in general, an ergodic component for the $\mathbb{R}$-action corresponding to A can fail to be ergodic under the $\mathbb{Z}$-action generated by $a(1)$, but the entropy of this $\mathbb{R}$-ergodic component is equal to the entropy of almost every $\mathbb{Z}$-ergodic subcomponent.

In essence, this theorem is a corollary of a theorem of Ledrappier and Young ([LY85, Th. B]). Strictly speaking, however, the results of that paper which deals with smooth actions on smooth compact manifolds do not apply here. In the S-algebraic context a suitable variant of this theory can be found in §9 of [MT94]. With slightly more work, Theorem 7.6 (which is the only place where S-algebraicity is used in the proof of Theorem 7.1) can be proved for general locally compact L, but it is not clear how useful such an extension would be.

Proof of Theorem 7.6. Let α be the map $x \mapsto xa(1)$, and $\mu = \int \mu_\xi^{\mathcal{E}} d\mu(\xi)$ be the ergodic decomposition of μ with respect to α (see §5), and let $h_\alpha(\mu_\xi^{\mathcal{E}})$ denote the entropy of multiplication from the right by $a(1)$ of the ergodic component $\mu_\xi^{\mathcal{E}}$.

We will show that μ-almost surely, if $h_\alpha(\mu_\xi^{\mathcal{E}}) > 0$ then for $\mu_\xi^{\mathcal{E}}$-almost every y, the conditional measure μ_{y,N^+} is infinite (recall that by Corollary

5.4(2), $\mu_{y,N^+} = (\mu_\xi^{\mathcal{E}})_{y,N^+})$. We will also show that conversely, if $h_\alpha(\mu_\xi^{\mathcal{E}}) = 0$ then for $\mu_\xi^{\mathcal{E}}$-almost every y the measure μ_{y,N^+} is finite, indeed equal to the delta measure at 0.

As a preliminary step, we note that the sets

$$E_1 := \left\{x : \mu_{x,N^+} \text{is finite } \right\} \supset \left\{x : \mu_{x,N^+} = \delta_0\right\} =: E_2$$

satisfy

$$\mu(E_1 \setminus E_2) = 0. \tag{7.12}$$

Indeed, define

$$r(x) = \begin{cases} \inf \left\{r > 0 : \mu_{x,N^+} B_r^{N^+} > \frac{1}{2}\mu_{x,N^+} B_\infty^{N^+}\right\} & \text{if } x \in E_1 \\ 0 & \text{otherwise.} \end{cases}$$

Then for μ-almost every x we have that $r(x) = e^{-1}r(\alpha(x))$. By Poincaré recurrence this implies that $r(x) = 0$ almost surely, which is equivalent to (7.12).

Let now $\nu = \mu_\xi^{\mathcal{E}}$ be an ergodic component. By [MT94, Prop. 9.2] there is a countably generated Borel sigma algebra $\mathcal{A}$ of subsets of X with the following properties:

(i) $\mathcal{A}$ is subordinate to N^+; i.e., for every x we have that there is some $r > 0$ so that $[x]_{\mathcal{A}} \subset B_r^{N^+}(x)$ and for ν-almost every x we have that there is some $\epsilon > 0$ so that $[x]_{\mathcal{A}} \supset B_\epsilon^{N^+}(x)$.

(ii) $\mathcal{A} < \alpha^{-1}(\mathcal{A})$

(iii) The mean conditional entropy $H_\nu(\mathcal{A} \,|\, \alpha\mathcal{A})$ is equal to the entropy $h_\alpha(\nu)$.

By definition, the mean conditional entropy is given by

$$\begin{aligned} H_\nu(\mathcal{A} \,|\, \alpha\mathcal{A}) &= -\int \log \nu_x^{\alpha\mathcal{A}}([x]_{\mathcal{A}}) d\nu(x) \\ &= -\int \log \frac{\nu_{x,N^+}[x]_{\alpha\mathcal{A}}}{\nu_{x,N^+}[x]_{\mathcal{A}}} d\nu(x). \end{aligned} \tag{7.13}$$

Since E_2 is α-invariant (up to a set of measure 0), for almost every ξ we have that $\nu(E_1) = \nu(E_2)$ can be either 0 or 1. In the case $\nu(E_2) = 1$, by (iii) and (7.13) we see that $h_\alpha(\nu) = 0$.

In the case $\nu(E_1) = 0$ we have that since for ν-almost every x the measure ν_{x,N^+} is infinite,

$$\log \frac{\nu_{x,N^+}([x]_{\alpha^k\mathcal{A}})}{\nu_{x,N^+}([x]_{\mathcal{A}})} = \sum_{i=0}^{k-1} \log \frac{\nu_{\alpha^{-i}x,N^+}([\alpha^{-i}x]_{\alpha\mathcal{A}})}{\nu_{\alpha^{-i}x,N^+}([\alpha^{-i}x]_{\mathcal{A}})} \to \infty.$$

Since ν is a α-invariant, the above equation implies that

$$\nu\left\{x : -\log\frac{\nu_{x,N^+}[x]_{\alpha\mathcal{A}}}{\nu_{x,N^+}[x]_{\mathcal{A}}} > 0\right\} > 0. \tag{7.14}$$

Thus if $\nu(E_1) = 0$ then the integral (7.13) is positive, and so is $h_\alpha(\nu)$. □

COROLLARY 7.7. *If the entropy of almost every A-ergodic component* $\mu_x^{\mathcal{E}}$ *with respect to the action of A is positive, then there is a* ρ *so that*

$$\mathcal{X}(\rho) = \left\{x : \mu_{x,N^+}(B_1^{N^+}) > 2\mu_{x,N^+}B_\rho^{N^+})\right\}$$

has $\mu(\mathcal{X}(\rho)) > 1 - \epsilon$.

Note that $\mathcal{R}_\rho(x)$ is related to $\mathcal{X}(\rho)$ by

$$\mathcal{R}_\rho(x) = \left\{e^{2t} : xa(-t) \in \mathcal{X}(\rho)\right\}. \tag{7.15}$$

We now let X_3 be a compact subset of the set X_2 defined as in §7.1, equation (7.4) with $\mu(X_3) \geq 1 - C_2\epsilon^{1/4}$ so that for every $x \in X_3$ and $\tau > 0$ and with ρ as in Corollary 7.7,

$$\begin{aligned}\tfrac{1}{\tau}\int_0^\tau 1_{X_2}(xa(s))ds &\geq (1-\epsilon^{1/4}),\\ \tfrac{1}{\tau}\int_{-\tau}^0 1_{X_2}(xa(s))ds &\geq (1-\epsilon^{1/4}),\\ \tfrac{1}{\tau}\int_0^\tau 1_{\mathcal{X}(\rho)}(xa(s))ds &\geq (1-\epsilon^{1/4}),\\ \tfrac{1}{\tau}\int_{-\tau}^0 1_{\mathcal{X}(\rho)}(xa(s))ds &\geq (1-\epsilon^{1/4}).\end{aligned} \tag{7.16}$$

The existence of such a set is guaranteed by the maximal ergodic theorem (this time in the classical, i.e. measure-preserving, context).

Now take $\delta > 0$ to be very small, and find $x, x' \in X_3$ so that $d(x, x') < \delta$ and $x \stackrel{T}{\sim} x'$ using Poincaré recurrence for T as in §7.1. Later, δ will be determined but in particular we demand that $\delta < \eta_0$ with η_0 as in Lemma 7.5 applied to the compact subset X_1.

The following lemma is simply a somewhat more quantitative version of the argument in the simplified proof of §7.1.

LEMMA 7.8. *Let X and* μ *be as in Theorem 7.1. Let* X_2 *be a compact subset of X as in* §7.1. *Then for any sufficiently small* $\delta > 0$, *and any* $C > 0$ *if* $x, x' \in X_2$ *satisfy*

(*-a) $d(x, x') < \delta$,

(*-b) $x \stackrel{T}{\sim} x'$,

(*-c) *x is not in the same N^+-leaf as x',*

(*-d) $D_{\rho,C,\gamma}(x,x') \cap \mathcal{R}_\rho(x) \neq \emptyset$,

then there is an $s \in \mathbb{R}$ and an s' with $C^{-1} < |s'| < C$ so that:

(*-1) *$y = xn_+(s)$ and $y' = x'n_+(s)$ are both in X_1,*

(*-2) $y \in B_\gamma(y'n_+(s'))$,

(*-3) $\mu_{y,N^+} = \mu_{y',N^+}$.

Proof. We first remark that *-3 follows from *-1 since $y \overset{T}{\sim} y'$ and for any two T-equivalent points in X_1,

$$\mu_{y,N^+} = \mu_{y',N^+}.$$

Thus we need only to prove we can find $s \in \mathbb{R}$ so that both *-1 and *-2 hold.

As in §7.1, equation (7.5), we set

$$G_1 = \{s \in \mathbb{R} : xn_+(s) \in X_1\},$$
$$G_2 = \left\{s \in \mathbb{R} : x'n_+(s) \in X_1\right\},$$

and note once more that since $x, x' \in X_1$, we have that $\mu_{x,N^+} = \mu_{x',N^+}$. Let $a \in D_{\rho,C}(x,x') \cap \mathcal{R}_\rho(x)$; clearly if δ is small $a \gg 1$. Since $x, x' \in X_2$ we have that for $i = 1, 2$,

$$\mu_{x,N^+}(\{s : \rho a < |s| < a\} \setminus G_i) \leq \epsilon^{1/2} \mu_{x,N^+}(B_a^{N^+}); \tag{7.17}$$

and since $a \in \mathcal{R}_\rho(x)$ we get

$$(7.17) \leq 2\epsilon^{1/2} \mu_{x,N^+}(\{s : \rho a < |s| < a\}).$$

This implies (as long as $\epsilon < 0.01$) that there is some

$$s_0 \in \{s : \rho a < |s| < a\} \cap G_1 \cap G_2.$$

Set $y = xn_+(s_0)$, $y' = x'n_+(s_0)$. By our choice of s_0, both y and y' are in X_1. Since $a \in D_{\rho,C}(x,x')$ we have that

$$y' \in B_\gamma(yn_+(s'))$$

with $C^{-1} < |s'| < C$ and we are done. □

Lemma 7.9. *Let $\rho \in (0,1)$ be arbitrary. Then for any sufficiently small $\delta > 0$, for any $x, x' \in X_1$ with $d(x,x') < \delta$ at least one of the following holds, for some constant C_0 that does not depend on δ*:

(1) *There is some $\xi_1 > C_0^{-1}\delta^{-1/2}$ so that for all $0 < t < \kappa\,|\ln \xi_1|$,*

$$\xi_1 \in D_{\rho,C_0,\delta^{1/4}}(xa(-t), x'a(-t))$$

for some fixed absolute constant $\kappa > 0$.

(2) *There is some* $\xi_1 > C_0^{-1}\delta^{-1/2}$ *so that for all*

$$\kappa' \left|\ln \xi_1\right| < t < 2\kappa' \left|\ln \xi_1\right|,$$

$$e^{-t}\xi_1 \in D_{\rho,C_0,\delta^{1/4}}(xa(-t), x'a(-t))$$

where again $\kappa' > 0$ *is an absolute constant.*

Proof. Define s_a, s_+, s_- by

$$x' \overset{M}{\sim} xn_-(s_-)n_+(s_+)a(s_a), \quad d(x', xn_-(s_-)n_+(s_+)a(s_a)) < \delta \tag{7.18}$$

(since X_1 is compact, it is an immediate consequence of the definition of the metric on X that there are indeed such s_a, s_+, s_-). It also follows that $|s_a|, |s_+|, |s_-| < C\delta$ for some constant C. (We note that throughout this proof, C, C_1, etc. stand for some large constants that does not depend on δ, with the agreement that each constant can be taken as large as needed and may depend only on the constants that have appeared before.)

From (7.18) and the fact that H acts isometrically on the T-leaves of X it follows that

$$x'a(-\tau)n_+(\xi) \overset{M}{\sim} xn_-(s_-)n_+(s_+)a(s_a)a(-\tau)n_+(\xi), \quad d(x'a(-\tau)n_+(\xi), xn_-(s_-)n_+(s_+)a(s_a)a(-\tau)n_+(\xi)) < \delta. \tag{7.19}$$

Using the formula from Lemma 7.4, we see that assuming $|\xi| > 1, \tau > 0$ and

$$\left|\xi^2 e^{2\tau} s_-\right|, |2\xi s_a| \leq 1,$$

$$\begin{aligned} & xn_-(s_-)n_+(s_+)a(s_a)a(-\tau)n_+(\xi) \\ &\quad = xa(-\tau)n_-(e^{2\tau}s_-)n_+(e^{-2\tau}s_+ + e^{-2s_a}\xi)a(s_a) \\ &\quad \in xa(-\tau)n_+\left(\frac{e^{-2\tau}s_+ + e^{-2s_a}\xi}{1+e^{2\tau}s_-(e^{-2\tau}s_+ + e^{-2s_a}\xi)}\right)B^H_{C_1\xi e^{2\tau}|s_-|} \\ &\quad \in xa(-\tau)n_+(\xi - 2s_a\xi - e^{2\tau}s_-\xi^2)B^H_\sigma \end{aligned} \tag{7.20}$$

with

$$\sigma = C_2 \max(\xi e^{2\tau}\left|s_-\right|, \, e^{-2\tau}\left|s_+\right|, |\xi|^{-1}). \tag{7.21}$$

Combining (7.20) with (7.19) we get that

$$x'a(-\tau)n_-(\xi) \in B_{\max(\sigma,\delta)}(xa(-\tau)n_-(\xi')) \quad \text{with} \quad \xi' = \xi - 2s_a\xi - e^{2\tau}s_-\xi^2. \tag{7.22}$$

There are now two cases, corresponding to the two cases in the lemma:

Case 1. $|s_a| > |s_-|^{10/21}$. In this case we take $\xi_1 = |s_a|^{-1}$, and consider τ in the range

$$0 < \tau < \tau_0 = 0.01 \ln \xi_1.$$

Note that in particular $\xi_1 > C^{-1}\delta^{-1}$. Let ξ' be as in (7.22). Then for any ξ in the range $\rho\xi_1 < \xi < \xi_1$ we have that

$$\left|e^{2\tau} s_- \xi^2\right| \leq \xi_1^{2.02} |s_-| \leq \xi_1^{2.02} |s_a|^{2.1} \leq |s_a|^{0.08} \leq \delta^{0.08},$$

while on the other hand $|2s_a\xi| > 2\rho$. Thus for δ small enough, depending only on ρ,

$$\frac{\rho}{2} \leq \left|\xi' - \xi\right| = \left|2s_a\xi + e^{2\tau}s_-\xi^2\right| \leq 2$$

and so for appropriate choice of C_0 by (7.22)

$$\xi_1 \in D_{\rho,C_0,\max(\sigma,\delta)}(xa(-\tau), x'a(-\tau)).$$

By (7.21)

$$\begin{aligned}\sigma &= C_2 \max(\xi e^{2\tau} |s_-|\,,\, e^{-2\tau} |s_+|\,, |\xi|^{-1}) \\ &\leq C_2 \max(|s_a|^{1.08}\,, C\delta) \leq C_3\delta,\end{aligned}$$

which is substantially better than the estimate $\leq \delta^{1/4}$ that we needed.

Case 2. $|s_a| \leq |s_-|^{10/21}$. In this case we take $\xi_1 = |s_-|^{-1/2}$, and consider τ in the range

$$0.05 \ln \xi_1 < \tau < 0.1 \ln \xi_1.$$

Then for any ξ in the range $\rho e^{-\tau}\xi_1 < \xi < e^{-t}\xi_1$

$$\rho \leq \left|e^{2\tau} s_- \xi^2\right| \leq 1$$

and

$$|s_a\xi| < e^{-\tau}\xi_1 |s_a| \leq \xi_1^{0.95} |s_a| \leq |s_-|^{0.475-10/21} \leq |C\delta|^{0.001}.$$

So once again if δ is small enough (depending only on ρ) and ξ' is as in (7.22)

$$\frac{\rho}{2} \left|\xi - \xi'\right| \leq 2;$$

i.e. $e^{-\tau}\xi_1 \in D_{\rho,C_0,\max(\sigma,\delta)}(xa(-\tau), x'a(-\tau))$.

We are left with estimating σ in this case:

$$\begin{aligned}\sigma &= C_2 \max(\xi e^{2\tau} |s_-|\,, e^{-2\tau} |s_+|\,, |\xi|^{-1}) \\ &\leq C_2(|s_-|^{0.475}\,, \delta, |s_-|^{0.5}) \leq C_3\delta^{0.475}\end{aligned}$$

which is again better than advertised. □

LEMMA 7.10. *Let ρ be as in Corollary 7.7, and $x, x' \in X_3$ so that $d(x,x') < \delta$ and $x \overset{T}{\sim} x'$ for a sufficiently small δ. Then if ϵ (the constant used in the definition of X_3) is smaller than some absolute constant there is a $\tau \geq 0$ so that*

$$D_{\rho,C_0,\delta^{1/4}}(xa(-\tau), x'a(-\tau)) \cap \mathcal{R}_\rho(xa(-\tau)) \neq \emptyset, \tag{7.23}$$

$$xa(-\tau) \in X_2, \tag{7.24}$$

$$x'a(-\tau) \in X_2. \tag{7.25}$$

Proof. There are two (very similar) cases corresponding to the two cases of Lemma 7.9 applied to x, x':

Case (1) *of Lemma* 7.9 *holds.* Let ξ_1 be as in Lemma 7.9.(1). We know that for all $\tau \in (0, \kappa \log \xi_1)$,

$$\xi_1 \in D_{\rho,C_0,\delta^{1/4}}(xa(-\tau), x'a(-t'))$$

so we need to check that there is some τ in the above range for which simultaneously (7.24), (7.25) and

$$\xi_1 \in \mathcal{R}_\rho(xa(-\tau)) \tag{7.26}$$

all hold. We can rewrite (7.26) using (7.15) as

$$xa(-\tau - \tfrac{1}{2}\ln \xi_1) \in \mathcal{X}(\rho). \tag{7.27}$$

Using (7.16), since $x, x' \in X_3$, we know that

$$\int_0^{\kappa \ln \xi_1} 1_{X_2}(xa(-s)) 1_{X_2}(x'a(-s))\, ds \geq (1 - 2\epsilon^{1/4}) \kappa \ln \xi_1. \tag{7.28}$$

On the other hand, using the same equation

$$\int_{-(\frac{1}{2}+\kappa)\ln \xi_1}^{0} 1_{X \setminus \mathcal{X}(\rho)}(xa(s))\, ds \leq \epsilon^{1/4} \ln \xi_1.$$

So in particular

$$\int_0^{\kappa \ln \xi_1} 1_{X \setminus \mathcal{X}(\rho)}(xa(-s - \tfrac{1}{2}\ln \xi_1))\, ds \leq \epsilon^{1/4} \ln \xi_1. \tag{7.29}$$

Combining (7.28) with (7.29) we see that as long as

$$\epsilon^{1/4}(2\kappa + 1) < \kappa$$

(which certainly holds for ϵ less than some absolute constant) there is a τ as in the statement of Lemma 7.10.

Case (2) *of Lemma* 7.9 *holds.* Again let ξ_1 be as in Lemma 7.9.(2). We know that for all $\tau \in (\kappa' \log \xi_1, 2\kappa' \log \xi_1)$,

$$e^{-\tau}\xi_1 \in D_{\rho,C_0,\delta^{1/4}}(xa(-\tau), x'a(-t'))$$

so again we need to check that there is some τ in the above range for which simultaneously (7.24), (7.25) and

$$e^{-\tau}\xi_1 \in \mathcal{R}_\rho(xa(-\tau)) \tag{7.30}$$

hold; i.e.,

$$xa(-\tfrac{1}{2}\tau - \tfrac{1}{2}\ln\xi_1) \in \mathcal{X}(\rho).$$

Similarly to the previous case, we can estimate the measure of the parameters τ in the required range which fails to satisfy one of the assumptions of Lemma 7.10:

$$\int_{\kappa'\ln\xi_1}^{2\kappa'\ln\xi_1} 1_{X_2}(xa(-s))1_{X_2}(x'a(-s))\,ds \geq (1-2\epsilon^{1/4})2\kappa'\ln\xi_1.$$

Next,

$$\int_{\kappa'\ln\xi_1}^{2\kappa'\ln\xi_1} 1_{X\setminus\mathcal{X}(\rho)}(xa(-\tfrac{1}{2}s - \tfrac{1}{2}\ln\xi_1)) \leq \epsilon^{1/4}\ln\xi_1.$$

It is again clear that if ϵ is smaller than some absolute constant there will be a parameter τ satisfying all the conditions of this lemma. □

Conclusion of the proof of Theorem 7.1. We have already shown that for any $\delta > 0$ we can find a pair of points $x, x' \in X_3$ with $x \overset{T}{\sim} x'$ and $d(x, x') < \delta$.

By Lemma 7.10 there is some τ so that

$$D_{\rho,C_0,\delta^{1/4}}(xa(-\tau), x'a(-\tau)) \cap \mathcal{R}_\rho(a(-\tau)x) \neq \emptyset,$$
$$xa(-\tau) \in X_2$$
$$x'a(-\tau) \in X_2.$$

By Lemma 7.8 there is some s so that for some s' in a fixed bounded closed subset $S \subset \mathbb{R} \setminus \{0\}$

$$y := xa(-\tau)n_+(s) \in X_1,$$
$$y' := x'a(-\tau)n_+(s) \in X_1,$$
$$y \in B_{\delta^{1/4}}(yn_+(s')),$$
$$\mu_{y,N^+} = \mu_{y',N^+}.$$

Since $z \mapsto \mu_{z,N^+}$ is continuous on X_1, X_1 is compact, and δ arbitrarily small, we see that there must be points $z, z' \in X_1$ with

$$z = z'n_+(s') \qquad \text{for some } s' \in S, \quad \mu_{z,N^+} = \mu_{z',N^+},$$

a contradiction to the definition of X_1. □

8. Hecke Maas forms and recurrent measures

In this section we take $\mathbb{G}$ to be the linear algebraic group of invertible elements in a quaternion division algebra defined over $\mathbb{Q}$. Assume that $\mathbb{G}$ is unramified over $\mathbb{R}$ and $\mathbb{Q}_p$, and take $G = \mathbb{G}(\mathbb{R}) \times \mathbb{G}(\mathbb{Q}_p)$. Take $\Gamma = \mathbb{G}(\mathbb{Z}[\frac{1}{p}])$, or more precisely the diagonal embedding of this group in G. Then as is well-known, Γ is a lattice in G. More generally, one may take a congruence sub group of this lattice of order relatively prime to p – everything mentioned below is equally valid for such a lattice, and except for minor notational nuisances the arguments need not be modified.

We take $K_\infty < \mathbb{G}(\mathbb{R})$ and $K_p = \mathbb{G}(\mathbb{Z}_p) < \mathbb{G}(\mathbb{Q}_p)$ to be the respective maximal compact subgroups, and take $K = K_\infty \times K_p$. Let C denote the center of $\mathbb{G}(\mathbb{R})$, considered as a subgroup of G. As is well-known, $M = C\Gamma\backslash G/K$ can be identified as a compact quotient of the hyperbolic half plane $\mathbb{H}$, and $X = C\Gamma\backslash G/K_p$ a compact quotient of $\mathrm{SL}(2,\mathbb{R})$. Finally, set $\tilde{X} = C\Gamma\backslash G$, π_p be the projection $x \mapsto xK_p$, and let π_∞ be the projection $x \mapsto xK_\infty$, and $\pi_{p,\infty} = \pi_p \circ \pi_\infty$.

Let $C_p = \mathbb{G}(\mathbb{Q}_p) \cap C\Gamma$, which we identify between $\mathbb{G}(\mathbb{Q}_p)$ and its image in G. This is always a subgroup of the center of $\mathbb{G}(\mathbb{Q}_p)$; indeed, this is just the multiplicative group of nonzero rationals viewed as a subgroup of the nonzero quaternions. Thus $\mathbb{G}(\mathbb{Q}_p)/C_p$ is a group which acts freely and continuously on $\tilde{X}$. This group no longer acts on M or X; however, this action has not completely disappeared: if one takes a $\mathbb{G}(\mathbb{Q}_p)$ orbit $x\mathbb{G}(\mathbb{Q}_p) \subset C\Gamma\backslash G$ then for any $x \in \tilde{X}$, the map $t[x] : gC_pK_p \mapsto \pi_p(xg)$ is an embedding of $T = \mathbb{G}(\mathbb{Q}_p)/C_p\mathbb{G}(\mathbb{Z}_p)$ (i.e. of a $p+1$-regular tree with some additional algebraic structure) in X. What is more, if $y = xg \in x\mathbb{G}(\mathbb{Q}_p)$, then $t[x](T) = t[y](T)$ and $t[y]^{-1} \circ t[x]$ is a tree automorphism: indeed, it is simply the map $qC_pK_p \mapsto g^{-1}qC_pK_p$. Finally, for any $y \in X$ one can find a neighborhood $y \in U \subset X$ in which there is a continuous section τ_U of the bundle $\tilde{X} \to X$, which gives us a map $t_U : U \times T \to X$ defined by

$$t_U(y', q) = t[\tau_U(y')](q).$$

In this way we see that X has a natural T-space structure. Take $\mathfrak{T}$ to be some open cover of X with sets U as above. Since T can be naturally identified with the tree it is natural to take the metric on T to be normalized so that the distance between nearest neighbors is 1; with this normalization,[3] for every $g \in \mathbb{G}(\mathbb{Q}_p)/C_p$ we have that

$$d_T(gK_p, g\begin{pmatrix} p^l & 0 \\ 0 & 1 \end{pmatrix} K_p) = l.$$

[3] and identifying $\mathbb{G}(\mathbb{Q}_p)$ with $\mathrm{GL}(2,\mathbb{Q}_p)$

This structure as a T-space is intimately connected with the Hecke operators T_p. Indeed, let $q_1, q_2, \dots, q_{p+1}$ be the nearest neighbors of the distinguished point $e \in T$. Then for any function f on X one can define $T_p f$ by

$$T_p f(x) = \sum_{i=1}^{p+1} f(t_U(x, q_i)),$$

where $U \in \mathfrak{T}$ is a neighborhood of x (this does not depend on U).

THEOREM 8.1. *Let Φ_i be a sequence of eigenfunctions of T_p in $L^2(X) \cap C(X)$, with $\|\Phi_i\|_2 = 1$. Suppose that the probability measures $|\Phi_i|^2 \, d\,\mathrm{vol}$ converge in the weak star topology to a measure μ. Then μ is T-recurrent.*

Remark. If X is not compact, it is not necessarily true that μ is a probability measure. If μ is the trivial 0 measure, then either agree to call it T-recurrent or exclude this case from the theorem.

In [Wol01], [Lin01a] it was shown that every arithmetic quantum limit can be realized as a weak star limit of $|\Phi_i|^2 \, d\,\mathrm{vol}$ with Φ_i Hecke eigenfunctions in $L^2(X) \cap C(X)$ as above; hence the following is a direct corollary of Theorem 8.1:

COROLLARY 8.2. *Let X, p and T be as above. Then every arithmetic quantum limit on X is T-recurrent.*

If f is a function $f : T \to \mathbb{C}$, we let

$$S_p f(x) = \sum_{d_T(x,y)=1} f(y),$$

more generally, set $S_{p^k} f(x) = \sum_{d_T(x,y)=k} f(y)$.

The following easy estimate (very similar to the one used in[BL03]) is the heart of the proof of Theorem 8.1.

LEMMA 8.3. *If $S_T f = \lambda f$ for $f : T \to \mathbb{C}$ and $\lambda \in \mathbb{R}$, then for all $n \geq 0$,*

$$\sum_{y \in B_n^T} |f(y)|^2 \geq C_0 n \, |f(e)|^2 \,, \tag{8.1}$$

with C_0 an absolute constant that does not depend on λ or even on p.

Proof. There are two cases: $|\lambda| > 2p^{1/2}$ and $|\lambda| \leq 2p^{1/2}$. We begin with the former case. Since S_{p^k} can be expressed as a polynomial in S_p, we get that f is an eigenfunction of S_{p^k}. Let λ_{p^k} be the corresponding eigenvalue. As one may verify, e.g. by induction, if we set $\cosh \alpha = \left| \frac{\lambda}{2p^{1/2}} \right|$ then

$$\sum_{k=0}^{n} \lambda_{p^{2k}} = p^n \frac{\sinh(2n+1)\alpha}{\sinh \alpha} \geq (2n+1) p^n.$$

In other words,

$$\left| \sum_{d_T(e,y)\in\{0,2,\dots,2n\}} f(y) \right| \geq (2n+1)p^n f(e).$$

Applying Cauchy-Schwartz, we get

$$\sum_{d_T(e,y)\in\{0,2,\dots,2n\}} |f(y)|^2 \geq n^2 |f(e)|^2 .$$

We now turn to the case $|\lambda| \leq 2p^{1/2}$. We proceed similarly to the previous case: we set $\cos\theta = \frac{\lambda}{2p^{1/2}}$, and use the identity

$$\sum_{k=0}^{n} \lambda_{p^{2k}} = p^n \frac{\sin(2n+1)\theta}{\sin\theta}. \tag{8.2}$$

Subtracting (8.2) with $n = k-1$ from the same equation for $n = k$, and using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
\sum_{d(e,y)=2k} |f(y)|^2 &\geq \frac{\left|\sum_{d(e,y)=2k} f(y)\right|^2}{(p+1)p^{2k-1}} \\
&= \frac{\left|\lambda_{p^{2k}}\right|^2 |f(e)|^2}{(p+1)p^{2k-1}} \\
&\geq \frac{1}{2}\left[\frac{\sin(2k+1)\theta}{\sin\theta} - \frac{\sin(2k-1)\theta}{p\sin\theta}\right]^2 \\
&\geq c\,|f(e)|^2 \qquad \text{if } (2k+1)\theta \bmod \pi \in [2\pi/5, 3\pi/5].
\end{aligned}$$

Since it is easy to see that if $n > c_1/\theta$ then

$$\sum_{k=1}^{n} \chi_{[\frac{2\pi}{5},\frac{3\pi}{5}]}((2k+1)\theta \bmod \pi) > c_2 n,$$

we get that (8.1) holds for $n > c_1/\theta$.

On the other hand, if $n \leq c_3/\theta$ for a sufficiently small absolute constant c_3 one has that $\frac{\sin(2n+1)\theta}{\sin\theta} \geq n$; so by (8.2) we have that for such n

$$\left|\sum_{k=0}^{n} \lambda_{p^{2k}}\right| \geq np^n$$

and so

$$\sum_{y\in B_{2n}^T} |f(y)|^2 \geq cn^2 |f(e)|^2 \geq cn\,|f(e)|^2 .$$

By suitably choosing C_0 in (8.1) the bounds we obtained for $n > c_1/\theta$ and $n < c_3/\theta$ suffice to prove this equation in all cases. □

Fix some left invariant metric on $\mathbb{G}(\mathbb{R})$; since it is left invariant, it gives rise to a well-defined metric $d_X(\cdot,\cdot)$ on X. Define the injectivity radius r_{inj} as

$$r_{\text{inj}} = \min\left\{d_{\mathbb{G}(\mathbb{R})}(g_1,g_2) : g_1,g_2 \in \mathbb{G}(\mathbb{R}) \text{ with } \pi_p(g_1)=\pi_p(g_2)\right\}.$$

COROLLARY 8.4. *Let μ be a measure on X as in Theorem* 8.1. *Let $n\in\mathbb{N}$ and $x\in V\in\mathfrak{T}$ be arbitrary, and take $0<r<r_{inj}/3$ so that $B_r^X(x)\subset V$. Then*

$$\sum_{y\in t(x,B_n^T)} \mu(\overline{B_r^X(y)}) \ge C_0 n\mu(B_r^X(x)). \tag{8.3}$$

Remark. The restriction $B_r^X(x)\subset V$ is not essential. It is used merely to simplify notation, and is not really a limitation since we will only be interested in small balls.

Proof. X is a T-space with the additional nice property that $t_V(\cdot,q): V\to X$ is an isometry for every $V\in\mathfrak{T}$ and $q\in T$. This in particular implies that for any $y=t_V(x,q)$ and any $f\in L^1(X)$,

$$\int_{B_r^X(y)} f(z)d\operatorname{vol}(z) = \int_{B_r^X(x)} f(t_V(z,q))d\operatorname{vol}(z).$$

Now let $\Phi_i\in C(X)\cap L_1^2(X)$ be an eigenfunction of the Hecke operator T_p. Let μ_i be the measure defined by $\mu_i(A)=\int_A |\Phi_i(z)|^2\,d\operatorname{vol}(z)$. Then

$$\begin{aligned}\sum_{y\in t_V(x,B_n^T)} \mu_i(B_r^X(y)) &= \sum_{y\in t_V(x,B_n^T)} \int_{B_r^X(y)} |\Phi_i(z)|^2\,d\operatorname{vol}(z)\\ &= \int_{B_r^X(x)} \sum_{q\in B_n^T} |\Phi_i(t_V(z,q))|^2\,d\operatorname{vol}(z).\end{aligned} \tag{8.4}$$

Now since Φ_i is an eigenfunction of T_p, for every $z\in V$ the map $q\mapsto \Phi_i(t_V(z,q))$ is an eigenfunction of S_p and we may apply Lemma 8.3 to get

$$(8.4) \ge C_0 n \int_{B_r^X(x)} |\Phi_i(z)|^2\,d\operatorname{vol}(z) = \mu_i(B_r^X(x)). \tag{8.5}$$

By definition, $\mu_i \xrightarrow{w} \mu$, so for any open set $U\subset X$ we have that

$$\mu(U) \le \underline{\lim}\,\mu_i(U) \le \overline{\lim}\,\mu_i(U) \le \mu(\overline{U}).$$

Applying this to (8.5) one gets (8.3). □

Proof of Theorem 8.1. Let $\epsilon>0$ be arbitrary and $n_0>(C_0\epsilon)^{-1}$.

Let $x\in X$ and r be sufficiently small so that all the balls $B_r^X(y)$ with $y\in t(x,B_n^T)$ are pairwise disjoint. Without loss of generality we may also assume that $r<r_{\text{inj}}/3$, and that there is some $V\in\mathfrak{T}$ so that $B_r^X(x)\subset V$.

Set $U=\bigcup_{y\in t(x,B_n^T)} B_r^X(y)$, and take $\mathcal{A}$ to be the measurable partition whose atoms are precisely the sets $t(y,B_n^T)$ for $y\in B_r^X(x)$. If $\mathcal{C}_1$ is a countable algebra of Borel subsets of $B_r^X(x)$ generating the sigma ring of Borel measurable subsets of $B_r^X(x)$ then

$$\mathcal{C} = \left\{ \bigcup_{V \in \mathfrak{T}} t_V(C \cap V, B_n^T) : C \in \mathcal{C}_1 \right\}$$

is a countable algebra of Borel subsets of U generating $\mathcal{A}$. Since the topology on T is the discrete topology, $\mathcal{A}$ satisfies the conditions of part (2) of Theorem 3.6: every atom of $\mathcal{A}$ is clearly an open T-plaque.

Decompose the measure $\mu|_U := \mu(\cdot \cap U)$ according to the sigma ring $\mathcal{A}$, obtaining a system of conditional measures $\mu_y^{\mathcal{A}}$ (each supported on a finite subset of U) so that for any $B \subset U$

$$\mu(B) = \int_U \mu_y^{\mathcal{A}}(B \cap [y]_{\mathcal{A}}) d\mu(y). \tag{8.6}$$

Define $a : U \to B_r^X(x)$ by

$$a(y) = [y]_{\mathcal{A}} \cap B_r^X(x) \tag{8.7}$$

(more precisely, $a(y)$ is a unique element of the set on the right-hand side of (8.7)). Set $\nu = a_*(\mu|_U)$ and for every $q \in B_n^T$ set

$$\nu_q = t_V(\cdot, q)^{-1}{}_*(\mu|_{B_r^X(t_V(x,q))}).$$

Thus ν and all ν_q are measures supported on $B_r^X(x)$ and $\nu_e = \mu|_{B_r^X(x)}$. Note also that $\nu = \sum_q \nu_q$. In particular for every $q \in B_n^T$ we have that $\nu_q \ll \nu$, and we set ρ_q to be the Radon-Nikodym derivative $\rho_q = \frac{\nu_q}{\nu}$.

Using this we can write for any $B \subset U$

$$\begin{aligned} \mu(B) &= \sum_{q \in B_n^T} [t_V(\cdot, q)_* \nu_q](B \cap B_r^X(t_V(x,q))) \\ &= \int_{B_r^X(x)} \sum_{q \in B_n^T} \rho_q(y) \chi_B(t_V(y,q)) d\nu(y). \end{aligned} \tag{8.8}$$

Comparing (8.6) with (8.8) we see that for ν-almost every y

$$\mu_x^{\mathcal{A}}(\{t_V(y,q)\}) = \rho_q(y).$$

By the theorems on differentiation of measures [Mat95]) for ν-almost every y

$$\rho_q(y) = \lim_{s \to 0} \frac{\nu_q(B_s^X(y))}{\nu(B_s^X(y))}.$$

Also note that except for a countable set of radii s, we have that $\nu_q(\partial B_s^X) = 0$. Furthermore, Lemma 3.7 implies that $\rho_e \neq 0$ almost surely. Using this and

Corollary 8.4 we see that for ν-almost every y

$$\begin{aligned}\frac{\mu_x^{\mathcal{A}}([y]_{\mathcal{A}})}{\mu_x^{\mathcal{A}}(\{y\})} &= \frac{\sum_{q\in B_n^T}\rho_q(y)}{\rho_e(y)}\\ &= \lim_{s\to 0}\frac{\sum_{q\in B_n^T}\nu_q(B_s^X(y))}{\nu_e(B_s^X(y))}\\ &= \underline{\lim}_{s\to 0}\frac{\sum_{q\in B_n^T}\nu_q(\overline{B_s^X(y)})}{\nu_e(B_s^X(y))}\\ &\geq C_0 n.\end{aligned}$$

It follows from part (2) in Theorem 3.6 that for μ-almost every y we have that

$$\mu_{x,T}^V(B_n^T)\geq C_0 n.$$

In other words, μ is T-recurrent in a rather quantitative and uniform way! □

Appendix A. A maximal ergodic theorem for noninvariant measures (joint with D. Rudolph)

The maximal ergodic theorem states that for any probability measure μ on the space X invariant under an $\mathbb{R}^d$-action $x\mapsto t_{;\mathbb{R}^d}(x,s)$, if we define for any function f on X

$$M(f)[x]=\sup_{r>0}\frac{1}{vol(B_r)}\int_{B_r}\left|f(t_{;\mathbb{R}^d}(x,s))\right|ds,$$

then for any $f\in L^1(X,\mu)$

$$\mu\{x:M(f)[x]>R\}<\frac{C_d\left\|f\right\|_1}{R},$$

with C_d a universal constant depending on d.[4]

In 1944 W. Hurewicz [Hur44] proved a version of the pointwise ergodic theorem, using a maximal ergodic theorem, valid for a general recurrent measurable $\mathbb{Z}$-action on a probability measure space. It is most often quoted today with the additional assumption that the action is measure-class-preserving; however this assumption, which was not made in the original paper, is not a natural one for the purposes of this paper.

Hurewicz also claimed to have a similar theorem for $\mathbb{R}$-actions (which is the case used in the proof of Theorem 1.1) but neither the statement nor the proof of this theorem appears to have been written.

[4]The maximal ergodic theorem is known in much greater generality for actions of general amenable groups (see [Lin01b]). We do not know if our results here can also be similarly extended.

The main result of this appendix is the following version of a maximal ergodic theorem in the non-measure-preserving setting. In what follows, we take T to be $\mathbb{R}^d$ or more generally any (locally compact, second countable) metric space with a transitive metric-preserving action on which the Besicovitch covering theorem holds (see [Mat95, Th. 2.7]). More precisely, we need some number $P(T)$ so that for any bounded subset $A \subset T$ and family of closed balls $\mathcal{B}$ so that every point of A is a center of some ball of $\mathcal{B}$ there is a finite or countable collection of balls $\overline{B}_i \in \mathcal{B}$ such that they cover A and every point of T belongs to at most $p(T)$ balls $\overline{B}_i$.

Theorem A.1. *Let T be a metric space satisfying the Besicovitch covering theorem, and let X be an* $(\mathrm{Isom}(T), T)$*-space, and $\alpha : X \to X$ be a homeomorphism that uniformly expands the T-leaves. Suppose that μ is an α invariant probability measure on X, and that for μ-almost every x its T-leaf is embedded. Define*

$$M_\mu(f)[x] = \sup_{r>0} \frac{1}{\mu_{x;T}(\overline{B}_r)} \int_{\overline{B}_r} |f(t_{;T}(x,s))| \, d\mu_{x;T}(s).$$

Then

$$\mu\{x : M_\mu(f)[x] > R\} < \frac{C_T \|f\|_1}{R},$$

with C_T a universal constant depending only on T.

The main novelty in the (proof of the) above theorem is the introduction of the Besicovitch covering theorem to this context. This allows us, in particular, to treat non-measure-preserving $\mathbb{R}^n$-actions, for which relatively little seems to have been done. We note that the assumption regarding the existence of a measure-preserving leaf expanding homeomorphism α is not needed; we have not made an effort to prove an optimal theorem (deferring this to a later paper) but a theorem sufficient for the purposes of this paper and probable generalizations.

Added in Proof. J. Feldman has brought to our attention the paper [Bec83] which has related results, and in particular contains the key idea of using the Besicovitch covering lemma to prove more general ergodic theorems.

The following lemma allows us to translate Theorem A.1 to a question about covers of T.

Lemma A.2. *Let X be as in Theorem* A.1. *For every $r, \delta > 0$ there are a subset X' and a sigma ring $\mathcal{A}$ of subsets of X' so that*

$$[x]_{\mathcal{A}} \subset B^T_\infty(x) \qquad \textit{for every } x \in X' \tag{A.1}$$

$$\mu\left\{x \in X' : B^T_r(x) \subset [x]_{\mathcal{A}}\right\} > 1 - \delta. \tag{A.2}$$

Proof. First we show that there are a subset X'' with $\mu(X'') > 1-\delta$, an $r' > 0$, and a sigma ring $\mathcal{A}'$ with $\cup\mathcal{A}' \supset X''$ so that (A.1) and (A.2) hold for $x \in X''$, $\mathcal{A}'$ and r'. This does not use α-invariance of μ.

Indeed, let $K \subset X$ be a compact set with $\mu(K) \geq 1-\delta/2$ so that every $x \in K$ has an embedded T-leaf. We use Corollary 3.5 to construct finitely many $1, T$-flowers, say $\{(\mathcal{A}_i, U_i)\}_{i=1,\dots,N}$ with centers $\{B_i\}_{i=1,\dots,N}$, so that the centers B_i cover K (see Definition 3.4). Define, for every $0 < a \leq r$

$$U_{i,a} = \left\{x : B_a^T(x) \subset U_i\right\}, \qquad \mathcal{A}_{i,a} = \{A \cap U_{i,a} : A \in \mathcal{A}\}.$$

Notice that by ♣-3 in Definition 3.4 we have $B_i \subset U_{i,r}$.

Set $r' = \delta/4N$. Since $\mu(U_i) < 1$, there must be an $r' < a(i) \leq r - r'$ so that

$$\mu(U_{i,a-r'} \setminus U_{i,a+r'}) < 2r'.$$

Now, take

$$\mathcal{A}' = \bigvee_{i=1}^{N} \mathcal{A}_{i,a(i)}$$

(i.e. $\mathcal{A}'$ is the sigma ring generated by the union $\bigcup_{i=1}^{N} \mathcal{A}_{i,a(i)}$) and set $X'' = \bigcup_{i=1}^{N}(U_{i,a(i)})$.

It is clear that for every $x \in X''$, the atom $[x]_{\mathcal{A}'} \subset B_\infty^T(x)$; so we only need estimate

$$\mu\left\{x \in X'' : B_{r'}^T(x) \not\subset [x]_{\mathcal{A}'}\right\}. \tag{A.3}$$

So when is $B_{r'}^T(x) \not\subset [x]_{\mathcal{A}'}$? Only if for some i there is an $A \in \mathcal{A}_{i,a(i)}$ so that either

- $x \in A$ but $B_{r'}^T \not\subset A$ or
- $x \in X'' \setminus A$ but $B_{r'}^T \cap A \neq \emptyset$.

In either case, $x \in U_{i,a(i)-r'} \setminus U_{i,a(i)+r'}$.

Thus we see that

$$\text{(A.3)} \leq \sum_{i=1}^{N} \mu(U_{i,a(i)-r'} \setminus U_{i,a(i)+r'}) \leq \delta/2,$$

and r', A' and X'' satisfy (A.2).

Suppose α expands the T-leaves by at least a factor of $c > 1$. Then for any $x \in \cup\mathcal{A}'$

$$\partial_{c^n r'}^T[\alpha^n x]_{\alpha^n(\mathcal{A}')} \subset \alpha^n(\partial_{r'}^T[x]_{\mathcal{A}'}).$$

Take n large enough so that $c^n r' > r$ and set $\mathcal{A} = \alpha^n(\mathcal{A}')$, $X' = \alpha^n X''$. Then (A.1) and (A.2) for $\mathcal{A}'$, X'' and r' imply the same for $\mathcal{A}$, X' and r. □

Proof of Theorem A.1. Let $Y = \{x \in X : M_\mu(f)[x] > R\}$, and for any $r > 0$ define

$$M_{\mu,r}(f)[x] = \sup_{0<\rho<r} \frac{1}{\mu_{x,T}(\bar{B}_\rho)} \int_{\bar{B}_\rho} |f(t_{;T}(x,s))| \, d\mu_{x;T}(s).$$

Let r be sufficiently large so that

$$Y' = \{x \in X : M_{\mu,r}(f)[x] > R/2\}$$

satisfies $\mu(Y') > \mu(Y)/2$. Let $\mathcal{A}$ and X' be as in Lemma A.2 for $\delta = \mu(Y)/4$, and set

$$Y'' = Y' \cap \left\{x \in X' : B_r^T(x) \subset [x]_{\mathcal{A}}\right\},$$

so that in particular $\mu(Y'') \geq \mu(Y)/4$.

Choose $x \in X'$, and let $Y_x = Y'' \cap [x]_{\mathcal{A}}$. For every $y \in Y_x$ there is an $r_y < r$ so that

$$\int_{\bar{B}^T_{r_y}(y)} |f(z)| \, d\mu_x^{\mathcal{A}}(z) > R\mu_x^{\mathcal{A}}(B^T_{r_y}(y))/2.$$

Note that since $y \in Y''$ and $r_y < r$ we have $B^T_{r_y}(y) \subset [x]_{\mathcal{A}}$. Find, using the Besicovitch covering theorem, a countable subcollection $\mathcal{F} = \left\{\bar{B}^T_{r_i}(y_i)\right\}$ of the collection $\left\{\bar{B}^T_{r_y}(y) : y \in Y_x\right\}$ so that $Y_x \subset \cup\mathcal{F}$ but no point in $[x]_{\mathcal{A}}$ is contained in more than $P(T)$ balls from the collection $\mathcal{F}$. Then

$$\begin{aligned}\int |f(y)| \, d\mu_x^{\mathcal{A}}(y) &\geq P(T)^{-1} \sum_{B \in \mathcal{F}} \int_B |f(y)| \, d\mu_x^{\mathcal{A}}(y) \\ &\geq \frac{R}{2P(T)} \sum_{B \in \mathcal{F}} \mu_x^{\mathcal{A}}(B) \\ &\geq \frac{R}{2P(T)} \mu_x^{\mathcal{A}}(Y_x).\end{aligned}$$

We now integrate over $x \in X'$ to get

$$\begin{aligned}\int_{X'} |f(y)| \, d\mu(y) &= \int_{X'} \int |f(y)| \, d\mu_x^{\mathcal{A}}(y) d\mu(x) \\ &\geq \frac{R}{2P(T)} \int_{X'} \int \mu_x^{\mathcal{A}}(Y_x) d\mu(x) \\ &= \frac{R}{2P(T)} \mu(Y''),\end{aligned}$$

and so we indeed get the maximal inequality

$$\mu(Y) \leq \frac{8P(T)\,\|f\|_1}{R}. \qquad \square$$

Princeton University, Princeton, NJ
E-mail address: elonl@math.princeton.edu

References

[Aar97] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, volume 50 of *Mathematical Surveys and Monographs*, A.M.S., Providence, RI, 1997.

[Bec83] M. Becker, A ratio ergodic theorem for groups of measure-preserving transformations, *Illinois J. Math.* **27** (1983), 562–570.

[BL03] J. Bourgain and E. Lindenstrauss, Entropy of quantum limits, *Comm. Math. Phys.* **233** (2003), 153–171.

[BL04] ———, Corrections and additions to "Entropy of quantum limits": http://www.math.princeton.edu/~elonl/Publications/index.html.

[CdV85] Y. Colin de Verdière, Ergodicité et fonctions propres du laplacien. *Comm. Math. Phys.* **102** (1985), 497–502.

[EK03] M. Einsiedler and A. Katok, Invariant measures on G/Γ for split simple Lie groups G, Dedicated to the memory of Jürgen K. Moser, *Comm. Pure Appl. Math.* **56** (2003), 1184–1221.

[EKL06] M. Einsiedler, A. Katok, and E. Lindenstrauss, Invariant measures and the set of exceptions to Littlewood's conjecture, *Ann. of Math.* **164** (2006), to appear.

[EL03] M. Einsiedler and E. Lindenstrauss, Rigidity properties of Z^d-actions on tori and solenoids, *Electron. Res. Announc. Amer. Math. Soc.* **9** (2003), 99–110.

[Fed69] H. Federer, *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag Inc., New York, 1969.

[Hal50] Paul R. Halmos, *Measure Theory*, D. Van Nostrand Company, Inc., New York, N.Y., 1950.

[Hos95] B. Host, Nombres normaux, entropie, translations, *Israel J. Math.* **91** (1995), 419–428.

[Hos00] ———, Some results of uniform distribution in the multidimensional torus, *Ergodic Theory Dynam. Systems* **20** (2000), 439–452.

[Hu93] H.-Y. Hu, Some ergodic properties of commuting diffeomorphisms, *Ergodic Theory Dynam. Systems* **13** (1993), 73–100.

[Hur44] W. Hurewicz, Ergodic theorem without invariant measure, *Ann. of Math.* **45** (1944), 192–206.

[Joh92] A. S. A. Johnson, Measures on the circle invariant under multiplication by a nonlacunary subsemigroup of the integers, *Israel J. Math.* **77** (1992), 211–240.

[KH95] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems* (With a supplementary chapter by A. Katok and L. Mendoza) **54**, *Encyclopedia of Mathematics and its Applications*, Cambridge Univ. Press, Cambridge, 1995.

[KS96] A. Katok and R. J. Spatzier, Invariant measures for higher-rank hyperbolic abelian actions, *Ergodic Theory Dynam. Systems* **16** (1996), 751–778.

[Lin01a] E. Lindenstrauss, On quantum unique ergodicity for $\Gamma\backslash\mathbb{H}\times\mathbb{H}$, *Internat. Math. Res. Notices* **2001**, no. 17, 913–933.

[Lin01b] ———, Pointwise theorems for amenable groups, *Invent. Math.* **146** (2001), 259–295.

[Lin03] E. Lindenstrauss, Rigidity of multiparameter actions, preprint (26 pages), 2003.

[Lin04] ———, Arithmetic quantum unique ergodicity and adelic dynamics, in preparation, 2004.

[LL03] F. Ledrappier and E. Lindenstrauss, On the projections of measures invariant under the geodesic flow, *Internat. Math. Res. Notices* **2003**, no. 9, 511–526.

[LMP99] E. Lindenstrauss, D. Meiri, and Yu. Peres, Entropy of convolutions on the circle, *Ann. of Math.* **149** (1999), 871–904.

[LS04] E. Lindenstrauss and K. Schmidt, Invariant sets and measures of nonexpansive group automorphisms, *Israel J. Math* **144** (2004), 29–60.

[LW01] E. Lindenstrauss and B. Weiss, On sets invariant under the action of the diagonal group, *Ergodic Theory Dynam. Systems* **21** (2001), 1481–1500.

[LY85] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula, *Ann. of Math.* **122** (1985), 509–539.

[Lyo88] R. Lyons, On measures simultaneously 2- and 3-invariant, *Israel J. Math.* **61** (1988), 219–224.

[Mat95] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, in *Cambridge Studies in Advanced Math.* **44**, Cambridge Univ. Press, Cambridge, 1995.

[Mor05] D. W. Morris, *Ratner's Theorems on Unipotent Flows*, *Chicago Lectures in Mathematics Series*, University of Chicago Press, Chicago, IL, 2005.

[MT94] G. A. Margulis and G. M. Tomanov, Invariant measures for actions of unipotent groups over local fields on homogeneous spaces, *Invent. Math.* **116** (1994), 347–392.

[Ra82] M. Ratner, Factors of horocycle flows, *Ergodic Theory Dynam. Systems* **2** (1982), 465–489.

[Ra83] ———, Horocycle flows, joinings and rigidity of products, *Ann. of Math.* **118** (1983), 277–313.

[Ra90a] ———, On measure rigidity of unipotent subgroups of semisimple groups, *Acta Math.* **165** (1990), 229–309.

[Ra90b] ———, Strict measure rigidity for unipotent subgroups of solvable groups, *Invent. Math.* **101** (1990), 449–482.

[Ra91] ———, On Raghunathan's measure conjecture, *Ann. of Math.* **134** (1991), 545–607.

[Ra92] ———, Raghunathan's conjectures for $\mathrm{SL}(2,\mathbb{R})$, *Israel J. Math.* **80** (1992), 1–31.

[Ra95] ———, Raghunathan's conjectures for Cartesian products of real and p-adic Lie groups, *Duke Math. J.* **77** (1995), 275–382.

[Ra98] ———, On the p-adic and S-arithmetic generalizations of Raghunathan's conjectures, in *Lie Groups and Ergodic Theory* (Mumbai, 1996) **14**, *Tata Inst. Fund. Res. Stud. Math.*, 167–202, Tata Inst. Fund. Res., Bombay, 1998.

[RS94] Z. Rudnick and P. Sarnak, The behaviour of eigenstates of arithmetic hyperbolic manifolds, *Comm. Math. Phys.* **161** (1994), 195–213.

[Rud82] D. J. Rudolph, Ergodic behaviour of Sullivan's geometric measure on a geometrically finite hyperbolic manifold, *Ergodic Theory Dynam. Systems* **2** (1982), 491–512.

[Rud90a] ———, *Fundamentals of Measurable Dynamics*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1990.

[Rud90b] ———, $\times 2$ and $\times 3$ invariant measures and entropy, *Ergodic Theory Dynam. Systems* **10** (1990), 395–406.

[Ser85] C. Series, The modular surface and continued fractions, *J. London Math. Soc.* **31** (1985), 69–80.

[Šni74] A. I. Šnirel′man, Ergodic properties of eigenfunctions, *Uspehi Mat. Nauk* **29** (1974), 181–182.

[Thu97] W. P. Thurston, *Three-dimensional Geometry and Topology. Vol.* 1 (Silvio Levy, ed.), *Princeton Math. Series* **35**, Princeton University Press, Princeton, NJ, 1997.

[Wat01] T. Watson, Rankin triple products and quantum chaos, Ph.D. thesis, Princeton University, 2001.

[Wei67] A. Weil, *Basic Number Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 144, Springer-Verlag, New York, 1967.

[Wol01] S. A. Wolpert, The modulus of continuity for $\Gamma_0(m)\backslash\mathbb{H}$ semi-classical limits, *Comm. Math. Phys.* **216** (2001), 313–323.

[Zel87] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces, *Duke Math. J.* **55** (1987), 919–941.

(Received March 29, 2003)