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Universal bounds for hyperbolic Dehn surgery

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Abstract

This paper gives a quantitative version of Thurston's hyperbolic Dehn surgery theorem. Applications include the first universal bounds on the number of nonhyperbolic Dehn fillings on a cusped hyperbolic 3-manifold, and estimates on the changes in volume and core geodesic length during hyperbolic Dehn filling. The proofs involve the construction of a family of hyperbolic cone-manifold structures, using infinitesimal harmonic deformations and analysis of geometric limits.

1. Introduction

If X is a noncompact, finite volume, orientable, hyperbolic 3-manifold, it is the interior of a compact 3-manifold with a finite number of torus boundary components. For each torus, there are an infinite number of topologically distinct ways to attach a solid torus. Such "Dehn fillings" are parametrized by pairs of relatively prime integers, once a basis for the fundamental group of the torus is chosen. If each torus is filled, the resulting manifold is closed. A fundamental theorem of Thurston ([43]) states that, for all but a finite number of Dehn surgeries on each boundary component, the resulting closed 3-manifold has a hyperbolic structure. However, it was unknown whether or not the number of such nonhyperbolic surgeries was bounded independent of the original noncompact hyperbolic manifold.

In this paper we obtain a universal upper bound on the number of nonhyperbolic Dehn surgeries per boundary torus, independent of the manifold X. There are at most 60 nonhyperbolic Dehn surgeries if there is only one cusp; if there are multiple cusps, at most 114 surgery curves must be excluded from each boundary torus.

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These results should be compared with the known bounds on the number of Dehn surgeries which yield manifolds which fail to be either irreducible or atoroidal or fail to have infinite fundamental group. These are all necessary conditions for a 3-manifold to be hyperbolic. The hyperbolic geometry part of Thurston's geometrization conjecture states that these conditions should also be sufficient; i.e., that the interior of a compact, orientable 3-manifold has a complete hyperbolic structure if and only if it is irreducible, atoroidal, and has infinite fundamental group.

It follows from the work of Gromov-Thurston ([26], see also [5]) that all but a universal number of surgeries on each torus yield 3-manifolds which admit negatively curved metrics. More recent work by Lackenby [33] and, independently, by Agol [2], similarly shows that for all but a universally bounded number of surgeries on each torus the resulting manifolds are irreducible with infinite word hyperbolic fundamental group. Similar types of bounds using techniques less comparable to those in this paper have been obtained by Gordon, Luecke, Wu, Culler, Shalen, Boyer, Zhang and many others. (See, for example, [13], [7] and the survey articles [21], [22].) Negatively curved 3-manifolds are irreducible, atoroidal and have infinite fundamental groups. If the geometrization conjecture were known to be true, it would imply that these manifolds actually have hyperbolic metrics. The same is true for irreducible 3-manifolds with infinite word hyperbolic fundamental group. Thus, the above results would provide a universal bound on the number of nonhyperbolic Dehn fillings. However, without first establishing the geometrization conjecture, no such conclusion is possible and other methods are required.

The bound on the number of Dehn surgeries that fail to be negatively curved comes from what is usually referred to as the "2π-theorem". It can be stated as follows: Given a cusp in a complete, orientable hyperbolic 3-manifold X, remove a horoball neighborhood of the cusp, leaving a manifold with a boundary torus which has a flat metric. Let γ be an isotopy class of simple closed curve on this torus and let $X(\gamma)$ denote X filled in so that γ bounds a disk. Then the 2π-theorem states that, if the flat geodesic length of γ on the torus is greater than 2π, then $X(\gamma)$ can be given a metric of negative curvature which agrees with the hyperbolic metric in the region outside the horoball. The bound then follows from the fact that it is always possible to find an embedded horoball neighborhood with boundary torus whose shortest geodesic has length at least 1. On such a torus there are a bounded number of isotopy classes of geodesics with length less than or equal to 2π.

Similarly, Lackenby and Agol show that, if the flat geodesic length is greater than 6, then the Dehn filled manifold is irreducible with infinite word hyperbolic fundamental group. Agol then uses the recent work of Cao-Meyerhoff ([11]), which provides an improved lower bound on the area of the maximal embedded horotorus, to conclude that, when there is a single cusp, at most 12 surgeries fail to be irreducible or infinite word hyperbolic. This is remarkably close to the the largest known number of nonhyperbolic Dehn surgeries which is 10, occurring for the complement of the figure-8 knot.

Our criterion for those surgery curves whose corresponding filled manifold is guaranteed to be hyperbolic is similar. We consider the *normalized length* of curves on the torus, measured after rescaling the metric on the torus to have area 1, i.e. normalized length = (geodesic length)/$\sqrt{\text{torus area}}$. Our main result shows that, if the normalized length of γ on the torus is sufficiently long, then it is possible to deform the complete hyperbolic structure through cone-manifold structures on $X(\gamma)$ with γ bounding a singular meridian disk until the cone angle reaches 2π. This gives a smooth hyperbolic structure on $X(\gamma)$. The important point here is that "sufficiently long" is universal, independent of X. As before, it is straightforward to show that all but a universal number of isotopy classes of simple closed curves satisfy this normalized length condition.

The condition in this case that the normalized length, rather than just the flat geodesic length, be long is probably not necessary, but is an artifact of the proof.

We will now give a rough outline of the proof.

We begin with a noncompact, finite volume hyperbolic 3-manifold X, which, for simplicity, we assume has a single cusp. In the general case the cusps are handled independently. The manifold X is the interior of a compact manifold which has a single torus boundary. Choose a simple closed curve γ on the torus. We wish to put a hyperbolic structure on the closed manifold $X(\gamma)$ obtained by Dehn filling. The metric on the open manifold X is deformed through incomplete metrics whose metric completion is a singular metric on $X(\gamma)$, called a cone metric. (See [28] for a detailed description of such metrics.) The singular set is a simple closed geodesic at the core of the added solid torus. For any plane orthogonal to this geodesic the disks of small radius around the intersection with the geodesic have the metric of a 2-dimensional hyperbolic cone with angle α. The angle α is the same at every point along the singular geodesic Σ and is called the cone angle at Σ. The complete structure can be considered as a cone-manifold with angle 0. The cone angle is increased monotonically, and, if the angle of 2π is reached, this defines a smooth hyperbolic metric on $X(\gamma)$.

The theory developed in [28] shows that it is always possible to change the cone angle a small amount, either increase it or decrease it. Furthermore, this can be done in a unique way, at least locally. The cone angles locally parametrize the set of cone-manifold structures on $X(\gamma)$. In particular, there are no variations of the hyperbolic metric which leave the cone angle fixed. This property is referred to as *local rigidity rel cone angles*. Thus, to choose a 1-parameter family of cone angles is to choose a well-defined family of singular hyperbolic metrics on $X(\gamma)$ of this type.

Although there are always local variations of the cone-manifold structure, the structure may degenerate in various ways as a family of angles reaches a limit. In order to find a smooth hyperbolic metric on $X(\gamma)$ it is necessary to show that no degeneration occurs before the angle 2π is attained.

The proof has two main parts, involving rather different types of arguments. One part is fairly analytic, showing that under the normalized length hypothesis on γ, there is a lower bound to the tube radius for any of the cone-manifold structures on $X(\gamma)$ with angle at most 2π. The second part consists of showing that, under certain geometric conditions, most importantly the lower bound on the tube radius, no degeneration of the hyperbolic structure is possible. This involves studying possible geometric limits where the tube radius condition restricts such limits to fairly tractable and well-understood types.

The argument showing that there is a lower bound to the tube radius is based on the local rigidity theory for cone-manifolds developed in [28]. Indeed, the key estimates are best viewed as effective versions of local rigidity of cone-manifolds. We choose a smooth parametrization of the increasing family of cone angles, which uniquely determines a family of cone-manifold structures. We then need to control the global behavior of these metrics. The idea is first to form a model for the deformation in a neighborhood of the singular locus which changes the cone angle in the prescribed fashion and then find estimates which bound the deviation of the actual deformation from the model.

The main goal is to estimate the actual behavior of the holonomy of the fundamental group elements corresponding to the boundary torus. The holonomy representation of the meridian is simply an elliptic element which rotates by the cone angle so it suffices to understand the longitudinal holonomy. We derive some estimates on the complex length of the longitude in terms of the cone angle which depend on the original geometry of the horospherical torus, including the length of the meridian on the torus. These results may be of independent interest.

The estimates are derived by analyzing boundary terms in a Weitzenböck formula for the infinitesimal change of metric which arises from differentiating our family of cone metrics. This formula is the basis for local rigidity of hyperbolic metrics in dimensions 3 and higher ([9], [46]) and of hyperbolic cone-manifolds in dimension 3 ([28]). Our estimates ultimately provide a bound on the derivative of the ratio of the cone angle to the hyperbolic length of the singular core curve of the cone-manifold. The bound depends on the tube radius. On the other hand, a geometric packing argument shows that the change in the tube radius can be controlled when the product of the cone angle and the core length is small.

Putting these results together, we arrive at differential inequalities which provide strong control on the change in the geometry of the maximal tube around the singular geodesic, including the tube radius. The value of the normalized flat length of the surgery curve on the maximal cusp torus for the complete structure gives the initial condition for the ratio of the cone angle to the core length. (Note: The *ratio* of the cone angle to the core length approaches a finite, nonzero value even though they individually approach zero at the complete structure.)

The conclusion is that, if the initial value of the ratio is large, then it will remain large and the product of the cone angle and the core length will remain small. The packing argument then shows that there will be a lower bound to the tube radius.

This gives a proof of the following theorem:

THEOREM 1.1. *Let X be a complete, finite volume, orientable, hyperbolic 3-manifold with one cusp and let T be a horospherical torus which is embedded as a cross-section to the cusp. Let γ be a simple closed curve on T and $X(\gamma)$ be the Dehn filling with γ as meridian. Let $X_\alpha(\gamma)$ be a cone-manifold structure on $X(\gamma)$ with cone angle α along the core, Σ, of the added solid torus, obtained by increasing the angle from the complete structure. If the normalized length of γ on T is at least 7.515, then there is a positive lower bound to the tube radius around Σ for all $2\pi \geq \alpha \geq 0$.*

This theorem does not guarantee that cone angle 2π can actually be reached, just that there is a lower bound to the tube radius over all angles less than or equal to 2π that *are* attained. That 2π can actually be attained follows from the next theorem.

THEOREM 1.2. *Let M_t, $t \in [0, t_\infty)$, be a smooth path of closed hyperbolic cone-manifold structures on (M, Σ) with cone angle α_t along the singular locus Σ. Suppose $\alpha_t \to \alpha \geq 0$ as $t \to t_\infty$, that the volumes of the M_t are bounded above by V_0, and that there is a positive constant R_0 such that there is an embedded tube of radius at least R_0 around Σ for all t. Then the path extends continuously to $t = t_\infty$ so that as $t \to t_\infty$, M_t converges in the bilipschitz topology to a cone-manifold structure M_∞ on M with cone angles α along Σ.*

Given X and T as in Theorem 1.1, choose *any* nontrivial simple closed curve γ on T. There is a maximal sub-interval $J \subset [0, 2\pi]$ containing 0 such that there is a smooth family M_α, where $\alpha \in J$, of hyperbolic cone-manifold structures on $X(\gamma)$ with cone angle α. Thurston's Dehn surgery theorem ([43]) implies that J is nonempty and [28, Theorem 4.8] implies that it is open. Theorem 1.2 implies that, with a lower bound on the tube radii and an upper bound on the volume, the path of M_α's can be extended continuously to the endpoint of J. Again, [28, Theorem 4.8] implies that this extension can be made to be smooth. Hence, under these conditions J will be closed. By Schläfli's formula (23, Section 2) the volumes decrease as the cone angles increase, so that they will clearly be bounded above. Theorem 1.1 provides initial conditions on γ which guarantee that there will be a lower bound on the tube radii for all $\alpha \in J$. Thus, assuming Theorems 1.1 and 1.2, we have proved:

THEOREM 1.3. *Let X be a complete, finite volume, orientable, hyperbolic 3-manifold with one cusp, and let T be a horospherical torus which is embedded as a cross-section to the cusp of X. Let γ be a simple closed curve on T whose Euclidean geodesic length on T is denoted by L. If the normalized length of γ, $\hat{L} = \dfrac{L}{\sqrt{\mathrm{area}(T)}}$, is at least 7.515, then the closed manifold $X(\gamma)$ obtained by Dehn filling along γ is hyperbolic.*

This result also gives a universal bound on the number of nonhyperbolic Dehn fillings on a cusped hyperbolic 3-manifold X, independent of X.

COROLLARY 1.4. *Let X be a complete, orientable, hyperbolic 3-manifold with one cusp. Then at most 60 Dehn fillings on X yield manifolds which admit no complete hyperbolic metric.*

When there are multiple cusps the results (Theorem 5.12) are only slightly weaker. Theorem 1.2 holds without change. If there are k cusps, the cone angles α_t and α are simply interpreted as k-tuples of angles. Having tube radius at least R is interpreted as meaning that there are disjoint, embedded tubes of radius R around all components of the singular locus. The conclusion of Theorem 1.1 and hence of Theorem 1.3 holds when there are multiple cusps as long as the normalized lengths of all the meridian curves are at least $\sqrt{2}\,7.515 \approx 10.6273$. At most 114 curves from each cusp need to be excluded. In fact, this can be refined to say that at most 60 curves need to be excluded from one cusp and at most 114 excluded from the remaining cusps. The rest of the Dehn filled manifolds are hyperbolic.

In the final section of the paper (Section 6), we prove that every closed hyperbolic 3-manifold with a sufficiently short (length less than .111) closed geodesic can be obtained by the process studied in this paper. Specifically, if one removes a simple closed geodesic from a closed hyperbolic 3-manifold, the resulting manifold can be seen to have a complete, finite volume hyperbolic structure. We prove that, if the removed geodesic had length less than .111, then the hyperbolic structure on the closed manifold and that of the complement of the geodesic can be connected by a smooth family of hyperbolic cone-manifolds, with angles varying monotonically from 2π to 0.

Also in that section (Theorem 6.5), we prove inequalities bounding the difference between the volume of a complete hyperbolic 3-manifold and certain closed hyperbolic 3-manifolds obtained from it by Dehn filling. We see (Corollary 6.7) that, for the manifolds constructed in Theorem 1.3, this difference is at most 0.329. Similarly, using known bounds on the volume of cusped hyperbolic 3-manifolds, we prove (Corollary 6.8) that every closed 3-manifold with a closed geodesic of length at most 0.162 has volume at least 1.701.

This paper is organized as follows: In Section 2 we recall basic definitions for deformations of hyperbolic structures and some necessary results from a previous paper ([28]). We use these to derive our fundamental inequality (Theorem 2.7) for the variation of the length of the singular locus as the cone angle is changed. Section 3 analyzes the limiting behavior of sequences of hyperbolic cone-manifolds under the hypothesis of a lower bound to the tube radius around the singular locus. The proof of Theorem 1.2 is given in that section. It is, for the most part, independent of the rest of the paper. In Section 4 we use a packing argument to relate the tube radius to the length of the singular locus. In Section 5 we combine this relation with the inequality from Section 2 to derive initial conditions that ensure that there will be a lower bound to the tube radius for all cone angles between 0 and 2π. In particular, the proof of Theorem 1.1 is completed in that section.

2. Deformation models and changes in holonomy

In this section we recall the description of an infinitesimal change of hyperbolic structure in terms of bundle-valued 1-forms and the Weitzenböck formula satisfied by such a form when it is harmonic in a suitable sense. We compute the boundary term for this formula in some specific cases which will allow us to estimate the infinitesimal changes in the holonomy representations of peripheral elements of the fundamental group.

In order to discuss the analytic and geometric objects associated to an infinitesimal deformation of a hyperbolic structure, we need first to describe what we mean by a 1-parameter family of hyperbolic structures.

A hyperbolic structure on an n-manifold X is determined by local charts modelled on $\mathbb{H}^n$ whose overlap maps are restrictions of global isometries of $\mathbb{H}^n$. These determine, via analytic continuation, a map $\Phi : \tilde{X} \to \mathbb{H}^n$ from the universal cover $\tilde{X}$ of X to $\mathbb{H}^n$, called the *developing map*, which is determined uniquely up to post-multiplication by an element of $G = \text{isom}(\mathbb{H}^n)$. The developing map satisfies the equivariance property $\Phi(\gamma m) = \rho(\gamma)\Phi(m)$, for all $m \in \tilde{X}$, $\gamma \in \pi_1(X)$, where $\pi_1(X)$ acts on $\tilde{X}$ by covering transformations, and $\rho : \pi_1(X) \to G$ is the *holonomy representation* of the structure. The developing map also determines the hyperbolic metric on $\tilde{X}$ by pulling back the hyperbolic metric on $\mathbb{H}^n$. (See [44] and [42] for a complete discussion of these ideas.)

We say that two hyperbolic structures are *equivalent* if there is a diffeomorphism f, isotopic to the identity, from X to itself taking one structure to the other. We will use the term "hyperbolic structure" to mean such an equivalence class. A 1-*parameter family*, X_t, of hyperbolic structures defines a 1-parameter family of developing maps $\Phi_t : \tilde{X} \to \mathbb{H}^n$, where two families are equivalent under the relation $\Phi_t \equiv k_t \Phi_t \tilde{f}_t$ where k_t are isometries of $\mathbb{H}^n$ and $\tilde{f}_t$ are lifts of diffeomorphisms f_t from X to itself. We assume that k_0 and $\tilde{f}_0$ are the identity, and write $\Phi_0 = \Phi$. All of the maps here are assumed to be smooth and to vary smoothly with respect to t.

The tangent vector to a smooth family of hyperbolic structures will be called an *infinitesimal deformation*. The derivative at $t = 0$ of a 1-parameter family of developing maps $\Phi_t : \tilde{X} \to \mathbb{H}^n$ defines a map $\dot{\Phi} : \tilde{X} \to T\mathbb{H}^n$. For any point $m \in \tilde{X}$, $\Phi_t(m)$ is a curve in $\mathbb{H}^n$ describing how the image of m is moving under the developing maps; $\dot{\Phi}(m)$ is the initial tangent vector to the curve.

We will identify $\tilde{X}$ locally with $\mathbb{H}^n$ and $T\tilde{X}$ locally with $T\mathbb{H}^n$ via the initial developing map Φ. Note that this identification is generally not a homeomorphism unless the hyperbolic structure is complete. However, it is a local diffeomorphism, providing identification of small open sets in $\tilde{X}$ with ones in $\mathbb{H}^n$.

In particular, each point $m \in \tilde{X}$ has a neighborhood U where $\Psi_t = \Phi^{-1} \circ \Phi_t : U \to \tilde{X}$ is defined, and the derivative at $t = 0$ defines a vector field on $\tilde{X}$, $v = \dot{\Psi} : \tilde{X} \to T\tilde{X}$. This vector field determines the variation in developing maps since $\dot{\Phi} = d\Phi \circ v$, and also determines the variation in the metric as follows. Let g_t be the hyperbolic metric on $\tilde{X}$ obtained by pulling back the hyperbolic metric on $\mathbb{H}^n$ via Φ_t and put $g_0 = g$. Then $g_t = \Psi_t^* g$ and the variation in metrics $\dot{g} = \frac{dg_t}{dt}|_{t=0}$ is the Lie derivative, $\mathcal{L}_v g$, of the initial metric g along v.

Covariant differentiation of the vector field v gives a $T\tilde{X}$ valued 1-form on $\tilde{X}$, $\nabla v : T\tilde{X} \to T\tilde{X}$, defined by $\nabla v(x) = \nabla_x v$ for $x \in T\tilde{X}$. We can decompose ∇v at each point into a symmetric part and a skew-symmetric part. The *symmetric part*, $\tilde{\eta} = (\nabla v)_{\mathrm{sym}}$, represents the infinitesimal change in metric, since

$$\dot{g}(x, y) = \mathcal{L}_v g(x, y) = g(\nabla_x v, y) + g(x, \nabla_y v) = 2g(\tilde{\eta}(x), y)$$

for $x, y \in T\tilde{X}$. In particular, $\tilde{\eta}$ descends to a well-defined TX-valued 1-form η on X. The *skew-symmetric* part $(\nabla v)_{\mathrm{skew}}$ is the *curl* of the vector field v, and its value at $m \in \tilde{X}$ represents the effect of an infinitesimal rotation about m.

To connect this discussion of infinitesimal deformations with cohomology theory, we consider the Lie algebra $\mathfrak{g}$ of $G = \mathrm{isom}(\mathbb{H}^n)$ as vector fields on $\mathbb{H}^n$ representing infinitesimal isometries of $\mathbb{H}^n$. Pulling back these vector fields via the initial developing map Φ gives locally defined infinitesimal isometries on $\tilde{X}$ and on X.

Let $\tilde{E}, E$ denote the vector bundles over $\tilde{X}, X$ respectively of (germs of) infinitesimal isometries. Then we can regard $\tilde{E}$ as the product bundle with total space $\tilde{X} \times \mathfrak{g}$, and E as isomorphic to $(\tilde{X} \times \mathfrak{g})/\!\sim$ where $(m, v) \sim (\gamma m, \mathrm{Ad}\rho(\gamma) \cdot v)$

with $\gamma \in \pi_1(X)$ acting on $\tilde{X}$ by covering transformations and on $\mathfrak{g}$ by the adjoint action of the holonomy $\rho(\gamma)$. At each point p of $\tilde{X}$, the fiber of $\tilde{E}$ splits as a direct sum of infinitesimal pure translations and infinitesimal pure rotations about p; these can be identified with $T_p\tilde{X}$ and $\mathrm{so}(n)$ respectively.

We now lift v to a section $s : \tilde{X} \to \tilde{E}$ by choosing an "osculating" infinitesimal isometry $s(m)$ which best approximates the vector field v at each point $m \in \tilde{X}$. Thus $s(m)$ is the unique infinitesimal isometry whose translational part and rotational part at m agree with the values of v and *curl* v at m. (This is the "canonical lift" as defined in [28].) In particular, if v is itself an infinitesimal isometry of $\tilde{X}$ then s will be a constant function.

By the equivariance property of the developing maps it follows that s satisfies an "automorphic" property: $s(\gamma m) - \mathrm{Ad}\rho(\gamma)s(m)$ is a *constant* infinitesimal isometry, given by the variation $\dot{\rho}(\gamma)$ of holonomy isometries $\rho_t(\gamma) \in G$ (see Prop. 2.3(a) of [28]). Here $\dot{\rho} : \pi_1(X) \to \mathfrak{g}$ satisfies the cocyle condition $\dot{\rho}(\gamma_1\gamma_2) = \dot{\rho}(\gamma_1) + \mathrm{Ad}\rho(\gamma_1)\dot{\rho}(\gamma_2)$, and so represents a class in group cohomology $[\dot{\rho}] \in H^1(\pi_1(X); \mathrm{Ad}\rho)$, describing the variation of holonomy representations ρ_t.

When s is a vector-valued function with values in the vector space $\mathfrak{g}$, its differential $\tilde{\omega} = ds$ satisfies $\tilde{\omega}(\gamma m) = \mathrm{Ad}\rho(\gamma)\tilde{\omega}(m)$ so it descends to a closed 1-form ω on X with values in the bundle E. Hence it determines a de Rham cohomology class $[\omega] \in H^1(X; E)$. This agrees with the cohomology class $[\dot{\rho}]$ under the de Rham isomorphism $H^1(X; E) \cong H^1(\pi_1(X); \mathrm{Ad}\rho)$. Also, we note that the translational part of ω can be regarded as a TX-valued 1-form on X. This is exactly the form η defined above (see Prop. 2.3(b) of [28]), describing the infinitesimal change in metric on X.

On the other hand, a family of hyperbolic structures determines only an equivalence class of families of developing maps and we need to see how replacing one family by an equivalent family changes the cocycles. Recall that a family equivalent to Φ_t is of the form $k_t\Phi_t\tilde{f}_t$ where k_t are isometries of $\mathbb{H}^n$ and $\tilde{f}_t$ are lifts of diffeomorphisms f_t from X to itself. We assume that k_0 and $\tilde{f}_0$ are the identity.

The k_t term changes the path ρ_t of holonomy representations by conjugating by k_t. Infinitesimally, this changes the cocycle $\dot{\rho}$ by a coboundary in the sense of group cohomology. Thus it leaves the class in $H^1(\pi_1(X); \mathrm{Ad}\rho)$ unchanged. The diffeomorphisms f_t amount to a different map from X_0 to X_t. But f_t is isotopic to $f_0 =$ identity, so the lifts $\tilde{f}_t$ do not change the group cocycle at all. It follows that equivalent families of hyperbolic structures determine the same group cohomology class.

If, instead, we view the infinitesimal deformation as represented by the E-valued 1-form ω, we note that the infinitesimal effect of the isometries k_t is to add a constant to $s : \tilde{X} \to \tilde{E}$. Thus, ds, its projection ω, and the infinitesimal variation of metric are all unchanged. However, the infinitesimal effect of the $\tilde{f}_t$ is to change the vector field on $\tilde{X}$ by the lift of a globally defined vector field on X. This changes ω by the derivative of a *globally defined* section of E. Hence, it does not affect the de Rham cohomology class in $H^1(X;E)$. The corresponding infinitesimal change of metric is altered by the Lie derivative of a globally defined vector field on X.

Since, within an equivalence class of an infinitesimal deformation, we are free to choose an identification of X_0 with X_t, we can try to find a canonical choice with particularly nice analytic properties. A natural choice would be a harmonic map. At the infinitesimal level, this corresponds to choosing a Hodge representative for the de Rham cohomology class in $H^1(X;E)$. The translational part, which describes the infinitesimal change in metric, is a *harmonic* TX-valued 1-form. These are studied in detail for the case of cone-manifolds in [28]. They correspond to variations of metric which are L^2-orthogonal to the trivial variations given by the Lie derivative of compactly supported vector fields on X.

One special feature of the 3-dimensional case is the *complex structure* on the Lie algebra $\mathfrak{g} \cong sl_2\mathbb{C}$ of infinitesimal isometries of $\mathbb{H}^3$. The infinitesimal rotations fixing a point $p \in \mathbb{H}^3$ can be identified with $\mathrm{su}(2) \cong \mathrm{so}(3)$; then the infinitesimal pure translations at p correspond to $i\,\mathrm{su}(2) \cong T_p\mathbb{H}^3$. Geometrically, if $t \in T_p\mathbb{H}^3$ represents an infinitesimal translation, then it represents an infinitesimal rotation with axis in the direction of t. Thus, on a hyperbolic 3-manifold X we can identify the bundle E of (germs of) infinitesimal isometries with the *complexified* tangent bundle $TX \otimes \mathbb{C}$.

We now specialize to the case of interest in this paper, 3-dimensional hyperbolic cone-manifolds. We recall some of the results and computations derived in [28]. The reader is referred to that paper for further details.

Let M_t be a smooth family of hyperbolic cone-manifold structures on M with cone angles α_t along Σ, where $0 \leq \alpha_t \leq 2\pi$. Note that, locally, M_t is uniquely determined by α_t, by the local rigidity results of [28]. Let $U = U_R$ denote an embedded tube consisting of points distance at most $R = R_t$ from the singular locus Σ.

By the Hodge theorem proved in [28], the infinitesimal deformation of hyperbolic structures ("$\frac{d}{dt}(M_t)$") can be represented by a unique harmonic TX-valued 1-form η on $X = M - \Sigma$ such that

$$D^*\eta = 0, \ D^*D\eta = -\eta,$$

where D is the exterior covariant derivative on such forms and D^* is its adjoint. In addition, η and $D\eta$ are symmetric and traceless, and inside U we can write

$$\eta = \eta_0 + \eta_c$$

where η_0 is a "standard" (non-L^2) form, and η_c is a correction term with η_c, $D\eta_c$ in L^2. Further, only η_0 changes the holonomy of the meridian and longitude on the torus $T_R = \partial U_R$.

Alternatively, we can represent the infinitesimal deformation by a 1-form with values in the infinitesimal local isometries of X:

$$\omega = \eta + i *D\eta. \tag{1}$$

There is an analogous decomposition of ω in the neighborhood U as $\omega = \omega_0 + \omega_c$ where only ω_0 changes the holonomy and ω_c is in L^2.

The tubular neighborhood U of the singular locus will be mapped by the developing map into a neighborhood in $\mathbb{H}^3$ of a geodesic. If we use cylindrical coordinates, (r, θ, ζ), the hyperbolic metric is $dr^2 + \sinh^2 r\, d\theta^2 + \cosh^2 r\, d\zeta^2$, where the angle θ is defined modulo the cone angle α. We denote the moving co-frame adapted to this coordinate system by $(\omega_1, \omega_2, \omega_3) = (dr, \sinh r\, d\theta, \cosh r\, d\zeta)$.

To define our standard forms, we use the cylindrical coordinates on U defined above, and we denote by e_1, e_2, e_3 the orthonormal frame in U dual to the co-frame $\omega_1, \omega_2, \omega_3$. In particular, e_2 is tangent to the meridian and e_3 is tangent to the singular locus, which is homotopic in the cone-manifold to the longitude. We can interpret an E-valued 1-form as a complex-valued section of $TX \otimes T^*X \cong \mathrm{Hom}(TX, TX)$. Then an element of $TX \otimes T^*X$ can be described as a matrix whose (i, j) entry is the coefficient of $e_i \otimes \omega_j$.

Explicitly, ω_0 is a linear combination of the forms given in (23) and (24) of [28]. The form $\omega_m = \eta_m + i *D\eta_m$ below is a "standard" closed and co-closed (non-L^2) form which represents an infinitesimal deformation which decreases the cone angle but does not change the real part of the complex length of the meridian. It preserves the property that the meridian is elliptic and, hence, that there is a cone-manifold structure.

$$\omega_m = \begin{bmatrix} \frac{-1}{\cosh^2(r)\sinh^2(r)} & 0 & 0 \\ 0 & \frac{1}{\sinh^2(r)} & \frac{-i}{\cosh(r)\sinh(r)} \\ 0 & \frac{-i}{\cosh(r)\sinh(r)} & \frac{-1}{\cosh^2(r)} \end{bmatrix} \tag{2}$$

The form $\omega_l = \eta_l + i *D\eta_l$ below is a "standard" closed and co-closed, L^2 form which stretches the singular locus, but leaves the holonomy of the meridian (hence the cone angle) unchanged.

$$\omega_l = \begin{bmatrix} \frac{-1}{\cosh^2(r)} & 0 & 0 \\ 0 & -1 & \frac{-i\sinh(r)}{\cosh(r)} \\ 0 & \frac{-i\sinh(r)}{\cosh(r)} & \frac{\cosh(r)^2+1}{\cosh(r)^2} \end{bmatrix} \tag{3}$$

The effect of ω_m and ω_l on the complex lengths of the group elements on the boundary torus was computed in [28] (pages 32–33). For a detailed explanation for these computations we refer to this reference. We merely record the results here.

Lemma 2.1. *The effects of the infinitesimal deformations given by the standard forms on the complex length, $\mathcal{L}$, of any peripheral curve are as follows.*

(a) *For ω_m,*

$$\frac{d}{dt}(\mathcal{L}) = -2\mathcal{L}.$$

(b) *For ω_l,*

$$\frac{d}{dt}(\mathcal{L}) = 2\,\mathrm{Re}(\mathcal{L}),$$

where $\mathrm{Re}(\mathcal{L})$ *denotes the real length of the curve.*

Remark 2.2. A *meridian* curve has complex length $i\alpha$. So the effect of ω_m on its derivative is $-2i\alpha$. This shows that the meridian remains elliptic and that the derivative of the cone angle α is -2α. Similarly, for ω_l, the complex length of the meridian has derivative zero.

If $\mathcal{L}$ denotes the complex length of the *longitude*, then the real part of $\mathcal{L}$ is the length ℓ of the singular locus. Thus for ω_m, the derivative of ℓ is -2ℓ. For ω_l, the derivative of ℓ is 2ℓ.

The infinitesimal changes in the complex lengths of the elements of the fundamental group of the torus uniquely determine a complex linear combination of ω_m and ω_l and conversely any such linear combination determines the infinitesimal changes in these complex lengths. The coefficient of ω_m uniquely determines and is determined by the change in the meridian since ω_l leaves the complex length of the meridian unchanged. By our computations above, the length of the meridian remains pure imaginary (i.e. an elliptic element) precisely when the coefficient is real.

The smooth family of structures M_t is determined by a choice of parametrization of the cone angles α_t and we are free to choose this as we wish. The value of the coefficient for ω_m is determined by the derivative of the cone angle. It turns out to be useful to parametrize the cone-manifolds by the *square* of the cone angle; i.e., we will let $t = \alpha^2$. Since the derivative of the square of the cone angle is 1 and the derivative of α under ω_m is -2α, we have

$$\omega_0 = \frac{-1}{4\alpha^2}\omega_m + (x + iy)\omega_l \tag{4}$$

for some real constants x and y. One of the goals of this section is to estimate the values of x and y. This will allow us to estimate the infinitesimal change in all of the complex lengths of curves on the torus. In particular, we can estimate the change in the length of the singular locus.

The estimates in this section can be viewed as effective versions of local rigidity arguments. The basic idea behind local rigidity is to represent an infinitesimal deformation by a harmonic representative in the cohomology group

$H^1(X;E)$. The symmetric real part of this representative is a 1-form with values in the tangent bundle of X. Harmonicity, and the fact that it will be volume preserving (this takes a separate argument), imply that the 1-form satisfies a Weitzenböck-type formula:

$$D^*D\eta \ + \ \eta \ = \ 0$$

where D is the exterior covariant derivative on such forms and D^* is its adjoint. Taking the L^2 inner product of this formula with η and integrating by parts we obtain the formula

$$||D\eta||_X^2 \ + \ ||\eta||_X^2 \ = \ 0$$

when X is closed. (Here $||\eta||_X^2$ denotes the square of the L^2 norm of η on X. The pointwise L^2 norm is denoted simply by $||\eta||$.) Thus $\eta = 0$ and the deformation is trivial. This is the proof of local rigidity for closed hyperbolic 3-manifolds.

When X has boundary or is noncompact, there will be a boundary term b:

$$||D\eta||_X^2 \ + \ ||\eta||_X^2 \ = \ b.$$

If the boundary term is nonpositive, the same conclusion holds: the deformation is trivial. When $X = M - \Sigma$, where M is a hyperbolic cone-manifold with cone angles at most 2π along its singular set Σ, it was shown in [28] that, for a deformation which leaves the cone angle fixed, it is possible to find a representative as above for which the boundary term goes to zero on the boundary of tubes around the singular locus whose radii go to zero. Again, such an infinitesimal deformation must be trivial. This proves local rigidity rel cone angles.

The argument for local rigidity rel cone angles actually shows that the boundary term is negative when the cone angle is unchanged. Note that leaving the cone angle unchanged is equivalent to the vanishing of the coefficient of ω_m. As we shall see below the boundary term for ω_m by itself is positive. Roughly speaking, ω_m contributes positive quantities to the boundary term, while everything else gives negative contributions. (There are, of course, also some cross-terms.) We think of $\frac{-1}{4\alpha^2}\omega_m$ as a preliminary model for the infinitesimal deformation in a tube around the singular locus. Then this is "corrected" by adding $(x+iy)\omega_l$ to get the actual change in complex lengths and then by adding a further term ω_c that does not change the holonomy at all. The requirement that the boundary term for the actual representative (model plus the other terms) be positive puts strong restrictions on these "correction" terms. This is the underlying philosophy for the estimates in this section.

In order to implement these ideas we need to derive a formula for the boundary term. For details we refer to [28].

The Hodge Theorem ([28]) for cone-manifolds gives a closed and co-closed E-valued form $\omega = \eta + i *D\eta$ satisfying $D^*D\eta = -\eta$. Integration by parts, as in [28, Prop. 1.3 and p. 36], over any sub-manifold N of X with boundary ∂N gives:

LEMMA 2.3. *For any closed and co-closed form $\omega = \eta + i * D\eta$ satisfying $D^* D\eta = -\eta$, and any submanifold N with boundary ∂N oriented by the* outward *normal*,

$$0 = \int_N (||\eta||^2 + ||*D\eta||^2) + \int_{\partial N} *D\eta \wedge \eta. \tag{5}$$

Note that in these integrals, $\alpha \wedge \beta$ denotes the real valued 2-form obtained using the wedge product of the form parts, and the geometrically defined inner product on vector-valued parts.

Denote by U_r the tubular neighborhood of points at distance less than or equal to r from the singular locus. It will always be assumed that r is small enough so that U_r will be embedded. Let T_r denote the boundary torus of U_r, oriented with $\frac{\partial}{\partial r}$ as outward normal. We define

$$b_r(\alpha, \beta) = \int_{T_r} *D\alpha \wedge \beta. \tag{6}$$

We emphasize that T_r is oriented as above, so that $\omega_2 \wedge \omega_3 = \sinh r \, \cosh r \, d\theta \wedge d\zeta$ is the oriented area form.

Fix a value R for the radius and let $N = X - U_R$. Then $\partial N = -T_R$, where the minus sign denotes the opposite orientation (since $-\frac{\partial}{\partial r}$ is the outward normal for N). Applying (5) in this case, we obtain:

COROLLARY 2.4. *Let $N = X - U_R$ be the complement of the tubular neighborhood of radius R around the singular locus. Then, for any closed and co-closed form $\omega = \eta + i * D\eta$ satisfying $D^* D\eta = -\eta$,*

$$b_R(\eta, \eta) = ||\eta||_N^2 + ||*D\eta||_N^2 = ||\omega||_N^2. \tag{7}$$

In particular, we see that the boundary term $b_R(\eta, \eta)$ is *nonnegative*. Writing $\eta = \eta_0 + \eta_c$ as before, we analyze the contribution from each part. First, we note that the cross-terms vanish so that the boundary term is simply the sum of two boundary terms:

LEMMA 2.5. $b_R(\eta, \eta) = b_R(\eta_0, \eta_0) + b_R(\eta_c, \eta_c)$.

Proof. Expanding this, we have that $b_R(\eta, \eta) = b_R(\eta_0 + \eta_c, \eta_0 + \eta_c) = b_R(\eta_0, \eta_0) + b_R(\eta_c, \eta_c) + b_r(\eta_0, \eta_c) + b_r(\eta_c, \eta_0)$. So it suffices to show that $b_r(\eta_0, \eta_c) = b_r(\eta_c, \eta_0) = 0$.

This follows from the Fourier decomposition for η_c obtained in [28]. The term η_c is the infinitesimal change of metric induced by a vector field that satisfies a harmonicity condition in a neighborhood of the singular locus. The main point is that η_c has no purely radial terms. This can be seen from Proposition 3.2 of that paper, where the purely radial solutions correspond, in the notation used there, to the case $a = b = 0$. There is a 3-dimensional solution space allowed by the chosen domain for the harmonicity equations (equations (21) in that paper). It becomes 2-dimensional after the conclusion that the deformation is volume-preserving. However, there is an obvious 2-dimensional space of radial solutions coming from the infinitesimal rotations and translations along the axis corresponding to the singular locus. Since these are isometries, they do not contribute anything to the change of metric, η_c.

On the other hand, η_0 only depends on r by definition, so that each term in the integrands for $b_r(\eta_0, \eta_c)$ and $b_r(\eta_c, \eta_0)$ has a trigonometric factor which integrates to zero over the torus T_r. □

Next, we show that the contribution, $b_R(\eta_c, \eta_c)$, from the part of the "correction term" that does not affect the holonomy is *nonpositive*. In fact,

LEMMA 2.6.

$$b_R(\eta_c, \eta_c) = -(||\eta_c||^2_{U_R} + ||{*D\eta_c}||^2_{U_R}) = -||\omega_c||^2_{U_R}. \tag{8}$$

Proof. Consider a region $N = U_{r_1,r_2}$ in U_R bounded by the tori T_{r_1} and T_{r_2} where $0 < r_1 < r_2 \leq R$. Then $\partial N = T_{r_2} \cup -T_{r_1}$ where, as before, the minus sign denotes the opposite orientation.

The equation (5), applied to this region with $\eta = \eta_c$, gives

$$0 = \int_{U_{r_1,r_2}} (||\eta_c||^2 + ||{*D\eta_c}||^2) + \int_{T_{r_2}} *D\eta_c \wedge \eta_c - \int_{T_{r_1}} *D\eta_c \wedge \eta_c,$$

or

$$b_{r_2}(\eta_c, \eta_c) - b_{r_1}(\eta_c, \eta_c) \;=\; -\int_{U_{r_1,r_2}} (||\eta_c||^2 + ||{*D\eta_c}||^2). \tag{9}$$

The main point here is that $\lim_{r \to 0} b_r(\eta_c, \eta_c) = 0$. This is a restatement of the main result in section 3 of [28], since η_c represents an infinitesimal deformation which does not change the cone angle.

Applying (9), with $r_2 = R$ and taking the limit as $r_1 \to 0$ we obtain the desired result. □

Combining Lemma 2.5 with (7) and (8), we obtain:

$$b_R(\eta_0, \eta_0) = ||\omega||^2_{X-U_R} + ||\omega_c||^2_{U_R}. \tag{10}$$

In particular, this shows that

$$b_R(\eta_0, \eta_0) \geq 0, \tag{11}$$

Remark. This positivity is the only application of formula (10) we will use in this paper. However, we note here for future reference that an upper bound on $b_R(\eta_0, \eta_0)$ provides an upper bound on the L^2 norm of ω on the complement of the tubular neighborhood of the singular locus. Such a bound can be used to bound the infinitesimal change in geometric quantities, like lengths of geodesics, away from the singular locus. Similarly, an upper bound on $b_R(\eta_0, \eta_0)$ provides an upper bound on the L^2 norm of the correction term ω_c in the tubular neighborhood itself. This can be used to bound changes in the geometry of the tubular neighborhood that are not detected simply by the holonomy of group elements on the boundary torus.

In the remainder of this section we will use the inequality (11) to find bounds on the infinitesimal variation of the holonomy of the peripheral elements. Of particular interest will be bounding the variation in the length of the singular locus (which equals the real part of the complex length of any longitude of the boundary torus). To this end, we further decompose η_0 as a sum of a component that changes the cone angle and ones that leave it unchanged.

Recall that $\omega_0 = \frac{-1}{4\alpha^2}\omega_m + (x+iy)\omega_l$ so that

$$\eta_0 = \mathrm{Re}(\omega_0) = \frac{-1}{4\alpha^2}\eta_m + x\eta_l - y * D\eta_l.$$

The basic principle here is that the contribution of the η_m term to $b_R(\eta_0, \eta_0)$ is positive, while those of the η_l and $*D\eta_l$ terms are negative. (The cross-terms only complicate matters slightly.) The coefficient of the η_m term is fixed by the choice of parametrization of the family of cone-manifolds by $t = \alpha^2$. Thus, the fact that $b_R(\eta_0, \eta_0)$ is positive will provide a bound on the coefficients x and y.

We calculate

$$\begin{aligned} b_R(\eta_0, \eta_0) = & \frac{1}{16\alpha^4} b_R(\eta_m, \eta_m) + x^2 b_R(\eta_l, \eta_l) + y^2 b_R(*D\eta_l, *D\eta_l) \\ & - \frac{x}{4\alpha^2}(b_R(\eta_m, \eta_l) + b_R(\eta_l, \eta_m)) + \frac{y}{4\alpha^2}(b_R(\eta_m, *D\eta_l) \\ & + b_R(*D\eta_l, \eta_m)) - xy(b_R(\eta_l, *D\eta_l) + b_R(*D\eta_l, \eta_l)). \end{aligned}$$

Now, using the explicit formulas for η_m and η_l, we find

$$b_R(\eta_m, \eta_m) = \frac{1}{\sinh(R)\cosh(R)}\left(\frac{1}{\sinh^2(R)} + \frac{1}{\cosh^2(R)}\right)\mathrm{area}(T_R), \tag{12}$$

$$b_R(\eta_l, \eta_l) = b_R(*D\eta_l, *D\eta_l) = \frac{-\sinh(R)}{\cosh(R)}\left(2 + \frac{1}{\cosh^2(R)}\right)\mathrm{area}(T_R), \tag{13}$$

$$b_R(\eta_m, \eta_l) = \frac{-1}{\sinh(R)\cosh(R)}\left(2 + \frac{1}{\cosh^2(R)}\right)\mathrm{area}(T_R), \tag{14}$$

$$b_R(\eta_l,\eta_m)=\frac{\sinh(R)}{\cosh(R)}\left(\frac{1}{\sinh^2(R)}+\frac{1}{\cosh^2(R)}\right)\operatorname{area}(T_R), \tag{15}$$

and the other terms vanish.

It simplifies matters slightly and is somewhat illuminating to rewrite the value of the boundary term $b_R(\eta_0,\eta_0)$ using the geodesic length m of the meridian on the flat boundary torus T_R. Recall that

$$m \;=\; \alpha \sinh(R).$$

Then we obtain

$$b_R(\eta_0,\eta_0)/\operatorname{area}(T_R)=a(x^2+y^2)+bx+c,$$

where

$$a=\frac{-\sinh(R)}{\cosh(R)}\left(2+\frac{1}{\cosh^2(R)}\right)=-\tanh(R)\;\frac{2\cosh^2(R)+1}{\cosh^2(R)},$$
$$b=\frac{1}{4\alpha^2}\left(\frac{2}{\cosh^3(R)\sinh(R)}\right)=\frac{1}{m^2}\;\;\frac{\tanh(R)}{2\cosh^2(R)},$$
$$c=\frac{1}{16\alpha^4}\frac{1}{\sinh(R)\cosh(R)}\left(\frac{1}{\sinh^2(R)}+\frac{1}{\cosh^2(R)}\right)=\frac{1}{m^4}\;\;\frac{\tanh(R)+\tanh^3(R)}{16}.$$

Completing the squares gives

$$\begin{aligned}b_R(\eta_0,\eta_0)/\operatorname{area}(T_R)&=a(x^2+y^2)+bx+c\\&=a\left(\left(x+\frac{b}{2a}\right)^2+y^2\right)+\frac{4ac-b^2}{4a}.\end{aligned}$$

By direct computation we see that

$$b^2-4ac \;=\; \frac{\tanh^2(R)}{m^4}. \tag{16}$$

Since a is negative, we obtain the following estimate for the boundary term $b_R(\eta_0,\eta_0)$. As noted before, we will not use this estimate here, but rather record it for future reference.

$$b_R(\eta_0,\eta_0)/\operatorname{area}(T_R)\le\frac{4ac-b^2}{4a}\;=\;\frac{1}{4m^4}\;\frac{\sinh(R)\cosh(R)}{2\cosh^2(R)+1}. \tag{17}$$

Our main application of the positivity (11) of $b_R(\eta_0,\eta_0)$ is that, using (16), we can conclude that:

$$\left(x+\frac{b}{2a}\right)^2+y^2\le\frac{b^2-4ac}{4a^2}=\frac{1}{4m^4}\;\frac{\cosh^4(R)}{(2\cosh^2(R)+1)^2}. \tag{18}$$

This implies, in particular, that x lies in the interval of radius

$$\frac{1}{2m^2}\;\frac{\cosh^2(R)}{2\cosh^2(R)+1}$$

around

$$\frac{-b}{2a} = \frac{1}{4m^2}\,\frac{1}{2\cosh(R)^2+1}.$$

In other words, x lies in the interval $[x_1, x_2]$ where

$$x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-1}{4m^2}\,\frac{2\cosh^2(R)-1}{2\cosh(R)^2+1},$$

and

$$x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{1}{4m^2}.$$

Remark. It is useful to rewrite the factor in the formula above for x_1 as

$$\frac{2\cosh^2(R)-1}{2\cosh(R)^2+1} = \frac{2\sinh^2(R)+1}{2\sinh^2(R)+3}.$$

Note that this is monotonic increasing in R, taking on values between $\frac{1}{3}$ and 1.

By Lemma 2.1 the effect of ω_0 on the complex length, $\mathcal{L}$, of any peripheral curve is given by

$$\frac{d}{dt}(\mathcal{L}) = \frac{-1}{4\alpha^2}(-2\mathcal{L}) + (x+iy)(2\operatorname{Re}(\mathcal{L})), \tag{19}$$

where $\operatorname{Re}(\mathcal{L})$ denotes the real length of the curve.

In particular, the derivative of the real length ℓ of the longitude (the length of the singular locus) satisfies

$$\frac{d\ell}{dt} = \frac{\ell}{2\alpha^2}(1+4\alpha^2 x). \tag{20}$$

Since $t=\alpha^2$, we conclude that

$$\frac{d\ell}{d\alpha} = \frac{\ell}{\alpha}(1+4\alpha^2 x). \tag{21}$$

Putting this formula for the derivative of the length of the singular locus together with the estimates above for the coefficient x (and recalling that $m = \alpha\sinh(R)$), we obtain the main result of this section:

THEOREM 2.7. *Consider any smooth family of hyperbolic cone structures on M, all of whose cone angles are at most 2π. For any component of the singular set, let ℓ denote its length and α its cone angle. Suppose there is an embedded tube of radius R around that component. Then*

$$\frac{d\ell}{d\alpha} = \frac{\ell}{\alpha}(1+4\alpha^2 x),$$

where

$$\frac{-1}{\sinh^2(R)}\left(\frac{2\sinh^2(R)+1}{2\sinh^2(R)+3}\right) \le 4\alpha^2 x \le \frac{1}{\sinh^2(R)}. \tag{22}$$

Remark 2.8. This implies that ℓ is an increasing function of α provided the tube radius R is large enough. Explicitly

$$\frac{d\ell}{d\alpha} \geq 0$$

provided

$$\frac{1}{\sinh^2(R)}\left(\frac{2\sinh^2(R)+1}{2\sinh^2(R)+3}\right) \leq 1$$

which simplifies to

$$R \geq \operatorname{arcsinh}(\frac{1}{\sqrt{2}}) \approx 0.65848.$$

This has implications concerning the variation of the volume V of a family of cone-manifolds due to the Schläfli formula (see [27], [12, Theorem 3.20]):

$$\frac{dV}{d\alpha} = -\frac{1}{2}\ell. \tag{23}$$

Since

$$\frac{d^2V}{d\alpha^2} = -\frac{1}{2}\frac{d\ell}{d\alpha} \leq 0$$

for these values of R, the volume function will be a concave function of α as long as the tube radius is sufficiently large.

More specifically, if one considers a family of cone-manifolds with a single component of the singular locus in which the cone angle is *decreasing*, the total change, ΔV, in the volume will be positive. If $R \geq \operatorname{arcsinh}(\frac{1}{\sqrt{2}})$ throughout the deformation, then we obtain the inequality

$$\Delta V \leq \frac{|\Delta\alpha|}{2}\ell_0, \tag{24}$$

where $\Delta\alpha$ denotes the total change in cone angle and ℓ_0 denotes the initial length of the singular locus.

In Section 5, we will see how to control the tube radius by controlling the length of the singular locus. This will lead to sharper estimates for the change in volume by integrating the more detailed estimates for $\frac{d\ell}{d\alpha}$ which are derived there. However, it seems worthwhile to note that the above estimates follow immediately from (22).

3. Geometric limits of cone-manifolds

This section is primarily devoted to the proof of Theorem 1.2.

In general, the limiting behavior of a sequence of hyperbolic cone-manifolds can be quite complicated. In particular, it can collapse to a lower dimensional object or the singular locus can converge to something of higher complexity. However, by the results of Section 5, we will be able to assume that there is a lower bound to the tube radius around each component of Σ and that the geometry of the boundary of the tube does not degenerate. This greatly simplifies matters, essentially reducing them to the manifold case.

Given a sequence of hyperbolic cone-manifold structures M_i on (M, Σ), remove disjoint, embedded equidistant tubes around each component of Σ. The result is a sequence of smooth, hyperbolic manifolds N_i with torus boundary components, each of which has an intrinsic flat metric. Furthermore, the principal normal curvatures are constant on each component, equalling $\kappa, \frac{1}{\kappa}$ (we assume that $\kappa \geq 1$). When $\kappa > 1$ the lines of curvature are geodesics in the flat metric corresponding to the meridional and longitudinal directions, respectively. Note that the normal curvatures and the tube radius, R, are related by $\coth R = \kappa$ and so they determine each other.

We now formalize the structure of this type of boundary torus. Let $\mathbb{H}^3_R$ denote 3-dimensional hyperbolic space minus the open tube of points distance less than R from a geodesic. We allow the values $0 < R \leq \infty$, where $\mathbb{H}^3_\infty$ denotes the complement of an open horoball based at a point at infinity. We say that a torus boundary component of a hyperbolic 3-manifold is *locally modelled on* $\mathbb{H}^3_R$ if, for some fixed R, each point on the boundary torus has a neighborhood isometric to a neighborhood of a point on the boundary of $\mathbb{H}^3_R$. The overlap maps are required to be restrictions of 3-dimensional hyperbolic isometries. This is equivalent to the condition that the torus have an induced flat metric and have normal curvatures and lines of curvature as in the previous paragraph. Note that normal curvatures all equal to 1 corresponds to the case $R = \infty$.

Definition 3.1. A hyperbolic 3-manifold is said to have *tubular boundary* if its boundary consists of tori that are each locally modelled on $\mathbb{H}^3_R$ for some $0 < R \leq \infty$. (The value of R is allowed to be different on different components of the boundary.)

As noted above, one way these arise is when one removes tubular neighborhoods of the components of the singular locus of a cone-manifold. On the other hand, we will see below that there is a canonical way to fill in any tubular boundary component. If a hyperbolic 3-manifold with tubular boundary came from a hyperbolic cone-manifold by removing tubular neighborhoods, the filling process recovers the same cone-manifold.

To see this, first note that when $R = \infty$ the boundary torus has all normal curvatures equal to 1, so it can be identified with a horosphere modulo a group of parabolic isometries fixing the corresponding point at infinity. This group action extends canonically to an action on the horoball bounded by the horosphere. In this case, the boundary is "filled in" with a cusp. This is interpreted as a cone-manifold structure with cone angle 0. If the tubular boundary came from removing a tubular neighborhood of the "singular locus", it must actually have been a cusp because the normal curvatures all equal 1. Furthermore, since the structure of the cusp is determined by the flat structure on the boundary, the cusp replaced must be isometric to the one removed.

To analyze the case of finite R, we note that the universal cover of the complement of a geodesic in $\mathbb{H}^3$ is isometric to $\mathbb{R}^3$ with metric in cylindrical coordinates (r, θ, ζ), where $0 < r$, given by

$$dr^2 \ + \ \sinh^2 r \, d\theta^2 \ + \ \cosh^2 r \, d\zeta^2. \tag{25}$$

A neighborhood of the tubular boundary is given by dividing out a neighborhood of the plane $r = R$ in $\mathbb{R}^3$ by a $\mathbb{Z} \oplus \mathbb{Z}$ lattice in the (θ, ζ)-plane. The above metric descends to the metric in a neighborhood of the tubular boundary. In particular, the boundary is the image of $r = R$ and the principal curvatures, $\kappa, 1/\kappa$, are in the θ, ζ directions, respectively. The metric on the tubular boundary can be canonically extended by adding the quotient of the region $r \in (0, R]$ by the (θ, ζ) lattice group. This metric is incomplete. In general its completion is singular, resulting in a hyperbolic structure "with Dehn surgery singularities" (see Thurston [43] for further discussion). This structure includes cone-manifolds as a special case. We will not be concerned with the more general type of singularity here, but rather see below that the cone-manifold structures can be identified from the structure on the tubular boundary.

If one removes a tubular neighborhood of a component of the singular locus of a cone-manifold with cone angle α, the boundary torus has a closed geodesic in the meridian ($\zeta =$ constant) direction which is the boundary of a totally geodesic, singular disc with cone angle α perpendicular to the core geodesic. Conversely, we claim that if there is such a closed geodesic, the completion defined above will be a cone-manifold. To see this, note that there is a closed meridian on the boundary torus if and only if the lattice in (θ, ζ) can be chosen to have one generator of the form $(\alpha, 0)$. We denote by (τ, ℓ) the other generator, where necessarily $\ell \neq 0$. This corresponds to the first generator being a rotation by angle α around the removed geodesic. The second generator translates distance ℓ along the removed geodesic and rotates by angle τ; i.e., it has complex length $\ell + i\tau$. Then the completion is obtained by adding in the quotient of the removed geodesic (corresponding to $r = 0$) under the action.

This is easily seen to be a cone-manifold with cone angle α. In particular, the singular locus is the geodesic added in the completion and the meridian, which is a closed geodesic in the flat metric on the original tubular boundary, bounds a singular, totally geodesic disk intersecting the singular locus in a single point. The flat structure on the tubular boundary can be constructed by taking a flat cylinder of circumference m and height h and attaching it with a twist of distance tw. The cone angle, α, and the complex length $\ell \ + \ i\tau$ are related to these quantities by the equations:

$$\begin{aligned} m &= \alpha \sinh R, \\ h &= \ell \cosh R, \\ \mathrm{tw} &= \tau \sinh R. \end{aligned}$$

This implies that the region added is canonically determined by the geometry of the boundary torus, the value of R, and the fact that there is a closed geodesic in the meridian (principal curvature $\kappa > 1$) direction. Thus, if the tubular boundary structure arose from removing a tubular neighborhood of a component of the singular locus of a cone-manifold, the filling in process would recover the same cone-manifold structure.

The results proved in this section concern bilipschitz limits of sequences of hyperbolic manifolds with tubular boundary. The above analysis implies that if the members of the sequence all arise from cone-manifold structures, and if the limit is a hyperbolic manifold with tubular boundary, then it can be filled in to be a cone-manifold also, and the results can be viewed in terms of bilipschitz limits of cone-manifolds.

There are two advantages to considering sequences of hyperbolic structures with such boundary data rather than studying sequences of hyperbolic cone-manifolds directly. First, the analysis of geometric limits is much simpler in the manifold setting. Though the boundary does introduce complications similar to those that arise for cone-manifolds, it is easier to isolate them if the singular locus is removed. Secondly, the results of this section will apply to more general singular structures than cone-manifolds. In particular, they will apply to a sequence of hyperbolic structures with Dehn surgery singularities as long as there is a lower bound to the radii of disjoint tubes around the singularities. We expect to use this application in a future paper.

A topological ball in a hyperbolic manifold with tubular boundary will be called *standard* if it is isometric to a ball of radius $r > 0$ in $\mathbb{H}^3$ or to a ball of radius $r > 0$ about a point on the boundary of $\mathbb{H}^3_R$. In the latter case, we further require that $r < R$. This corresponds to the geometric condition that if the tube of radius R were added back to $\mathbb{H}^3_R$ and the ball extended to a ball in $\mathbb{H}^3$, then the extended ball would be disjoint from the geodesic core of the added tube.

The *injectivity radius* at a point x in a hyperbolic manifold, N, with tubular boundary is

$$\mathrm{inj}(x, N) = \sup\{r \mid B_r(x) \subset \text{a standard ball in } N\}.$$

Here $B_r(x)$ simply denotes the set of points in N distance less than r from x; there is no assumption on its topology. We will write $\mathrm{inj}(N)$ to denote $\inf_{x \in N}(\mathrm{inj}(x, N))$.

Note that we *do not* assume that the standard neighborhood is *centered* at the point x. This is to avoid difficulties near the boundary: a point x near, but not on, the boundary has only a small standard ball centered at x, with radius at most the distance to the boundary. However, there may be much larger standard balls which contain x that are centered at a point on the boundary.

It is important also to notice that because of the condition that $R > r$ for a standard ball of radius r centered at a point on a boundary torus locally modelled on $\mathbb{H}^3_R$, a lower bound on the injectivity radius of N implies a lower bound on the tube radii of all the boundary components.

The goal of this section is to find conditions on a family of hyperbolic 3-manifolds with tubular boundary that ensure that they converge to a diffeomorphic manifold with such a structure. The notion of convergence that we will use is based on a distance between metric spaces defined using bilipschitz mappings.

Definition 3.2. The *bilipschitz distance* between two metric spaces X, Y is the infimum of the numbers

$$|\log \operatorname{lip}(f)| \; + \; |\log \operatorname{lip}(f^{-1})| \tag{26}$$

where f ranges over all bilipschitz mappings from X to Y and $\operatorname{lip}(f)$ denotes the lipschitz constant of f.

The bilipschitz distance between X and Y is defined to be ∞ if there is no bilipschitz map between them. In particular, metric spaces that are a finite distance apart are necessarily homeomorphic. It is not hard to show that two compact metric spaces are bilipschitz distance 0 apart if and only if they are isometric.

For noncompact spaces, bilipschitz distance is not very useful because it is so often infinite. For many purposes, it is important to allow a more flexible idea of convergence of sequences of metric spaces than that induced simply by bilipschitz distance. To make this idea precise, it is necessary to choose a basepoint in each metric space.

Definition 3.3. A sequence, $\{(Y_i, y_i)\}$, of metric spaces with basepoint *converges* to (Y, y) in the *pointed bilipschitz topology* if, for each fixed $R > 0$, the radius R neighborhood of y_i in Y_i converges with respect to the bilipschitz distance to the radius R neighborhood of $y \in Y$.

Note that with this notion of convergence, a sequence of compact spaces can converge to a noncompact space. In particular, there is no requirement that the Y_i in a convergent sequence be eventually homeomorphic. Convergence in the pointed bilipschitz topology means that the metric spaces are becoming closer and closer to being isometric on larger and larger diameter subsets.

However, when there is a uniform bound to the diameter of all the spaces in the sequence, convergence is independent of the choice of basepoint and is just convergence with respect to the bilipschitz metric.

Our beginning point in the study of convergence of hyperbolic 3-manifolds with tubular boundary is a seminal and general theorem due to Gromov. It says that, under very mild conditions (pinched curvature and bounded injectivity radius at the basepoint), a sequence of complete, pointed Riemannian manifolds will have a convergent subsequence in this topology. This theorem is actually a corollary of an even broader compactness theorem, involving a much more general notion of convergence of metric spaces, usually referred to as Gromov-Hausdorff convergence. However, Gromov shows that, when applied to various classes of Riemannian manifolds, this general notion of convergence implies convergence in the pointed bilipschitz topology. We will not need to use the concept of Gromov-Hausdorff convergence in this paper, but rather begin with its application to Riemannian manifolds.

Theorem 3.4 ([24, Theorem 8.25], [25, Theorem 8.20]). *Consider a sequence of complete, pointed Riemannian manifolds* (N_i, v_i) *with pinched sectional curvatures* $|k| \leq K$ *and injectivity radius at the basepoints*, v_i, *bounded below by* $c > 0$. *Then there is a pointed Riemannian manifold* (N, v), *together with a subsequence of the* (N_i, v_i) *which converges in the pointed bilipschitz topology to* (N, v). *Furthermore, if there is a* $D > 0$ *so that the diameters of the* N_i *are less than* D *for all* i, *then the* N_i *in the convergent subsequence will be diffeomorphic to* N *for* i *sufficiently large.*

The fact that convergence in the metric is only lipschitz means that, *a priori*, the limit metric is only C^0. In [24] and [25], it is explained how a somewhat higher level of regularity can be achieved by consideration of harmonic coordinates. For closed manifolds, a complete proof along the lines sketched there appears in [30]. Proofs along somewhat different lines appear in [23] and [39]; these references also provide simple examples showing why the limit metric won't be C^2 in general. However, if all the metrics in the sequence are of a special type, much stronger conclusions are possible. As explained in [40, p. 307], if the approximating metrics are Einstein, then use of the Einstein equation and elliptic regularity allows one to bootstrap the regularity of convergence to any number of derivatives and the limit metric will also be Einstein.

In our situation with constant curvature, things are vastly simpler. The regularity issues discussed above are all local. The regularity of the convergence and of the limit metric follow from local analysis on embedded balls of fixed radius. In general, simply bounding the injectivity radius and curvature of a sequence of metrics does not bound derivatives of the curvature and smoothness may be lost in the limit, even locally. However, since all metric balls of a fixed radius in hyperbolic n-space are isometric, the bilipschitz limit of a sequence of hyperbolic n-balls of fixed radius will automatically be hyperbolic. Thus, in the theorem above, if the approximating manifolds are all hyperbolic, the limit manifold will be also.

The fact that we are considering manifolds with boundary means that we can't immediately apply Theorem 3.4 above. Indeed, a few extra conditions on the boundary are necessary, for example, to keep the boundary from collapsing to a point or to keep two components on the boundary from colliding in the limit. This has been worked out in [31], where Gromov's theorem is extended to manifolds with boundary if one has the added conditions that the principal curvatures and intrinsic diameters of the components of the boundary are bounded above and below and that there is a lower bound to the width of an embedded tubular neighborhood of the boundary. We see in the proof below that, with our definition of the injectivity radius, these conditions hold for manifolds with tubular boundary if the injectivity radius is bounded below for points on the boundary and the volume of the entire manifold is bounded above.

THEOREM 3.5. *Let (N_i, v_i) be a sequence of hyperbolic 3-manifolds with tubular boundary with basepoints v_i on ∂N_i. Assume there are constants $c, V > 0$, such that, for all i, $\mathrm{inj}(x, N_i) \geq c$ for all $x \in \partial N_i$ and $\mathrm{vol}(N_i) < V$. Then there is subsequence converging in the pointed bilipschitz topology to a pointed hyperbolic 3-manifold with tubular boundary, (N_∞, v_∞). Furthermore, if the diameters of all the N_i are uniformly bounded, then all the N_i in the subsequence will be diffeomorphic to N_∞ for sufficiently large i.*

Remark 3.6. The bound on the volume will only be used to conclude that the intrinsic diameters of the boundary components of all the N_i's are uniformly bounded. Thus, the theorem remains true with the volume condition replaced by such a bound on these intrinsic diameters.

Proof. In order to apply the generalization in [31] of Theorem 3.4 we need to check the required conditions on the boundary. Recall that points on the tubular boundary are locally modelled on $\mathbb{H}^3_R$ and that the definition of injectivity radius implies that $\mathrm{inj}(x, N_i) < R$ for such points. Since the principal curvatures on the boundary equal $\kappa, 1/\kappa$, where $\kappa = \coth R$, a lower bound on the injectivity radius for boundary points immediately bounds the principal curvatures above and below.

The definition of injectivity radius at a point requires that there will be a standard ball containing the set of points distance r from the point, for any r less than the injectivity radius. The radius of the standard ball must be at least equal to this r. But any standard ball in $\mathbb{H}^3_R$ containing a boundary point must be centered at some (possibly different) point on the boundary of $\mathbb{H}^3_R$.

This implies that there is a tubular neighborhood around the boundary with a lower bound to its width.

Finally, we need to see that the intrinsic diameters of the boundary components are bounded above. The boundaries all have flat metrics. By hypothesis, the injectivity radii of all points on the boundary are all bounded below so the intrinsic injectivity radii of boundary tori with respect to the flat metrics will also be bounded below. To see that their intrinsic diameters are bounded above, it suffices to show that their areas are bounded above. There are collar neighborhoods of each boundary component with a lower bound on their width and the normal curvatures are bounded above. Thus, if the areas of the boundary were unbounded, the volumes of the collar neighborhood would be unbounded. Since the volumes are assumed bounded, the areas, hence the diameters, are bounded.

The theorems in [31] have the extra hypotheses that the injectivity radius of all points in the manifold be bounded below, not just boundary points. Also, the diameters of the N_i are required to be uniformly bounded above. However, the injectivity radius at a point x changes continuously with x and the rate at which it can go to zero as a function of distance is uniformly bounded depending only on the curvature (Proposition 8.22 in [24] or Theorem 8.5 in [25]). This is often referred to as "bounded decay of injectivity radius". It follows that, if the diameters of the N_i are uniformly bounded above, then the injectivity radius bound on the boundary gives a uniform lower bound to the injectivity radius over all of the N_i. The results in [31] apply directly.

In general, the bounded decay of injectivity radius implies that, if the injectivity radius at the basepoints of the N_i are bounded below, then, for any fixed distance ρ, the injectivity radius over the neighborhood of radius ρ will be uniformly bounded below. The convergence results for manifolds with bounded diameter give a convergent subsequence for each ρ. The usual diagonal argument gives a subsequence converging for any fixed ρ which is the definition of bilipschitz convergence.

Finally, we need to check that the limit manifold is hyperbolic with tubular boundary. Any interior point in the limit has a neighborhood that is the bilipschitz limit of a sequence of embedded balls in $\mathbb{H}^3$ with fixed radius. The limit will be isometric to such a ball and so N_∞ will be hyperbolic at such a point. A boundary point will have a neighborhood that is the bilipschitz limit of a sequence of embedded balls on the boundary of $\mathbb{H}^3_{R_i}$ with fixed radius. Since the R_i are bounded below there will be a subsequence which converges to some R, where possibly $R = \infty$. The limit neighborhood will be isometric to such a ball in $\mathbb{H}^3_R$ so N_∞ will have tubular boundary. □

Remark 3.7. Although we have based our proof of Theorem 3.5 on the very general theorems of Gromov and others, there is a much more direct proof, following the proof of the compactness result of Jørgensen-Thurston in [43, Theorem 5.11.2]. A sketch of the argument is as follows: For fixed ε, let $N_{[\varepsilon,\infty)}$ be the set points where the injectivity radius is at least ε. For sufficiently small δ (depending only on ε), there is a covering of $N_{[\varepsilon,\infty)}$ by embedded balls of radius δ so that the balls of radius $\delta/2$ with the same centers are disjoint. If N is a hyperbolic 3-manifold with tubular boundary with $\mathrm{vol}(N) < V$, then the number of such disjoint balls is bounded in terms of V. Thus, there are finitely many intersection patterns of the larger balls that cover, and the hyperbolic structures on $N_{[\varepsilon,\infty)}$ are completely determined by the relative positions of the centers of the balls. The space of choices of such relative positions is compact. On the other hand, an application of the Margulis lemma, extended to allow tubular boundary, implies that, for sufficiently small ε (universal over all hyperbolic 3-manifolds), the regions where the injectivity radius is less than ε is a finite disjoint union of tubular neighborhoods of short geodesics or of cusps. In the discussion above of canonically filling in tubular boundaries, we showed that these regions are determined isometrically by their boundary data. This implies Theorem 3.5.

Rather than filling in the details of this argument, we have chosen to base our proof on published results. However, some readers may find this argument clearer.

Theorem 3.5 allows for the possibility that, even if all the hyperbolic manifolds N_i are diffeomorphic, the limiting manifold N_∞ may not be. For this to occur the diameters must go to infinity. If this were to occur, then *a priori* a portion of the approximating manifolds might be pushed an infinite distance from the basepoint and be lost in the limit. This is a familiar occurrence for hyperbolic surfaces where the length of a geodesic can go to zero, creating a new cusp and a new diffeomorphism type.

We prove below that this is not possible for sequences of 3-manifolds with tubular boundary having bounded volume and a lower bound for injectivity radius at boundary points. First we need to establish the fact that the ends of a finite volume hyperbolic 3-manifold with tubular boundary have the same structure as those of a complete, finite volume hyperbolic 3-manifold. They are cusp neighborhoods, diffeomorphic to $T^2 \times (0,\infty)$, formed by dividing out a horoball by a discrete $\mathbb{Z}\oplus\mathbb{Z}$ lattice. The usual proof that this is the structure of the ends of a complete, finite volume hyperbolic 3-manifold uses a refined version of the Margulis lemma and relies on discreteness of the holonomy group. The holonomy groups of hyperbolic 3-manifolds with tubular boundary are usually not discrete so the proof does not immediately apply. It is possible to give a direct geometric proof for the case with tubular boundary as in Gromov's extension of the Margulis lemma ([24, Prop. 8.51]). Instead we use known results about the ends of finite volume manifolds with pinched negative curvature, due to Eberlein ([14]).

To apply these results we first prove the following lemma:

LEMMA 3.8. *The metric on a hyperbolic manifold N with tubular boundary can be extended to a complete metric with pinched negative curvature on a manifold X diffeomorphic to the interior of N. N embeds isometrically in X in this metric and the volume of its complement $X - N$ is finite.*

Proof. The idea of the proof is simply to attach to each component of the tubular boundary a space diffeomorphic to $T^2 \times (-\infty, 0]$, with $T^2 \times 0$ attached to the boundary. The result is clearly diffeomorphic to the interior of N. Furthermore, the metric on each of the $T^2 \times (-\infty, 0]$ pieces will have pinched negative curvature, finite volume, and agree with the metric on N in a neighborhood of the tubular boundary.

If $R = \infty$ for a boundary component of N, then, as discussed above, the canonical extension of the boundary metric results in a finite volume cusp. In this case, the attached piece has *constant* curvature -1.

If R is finite, we use the fact that the metric in a neighborhood of the tubular boundary is induced from the metric (25) in a neighborhood of $r = R$ by dividing out by the action of a (θ, ζ) lattice. We alter the metric, keeping it of the form

$$dr^2 \ + \ f(r)^2 \, d\theta^2 \ + \ g(r)^2 \, d\zeta^2, \tag{27}$$

where $f(r), g(r)$ are defined on $(-\infty, R]$ and agree with $\sinh r, \cosh r$, respectively near $r = R$. Furthermore, we want $f(r), f'(r), f''(r), g(r), g'(r), g''(r)$ to be positive on $(-\infty, R]$. Such a metric is complete and has negative curvature. From the explicit formulae for the curvatures, it is not hard to see that the sectional curvatures can be pinched between two negative constants. (See [3] or [32] for details of the curvature computation.)

Since the functions $f(r), g(r)$ depend only on r, such a metric is invariant under any (θ, ζ) lattice and so it descends to a pinched negatively curved metric on $T^2 \times (-\infty, R]$ which can be attached to the boundary of N. Further, choosing the functions so that $\int_{-\infty}^{R} f(r)\, g(r)\, dr < \infty$ ensures that the volume will be finite. □

PROPOSITION 3.9. *Each end of a complete, finite volume hyperbolic 3-manifold with tubular boundary is diffeomorphic to $T^2 \times (0, \infty)$ and is isometric to a horoball in $\mathbb{H}^3$ divided by a parabolic $\mathbb{Z} \oplus \mathbb{Z}$ lattice.*

Proof. In [14] it is proved that for complete, finite volume n-manifolds with pinched negative curvature, there will be a finite number of ends, each of the form $W \times (0, \infty)$ where W is an $(n-1)$-manifold with virtually nilpotent fundamental group. Since our manifolds are orientable and 3-dimensional, W is an orientable surface and the only possibility is a torus. It is further shown that the end is isometric to a horoball divided by a parabolic lattice isomorphic to the fundamental group of W. In the general negatively curved context, horoballs are defined in terms of Busemann functions. However, since the ends of the negatively curved manifold constructed in Lemma 3.8 that come from the original hyperbolic manifold with tubular boundary all have constant curvature, a horoball sufficiently far out in the end defined by a Busemann function will agree with the usual definition in hyperbolic geometry. □

We are now in a position to prove a compactness result for the set of hyperbolic structures with tubular boundary on a fixed compact 3-manifold.

THEOREM 3.10. *The set of hyperbolic structures with tubular boundary on a fixed compact* 3*-manifold* N *with volumes bounded above and injectivity radius on the boundary bounded below is compact in the bilipschitz topology. In other words, suppose that* N_i *is a sequence of hyperbolic manifolds with tubular boundary, all diffeomorphic to* N. *Assume there are constants* $c, V > 0$ *such that, for all* i, $\mathrm{inj}(x, N_i) \geq c$ *for all* $x \in \partial N_i$ *and* $\mathrm{vol}(N_i) \leq V$. *Then there is a subsequence which converges in the bilipschitz topology to a hyperbolic structure on* N *with tubular boundary.*

Proof. By Theorem 3.5, it suffices to show that the diameters of the N_i are uniformly bounded. Choose a basepoint $x_i \in \partial N_i$ for all i. Again, by Theorem 3.5, there will always be a subsequence of (N_i, x_i) with a limit (N_∞, x_∞) in the pointed bilipschitz topology which is again a pointed hyperbolic 3-manifold with tubular boundary.

Suppose that the diameters of the N_i are not bounded above. By definition of convergence in the bilipschitz topology the limit will be noncompact and will have finite volume. It will have at least one end, and each end is a cusp with a horospherical Euclidean torus as cross section by Proposition 3.9.

Convergence in the bilipschitz topology further implies that we get a sequence of bilipschitz maps of larger and larger radius neighborhoods of $x_\infty \in N_\infty$ into N_i which, for any fixed radius, are becoming arbitrarily close to an isometry onto their images. For a sufficiently large radius, the topology of these neighborhoods will be constant and equal to a manifold W with torus boundary components whose interior is diffeomorphic to N_∞. We identify the N_i with N and the large radius neighborhoods with W and consider the bilipschitz maps as maps $\phi_i : W \to N$. Under the identification of the interior of W with N_∞ certain of the boundary tori of W correspond to cusps of N_∞. We will refer to these tori as "cusp tori".

The hyperbolic structures, N_i, on N induce holonomy representations $\rho_i : \pi_1 N \to G$, where G is the group of isometries of $\mathbb{H}^3$. The representations are well-defined up to conjugation by elements in G. Similarly, the hyperbolic structure N_∞, viewed as a structure on the interior of W, induces a representation $\rho : \pi_1 W \to G$. The fact that the bilipschitz maps converge on compact sets implies the convergence of the holonomy representations of any finite set of group elements, at least after conjugating the representations. Since $\pi_1 W$ is finitely generated, this implies that, perhaps after conjugating the ρ_i by elements of G, we obtain

$$\rho_i \circ (\phi_i)_* \to \rho. \tag{28}$$

By Proposition 3.9 the fundamental group of the torus cross-sections of the cusp ends of N_∞ inject into the fundamental group of N_∞. Since N_∞ is diffeomorphic to the interior of W, it follows that the fundamental group of each cusp torus of W injects into the fundamental group of W. Choose any cusp torus and denote it by T. We wish to show that, for i sufficiently large, the fundamental group of T must inject under $(\phi_i)_*$ into the fundamental group of N. Furthermore, T will not be peripheral in N. This will contradict the fact that N is atoroidal, implying that the diameters of the N_i must have been uniformly bounded above.

For each value of i we denote by W_i the homeomorphic image of W in N under ϕ_i and by T_i the homeomorphic image of T. Suppose, for some i, the torus $T_i \subset W_i$ is compressible in N. Since N is irreducible, the torus must either bound a solid torus outside W_i or be contained in a 3-ball in N. For any element $\gamma \in \pi_1 T$ we have $\rho_i \circ (\phi_i)_*(\gamma) \to \rho(\gamma)$. Since, for any nontrivial γ, $\rho(\gamma)$ is a nontrivial parabolic element, this implies that $\rho_i \circ (\phi_i)_*(\gamma)$ is nontrivial for sufficiently large i. Hence, $\pi_1 T$ at least maps nontrivially under $(\phi_i)_*$. Therefore, no cusp torus is contained in a 3-ball and so all the cusp tori must bound solid tori outside W_i. Since this is true for all of the cusp tori in W, it follows that, for all sufficiently large i, adding $N - W_i$ to $W_i \subset N$ corresponds to obtaining N by Dehn filling on W.

Let γ_i denote a curve on a cusp torus T of W which bounds a disk when mapped into N by ϕ_i. As above, for any fixed nontrivial element $\gamma \in \pi_1 T$, $\rho_i \circ (\phi_i)_*(\gamma)$ will be nontrivial for sufficiently large i (where "sufficiently large" generally depends on γ). Since $(\phi_i)_*(\gamma_i) = e$, its holonomy representation is trivial. Thus, γ_i can represent a fixed element of $\pi_1 T$ for only *finitely many* values of i. Since this argument holds for each cusp torus, it implies that N can be obtained by Dehn fillings on W using infinitely many distinct filling curves on each cusp torus. We will show that this is impossible by Thurston's theory of hyperbolic Dehn surgery.

First, note that, since N has a complete metric of pinched negative curvature, it is irreducible and atoroidal ([14]). It is the interior of the compact manifold W with nonempty boundary which is therefore Haken. By Thurston's Geometrization Theorem for Haken manifolds ([45], [37], [38], [29]), N supports a complete, finite volume metric of constant negative curvature. Thurston's hyperbolic Dehn surgery theorem says that, when considering all possible Dehn fillings of such a 3-manifold, for all but finitely many choices of filling curve on each cusp torus, the result is hyperbolic. Thus, for i sufficiently large, all the manifolds obtained above by Dehn filling W are hyperbolic. Furthermore, they have volumes converging from below to the volume of the complete hyperbolic structure on N. But, since the resulting 3-manifold is always diffeomorphic to N and the hyperbolic volume of N is a topological invariant, this is a contradiction. □

Remark 3.11. The above result generalizes to the case when N has cusps. To do this, one shows (using, for example, the packing results of the next section), that it is possible to remove neighborhoods of the cusps in such a way that the injectivity radii of the new boundary components of the resulting compact hyperbolic manifold with tubular boundary are also bounded below.

We are now in a position to prove our main convergence result, referred to in the introduction as Theorem 1.2.

THEOREM 3.12. *Let M_t, $t \in [0, t_\infty)$, be a smooth path of closed hyperbolic cone-manifold structures on (M, Σ) with cone angle α_t along the singular locus Σ. Suppose that $\alpha_t \to \alpha \geq 0$ as $t \to t_\infty$, that the volumes of the M_t are bounded above by V_0, and that there is a positive constant R_0 such that there is an embedded tube of radius at least R_0 around Σ for all t. Then the path extends continuously to $t = t_\infty$ so that as $t \to t_\infty$, M_t converges in the bilipschitz topology to a cone-manifold structure M_∞ on M with cone angles α along Σ.*

Proof. Removing disjoint tubular neighborhoods of the singular locus, we obtain a smooth path of hyperbolic manifolds N_t with tubular boundary, with all the N_t diffeomorphic to a fixed compact 3-manifold N. The volumes are bounded above since they are smaller than the volumes of the cone-manifolds M_t which are bounded above by hypothesis.

To apply Theorem 3.10 we need to show that there is a lower bound to the injectivity radii on the boundary of the N_t. That will imply that there is a subsequence of the N_t converging to a hyperbolic manifold N_∞ with tubular boundary. The boundary can then be filled in canonically to obtain a hyperbolic cone-manifold M_∞.

By definition, the injectivity radius at a boundary point is less than its distance to the singular locus in the corresponding hyperbolic cone-manifold structure. Similarly, it is less than its distance to any other boundary components besides the one it is on. Since the tube radii of the hyperbolic cone-manifolds are bounded below by hypothesis, the tubular neighborhoods that are removed can be chosen so that both the distances to the singular locus and to other boundary components are bounded below. Furthermore, we will see in the next section that, on a boundary torus of radius R, there is always an embedded ellipse with minor axis lengths given by (33). This implies that the injectivity radii are bounded below as desired.

Take any sequence N_{t_j}, where $t_j \in [0, t_\infty), t_j \to t_\infty$. We can apply Theorem 3.10 to conclude that there is a subsequence N_i which converges in the bilipschitz topology to a hyperbolic manifold, N_∞, with tubular boundary. It is also diffeomorphic to N. As in the proof of the previous theorem, the hyperbolic structures N_i and N_∞ give rise to holonomy representations ρ_i and ρ respectively from $\pi_1 N$ to the group G of isometries of $\mathbb{H}^3$. Since the diameters of the N_i are uniformly bounded, convergence in the bilipschitz topology provides basepoint-preserving bilipschitz homeomorphisms from N_∞ to the N_i which, under the identifications of both the domain and range with N, give basepoint-preserving homeomorphisms $\phi_i : N \to N$. As in the proof of the previous theorem, it is possible to choose conjugacy classes of the holonomy representations so that $\rho_i \circ (\phi_i)_* \to \rho$.

Since the $\phi_i : N \to N$ are basepoint-preserving homeomorphisms, the induced maps $(\phi_i)_*$ on $\pi_1 N$ are automorphisms. We saw in the proof of Theorem 3.10 that N has a complete, finite volume hyperbolic metric on its interior. This implies that the outer automorphism group of $\pi_1 N$ is finite because Mostow rigidity says that any outer automorphism is homotopic to an isometry of the complete finite volume metric on the interior of N. The group of such isometries is finite. (See [43] for a more detailed version of this argument.) Since there are only finitely many choices for $(\phi_i)_*$ up to conjugacy, there is a further subsequence so that $(\phi_i)_*$ is constant and, hence, that $\rho_i \circ (\phi)_* \to \rho$ for a fixed automorphism $(\phi)_*$ of $\pi_1 N$. This implies that the holonomy representations ρ_i converge in the representation variety (representations of $\pi_1 N$ to G modulo conjugation) to $\rho \circ (\phi)_*^{-1} = \hat{\rho}$.

The hyperbolic structure N_∞ with tubular boundary has $\hat{\rho}$ as a holonomy representation. The boundary data of the N_i determine the canonical completion to the hyperbolic cone-manifold structures M_i. Since these boundary data converge to that of N_∞, its canonical completion is a hyperbolic cone-manifold structure M_∞ with cone angle α along its singular locus Σ. Under the isomorphism $\pi_1 N \cong \pi_1(M_\infty - \Sigma)$ the holonomy representation of $M_\infty - \Sigma$ can be identified with $\hat{\rho}$.

The local rigidity theorem of [28] implies that the hyperbolic cone-manifold structures on (M_∞, Σ) with a fixed cone angle (with angle at most 2π) are isolated. The above analysis applies to any convergent subsequence of the M_t. If we view the path M_t as a path ρ_t in the representation variety, this implies that any accumulation point of ρ_t as $t \to t_\infty$ corresponds to a hyperbolic cone-manifold structure on (M_∞, Σ) with cone angle α. Since these are isolated and the set of accumulation points is connected, there can be only a single accumulation point. It follows that the path ρ_t extends continuously to t_∞ and that the M_t converge in the bilipschitz topology to M_∞. $\square$

4. A packing argument

Let M be a 3-dimensional hyperbolic cone-manifold with a link Σ as singular locus. Let R be the radius of the maximal embedded tube in M around Σ and denote this tube by U_R. If Σ has multiple components, this is to be interpreted as meaning that the radii of the tubes around all of the components are the same, equal to R. In this section we will find lower bounds for the area of each component of the boundary of U_R via a packing argument analogous to the usual horoball packing arguments for nonsingular cusped hyperbolic 3-manifolds (cf. [34], [1]). For nonsingular hyperbolic 3-manifolds, similar tube packing arguments are used in [15].

Denote by $\tilde{X}$ the universal cover of $X = M - \Sigma$, equipped with the lift of the metric on X. The developing map $\tilde{X} \to \mathbb{H}^3$ can be extended by completion to the lifts of the singular locus, giving a map $D : \hat{M} \to \mathbb{H}^3$ where $\hat{M}$ is the metric completion of $\tilde{X}$. Further the covering projection $\tilde{X} \to X$ extends by completion to a map $p : \hat{M} \to M$. ($\hat{M}$ can be regarded as the universal branched covering of M, branched over Σ.)

Choose a component, Σ_0, of the lift of a component of the singular locus to $\hat{M}$. Under the developing map Σ_0 maps to a geodesic, g, in $\mathbb{H}^3$. The universal cover of $\mathbb{H}^3 - g$ can be completed by adding a geodesic, $\hat{g}$, which projects to g in $\mathbb{H}^3$. (This can be thought of as the infinite cyclic branched cover of $\mathbb{H}^3$ branched over the geodesic g.) Let $\hat{\mathbb{H}}^3$ denote this completion and let $\hat{U}_r$ denote the neighborhood of radius r about $\hat{g}$ in $\hat{\mathbb{H}}^3$. Then for each $r < R$, the r-neighborhood U_r of each component of Σ in M is isometric to the quotient of $\hat{U}_r$ by a discrete group $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$ of isometries of $\hat{\mathbb{H}}^3$ preserving the axis $\hat{g}$.

We can also regard $\hat{\mathbb{H}}^3$ as the "normal bundle" to Σ_0 in $\hat{M}$ and there is an exponential map $E : \hat{U}_{2R} \to \hat{M}$ defined by extending geodesics orthogonally from Σ_0. This gives an isometric embedding from $\hat{U}_{2R}$ onto the neighborhood of radius $2R$ about Σ_0 in $\hat{M}$.

Because R is the maximal tube radius, there is a geodesic arc τ of length $2R$ in M going from Σ to itself which is perpendicular to Σ at both endpoints. It is a shortest geodesic arc from Σ to itself not entirely contained in Σ. The radius R tube around Σ, U_R, has a self-tangency at the midpoint of τ. Now consider all the lifts to $\hat{M}$ of arcs of length $2R$ from Σ to itself, beginning at Σ_0. They end at points q_i lying on other lifts of Σ. These points can be identified, via the inverse of the exponential map E with points, also denoted by q_i, in $\hat{U}_{2R}$.

Lemma 4.1. *Let $\{q_i\}$ be the set of all endpoints of such lifts of arcs of length $2R$ from Σ to itself. Then the distance between q_i and q_j in $\hat{\mathbb{H}}^3$ satisfies $d(q_i, q_j) \geq 2R$ for all $i \neq j$.*

Proof. Consider two points q_i, q_j with $i \neq j$. These lie on the boundary of the set $\hat{U}_{2R}$ in $\hat{\mathbb{H}}^3$, which is convex since the distance to a geodesic is a convex function. Thus, the shortest geodesic γ from q_i to q_j in $\hat{\mathbb{H}}^3$ lies inside $\hat{U}_{2R}$. Composing γ with the exponential map $E : \hat{U}_{2R} \to \hat{M}$ and the (branched) covering projection $p : \hat{M} \to M$ gives a geodesic $\bar{\gamma}$ in M joining Σ to itself. Since $\bar{\gamma}$ is not entirely contained in Σ it has length at least $2R$. Hence $d(q_i, q_j) \geq 2R$. □

For each i, let B_i denote the ball in $\hat{\mathbb{H}}^3$ of radius R about q_i. We project the balls for all the q_i orthogonally onto the surface $\partial \hat{U}_R$ in $\hat{\mathbb{H}}^3$ at radius R from the singular set. The fact that the balls B_i are disjoint implies that their projections P_i are also disjoint. This follows easily from the facts that the centers of the B_i all have the same radial coordinate and all of the B_i have the same radius.

Next we will estimate the area of each P_i and use this to estimate the area of T_R. But first we prove some preliminary geometric results.

Let (r, θ, ζ) denote hyperbolic cylindrical coordinates on $\mathbb{H}^3$ about a geodesic g. These can also be regarded as cylindrical coordinates on $\hat{\mathbb{H}}^3$ about the geodesic $\hat{g}$ covering g, but the angle θ is no longer measured modulo 2π, but rather as a real number.

LEMMA 4.2. *The distance d between two points p_1, p_2 in $\hat{\mathbb{H}}^3$ with cylindrical coordinates (r_1, θ_1, ζ_1) and (r_2, θ_2, ζ_2) with $|\theta_1 - \theta_2| \leq \pi$ is given by*

$$\cosh d = \cosh(\zeta_1 - \zeta_2) \cosh r_1 \cosh r_2 - \cos(\theta_1 - \theta_2) \sinh r_1 \sinh r_2.$$

Proof. See [15, Lemma 2.1]. □

We now study the projection of a ball onto a hyperbolic cylinder.

LEMMA 4.3. *Consider a ball of radius d centered at the point with cylindrical coordinates $(r, \theta, \zeta) = (r_0, 0, 0)$ with $d < r_0$. The projection of this ball to the (θ, ζ)-plane has equation*

$$\sinh^2 \zeta \cosh^2 r_0 + \sin^2 \theta \sinh^2 r_0 \leq \sinh^2 d.$$

Proof. From the distance formula in cylindrical coordinates (Lemma 4.2), the ball has equation

$$\cosh \zeta \cosh r_0 \cosh r - \cos \theta \sinh r_0 \sinh r \leq \cosh d.$$

Writing $\cosh r$ and $\sinh r$ as exponentials gives

$$\cosh \zeta \cosh r_0 (e^r + e^{-r}) - \cos \theta \sinh r_0 (e^r - e^{-r}) - 2 \cosh d \leq 0$$

or

$$e^{2r}(\cosh \zeta \cosh r_0 - \cos \theta \sinh r_0) - 2e^r \cosh d + (\cosh \zeta \cosh r_0 + \cos \theta \sinh r_0) \leq 0.$$

Given (θ,ζ) this quadratic for e^r has a real solution if and only if the discriminant is nonnegative, i.e.

$$(2\cosh d)^2 - 4(\cosh\zeta\cosh r_0 - \cos\theta\sinh r_0)(\cosh\zeta\cosh r_0 + \cos\theta\sinh r_0) \geq 0,$$

or

$$\cosh^2\zeta\cosh^2 r_0 - \cos^2\theta\sinh^2 r_0 \leq \cosh^2 d.$$

Rewriting this, using $\cosh^2\zeta = \sinh^2\zeta + 1$ and $\cos^2\theta = 1 - \sin^2\theta$, we have

$$\sinh^2\zeta\cosh^2 r_0 + \sin^2\theta\sinh^2 r_0 \leq \sinh^2 d. \qquad \square$$

Each ball B_i has radius R and its center is at distance $2R$ from the geodesic $\hat{g}$ in $\hat{\mathbb{H}}^3$. We choose coordinates so that the center of a ball B_i has coordinates $(r,\theta,\zeta) = (2R,0,0)$. From Lemma 4.3, the projection of B_i onto the (θ,ζ)-plane satisfies the equation:

$$f(\zeta,\theta) = \sinh^2\zeta\cosh^2 2R + \sin^2\theta\sinh^2 2R \leq \sinh^2 R. \tag{29}$$

Ignoring the self-tangencies, the boundary of the maximal tube, U_R, in the cone-manifold is a torus T_R with an induced Euclidean structure. The Euclidean structure is induced from the set of points in $\hat{\mathbb{H}}^3$ at distance R from $\hat{g}$ modulo the group Γ. Since the projection of each B_i onto the (θ,ζ)-plane is disjoint from its translates under Γ, the corresponding set P_i with radial coordinate R is disjoint from its translates. This implies that it embeds in T_R under the quotient map from the action of Γ. Further the collection of P_i contains at least two distinct Γ-orbits if Σ consists of a single component.

Next we estimate the area of each P_i and use this to estimate the area of T_R.

THEOREM 4.4. *The area of the torus T_R at distance R from Σ satisfies*

$$\operatorname{area}(T_R) \geq 3.3957\frac{\sinh^2 R}{\cosh(2R)} \tag{30}$$

if Σ is connected. If Σ has multiple components, then one component satisfies (30), *while the other components satisfy*

$$\operatorname{area}(T_R) \geq 1.6978\frac{\sinh^2 R}{\cosh(2R)}. \tag{31}$$

Proof. Equation (29) gives bounds on ζ and θ:

$$|\sinh\zeta| \leq \frac{\sinh(R)}{\cosh(2R)} \quad\text{and}\quad |\sin\theta| \leq \frac{\sinh(R)}{\sinh(2R)}.$$

Now

$$\frac{\sinh R}{\cosh(2R)} = \frac{s}{1+2s^2}$$

where $s = \sinh(R)$. By the arithmetic-geometric mean inequality we have $\sqrt{2}s = \sqrt{2s^2} \leq \frac{1+2s^2}{2}$; hence $\frac{s}{1+2s^2} \leq \frac{1}{2\sqrt{2}}$ for all $s \geq 0$, with equality attained exactly when $1 = 2s^2$; i.e. $R = R_0$ where $\sinh(R_0) = \frac{1}{\sqrt{2}}$.

So for such ζ we have $|\sinh\zeta| \leq \frac{1}{2\sqrt{2}}$. Since $\sinh\zeta$ is a convex function for positive values of ζ, we obtain $\left|\frac{\sinh\zeta}{\zeta}\right| \leq S$ where

$$S = \frac{\frac{1}{2\sqrt{2}}}{\operatorname{arcsinh}(\frac{1}{2\sqrt{2}})} \approx \frac{1}{0.980258}.$$

Since $|\frac{\sin\theta}{\theta}| \leq 1$, we deduce that

$$f(\zeta,\theta) \leq (S\zeta)^2\cosh^2(2R) + \theta^2\sinh^2(2R).$$

Thus the projected ball defined by equation (29) contains the region

$$(S\zeta)^2\cosh^2(2R) + \theta^2\sinh^2(2R) \leq \sinh^2 R$$

or

$$\left(\frac{S\cosh(2R)}{\cosh R\sinh R}\right)^2(\zeta\cosh R)^2 + \left(\frac{\sinh(2R)}{\sinh^2 R}\right)^2(\theta\sinh R)^2 \leq 1. \tag{32}$$

Since $\zeta\cosh R$ and $\theta\sinh R$ are Euclidean coordinates on the torus at radius R, equation (32) describes an ellipse with semi-major axes

$$a = \frac{\cosh R\sinh R}{S\cosh(2R)} \quad \text{and} \quad b = \frac{\sinh^2 R}{\sinh(2R)} \tag{33}$$

and area

$$\pi ab = \frac{\pi\sinh^2 R}{2S\cosh(2R)}.$$

The axes of all of the ellipses are parallel to the θ and ζ axes. By an area-preserving affine transformation of the torus, we can arrange that all the inscribed ellipses simultaneously become circles of the same radius. It follows that the packing density of the ellipses is at most the maximum packing density of circles, namely $\frac{\pi}{2\sqrt{3}}$.

Furthermore, if Σ is *connected*, the torus T_R at radius R contains at least *two* disjoint ellipses (see, for example, [1]), so that its area satisfies:

$$\operatorname{area}(T_R) \geq \frac{2\sqrt{3}}{\pi}2\pi ab = \sqrt{3}ab = \frac{2\sqrt{3}\sinh^2 R}{S\cosh(2R)},$$

and so

$$\operatorname{area}(T_R) \geq 3.3957\frac{\sinh^2 R}{\cosh(2R)}.$$

When there are multiple components, it can be arranged so that there will still be two disjoint embedded ellipses on one boundary torus but perhaps only one on the remaining boundary tori. (See [5] for this argument.) Hence, for one component we obtain the same lower bound for $\operatorname{area}(T_R)$, while for the other components the lower bound for $\operatorname{area}(T_R)$ is half as large. $\square$

5. Controlling the tube radius

In this section we will use the information derived in Sections 2 and 4 to control the change in the radius of the maximal embedded tube around the singular locus. This will allow us to complete the proof of Theorem 1.1. Finally we combine this with Theorem 1.2 to prove Theorems 1.3 and 1.4.

Rather than studying the tube radius directly, we will derive information about it by studying the geometry of the torus on the boundary of the maximal tube. The boundary torus has an intrinsic flat metric. We will denote by m the length in this metric of the geodesic in the homotopy class of the meridian. The height of the maximal annulus with the meridian as its core will be denoted by h. Thus, the area of the torus, denoted by A, will equal mh. If the radius of the tube is R, then m, h and A are related to the cone angle α and the length ℓ of the singular locus by the formulae:

$$\begin{aligned} m &= \alpha \sinh R, \\ h &= \ell \cosh R, \\ A &= \alpha\ell \sinh R \cosh R. \end{aligned}$$

Theorem 4.4 implies that the area A of the flat torus satisfies

$$A \geq 3.3957 \frac{\sinh^2 R}{\cosh(2R)}.$$

Dividing by $\sinh R \cosh R$ provides the following key estimate.

Corollary 5.1. *Suppose the singular set Σ has length ℓ and cone angle α. Then the radius R of a maximal embedded tube about Σ satisfies*

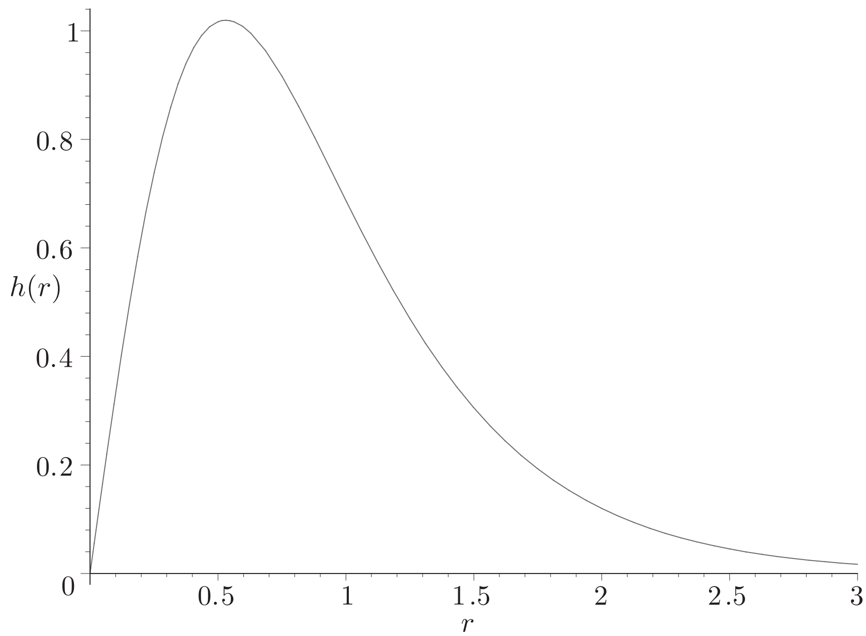
$$\alpha\ell \geq h(R) = 3.3957 \frac{\tanh R}{\cosh(2R)}. \tag{34}$$

Remark. In the case of a closed geodesic in a nonsingular hyperbolic 3-manifold we have $\alpha = 2\pi$, and this gives

$$\ell \geq 0.5404 \frac{\tanh R}{\cosh(2R)}.$$

This seems to be very close to the estimate given in Proposition 3.1 of [15].

The qualitative behavior of the function $h(r) = 3.3957\frac{\tanh r}{\cosh(2r)}$, whose graph is pictured below, is very important and will influence the form of all of our arguments.



The inequality (34) implies that, for a given tube radius, there is a lower bound to the product, $\alpha\ell$. Hence, for a given tube radius and cone angle, there is a lower bound to the length of the core curve. Instead, we would like to bound the tube radius in terms of $\alpha\ell$. This is, in fact, not literally possible and is reflected by the graph of $h(r)$ as it drops down to 0 as $r \to 0$.

However, note that $h(r)$ appears to have a single maximum near $r = .5$ (more precise values are given below) and to be strictly decreasing for values of r larger than this. In particular, it appears to be invertible for such values of r. Thus, if the tube radius R is known to be larger than this value and if $\alpha\ell$ is smaller than the maximum value of $h(r)$, then the value $h^{-1}(\alpha\ell)$ of the inverse function will provide a further lower bound for R. This lower bound goes to infinity as $\alpha\ell$ goes to zero.

In our situation, we will be starting with a complete structure, for which the tube radius is infinite and $\alpha = \ell = 0$. In particular, as we try to increase the cone angle, we begin with values of the tube radius and $\alpha\ell$ for which the inverse of the function $h(r)$ provides a lower bound to the tube radius. As long as the value of $\alpha\ell$ remains below the maximum value of $h(r)$, the tube radius is bounded below and the results of section 3 imply that there can be no degeneration.

The goal of this section is to provide initial conditions on the surgery curve that will guarantee that $\alpha\ell$ remains below this maximum value until the cone angle reaches 2π.

Remark. For smooth structures, i.e. when $\alpha = 2\pi$, the results of [35] imply that, for sufficiently short geodesics, there is a lower bound to the tube radius. This result uses Jørgensen's inequality, which has no literal analogue for cone-manifolds. To see that there is no such lower bound for the tube radius around short core curves in a general cone-manifold, one can consider the figure eight knot complement and choose the standard meridian as the surgery curve. As the cone angle increases, the length of the core geodesic increases for a while (enough for $\alpha\ell$ to become larger than the maximum of $h(r)$), but then goes to 0 as the cone angle approaches $\alpha = \frac{2\pi}{3}$. In fact, the hyperbolic structures degenerate in such a way, that, if they are rescaled to have volume 1, they converge to a Euclidean orbifold at $\alpha = \frac{2\pi}{3}$.

The following lemma shows that the qualitative behavior of the function $h(r)$ which was presumed in the previous discussion is as desired. It also provides an accurate value for the maximum of $h(r)$ and for the value of r at which it is attained.

Lemma 5.2. *The function $h(r)$ is a decreasing function of r for $r \geq 0.531$ with an inverse $h^{-1}(a)$ defined for $0 \leq a \leq h_{\max} = h(0.531) \approx 1.019675$ such that $h^{-1}(a) = r$ if and only if $h(r) = a$ and $r \geq 0.531$.*

Proof. Writing the function h in terms of $\zeta = \tanh r$, we have

$$h(r) = 3.3957 \tanh r \, \frac{\cosh^2 r - \sinh^2 r}{\cosh^2 r + \sinh^2 r} = 3.3957 \, \frac{\zeta(1-\zeta^2)}{1+\zeta^2}.$$

If we put $f(\zeta) = \dfrac{\zeta(1-\zeta^2)}{1+\zeta^2}$, then $f'(\zeta) = \dfrac{1-4\zeta^2-\zeta^4}{(1+\zeta^2)^2}$. Hence $f(\zeta)$ has a unique maximum for $0 < \zeta < 1$ when $1 - 4\zeta^2 - \zeta^4 = 0$, or $\zeta^2 = \sqrt{5} - 2$. Then $\zeta \approx 0.485868$, $r = \operatorname{arctanh}(\zeta) \approx 0.5306375$ and $h(r) = 3.3957 \, f(\zeta) \approx 1.0196755$. The result follows immediately. □

From this lemma we deduce that the estimate (34) gives a lower bound for the tube radius in terms of $\alpha\ell$:

Proposition 5.3. *The tube radius R satisfies*

$$R \geq h^{-1}(\alpha\ell) \text{ when } \alpha\ell \leq h_{\max} \approx 1.019675 \text{ and } R \geq 0.531. \tag{35}$$

Together with the nondegeneration results of Section 3 we immediately have the following theorem:

Theorem 5.4. *Let M_s be a smooth family of finite volume 3-dimensional hyperbolic cone-manifolds, with cone angles $\alpha_s, 0 \leq s < 1$, where $\lim_{s\to 1} \alpha_s = \alpha_1$. Suppose the tube radius R satisfies $R \geq 0.531$ for $s = 0$ and $\alpha_s \ell_s \leq h_{\max}$ holds for all s, where ℓ_s denotes the length of the singular geodesic. If the* *volumes of the M_s remain bounded, then the M_s converge geometrically to a cone-manifold M_1 with cone angle α_1. In particular, this conclusion holds if M_0 is complete ($\alpha_0 = 0$), α_s is increasing and $0 < \alpha_s \ell_s \le h_{\max}$ for all s.*

Proof. Proposition 5.3 implies that, if the initial tube radius is at least 0.531, then, since $h^{-1}(\alpha\ell) \ge 0.531$ by definition, the tube radius will remain at least 0.531 as long as h^{-1} is defined. This will be the case as long as $\alpha_s \ell_s \le h_{\max}$. The first statement now follows immediately from Theorem 1.2. In the special case when M_0 is complete, the tube radius is infinite, hence bigger than 0.531, for $s = 0$. From the Schläfli differential formula (23), the volume decreases as the cone angle increases. Hence the volumes are uniformly bounded throughout the deformation and this special case follows from the general case. □

In light of the above theorem, we would like to find a method to bound the quantity $\alpha\ell$ from above throughout a deformation. Since $t = \alpha^2$ is our parameter, this amounts to controlling the growth of the core length ℓ. Our estimates from Section 2 provide control of the change in ℓ in terms of α provided that the tube radius is bounded below. Specifically, recall that equation (21) gives

$$\frac{d\ell}{d\alpha} = \frac{\ell}{\alpha}(1 + 4\alpha^2 x),$$

and that we have the estimate (22)

$$\frac{-1}{\sinh^2(R)}\left(\frac{2\sinh^2(R)+1}{2\sinh^2(R)+3}\right) \le 4\alpha^2 x \le \frac{1}{\sinh^2(R)}.$$

Using Proposition (5.3) we can, in turn, bound R in terms of $\alpha\ell$. Because of its importance in what follows, we introduce the new variable

$$\rho = h^{-1}(\alpha\ell).$$

Note that ρ is defined whenever $\alpha\ell = h(\rho) \le h_{\max}$ and, if $R \ge 0.531$ also, it satisfies $0.531 \le \rho \le R$. This allows us to replace R with ρ in the estimate (22):

Proposition 5.5. *Whenever $\alpha\ell \le h_{\max}$ and $R \ge 0.531$ the following inequality holds*:

$$\frac{-1}{\sinh^2(\rho)}\left(\frac{2\sinh^2(\rho)+1}{2\sinh^2(\rho)+3}\right) \le 4\alpha^2 x \le \frac{1}{\sinh^2(\rho)}. \tag{36}$$

Proof. Proposition 5.3 implies that $\rho \le R$. The result follows immediately once it is noted that both $\frac{1}{\sinh^2(r)}$ and $\frac{1}{\sinh^2(r)}\left(\frac{2\sinh^2(r)+1}{2\sinh^2(r)+3}\right)$ are decreasing in r. That the first is decreasing is obvious; that the second is decreasing can be seen easily by rewriting it as $\frac{2+\frac{1}{\sinh^2(r)}}{2\sinh^2(r)+3}$ so that the numerator is decreasing and the denominator increasing. □

The significance of putting the inequality in the form (36), as opposed to that of (22) is that, since ρ is a function of $\alpha\ell$, the inequality bounds the derivative of the core length ℓ purely in terms of α and ℓ. Since α^2 is our parameter, this will allow us to bound the value $\alpha\ell$ by integration, after some algebraic manipulation and separation of variables.

Now put

$$u = \frac{\alpha}{\ell}.$$

This turns out to be an important and useful function of α and ℓ. It approaches a finite, nonzero value as one approaches the cusp case, even though ℓ and α both approach 0. Recall that the meridian length m and annulus height h in the flat metric on the boundary of a tube of radius R around the singular locus satisfy $m = \alpha \sinh R,\ h = \ell \cosh R$. Thus, as $R \to \infty$, the ratio of α to ℓ approaches that of m to h. This implies that:

$$\lim_{R\to\infty} u = \lim_{R\to\infty} \frac{m}{h} = \lim_{R\to\infty} \frac{m^2}{A} = \hat{L}^2, \tag{37}$$

where $\hat{L}$ is the normalized length of the meridian curve on the torus boundary of the tube around the cusp.

This provides an initial condition for u in terms of the normalized length of the chosen surgery curve. To control the value of $\alpha\ell$, it suffices to control the value of u. The derivative of u can be computed by:

$$\frac{du}{d\alpha} = \frac{1}{\ell} - \frac{\alpha}{\ell^2}\frac{d\ell}{d\alpha} = \frac{\alpha}{\ell^2}\left(\frac{\ell}{\alpha} - \frac{d\ell}{d\alpha}\right) = \frac{\alpha}{\ell^2}\left(-4\alpha^2 x\frac{\ell}{\alpha}\right) = -\frac{1}{\ell}(4\alpha^2 x),$$

or

$$\frac{du}{dt} = \frac{1}{2\alpha}\frac{du}{d\alpha} = -\frac{1}{2\alpha\ell}(4\alpha^2 x),$$

where $t = \alpha^2$.

Using (36) and the fact that $h(\rho) = \alpha\ell$ we obtain upper and lower bounds on the derivative of u in terms of ρ. The expressions for these bounds become simpler if we use the variable:

$$z = \tanh\rho.$$

Then, as derived in the proof of Lemma 5.2, $h(\rho) = 3.3957\frac{z(1-z^2)}{1+z^2}$. We define the function

$$H(z) = \frac{1}{\alpha\ell} = \frac{1}{h(\rho)} = \frac{1+z^2}{3.3957z(1-z^2)}. \tag{38}$$

Noting that $\sinh^2(\rho) = \frac{z^2}{1-z^2}$ we can rewrite the inequality (36) in terms of z:

$$-\left(\frac{(1-z^2)(1+z^2)}{z^2(3-z^2)}\right) \le 4\alpha^2 x \le \frac{1-z^2}{z^2}. \tag{39}$$

We introduce the functions:

$$G(z) = \frac{H(z)}{2}\frac{1-z^2}{z^2} = \frac{1+z^2}{6.7914\ z^3} \tag{40}$$

and

$$\tilde{G}(z) = \frac{H(z)}{2}\frac{(1-z^2)(1+z^2)}{z^2(3-z^2)} = \frac{(1+z^2)^2}{6.7914\ z^3\,(3-z^2)}. \tag{41}$$

Since $\frac{du}{dt} = -\frac{1}{2\alpha\ell}(4\alpha^2 x) = -\frac{H(z)}{2}(4\alpha^2 x)$, the inequality (39) provides inequalities for $\frac{du}{dt}$ expressed purely in terms of z. Using the functions defined above, the inequalities can be written simply as

$$-G(z) \leq \frac{du}{dt} \leq \tilde{G}(z).$$

These inequalities hold as long as $R \geq 0.531$ and $\alpha\ell < h_{\max}$. The latter holds, by definition of h^{-1}, as long as $h^{-1}(\alpha\ell) = \rho \geq \rho_1 = 0.531$, or, since $z = \tanh(\rho)$ is increasing in ρ, as long as $z \geq \tanh\rho_1 \approx 0.4862 = z_1$. If the initial tube radius is at least 0.531 then it will remain so as long as $\rho \geq \rho_1$. Thus, in this case, as long as $z \geq z_1$, the inequalities are valid. We record this fact as a proposition.

Proposition 5.6. *For any smooth family of hyperbolic cone-manifolds whose initial tube radius is at least* 0.531, *the following differential inequalities hold as long as* $z \geq z_1 = 0.4862$:

$$-G(z) \leq \frac{du}{dt} \leq \tilde{G}(z), \tag{42}$$

with the functions $G(z)$ *and* $\tilde{G}(z)$ *defined by* (40) *and* (41), *respectively.*

We will only use the lower bound in this section. The upper bound will be used in the final section.

In order to solve this differential inequality, we note that $u = \frac{\alpha}{\ell} = \frac{t}{\alpha\ell}$, where $t = \alpha^2$ is our variational parameter. By definition, $H(z) = \frac{1}{\alpha\ell}$ and this becomes

$$u = tH(z). \tag{43}$$

From the inequality (42) we obtain

$$\frac{d}{dt}(H(z)t) \geq -G(z)$$

or

$$t\frac{dH}{dz}\frac{dz}{dt} \geq -(H(z)+G(z)). \tag{44}$$

Denoting $\frac{dH}{dz}$ by $H'(z)$ we obtain the inequality:

$$\frac{dz}{dt} \geq \frac{-(H(z)+G(z))}{t\ H'(z)}. \tag{45}$$

Again, if the initial structure has tube radius at least 0.531, this inequality is valid as long as $z > z_1$. Observe that $H'(z)$ is *positive* since, by Lemma 5.2, $h(\rho) = \frac{1}{H(z)}$ is decreasing for these values of $z = \tanh \rho$.

Since this inequality bounds the change in z, if we start with a complete structure, where $z = 1$, it should provide conditions under which this inequality will be maintained until $t = (2\pi)^2$. In particular, we will have $z \geq z_1$; hence $\alpha\ell < h_{\max}$, throughout the deformation, implying, by Theorem 5.4, that the smooth structure with cone angle 2π can be reached without any degeneration. To do this explicitly we will use separation of variables.

By algebraic manipulation we obtain

$$\frac{H'(z)}{H(z)+G(z)}\,\frac{dz}{dt} \geq -\frac{1}{t}. \tag{46}$$

However this separation of variables is only valid away from the complete structure because both sides of the new inequality blow up as $t \to 0$ and $z \to 1$. It cannot be applied directly for initial conditions at the complete structure. Some care must be taken to analyze the rate at which the left side goes to infinity as $t \to 0$.

We compute that

$$\frac{H'(z)}{H(z)+G(z)} = F(z) + \frac{1}{1-z}$$

where

$$F(z) = -\frac{(1+4z+6z^2+z^4)}{(z+1)(1+z^2)^2},$$

and F is integrable on the interval $0 \leq z \leq 1$. Recall that $z(t)$ is a smooth function of t which approaches 1 as t approaches 0. For any sufficiently small value of $t > 0$, $z(t) < 1$ will be larger than $z_1 = .4862$ and the differential inequality (46) holds. Choose $0 < t_0 < \tau$ so that $z_1 < z(t) < 1$ for all $0 < t < \tau$, and denote $z(t_0)$ by z_0. Integrating the inequality over the interval $0 < t < \tau$ and changing the variable to $w = z(t)$, we obtain

$$\int_{z_0}^{z(\tau)} F(w)\,dw \;+\; \log(1-z_0) - \log(1-z(\tau)) \geq \log(t_0) - \log(\tau)$$

or

$$\exp\left(\int_{z_0}^{z(\tau)} F(w)\,dw\right) \geq \frac{t_0}{1-z_0}\,\frac{1-z(\tau)}{\tau}.$$

To compute the limit of $\frac{t_0}{1-z_0}$ as $t_0 \to 0$, multiply the numerator and denominator by $H(z_0)$. Since $u(t) = H(z(t))t$, this becomes $\frac{u(t_0)}{(1-z_0)H(z_0)}$. Now as $t_0 \to 0$, $z_0 \to 1$ and from the formula (38) it is clear that $H(z_0)(1-z_0) \to \frac{1}{3.3957}$. From (37) we know that $\lim_{R\to\infty} u = \hat{L}^2$. Since $R \to \infty$ as $t \to 0$, it follows that $\lim_{t\to 0} u(t) = \hat{L}^2$.

We conclude that

$$\exp\left(\int_1^{z(\tau)} F(w)\,dw\right) \geq 3.3957\ \hat{L}^2\ \frac{1-z(\tau)}{\tau}.$$

This inequality holds for any time τ during a deformation through cone-manifolds which begins at a complete structure (where $z(0) = 1$); we use a surgery curve of normalized length $\hat{L}$, as long as $z(t)$ is larger than z_1 throughout the deformation. This provides information about the times t at which various values of $z(t)$ can be attained. In particular, this implies, for any $z \geq z_1$, the following inequality for the first time t at which $z(t) = z$:

$$t \geq 3.3957\ \hat{L}^2\ (1-z)\ \exp\left(-\int_1^{z} F(w)\,dw\right). \tag{47}$$

We conclude that we can increase the cone angle α from 0 to 2π, while maintaining the condition $z = \tanh\rho \geq z_1 > \tanh(\rho_1)$, hence keeping the tube radius $R \geq \rho \geq \rho_1 = 0.531$ and $\alpha\ell \leq h_{\max}$, provided

$$3.3957\ \hat{L}^2\ (1-z_1)\ \exp\left(-\int_1^{z_1} F(w)dw\right)\ \geq\ (2\pi)^2$$

or

$$\hat{L}^2\ \geq\ \frac{(2\pi)^2}{3.3957(1-z_1)}\ \exp\left(\int_1^{z_1} F(w)dw\right) \approx 56.4696$$

or

$$\hat{L}\ \geq\ \sqrt{56.4696} \approx 7.5146.$$

Thus, we have shown that as long as the normalized Euclidean geodesic length $\hat{L}$ of the surgery curve satisfies this inequality then there is a lower bound to the tube radius. This completes the proof of Theorem 1.1 which we restate here for convenience.

Theorem 5.7. *Let X be a complete, finite volume, hyperbolic 3-manifold with one cusp and let T be a horospherical torus which is embedded as a cross-section to the cusp. Let γ be a simple closed curve on T and $X(\gamma)$ be the Dehn filling with γ as meridian. Let $X_\alpha(\gamma)$ be a cone-manifold structure on $X(\gamma)$ with cone angle α along the core, Σ, of the added solid torus, obtained by increasing the angle from the complete structure. If the normalized length of γ on T is at least 7.515, then there is a positive lower bound to the tube radius around Σ for all $2\pi \geq \alpha \geq 0$.*

Remark 5.8. The proof of this theorem shows that $\alpha\ell \leq h_{\max}$ and $R \geq 0.531$ for $0 \leq \alpha \leq 2\pi$, where ℓ and R denote the length and tube radius of the singular geodesic Σ. In particular, the core geodesic of the nonsingular hyperbolic structure on $X(\gamma)$ has length $\ell \leq \frac{h_{\max}}{2\pi} \approx 0.162$ and tube radius $R \geq 0.531$.

Theorem 1.1, together with Theorem 1.2, implies our main result, Theorem 1.3. (Note also that Theorem 5.4 (which depends on Theorem 1.2) together with (47) immediately implies Theorem 1.3.)

THEOREM 5.9. *Let X be a complete, orientable hyperbolic 3-manifold with one cusp, and let T be a horospherical torus which is embedded as a cross-section to the cusp of X. Let γ be a simple closed curve on T whose normalized Euclidean geodesic length $\hat{L}$ is at least 7.515. Then the closed manifold $X(\gamma)$ obtained by Dehn filling along γ is hyperbolic.*

This result also gives a universal bound on the number of nonhyperbolic Dehn fillings on a cusped hyperbolic 3-manifold.

COROLLARY 5.10. *Let X be a complete, orientable hyperbolic 3-manifold with one cusp. Then at most 60 Dehn fillings on X yield manifolds which admit no complete hyperbolic metric.*

Proof. The arguments of Agol [2] give the following result.

LEMMA 5.11. *The number of slopes of length less than L on a Euclidean torus of area A is at most $p+1$, where p is the smallest prime number larger than L^2/A.*

Now let T be the Euclidean torus obtained by taking a horospherical cusp cross-section in X and rescaling the metric so that T has area $A = 1$. Then the slopes β with X_β nonhyperbolic have length on T less than $L = 7.515$. Since $56 < L^2/A = (7.515)^2 < 57$, application of Agol's lemma with $p = 59$ shows that the number of exceptional slopes is at most $p + 1 = 60$. □

If there are multiple cusps, we must vary the rate at which the cone angles are increased to ensure that the upper bound in (22) is satisfied for each boundary component. The argument then proceeds as before except that we can no longer use the larger area bound (30) from the packing theorem (Theorem 4.4). When there are multiple cusps, Theorem 4.4 gives an area bound (30) for one boundary torus and an area bound (31) that is half as large for the remaining boundary tori.

This implies that the function corresponding to $h(r)$ (see (34)) on the remaining boundary tori is half as big. It follows that the functions corresponding to $H(z), G(z), \tilde{G}(z)$ are twice as big. Everything else remains the same. The effect on the differential inequalities is that the inequality (42) is replaced by one in which $G(z)$ and $\tilde{G}(z)$ are twice as large. However, the key inequality (44) relating the change in $z = \tanh(\rho)$ to that of $t = \alpha^2$ remains exactly the same because it involves the ratio of functions, each of which is twice as large. The only change in the analysis arising from that inequality is that the limit as $z \to 1$ of $H(z)(1-z)$ is twice as large. In other words, $H(z)(1-z) \to \frac{2}{3.3957}$ and the coefficient 3.3957 in inequality (47) is replaced by $\frac{3.3957}{2}$. So, to guarantee that angle $\alpha = 2\pi$ is reached, we need to assume that the normalized Euclidean geodesic lengths of all of the surgery curves satisfy

$$\hat{L} \ \geq \ \sqrt{2\ (56.4696)} \approx 10.6273.$$

Since the first prime number larger than 2 (56.4696) $\approx$ 112.939 is 113, the bound on the number of exceptional slopes per cusp, for all but one cusp, becomes $113 + 1 = 114$. The bound for the other cusp will still be 60.

Thus, we have proved the following:

THEOREM 5.12. *Let X be a complete, finite volume orientable hyperbolic 3-manifold with more than one cusp, and let T_i be a horospherical torus which is embedded as a cross-section to the i^{th} cusp of X. Let $\{\gamma_i\}$ be simple closed curves on the T_i and suppose that, for all $i > 1$, the normalized Euclidean geodesic length of γ_i on T_i is at least 10.628 and for $i = 1$ it is at least 7.515. Then the closed manifold $X(\gamma)$ obtained by Dehn filling along $\gamma = \{\gamma_i\}$ is hyperbolic. In particular, there are at most 60 choices of γ_1 on the first cusp and 114 choices of γ_i on the remaining cusps so that $X(\gamma)$ can fail to have a hyperbolic metric.*

Remark 5.13. The results in this section provide initial conditions which guarantee that any particular collection of cone angles, all at most 2π, can be realized by hyperbolic cone-manifold structures on $X(\gamma)$. In particular, they imply the existence of hyperbolic structures on *orbifolds* when the singular locus is a link. In this case, the cone angles are all of the form $\frac{2\pi}{n}, n \in \mathbb{Z}$. The conditions on the the normalized Euclidean geodesic lengths are replaced by the same condition on n times the length. Similarly, the results of Section 6, concerning volumes and lengths of the singular locus, will also apply in this case.

6. Geometry comparison

6.1. *Decreasing the cone angle.* It is natural to ask how general this process of constructing a closed hyperbolic manifold is. Can every closed hyperbolic 3-manifold be obtained by starting with a noncompact, finite volume 3-manifold with one cusp and increasing the cone angle from 0 to 2π? Specifically, given a simple closed geodesic τ in a closed hyperbolic 3-manifold N, can the cone angle be *decreased* from 2π (at the smooth structure) back to angle 0? There is no topological obstruction to doing this. It can be shown (see [32, Theorem 1.2.1], [3]) that $N - \tau$ can be given a complete finite volume metric with pinched negative curvature so that it will be irreducible, atoroidal, and have infinite fundamental group. In fact, since it is the interior of a manifold with nonempty boundary, it is Haken, so that Thurston's geometrization theorem for Haken manifolds implies that it can be given a hyperbolic structure. The issue is whether or not the hyperbolic structures on N and on $N - \tau$ can be connected by a family of hyperbolic cone-manifolds.

In this section we apply our techniques to show that, as long as τ is sufficiently short, with length less than a universal constant independent of N, then N can be constructed in this manner. The cone angle can be decreased back from 2π to 0.

To see why the condition that τ be short might arise from the techniques of the previous section, note that all of the closed hyperbolic manifolds constructed in the proof of Theorem 1.3 have a short geodesic, which was the singular set throughout the deformation through cone-manifolds. It is short because the control of the tube radius using the inverse function $h^{-1}(\alpha\ell)$ only held as long as $\alpha\ell \leq h_{\max} \approx 1.019675$. When $\alpha = 2\pi$ this holds if $\ell \leq 0.162$.

In order to show that it is possible to decrease the cone angle back to 0, the main step is again to show that the tube radius is bounded below. By Theorem 5.4 it suffices to show that the initial tube radius R satisfies $R \geq 0.531$, that the volumes remain bounded, and that, if $\alpha\ell \leq h_{\max}$ at the beginning of the deformation, it will remain so throughout.

Lemma 6.1. *$\alpha\ell$ is an increasing function of α provided the tube radius R satisfies $R \geq 0.4407$ and $\alpha \leq 2\pi$.*

Proof. Using equation (21) and estimate (22) (the proof of the estimate using [28] requires that $\alpha \leq 2\pi$, though this is probably unnecessary), we have

$$\frac{1}{\alpha}\frac{d(\alpha\ell)}{d\alpha} = \frac{d\ell}{d\alpha} + \frac{\ell}{\alpha} = \frac{\ell}{\alpha}(2 + 4\alpha^2 x) \geq \frac{\ell}{\alpha}\left(2 - \frac{1}{\sinh^2 R}\left(\frac{2\sinh^2 R + 1}{2\sinh^2 R + 3}\right)\right) \geq 0$$

provided $\frac{1}{s^2}(\frac{2s^2+1}{2s^2+3}) \leq 2$ or $4s^4 + 4s^2 - 1 \geq 0$ where $s = \sinh R$. This holds provided $(2s^2+1)^2 \geq 2$, i.e. $s^2 \geq \frac{\sqrt{2}-1}{2}$ or $R \geq 0.4407$. $\square$

Theorem 6.2. *Let M be a closed hyperbolic 3-manifold and τ a simple closed geodesic in M having length $l(\tau) \leq h_{\max}/(2\pi) \approx 0.1623$ and tube radius $R \geq 0.531$. Then the hyperbolic structure on M can be deformed to a complete hyperbolic structure on $M - \tau$ by decreasing the cone angle along τ from 2π to 0.*

Proof. For $\alpha = 2\pi$ we have $\alpha\ell \leq h_{\max}$ and $R \geq 0.531 > 0.4407$. By Lemma 6.1, $\alpha\ell \leq h_{\max}$ throughout any deformation decreasing the cone angles. Since the volume is increasing, it is not immediate that the volumes are bounded above. However, it is not difficult to show by general arguments, for example by using the Gromov norm, that an upper bound exists. (See, for example, [32, Prop. 1.3.2] or [3].) For our specific situation, we can appeal to the next section in which we give explicit bounds on the change in volume as the cone angle is changed. In particular, this provides an upper bound over the family of cone-manifolds.

Hence there can be no degeneration by Theorem 5.4. □

The previous theorem requires conditions on both the length of the geodesic and on its tube radius. As noted previously, for a general cone-manifold, it is not true that a sufficiently short singular locus provides a lower bound on the tube radius. However, for smooth hyperbolic manifolds such a lower bound does exist. This follows from the Margulis Lemma or the Jørgensen inequality.

An explicit formula giving a lower bound to the tube radius around sufficiently short closed geodesics in closed hyperbolic 3-manifolds was derived by Meyerhoff and Zagier [35] and sharpened by Cao, Gehring, Martin [10] (see also [15, Theorem 3.2]). By combining this bound with a tube-packing estimate, similar to the estimate $\alpha\ell \geq h(r)$ from the previous section, Gabai-Milley-Meyerhoff obtain an improved bound on the tube radius of short geodesics ([15, Theorem 3.1]). In particular, their formula implies that if τ is a closed geodesic in a smooth hyperbolic 3-manifold and if its length satisfies $\ell(\tau) \leq 0.111$, then it has tube radius $R \geq 0.982 \geq 0.531$. Hence, the previous theorem applies. The conclusion is:

Corollary 6.3. *Let M be a closed hyperbolic 3-manifold and τ a simple closed geodesic in M having length $\ell(\tau) \leq 0.111$. Then the hyperbolic structure on M can be deformed to a complete hyperbolic structure on $M - \tau$ by decreasing the cone angle along τ from 2π to zero.*

Suppose τ is actually a *shortest* closest geodesic in a closed hyperbolic 3-manifold M. Then τ is a simple closed curve, and the results of Gabai-Meyerhoff-Thurston [16] show that either τ has tube radius $R \geq \log(3)/2 > 0.531$ or τ has length > 0.831. Thus if τ has length ≤ 0.162 then the hypotheses of Theorem 6.2 are again satisfied. This proves

Corollary 6.4. *Let M be a closed hyperbolic manifold and let τ be a shortest closed geodesic in M having length $\ell(\tau) \leq 0.162$. Then the hyperbolic structure on M can be deformed to a complete hyperbolic structure on $M - \tau$ by decreasing the cone angle along τ from 2π to 0.*

6.2. *Volume estimates.* The rigidity theorem of Mostow and Prasad shows that geometric invariants of finite volume hyperbolic 3-manifolds are actually topological invariants. Perhaps the most useful such invariant is the hyperbolic volume. This volume has proved to be a good way of distinguishing 3-manifolds, and is a very good measure of the complexity of a manifold.

Thurston and Jørgensen [43] proved that the set of volumes of complete, finite volume, orientable, hyperbolic 3-manifolds is a well-ordered, closed subset of $\mathbb{R}$ of order type ω^ω, and that there are finitely many manifolds of any given volume. Thus the volumes can be arranged:

$$0 < v_1 < v_2 < \cdots < v_\omega < v_{\omega+1} < \cdots < v_{2\omega} < \cdots < v_{3\omega} < \cdots < v_{\omega^2} < \ldots .$$

The smallest volume v_1 is the volume of a closed hyperbolic 3-manifold, and the first limit volume v_ω represents the volume of the smallest cusped hyperbolic 3-manifold.

In general, the volume v of each cusped hyperbolic 3-manifold M is a limit point: performing Dehn filling on M and produces a collection of closed hyperbolic manifolds converging geometrically to the cusped manifold, and their volumes converge to v from below. Thus the decrease in volume during Dehn filling is an indication of how close the filled manifold is geometrically to the cusped manifold.

A few of the lowest volumes are now known. Adams [1] has shown the smallest *nonorientable* cusped hyperbolic 3-manifold has volume $1.01494\ldots$; this is the volume of the Gieseking manifold, a nonorientable manifold double covered by the figure eight knot complement. Recently, Cao-Meyerhoff [11] showed that the *orientable* cusped hyperbolic 3-manifolds of smallest volume are the figure eight knot complement and another closely related manifold, with volume $v_\omega = 2.02988\ldots$.

For closed manifolds, much less is known. The best current estimate for v_1 is that $0.32 < v_1 \leq 0.9427\ldots$, where the right-hand side represents the volume of the "Weeks manifold" obtained by $(5,-1),(5,2)$ surgery on the Whitehead link. The left-hand side is an estimate obtained by Agol [3], improving earlier results of Meyerhoff [35] [34], Gabai-Meyerhoff-Thurston [16], Gehring-Martin [18], [19] and Przeworski [41].

Since the smallest cusped manifold volume is known but the smallest closed manifold volume is not known, we could try to study volumes of closed hyperbolic 3-manifolds by regarding them as Dehn fillings on cusped manifolds. Our work in Sections 2 and 5, gives good control on the change in length of the core geodesic during Dehn filling. We now show that this leads to good estimates on the change in hyperbolic volume during Dehn filling.

THEOREM 6.5. *Let X be a cusped hyperbolic 3-manifold and M a closed hyperbolic 3-manifold which can be joined by a smooth family of hyperbolic cone-manifolds with cone angles $0 \leq \alpha \leq 2\pi$ along a knot Σ. Suppose that $\alpha\ell \leq h_{\max} \approx 1.019675$ holds throughout the deformation, where ℓ denotes the length of Σ. Then the difference in volume*

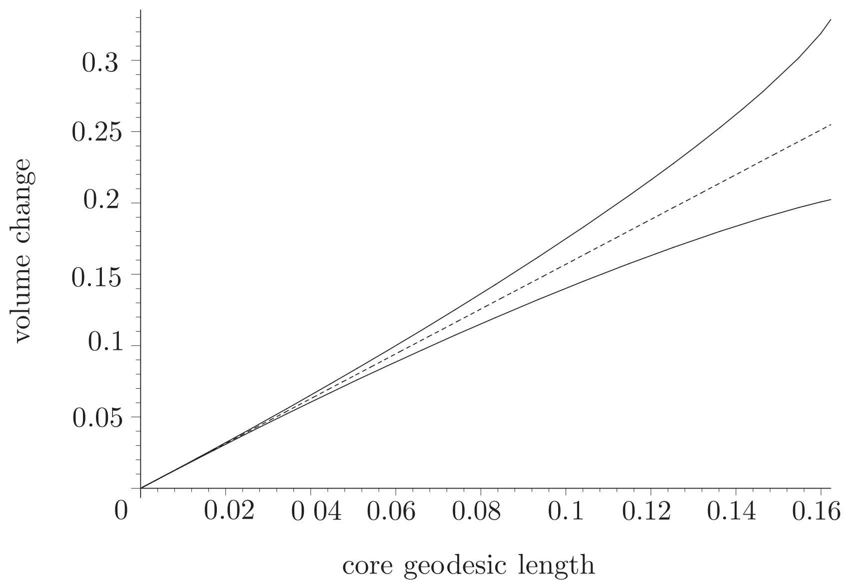
$$\Delta V = \text{Volume}(X) - \text{Volume}(M)$$

satisfies

$$\int_{\hat{z}}^{1} \frac{H'(z)dz}{4H(z)(H(z)+G(z))} \leq \Delta V \leq \int_{\hat{z}}^{1} \frac{H'(z)dz}{4H(z)(H(z)-\tilde{G}(z))} \qquad (48)$$

where $\hat{z} = \tanh(\hat{\rho})$, $\hat{\rho}$ *is the unique solution of* $h(\hat{\rho}) = 2\pi\hat{\ell}$ *with* $\hat{\rho} \geq 0.531$, *and* $\hat{\ell}$ *is the length of* Σ *in* M *(i.e. when* $\alpha = 2\pi$*).*

Remark 6.6. The graph below shows the upper and lower bounds for ΔV given by this theorem as a function of the core geodesic length ℓ in M, for $\ell < 0.162$. The dotted line shows the asymptotic formula $\Delta V \sim \frac{\pi}{2}\ell$ as $\ell \to 0$ of Neumann-Zagier [36].



Proof. We write $t = \alpha^2$ and use the notation from Section 5. From the Schläfli formula (23), the change in volume V of a hyperbolic cone-manifold during a deformation satisfies

$$dV = -\frac{1}{2}\ell d\alpha = -\frac{\alpha d\alpha}{2u} = -\frac{dt}{4u} \tag{49}$$

since $\ell = \frac{\alpha}{u}$ and $dt = d(\alpha^2) = 2\alpha d\alpha$. Recalling that $u = H(z)t$ we can rewrite this as:

$$\frac{dV}{dt} = -\frac{1}{t}\frac{1}{4H(z)}. \tag{50}$$

Since X is a cusped manifold, the condition $\alpha\ell \leq h_{\max}$ guarantees that the tube radius satisfies $R \geq \rho_1 = 0.531$ throughout the deformation (see Theorem 5.4). From equation (42) we have

$$-G(z) \leq \frac{du}{dt} \leq \tilde{G}(z),$$

with $G(z)$ and $\tilde{G}(z)$ defined as in (40) and (41). Again, since $u = H(z)t$, it follows that $\frac{du}{dt} = H'(z)t\frac{dz}{dt} + H(z)$. Hence, we obtain:

$$-(G(z) + H(z)) \leq H'(z)t\frac{dz}{dt} \leq \tilde{G}(z) - H(z).$$

With algebraic manipulation to separate the variables as in Section 5 this becomes:

$$\frac{H'(z)}{G(z)+H(z)}\,\frac{dz}{dt} \geq -\frac{1}{t} \geq \frac{H'(z)}{H(z)-\tilde{G}(z)}\,\frac{dz}{dt}. \tag{51}$$

To verify the direction of the inequalities, note as before that $H'(z)$ is positive for all $z > z_1$. It is also true, for such values of z, that $H(z) - \tilde{G}(z)$ is positive. To see this, recall that $\tilde{G}(z) = \frac{H(z)}{2}\left(\frac{(1-z^2)(1+z^2)}{z^2(3-z^2)}\right)$. Hence, it suffices to check that $\frac{(1-z^2)(1+z^2)}{z^2(3-z^2)} < 2$. But $\frac{(1-z^2)(1+z^2)}{z^2(3-z^2)} = \frac{1}{\sinh^2\rho}\left(\frac{2\sinh^2\rho+1}{2\sinh^2\rho+3}\right)$ and we computed in the proof of Lemma 6.1 that this is less than 2 as long as $\rho > 0.4407$. Since $\rho \geq \rho_1 = 0.531$ the inequality holds.

Putting together equation (50) with inequality (51), we obtain:

$$\frac{H'(z)}{4H(z)(H(z)+G(z))}\,\frac{dz}{dt} \;\geq\; \frac{dV}{dt} \;\geq\; \frac{H'(z)}{4H(z)(H(z)-\tilde{G}(z))}\,\frac{dz}{dt}. \tag{52}$$

We now integrate over the interval $0 \leq t \leq \hat{t} = (2\pi)^2$, and change variable from t to $z(t)$. As t increases, the values of z *decrease* from $z(0) = 1$ to a value $\hat{z} = z(\hat{t})$, satisfying $\hat{z} > z_1 = \tanh\rho_1$, so that the *decrease* in volume satisfies:

$$\int_{\hat{z}}^{1} \frac{H'(z)}{4H(z)(H(z)+G(z))}\,dz \;\leq\; \Delta V \;\leq \int_{\hat{z}}^{1} \frac{H'(z)}{4H(z)(H(z)-\tilde{G}(z))}\,dz.$$

(Note that the integrands are positive.) □

The results of Section 5 (Remark 5.8) show that Theorem 6.5 applies when $M = X(\gamma)$ is obtained from a cusped manifold X by Dehn filling along a surgery curve γ with normalized length $\hat{L} \geq 7.515$. This gives

Corollary 6.7. *Let X be a complete, finite volume, hyperbolic manifold with one cusp and γ a surgery curve with normalized length $\hat{L} \geq 7.515$. Then $X(\gamma)$ is hyperbolic and its volume satisfies*:

$$\text{Volume}(X(\gamma)) \geq \text{Volume}(X) - 0.329.$$

In particular,

$$\text{Volume}(X(\gamma)) \geq 1.701.$$

Proof. For surgery curves γ with normalized length $\hat{L} \geq 7.515$, we have $\alpha\ell \leq h_{\max}$ and tube radius $R \geq \rho_1 = 0.531$ as the cone angle is increased from 0 to 2π. The values of z *decrease* from 1 to a value $\hat{z}$, satisfying $\hat{z} > z_1 = \tanh \rho_1$; so the *decrease* in volume is at most:

$$\int_{\hat{z}}^{1} \frac{H'(z)dz}{4H(z)(H(z) - \tilde{G}(z))} \leq \int_{z_1}^{1} \frac{H'(z)dz}{4H(z)(H(z) - \tilde{G}(z))} < 0.3287,$$

i.e.

$$\text{Volume}(X(\gamma)) \geq \text{Volume}(X) - 0.3287.$$

The results of Cao-Meyerhoff [11] show that the figure eight knot complement and its sister are the cusped orientable hyperbolic 3-manifolds of minimal volume ≈ 2.02988. So we conclude that any surgery on a cusped hyperbolic 3-manifold along a surgery curve with $\hat{L} \geq 7.515$ gives a hyperbolic manifold with volume at least $2.0298 - 0.3287 = 1.701$. □

The results of Section 6.1 show that we can also apply Theorem 6.5 when M is a closed hyperbolic 3-manifold and $X = M - \tau$ is obtained by removing a sufficiently short simple closed geodesic τ. For example, the proof of Corollary 6.4 shows that the hypotheses of Theorem 6.5 are satisfied for any *shortest* closed geodesic τ of length at most 0.162. By the same argument as above, we then obtain the following estimate on volumes of closed 3-manifolds containing short geodesics.

COROLLARY 6.8. *Let M be a closed hyperbolic 3-manifold and let τ be a shortest closed geodesic in M having length $\ell(\tau) \leq 0.162$. Then $M - \tau$ has a finite volume hyperbolic structure and*

$$\text{Volume}(M) \geq \text{Volume}(M - \tau) - 0.329.$$

In particular,

$$\text{Volume}(M) \geq 1.701.$$

Remark 6.9. It is interesting to compare this with the result of Agol in [3], which shows that any closed hyperbolic 3-manifold with shortest geodesic length less than 0.244 has volume greater than the volume of the Weeks manifold $(0.9427\dots)$. That there is such a large gap between the volume estimate above (1.701) and the volume of the Weeks manifold suggests that one ought to be able to improve significantly our bound on the geodesic length. Unfortunately, the fact that all our arguments using the function $h(r)$ break down once the geodesic length gets much bigger means that such an improvement would require further methods.

Remark 6.10. We saw in Remark 2.8 that for a hyperbolic cone-manifold, if the tube radius R satisfies $R \geq \operatorname{arcsinh}(\frac{1}{\sqrt{2}}) \approx 0.6584$, then the core length ℓ decreases as α decreases. If the length of a closed geodesic in a closed hyperbolic

3-manifold satisfies $\ell(\tau) \leq 0.111$, then τ has tube radius $R \geq \rho(\tau) > 0.98 > 0.6584$. Furthermore, we can decrease the cone angle α along τ from 2π to zero, keeping the tube radius larger than this value throughout the deformation. As discussed in Remark 2.8, this implies, by Schläfli's formula, that

$$\text{Volume}(M - \tau) \leq \text{Volume}(M) + \pi\ell(\tau).$$

However, it is not hard to see that the estimate given in Theorem 6.5 is considerably stronger. (Compare the figure in Remark 6.6.)

Bridgeman ([8]) showed that such an estimate holds for a certain nice class of geodesics in hyperbolic 3-manifolds. However, the estimate does not hold in general. Using Oliver Goodman's "Tube" program [20], Ian Agol has observed that this estimate is violated for several closed geodesics τ in the Weeks manifold (see [3]).

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