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Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes

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Abstract

The infinite-dimensional unitary group $U(\infty)$ is the inductive limit of growing compact unitary groups $U(N)$. In this paper we solve a problem of harmonic analysis on $U(\infty)$ stated in [Ol3]. The problem consists in computing spectral decomposition for a remarkable 4-parameter family of characters of $U(\infty)$. These characters generate representations which should be viewed as analogs of nonexisting regular representation of $U(\infty)$.

The spectral decomposition of a character of $U(\infty)$ is described by the spectral measure which lives on an infinite-dimensional space Ω of indecomposable characters. The key idea which allows us to solve the problem is to embed Ω into the space of point configurations on the real line without two points. This turns the spectral measure into a stochastic point process on the real line. The main result of the paper is a complete description of the processes corresponding to our concrete family of characters. We prove that each of the processes is a determinantal point process. That is, its correlation functions have determinantal form with a certain kernel. Our kernels have a special 'integrable' form and are expressed through the Gauss hypergeometric function.

From the analytic point of view, the problem of computing the correlation kernels can be reduced to a problem of evaluating uniform asymptotics of certain discrete orthogonal polynomials studied earlier by Richard Askey and Peter Lesky. One difficulty lies in the fact that we need to compute the asymptotics in the oscillatory regime with the period of oscillations tending to 0. We do this by expressing the polynomials in terms of a solution of a discrete Riemann-Hilbert problem and computing the (nonoscillatory) asymptotics of this solution.

From the point of view of statistical physics, we study thermodynamic limit of a discrete log-gas system. An interesting feature of this log-gas is that its density function is asymptotically equal to the characteristic function of an interval. Our point processes describe how different the random particle configuration is from the typical 'densely packed' configuration.

In simpler situations of harmonic analysis on infinite symmetric groups and harmonic analysis of unitarily invariant measures on infinite hermitian matrices, similar results were obtained in our papers [BO1], [BO2], [BO4].

Contents

Introduction

(a) *Preface.* We tried to make this work accessible and interesting for a wide category of readers. So we start with a brief explanation of the concepts that enter in the title.

The purpose of harmonic analysis is to decompose natural representations of a given group on irreducible representations. By natural representations we mean those representations that are produced, in a natural way, from the group itself. For instance, this can be the regular representation, which is realized in the L^2 space on the group, or a quasiregular representation, which is built from the action of the group on a homogeneous space.

In practice, a natural representation often comes together with a distinguished cyclic vector. Then the decomposition into irreducibles is governed by a measure, which may be called the *spectral measure.* The spectral measure lives on the dual space to the group, the points of the dual being the irreducible unitary representations. There is a useful analogy in analysis: expanding a given function on eigenfunctions of a self-adjoint operator. Here the spectrum of the operator is a counterpart of the dual space.

If our distinguished vector lies in the Hilbert space of the representation, then the spectral measure has finite mass and can be made a probability measure.[1]

Now let us turn to *point processes* (or random point fields), which form a special class of stochastic processes. In general, a stochastic process is a discrete or continual family of random variables, while a point process (or random point field) is a random point configuration. By a (nonrandom) point configuration we mean an unordered collection of points in a locally compact space $\mathfrak{X}$. This collection may be finite or countably infinite, but it cannot have accumulation points in $\mathfrak{X}$. To define a point process on $\mathfrak{X}$, we have to specify a probability measure on $\operatorname{Conf}(\mathfrak{X})$, the set of all point configurations.

The classical example is the Poisson process, which is employed in a lot of probabilistic models and constructions. Another important example (or rather a class of examples) comes from random matrix theory. Given a probability measure on a space of $N \times N$ matrices, we pass to the matrix eigenvalues and get in this way a random N-point configuration. In a suitable scaling limit transition (as $N \to \infty$), it turns into a point process living on infinite point configurations.

As long as we are dealing with 'conventional' groups (finite groups, compact groups, real or p-adic reductive groups, etc.), representation theory seems to have nothing in common with point processes. However, the situation drastically changes when we turn to 'big' groups whose irreducible representations depend on infinitely many parameters. Two basic examples are the infinite symmetric group $S(\infty)$ and the infinite-dimensional unitary group $\mathrm{U}(\infty)$, which are defined as the unions of the ascending chains of finite or compact groups

$$S(1) \subset S(2) \subset S(3) \subset \dots, \qquad \mathrm{U}(1) \subset \mathrm{U}(2) \subset \mathrm{U}(3) \subset \dots,$$

respectively. It turns out that for such groups, the clue to the problem of harmonic analysis can be found in the theory of point processes.

The idea is to convert any infinite collection of parameters, which corresponds to an irreducible representation, to a point configuration. Then the spectral measure defines a point process, and one may try to describe this process (hence the initial measure) using appropriate probabilistic tools.

In [B1], [B2], [BO1], [P.I]–[P.V] we applied this approach to the group $S(\infty)$. In the present paper we study the more complicated group $\mathrm{U}(\infty)$.

[1] It may well happen that the distinguished vector belongs to an extension of the Hilbert space (just as in analysis, one may well be interested in expanding a function which is not square integrable). For instance, in the case of the regular representation of a Lie group one usually takes the delta function at the unity of the group, which is not an element of L^2. In such a situation the spectral measure is infinite. However, we shall deal with finite spectral measures only.

Notice that the point processes arising from the spectral measures do not resemble the Poisson process but are close to the processes of the random matrix theory.

Now we proceed to a detailed description of the content of the paper.

(b) *From harmonic analysis on* $\mathrm{U}(\infty)$ *to a random matrix type asymptotic problem.* Here we summarize the necessary preliminary results established in [Ol3]. For a more detailed review see Section 1–3 below.

The conventional definition of the regular representation is not applicable to the group $\mathrm{U}(\infty)$: one cannot define the L^2 space on this group, because $\mathrm{U}(\infty)$ is not locally compact and hence does not possess an invariant measure. To surpass this difficulty we embed $\mathrm{U}(\infty)$ into a larger space $\mathfrak{U}$, which can be defined as a *projective limit* of the spaces $\mathrm{U}(N)$ as $N \to \infty$. The space $\mathfrak{U}$ is no longer a group but is still a $\mathrm{U}(\infty) \times \mathrm{U}(\infty)$-space. That is, the two-sided action of $\mathrm{U}(\infty)$ on itself can be extended to an action on the space $\mathfrak{U}$. In contrast to $\mathrm{U}(\infty)$, the space $\mathfrak{U}$ possesses a biinvariant finite measure, which should be viewed as a substitute for the nonexisting Haar measure. Moreover, this biinvariant measure is included into a whole family $\{\mu^{(s)}\}_{s\in\mathbb{C}}$ of measures with good transformation properties.[2] Using the measures $\mu^{(s)}$ we explicitly construct a family $\{T_{zw}\}_{z,w\in\mathbb{C}}$ of representations, which seem to be a good substitute for the nonexisting regular representation.[3] In our understanding, the T_{zw}'s are 'natural representations', and we state the problem of harmonic analysis on $\mathrm{U}(\infty)$ as follows:

Problem 1. Decompose the representations T_{zw} on irreducible representations.

This initial formulation then undergoes a few changes.

The first step follows a very general principle of representation theory: reduce the spectral decomposition of representations to the decomposition on extreme points in a convex set $\mathcal{X}$ consisting of certain positive definite functions on the group.

In our concrete situation, the elements of the set $\mathcal{X}$ are positive definite functions on $\mathrm{U}(\infty)$, constant on conjugacy classes and taking the value 1 at the

[2]The idea to enlarge an infinite-dimensional space in order to build measures with good transformation properties is well known. This is a standard device in measure theory on linear spaces, but there are not so many works where it is applied to 'curved' spaces (see, however, [Pi1], [Ner]). For the history of the measures $\mu^{(s)}$ we refer to [Ol3] and [BO4]. A parallel construction for the symmetric group case is given in [KOV].

[3]More precisely, the T_{zw}'s are representations of the group $\mathrm{U}(\infty) \times \mathrm{U}(\infty)$. Thus, they are a substitute for the *biregular* representation. The reason why we are dealing with the group $\mathrm{U}(\infty) \times \mathrm{U}(\infty)$ and not $\mathrm{U}(\infty)$ is explained in [Ol1], [Ol2]. We also give in [Ol3] an alternative construction of the representations T_{zw}.

unity. These functions are called *characters* of $\mathrm{U}(\infty)$. The extreme points of $\mathcal{X}$, or *extreme characters,* are known. They are in a one-to-one correspondence, $\chi^{(\omega)} \leftrightarrow \omega$, with the points ω of an infinite-dimensional region Ω (the set Ω and the extreme characters $\chi^{(\omega)}$ are described in Section 1 below). An arbitrary character $\chi \in \mathcal{X}$ can be written in the form

$$\chi = \int_\Omega \chi^{(\omega)} P(d\omega),$$

where P is a probability measure on Ω. The measure P is defined uniquely, it is called the *spectral measure* of the character χ.

Now let us return to the representations T_{zw}. We focus on the case when the parameters z, w satisfy the condition $\Re(z + w) > -\frac{1}{2}$. Under this restriction, our construction provides a distinguished vector in T_{zw}. The matrix coefficient corresponding to this vector can be viewed as a character χ_{zw} of the group $\mathrm{U}(\infty)$. The spectral measure of χ_{zw} is also the spectral measure of the representation T_{zw} provided that z and w are not integral.[4]

Furthermore, we remark that the explicit expression of χ_{zw}, viewed as a function in four parameters z, $z' = \bar{z}$, w, $w' = \bar{w}$, correctly defines a character $\chi_{z,z',w,w'}$ for a wider set $\mathcal{D}_{\mathrm{adm}} \subset \mathbb{C}^4$ of 'admissible' quadruples (z, z', w, w'). The set $\mathcal{D}_{\mathrm{adm}}$ is defined by the inequality $\Re(z+z'+w+w') > -1$ and some extra restrictions; see Definition 3.4 below. Actually, the 'admissible' quadruples depend on four *real* parameters.

This leads us to the following reformulation of Problem 1:

Problem 2. For any $(z, z', w, w') \in \mathcal{D}_{\mathrm{adm}}$, compute the spectral measure of the character $\chi_{z,z',w,w'}$.

To proceed further we need to explain in what form we express the characters. Rather than write them directly as functions on the group $\mathrm{U}(\infty)$ we prefer to work with their 'Fourier coefficients'. Let us explain what this means.

Recall that the irreducible representations of the compact group $\mathrm{U}(N)$ are labeled by the dominant highest weights, which are nothing but N-tuples of nonincreasing integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$. For the reasons which are explained in the text we denote the set of all these λ's by $\mathbb{GT}_N$ (here 'GT' is the abbreviation of 'Gelfand-Tsetlin'). For each $\lambda \in \mathbb{GT}_N$ we denote by $\widetilde{\chi}^\lambda$ the normalized character of the irreducible representation with highest weight λ. Here the term 'character' has the conventional meaning, and normalization means division by the degree, so that $\widetilde{\chi}^\lambda(1) = 1$. Given a character $\chi \in \mathcal{X}$, we restrict it to the subgroup $\mathrm{U}(N) \subset \mathrm{U}(\infty)$. Then we get a positive definite function on $\mathrm{U}(N)$, constant on conjugacy classes and normalized at $1 \in \mathrm{U}(N)$.

[4] If z or w is integral then the distinguished vector is not cyclic, and the spectral measure of χ_{zw} governs the decomposition of a proper subrepresentation of T_{zw}.

Hence it can be expanded on the functions $\widetilde{\chi}^{\lambda}$, where the coefficients (these are the 'Fourier coefficients' in question) are nonnegative numbers whose sum equals 1:

$$\chi\,|_{\mathrm{U}(N)}=\sum_{\lambda\in\mathbb{GT}_N} P_N(\lambda)\widetilde{\chi}^{\lambda}; \qquad P_N(\lambda)\ge 0, \qquad \sum_{\lambda\in\mathbb{GT}_N} P_N(\lambda)=1; \quad N=1,2,\dots .$$

Thus, χ produces, for any $N=1,2,\dots$, a probability measure P_N on the discrete set $\mathbb{GT}_N$. This fact plays an important role in what follows.

For any character $\chi=\chi_{z,z',w,w'}$ we dispose of an exact expression for the 'Fourier coefficients' $P_N(\lambda)=P_N(\lambda\mid z,z',w,w')$:

(0.1)

$$P_N(\lambda\mid z,z',w,w')=(\text{normalization constant})\cdot\prod_{1\le i<j\le N}(\lambda_i-\lambda_j-i+j)^2$$
$$\times\prod_{i=1}^{N}\frac{1}{\Gamma(z-\lambda_i+i)\Gamma(z'-\lambda_i+i)\Gamma(w+N+1+\lambda_i-i)\Gamma(w'+N+1+\lambda_i-i)}\,.$$

Hence we explicitly know the corresponding measures $P_N=P_N(\,\cdot\mid z,z',w,w')$ on the sets $\mathbb{GT}_N$. Formula (0.1) is the starting point of the present paper.

In [Ol3] we prove that for any character $\chi\in\mathcal{X}$, its spectral measure P can be obtained as a limit of the measures P_N as $N\to\infty$. More precisely, we define embeddings $\mathbb{GT}_N\hookrightarrow\Omega$ and we show that the pushforwards of the P_N's weakly converge to P. [5]

By virtue of this general result, Problem 2 is now reduced to the following:

Problem 3. For any 'admissible' quadruple of parameters (z,z',w,w'), compute the limit of the measures $P_N(\,\cdot\mid z,z',w,w')$, given by formula (0.1), as $N\to\infty$.

This is exactly the problem we are dealing with in the present paper. There is a remarkable analogy between Problem 3 and asymptotic problems of random matrix theory. We think this fact is important, so that we discuss it below in detail. From now on the reader may forget about the initial representation-theoretic motivation: we switch to another language.

(c) *Random matrix ensembles*, *log-gas systems*, *and determinantal processes.* Assume there are a sequence of measures $\mu_1,\mu_2,\dots$ on $\mathbb{R}$ and a parameter $\beta>0$. For any $N=1,2,\dots$, we introduce a probability distribution P_N on the space of ordered N-tuples of real numbers $\{x_1>\dots>x_N\}$

[5] The definition of the embeddings $\mathbb{GT}_N\hookrightarrow\Omega$ is given in §2(c) below.

by

$$(0.2)\quad P_N\left(\prod_{i=1}^{N}[x_i, x_i+dx_i]\right) = (\text{normalization constant})\cdot \prod_{1\le i<j\le N}|x_i-x_j|^\beta \cdot \prod_{i=1}^{N}\mu_N(dx_i).$$

Important examples of such distributions come from random matrix ensembles (E_N, μ_N), where E_N is a vector space of matrices (say, of order N) and μ_N is a probability measure on E_N. Then $x_1, \dots, x_N$ are interpreted as the eigenvalues of an $N\times N$ matrix, and the distribution P_N is induced by the measure μ_N. As for the parameter β, it takes values $1, 2, 4$, depending on the base field.

For instance, in the *Gaussian ensemble*, E_N is the space of real symmetric, complex Hermitian or quaternion Hermitian matrices of order N, and μ_N is a Gaussian measure invariant under the action of the compact group $\mathrm{O}(N)$, $\mathrm{U}(N)$ or $\mathrm{Sp}(N)$, respectively. Then $\beta = 1, 2, 4$, respectively.

If μ_N is absolutely continuous with respect to the Lebesgue measure then the distribution (0.2) is also absolutely continuous, and its density can be written in the form

$$(0.3)\quad F_N(x_1,\dots,x_N) = (\text{constant})\cdot \exp\left\{-\beta\left(\sum_{1\le i<j\le N}\log|x_i-x_j|^{-1} + \sum_{i=1}^{N}V_N(x_i)\right)\right\}.$$

This can interpreted as the Gibbs measure of a system of N repelling particles interacting through a logarithmic Coulomb potential and confined by an external potential V_N. In mathematical physics literature such a system is called a *log-gas system*; see, e.g., [Dy].

Given a distribution of form (0.2) or (0.3), one is interested in the statistical properties of the random configuration $x = (x_i)$ as N goes to infinity. A typical question concerns the asymptotic behavior of the correlation functions. The *n-particle correlation function*, $\rho_n^{(N)}(y_1, \dots, y_n)$, can be defined as the density of the probability of finding a 'particle' of the random configuration in each of n infinitesimal intervals $[y_i, y_i + dy_i]$.[6]

One can believe that under a suitable limit transition the N-particle system 'converges' to a point process — a probability distribution on infinite configurations of particles. The limit distribution cannot be given by a formula of type (0.2) or (0.3). However, it can be characterized by its correlation

[6]This is an intuitive definition only. In a rigorous approach one defines the correlation *measures*; see, e.g. [Len], [DVJ] and also the beginning of Section 4 below.

functions, which presumably are limits of the functions $\rho_n^{(N)}$ as $N \to \infty$. The limit transition is usually accompanied by a scaling (a change of variables depending on N), and the final result may depend on the scaling. See, e.g., [TW].

The special case $\beta = 2$ offers many more possibilities for analysis than the general one. This is due to the fact that for $\beta = 2$, the correlation functions before the limit transition are readily expressed through the orthogonal polynomials $p_0, p_1, \dots$ with weight μ_N. Namely, let $S^{(N)}(y', y'')$ denote the N^{th} Christoffel-Darboux kernel,

$$\begin{aligned} S^{(N)}(y', y'') &= \sum_{i=0}^{N-1} \frac{p_i(y')p_i(y'')}{\|p_i\|^2} \\ &= (\text{a constant}) \cdot \frac{p_N(y')p_{N-1}(y'') - p_{N-1}(y')p_N(y'')}{y' - y''}, \quad y', y'' \in \mathbb{R}, \end{aligned}$$

and assume (for the sake of simplicity only) that μ_N has a density $f_N(x)$. Then the correlation functions are given by a simple determinantal formula

$$\rho_n^{(N)}(y_1, \dots, y_n) = \det \left[S^{(N)}(y_i, y_j) \sqrt{f_N(y_i) f_N(y_j)} \right]_{1 \le i,j \le n}, \qquad n = 1, 2, \dots .$$

If the kernel $S^{(N)}(y', y'')\sqrt{f_N(y') f_N(y'')}$ has a limit $K(x', x'')$ under a scaling limit transition then the limit correlation functions also have a determinantal form,

$$\rho_n(x_1, \dots, x_n) = \det \left[K(x_i, x_j) \right]_{1 \le i,j \le n}, \qquad n = 1, 2 \dots . \tag{0.4}$$

The limit kernel can be evaluated if one disposes of appropriate information about the asymptotic properties of the orthogonal polynomials.

A point process whose correlation functions have the form (0.4) is called *determinantal,* and the corresponding kernel K is called the *correlation kernel.* Finite log-gas systems and their scaling limits are examples of determinantal point processes. In these examples, the correlation kernel is symmetric, but this property is not necessary. Our study leads to processes with nonsymmetric correlation kernels (see (k) below). A comprehensive survey on determinantal point processes is given in [So].

(d) *Lattice log-gas system defined by* (0.1). Note that the expression (0.1) can be transformed to the form (0.2). Indeed, given $\lambda \in \mathbb{GT}_N$, set $l = \lambda + \rho$, where

$$\rho = \left(\tfrac{N-1}{2}, \tfrac{N-3}{2}, \dots, -\tfrac{N-3}{2}, -\tfrac{N-1}{2}\right)$$

is the half-sum of positive roots for $\mathrm{GL}(N)$. That is,

$$l_i = \lambda_i + \tfrac{N+1}{2} - i, \qquad i = 1, \dots, N.$$

Then $\mathcal{L} = \{l_1, \ldots, l_N\}$ is an N-tuple of distinct numbers belonging to the lattice

$$\mathfrak{X}^{(N)} = \begin{cases} \mathbb{Z}, & N \text{ odd}, \\ \mathbb{Z} + \frac{1}{2}, & N \text{ even}. \end{cases}$$

The measure (0.1) on λ's induces a probability measure on $\mathcal{L}$'s such that

$$\text{(Probability of } \mathcal{L}) = (\text{a constant}) \cdot \prod_{1 \le i < j \le N} (l_i - l_j)^2 \cdot \prod_{i=1}^{N} f_N(l_i), \tag{0.5}$$

where, for any $x \in \mathfrak{X}^{(N)}$,

(0.6)

$$f_N(x) = \frac{1}{\Gamma\big(z - x + \frac{N+1}{2}\big)\,\Gamma\big(z' - x + \frac{N+1}{2}\big)\,\Gamma\big(w + x + \frac{N+1}{2}\big)\,\Gamma\big(w' + x + \frac{N+1}{2}\big)}.$$

Now we see that (0.5) may be viewed as a *discrete log-gas system* living on the lattice $\mathfrak{X}^{(N)}$.

(e) *Askey-Lesky orthogonal polynomials.* The orthogonal polynomials defined by the weight function (0.6) on $\mathfrak{X}^{(N)}$ are rather interesting. To our knowledge, they appeared for the first time in Askey's paper [As]. Then they were examined in Lesky's papers [Les1], [Les2]. We propose to call them the *Askey-Lesky polynomials.* More precisely, we reserve this term for the orthogonal polynomials defined by a weight function on $\mathbb{Z}$ of the form

$$\frac{1}{\Gamma(A - x)\Gamma(B - x)\Gamma(C + x)\Gamma(D + x)}, \tag{0.7}$$

where A, B, C, D are any complex parameters such that (0.7) is nonnegative on $\mathbb{Z}$.

The Askey-Lesky polynomials are orthogonal polynomials of hypergeometric type in the sense of [NSU]. That is, they are eigenfunctions of a difference analog of the hypergeometric differential operator.

In contrast to classical orthogonal polynomials, the Askey-Lesky polynomials form a *finite* system. This is caused by the fact that (for nonintegral parameters A, B, C, D) the weight function has slow decay as x goes to $\pm\infty$, so that only finitely many moments exist.

The Askey-Lesky polynomials admit an explicit expression in terms of the generalized hypergeometric series ${}_3F_2(a, b, c; e, f; 1)$ with unit argument: the parameters A, B, C, D are inserted, in a certain way, in the indices a, b, c, e, f of the series. This allows us to explicitly express the Christoffel-Darboux kernel in terms of the ${}_3F_2(1)$ series.

(f) *The two-component gas system.* We have just explained how to reduce (0.1) to a lattice log-gas system (0.5), for which we are able to evaluate the correlation functions. To solve Problem 3, we must then pass to the large N

limit. However, the limit transition that we need here is qualitatively different from typical scaling limits of Random Matrix Theory. It can be shown that, as N gets large, almost all N particles occupy positions inside $(-\frac{N}{2}, \frac{N}{2})$. Note that there are exactly N lattice points in this interval, hence, almost all of them are occupied by particles. More precisely, for any $\varepsilon > 0$, as $N \to \infty$, the number of particles outside $\big(-(\frac{1}{2}+\varepsilon)N, (\frac{1}{2}+\varepsilon)N\big)$ remains finite almost surely. In other words, this means that the density function of our discrete log-gas is asymptotically equal to the characteristic function of the N-point set of lattice points inside $(-\frac{N}{2}, \frac{N}{2})$.

At first glance, this picture looks discouraging. Indeed, we know that in the limit all the particles are densely packed inside $(-\frac{N}{2}, \frac{N}{2})$, and there seems to exist no nontrivial limit point process. However, the representation theoretic origin of the problem leads to the following modification of the model which possesses a meaningful scaling limit.

Let us divide the lattice $\mathfrak{X}^{(N)}$ into two parts, which will be denoted by $\mathfrak{X}^{(N)}_{\text{in}}$ and $\mathfrak{X}^{(N)}_{\text{out}}$:

$$\mathfrak{X}^{(N)}_{\text{in}} = \left\{-\tfrac{N-1}{2}, -\tfrac{N-3}{2}, \dots, \tfrac{N-3}{2}, \tfrac{N-1}{2}\right\},$$
$$\mathfrak{X}^{(N)}_{\text{out}} = \left\{\dots, -\tfrac{N+3}{2}, -\tfrac{N+1}{2}\right\} \cup \left\{\tfrac{N+1}{2}, \tfrac{N+3}{2}, \dots\right\}.$$

Here $\mathfrak{X}^{(N)}_{\text{in}}$, the 'inner' part, consists of N points of the lattice that lie on the interval $(-\frac{N}{2}, \frac{N}{2})$, while $\mathfrak{X}^{(N)}_{\text{out}}$, the 'outer' part, is its complement in $\mathfrak{X}^{(N)}$, consisting of the points outside this interval. .

Given a configuration $\mathcal{L}$ of N particles sitting at points $l_1, \dots, l_N$ of the lattice $\mathfrak{X}^{(N)}$, we assign to it another configuration, X, formed by the particles in $\mathfrak{X}^{(N)}_{\text{out}}$ and the *holes* (i.e., the unoccupied positions) in $\mathfrak{X}^{(N)}_{\text{in}}$. Note that X is a finite configuration, too. Since the 'interior' part consists of exactly N points, we see that in X, there are equally many particles and holes. However, their number is no longer fixed; it varies between 0 and $2N$, depending on the mutual location of $\mathcal{L}$ and $\mathfrak{X}^{(N)}_{\text{in}}$. For instance, if these two sets coincide then X is the empty configuration, and if they do not intersect then $|X| = 2N$.

Under the correspondence $\mathcal{L} \mapsto X$ our random N-particle system turns into a random system of particles and holes. Note that $\mathcal{L} \mapsto X$ is reversible, so that both systems are equivalent.

Rewriting (0.5) in terms of the configurations X one sees that the new system can be viewed as a *discrete two-component log-gas system* consisting of oppositely signed charges. Systems of such a type were earlier investigated in the mathematical physics literature (see [AF], [CJ1], [CJ2], [G], [F1]–[F3] and references therein). However, the known concrete models are quite different from our system.

From what was said above it follows that all but finitely many particles of the new system concentrate, for large N, near the points $\pm\frac{N}{2}$. This suggests that if we shrink our phase space $\mathfrak{X}^{(N)}$ by the factor of N (so that the points $\pm\frac{N}{2}$ turn into $\pm\frac{1}{2}$) then our two-component log-gas should have a well-defined scaling limit. We prove that such a limit exists and it constitutes a point process on $\mathbb{R}\setminus\{\pm\frac{1}{2}\}$ which we will denote by $\mathcal{P}$.

As a matter of fact, the process $\mathcal{P}$ can be defined directly from the spectral measure P of the character $\chi_{z,z',w,w'}$ as we explain in Section 9. Moreover, knowing $\mathcal{P}$ is almost equivalent to knowing P; see the discussion before Proposition 9.7. Thus, we may restate Problem 3 as

Problem 4. Describe the point process $\mathcal{P}$.

It turns out that the most convenient way to describe this point process is to compute its correlation functions. Since the correlation functions of $\mathcal{P}$ define $\mathcal{P}$ uniquely, we will be solving

Problem 4$'$. Find the correlation functions of $\mathcal{P}$.

(g) *Two correlation kernels of the two-component log-gas.* There are two ways of computing the correlation functions of the two-component log-gas system introduced above. The first one is based on the *complementation principle*, see [BOO, Appendix] and §5(c) below, which says that if we have a determinantal point process defined on a discrete set $\mathfrak{Y}=\mathfrak{Y}_1\sqcup\mathfrak{Y}_2$ then a new process whose point configurations consist of particles in $\mathfrak{Y}_1$ and holes in $\mathfrak{Y}_2$, is also determinantal. Furthermore, the correlation kernel of this new process is easily expressed through the correlation kernel of the original process. Thus, one way to obtain the correlation functions for the two-component log-gas is to apply the complementation principle to the (one-component) log-gas (0.1), whose correlation kernel is, essentially, the Christoffel-Darboux kernel for Askey-Lesky orthogonal polynomials. Let us denote by $K^{(N)}_{\text{compl}}$ the correlation kernel for the two-component log-gas obtained in this way.

Another way to compute the correlation functions of our two-component log-gas is to notice that this system belongs to the class of point processes with the following property:
The probability of a given point configuration $X=\{x_1,\dots,x_n\}$ is given by

$$\operatorname{Prob}\{X\}=\text{const}\cdot\det[L^{(N)}(x_i,x_j)]_{i,j=1}^n$$

where $L^{(N)}$ is a $\mathfrak{X}^{(N)}\times\mathfrak{X}^{(N)}$ matrix (see §6). A simple general theorem shows that any point process with this property is determinantal, and its correlation kernels $K^{(N)}$ is given by the relation $K^{(N)}=L^{(N)}(1+L^{(N)})^{-1}$.

Thus, we end up with two correlation kernes $K^{(N)}_{\text{compl}}$ and $K^{(N)}$ of the same point process. These two kernels must not coincide. For example, they may be related by conjugation:

$$K^{(N)}_{\text{compl}}(x,y)=\frac{\phi(x)}{\phi(y)}\,K^{(N)}(x,y)$$

where $\phi(\,\cdot\,)$ is an arbitrary nonvanishing function on $\mathfrak{X}^{(N)}$. (The determinants of the form $\det[K(x_i,x_j)]$ for two conjugate kernels are always equal.) We show that this is indeed the case, and that the function ϕ takes values ±1. Moreover, we prove this statement in a more general setting of a two-component log-gas system obtained in a similar way by particles-holes exchange from an *arbitrary* $\beta=2$ discrete log-gas system on the real line.

(h) *Asymptotics.* In our concrete situation the function ϕ is identically equal to 1 on the set $\mathfrak{X}^{(N)}_{\mathrm{out}}$ and is equal to $(-1)^{x-\frac{N-1}{2}}$ on the set $\mathfrak{X}^{(N)}_{\mathrm{in}}$. This means that if we want to compute the scaling limit of the correlation functions of our two-component log-gas system as $N\to\infty$, then only one of the kernels $K^{(N)}_{\mathrm{compl}}$ and $K^{(N)}$ may be used for this purpose, because the function ϕ does not have a scaling limit. It is not hard to guess which kernel is 'the right one' from the asymptotic point of view.

Indeed, it is easy to verify that the kernel $L^{(N)}$ mentioned above has a well-defined scaling limit which we will denote by L. It is a (smooth) kernel on $\mathbb{R}\setminus\{\pm\frac12\}$. It is then quite natural to assume that the kernel $K^{(N)}=L^{(N)}(1+L^{(N)})^{-1}$ also has a scaling limit K such that $K=L(1+L)^{-1}$. Although this argument is only partially correct (the kernel L does not always define a bounded operator in $L^2(\mathbb{R})$), it provides good intuition. We prove that for all admissible values of the parameters z,z',w,w', the kernel $K^{(N)}$ has a scaling limit K, and this limit kernel is the correlation kernel for the point process $\mathcal{P}$.

Explicit computation of the kernel K is our main result, and we state it in Section 10.

(i) *Overcoming technical difficulties*: *The Riemann-Hilbert approach.* The task of computing the scaling limit of $K^{(N)}$ as $N\to\infty$ is by no means easy. As was explained above, this kernel coincides, up to a sign, with $K^{(N)}_{\mathrm{compl}}$ which, in turn, is easily expressible through the Christoffel-Darboux kernel for the Askey-Lesky orthogonal polynomials. Thus, Problem 4 (or 4′) may be restated as

Problem 5. Compute the asymptotics of the Askey-Lesky orthogonal polynomials.

Since it is known how to express these polynomials through the ${}_3F_2$ hypergeometric series, one might expect that the remaining part is rather smooth and is similar to the situation arising in most $\beta=2$ random matrix models. That is, in the chosen scaling the polynomials converge with all the derivatives to nice analytic functions (like sine or Airy) which then enter in the formula for the limit kernel. As a matter of fact, this is indeed how things look on $\mathfrak{X}^{(N)}_{\mathrm{out}}$. The limit kernel K is not hard to compute and it is expressed through the Gauss hypergeometric function ${}_2F_1$.

The problem becomes much more complicated when we look at $\mathfrak{X}^{(N)}_{\mathrm{in}}$. The basic reason is that this is the oscillatory zone for our orthogonal polynomials, and in the scaling limit that we need the period of oscillations tends to zero. Of course, one cannot expect to see any uniform convergence in this situation.

Let us recall, however, that all we need is the asymptotics *on the lattice*. This remark is crucial. The way we compute the asymptotics on the lattice is, roughly speaking, as follows. We find meromorphic functions with poles in $\mathfrak{X}_{\text{out}}^{(N)}$ which coincide, up to a sign, with our orthogonal polynomials on $\mathfrak{X}_{\text{in}}^{(N)}$. These functions are also expressed through the ${}_3F_2$ series and look more complicated than the polynomials themselves. However, they possess a well-defined limit (convergence with all the derivatives) which is again expressed through the Gauss hypergeometric function. This completes the computation of the asymptotics.

The question is: how did we find these convenient meromorphic functions? The answer lies in the definition of the kernel $K^{(N)}$ as $L^{(N)}(1+L^{(N)})^{-1}$. It is not hard to see that the kernel $L^{(N)}$ belongs to the class of (discrete) integrable operators (see [B3]). This implies that the kernel $K^{(N)}$ can be expressed through a solution of a (discrete) Riemann-Hilbert problem (RHP, for short); see [B3, Prop. 4.3]. It is the solution of this Riemann-Hilbert problem that yields the needed meromorphic functions.

The problem of finding this solution explicitly requires additional efforts. The key fact here is that the jump matrix of this RHP can be reduced to a constant jump matrix by conjugation. It is a very general idea of the inverse scattering method that in such a situation the solution of the RHP must satisfy a difference (differential, in the case of continuous RHP) equation. Finding this equation and solving it in meromorphic functions yields the desired solution.

It is worth noting that even though the correct formula for the limit correlation kernel K can be guessed from just knowing the Askey-Lesky orthogonal polynomials, the needed convergence of the kernels $K^{(N)}$ was only possible to achieve through solving the RHP mentioned above.

Let us also note that computing the limit of the solution of our RHP is not completely trivial as well. The difficulty here lies in finding, by making use of numerous known transformation formulas for the ${}_3F_2$ series, a presentation of the solution that would be convenient for the limit transition.

(j) *The main result.* In (f) above we explained how to reduce our problem of harmonic analysis on $\mathrm{U}(\infty)$ to the problem of computing the correlation functions of the process $\mathcal{P}$. In this paper we prove that the n^{th} correlation function $\rho_n(x_1,\dots,x_n)$ of $\mathcal{P}$ has the determinantal form

$$\rho_n(x_1,\dots,x_n)=\det[K(x_i,x_j)]_{i,j=1}^n,\qquad n=1,2,\dots.$$

Here $K(x,y)$ is a kernel on $\mathbb{R}\setminus\{\pm\frac12\}$ which can be written in the form

$$K(x,y)=\frac{F_1(x)G_1(y)+F_2(x)G_2(y)}{x-y}\,,\qquad x,y\in\mathbb{R}\setminus\{\pm\tfrac12\},$$

where the functions F_1, G_1, F_2, G_2 can be expressed through the Gauss hypergeometric function ${}_2F_1$. In particular, if $x > \frac{1}{2}$ and $y > \frac{1}{2}$ we have

$$F_1(x) = -G_2(x) = \frac{\sin(\pi z)\sin(\pi z')}{\pi^2} \times \left(x - \frac{1}{2}\right)^{-(\frac{z+z'}{2}+w')} \left(x + \frac{1}{2}\right)^{\frac{w'-w}{2}} {}_2F_1\left[\begin{matrix} z+w',\, z'+w' \\ z+z'+w+w' \end{matrix} \middle| \frac{1}{\frac{1}{2}-x}\right],$$

$$G_1(x) = F_2(x) = \frac{\Gamma(z+w+1)\Gamma(z+w'+1)\Gamma(z'+w+1)\Gamma(z'+w'+1)}{\Gamma(z+z'+w+w'+1)\Gamma(z+z'+w+w'+2)} \times \left(x - \frac{1}{2}\right)^{-(\frac{z+z'}{2}+w'+1)} \left(x + \frac{1}{2}\right)^{\frac{w'-w}{2}} \times {}_2F_1\left[\begin{matrix} z+w'+1,\, z'+w'+1 \\ z+z'+w+w'+2 \end{matrix} \middle| \frac{1}{\frac{1}{2}-x}\right].$$

A complete statement of the result can be found in Theorem 10.1 below.

(k) *Symmetry of the kernel.* The correlation kernel $K(x, y)$ introduced above satisfies the following symmetry relations:

$$K(x,y) = \begin{cases} K(y,x) & \text{if } \left(|x| > \frac{1}{2},\ |y| > \frac{1}{2}\right) \text{ or } \left(|x| < \frac{1}{2}, |y| < \frac{1}{2}\right), \\ -K(y,x) & \text{if } \left(|x| > \frac{1}{2},\ |y| < \frac{1}{2}\right) \text{ or } \left(|x| < \frac{1}{2}, |y| > \frac{1}{2}\right). \end{cases}$$

Moreover, the kernel is real-valued. This implies that the restrictions of K to $(-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ and $\left(\mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]\right) \times \left(\mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]\right)$ are Hermitian kernels, while the kernel K on the whole line is a J-Hermitian[7] kernel.

We have encountered certain J-Hermitian kernels in our work on harmonic analysis on the infinite symmetric group, see [BO1], [P.I]–[P.V]. At that time we were not aware of the fact that examples of J-Hermitian correlation kernels had appeared before in works of mathematical physicists on solvable models of systems with positive and negative charged particles, see [AF], [CJ1], [CJ2], [G], [F1]–[F3] and references therein.

As was explained in (f), our system also contains ‘particles of opposite charges.’ The property of J-symmetry is closely related to this fact; see Section 5(f),(g) for more details.

(l) *Further development*: *Painlevé* VI. It is well known that for a determinantal point process with a correlation kernel K, the probability of having

[7] I.e., Hermitian with respect to the indefinite inner product defined by the matrix $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

no particles in a region I is equal to the Fredholm determinant $\det(1-K_I)$, where K_I is the restriction of K to $I \times I$. It often happens that such a *gap probability* can be expressed through a solution of a (second order nonlinear ordinary differential) Painlevé equation. One of the main results of [BD] is the following statement.

Let K_s be the restriction of the kernel $K(x,y)$ of (j) above to $(s,+\infty) \times (s,+\infty)$. Set

$$\nu_1 = \frac{z+z'+w+w'}{2}, \quad \nu_3 = \frac{z-z'+w-w'}{2}, \quad \nu_4 = \frac{z-z'-w+w'}{2},$$

$$\sigma(s) = \left(s^2 - \tfrac{1}{4}\right) \frac{d \ln \det(1-K_s)}{ds} - \nu_1^2\, s + \frac{\nu_3 \nu_4}{2}.$$

Then $\sigma(s)$ satisfies the differential equation

$$-\sigma' \left(\left(s^2 - \tfrac{1}{4}\right) \sigma''\right)^2 = \left(2\left(s\sigma' - \sigma\right)\sigma' - \nu_1^2 \nu_3 \nu_4\right)^2 - (\sigma' + \nu_1^2)^2 (\sigma' + \nu_3^2)(\sigma' + \nu_4^2).$$

This differential equation is the so-called σ-form of the Painlevé VI equation. We refer to [BD, Introduction] for a brief historical introduction and references on this subject. [BD] also contains proofs of several important properties of the kernel $K(x,y)$ which we list at the end of Section 10 below.

(m) *Connection with previous work.* In [BO1], [BO2], [B1], [B2], [BO4] we worked out two other problems of harmonic analysis in the situations when spectral measures live on infinite-dimensional spaces. We will describe them in more detail and compare them to the problem of the present paper.

The problem of harmonic analysis on the group $S(\infty)$ was initially formulated in [KOV]. It consists in decomposing certain 'natural' (generalized regular) unitary representations T_z of the group $S(\infty) \times S(\infty)$, depending on a complex parameter z. In [KOV], the problem was solved in the case when the parameter z takes integral values (then the spectral measure has finite-dimensional support). The general case presents more difficulties and we studied it in a cycle of papers [P.I]–[P.V], [BO1]–[BO3], [B1], [B2]. Our main result is that the spectral measure governing the decomposition of T_z can be described in terms of a determinantal point process on the real line with one punctured point. The correlation kernel was explicitly computed; it is expressed through a confluent hypergeometric function (specifically, through the W-Whittaker function).

The second problem deals with decomposition of a family of unitarily invariant probability measures on the space of all infinite Hermitian matrices on ergodic components. The measures depend on one complex parameter and essentially coincide with the measures $\{\mu^{(s)}\}$ mentioned in the beginning of (b) above. The problem of decomposition on ergodic components can be also viewed as a problem of harmonic analysis on an infinite-dimensional Cartan motion group. The main result of [BO4] states that the spectral measures in this case can be interpreted as determinantal point processes on the real line with a correlation kernel expressed through a confluent hypergeometric function (this time, this is the M-Whittaker function).

These two problems and the problem that we deal with in this paper have a number of similarities. Already the descriptions of the spaces of irreducible objects (see Thoma [Th] for $S(\infty)$, Pickrell [Pi1] and Olshanski-Vershik [OV] for measures on Hermitian matrices, and Voiculescu [Vo] for $\mathrm{U}(\infty)$) are quite similar. Furthermore, all three models have some sort of an approximation procedure using finite-dimensional objects, see [VK1], [OV], [VK2], [OkOl]. The form of the correlation kernels is also essentially the same, with different special functions involved in different problems.

It is worth noting that the similarity of theories for the two groups $S(\infty)$ and $\mathrm{U}(\infty)$ seems to be a striking phenomenon. In addition, as mentioned above, this can be traced in the geometric construction of the 'natural' representations and in probabilistic properties of the corresponding point processes. At present we cannot completely explain the nature of this parallelism (it looks quite different from the well-known classical connection between the representations of the groups $S(n)$ and $\mathrm{U}(N)$).

However, the differences among all these problems should not be underestimated. Indeed, the problem of harmonic analysis on $S(\infty)$ is a problem of asymptotic combinatorics consisting in controlling the asymptotics of certain explicit probability distributions on partitions of n as $n \to \infty$. One consequence of such asymptotic analysis is a simple proof and generalization of the Baik-Deift-Johansson theorem [BDJ] on longest increasing subsequences of large random permutations, see [BOO] and [BO3]. The problem of decomposing measures on Hermitian matrices on ergodic components is of a different nature. It belongs to Random Matrix Theory which deals with asymptotics of probability distributions on large matrices. In fact, for a specific value of the parameter, the result of [BO4] reproduces one of the basic computations of Random Matrix Theory – that of the scaling limit of Dyson's circular ensemble.

The problem solved in the present paper is more general compared to both problems described above. Our model here depends on a larger number of parameters, it deals with a more complicated group and representation structure, and the analysis requires a substantial amount of new ideas. Moreover, in appropriate limits this model degenerates to both models studied earlier. The limits, of course, are very different. On the level of correlation kernels this leads to two different degenerations of the Gauss hypergeometric function to confluent hypergeometric functions. We view the $\mathrm{U}(\infty)$-model as a unifying object for the combinatorial and random matrix models, and we think that it sheds some light on the nature of the recently discovered remarkable connections between different models of these two kinds.

The model of the present paper can be also viewed as the top of a hierarchy of (discrete and continuous) probabilistic models leading to determinantal point processes with 'integrable' correlation kernels. In the language of kernels this looks very much like the hierarchy of the classical special functions. A description of the '$S(\infty)$-part' of the hierarchy can be found in [BO3]. The subject of degenerating the $\mathrm{U}(\infty)$-model to simpler models (in particular, to the two models discussed above) will be addressed in a later publication.

(n) *Organization of the paper.* In Section 1 we give a brief introduction to representation theory and harmonic analysis of the infinite-dimensional unitary group $\mathrm{U}(\infty)$. Section 2 explains how spectral decompositions of representations of $\mathrm{U}(\infty)$ can be approximated by those for finite-dimensional groups $\mathrm{U}(N)$. In Section 3 we introduce a remarkable family of characters of $\mathrm{U}(\infty)$ which we study in this paper. In Section 4 we reformulate the problem of harmonic analysis of these characters in the language of random point processes. Section 5 is the heart of the paper: there we develop general theory of discrete determinantal point processes which will later enable us to compute the correlation functions of our concrete processes. In Section 6 we show that the point processes introduced in Section 4 are determinantal. In Section 7 we derive discrete orthogonal polynomials on $\mathbb{Z}$ with the weight function (0.7). This allows us to write out a correlation kernel for approximating point processes associated with $\mathrm{U}(N)$'s. Section 8 is essentially dedicated to representing this correlation kernel in a form suitable for the limit transition $N \to \infty$. The main tool here is the discrete Riemann-Hilbert problem. Section 9 establishes certain general facts about scaling limits of point processes associated with restrictions of characters of $\mathrm{U}(\infty)$ to $\mathrm{U}(N)$. The main result here is that an appropriate scaling limit yields the spectral measure for the initial character of $\mathrm{U}(\infty)$. In Section 10 we perform such a scaling limit for our concrete family of characters. Section 11 describes a nice combinatorial degeneration of our characters. In this degeneration the spectral measure loses its infinite-dimensional support and turns into a Jacobi polynomial ensemble. Finally, the appendix contains proofs of transformation formulas for the hypergeometric series ${}_3F_2$ which were used in the computations.

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1. Characters of the group $\mathrm{U}(\infty)$

(a) *Extreme characters.* Let $\mathrm{U}(N)$ be the group of unitary matrices of order N. For any $N \geq 2$ we identify $\mathrm{U}(N-1)$ with the subgroup in $\mathrm{U}(N)$ fixing the N^{th} basis vector, and we set

$$\mathrm{U}(\infty) = \varinjlim \mathrm{U}(N).$$

One can view $\mathrm{U}(\infty)$ as a group of matrices $U = [U_{ij}]_{i,j=1}^{\infty}$ such that there are finitely many matrix elements U_{ij} not equal to δ_{ij}, and $U^* = U^{-1}$.

A *character* of $\mathrm{U}(\infty)$ is a function $\chi : \mathrm{U}(\infty) \to \mathbb{C}$ which is constant on conjugacy classes, positive definite, and normalized at the unity ($\chi(e) = 1$). We also assume that χ is continuous on each subgroup $\mathrm{U}(N) \subset \mathrm{U}(\infty)$. The characters form a convex set. The extreme points of this convex set are called the *extreme characters.*

A fundamental result of the representation theory of the group $\mathrm{U}(\infty)$ is a complete description of extreme characters. To state it we need some notation.

Let $\mathbb{R}^\infty$ denote the product of countably many copies of $\mathbb{R}$, and set

$$\mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R}.$$

Let $\Omega \subset \mathbb{R}^{4\infty+2}$ be the subset of sextuples

$$\omega = (\alpha^+, \beta^+; \alpha^-, \beta^-; \delta^+, \delta^-)$$

such that

$$\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^{\infty} (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm, \quad \beta_1^+ + \beta_1^- \leq 1.$$

Set

$$\gamma^\pm = \delta^\pm - \sum_{i=1}^{\infty} (\alpha_i^\pm + \beta_i^\pm)$$

and note that γ^+, γ^- are nonnegative.

To any $\omega \in \Omega$ we assign a function $\chi^{(\omega)}$ on $\mathrm{U}(\infty)$:

$$\chi^{(\omega)}(U) = \prod_{u \in \mathrm{Spectrum}(U)} \left\{ e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^{\infty} \frac{1+\beta_i^+(u-1)}{1-\alpha_i^+(u-1)} \, \frac{1+\beta_i^-(u^{-1}-1)}{1-\alpha_i^-(u^{-1}-1)} \right\}.$$

Here U is a matrix from $\mathrm{U}(\infty)$ and u ranges over the set of its eigenvalues. All but finitely many u's equal 1, so that the product over u is actually finite. The product over i is convergent, because the sum of the parameters is finite. Note also that different ω's correspond to different functions; here the condition $\beta_1^+ + \beta_1^- \leq 1$ plays the decisive role; see [Ol3, Remark 1.6].

THEOREM 1.1. *The functions* $\chi^{(\omega)}$, *where* ω *ranges over* Ω, *are exactly the extreme characters of the group* $\mathrm{U}(\infty)$.

Proof. The fact that any $\chi^{(\omega)}$ is an extreme character is due to Voiculescu [Vo]. The fact that the extreme characters are exhausted by the $\chi^{(\omega)}$'s can be proved in two ways: by reduction to an old theorem due to Edrei [Ed] (see [Boy] and [VK2]) and by Vershik-Kerov's asymptotic method (see [VK2] and [OkOl]). □

The coordinates $\alpha_i^{\pm}$, $\beta_i^{\pm}$, and $\gamma^{\pm}$ (or $\delta^{\pm}$) are called the *Voiculescu parameters* of the extreme character $\chi^{(\omega)}$. Theorem 1.1 is similar to Thoma's theorem which describes the extreme characters of the infinite symmetric group, see [Th], [VK1], [Wa], [KOO]. Another analogous result is the classification of invariant ergodic measures on the space of infinite Hermitian matrices (see [OV] and [Pi2]).

(b) *Spectral measures.* Equip $\mathbb{R}^{4\infty+2}$ with the product topology. It induces a topology on Ω. In this topology, Ω is a locally compact separable space. On the other hand, we equip the set of characters with the topology of uniform convergence on the subgroups $\mathrm{U}(N) \subset \mathrm{U}(\infty)$, $N = 1, 2, \dots$. One can prove that the bijection $\omega \longleftrightarrow \chi^{(\omega)}$ is a homeomorphism with respect to these two topologies (see [Ol3, §8]). In particular, $\chi^{(\omega)}(U)$ is a continuous function of ω for any fixed $\mathrm{U} \in \mathrm{U}(\infty)$.

THEOREM 1.2. *For any character* χ *of the group* $\mathrm{U}(\infty)$ *there exists a probability measure* P *on the space* Ω *such that*

$$\chi(U) = \int_{\Omega} \chi^{(\omega)}(U)\, P(d\omega), \qquad U \in \mathrm{U}(\infty).$$

Such a measure P *is unique. The correspondence* $\chi \mapsto P$ *is a bijection between the set of all characters and the set of all probability measures on* Ω.

Here and in what follows, by a measure on Ω we mean a Borel measure. We call P the *spectral measure* of χ.

Proof. See [Ol3, Th. 9.1]. □

Similar results hold for the infinite symmetric group (see [KOO]) and for invariant measures on infinite Hermitian matrices (see [BO4]).

(c) *Signatures.* Define a *signature* λ of length N as an ordered sequence of integers with N members:

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \mid \lambda_i \in \mathbb{Z}).$$

Signatures of length N are naturally identified with highest weights of irreducible representations of the group $\mathrm{U}(N)$; see, e.g., [Zh]. Thus, there is a natural bijection $\lambda \longleftrightarrow \chi^\lambda$ between signatures of length N and irreducible characters of $\mathrm{U}(N)$ (here we use the term "character" in its conventional sense). The character χ^λ can be viewed as a *rational Schur function* (Weyl's character formula)

$$\chi^\lambda(u_1,\dots,u_N)=\frac{\det[u_i^{\lambda_j+N-j}]_{i,j=1,\dots,N}}{\det[u_i^{N-j}]_{i,j=1,\dots,N}}\,.$$

Here the collection $(u_1,\dots,u_N)$ stands for the spectrum of a matrix in $\mathrm{U}(N)$.

We will represent a signature λ as a pair of Young diagrams (λ^+,λ^-): one consists of positive λ_i's, the other consists of minus negative λ_i's; zeros can go in either of the two:

$$\lambda=(\lambda_1^+,\lambda_2^+,\dots,-\lambda_2^-,-\lambda_1^-).$$

Let $d^+=d(\lambda)$ and $d^-=d(\lambda^-)$, where the symbol $d(\,\cdot\,)$ denotes the number of diagonal boxes of a Young diagram. Write the diagrams λ^+ and λ^- in Frobenius notation:

$$\lambda^\pm=(p_1^\pm,\dots,p_{d^\pm}^\pm\mid q_1^\pm,\dots,q_{d^\pm}^\pm).$$

We recall that the Frobenius coordinates p_i, q_i of a Young diagram ν are defined by

$$p_i=\nu_i-i,\quad q_i=(\nu')_i-i,\qquad i=1,\dots,d(\nu),$$

where ν' stands for the transposed diagram. Following Vershik-Kerov [VK1], we introduce the *modified Frobenius coordinates* of ν by

$$\widetilde{p}_i=p_i+\tfrac12,\quad \widetilde{q}_i=q_i+\tfrac12.$$

Note that $\sum(\widetilde{p}_i+\widetilde{q}_i)=|\nu|$, where $|\nu|$ denotes the number of boxes in ν.

We agree that

$$\widetilde{p}_i=\widetilde{q}_i=0,\qquad i>d(\nu).$$

(d) *Approximation of extreme characters.* Recall that the dimension of the irreducible representation of $\mathrm{U}(N)$ indexed by λ is given by Weyl's dimension formula

$$\mathrm{Dim}_N\,\lambda=\chi^\lambda(\underbrace{1,\dots,1}_{N})=\prod_{i\le i<j\le N}\frac{\lambda_i-i-\lambda_j+j}{j-i}\,.$$

Define the *normalized* irreducible character indexed by λ as follows

$$\widetilde{\chi}^\lambda=\frac{1}{\mathrm{Dim}_N\,\lambda}\,\chi^\lambda.$$

Clearly, $\widetilde{\chi}^\lambda(e)=1$.

Given a sequence $\{f_N\}_{N=1,2,\dots}$ of functions on the groups $\mathrm{U}(N)$, we say that f_N's *approximate* a function f defined on the group $\mathrm{U}(\infty)$ if, for any fixed $N_0=1,2,\dots$, the restrictions of the functions f_N (where $N\ge N_0$) to the subgroup $\mathrm{U}(N_0)$ uniformly tend, as $N\to\infty$, to the restriction of f to $\mathrm{U}(N_0)$.

THEOREM 1.3. *Any extreme character χ of* $\mathrm{U}(\infty)$ *can be approximated by a sequence* $\widetilde{\chi}^{(N)}$ *of normalized irreducible characters of the groups* $\mathrm{U}(N)$.

In more detail, write $\widetilde{\chi}^{(N)} = \widetilde{\chi}^{\lambda(N)}$, *where* $\{\lambda(N)\}_{N=1,2,\dots}$ *is a sequence of signatures, and let* $\widetilde{p}_i^{\pm}(N)$ *and* $\widetilde{q}_i^{\pm}(N)$ *stand for the modified Frobenius coordinates of* $(\lambda(N))^{\pm}$. *Then the functions* $\widetilde{\chi}^{(N)}$ *approximate* χ *if and only if the following conditions hold*:

$$\lim_{N\to\infty} \frac{\widetilde{p}_i^{\pm}(N)}{N} = \alpha_i^{\pm}, \quad \lim_{N\to\infty} \frac{\widetilde{q}_i^{\pm}(N)}{N} = \beta_i^{\pm}, \quad \lim_{N\to\infty} \frac{|(\lambda(N))^{\pm}|}{N} = \delta^{\pm},$$

where $i = 1, 2, \dots$, *and* $\alpha_i^{\pm}$, $\beta_i^{\pm}$, $\delta^{\pm}$ *are the Voiculescu parameters of the character* χ.

This claim reveals the asymptotic meaning of the Voiculescu parameters. Note that for any $\omega = (\alpha^+, \beta^+; \alpha^-, \beta^-; \delta^+, \delta^-) \in \Omega$, there exists a sequence of signatures satisfying the above conditions, hence any extreme character indeed admits an approximation.

Proof. This result is due to Vershik and Kerov; see their announcement [VK2]. A detailed proof is contained in [OkOl]. □

For analogous results, see [VK1], [OV].

2. Approximation of spectral measures

(a) *The graph* $\mathbb{GT}$. For two signatures ν and λ, of length $N-1$ and N, respectively, write $\nu \prec \lambda$ if

$$\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots \geq \nu_{N-1} \geq \lambda_N \,.$$

The relation $\nu \prec \lambda$ appears in the *Gelfand-Tsetlin branching rule* for the irreducible characters of the unitary groups, see, e.g., [Zh]:

$$\chi^{\lambda}(u_1, \dots, u_{N-1}, 1) = \sum_{\nu:\, \nu \prec \lambda} \chi^{\nu} \,.$$

The *Gelfand-Tsetlin graph* $\mathbb{GT}$ is a $\mathbb{Z}_+$-graded graph whose N^{th} level $\mathbb{GT}_N$ consists of signatures of length N. Two vertices $\nu \in \mathbb{GT}_{N-1}$ and $\lambda \in \mathbb{GT}_N$ are joined by an edge if $\nu \prec \lambda$. This graph is a counterpart of the Young graph associated with the symmetric group characters [VK1], [KOO].

(b) *Coherent systems of distributions.* For $\nu \in \mathbb{GT}_{N-1}$ and $\lambda \in \mathbb{GT}_N$, set

$$q(\nu, \lambda) = \begin{cases} \dfrac{\operatorname{Dim}_{N-1} \nu}{\operatorname{Dim}_N \lambda}\,, & \nu \prec \lambda, \\ 0, & \nu \not\prec \lambda. \end{cases}$$

This is the *cotransition probability function* of the Gelfand-Tsetlin graph. It satisfies the relation

$$\sum_{\nu\in\mathbb{GT}_{N-1}} q(\nu,\lambda)=1, \qquad \forall\,\lambda\in\mathbb{GT}_N\,.$$

Assume that for each $N=1,2,\dots$ we are given a probability measure P_N on the discrete set $\mathbb{GT}_N$. Then the family $\{P_N\}_{N=1,2,\dots}$ is called a *coherent system* if

$$P_{N-1}(\nu)=\sum_{\lambda\in\mathbb{GT}_N} q(\nu,\lambda)P_N(\lambda), \qquad \forall\,N=2,3,\dots, \quad \forall\,\nu\in\mathbb{GT}_{N-1}\,.$$

Note that if P_N is an arbitrary probability measure on $\mathbb{GT}_N$ then this formula defines a probability measure on $\mathbb{GT}_{N-1}$ (indeed, this follows at once from the above relation for $q(\nu,\lambda)$). Thus, in a coherent system $\{P_N\}_{N=1,2,\dots}$, the N^{th} term is a refinement of the $(N-1)$st one.

PROPOSITION 2.1. *There is a natural bijective correspondence* $\chi\longleftrightarrow\{P_N\}$ *between characters of the group* $\mathrm{U}(\infty)$ *and coherent systems*, *defined by the relations*

$$\chi\,|_{\mathrm{U}(N)}=\sum_{\lambda\in\mathbb{GT}_N} P_N(\lambda)\widetilde{\chi}^{\lambda}, \qquad N=1,2,\dots\,.$$

Proof. See [Ol3, Prop. 7.4]. □

A similar claim holds for the infinite symmetric group $S(\infty)$, see [VK1], [KOO], and for the infinite-dimensional Cartan motion group, see [OV]. Note that $\{P_N\}$ can be viewed as a kind of Fourier transform of the corresponding character.

The concept of a coherent system $\{P_N\}$ is important for two reasons. First, we are unable to calculate directly the "natural" nonextreme characters but we dispose of nice closed expressions for their "Fourier coefficients" $P_N(\lambda)$; see the next section. Note that in the symmetric group case the situation is just the same, see [KOV], [BO1]–[BO3]. Second, the measures P_N approximate the spectral measure P; see below.

(c) *Approximation* $P_N\to P$. Let χ be a character of $\mathrm{U}(\infty)$ and let P and $\{P_N\}$ be the corresponding spectral measure and coherent system.

For any $N=1,2,\dots$, we embed the set $\mathbb{GT}_N$ into $\Omega\subset\mathbb{R}^{4\infty+2}$ as follows:

$$\mathbb{GT}_N\ni\lambda\longmapsto(a^+,b^+;a^-,b^-;c^+,c^-)\in\mathbb{R}^{4\infty+2},$$
$$a_i^\pm=\frac{\widetilde{p}_i^{\,\pm}}{N},\quad b_i^\pm=\frac{\widetilde{q}_i^{\,\pm}}{N},\quad c^\pm=\frac{|\lambda^\pm|}{N}\,,$$

where $i=1,2,\dots$, and $\widetilde{p}_i^{\,\pm}$, $\widetilde{q}_i^{\,\pm}$ are the modified Frobenius coordinates of $\lambda^\pm$.

Let $\underline{P}_N$ be the pushforward of P_N under this embedding. Then $\underline{P}_N$ is a probability measure on Ω.

Theorem 2.2. *As $N \to \infty$, the measures $\underline{P}_N$ weakly tend to the measure P. That is, for any bounded continuous function F on Ω,*

$$\lim_{N\to\infty} \langle F, \underline{P}_N \rangle = \langle F, P \rangle.$$

Proof. See [Ol3, Th. 10.2]. □

This result should be compared with [KOO, Proof of Theorem B in §8] and [BO4, Th. 5.3]. Its proof is quite similar to that of [BO4, Th. 5.3].

Theorem 2.2 shows that the spectral measure can be, in principle, computed if one knows the coherent system $\{P_N\}$.

3. ZW-Measures

The goal of this section is to introduce a family of characters χ of the group $U(\infty)$, for which we solve the problem of harmonic analysis. We describe these characters in terms of the corresponding coherent systems $\{P_N\}$. For detailed proofs we refer to [Ol3].

Let z, z', w, w' be complex parameters. For any $N = 1, 2, \dots$ and any $\lambda \in \mathbb{GT}_N$ set

$$P'_N(\lambda \mid z, z', w, w') = \mathrm{Dim}_N^2(\lambda) \times \prod_{i=1}^{N} \frac{1}{\Gamma(z-\lambda_i+i)\Gamma(z'-\lambda_i+i)\Gamma(w+N+1+\lambda_i-i)\Gamma(w'+N+1+\lambda_i-i)},$$

where $\mathrm{Dim}_N \lambda$ is as defined in Section 1. Clearly, for any fixed N and λ, $P'_N(\lambda \mid z, z', w, w')$ is an entire function on $\mathbb{C}^4$. Set

$$\mathcal{D} = \{(z, z', w, w') \in \mathbb{C}^4 \mid \Re(z + z' + w + w') > -1\}.$$

This is a domain in $\mathbb{C}^4$.

Proposition 3.1. *Fix an arbitrary $N = 1, 2, \dots$. The series of entire functions*

$$\sum_{\lambda \in \mathbb{GT}_N} P'_N(\lambda \mid z, z', w, w')$$

converges in the domain $\mathcal{D}$, uniformly on compact sets. Its sum is equal to

$$S_N(z, z', w, w') = \prod_{i=1}^{N} \frac{\Gamma(z + z' + w + w' + i)}{\Gamma(z + w + i)\Gamma(z + w' + i)\Gamma(z' + w + i)\Gamma(z' + w' + i)\Gamma(i)}.$$

Proof. See [Ol3, Prop. 7.5]. □

Note that in the special case $N=1$, the set $\mathbb{GT}_1$ is simply $\mathbb{Z}$ and the identity

$$\sum_{\lambda\in\mathbb{GT}_1} P_1'(\lambda\mid z,z',w,w')=S_1(z,z',w,w')$$

is equivalent to Dougall's well-known formula (see [Er, vol. 1, §1.4]).

Consider the subdomain

$$\begin{aligned}\mathcal{D}_0 &= \{(z,z',w,w')\in\mathcal{D}\mid z+w,\ z+w',\ z'+w,\ z'+w'\neq -1,-2,\dots\}\\ &= \{(z,z',w,w')\in\mathcal{D}\mid S_N(z,z',w,w')\neq 0\}.\end{aligned}$$

For any $(z,z',w,w')\in\mathcal{D}_0$ we set

$$P_N(\lambda\mid z,z',w,w')=\frac{P_N'(\lambda\mid z,z',w,w')}{S_N(z,z',w,w')}\,,\qquad N=1,2,\dots,\quad \lambda\in\mathbb{GT}_N.$$

Then, by Proposition 3.1,

$$\sum_{\lambda\in\mathbb{GT}_N} P_N(\lambda\mid z,z',w,w')=1,\qquad (z,z',w,w')\in\mathcal{D}_0\,,$$

uniformly on compact sets in $\mathcal{D}_0$.

Proposition 3.2. *Let* $(z,z',w,w')\in\mathcal{D}_0$. *For any* $N=2,3,\dots,$ *the coherency relation of* §2(b) *is satisfied*,

$$P_{N-1}(\nu\mid z,z',w,w')=\sum_{\lambda\in\mathbb{GT}_N} q(\nu,\lambda)P(\lambda\mid z,z',w,w').$$

Proof. See [Ol3, Prop. 7.7]. □

Combining this with Proposition 2.1 we conclude that $\{P_N(\,\cdot\mid z,z',w,w')\}$, where $N=1,2,\dots,$ is a coherent system provided that $(z,z',w,w')\in\mathcal{D}_0$ satisfies the *positivity condition*: for any $N=1,2,\dots,$ the expression $P_N'(\lambda\mid z,z',w,w')$ is nonnegative for all $\lambda\in\mathbb{GT}_N$. (Note that there always exists λ for which $P_N'(\lambda\mid z,z',w,w')\neq 0$, because the sum over λ's is not 0.) We proceed to describe a set of quadruples $(z,z',w,w')\in\mathcal{D}_0$ satisfying the positivity condition.

Define the subset $\mathcal{Z}\subset\mathbb{C}^2$ as follows:

$$\begin{aligned}\mathcal{Z} &= \mathcal{Z}_{\text{princ}}\sqcup\mathcal{Z}_{\text{compl}}\sqcup\mathcal{Z}_{\text{degen}},\\ \mathcal{Z}_{\text{princ}} &= \{(z,z')\in\mathbb{C}^2\setminus\mathbb{R}^2\mid z'=\bar z\},\\ \mathcal{Z}_{\text{compl}} &= \{(z,z')\in\mathbb{R}^2\mid \exists m\in\mathbb{Z},\ m<z,z<m+1\},\\ \mathcal{Z}_{\text{degen}} &= \bigsqcup_{m\in\mathbb{Z}}\mathcal{Z}_{\text{degen},m},\\ \mathcal{Z}_{\text{degen},m} &= \{(z,z')\in\mathbb{R}^2\mid z=m,\ z'>m-1,\quad\text{or}\quad z'=m,\ z>m-1\},\end{aligned}$$

where "princ", "compl", and "degen" are abbreviations for "principal", "complementary", and "degenerate", respectively. For an explanation of this terminology, see [Ol3].

PROPOSITION 3.3. *Let* $(z, z') \in \mathbb{C}^2$.

(i) *The expression* $(\Gamma(z-k)\Gamma(z'-k))^{-1}$ *is nonnegative for all* $k \in \mathbb{Z}$ *if and only if* $(z, z') \in \mathcal{Z}$.

(ii) *If* $(z, z') \in \mathcal{Z}_{\text{princ}} \sqcup \mathcal{Z}_{\text{compl}}$ *then this expression is strictly positive for all* $k \in \mathbb{Z}$.

(iii) *If* $(z, z') \in \mathcal{Z}_{\text{degen},m}$ *then this expression vanishes for* $k = m, m+1, \dots$ *and is strictly positive for* $k = m-1, m-2, \dots$.

Proof. See [Ol3, Lemma 7.9]. □

Definition 3.4. The *set of admissible values* of the parameters z, z', w, w' is the subset $\mathcal{D}_{\text{adm}} \subset \mathcal{D}$ of quadruples (z, z', w, w') such that both (z, z') and (w, w') belong to $\mathcal{Z}$. When both (z, z') and (w, w') are in $\mathcal{Z}_{\text{degen}}$, an extra condition is added: let k, l be such that $(z, z') \in \mathcal{Z}_{\text{degen},k}$ and $(w, w') \in \mathcal{Z}_{\text{degen},l}$; then we require $k + l \geq 0$. A quadruple (z, z', w, w') will be called *admissible* if it belongs to the set $\mathcal{D}_{\text{adm}}$.

Note that in this definition we do not assume *a priori* that (z, z', w, w') belongs to the subdomain $\mathcal{D}_0 \subset \mathcal{D}$. However the conditions imposed on (z, z', w, w') imply that $\mathcal{D}_{\text{adm}} \subset \mathcal{D}_0$; see below.

PROPOSITION 3.5. *Let* $(z, z', w, w') \in \mathcal{D}_{\text{adm}}$ *and let* $N = 1, 2, \dots$. *Then* $P'_N(\lambda \mid z, z', w, w') \geq 0$ *for any* $\lambda \in \mathbb{GT}_N$, *and there exists* $\lambda \in \mathbb{GT}_N$ *for which the above inequality is strict.*

Proof. The first claim follows from Proposition 3.3 (i). Now we shall describe the set of those $\lambda \in \mathbb{GT}_N$ for which $P'_N(\lambda \mid z, z', w, w') > 0$.

When both (z, z') and (w, w') are in $\mathcal{Z}_{\text{princ}} \sqcup \mathcal{Z}_{\text{compl}}$ then, by Proposition 3.3 (ii), this is the whole $\mathbb{GT}_N$.

When $(w, w') \in \mathcal{Z}_{\text{princ}} \sqcup \mathcal{Z}_{\text{compl}}$ and $(z, z') \in \mathcal{Z}_{\text{degen}}$, say, $(z, z') \in \mathcal{Z}_{\text{degen},m}$, then this set is formed by λ's satisfying the condition $\lambda_1 \leq m$. Indeed, this readily follows from claims (ii) and (iii) of Proposition 3.3.

Likewise, when $(z, z') \in \mathcal{Z}_{\text{princ}} \sqcup \mathcal{Z}_{\text{compl}}$ and $(w, w') \in \mathcal{Z}_{\text{degen},m}$, then the condition takes the form $\lambda_N \geq -m$.

Finally, when both (z, z') and (w, w') are in $\mathcal{Z}_{\text{degen}}$, say $(z, z') \in \mathcal{Z}_{\text{degen},k}$ and $(w, w') \in \mathcal{Z}_{\text{degen},l}$, then the set in question is described by the conditions $\lambda_1 \leq k$, $\lambda_N \geq -l$. The set is nonempty provided that $k \geq -l$, which is exactly the extra condition from Definition 3.4.

Note that if $k = -l$ then this set consists of a single element $\lambda = (k, \dots, k)$. □

Proposition 3.5 implies that $\mathcal{D}_{\mathrm{adm}} \subset \mathcal{D}_0$. Of course, this can be checked directly, but the claim is not entirely obvious, for instance, when both (z, z') and (w, w') are in $\mathcal{Z}_{\mathrm{compl}}$.

Now we can summarize the above definitions and results in the following theorem.

THEOREM 3.6. *For any admissible quadruple* (z, z', w, w'), *the family* $\{P_N(\,\cdot \mid z, z', w, w')\}$, *where* $N = 1, 2, \dots$, *is a coherent system, so that it determines a character* $\chi_{z,z',w,w'}$ *of the group* $\mathrm{U}(\infty)$.

Proof. Indeed, let (z, z', w, w') be admissible. Since (z, z', w, w') is in $\mathcal{D}_0$, the definition of $P_N(\lambda \mid z, z', w, w')$'s makes sense. By Proposition 3.5, for any N, $P_N(\,\cdot \mid z, z', w, w')$ is a probability distribution on $\mathbb{GT}_N$. By Proposition 3.2, the family $\{P_N(\,\cdot \mid z, z', w, w')\}_{N=1,2,\dots}$ is a coherent system. By Proposition 2.1, it defines a character of $\mathrm{U}(\infty)$. □

Remark 3.7. The set of characters of the form $\chi_{z,z',w,w'}$ is stable under tensoring with one-dimensional characters $(\det(\,\cdot\,))^k$, where $k \in \mathbb{Z}$. Indeed, the sets $\mathcal{D}$, $\mathcal{D}_0$, and $\mathcal{D}_{\mathrm{adm}}$ are invariant under the shift

$$(z, z', w, w') \mapsto (z + k, z' + k, w - k, w' - k),$$

and we have

$$P_N(\lambda + (k, \dots, k) \mid z, z', w, w') = P_N(\lambda \mid z + k, z' + k, w - k, w' - k).$$

On the other hand, in terms of coherent systems, tensoring with $(\det(\,\cdot\,))^k$ is equivalent to shifting λ by $(k, \dots, k)$.

Remark 3.8. In the special case when both (z, z') and (w, w') are in $\mathcal{Z}_{\mathrm{degen}}$, a detailed study of the distributions $P_N(\,\cdot \mid z, z', w, w')$ from a combinatorial point of view was given by Kerov [Ke].

Remark 3.9. As we see, the structure of the set of all admissible parameters is fairly complicated. However, all the major formulas that will be obtained below hold for all admissible parameters. The explanation of this phenomenon is rather simple: the quantities in question (like correlation functions) can usually be defined for the parameters varying in the domain which is much larger than $\mathcal{D}_{\mathrm{adm}}$; see e.g. Propositions 3.1 and 3.2 above. Thus, the formulas for these quantities usually hold on an open subset of $\mathbb{C}^4$ containing $\mathcal{D}_{\mathrm{adm}}$. It is only when we require certain quantities to be positive in order to fit our computations in the framework of probability theory, that we need to restrict ourselves to the smaller set of admissible parameters.

4. Two discrete point processes

In this section we will explain two different ways to associate to the measure P_N introduced in the previous section, a discrete point process. We also show how the two resulting processes can be obtained one from the other.

First, we recall the general definition of a random point process.

Let $\mathfrak{X}$ be a locally compact separable topological space. A *multiset* X in $\mathfrak{X}$ is a collection of points with possible multiplicities and with no ordering imposed. A *locally finite point configuration* (*configuration,* for short) is a multiset X such that for any compact set $A \subset \mathfrak{X}$ the intersection $X \cap A$ is finite (with multiplicities counted). This implies that X itself is either finite or countably infinite.

The set of all configurations in $\mathfrak{X}$ is denoted by $\mathrm{Conf}(\mathfrak{X})$. Given a relatively compact Borel set $A \subset \mathfrak{X}$, we introduce a function $\mathcal{N}_A$ on $\mathrm{Conf}(\mathfrak{X})$ by setting $\mathcal{N}_A(X) = |X \cap A|$. We equip $\mathrm{Conf}(\mathfrak{X})$ with the Borel structure generated by all functions of this form.

A *random point process* on $\mathfrak{X}$ (point process, for short; another term is random point field) is a probability Borel measure $\mathcal{P}$ on the space $\mathrm{Conf}(\mathfrak{X})$.

We do not need the full generality of the definitions in this section. Here the situation is rather simple: all our processes are discrete (that is, the space $\mathfrak{X}$ is discrete), and the point configurations are finite. However, in Section 9 we will consider a continuous point process with infinitely many particles, and then we will need the above definitions.

Consider the lattice

$$\mathfrak{X} = \mathfrak{X}^{(N)} = \begin{cases} \mathbb{Z}, & N \text{ is odd,} \\ \mathbb{Z} + \frac{1}{2}, & N \text{ is even,} \end{cases}$$

and divide it into two parts

$$\begin{aligned} &\mathfrak{X} = \mathfrak{X}_{\mathrm{in}} \sqcup \mathfrak{X}_{\mathrm{out}}, \\ &\mathfrak{X}_{\mathrm{in}} = \left\{ -\frac{N-1}{2}, -\frac{N-3}{2}, \dots, \frac{N-3}{2}, \frac{N-1}{2} \right\}, \quad |\mathfrak{X}_{\mathrm{in}}| = N, \\ &\mathfrak{X}_{\mathrm{out}} = \left\{ \dots, -\frac{N+3}{2}, -\frac{N+1}{2} \right\} \sqcup \left\{ \frac{N+1}{2}, \frac{N+3}{2}, \dots \right\}, \quad |\mathfrak{X}_{\mathrm{out}}| = \infty. \end{aligned}$$

Let $\rho_i = \frac{N+1}{2} - i$, $i = 1, \dots, N$. For any $\lambda \in \mathbb{GT}_N$ we set

$$\mathcal{L}(\lambda) = \{\lambda_1 + \rho_1, \dots, \lambda_N + \rho_N\}.$$

Clearly, $\lambda \mapsto \mathcal{L}(\lambda)$ defines a bijection between $\mathbb{GT}_N$ and the set of N-point multiplicity-free configurations on $\mathfrak{X}$.

Now we define another correspondence between signatures and point configurations. Let us represent a signature λ as a pair of Young diagrams (λ^+, λ^-); see §1(c).

Finally, we define a point configuration as

$$X(\lambda) = \left\{p_i^+ + \frac{N+1}{2}\right\} \sqcup \left\{\frac{N-1}{2} - q_i^+\right\} \sqcup \left\{-p_j^- - \frac{N+1}{2}\right\} \sqcup \left\{-\frac{N-1}{2} + q_j^-\right\}, \tag{4.1}$$

where $i = 1, \dots, d^+$ and $j = 1, \dots, d^-$, see §1(c) for the notation. Note that if $\lambda = 0$ then the configuration is empty.

From the inequalities

$$\begin{gathered} p_1^+ > \cdots > p_{d^+}^+ \geq 0, \quad q_1^+ > \cdots > q_{d^+}^+ \geq 0, \\ p_1^- > \cdots > p_{d^-}^- \geq 0, \quad q_1^- > \cdots > q_{d^-}^- \geq 0, \\ d^+ + d^- \leq N \end{gathered}$$

it follows that $X(\lambda)$ consists of an even number of distinct points (equal to $2(d^+ + d^-)$), of which half lie in $\mathfrak{X}_{\text{out}}$ while another half lie in $\mathfrak{X}_{\text{in}}$. Finite point configurations with this property will be called *balanced*.

Conversely, each balanced, multiplicity-free configuration on $\mathfrak{X}$ is of the form $X(\lambda)$ for one and only one signature $\lambda \in \mathbb{GT}_N$. Thus, the map $\lambda \mapsto X(\lambda)$ defines a bijection between $\mathbb{GT}_N$ and the set of finite balanced configurations on $\mathfrak{X}$ with no multiplicities.

Define an involution on the set $\operatorname{Conf}(\mathfrak{X})$ of multiplicity-free point configurations on $\mathfrak{X}$ by

$$X \mapsto X^{\triangle} = X \triangle \mathfrak{X}_{\text{in}} = (X \cap \mathfrak{X}_{\text{out}}) \cup (\mathfrak{X}_{\text{in}} \setminus X).$$

Since $|\mathfrak{X}_{\text{in}}| = N$, this involution defines a bijection between N-point configurations and finite balanced configurations.

Proposition 4.1. *In the above notation*, $X(\lambda) = \mathcal{L}(\lambda)^{\triangle}$ *for any signature* $\lambda \in \mathbb{GT}_N$.

For instance, let $N = 7$ and $\lambda = (4, 2, 2, 0, -1, -2, -2)$. Then

$$\mathcal{L}(\lambda) = \{7, 4, 3, 0, -2, -4, -5\}$$

and, since $\mathfrak{X}_{\text{in}} = \{3, 2, 1, 0, -1, -2, -3\}$, we have

$$X(\lambda) = \mathcal{L}(\lambda)^{\triangle} = \{7, 4, 2, 1, -1, -3, -4, -5\}.$$

On the other hand, $\lambda^+ = (4, 2, 2) = (3, 0 \,|\, 2, 1)$, $\lambda^- = (2, 2, 1) = (1, 0 \,|\, 2, 0)$, and (4.1) gives the same $X(\lambda)$.

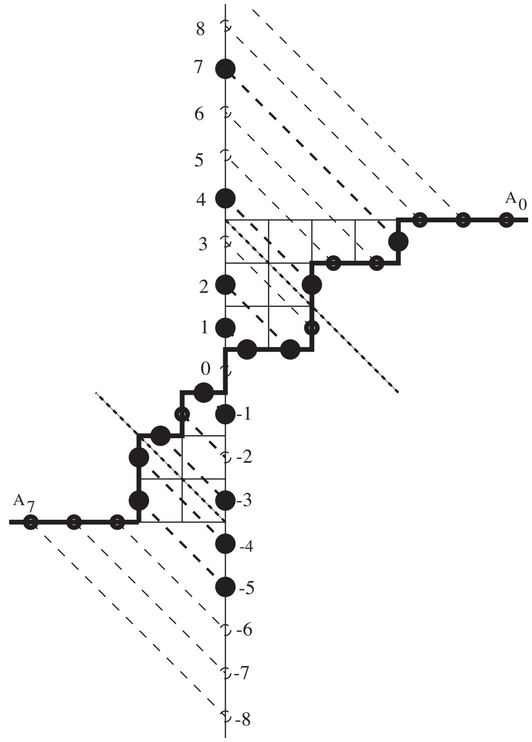

Figure 1 (Proposition 4.1)

Another example: for the zero signature $\underline{0}$ we have $\mathcal{L}(\underline{0}) = \mathfrak{X}_{\mathrm{in}}$ and $X(\underline{0}) = \varnothing$.

Proof of Proposition 4.1. See Figure 1 where geometric constructions described below are illustrated. Consider a plane with Cartesian coordinates (x, y) and put the lattice $\mathfrak{X} = \mathfrak{X}^{(N)}$ on the vertical axis $x = 0$, so that each $a \in \mathfrak{X}$ is identified with the point $(0, a)$ of the plane. Draw a square grid in the plane, formed by the horizontal lines $y = a + \frac{1}{2}$, where a ranges over $\mathfrak{X}$, and by the vertical lines $x = b$, where b ranges over $\mathbb{Z}$. We represent λ by an infinite polygonal line $\mathbb{L}$ on the grid, as follows.

Denote by $A_0, \ldots, A_N$ the horizontal lines defined by $y = \frac{N}{2}$, $y = \frac{N}{2} - 1$, $\ldots$, $y = -\frac{N}{2}$, respectively. We remark that these lines belong to the grid: indeed, $\mathfrak{X}$ coincides with $\mathbb{Z}$ shifted by $\frac{N-1}{2}$, so that the points $\frac{N}{2}, \frac{N}{2} - 1, \ldots, -\frac{N}{2}$ belong to the shift of $\mathfrak{X}$ by $\frac{1}{2}$.

The polygonal line $\mathbb{L}$ first goes along A_0, from right to left, starting at $x = +\infty$, up to the point with the coordinate $x = \lambda_1$. Then it changes the direction and goes downwards until it meets the next horizontal line A_1. Then it goes along A_1, again from right to left, up to the point with the coordinate

$x = \lambda_2$, etc. Finally, after reaching the lowest line A_N at the point with the coordinate $x = \lambda_N$, it goes only to the left, along this line.

Further, we define a bijective correspondence $a \leftrightarrow s$ between the points $a \in \mathfrak{X}$ and the sides s of $\mathbb{L}$, as follows. Given a, we draw the line $x + y = a$, it intersects $\mathbb{L}$ at the midpoint of a side, which is, by definition, s. Let us call a a "v-point" or an "h-point" according to whether the corresponding side s is vertical or horizontal. Thus, the whole set $\mathfrak{X}$ is partitioned into "v-points" and "h-points".

The "v-points" of $\mathfrak{X}$ are exactly those of the configuration $\mathcal{L}(\lambda)$. Consequently, the collection $\big(\mathcal{L}(\lambda) \cap \mathfrak{X}_{\text{out}}\big) \sqcup \big(\mathfrak{X}_{\text{in}} \setminus \mathcal{L}(\lambda)\big)$ is formed by the "v-points" from $\mathfrak{X}_{\text{out}}$ and the "h-points" from $\mathfrak{X}_{\text{in}}$.

On the other hand, the correspondence $a \leftrightarrow s$ makes it possible to interpret the same collection of points in terms of the Frobenius coordinates of the diagrams λ^+ and λ^-. Indeed, the diagram λ^+ can be identified with the figure bounded by the horizontal line A_0, the vertical line $x = 0$, and by $\mathbb{L}$. Then the line $x + y = \frac{N}{2}$ coincides with the diagonal of λ^+. Above this line, there are d^+ vertical sides of $\mathbb{L}$, say, $s_1, \dots, s_{d^+}$, which lie in the rows of λ^+ with numbers $1, \dots, d^+$. The corresponding Frobenius coordinates are $p_i^+ = \lambda_i - i$, where $i = 1, \dots, d^+$. It easily follows that the midpoint of the side s_i lies on the line $x + y = \frac{N+1}{2} + p_i^+$; i.e., s_i corresponds to $\frac{N+1}{2} + p_i^+$. In this way we get the first component $\big\{p_i^+ + \frac{N+1}{2}\big\} \subset X(\lambda)$, see (4.1). The remaining three components are interpreted similarly. □

Fix any admissible quadruple (z, z', w, w') of parameters and consider the corresponding probability measure P_N on $\mathbb{GT}_N$; see Section 3. Taking the pushforwards of the measure P_N under the maps $\lambda \mapsto \mathcal{L}(\lambda)$ and $\lambda \mapsto X(\lambda)$ we get two point processes on the lattice $\mathfrak{X}$, which we denote by $\widetilde{\mathcal{P}}^{(N)}$ and $\mathcal{P}^{(N)}$, respectively. We are mainly interested in the process $\mathcal{P}^{(N)}$, which is defined by $\lambda \mapsto X(\lambda)$; the process $\widetilde{\mathcal{P}}^{(N)}$ defined by $\lambda \mapsto \mathcal{L}(\lambda)$ will play an auxiliary role. Proposition 4.1 implies that $\widetilde{\mathcal{P}}^{(N)}(X) = \mathcal{P}^{(N)}(X^{\triangle})$ for any finite configuration X.

5. Determinantal point processes. General theory

(a) *Correlation measures.* Let $\mathcal{P}$ be a point process on $\mathfrak{X}$ (see the definition in the beginning of §4), and let A denote an arbitrary relatively compact Borel subset of $\mathfrak{X}$. Then $\mathcal{N}_A$ is a random variable with values in $\{0, 1, 2, \dots\}$. We assume that for any A as above, $\mathcal{N}_A$ has finite moments of all orders.

Let n range over $\{1, 2, \dots\}$. The n^{th} *correlation measure* of $\mathcal{P}$, denoted as ρ_n, is a Borel measure on $\mathfrak{X}^n$, uniquely defined by

$$\rho_n(A^n) = \mathbb{E}[\mathcal{N}_A(\mathcal{N}_A - 1) \dots (\mathcal{N}_A - n + 1)],$$

where the symbol $\mathbb{E}$ means expectation with respect to the probability space $(\mathrm{Conf}(\mathfrak{X}), \mathcal{P})$.

Equivalently, for any bounded compactly supported Borel function F on $\mathfrak{X}^n$,

$$\langle F, \rho_n \rangle = \int_{X \in \mathrm{Conf}(X)} \left(\sum_{\substack{x_1, \ldots, x_n \in X \\ \text{pairwise distinct}}} F(x_1, \ldots, x_n) \right) \mathcal{P}(dX),$$

where the summation is taken over all ordered n-tuples of pairwise distinct points taken from the (random) configuration X (here a multiple point is viewed as a collection of different elements).

The measure ρ_n takes finite values on the compact subsets of $\mathfrak{X}^n$. The measure ρ_n is symmetric with respect to the permutations of the arguments.

Under mild assumptions about the growth of $\rho_n(A^n)$ as $n \to \infty$ (here A is an arbitrary compact set), the collection of the correlation measures $\rho_1, \rho_2, \ldots$ defines the initial process $\mathcal{P}$ uniquely. See [Len] and [So, (1.6)].

When there is a "natural" reference measure μ on $\mathfrak{X}$ such that, for any n, ρ_n is absolutely continuous with respect to the product measure $\mu^{\otimes n}$, the density of ρ_n is called the n^{th} *correlation function*. For instance, this always holds if the space $\mathfrak{X}$ is discrete: then as μ one takes the counting measure on $\mathfrak{X}$. The correlation functions are denoted as $\rho_n(x_1, \ldots, x_n)$.

If the space $\mathfrak{X}$ is discrete and the process is multiplicity-free then $\rho_n(x_1, \ldots, x_n)$ is the probability that the random point configuration contains the points $x_1, \ldots, x_n$ (here x_i's are pairwise distinct, otherwise $\rho_n(x_1, \ldots, x_n) = 0$).

For a general discrete process, $\rho_n(x_1, \ldots, x_n)$ is equal to the sum of weights of the point configurations with certain combinatorial prefactors computed as follows: if x has multiplicity k in the multiset $(x_1, \ldots, x_n)$ and has multiplicity m in the point configuration in question, then this produces the prefactor $m(m-1)\cdots(m-k+1)$ (such a prefactor is computed for every element of the set $\{x_1, \ldots, x_n\}$). Note that this prefactor vanishes unless $m \geq k$ for every $x \in \{x_1, \ldots, x_n\}$.

(b) *Determinantal processes.* A point process is called *determinantal* if there exists a function $K(x, y)$ on $\mathfrak{X} \times \mathfrak{X}$ such that, for an appropriate reference measure μ, the correlation functions are given by the determinantal formula

$$\rho_n(x_1, \ldots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^n, \qquad n = 1, 2, \ldots .$$

The function K is called the *correlation kernel* of the process. It is not unique: replacing $K(x, y)$ by $f(x)K(x, y)f(y)^{-1}$, where f is an arbitrary nonzero function on $\mathfrak{X}$, leaves the above expression for the correlation functions intact.

If the reference measure is multiplied by a positive function f then the correlation kernel should be appropriately transformed. For instance, one can multiply it by $(f(x)f(y))^{-1/2}$.

It is often useful to view $K(x, y)$ as the kernel of an integral operator acting in the Hilbert space $L^2(\mathfrak{X}, \mu)$. We will denote this operator by the same symbol K.

Assume that a function $K(x, y)$ is Hermitian symmetric (i.e., $K(x, y) = \overline{K(y, x)}$) and locally of trace class (i.e., its restriction to any compact set $A \subset \mathfrak{X}$ defines a trace class operator in $L^2(A, \mu)$, where μ is a fixed reference measure). Then $K(x, y)$ is the correlation kernel of a determinantal point process if and only if the operator K in $L^2(\mathfrak{X}, \mu)$ satisfies the condition $0 \le K \le 1$; see [So]. However, there are important examples of correlation kernels which are not Hermitian symmetric, see below.

If $\mathfrak{X}$ is a discrete countably infinite space then any multiset with finite multiplicities is a configuration. As μ we will always take the counting measure. A correlation kernel is simply an infinite matrix with the rows and columns labeled by the points of $\mathfrak{X}$. For any determinantal process on $\mathfrak{X}$ the random configuration is multiplicity free with probability 1. Indeed, if any two arguments of the n^{th} correlation function coincide then the defining determinant above vanishes.

(c) *The complementation principle.* Assume that $\mathfrak{X}$ is discrete and fix a subset $Z \subseteq \mathfrak{X}$. For a subset X in $\mathfrak{X}$ let $X \triangle Z$ denote its symmetric difference with Z, i.e., $X \triangle Z = (X \cap \bar{Z}) \cup (Z \setminus X)$, where $\bar{Z} = \mathfrak{X} \setminus Z$. The map $X \mapsto X \triangle Z$, which we will denote by the symbol $\triangle$, is an involution on multiplicity-free configurations. If the process $\mathcal{P}$ lives on the multiplicity-free configurations, we can define its image $\mathcal{P}^{\triangle}$ under $\triangle$.

Assume further that $\mathcal{P}$ is determinantal and let K be its correlation kernel. Then the process $\mathcal{P}^{\triangle}$ is also determinantal. Its correlation kernel $K^{\triangle}$ can be obtained from K as follows:

$$K^{\triangle}(x, y) = \begin{cases} K(x, y), & x \in \bar{Z}, \\ \delta_{xy} - K(x, y), & x \in Z, \end{cases} \tag{5.1}$$

where δ_{xy} is the Kronecker symbol. See [BOO, §A.3].

Note that one could equally well use the formula

$$K^{\triangle}(x, y) = \begin{cases} K(x, y), & y \in \bar{Z}, \\ \delta_{xy} - K(x, y), & y \in Z, \end{cases}$$

obtained from (5.1) by multiplying the kernel by the function $\varepsilon(x)\varepsilon(y)$, where $\varepsilon(\,\cdot\,)$ is equal to 1 on $\bar{Z}$ and to -1 on Z. This operation does not affect the correlation functions; see Section 5(b).

We call the passage from the process $\mathcal{P}$ to the process $\mathcal{P}^{\triangle}$, together with formula (5.1) the *complementation principle.* The idea was borrowed from unpublished work notes by Sergei Kerov connected with an early version of [BOO].

Note that Proposition 4.1 can now be restated as follows:

$$\widetilde{\mathcal{P}}^{(N)} = (\mathcal{P}^{(N)})^{\triangle},$$

where the role of the set Z is played by $\mathfrak{X}_{\text{in}}$.

(d) *Discrete polynomial ensembles.* Here we assume that $\mathfrak{X}$ is a finite or countably infinite subset of $\mathbb{R}$ without limit points.

Assume that we are given a nonnegative function $f(x)$ on $\mathfrak{X}$. Fix a natural number N. We consider f as a weight function: denoting by μ the counting measure on $\mathfrak{X}$ we assign to f the measure $f\mu$ on $\mathfrak{X}$.

We impose on f two basic assumptions:

($*$) f has finite moments at least up to order $2N-2$, i.e.,

$$\sum_{x\in\mathfrak{X}} x^{2N-2} f(x) < \infty.$$

($**$) f does not vanish at least at N distinct points.

Under these assumptions the functions $1, x, \dots, x^{N-1}$ on $\mathfrak{X}$ are linearly independent and lie in the Hilbert space $L^2(\mathfrak{X}, f\mu)$. Let $p_0 = 1$, p_1, $\dots$, p_{N-1} be the monic polynomials obtained by orthogonalizing the system $(1, x, \dots, x^{N-1})$ in $L^2(\mathfrak{X}, f\mu)$.

We set

$$h_n = (p_n, p_n)_{L^2(\mathfrak{X}, f\mu)} = \sum_{x\in\mathfrak{X}} p_n^2(x) f(x), \qquad n = 0, \dots, N-1,$$

and consider the *Christoffel-Darboux kernel*

$$\sum_{n=0}^{N-1} \frac{p_n(x)p_n(y)}{h_n}, \qquad x, y \in \mathfrak{X}.$$

This kernel defines an orthogonal projection operator in $L^2(\mathfrak{X}, f\mu)$; its range is the N-dimensional subspace spanned by $1, x, \dots, x^{N-1}$.

Consider an isometric embedding $L^2(\mathfrak{X}, f\mu) \to \ell^2(\mathfrak{X})$ which is defined as multiplication by $\sqrt{f(\,\cdot\,)}$. Under this isomorphism the Christoffel-Darboux kernel turns into another kernel which we will call the *normalized* Christoffel-Darboux kernel and denote as K^{CD}:

$$K^{\text{CD}}(x, y) = \sqrt{f(x)f(y)} \cdot \sum_{n=0}^{N-1} \frac{p_n(x)p_n(y)}{h_n}, \qquad x, y \in \mathfrak{X}. \tag{5.2}$$

This kernel defines a projection operator in $\ell^2(\mathfrak{X})$ of rank N.

Let $\mathrm{Conf}_N(\mathfrak{X})$ denote the set of N-point multiplicity-free configurations (subsets) in $\mathfrak{X}$. For $X \in \mathrm{Conf}_N(\mathfrak{X})$ we set

$$V^2(X) = \prod_{1 \le i < j \le N} (x_i - x_j)^2,$$

where $x_1, \dots, x_N$ are the points of X written in any order.

Under the assumptions $(*)$ and $(**)$ we have

$$0 < \sum_{X \in \mathrm{Conf}_N(\mathfrak{X})} \left(\prod_{x \in X} f(x) \cdot V^2(X) \right) < \infty.$$

Therefore, we can form a point process on $\mathfrak{X}$ which lives on $\mathrm{Conf}_N(\mathfrak{X})$ and for which the probability of a configuration X is given by

$$\mathrm{Prob}(X) = \mathrm{const} \cdot \prod_{x \in X} f(x) \cdot V^2(X), \qquad X \in \mathrm{Conf}_N(\mathfrak{X}), \tag{5.3}$$

where const is the normalizing constant. This process is called the N-point *polynomial ensemble* with the weight function f.

Proposition 5.1. *Let $\mathfrak{X}$ and f be as above, where f satisfies the assumptions* $(*)$, $(**)$. *Then the N-point polynomial ensemble with the weight function f is a determinantal point process whose correlation kernel is the normalized Christoffel-Darboux kernel* (5.2).

Proof. A standard argument from the Random Matrix Theory, see, e.g., [Me, §5.2]. □

Remark 5.2. Under a stronger than $(*)$ condition

$$\sum_{x \in \mathfrak{X}} |x|^{2N-1} f(x) < \infty,$$

there exists a monic polynomial p_N of degree N, orthogonal to $1, x, \dots, x^{N-1}$ in $L^2(\mathfrak{X}, f\mu)$. Then the Christoffel-Darboux kernel can be written as

$$\frac{1}{h_{N-1}} \frac{p_N(x)p_{N-1}(y) - p_{N-1}p_N(y)}{x-y}.$$

The value at the diagonal $x = y$ is determined via L'Hospital's rule.

According to this, the normalized Christoffel-Darboux kernel can be written in the form

$$K^{\mathrm{CD}}(x,y) = \frac{\sqrt{f(x)f(y)}}{h_{N-1}} \frac{p_N(x)p_{N-1}(y) - p_{N-1}p_N(y)}{x-y}. \tag{5.4}$$

(e) *L-ensembles.* Let $\mathfrak{X}$ be an arbitrary discrete space (finite or countably infinite). We are dealing with the Hilbert space $\ell^2(\mathfrak{X}) = L^2(\mathfrak{X}, \mu)$, where, as usual, μ denotes the counting measure on $\mathfrak{X}$. Let $\operatorname{Conf}_{\text{fin}}(\mathfrak{X})$ denote the set of all finite, multiplicity-free configurations in $\mathfrak{X}$ (i.e., simply finite subsets).

Let L be an operator in $\ell^2(\mathfrak{X})$ and $L(x,y)$ be its matrix ($x, y \in \mathfrak{X}$). For $X \in \operatorname{Conf}_{\text{fin}}(\mathfrak{X})$ we denote by $L_X(x,y)$ the submatrix of $L(x,y)$ of order $|X|$ whose rows and columns are indexed by the points $x \in X$. The determinants $\det L_X$ are exactly the diagonal minors of the matrix $L(x,y)$.

We impose on L the following two conditions:

($*$) L is of trace class.

($**$) All finite diagonal minors $\det L_X$ are nonnegative.

Under these assumptions we have

$$\sum_{X \in \operatorname{Conf}_{\text{fin}}(\mathfrak{X})} \det L_X = \det(1+L) < \infty.$$

We agree that $\det L_{\varnothing} = 1$. Hence, the sum above is always strictly positive.

Now we form a point process on $\mathfrak{X}$ living on the finite multiplicity-free configurations $X \in \operatorname{Conf}_{\text{fin}}(\mathfrak{X})$ with the probabilities given by

$$\operatorname{Prob}(X) = (\det(1+L))^{-1} \det L_X, \qquad X \in \operatorname{Conf}_{\text{fin}}(\mathfrak{X}). \tag{5.5}$$

It is convenient to have a name for the processes obtained in this way; let us call them the *L-ensembles.*

PROPOSITION 5.3. *Let L satisfy the conditions ($*$) and ($**$) above. Then the associated L-ensemble is a determinantal process with the correlation kernel $K = L(1+L)^{-1}$.*

Proof. See [DVJ, Exercise 4.7], [BO2, Prop. 2.1], [BOO, Appendix]. □

The condition ($*$) can be slightly relaxed, see [BOO, Appendix]. The condition ($**$) holds, for instance, when L is Hermitian nonnegative. However, this is by no means necessary, see §5(f) below.

The relation between L and K can also be written in the form

$$1 - K = (1+L)^{-1}.$$

Remark 5.4. Assume that K is a finite-dimensional orthogonal projection operator in $\ell^2(\mathfrak{X})$ (for instance, $K(x,y) = K^{\text{CD}}(x,y)$ as in §5(d)). One can prove that there exists a determinantal point process $\mathcal{P}$ for which K serves as the correlation kernel. $\mathcal{P}$ is not an L-ensemble, because $1-K$ is not invertible (except $K = 0$). However, $\mathcal{P}$ can be approximated by certain L-ensembles. To see this, replace K by $K_\varepsilon = \varepsilon K$, where $0 < \varepsilon < 1$. The matrices $L_\varepsilon = (1-K_\varepsilon)^{-1} - 1$ satisfy both ($*$) and ($**$). The process $\mathcal{P}$ arises in the limit of the L-ensembles associated with the matrices L_ε as $\varepsilon \nearrow 1$. One can check that the probabilities

$$\lim_{\varepsilon \nearrow 1} \det(1+L_\varepsilon)^{-1} \det(L_\varepsilon)_X\,, \qquad X \in \mathrm{Conf}_{\mathrm{fin}}(\mathfrak{X}),$$

are correctly defined.

For a special class of matrices L there exists a complex analytic problem the solution of which yields the resolvent matrix K.

We will follow the exposition of [B3].

Let $\mathfrak{X}$ be a discrete locally finite subset of $\mathbb{C}$. We call an operator L acting in $\ell^2(\mathfrak{X})$ *integrable* if its matrix has the form

$$L(x,x') = \begin{cases} \dfrac{\sum_{j=1}^M f_j(x) g_j(x')}{x-x'}, & x \neq x', \\ 0, & x = x', \end{cases} \tag{5.6}$$

for some functions $f_j,\ g_j$ on $\mathfrak{X}$, $j=1,\dots,M$, satisfying the relation

$$\sum_{j=1}^M f_j(x) g_j(x) = 0, \quad x \in \mathfrak{X}. \tag{5.7}$$

We will assume that $f_j, g_j \in \ell^2(\mathfrak{X})$ for all j.

Set

$$f = (f_1,\dots,f_M)^t, \quad g = (g_1,\dots,g_M)^t.$$

Then (5.7) can be rewritten as $g^t(x)f(x)=0$. We will also assume that the operator

$$(Th)(x) = \sum_{x' \in \mathfrak{X},\, x' \neq x} \frac{h(x')}{x-x'} \tag{5.8}$$

is a bounded operator in $\ell^2(\mathfrak{X})$. For example, this holds for $\mathfrak{X} = \mathbb{Z}+c$ for any $c \in \mathbb{C}$.[8] Under these assumptions, it is easy to see that L is a bounded operator in $\ell^2(\mathfrak{X})$.

Now we introduce the complex analytic object.

Let w be a map from $\mathfrak{X}$ to $\mathrm{Mat}(k,\mathbb{C})$, with k a fixed integer.

We say that a matrix function $m : \mathbb{C} \setminus \mathfrak{X} \to \mathrm{Mat}(k,\mathbb{C})$ with simple poles at the points $x \in \mathfrak{X}$ is a solution of the *discrete Riemann-Hilbert problem*[9] $(\mathfrak{X}, w)$ if the following conditions are satisfied

- $m(\zeta)$ is analytic in $\mathbb{C} \setminus \mathfrak{X}$,

[8] Indeed, then T is a Toeplitz operator with the symbol $\sum_{n\neq 0} \dfrac{u^n}{n} \in L^\infty(S^1)$.

[9] DRHP, for short

- $\operatorname{Res}_{\zeta=x} m(\zeta) = \lim_{\zeta\to x} \left(m(\zeta)w(x)\right), \quad x \in \mathfrak{X}$,
- $m(\zeta) \to I$ as $\zeta \to \infty$.

Here I is the $k \times k$ identity matrix. The matrix $w(x)$ is called the *jump matrix*.

If the set $\mathfrak{X}$ is infinite, the last condition must be made more precise. Indeed, a function with poles accumulating at infinity cannot have asymptotics at infinity. One way to make this condition precise is to require the uniform asymptotics on a sequence of expanding contours, for example, on a sequence of circles $|\zeta| = a_k$, $a_k \to +\infty$.

In order to guarantee the uniqueness of solutions of the DRHPs considered below, we always assume that there exists a sequence of expanding contours such that the distance from these contours to the set $\mathfrak{X}$ is bounded from zero, and we will require a solution $m(\zeta)$ to uniformly converge to I on these contours.

The setting of the DRHP above is very similar to the pure soliton case in the inverse scattering method, see [BC], [BDT], [NMPZ, Ch. III].

PROPOSITION 5.5 ([B3, Prop. 4.3]). *Let L be an integrable operator as described above such that the operator $(1 + L)$ is invertible, and let $m(\zeta)$ be a solution of the* DRHP $(\mathfrak{X}, w)$ *with*

$$w(x) = -f(x)g(x)^t \in \operatorname{Mat}(M, \mathbb{C}).$$

Then the matrix $K = L(1 + L)^{-1}$ has the form

$$K(x, x') = \begin{cases} \dfrac{G^t(x')F(x)}{x - x'}, & x \neq x', \\ G^t(x) \lim_{\zeta\to x} \left(m'(\zeta)\, f(x)\right), & x = x', \end{cases}$$

where $m'(\zeta) = \dfrac{dm(\zeta)}{d\zeta}$, and

$$F(x) = \lim_{\zeta\to x} \left(m(\zeta)\, f(x)\right), \qquad G(x) = \lim_{\zeta\to x} \left((m^t(\zeta))^{-1}\, g(x)\right).$$

Comments. 1) The continuous analog of this result was originally proved in [IIKS], see also [De] and [KBI].

2) It can be proved that the solution of the DRHP stated in Proposition 5.5 exists and is unique, see [B3, (4.9)] for the existence and [B3, Lemma 4.7] for the uniqueness.

3) The requirement of matrix L's vanishing on the diagonal can be substantially weakened, see [B3, Remark 4.2]. A statement similar to Proposition 5.5 can be proved if the diagonal elements of L are bounded from -1.

4) Proposition 5.5 holds *without* the assumptions $(*)$, $(**)$ stated in the beginning of this subsection.

5) If the operator L is bounded, has the form (5.6), but the functions f_j and g_j are not in $\ell^2(\mathfrak{X})$, then it may happen that the operator $K = L/(1+L)$ is well-defined while the corresponding DRHP fails to have a solution.

(f) *Special matrices L.* Let $\mathfrak{X}$ be a discrete space with a fixed splitting into the union of two disjoint subsets,

$$\mathfrak{X} = \mathfrak{X}_I \sqcup \mathfrak{X}_{II}.$$

The splitting induces an orthogonal decomposition of $\ell^2(\mathfrak{X})$,

$$\ell^2(\mathfrak{X}) = \ell^2(\mathfrak{X}_I) \oplus \ell^2(\mathfrak{X}_{II}).$$

According to this decomposition we will write operators in $\ell^2(\mathfrak{X})$ (or matrices of the format $\mathfrak{X} \times \mathfrak{X}$) in the block form. For instance,

$$L = \begin{bmatrix} L_{I,I} & L_{I,II} \\ L_{II,I} & L_{II,II} \end{bmatrix},$$

where $L_{I,I}$ acts from $\ell^2(\mathfrak{X}_I)$ to $\ell^2(\mathfrak{X}_I)$, $L_{I,II}$ acts from $\ell^2(\mathfrak{X}_{II})$ to $\ell^2(\mathfrak{X}_I)$, etc.

We are interested in the matrices L of the following special form:

$$L = \begin{bmatrix} 0 & A \\ -A^* & 0 \end{bmatrix}, \tag{5.9}$$

where A is an operator from $\ell^2(\mathfrak{X}_{II})$ to $\ell^2(\mathfrak{X}_I)$ and A^* is the adjoint operator.

For such L, the condition $(**)$ of §5(e) is satisfied, while the condition $(*)$ is equivalent to saying that A is of trace class. It can be shown that the construction of §5(e) holds even if A is a Hilbert-Schmidt operator; see [BOO, Appendix].

Note that the matrices of the form (5.9) are not Hermitian symmetric but J-symmetric. That is, the corresponding operator is Hermitian with respect to the indefinite inner product on the space $\ell^2(\mathfrak{X})$ defined by the matrix $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. It follows that the matrices $K = L(1+L)^{-1}$ are J-symmetric, too. This provides a class of determinantal processes whose correlation kernels are not Hermitian symmetric.

Now let us look at an even more special situation. Assume that $\mathfrak{X}$ is a locally finite subset of $\mathbb{C}$ such that the operator T defined by (5.8) is bounded.

Let $h_I(\,\cdot\,)$, $h_{II}(\,\cdot\,)$ be two functions defined on $\mathfrak{X}_I$ and $\mathfrak{X}_{II}$, respectively. We assume that $h_I \in \ell^2(\mathfrak{X}_I)$, $h_{II} \in \ell^2(\mathfrak{X}_{II})$. (The functions h_I, h_{II} should not be confused with the constants h_n attached to orthogonal polynomials, see §5(d).)

Set

$$L = \begin{bmatrix} 0 & A \\ -A^* & 0 \end{bmatrix}, \qquad \text{where} \quad A(x,y) = \frac{h_I(x)h_{II}(y)}{x-y}\,. \tag{5.10}$$

The matrix A is well defined, because x and y range over disjoint subsets $\mathfrak{X}_I$ and $\mathfrak{X}_{II}$ of $\mathfrak{X}$.

As is explained in [B3, §6], such an L is an integrable operator in the sense of §5(e) with $M = 2$. Let us assume that the functions h_I and h_{II} are real-valued. Then we have $L^* = -L$, and -1 cannot belong to the spectrum of L, that is, $(1 + L)$ is invertible. Thus, the DRHP of Proposition 5.5 has a unique solution.

Let us introduce a special notation for this solution $m(\zeta)$. We define four meromorphic functions R_I, S_I, R_{II}, S_{II} by the relation

$$m = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} R_I & -S_{II} \\ -S_I & R_{II} \end{bmatrix}.$$

Then the DRHP of Proposition 5.5 for our special L given by (5.10) can be restated as follows; see [B3, §6]:

- matrix elements $m_{11} = R_I$ and $m_{21} = -S_I$ are holomorphic in $\mathbb{C} \setminus \mathfrak{X}_{II}$;
- matrix elements $m_{12} = -S_{II}$ and $m_{22} = R_{II}$ are holomorphic in $\mathbb{C} \setminus \mathfrak{X}_I$;
- R_I and S_I have simple poles at the points of $\mathfrak{X}_{II}$, and for $x \in \mathfrak{X}_{II}$
$$\operatorname*{Res}_{\zeta=x} R_I(\zeta) = h_{II}^2(x) S_{II}(x),$$
$$\operatorname*{Res}_{\zeta=x} S_I(\zeta) = h_{II}^2(x) R_{II}(x);$$
- R_{II} and S_{II} have simple poles at the points of $\mathfrak{X}_I$, and for $x \in \mathfrak{X}_I$,
$$\operatorname*{Res}_{\zeta=x} R_{II}(\zeta) = h_I^2(x) S_I(x),$$
$$\operatorname*{Res}_{\zeta=x} S_{II}(\zeta) = h_I^2(x) R_I(x);$$
- $R_I, R_{II} \to 1$, $S_I, S_{II} \to 0$ as $\zeta \to \infty$.

As before, the last condition is understood as uniform convergence on a sequence of expanding contours such that the distance from these contours to the set $\mathfrak{X}$ is bounded from zero.

It can be proved that these conditions imply the relations

$$R_I(\zeta) = 1 - \sum_{y \in \mathfrak{X}_{II}} \frac{h_{II}^2(y) S_{II}(y)}{y - \zeta}, \quad S_I(\zeta) = -\sum_{y \in \mathfrak{X}_{II}} \frac{h_{II}^2(y) R_{II}(y)}{y - \zeta}, \tag{5.11}$$

$$R_{II}(\zeta) = 1 - \sum_{y \in \mathfrak{X}_I} \frac{h_I^2(y) S_I(y)}{y - \zeta} \qquad S_{II}(\zeta) = -\sum_{y \in \mathfrak{X}_I} \frac{h_I^2(y) R_I(y)}{y - \zeta}. \tag{5.12}$$

The inverse implication also holds if we know that the functions R_I, S_I, R_{II}, S_{II} have the needed asymptotics as $\zeta \to \infty$.

The next statement is a direct corollary of Proposition 5.5.

Proposition 5.6 ([B3, Prop. 6.1]). *Let*

$$m = \begin{bmatrix} R_I & -S_{II} \\ -S_I & R_{II} \end{bmatrix}$$

be a solution of the DRHP *stated above, where* $h_I \in \ell^2(\mathfrak{X}_I)$ *and* $h_{II} \in \ell^2(\mathfrak{X}_{II})$ *are real-valued. Then* $(1+L)$ *is invertible, and the matrix of the operator* $K = L/(1+L)$, *with respect to the splitting* $\mathfrak{X} = \mathfrak{X}_I \sqcup \mathfrak{X}_{II}$, *has the form*

$$\begin{aligned}
K_{I,I}(x,y) &= h_I(x)h_I(y)\,\frac{R_I(x)S_I(y) - S_I(x)R_I(y)}{x-y}\,,\\
K_{I,II}(x,y) &= h_I(x)h_{II}(y)\,\frac{R_I(x)R_{II}(y) - S_I(x)S_{II}(y)}{x-y}\,,\\
K_{II,I}(x,y) &= h_{II}(x)h_I(y)\,\frac{R_{II}(x)R_I(y) - S_{II}(x)S_I(y)}{x-y}\,,\\
K_{II,II}(x,y) &= h_{II}(x)h_{II}(y)\,\frac{R_{II}(x)S_{II}(y) - S_{II}(x)R_{II}(y)}{x-y}\,,
\end{aligned}$$

where the indeterminacy on the diagonal $x = y$ *is resolved by L'Hospital's rule*:

$$\begin{aligned}
K_{I,I}(x,x) &= h_I^2(x)\left((R_I)'(x)S_I(x) - (S_I)'(x)R_I(x)\right),\\
K_{II,II}(x,x) &= h_{II}^2(x)\left((R_{II})'(x)S_{II}(x) - (S_{II})'(x)R_{II}(x)\right).
\end{aligned}$$

(g) *Connection between discrete polynomial ensembles and L-ensembles.* Here we adopt the following assumptions:

- $\mathfrak{X} = \mathfrak{X}_I \sqcup \mathfrak{X}_{II}$ is a finite or countably infinite subset of $\mathbb{R}$ without limit points.
- The set $\mathfrak{X}_{II}$ is finite, $|\mathfrak{X}_{II}| = N$.
- h_I is a nonnegative function on $\mathfrak{X}_I$ such that

$$\sum_{x\in\mathfrak{X}_I} \frac{h_I^2(x)}{1+x^2} < \infty. \tag{5.13}$$

- h_{II} is a strictly positive function on $\mathfrak{X}_{II}$.

To these data we associate a function f on $\mathfrak{X}$ as follows:

$$f(x) = \begin{cases} \dfrac{h_I^2(x)}{\prod\limits_{y\in\mathfrak{X}_{II}} (x-y)^2}\,, & x \in \mathfrak{X}_I\,,\\[2ex] \dfrac{1}{h_{II}^2(x)\prod\limits_{\substack{y\in\mathfrak{X}_{II}\\ y\neq x}} (x-y)^2}\,, & x\in\mathfrak{X}_{II}\,. \end{cases} \tag{5.14}$$

We note that f is nonnegative on $\mathfrak{X}$, strictly positive on $\mathfrak{X}_{II}$, and its $(2N-2)$nd moment is finite,

$$\sum_{x\in\mathfrak{X}} x^{2N-2} f(x) < \infty.$$

Conversely, given f with such properties, we can define the functions h_I and h_{II} by inverting (5.14), and then the condition (5.13) will be satisfied.

Since the function f satisfies the two basic assumptions for a weight function stated in §5(d) (the moment of order $2N-2$ is finite, and f is strictly positive on an N-point subset), we can attach to it a discrete polynomial ensemble.

On the other hand, let us define a matrix L using (5.10). By virtue of (5.13), all the columns of A are vectors from $\ell^2(\mathfrak{X}_I)$. Since the total number of columns in A is finite, the trace class condition for A holds for trivial reasons. According to §5(f), such a matrix L defines a determinantal point process.

PROPOSITION 5.7. *Under the above assumptions, the orthogonal polynomial ensemble with the weight function f and the L-ensemble associated with the matrix* (5.10) *are connected by the involution $\triangle$ corresponding to $Z=\mathfrak{X}_{II}$.*

Proof. We will prove that for any balanced configuration X, the probability of X in the L-ensemble is equal to the probability of $X^{\triangle}$ in the orthogonal polynomial ensemble. We have to compare two expressions, (5.3) and (5.5), which both involve a normalizing constant. Since we know that we are dealing with probability measures, we may ignore constant factors.

Let X be a finite balanced configuration with no multiplicities. Write it as $A\sqcup B$, where $A=X\cap\mathfrak{X}_I=\{a_1,\dots,a_d\}$, $B=X\cap\mathfrak{X}_{II}=\{b_1,\dots,b_d\}$. In this notation, the probability of X in the L-ensemble is equal, up to a constant factor, to

$$\det L_{A\sqcup B}=\prod_{i=1}^{d}h_I^2(a_i)h_{II}^2(b_i)\cdot\det{}^2\left[\frac{1}{a_i-b_j}\right]_{1\le i,j\le d}. \tag{5.15}$$

For arbitrary finite configurations $C=\{c_1,\dots,c_m\}$ and $D=\{d_1,\dots,d_n\}$ we will abbreviate

$$V^2(C)=\prod_{1\le i<j\le m}(c_i-c_j)^2,\qquad V^2(C;D)=\prod_{i=1}^{m}\prod_{j=1}^{n}(c_i-d_j)^2.$$

By the well-known formula for Cauchy's determinant,

$$\det{}^2\left[\frac{1}{a_i-b_j}\right]=\frac{V^2(A)V^2(B)}{V^2(A;B)},$$

the expression (5.15) is equal to

$$\prod_{a\in A}h_I^2(a)\cdot\prod_{b\in B}h_{II}^2(b)\cdot\frac{V^2(A)V^2(B)}{V^2(A;B)}. \tag{5.16}$$

On the other hand, $X^{\triangle} = A \sqcup \bar{B}$, where $\bar{B} = \mathfrak{X}_{II} \setminus B$. The probability of $X^{\triangle}$ in the orthogonal polynomial ensemble is equal, up to a constant factor, to

$$\prod_{x \in X^{\triangle}} f(x) \cdot V^2(X^{\triangle}) = \prod_{a \in A} f(a) \cdot \prod_{\bar{b} \in \bar{B}} f(\bar{b}) \cdot V^2(A \sqcup \bar{B}). \tag{5.17}$$

Let us transform this expression. We have

$$\prod_{\bar{b} \in \bar{B}} f(\bar{b}) = \mathrm{const} \cdot \prod_{b \in B} \frac{1}{f(b)}, \quad \mathrm{const} = \prod_{x \in \mathfrak{X}_{II}} f(x).$$

Next,

$$\begin{aligned} V^2(A \sqcup \bar{B}) &= V^2(A) V^2(\bar{B}) V^2(A; \bar{B}) \\ &= \frac{V^2(A) V^2(B)}{V^2(A;B)} \cdot \frac{V^2(A;\bar{B}) V^2(A;B)}{V^4(B) V^2(B;\bar{B})} \cdot V^2(B) V^2(\bar{B}) V^2(B;\bar{B}) \\ &= \mathrm{const} \cdot \frac{V^2(A) V^2(B)}{V^2(A;B)} \cdot \frac{\prod_{a \in A} \prod_{y \in \mathfrak{X}_{II}} (a-y)^2}{\prod_{b \in B} \prod_{y \in \mathfrak{X}_{II} \setminus \{b\}} (b-y)^2}, \end{aligned}$$

where

$$\mathrm{const} = V^2(B) V^2(\bar{B}) V^2(B; \bar{B}) = V^2(\mathfrak{X}_{II}).$$

It follows that (5.17) is equal, up to a constant factor, to

$$\prod_{a \in A} \Bigg(f(a) \prod_{y \in \mathfrak{X}_{II}} (a-y)^2 \Bigg) \cdot \prod_{b \in B} \Bigg(f(b) \prod_{y \in \mathfrak{X}_{II} \setminus \{b\}} (b-y)^2 \Bigg)^{-1} \cdot \frac{V^2(A) V^2(B)}{V^2(A;B)}.$$

By virtue of the connection between f and $\{h_I, h_{II}\}$, see (5.14), this is equal to (5.16). □

(h) *Connection between two correlation kernels.* We keep the assumptions of §5(g). In particular, f is related to h_I and h_{II} by (5.14).

Recall that if $h_I \in \ell^2(\mathfrak{X}_I)$ and the operator T (see (5.8)) is bounded then the DRHP of §5(f) has a unique solution which defines the meromorphic functions R_I, S_I, R_{II}, S_{II}. Also, $h_I \in \ell^2(\mathfrak{X}_I)$ implies that the function f defined through (5.14) has a finite $(2N)^{\mathrm{th}}$ moment, and, hence, we can define monic, orthogonal with respect to the weight f polynomials $\{p_0, p_1, \dots\}$ at least up to the N^{th} one; see §5(d). Note that the condition $h_I \in \ell^2(\mathfrak{X}_I)$ is stronger than (5.13).

Proposition 5.8. *Under the assumptions of Section* 5(g), *if* $h_I \in \ell^2(\mathfrak{X}_I)$ *and the operator* T *is bounded, then we have*

$$R_I(\zeta) = \frac{p_N(\zeta)}{\prod_{y \in \mathfrak{X}_{II}} (\zeta - y)}, \qquad S_I(\zeta) = \frac{p_{N-1}(\zeta)}{h_{N-1} \prod_{y \in \mathfrak{X}_{II}} (\zeta - y)}. \tag{5.18}$$

Proof. Denote the right-hand sides of (5.18) by $\widetilde{R}_I$ and $\widetilde{S}_I$, respectively, and define

$$\widetilde{R}_{II}(\zeta) = 1 - \sum_{y\in\mathfrak{X}_I} \frac{h_I^2(y)\widetilde{S}_I(y)}{y-\zeta}, \qquad \widetilde{S}_{II}(\zeta) = -\sum_{y\in\mathfrak{X}_I} \frac{h_I^2(y)\widetilde{R}_I(y)}{y-\zeta};$$

cf. (5.12). We will show that the matrix

$$\widetilde{m} = \begin{bmatrix} \widetilde{R}_I & -\widetilde{S}_{II} \\ -\widetilde{S}_I & \widetilde{R}_{II} \end{bmatrix}$$

solves the DRHP of §5(f). By uniqueness of the solution we will conclude that $m = \widetilde{m}$.

The condition $h_I \in \ell^2(\mathfrak{X}_I)$ guarantees that the formulas above define meromorphic functions $\widetilde{R}_{II}$, $\widetilde{S}_{II}$ with needed asymptotics and location of poles. Thus, we only need to check the relations involving residues at the poles. The equalities

$$\operatorname*{Res}_{\zeta=x} \widetilde{R}_{II}(\zeta) = h_I^2(x)\widetilde{S}_I(x), \tag{5.19}$$

$$\operatorname*{Res}_{\zeta=x} \widetilde{S}_{II}(\zeta) = h_I^2(x)\widetilde{R}_I(x), \tag{5.20}$$

are obviously satisfied.

The relation $\operatorname*{Res}_{\zeta=x} \widetilde{S}_I(\zeta) = h_{II}^2(x)\widetilde{R}_{II}(x)$ is equivalent to the equality

$$-\frac{p_{N-1}(x)}{h_{N-1}\prod_{y\in\mathfrak{X}_{II},\, y\neq x}(x-y)} = -h_{II}^2(x)\left(1 - \sum_{y\in\mathfrak{X}_I}\frac{h_I^2(y)\widetilde{S}_I(y)}{y-x}\right) \tag{5.21}$$

which can be rewritten as ($x \in \mathfrak{X}_{II}$)

$$-p_{N-1}(x)f(x)\prod_{t\in\mathfrak{X}_{II},\, t\neq x}(x-t) = -h_{N-1} + \sum_{y\in\mathfrak{X}_I} p_{N-1}(y)f(y)\prod_{t\in\mathfrak{X}_{II},\, t\neq x}(y-t).$$

But this is the relation

$$\left\langle p_{N-1}(y), \prod_{t\in\mathfrak{X}_{II},\, t\neq x}(y-t)\right\rangle = h_{N-1}$$

which directly follows from the definition of the orthogonal polynomials.

The relation $\operatorname*{Res}_{\zeta=x} \widetilde{R}_I(\zeta) = h_{II}^2(x)\widetilde{S}_{II}(x)$ is equivalent to the equality

$$\frac{p_N(x)}{\prod_{y\in\mathfrak{X}_{II},\, y\neq x}(x-y)} = -h_{II}^2(x)\sum_{y\in\mathfrak{X}_I}\frac{h_I^2(y)\widetilde{R}_I(y)}{y-x}. \tag{5.22}$$

which is just the orthogonality relation

$$\left\langle p_N(y), \prod_{t\in\mathfrak{X}_{II},\, t\neq x}(y-t)\right\rangle = 0. \qquad \square$$

Corollary 5.9. *Under the assumptions of* §5(g), *if* $h_I \in \ell^2(\mathfrak{X})$ *and the operator* T *is bounded, then for any* $x \in \mathfrak{X}_{II}$,

(5.23)
$$R_{II}(x) = \frac{p_{N-1}(x)}{h_{N-1}\, h_{II}^2(x) \prod\limits_{y\in\mathfrak{X}_{II},\, y\neq x}(x-y)}\,, \quad S_{II}(x) = \frac{p_N(x)}{h_{II}^2(x) \prod\limits_{y\in\mathfrak{X}_{II},\, y\neq x}(x-y)}\,.$$

Proof. This follows from the relations
$$\operatorname*{Res}_{\zeta=x} R_I(\zeta) = h_{II}^2(x) S_{II}(x), \qquad \operatorname*{Res}_{\zeta=x} S_I(\zeta) = h_{II}^2(x) R_{II}(x),$$
and (5.18). □

For the next statement we drop the assumption $h_I \in \ell^2(\mathfrak{X}_I)$ and use the weaker assumption (5.13) instead.

Theorem 5.10. *Under the assumptions of* §5(g), *let* L *be given by* (5.10), $K = L(1+L)^{-1}$, *and* K^{CD} *be the* N^{th} *normalized Christoffel-Darboux kernel for the weight function* f, *as defined in* §5(d). *Introduce the following function on* $\mathfrak{X}$ *taking values in* $\{\pm 1\}$:
$$\varepsilon(x) = \begin{cases} \operatorname{sgn}\left(\prod\limits_{y\in\mathfrak{X}_{II}}(x-y)\right), & x\in\mathfrak{X}_I \\ \operatorname{sgn}\left(\prod\limits_{y\in\mathfrak{X}_{II}\setminus\{x\}}(x-y)\right), & x\in\mathfrak{X}_{II}. \end{cases}$$

Then
$$K(x,y) = \varepsilon(x)(K^{\mathrm{CD}})^{\triangle}(x,y)\varepsilon(y), \tag{5.24}$$
where the operation $(\,\cdot\,)^{\triangle}$ *is defined by* (5.1) *with* $Z = \mathfrak{X}_{II}$, $\bar{Z} = \mathfrak{X}_I$.

Before proceeding to the proof let us make a couple of comments.

Comments. 1) By Proposition 5.3 the kernel K describes the correlation functions of the L-ensemble in question. On the other hand, the same correlation functions are also expressed in terms of the kernel $(K^{\mathrm{CD}})^{\triangle}$, see Proposition 5.7 and §5(c). This does not mean that both kernels must coincide, because the correlation kernel of a determinantal point process is not defined uniquely; see Section 5(b). And indeed, we see that the kernels turn out to be conjugated by a nontrivial diagonal matrix. Note that conjugating by a diagonal matrix is the only possible "generic" transformation, because this is the only operation on the "generic" matrix which preserves all diagonal minors. In our situation, both kernels are real and possess a symmetry property (J-symmetry), so that it is not surprising that the diagonal entries of this diagonal matrix are equal to ± 1. However, the exact values of these ± 1's, as given in the theorem, are not evident.

2) The theorem makes it possible to calculate the kernel $K = L(1+L)^{-1}$ provided that we know the orthogonal polynomials with the weight function f. Both kernels, K and $(K^{\mathrm{CD}})^{\triangle}$, are suitable to describing the correlation functions of the L-ensembles. However, from the computations that follow in subsequent sections we will see that, for the particular L-ensemble we are interested in, the former kernel survives in a scaling limit procedure (see §10) while the latter kernel does not.

Proof of Theorem 5.10. Let us assume that the stronger condition $h_I \in \ell^2(\mathfrak{X}_I)$ is satisfied. Then the normalized Christoffel-Darboux kernel can be written in the form (5.4), which makes it possible to express $(K^{\mathrm{CD}})^{\triangle}$ as follows:

$$\begin{aligned}
(K^{\mathrm{CD}})^{\triangle}_{I,I}(x,y) &= \frac{\sqrt{f(x)f(y)}}{h_{N-1}}\,\frac{p_N(x)p_{N-1}(y)-p_{N-1}(x)p_N(y)}{x-y}\,,\\
(K^{\mathrm{CD}})^{\triangle}_{I,II}(x,y) &= \frac{\sqrt{f(x)f(y)}}{h_{N-1}}\,\frac{p_N(x)p_{N-1}(y)-p_{N-1}(x)p_N(y)}{x-y}\,,\\
(K^{\mathrm{CD}})^{\triangle}_{II,I}(x,y) &= \frac{\sqrt{f(x)f(y)}}{h_{N-1}}\,\frac{p_{N-1}(x)p_N(y)-p_N(x)p_{N-1}(y)}{x-y}\,,\\
(K^{\mathrm{CD}})^{\triangle}_{II,II}(x,y) &= \delta(x-y)\\
&\quad+\frac{\sqrt{f(x)f(y)}}{h_{N-1}}\,\frac{p_{N-1}(x)p_N(y)-p_N(x)p_{N-1}(y)}{x-y}\,.
\end{aligned}$$

Here f is the function defined in (5.14).

Assuming that the operator T of (5.8) is bounded, we see that (5.18), (5.23), and Proposition 5.6 directly imply the claim of the theorem everywhere except for the diagonal set $(x,x) \in \mathfrak{X} \times \mathfrak{X}$. But on this diagonal set it is immediately seen that both sides of the equality

$$K(x,x) = (K^{\mathrm{CD}})^{\triangle}(x,x)$$

represent the probability that the corresponding L-ensemble has a particle at the point x; see Comment 1 above.

Thus, we have verified (5.24) assuming $h_I \in \ell^2(\mathfrak{X}_I)$ and the boundedness of T. Now we will show how to get rid of these extra conditions.

Let H_I denote the set of all nonnegative functions $h_I(x)$ satisfying (5.13). We equip H_I with the weakest topology for which all the evaluations $h_I \mapsto h_I(x)$ $(x \in \mathfrak{X}_I)$ and the sum in (5.13) are continuous. Fix h_{II} and let h_I vary over H_I. We claim that for any fixed $x, y \in \mathfrak{X}$, both values $K^{\mathrm{CD}}(x,y)$ and $K(x,y)$ depend continuously on $h_I \in H_I$.

By the definition of the topology in H_I, the moments

$$c_j = \sum_{x\in\mathfrak{X}} x^j f(x), \qquad j = 0, \dots, 2N-2,$$

depend continuously on h_I. Let $G = [c_{i+j}]_{i,j=0}^{N-1}$ be the Gram matrix of the vectors $1, x, \dots, x^{N-1} \in L^2(\mathfrak{X}, f\mu)$. The Christoffel-Darboux kernel can be expressed through the moments as follows:

$$\sum_{i,j=0}^{N-1} (G^{-1})_{ij} x^i y^j.$$

(This can be derived from the classical determinantal expression for the orthogonal polynomials, see [Er, Vol. 2, 10.3(4)], or by making use of a simple direct argument, see, e.g., [B4].) Hence

$$K^{\mathrm{CD}}(x,y) = \sqrt{f(x)f(y)} \sum_{i,j=0}^{N-1} (G^{-1})_{ij} x^i y^j.$$

Clearly, this expression is a continuous function of h_I.

On the other hand, again by the definition of the topology, the columns of the matrix $A(x,y)$ (see (5.10)), viewed as vectors in $\ell^2(\mathfrak{X}_I)$, are continuous in h_I. It follows that the operators L and $K = L(1+L)^{-1}$ are continuous in h_I with respect to uniform operator topology. Hence $K(x,y)$ is continuous, too.

Thus, we have proved that both sides of the required equality (5.24) are continuous in $h_I \in H_I$.

Finally, let $H_I^0 \subset H_I$ be the subset of those functions which have finite support. Note that (5.24) holds for any $h_I \in H_I^0$. Indeed, (5.24) does not change if we replace $\mathfrak{X}_I$ by $\operatorname{supp} h_I$, and for a finite $\mathfrak{X}_I$ the extra conditions are obviously satisfied. Since H_I^0 is dense in H_I, we conclude that (5.24) holds for any $h_I \in H_I$. □

6. $\mathcal{P}^{(N)}$ and $\widetilde{\mathcal{P}}^{(N)}$ as determinantal point processes

Recall that in Section 4 we defined two discrete point processes $\widetilde{\mathcal{P}}^{(N)}$ and $\mathcal{P}^{(N)}$. These processes live on the lattice $\mathfrak{X}^{(N)}$ and depend on four parameters $(z, z', w, w') \in \mathcal{D}_{\mathrm{adm}}$. As was mentioned in §5(c), $\widetilde{\mathcal{P}}^{(N)}$ and $\mathcal{P}^{(N)}$ are related by the complementation principle: $\widetilde{\mathcal{P}}^{(N)} = (\mathcal{P}^{(N)})^{\triangle}$, where the special set Z is equal to $\mathfrak{X}_{\mathrm{in}}$. In this section we will show that $\widetilde{\mathcal{P}}^{(N)}$ is a discrete polynomial ensemble (as defined in §5(d)) and $\mathcal{P}^{(N)}$ is an L-ensemble (as defined in §5(e)).

Consider the following weight function on the lattice $\mathfrak{X}^{(N)}$:

(6.1)

$$f(x) = \frac{1}{\Gamma\left(z - x + \frac{N+1}{2}\right)\Gamma\left(z' - x + \frac{N+1}{2}\right)\Gamma\left(w + x + \frac{N+1}{2}\right)\Gamma\left(w' + x + \frac{N+1}{2}\right)}.$$

Here we assume that $(z, z', w, w') \in \mathcal{D}_{\mathrm{adm}}$. The expression (6.1) comes from the expression for $P'_N(\lambda \mid z, z', w, w')$; see Section 3. Namely, in the notation of Section 4, we have

$$P'_N(\lambda \mid z, z', w, w') = \mathrm{Dim}_N^2(\lambda) \cdot \prod_{x \in \mathcal{L}(\lambda)} f(x).$$

PROPOSITION 6.1. *Let* $(z, z', w, w') \in \mathcal{D}_{\mathrm{adm}}$. *The function* $f(x)$ *is nonnegative on* $\mathfrak{X}^{(N)}$ *and satisfies the assumptions* $(*)$ *and* $(**)$ *of* §5(d).

Proof. The nonnegativity of $f(x)$ follows from Proposition 3.5 and the definition of $\mathcal{D}_{\mathrm{adm}}$; see Definition 3.4.

The condition $(*)$ of §5(d) says that

$$\sum_{x \in \mathfrak{X}^{(N)}} x^{2N-2} f(x) < \infty.$$

This follows from the estimate

$$f(x) \leq \mathrm{const} \cdot (1 + |x|)^{-(z+z'+w+w'+2N)}$$

and the fact that $z + z' + w + w' > -1$ for $(z, z', w, w') \in \mathcal{D}_{\mathrm{adm}}$. As for the estimate above, it readily follows from the asymptotics of the gamma function; see [Ol3, (7.6)].

Finally, the condition $(**)$ of §5(d) says that $f(x)$ does not vanish at least on N distinct points. This follows from the fact that $P'_N(\lambda \mid z, z', w, w')$ does not vanish identically; see Proposition 3.5. □

Note that if $(z, z') \in \mathcal{Z}_{\mathrm{degen}}$ then $f(x)$ vanishes for positive large x. Similarly, if $(w, w') \in \mathcal{Z}_{\mathrm{degen}}$ then $f(x)$ vanishes for negative x such that $|x|$ is large enough.

COROLLARY 6.2. *Let* $(z, z', w, w') \in \mathcal{D}_{\mathrm{adm}}$. *Then* $\widetilde{\mathcal{P}}^{(N)}$ *is a discrete polynomial ensemble with the weight function* $f(x)$ *given by* (6.1). *That is, for any* N*-point configuration* $X = \{x_1, \ldots, x_N\}$

$$\widetilde{\mathcal{P}}^{(N)}(X) = \mathrm{const} \cdot \prod_{i=1}^{N} f(x_i) \cdot \prod_{1 \leq i < j \leq N} (x_i - x_j)^2.$$

Proof. Indeed, by Proposition 6.1, the assumptions of Section 5(d) are satisfied. The formula above is a direct corollary of the definition of $P_N(\lambda)$, see Section 3, and the formula for $\mathrm{Dim}_N \lambda$, see §1(d). □

Now let us turn to the process $\mathcal{P}^{(N)}$. We will apply the formalism of §5(g) for $\mathfrak{X} = \mathfrak{X}^{(N)}$, $\mathfrak{X}_I = \mathfrak{X}_{\mathrm{out}}$, $\mathfrak{X}_{II} = \mathfrak{X}_{\mathrm{in}}$. Recall that §5(g) relies on four assumptions. The first three of them hold for any quadruple $(z, z', w, w') \in \mathcal{D}_{\mathrm{adm}}$. As for the fourth assumption, it is equivalent to the strict positivity of the weight function $f(x)$ on $\mathfrak{X}_{\mathrm{in}}$; see Section 5(g). It may happen that for some admissible quadruples (z, z', w, w') this requirement is violated:

$f(x)$ vanishes at certain points of $\mathfrak{X}_{\text{in}}$. Specifically, this happens whenever (z, z') or (w, w') belongs to $\mathcal{Z}_{\text{degen},m}$ with $m < 0$. For this reason we have to impose an additional restriction on the parameters.

Definition 6.3. Let $\mathcal{D}'_{\text{adm}}$ denote the subset of $\mathcal{D}_{\text{adm}}$ formed by the quadruples $(z, z', w, w') \in \mathcal{D}_{\text{adm}}$ such that neither (z, z') nor (w, w') belongs to $\mathcal{Z}_{\text{degen},m}$ with $m < 0$.

In the rest of the section we assume that $(z, z', w, w') \in \mathcal{D}'_{\text{adm}}$, so that $f(x) > 0$ for any $x \in \mathfrak{X}_{\text{in}}$. Note that, in terms of signatures λ, this condition means that $P_N(\lambda \mid z, z', w, w')$ does not vanish at $\lambda = (0, \dots, 0)$.

Remark 6.4. Recall that in Remark 3.7 we introduced a natural shift on the set $\mathcal{D}_{\text{adm}}$,

$$(z, z', w, w') \mapsto (z + k, z' + k, w - k, w' - k), \quad k \in \mathbb{Z}.$$

This shift of the parameters is equivalent to the shift of all configurations of $\widetilde{\mathcal{P}}^{(N)}$ by k. The definition of $\mathcal{D}_{\text{adm}}$ implies that any quadruple from $\mathcal{D}_{\text{adm}} \setminus \mathcal{D}'_{\text{adm}}$ can be moved into $\mathcal{D}'_{\text{adm}}$ by an appropriate shift. Thus, for the study of the process $\widetilde{\mathcal{P}}^{(N)}$, the restriction of the admissible quadruples to $\mathcal{D}'_{\text{adm}}$ is not essential. This argument does not work for the process $\mathcal{P}^{(N)}$, because the shift above does not preserve the splitting $\mathfrak{X} = \mathfrak{X}_{\text{out}} \sqcup \mathfrak{X}_{\text{in}}$. However, as will be shown later (see the proof of Theorem 10.1), the effect of the shift is negligible in the limit transition as $N \to \infty$.

Let us define functions $\psi_{\text{in}}^{(N)}$ and $\psi_{\text{out}}^{(N)}$ on $\mathfrak{X}_{\text{in}}$ and $\mathfrak{X}_{\text{out}}$, respectively, by the formulas

$$\psi_{\text{in}}^{(N)}(x) = \Gamma\begin{bmatrix} -x + z + \frac{N+1}{2}, -x + z' + \frac{N+1}{2}, x + w + \frac{N+1}{2}, x + w' + \frac{N+1}{2} \\ -x + \frac{N+1}{2}, -x + \frac{N+1}{2}, x + \frac{N+1}{2}, x + \frac{N+1}{2} \end{bmatrix}, \tag{6.2}$$

$$\psi_{\text{out}}^{(N)}(x) = \frac{\left(\left(x - \frac{N-1}{2}\right)_N\right)^2}{\Gamma(-x + z + \frac{N+1}{2})\Gamma(-x + z' + \frac{N+1}{2})\Gamma(x + w + \frac{N+1}{2})\Gamma(x + w' + \frac{N+1}{2})}, \tag{6.3}$$

where we use the notation

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1), \quad \Gamma\begin{bmatrix} a,\, b,\, \dots \\ c,\, d,\, \dots \end{bmatrix} = \frac{\Gamma(a)\Gamma(b)\dots}{\Gamma(c)\Gamma(d)\dots}.$$

Note that

$$\psi^{(N)}_{\rm in}(x)=\frac{1}{\left(\Gamma\left(-x+\frac{N+1}{2}\right)\Gamma\left(x+\frac{N+1}{2}\right)\right)^2 f(x)}\,,$$

$$\psi^{(N)}_{\rm out}(x)=\begin{cases}\left(\dfrac{\Gamma\left(x+\frac{N+1}{2}\right)}{\Gamma\left(x-\frac{N-1}{2}\right)}\right)^2 f(x), & x\ge \frac{N+1}{2}\\ \left(\dfrac{\Gamma\left(-x+\frac{N+1}{2}\right)}{\Gamma\left(-x-\frac{N-1}{2}\right)}\right)^2 f(x), & x\le -\frac{N+1}{2}\,.\end{cases}$$

The function $\psi^{(N)}_{\rm out}$ is nonnegative and the function $\psi^{(N)}_{\rm in}$ is strictly positive. Note also that both functions are invariant with respect to the substitution

$$(z,z')\longleftrightarrow(w,w'),\quad x\longleftrightarrow -x.$$

Introduce a matrix $L^{(N)}$ of format $\mathfrak{X}\times\mathfrak{X}$ which in the block form corresponding to the splitting $\mathfrak{X}=\mathfrak{X}_{\rm out}\sqcup\mathfrak{X}_{\rm in}$ is given by

$$L^{(N)}_{\mathfrak{X}_{\rm out}\sqcup\mathfrak{X}_{\rm in}}=\begin{bmatrix}0 & \mathcal{A}^{(N)}\\ -(\mathcal{A}^{(N)})^* & 0\end{bmatrix},$$

cf. §5(f), where $\mathcal{A}^{(N)}$ is a matrix of format $\mathfrak{X}_{\rm out}\times\mathfrak{X}_{\rm in}$,

$$\mathcal{A}^{(N)}(a,b)=\frac{\sqrt{\psi^{(N)}_{\rm out}(a)\psi^{(N)}_{\rm in}(b)}}{a-b}\,,\qquad a\in\mathfrak{X}_{\rm out},\quad b\in\mathfrak{X}_{\rm in}.$$

PROPOSITION 6.5. *Let $(z,z',w,w')\in\mathcal{D}'_{\rm adm}$. The process $\mathcal{P}^{(N)}$ introduced in Section 4 is an L-ensemble with the matrix $L^{(N)}$ introduced above. That is, for any finite configuration X*

$$\mathcal{P}^{(N)}(X)=\frac{\det L^{(N)}_X}{\det(1+L^{(N)})}\,,$$

where $L^{(N)}_X$ denotes the submatrix of $L^{(N)}$ of finite format $X\times X$.

Proof. This is a direct corollary of Proposition 5.7 and Corollary 6.2, where for Proposition 5.7 we take

$$\mathfrak{X}_I=\mathfrak{X}_{\rm out},\quad \mathfrak{X}_{II}=\mathfrak{X}_{\rm in},\quad h_I^2=\psi^{(N)}_{\rm out},\quad h_{II}^2=\psi^{(N)}_{\rm in}.$$

The relations between the functions $\psi^{(N)}_{\rm out}$, $\psi^{(N)}_{\rm in}$, and f above exactly coincide with (5.14). □

7. The correlation kernel of the process $\widetilde{\mathcal{P}}^{(N)}$

The goal of this section is to compute the normalized Christoffel-Darboux kernel, see Section 5(d), associated with the weight function f on the lattice $\mathfrak{X}^{(N)}$ given by (6.1). According to Proposition 5.1, this kernel can be taken as a correlation kernel for the process $\widetilde{\mathcal{P}}^{(N)}$. We will denote this kernel by $\widetilde{K}^{(N)}$.

We will show below that the orthogonal polynomials with the weight f can be expressed through the hypergeometric function of type $(3,2)$. This is an analytic function in one complex variable u defined inside the unit circle by its Taylor series

$$ {}_3F_2\left[\begin{matrix} a,\, b,\, c \\ e,\, f \end{matrix} \,\Big|\, u\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{k!(e)_k (f)_k}\, u^k. $$

Here a, b, c, e, f are complex parameters, $e, f \notin \{0, -1, -2, \dots\}$. We will only need the value of this function at the point $u = 1$. Then the series above converges if $\Re(e + f - a - b - c) > 0$. Moreover, the function

$$ \frac{1}{\Gamma(e)\Gamma(f)\Gamma(e+f-a-b-c)}\, {}_3F_2\left[\begin{matrix} a,\, b,\, c \\ e,\, f \end{matrix} \,\Big|\, 1\right] $$

can be analytically continued to an entire function in five complex variables a, b, c, e, f.[10] Note also that if one of the parameters a, b, c is a nonpositive integer, say, $a \in \{0, -1, -2, \dots\}$, then the series above has only finitely many nonzero terms. It is easy to see that in such a case

$$ \frac{1}{\Gamma(e)\Gamma(f)}\, {}_3F_2\left[\begin{matrix} a,\, b,\, c \\ e,\, f \end{matrix} \,\Big|\, 1\right] $$

is an entire function in b, c, e, f.

Let us return to the process $\widetilde{\mathcal{P}}^{(N)}$. Set $\Sigma = z + z' + w + w'$.

Theorem 7.1. *For any $(z, z', w, w') \in \mathcal{D}_{\mathrm{adm}}$, the normalized Christoffel-Darboux kernel $\widetilde{K}^{(N)}$ is given by*

$$ \widetilde{K}^{(N)}(x,y) = \frac{1}{h_{N-1}} \frac{\mathfrak{p}_N(x)\mathfrak{p}_{N-1}(y) - \mathfrak{p}_{N-1}(x)\mathfrak{p}_N(y)}{x-y} \sqrt{f(x)f(y)}\,, \tag{7.1} $$

[10] We could not find a proof of this fact in the literature. We give our own proof at the end of the appendix.

where $x, y \in \mathfrak{X}^{(N)}$,

(7.2)

$$\mathfrak{p}_N(x) = \frac{\Gamma(x+w'+\frac{N+1}{2})}{\Gamma(x+w'-\frac{N-1}{2})}\, {}_3F_2\left[\begin{matrix}-N,\ z+w',\ z'+w'\\ \Sigma,\ x+w'-\frac{N-1}{2}\end{matrix}\middle|\, 1\right],$$

$$\mathfrak{p}_{N-1}(x) = \frac{\Gamma(x+w'+\frac{N+1}{2})}{\Gamma(x+w'-\frac{N-1}{2}+1)}\, {}_3F_2\left[\begin{matrix}-N+1,\ z+w'+1,\ z'+w'+1\\ \Sigma+2,\ x+w'-\frac{N-1}{2}+1\end{matrix}\middle|\, 1\right],$$

$$h_{N-1} = \Gamma\left[\begin{matrix}N,\ \Sigma+1,\ \Sigma+2\\ \Sigma+N+1,\ z+w+1,\ z+w'+1,\ z'+w+1,\ z'+w'+1\end{matrix}\right],$$

and $f(x)$ *is given by* (6.1).

Equivalently,

$$\widetilde{K}^{(N)}(x,y) = \frac{1}{h_{N-1}}\, \frac{\widetilde{\mathfrak{p}}_N(x)\mathfrak{p}_{N-1}(y) - \mathfrak{p}_{N-1}(x)\widetilde{\mathfrak{p}}_N(y)}{x-y}\, \sqrt{f(x)f(y)}\,, \tag{7.3}$$

where

$$\widetilde{\mathfrak{p}}_N(x) = \frac{\Gamma(x+w'+\frac{N+1}{2})}{\Gamma(x+w'-\frac{N-1}{2})}\, {}_3F_2\left[\begin{matrix}-N,\ z+w',\ z'+w'\\ \Sigma+1,\ x+w'-\frac{N-1}{2}\end{matrix}\middle|\, 1\right]. \tag{7.4}$$

Comment. 1) If $\Sigma = 0$ then the formula for $\mathfrak{p}_N$ above does not make sense because it involves a hypergeometric function with a zero lower index. The formula (7.3) gives an explicit continuation of the right-hand side of (7.1) to the set $\Sigma = 0$.

2) For any $x \in \mathbb{C}$, the values $\mathfrak{p}_{N-1}(x)$, $\widetilde{\mathfrak{p}}_N(x)$, as well as the constant h_{N-1}, are analytic functions in $(z, z', w, w') \in \mathcal{D}$. The value $\mathfrak{p}_N(x)$ is an analytic function in $(z, z', w, w') \in \mathcal{D} \setminus \{\Sigma = 0\}$. If $(z, z', w, w') \in \mathcal{D}_0$ then $h_{N-1} \neq 0$.

Proof of Theorem 7.1. It is convenient to set

$$t = x + \tfrac{N-1}{2}, \quad u = z + N - 1, \quad u' = z' + N - 1, \quad v = w, \quad v' = w'.$$

Then t ranges over the lattice $\mathbb{Z}$ and the function $f(x)$ turns into the function

$$g(t) = \frac{1}{\Gamma(u+1-t)\Gamma(u'+1-t)\Gamma(v+1+t)\Gamma(v'+1+t)}\,, \quad t \in \mathbb{Z}\,.$$

We consider $g(t)$ as a weight function on the lattice $\mathbb{Z}$. Note that

$$g(t) \le \text{const}\cdot(1+|t|)^{-u-u'-v-v'-2}, \quad |t| \to \infty.$$

Indeed, this follows from the estimate of the function f given in the beginning of the proof of Proposition 6.1.

We aim to study monic orthogonal polynomials $p_0 = 1, p_1, p_2, \dots$ corresponding to the weight function $g(t)$. In general, there are only finitely many such polynomials, because, for any (nonintegral) values of the parameters u, u', v, v', the weight function g has only finitely many moments. This is a major difference between our polynomial ensemble and polynomial ensembles which are usually considered in the literature; cf. [NW], [J].

The number of existing polynomials with the weight function $g(t)$ depends on the number of finite moments of $g(t)$, i.e., on $u + u' + v + v'$. Specifically, the $m^{\rm th}$ polynomial exists if $g(t)$ has finite moments up to the order $(2m - 1)$ (this follows, e.g., from the determinantal formula expressing orthogonal polynomials through the moments; see [Er, Vol. 2, 10.3(4)]). This condition is satisfied if $u + u' + v + v' > 2m - 2$. Let

$$(p_m, p_m) = \sum_{t\in\mathbb{Z}} p_m^2(t) g(t)$$

denote the square of the norm of the $m^{\rm th}$ polynomial. This quantity is well defined if $g(t)$ has finite $(2m)^{\rm th}$ moment, which holds if a slightly stronger condition is satisfied: $u + u' + v + v' > 2m - 1$. (Note that it may happen that p_m exists but $(p_m, p_m) = \infty$.)

Proposition 7.2. *Set* $\mathfrak{S} = u + u' + v + v'$ *and let* $m = 0, 1, \dots$.
If $\mathfrak{S} > 2m - 2$ *then*

$$p_m(t) = \frac{\Gamma(v' + 1 + t)}{\Gamma(v' + 1 + t - m)}\, {}_3F_2 \left[\left. \begin{matrix} -m,\ u + v' + 1 - m,\ u' + v' + 1 - m \\ \mathfrak{S} + 2 - 2m,\ v' + 1 + t - m \end{matrix} \right| 1 \right].$$

If $\mathfrak{S} > 2m - 1$ *then*

$$(p_m, p_m) \equiv \sum_{t\in\mathbb{Z}} p_m^2(t) g(t) =$$
$$\Gamma \left[\begin{matrix} m + 1, \mathfrak{S} + 1 - 2m, \mathfrak{S} + 2 - 2m \\ \mathfrak{S} - m + 2, u + v + 1 - m, u + v' + 1 - m, u' + v + 1 - m, u' + v' + 1 - m \end{matrix} \right].$$

We will give the proof of Proposition 7.2 at the end of this section. Now we proceed with the proof of Theorem 7.1.

Note that the condition $\Sigma = z + z' + w + w' > -1$ entering the definition of $\mathcal{D}_{\rm adm}$ is equivalent to the condition $\mathfrak{S} = u + u' + v + v' > 2N - 3$. This implies the existence of the $N^{\rm th}$ Christoffel-Darboux kernel associated with the weight function $g(t)$,

$$\sum_{m=0}^{N-1} \frac{p_m(t_1)\, p_m(t_2)}{(p_m, p_m)}, \qquad t_1, t_2 \in \mathbb{Z}.$$

Hence, the $N^{\rm th}$ Christoffel-Darboux kernel associated with the weight function f has the form

$$S^{(N)}(x,y) := \sum_{m=0}^{N-1} \frac{p_m(x+\frac{N-1}{2})\, p_m(y+\frac{N-1}{2})}{(p_m,p_m)}, \qquad x,y \in \mathfrak{X}^{(N)}.$$

By the general definition of the normalized Christoffel-Darboux kernel (see (5.2)),

$$\widetilde{K}^{(N)}(x,y) = S^{(N)}(x,y)\sqrt{f(x)f(y)}.$$

On the other hand, let $T_1^{(N)}(x,y)$ denote the right-hand side of (7.1) with the term $\sqrt{f(x)f(y)}$ removed. Similarly, let $T_2^{(N)}(x,y)$ denote the right-hand side of (7.3) with the term $\sqrt{f(x)f(y)}$ removed. It suffices to prove that

$$S^{(N)}(x,y) = T_1^{(N)}(x,y) = T_2^{(N)}(x,y).$$

LEMMA 7.3. *For any $x, y \in \mathfrak{X}^{(N)}$, the kernel $S^{(N)}(x,y)$ viewed as a function in (z,z',w,w'), can be extended to a holomorphic function on the domain $\mathcal{D}_0$.*

Recall that the domain $\mathcal{D}_0$ was introduced in §3.

Proof. Indeed, the claim of the lemma holds for $p_m(x+\frac{N-1}{2})$, $p_m(y+\frac{N-1}{2})$, and $(p_m,p_m)^{-1}$, where $m = 0,\dots,N-1$. This can be verified using explicit formulas of Proposition 7.2. □

Let us prove first that $S^{(N)}(x,y) = T_1^{(N)}(x,y)$ under the additional restriction $\Sigma > 0$. Then $\mathfrak{S} > 2N-2$, so that the $N^{\rm th}$ polynomial p_N exists. Hence, the kernel $S^{(N)}$ can be written in the form (see Remark 5.2)

$$S^{(N)}(x,y) = \frac{1}{(p_{N-1},p_{N-1})} \times \frac{p_N(x+\frac{N-1}{2})p_{N-1}(y+\frac{N-1}{2}) - p_{N-1}(x+\frac{N-1}{2})p_N(y+\frac{N-1}{2})}{x-y}.$$

By explicit formulas of Proposition 7.2,

$$p_N(x+\tfrac{N-1}{2}) = \mathfrak{p}_N(x), \quad p_{N-1}(x+\tfrac{N-1}{2}) = \mathfrak{p}_{N-1}(x), \quad (p_{N-1},p_{N-1}) = h_{N-1}.$$

This implies the desired equality when $\Sigma > 0$.

Next, observe that the expression $T_1^{(N)}(x,y)$ is well defined when (z,z',w,w') ranges over the domain $\mathcal{D}_0 \setminus \{\Sigma = 0\}$ and is a holomorphic function on this domain. Together with Lemma 7.3 this proves the equality $S^{(N)}(x,y) = T_1^{(N)}(x,y)$ provided that $(z,z',w,w') \in \mathcal{D}_0 \setminus \{\Sigma = 0\}$. As a consequence, we get (7.1) under the restriction $\Sigma \neq 0$.

Finally, the series representation for ${}_3F_2$ easily implies the identity

$${}_3F_2\left[\begin{matrix}a,\, b,\, c\\ e,\, f\end{matrix}\,\Big|\,1\right] = {}_3F_2\left[\begin{matrix}a,\, b,\, c\\ e+1,\, f\end{matrix}\,\Big|\,1\right] + \frac{abc}{e(e+1)f}\,{}_3F_2\left[\begin{matrix}a+1,\, b+1,\, c+1\\ e+2,\, f+1\end{matrix}\,\Big|\,1\right].$$

Applying this identity to $\mathfrak{p}_N$ we see that $\mathfrak{p}_N(x) = \widetilde{\mathfrak{p}}_N(x) + \operatorname{const}\mathfrak{p}_{N-1}(x)$. It follows that $T_1^{(N)}(x,y) = T_2^{(N)}(x,y)$ provided that $\mathfrak{p}_N(x)$ makes sense. Thus, we see that the singularity in $T_1^{(N)}(x,y)$ on the hyperplane $\Sigma = 0$ is removable. Specifically, this singularity is explicitly removed by means of the equality $T_1^{(N)}(x,y) = T_2^{(N)}(x,y)$ (equivalently, by means of (7.3)). This completes the proof of Theorem 7.1. □

Proof of Proposition 7.2. We apply the general formalism explained in [NSU, Ch. 2]. Consider the following difference equation on the lattice $\mathbb{Z}$:

$$\sigma(t)\Delta\nabla y(t) + \tau(t)\Delta\, y(t) + \gamma\, y(t) = 0, \tag{7.5$'$}$$

where

$$\sigma(t) = -(t+v)(t+v'), \tag{7.5$''$}$$
$$\tau(t) = \mathfrak{S}\, t + vv' - uu',$$
$$\nabla y(t) = y(t) - y(t-1), \quad \Delta y(t) = y(t+1) - y(t).$$

According to [NSU] this equation is of hypergeometric type, that is, $\sigma(t)$ is a polynomial of degree 2, $\tau(t)$ is a polynomial of degree 1, and γ is a constant. The crucial fact is that this equation can be rewritten in the self-adjoint form with the weight function $g(t)$:

$$(\Delta\circ\sigma g\circ\nabla)\, y + \gamma\, gy = 0,$$

which easily follows from the relation

$$\frac{g(t)}{g(t+1)} = \frac{\sigma(t+1)}{\sigma(t)+\tau(t)}\,.$$

(Note that in [NSU] the weight function is denoted by ρ and the spectral parameter γ is denoted by λ.)

We will seek p_m as a monic polynomial of degree m, which satisfies the difference equation above with an appropriate value of γ. According to [NSU], this value must be equal to

$$\gamma = \gamma_m = -m\tau' - \frac{m(m-1)}{2}\sigma'',$$

where the derivatives τ' and σ'' are constants, because τ has degree 1 and σ has degree 2. Further, the polynomial in question exists and is unique provided that

$$\mu_k := \gamma_m + k\tau' + \frac{k(k-1)}{2}\sigma'' \neq 0, \qquad k = 0,\dots,m-1.$$

In our case $\tau' = \mathfrak{S}$, $\sigma'' = -2$, so that

$$\gamma_m = m(m-1-\mathfrak{S})$$

and the nonvanishing condition $\mu_k \neq 0$ turns into

$$\mathfrak{S} \neq m+k-1, \qquad k = 0, \dots, m-1,$$

which is certainly satisfied if $\mathfrak{S} > 2m-2$.

Our additional condition $\Sigma > 0$ is equivalent to $\mathfrak{S} > 2N-2$. It follows that the desired polynomial solutions p_m certainly exist for $m \leq N$.

Following the notation of [NSU] set

$$\begin{aligned} A_{mm} &= m! \prod_{k=0}^{m-1} \left(\tau' + \tfrac{m+k-1}{2}\sigma''\right), \\ g_m(t) &= g(t) \prod_{l=1}^{m} \sigma(t+l), \\ S_m &= \sum_{t\in\mathbb{Z}} g_m(t). \end{aligned}$$

Then, according to [NSU],

$$\begin{aligned} p_m(x) &= \frac{(-1)^m\, m!}{A_{mm}} \sum_{k=0}^{m} \frac{(-m)_k g_m(x-m+k)}{k!\, g(x)}\,, \\ (p_m, p_m) &= \frac{(-1)^m (m!)^2 S_m}{A_{mm}}\,. \end{aligned}$$

Applying this recipe in our concrete case, we get

$$\begin{aligned} A_{mm} &= m!\, \frac{\Gamma(\mathfrak{S}-m+2)}{\Gamma(\mathfrak{S}-2m+2)}\,, \\ g_m(t) &= \frac{(-1)^m}{\Gamma(u+1-m-t)\Gamma(u'+1-m-t)\Gamma(v+1+t)\Gamma(v'+1+t)} \end{aligned}$$

and, by Dougall's formula [Er, Vol.1, 1.4(1)],

$$\begin{aligned} S_m &= \sum_{t\in\mathbb{Z}} g_m(t) \\ &= (-1)^m\, \Gamma \left[\begin{matrix} \mathfrak{S}+1-2m \\ u+v+1-m,\ u'+v'+1-m,\ u'+v+1-m\,, u+v'+1-m \end{matrix} \right]. \end{aligned}$$

This implies the expression for (p_m, p_m) given in Proposition 7.2.

Further, after simple transformations we get

$$\begin{aligned} p_m(t) &= \Gamma\left[\begin{matrix} \mathfrak{S}-2m+2,\, v+1+t,\, v'+1+t \\ \mathfrak{S}-m+2,\, v+1-m+t,\, v'+1-m+t \end{matrix}\right] \\ &\quad\times \sum_{k=0}^{m} \frac{(-m)_k (t-u)_k (t-u')_k}{(t+v+1-m)_k (t+v'+1-m)_k k!} \\ &= \Gamma\left[\begin{matrix} \mathfrak{S}-2m+2,\, v+1+t,\, v'+1+t \\ \mathfrak{S}-m+2,\, v+1-m+t,\, v'+1-m+t \end{matrix}\right] \\ &\quad\times {}_3F_2\left[\left.\begin{matrix} -m,\, t-u,\, t-u' \\ t+v+1-m,\, t+v'+1-m \end{matrix}\right| 1\right]. \end{aligned}$$

Applying the transformation

$$ {}_3F_2\left[\left.\begin{matrix} a,\, b,\, c \\ e,\, f \end{matrix}\right| 1\right] = \Gamma\left[\begin{matrix} e,\, e+f-a-b-c \\ e+f-b-c,\, e-a \end{matrix}\right] {}_3F_2\left[\left.\begin{matrix} a,\, f-b,\, f-c \\ e+f-b-c,\, f \end{matrix}\right| 1\right] \tag{7.6}$$

(see Appendix) to the last expression we arrive at the desired formula for $p_m(t)$. □

The orthogonal polynomials $\{p_m\}$ were discovered by R. Askey [As]. Later they were independently found by Peter A. Lesky (see [Les1], [Les2]). We are grateful to Tom Koornwinder for informing us about Lesky's work, and to Peter A. Lesky for sending us his preprint [Les2].

8. The correlation kernel of the process $\mathcal{P}^{(N)}$

In this section we study the process $\mathcal{P}^{(N)}$ on the lattice $\mathfrak{X}^{(N)}$ introduced in Section 4. In Section 6 we showed that $\mathcal{P}^{(N)}$ is an L-ensemble, hence, it is a determinantal point process. Denote by $K^{(N)}$ the $\mathfrak{X}^{(N)} \times \mathfrak{X}^{(N)}$ matrix defined by $K^{(N)} = L^{(N)}(1+L^{(N)})^{-1}$, where $L^{(N)}$ is as defined in Section 6. By Proposition 5.3, $K^{(N)}$ is a correlation kernel for $\mathcal{P}^{(N)}$. Our goal in this section is to provide analytic expressions for $K^{(N)}$ suitable for a future limit transition as $N \to \infty$.

Our analysis is based on the general results of §5(f)–(h), where we take $\mathfrak{X} = \mathfrak{X}^{(N)}$, $\mathfrak{X}_I = \mathfrak{X}_{\text{out}}$, $\mathfrak{X}_{II} = \mathfrak{X}_{\text{in}}$. To compute the kernel $K^{(N)}$ we use the method of §5(f). Proposition 5.6 expresses the kernel in terms of four functions R_I, R_{II}, S_I, S_{II}, which solve a discrete Riemann-Hilbert problem. In our concrete situation we will redenote these functions by $R^{(N)}_{\text{out}}$, $R^{(N)}_{\text{in}}$, $S^{(N)}_{\text{out}}$, $S^{(N)}_{\text{in}}$.

In order to apply this method we need to impose an additional restriction on the parameters. Specifically, we require that $(z, z', w, w') \in \mathcal{D}'_{\text{adm}} \cap \{\Sigma > 0\}$, where $\mathcal{D}'_{\text{adm}}$ was defined in Definition 6.3 and $\Sigma = z + z' + w + w'$. Later we show that the final expression for $K^{(N)}$ remains valid without the restriction $\Sigma > 0$.

Thanks to Proposition 5.8, the functions $R_{\mathrm{out}}^{(N)}$ and $S_{\mathrm{out}}^{(N)}$ are immediately expressed through the orthogonal polynomials $\mathfrak{p}_N$, $\mathfrak{p}_{N-1}$ evaluated in Section 7. The remaining two functions $R_{\mathrm{in}}^{(N)}$ and $S_{\mathrm{in}}^{(N)}$ are uniquely determined by their relations to $R_{\mathrm{out}}^{(N)}$ and $S_{\mathrm{out}}^{(N)}$; see (5.12) and (8.3) below. We provide certain explicit expressions for $R_{\mathrm{in}}^{(N)}$, $S_{\mathrm{in}}^{(N)}$ and check that they satisfy the needed relations. About the derivation of these expressions, see Remark 8.9 below.

To remove the restriction $\Sigma > 0$ we use Theorem 5.10 and Lemma 7.3. These results show that the kernel $K^{(N)}$ divided by a simple factor, admits a holomorphic continuation (as a function of the parameters) to a certain domain $\mathcal{D}'_0 \supset \mathcal{D}'_{\mathrm{adm}}$.

Now we proceed to the realization of the plan described above.

We assume that $(z, z', w, w') \in \mathcal{D}'_{\mathrm{adm}}$. As was mentioned in Section 6, this guarantees that the weight function $f(x)$ does not vanish on $\mathfrak{X}_{\mathrm{in}}$. Next, we temporarily assume that $\Sigma > 0$. This ensures that the function h_I belongs to $\ell^2(\mathfrak{X}_{\mathrm{in}})$ as required in §5(h).

Following Proposition 5.8 and using Theorem 7.1, we define two meromorphic functions $R_{\mathrm{out}}^{(N)}$ and $S_{\mathrm{out}}^{(N)}$ on the complex plane with poles in $\mathfrak{X}_{\mathrm{in}}$ as follows:

$$(8.1)\quad R_{\mathrm{out}}^{(N)}(x) = \frac{\mathfrak{p}_N(x)}{(x-\frac{N-1}{2})\cdots(x+\frac{N-1}{2})} = \Gamma\left[\begin{matrix} x-\frac{N-1}{2},\ x+w'+\frac{N+1}{2} \\ x+\frac{N+1}{2},\ x+w'-\frac{N-1}{2}\end{matrix}\right] {}_3F_2\left[\left.\begin{matrix}-N,\ z+w',\ z'+w' \\ \Sigma,\ x+w'-\frac{N-1}{2}\end{matrix}\right| 1\right],$$

and

$$(8.2)\quad S_{\mathrm{out}}^{(N)}(x) = \frac{\mathfrak{p}_{N-1}(x)}{h_{N-1}(x-\frac{N-1}{2})\cdots(x+\frac{N-1}{2})} = \frac{1}{h_{N-1}}\Gamma\left[\begin{matrix} x-\frac{N-1}{2},\ x+w'+\frac{N+1}{2} \\ x+\frac{N+1}{2},\ x+w'-\frac{N-3}{2}\end{matrix}\right] {}_3F_2\left[\left.\begin{matrix}-N+1,\ z+w'+1,\ z'+w'+1 \\ \Sigma+2,\ x+w'-\frac{N-3}{2}\end{matrix}\right| 1\right],$$

where h_{N-1} is as defined in (7.2). Observe that

$$R_{\mathrm{out}}^{(N)}(x) = 1 + \mathrm{O}(x^{-1}), \quad S_{\mathrm{out}}^{(N)} = \mathrm{O}(x^{-1}), \qquad x \to \infty.$$

Note that the right-hand side of (8.1) makes sense for any $x \in \mathbb{C} \setminus \mathfrak{X}_{\mathrm{in}}$ and $(z, z', w, w') \in \mathcal{D} \setminus \{\Sigma = 0\}$, and the right-hand side of (8.2) makes sense for any $x \in \mathbb{C} \setminus \mathfrak{X}_{\mathrm{in}}$ and $(z, z', w, w') \in \mathcal{D}_0$.

Using (5.12) as a prompt, we set

$$(8.3)\quad S_{\mathrm{in}}^{(N)}(x) = -\sum_{y \in \mathfrak{X}_{\mathrm{out}}} \frac{\psi_{\mathrm{out}}^{(N)}(y) R_{\mathrm{out}}^{(N)}(y)}{y - x}, \quad R_{\mathrm{in}}^{(N)}(x) = 1 - \sum_{y \in \mathfrak{X}_{\mathrm{out}}} \frac{\psi_{\mathrm{out}}^{(N)}(y) S_{\mathrm{out}}^{(N)}(y)}{y - x},$$

where the functions $\psi_{\mathrm{out}}^{(N)}$ and $\psi_{\mathrm{in}}^{(N)}$ are as introduced in Section 6.

Note that $\psi^{(N)}_{\mathrm{out}}(x) \le \mathrm{const}\,(1+|x|)^{-\Sigma}$ (this follows from the estimate of the weight function f, see §6). Since, by assumption, $\Sigma>0$, it follows that the series (8.3) converge and define meromorphic functions with poles in $\mathfrak{X}_{\mathrm{out}}$.

Proposition 8.1. *We have*

$$
\begin{aligned}
R^{(N)}_{\mathrm{in}}(x) = &-\frac{\sin\pi z}{\pi}\,\Gamma\begin{bmatrix} z'-z,\ z+w+1,\ z+w'+1 \\ \Sigma+1\end{bmatrix} \\
&\times\Gamma\begin{bmatrix} x+\frac{N+1}{2},\ -x+\frac{N+1}{2},\ N+1+\Sigma \\ -x+z'+\frac{N+1}{2},\ x+w+\frac{N+1}{2}, N+1+z+w'\end{bmatrix} \\
&\times{}_3F_2\left[\begin{matrix} z+w'+1,\ -z'-w,\ -x+z+\frac{N+1}{2} \\ z-z'+1,\ N+1+z+w'\end{matrix}\,\middle|\,1\right] \\
&-\{\ \textit{similar expression with } z \textit{ and } z' \textit{ interchanged}\ \},
\end{aligned}
\tag{8.4}
$$

and

$$
\begin{aligned}
S^{(N)}_{\mathrm{in}}(x) = &-\frac{\sin\pi z}{\pi}\,\Gamma\begin{bmatrix} z'-z,\ \Sigma \\ z'+w,\ z'+w'\end{bmatrix} \\
&\times\Gamma\begin{bmatrix} x+\frac{N+1}{2},\ -x+\frac{N+1}{2},\ N+1 \\ -x+z'+\frac{N+1}{2},\ x+w+\frac{N+1}{2}, N+1+z+w'\end{bmatrix} \\
&\times{}_3F_2\left[\begin{matrix} -z'-w+1,\ z+w',\ -x+z+\frac{N+1}{2} \\ z-z'+1,\ N+1+z+w'\end{matrix}\,\middle|\,1\right] \\
&-\{\ \textit{similar expression with } z \textit{ and } z' \textit{ interchanged}\ \}.
\end{aligned}
\tag{8.5}
$$

Singularities. Using the structure of singularities of a general ${}_3F_2$ function (see the beginning of §7) it is easy to verify that the formulas above make sense for $x\in\mathbb{C}\setminus\mathfrak{X}_{\mathrm{out}}$ and $(z,z',w,w')\in\mathcal{D}_0\setminus(\{\Sigma=0\}\cup\{z-z'\in\mathbb{Z}\})$. Also, $\{\Sigma=0\}$ is indeed a singular set for $S^{(N)}_{\mathrm{in}}$. The singularities $\{z-z'\in\mathbb{Z}\}$, however, are removable, as long as we are in $\mathcal{D}_0\setminus\{\Sigma=0\}$. For $\Sigma>0$ this follows from the definitions (8.3).

Proof. Since both parts of (8.4) and (8.5) are analytic in the parameters, it is enough to give a proof for nonintegral values of z,z',w,w', such that $z-z'\notin\mathbb{Z}$, $\Sigma\notin\mathbb{Z}$.

We start by rewriting the expressions above in a somewhat more suitable form and introduce a meromorphic function

$$F(x)=\frac{1}{\sin(\pi(z'-z))\sin(\pi(z+z'+w+w'))}\times\left(\frac{\sin(\pi(z+w))\sin(\pi(z+w'))\sin(\pi z')}{\sin(\pi(-x+z'+\frac{N+1}{2}))}-\frac{\sin(\pi(z'+w))\sin(\pi(z'+w'))\sin(\pi z)}{\sin(\pi(-x+z+\frac{N+1}{2}))}\right).$$

It is not difficult to see that if $x\in\mathfrak{X}$, that is, if $x-\frac{N-1}{2}\in\mathbb{Z}$, then $F(x)=(-1)^{x-\frac{N-1}{2}}$.

Let us also introduce a more detailed notation $h(N-1,z,z',w,w')$ for the constant h_{N-1} defined in (7.2).

LEMMA 8.2. *The right-hand side of* (8.4) *can be written in the form*

(8.6)
$$\Gamma\left[\begin{matrix}x+\frac{N+1}{2},\ x-z-\frac{N-1}{2}\\ x-\frac{N-1}{2},\ x-z+\frac{N+1}{2}\end{matrix}\right]{}_3F_2\left[\left.\begin{matrix}N,\ -z-w',\ -z-w\\ -\Sigma,\ x-z+\frac{N+1}{2}\end{matrix}\right|1\right]$$
$$+\Gamma\left[\begin{matrix}x+\frac{N+1}{2},\ -x+\frac{N+1}{2}\\ -x+z+\frac{N+1}{2},\ -x+z'+\frac{N+1}{2},\ x+w+\frac{N+1}{2},\ x+w'-\frac{N-3}{2}\end{matrix}\right]$$
$$\times\frac{F(x)}{h(N-1,z,z',w,w')}\ {}_3F_2\left[\left.\begin{matrix}-N+1,\ z+w'+1,\ z'+w'+1\\ \Sigma+2,\ x+w'-\frac{N-3}{2}\end{matrix}\right|1\right],$$

and the right-hand side of (8.5) *can be written in the form*

(8.7)
$$h\left(N,z-\frac{1}{2},z'-\frac{1}{2},w-\frac{1}{2},w'-\frac{1}{2}\right)\Gamma\left[\begin{matrix}x+\frac{N+1}{2},\ x-z-\frac{N-1}{2}\\ x-\frac{N-1}{2},\ x-z+\frac{N+3}{2}\end{matrix}\right]$$
$$\times{}_3F_2\left[\left.\begin{matrix}N+1,\ -z-w'+1,\ -z-w+1\\ -\Sigma+2,\ x-z+\frac{N+3}{2}\end{matrix}\right|1\right]$$
$$+\Gamma\left[\begin{matrix}x+\frac{N+1}{2},\ -x+\frac{N+1}{2}\\ -x+z+\frac{N+1}{2},\ -x+z'+\frac{N+1}{2},\ x+w+\frac{N+1}{2},\ x+w'-\frac{N-1}{2}\end{matrix}\right]$$
$$\times F(x)\ {}_3F_2\left[\left.\begin{matrix}-N,\ z+w',\ z'+w'\\ \Sigma,\ x+w'-\frac{N-1}{2}\end{matrix}\right|1\right].$$

Singularities. It looks as if (8.6) may have more singular points than the right-hand side of (8.4). For example, the first summand in (8.6) has poles at the points, where

$$x-u-\tfrac{N-1}{2}\in\{0,-1,\dots\},\quad u=z \text{ or } z',$$

and the same is true about the second summand. The fact that these poles cancel out is not obvious. Similar cancellations happen in (8.7).

The proof of the lemma can be found in the appendix. It is rather tedious and is based on the known transformation formulas for ${}_3F_2$ with the unit argument.

Our next step is to prove the following statement.

LEMMA 8.3. *The only singularities of the right-hand of* (8.4), *regarded as a function in $x\in\mathbb{C}$, are simple poles at the points of $\mathfrak{X}_{\rm out}$. The residue of the right-hand side of* (8.4) *at any point $x\in\mathfrak{X}_{\rm out}$ equals $\psi^{(N)}_{\rm out}(x)S^{(N)}_{\rm out}(x)$. Similarly, the only singularities of the right-hand of* (8.5), *regarded as a function in $x\in\mathbb{C}$, are simple poles at the points of $\mathfrak{X}_{\rm out}$. The residue of the right-hand side of* (8.5) *at any point $x\in\mathfrak{X}_{\rm out}$ equals $\psi^{(N)}_{\rm out}(x)R^{(N)}_{\rm out}(x)$.*

Proof. The location of poles follows from the general structure of singularities of ${}_3F_2$ (see the beginning of §7).

To evaluate the residue of the right-hand side of (8.4) we will use the formula (8.6). We easily see that first summand of (8.6) takes finite values on $\mathfrak{X}_{\rm out}$. As for the second summand of (8.6), the ${}_3F_2$ is a terminating series and it has no singularities in $\mathfrak{X}_{\rm out}$. Furthermore, $F(x)=(-1)^{x-\frac{N-1}{2}}$ for $x\in\mathfrak{X}$, in particular, for $x\in\mathfrak{X}_{\rm out}$. It remains to examine the residue of

$$\Gamma\left(x+\frac{N+1}{2}\right)\Gamma\left(-x+\frac{N+1}{2}\right).$$

Assume that $x\in\mathfrak{X}_{\rm out}$ and $x>\frac{N-1}{2}$. Then

$$\operatorname*{Res}_{u=x}\Gamma\left(-u+\tfrac{N+1}{2}\right)=\frac{(-1)^{x-\frac{N-1}{2}}}{\Gamma(x-\frac{N-1}{2})}.$$

Thus, the residue of (8.6) at x is equal to

$$\Gamma\left[\begin{matrix}x+\frac{N+1}{2}\\ x-\frac{N-1}{2},\ -x+z+\frac{N+1}{2},\ -x+z'+\frac{N+1}{2},\ x+w+\frac{N+1}{2},\ x+w'-\frac{N-3}{2}\end{matrix}\right]$$
$$\times\frac{1}{h(N-1,z,z',w,w')}\,{}_3F_2\left[\left.\begin{matrix}-N+1,\ z+w'+1,\ z'+w'+1\\ \Sigma+2,\ x+w'-\frac{N-3}{2}\end{matrix}\right|1\right].$$

By a direct comparison we see that this is equal to $\psi^{(N)}_{\rm out}(x)S^{(N)}_{\rm out}(x)$.

Similarly, if $x \in \mathfrak{X}_{\text{out}}$ and $x < -\frac{N-1}{2}$,

$$\operatorname*{Res}_{u=x} \Gamma\left(u + \tfrac{N+1}{2}\right) = \frac{(-1)^{-x-\frac{N+1}{2}}}{\Gamma(-x - \frac{N-1}{2})},$$

and the residue of (8.6) at x equals

$$\Gamma\begin{bmatrix} -x + \frac{N+1}{2} \\ -x-\frac{N-1}{2},\ -x+z+\frac{N+1}{2},\ -x+z'+\frac{N+1}{2},\ x+w+\frac{N+1}{2},\ x+w'-\frac{N-3}{2} \end{bmatrix}$$
$$\times \frac{(-1)^N}{h(N-1,z,z',w,w')}\ {}_3F_2\left[\left.\begin{matrix} -N+1,\ z+w'+1,\ z'+w'+1 \\ \Sigma+2,\ x+w'-\frac{N-3}{2} \end{matrix}\right| 1\right],$$

which is again equal to $\psi^{(N)}_{\text{out}}(x) S^{(N)}_{\text{out}}(x)$.

Thus, we have proved the first statement of the lemma. The proof of the second one is very similar. □

Lemma 8.3 shows that the right-hand sides of (8.3) and of (8.4), (8.5) have the same singularity structure. Clearly, the sums in the right-hand sides of (8.3) decay as $x \to \infty$ and x keeps finite distance from the points of $\mathfrak{X}_{\text{out}}$. We aim to prove that the right-hand sides of (8.4) and (8.5) have a similar property. Using Lemma 8.2, we may consider the expressions (8.6) and (8.7).

Consider the formula (8.6). We first assume that $\Re x \geq 0$. Let us examine the first summand. The gamma factors form a rational function which tends to 1 uniformly in $\arg(x)$ as $|x| \to \infty$.

In order to handle the asymptotics of the ${}_3F_2$ factor, let us apply the formula (7.6) to it with

$$a = -z - w, \quad b = -z - w', \quad c = N, \quad e = x - z + \frac{N+1}{2}, \quad f = -\Sigma.$$

We get

$${}_3F_2\left[\left.\begin{matrix} N,\ -z-w',\ -z-w \\ -\Sigma,\ x-z+\frac{N+1}{2} \end{matrix}\right| 1\right] = \Gamma\begin{bmatrix} x-z+\frac{N+1}{2},\ x-z'-\frac{N-1}{2} \\ x-z-z'-w-\frac{N-1}{2},\ x+w+\frac{N+1}{2} \end{bmatrix}$$
$$\times {}_3F_2\left[\left.\begin{matrix} -z-w,\ -z'-w,\ -\Sigma-N \\ -\Sigma,\ x-z-z'-w-\frac{N-1}{2} \end{matrix}\right| 1\right].$$

Using the asymptotic relation

$$\frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)} = x^{\alpha-\beta}(1 + \mathrm{O}(|x|^{-1})), \tag{8.8}$$

which is uniform in $\arg(x) \in (-\pi+\varepsilon, \pi-\varepsilon)$ for any $\varepsilon > 0$, we see that the gamma factors tend to 1 as $|x| \to \infty$.

As for the ${}_3F_2$, we see that the sum of the lower parameters minus the sum of the upper parameters is equal to $x + w + \frac{N+1}{2}$. Then if $\Re x \geq 0$ and $\Re w + \frac{N+1}{2} > 0$, the defining series for this ${}_3F_2$ converges uniformly in x, provided that the distance from x to the lattice $\mathbb{Z}+z+z'+w+\frac{N-1}{2}$ is bounded from zero. One proof of this fact can be obtained by applying the general estimate (see [Er, Vol. 1, 1.9(8)])

$$c_1\alpha^{\beta-\gamma} < \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\gamma)} < c_2\alpha^{\beta-\gamma}, \qquad \alpha>0, \quad \Re\beta,\, \Re\gamma > \text{const} > 0,$$

$c_1, c_2 > 0$ are some fixed constants, to the terms of the series.

Thus, we can pass to the limit $|x|\to\infty$ term-wise, and since all terms of the series starting from the second one converge to zero uniformly in $\arg(x)$, we conclude that the first summand of (8.6), as $|x|\to\infty$, converges to 1 uniformly in x such that $\Re x\ge 0$ and the distance from x to the lattice $\mathbb{Z}+z+z'+w+\frac{N-1}{2}$ is bounded from zero.

Let us proceed to the second summand of (8.6). Now the ${}_3F_2$ part is a rational function which tends to 1 as $|x|\to\infty$ uniformly in $\arg(x)$. The remaining part — the product of gamma factors and $F(x)$ — can be written in the form

$$\text{(8.9)} \qquad \text{const}_1\, \Gamma\begin{bmatrix} x+\frac{N+1}{2},\ -x+\frac{N+1}{2},\ x-z-\frac{N-1}{2} \\ -x+z'+\frac{N+1}{2},\ x+w+\frac{N+1}{2},\ x+w'-\frac{N-3}{2}\end{bmatrix}$$
$$+\,\text{const}_2\, \Gamma\begin{bmatrix} x+\frac{N+1}{2},\ -x+\frac{N+1}{2},\ x-z'-\frac{N-1}{2} \\ -x+z+\frac{N+1}{2},\ x+w+\frac{N+1}{2},\ x+w'-\frac{N-3}{2}\end{bmatrix},$$

where for $F(x)$ we used the formulas ($u=z$ or z')

$$\frac{1}{\sin(\pi(-x+u+\frac{N+1}{2}))} = \frac{\Gamma(-x+u+\frac{N+1}{2})\Gamma(x-u-\frac{N-1}{2})}{\pi}.$$

Further, using the relations ($u=z$ or z')

$$\frac{\Gamma\left(-x+\frac{N+1}{2}\right)}{\Gamma\left(-x+u+\frac{N+1}{2}\right)} = \frac{\sin(\pi(-x+u+\frac{N+1}{2}))\,\Gamma(x-u-\frac{N-1}{2})}{\sin(\pi(-x+\frac{N+1}{2}))\,\Gamma(x-\frac{N-1}{2})},$$

observing that the ratio of sines is bounded as long as x is bounded from the lattice $\mathfrak{X}=\mathbb{Z}+\frac{N-1}{2}$, and employing (8.8), we see that the absolute value of (8.9) is bounded by a constant times $|x|^{-\Sigma-1}$, and the bound is uniform in x such that $\arg(x)$ is bounded from $\pm\pi$ and x is bounded from $\mathfrak{X}$. Since $\Sigma+1>0$, the second term of (8.6) tends to zero.

We conclude that, under the condition $\Re w+\frac{N+1}{2}>0$, the expression (8.6) converges to 1 as $|x|\to\infty$ uniformly in x such that $\Re x\ge 0$ and the distance from x to the lattices $\mathbb{Z}+\frac{N-1}{2}$ and $\mathbb{Z}+z+z'+w+\frac{N-1}{2}$ is bounded from zero.

To extend this estimate to the domain $\Re x\le 0$ we use the following:

Lemma 8.4. *The expression* (8.6) *is invariant with respect to the following change of variable and parameters*:

$$x\mapsto -x, \qquad (z,z')\longleftrightarrow(w',w).$$

We give a proof of this lemma in the appendix. Similarly to the proof of Lemma 8.2, it is rather technical and is based on certain transformation formulas for ${}_3F_2$.

Lemma 8.4 immediately implies that, under the conditions $\Re w+\frac{N+1}{2}>0$, $\Re z'+\frac{N+1}{2}>0$, the expression (8.6) converges to 1 as $|x|\to\infty$ uniformly in x such that the distance from x to the lattices $\mathbb{Z}+\frac{N-1}{2}$, $\mathbb{Z}+z+z'+w+\frac{N-1}{2}$ and $\mathbb{Z}+z'+w+w'+\frac{N-1}{2}$ is bounded from zero.

This statement together with Lemma 8.3 shows that the right-hand side of the second formula of (8.3) and the right-hand side of (8.4) have the same singularities and asymptotics at infinity. Thus, they must be equal. This proves (8.4) under the additional conditions $\Re w+\frac{N+1}{2}>0$, $\Re z'+\frac{N+1}{2}>0$. Since both sides of (8.4) depend on the parameters z,z',w,w' analytically, the additional conditions may be removed. The proof of (8.4) is complete.

The proof of (8.5) is very similar. It is based on the following technical statement, which will also be addressed in the appendix.

Lemma 8.5. *The expression* (8.7) *is skew-invariant with respect to*

$$x\mapsto -x,\qquad (z,z')\longleftrightarrow(w',w).$$

The proof of Proposition 8.1 is now complete. □

Remark 8.6. Having proved the formulas (8.6), (8.7), it is easy to verify (5.23) directly. Indeed, if $x\in\mathfrak{X}_{\mathrm{in}}$ then the first summands of (8.6) and (8.7) vanish thanks to $\Gamma(x+\frac{N+1}{2})/\Gamma(x-\frac{N-1}{2})$, while in the second ones we have $F(x)=(-1)^{x-\frac{N-1}{2}}$. A direct comparison of the resulting expressions with (8.1) and (8.2) yields (5.23).

Theorem 8.7. *Let* $(z,z',w,w')\in\mathcal{D}'_{\mathrm{adm}}$ *(see Definition* (6.3)*),* $\mathcal{P}^{(N)}$ *be the corresponding point process defined in Section* 4, *and* $K^{(N)}=L^{(N)}(1+L^{(N)})^{-1}$, *where* $L^{(N)}$ *is as in Section* 6.

Then the kernel $K^{(N)}(x,y)$, *represented in the block form corresponding to the splitting* $\mathfrak{X}=\mathfrak{X}_{\mathrm{out}}\sqcup\mathfrak{X}_{\mathrm{in}}$, *is equal to*

$$K^{(N)}_{\mathrm{out,out}}(x,y)=\sqrt{\psi^{(N)}_{\mathrm{out}}(x)\psi^{(N)}_{\mathrm{out}}(y)}\,\frac{R^{(N)}_{\mathrm{out}}(x)S^{(N)}_{\mathrm{out}}(y)-S^{(N)}_{\mathrm{out}}(x)R^{(N)}_{\mathrm{out}}(y)}{x-y},$$

$$K^{(N)}_{\mathrm{out,in}}(x,y)=\sqrt{\psi^{(N)}_{\mathrm{out}}(x)\psi^{(N)}_{\mathrm{in}}(y)}\,\frac{R^{(N)}_{\mathrm{out}}(x)R^{(N)}_{\mathrm{in}}(y)-S^{(N)}_{\mathrm{out}}(x)S^{(N)}_{\mathrm{in}}(y)}{x-y},$$

$$K^{(N)}_{\mathrm{in,out}}(x,y)=\sqrt{\psi^{(N)}_{\mathrm{in}}(x)\psi^{(N)}_{\mathrm{out}}(y)}\,\frac{R^{(N)}_{\mathrm{in}}(x)R^{(N)}_{\mathrm{out}}(y)-S^{(N)}_{\mathrm{in}}(x)S^{(N)}_{\mathrm{out}}(y)}{x-y},$$

$$K^{(N)}_{\mathrm{in,in}}(x,y)=\sqrt{\psi^{(N)}_{\mathrm{in}}(x)\psi^{(N)}_{\mathrm{in}}(y)}\,\frac{R^{(N)}_{\mathrm{in}}(x)S^{(N)}_{\mathrm{in}}(y)-S^{(N)}_{\mathrm{in}}(x)R^{(N)}_{\mathrm{in}}(y)}{x-y},$$

with the functions $R_{\text{out}}^{(N)}$, $S_{\text{out}}^{(N)}$, $R_{\text{in}}^{(N)}$, $S_{\text{in}}^{(N)}$ *given by* (8.1), (8.2), (8.4), (8.5), *the functions* $\psi_{\text{out}}^{(N)}$, $\psi_{\text{in}}^{(N)}$ *given by* (6.2), (6.3), *and the indeterminacy on the diagonal* $x=y$ *resolved by L'Hospital's rule*:

$$K_{\text{out,out}}^{(N)}(x,x)=\psi_{\text{out}}^{(N)}(x)\left((R_{\text{out}}^{(N)})'(x)S_{\text{out}}^{(N)}(x)-(S_{\text{out}}^{(N)})'(x)R_{\text{out}}^{(N)}(x)\right),$$
$$K_{\text{in,in}}^{(N)}(x,x)=\psi_{\text{in}}^{(N)}(x)\left((R_{\text{in}}^{(N)})'(x)S_{\text{in}}^{(N)}(x)-(S_{\text{in}}^{(N)})'(x)R_{\text{in}}^{(N)}(x)\right).$$

Singularities. We know that the functions $R_{\text{out}}^{(N)}$ and $S_{\text{in}}^{(N)}$ are singular when $\Sigma=0$; see the beginning of the section. However, as will be clear from the proof, the value $K^{(N)}(x,y)$ is a well-defined continuous function on the whole $\mathcal{D}_{\text{adm}}$ including the set $\{\Sigma=0\}$, for any $x,y\in\mathfrak{X}^{(N)}$.

Proof. Under the additional restriction $\Sigma>0$ the statement of the theorem follows from the above results. Indeed, the functions $R_{\text{out}}^{(N)}$, $S_{\text{out}}^{(N)}$, $R_{\text{in}}^{(N)}$, $S_{\text{in}}^{(N)}$ defined by (8.1), (8.2), (8.3) solve the discrete Riemann-Hilbert problem associated with $L^{(N)}$; see (5.18) and (5.12). Proposition 5.6 explains how the kernel $K^{(N)}$ is written in terms of these functions. Proposition 8.1 provides explicit expressions (8.4), (8.5) for $R_{\text{in}}^{(N)}$, $S_{\text{in}}^{(N)}$, and this completes the proof when $\Sigma>0$.

Now we want to get rid of this restriction and introduce a function $\Psi^{(N)}$: $\mathfrak{X}^{(N)}\to\mathbb{C}$ by

$$\Psi^{(N)}(x)=\begin{cases}\psi_{\text{out}}^{(N)}(x), & x\in\mathfrak{X}_{\text{out}},\\ \dfrac{1}{\psi_{\text{in}}^{(N)}(x)}, & x\in\mathfrak{X}_{\text{in}}.\end{cases}$$

We have

$$\Psi^{(N)}(x)=f(x)\cdot\begin{cases}\left((x-\frac{N-1}{2})_N\right)^2, & x\in\mathfrak{X}_{\text{out}},\\ \left(\Gamma(-x+\frac{N+1}{2})\Gamma(x+\frac{N+1}{2})\right)^2, & x\in\mathfrak{X}_{\text{in}},\end{cases}\tag{8.10}$$

where $f(x)$ is as defined in (6.1). Note that $\Psi^{(N)}(x)$ is entire in (z,z',w,w') for any $x\in\mathfrak{X}$.

Set

$$\mathcal{D}_0'=\{(z,z',w,w')\in\mathcal{D}_0\mid z,z',w,w'\neq-1,-2,\dots\}.$$

Recall that $\mathcal{D}_0$ contains $\mathcal{D}_{\text{adm}}$ and note that $\mathcal{D}_0'\cap\mathcal{D}_{\text{adm}}=\mathcal{D}_{\text{adm}}'$. In particular, $\mathcal{D}_0'$ is a domain in $\mathbb{C}^4$ containing $\mathcal{D}_{\text{adm}}'$.

Lemma 8.8. *Let* $(z,z',w,w')\in\mathcal{D}_{\text{adm}}'$. *The kernel* $K^{(N)}(x,y)$ *can be written in the form*

$$K^{(N)}(x,y)=\sqrt{\Psi^{(N)}(x)\Psi^{(N)}(y)}\,\overset{\circ}{K}{}^{(N)}(x,y),\tag{8.11}$$

where $\overset{\circ}{K}{}^{(N)}(x,y)$ *can be extended (as a function in* (z,z',w,w')*) to a holomorphic function on the domain* $\mathcal{D}_0'$.

Proof of the lemma. Recall (see §7) that

$$\widetilde{K}^{(N)}(x,y) = S^{(N)}(x,y)\sqrt{f(x)f(y)},$$

where $S^{(N)}$ is the (ordinary, not normalized) Christoffel-Darboux kernel, which admits a holomorphic continuation to the domain $\mathcal{D}_0$ (see Lemma 7.3). Theorem 5.10 provides a connection between the kernels $K^{(N)}$ and $\widetilde{K}^{(N)}$. Specifically, (5.24) reads

$$K^{(N)}(x,y) = \varepsilon(x)(\widetilde{K}^{(N)})^{\triangle}(x,y)\varepsilon(y),$$

where the symbol $(\,\cdot\,)^{\triangle}$ is as explained in §5(c). Since $\varepsilon(\,\cdot\,)$ does not depend on (z,z',w,w'), it suffices to prove that the kernel $(\widetilde{K}^{(N)})^{\triangle}(x,y)$ can be represented as the product of $\sqrt{\Psi^{(N)}(x)\Psi^{(N)}(y)}$ and a holomorphic function on $\mathcal{D}_0'$.

By (8.10), $\Psi^{(N)}(x)$ differs from $f(x)$ by a positive factor not depending on (z,z',w,w'). Hence, it is enough to prove that $(\widetilde{K}^{(N)})^{\triangle}(x,y)$ can be written as the product of $\sqrt{f(x)f(y)}$ and a holomorphic function on $\mathcal{D}_0'$.

By (5.1) we have

$$(\widetilde{K}^{(N)})^{\triangle}(x,y) = \begin{cases} f(x)\left(\frac{1}{f(x)} - S^{(N)}(x,x)\right), & x = y \in \mathfrak{X}_{\rm in}\,, \\ \pm\sqrt{f(x)f(y)}S^{(N)}(x,y), & \text{otherwise}, \end{cases}$$

where the sign '$\pm$' does not depend on (z,z',w,w'). From the definition of $f(x)$ it follows that for $x \in \mathfrak{X}_{\rm in}$, $\frac{1}{f(x)}$ admits a holomorphic extension (as a function in the parameters) to $\mathcal{D}_0'$. Since $S^{(N)}(x,y)$ is holomorphic on $\mathcal{D}_0$, the lemma follows. □

Now we complete the proof of Theorem 8.7. Since we have already proved its claim when $\Sigma > 0$, we see that under this restriction

$$\begin{aligned}
\overset{\circ}{K}{}^{(N)}_{\rm out,out}(x,y) &= \frac{R^{(N)}_{\rm out}(x)S^{(N)}_{\rm out}(y) - S^{(N)}_{\rm out}(x)R^{(N)}_{\rm out}(y)}{x-y}\,,\\
\overset{\circ}{K}{}^{(N)}_{\rm out,in}(x,y) &= \Psi^{(N)}(y)\,\frac{R^{(N)}_{\rm out}(x)R^{(N)}_{\rm in}(y) - S^{(N)}_{\rm out}(x)S^{(N)}_{\rm in}(y)}{x-y}\,,\\
\overset{\circ}{K}{}^{(N)}_{\rm in,out}(x,y) &= \Psi^{(N)}(x)\,\frac{R^{(N)}_{\rm in}(x)R^{(N)}_{\rm out}(y) - S^{(N)}_{\rm in}(x)S^{(N)}_{\rm out}(y)}{x-y}\,,\\
\overset{\circ}{K}{}^{(N)}_{\rm in,in}(x,y) &= \Psi^{(N)}(x)\Psi^{(N)}(y)\,\frac{R^{(N)}_{\rm in}(x)S^{(N)}_{\rm in}(y) - S^{(N)}_{\rm in}(x)R^{(N)}_{\rm in}(y)}{x-y}\,.
\end{aligned}$$

Since the functions $R^{(N)}_{\rm out}$, $S^{(N)}_{\rm out}$, $R^{(N)}_{\rm in}$, $S^{(N)}_{\rm in}$, and $\Psi^{(N)}$ are all holomorphic on $\mathcal{D}_0' \setminus \{\Sigma = 0\}$, these formulas hold on $\mathcal{D}_0'$, and the singularity at $\Sigma = 0$ is removable. As $\mathcal{D}_0' \supset \mathcal{D}_{\rm adm}'$, the proof of Theorem 8.7 is complete. □

Remark 8.9. The reader might have noticed that the proof of Proposition 8.1 given above is a verification that our formulas give the right answer rather than a derivation of these formulas. Unfortunately, at this point we cannot suggest any derivation procedure for which we would be able to prove that it produces the formulas we want. However, we can explain how we obtained the answer.

Recall that in our treatment of the orthogonal polynomials in Section 7 the crucial role was played by the difference equation (7.5). Corollary 5.9 shows that if we know a difference equation for the orthogonal polynomials then we can write difference equations for the values of $R_{\mathrm{in}}^{(N)}$ and $S_{\mathrm{in}}^{(N)}$ on the lattice $\mathfrak{X}_{\mathrm{in}}$. At this point we make the assumption that the *meromorphic functions* $R_{\mathrm{in}}^{(N)}(\zeta)$ and $S_{\mathrm{in}}^{(N)}(\zeta)$ satisfy the same difference equations. *A priori*, it is not clear at all why this should be the case. However, the general philosophy of the Riemann-Hilbert problem suggests that $R_{\mathrm{in}}^{(N)}(\zeta)$ and $S_{\mathrm{in}}^{(N)}(\zeta)$ should satisfy some difference equations. So we proceed and find meromorphic solutions of the difference equation which we got from the lattice. This can be done using general methods of solving difference equations with polynomial coefficients, see [MT, Chap. XV]. We also want our solutions to be holomorphic in $\mathbb{C}\setminus\mathfrak{X}_{\mathrm{out}}$ and to have fixed asymptotics at infinity, and this leads (through heavy computations!) to the formulas of Lemma 8.2. Unfortunately, these formulas are not suitable for the limit transition $N\to\infty$. So we had to play around with the transformation formulas for ${}_3F_2$ to get the convenient formulas of Proposition 8.1.

9. The spectral measures and continuous point processes

Define a *continuous phase space*

$$\mathfrak{X}=\mathbb{R}\setminus\{\pm\tfrac12\}$$

and divide it into two parts

$$\mathfrak{X}=\mathfrak{X}_{\mathrm{in}}\sqcup\mathfrak{X}_{\mathrm{out}},$$

$$\mathfrak{X}_{\mathrm{in}}=(-\tfrac12,\tfrac12),\quad \mathfrak{X}_{\mathrm{out}}=(-\infty,-\tfrac12)\sqcup(\tfrac12,+\infty).$$

As in Section 4, by $\mathrm{Conf}(\mathfrak{X})$ we denote the space of configurations in $\mathfrak{X}$. We define a map $\iota:\Omega\to\mathrm{Conf}(\mathfrak{X})$ by

$$\begin{aligned}(9.1)\quad \omega\ &=\ (\alpha^+,\beta^+;\alpha^-,\beta^-;\delta^+,\delta^-)\\ &\overset{\iota}{\longmapsto}\mathcal{C}(\omega)=\{\alpha_i^++\tfrac12\}\sqcup\{\tfrac12-\beta_i^+\}\sqcup\{-\alpha_j^--\tfrac12\}\sqcup\{-\tfrac12+\beta_j^-\},\end{aligned}$$

where we omit possible 0's in $\alpha^+,\beta^+,\alpha^-,\beta^-$, and also omit possible 1's in β^+ or β^-.

Proposition 9.1. *The map* (9.1) *is a well-defined Borel map.*

Proof. We have to prove that, for any compact set $A \subset \mathfrak{X}$, the function $\mathcal{N}_A(\iota(\omega))$ takes finite values and is a Borel function; see the definitions at the beginning of Section 4. Without loss of generality, we may assume that A is a closed interval, entirely contained in $(-\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, +\infty)$ or $(-\infty, -\frac{1}{2})$.

Assume that $A = [x, y] \subset (-\frac{1}{2}, \frac{1}{2})$. Then

$$\begin{aligned}\mathcal{N}(\iota(\omega)) &= \operatorname{Card}\{i \mid \tfrac{1}{2} - y \le \beta_i^+ \le \tfrac{1}{2} - x\} + \operatorname{Card}\{j \mid \tfrac{1}{2} + x \le \beta_j^- \le \tfrac{1}{2} + y\} \\ &= \sum_{i=1}^{\infty} \mathbf{1}_{[1/2-y, 1/2-x]}(\beta_i^+) + \sum_{j=1}^{\infty} \mathbf{1}_{[1/2+x, 1/2+y]}(\beta_j^-),\end{aligned}$$

where $\mathbf{1}_{[\dots]}$ stands for the indicator of an interval. Since $\sum \beta_i^+ \le \delta^+$, $\sum \beta_j^- \le \delta^-$, and $\frac{1}{2} + x > 0$, $\frac{1}{2} - y > 0$, the sums of the indicator functions above are actually finite. Clearly, these are Borel functions in ω.

When A is contained in $(\frac{1}{2}, +\infty)$ or $(-\infty, \frac{1}{2})$, the argument is the same. □

Let χ be a character of the group $\mathrm{U}(\infty)$ and let P be its spectral measure. By Proposition 9.1, the pushforward $\iota(P)$ of P is a well-defined probability measure on the space $\operatorname{Conf}(\mathfrak{X})$. We view it as a point process and denote it by $\mathcal{P}$. Our purpose is to describe the spectral measure P (for concrete characters χ) in terms of the process $\mathcal{P}$. The map ι glues some points ω together, and it is natural to ask whether we lose any information about P by passing to $\mathcal{P}$. We discuss this issue at the end of the section.

Let $\{P_N\}$ be the coherent system corresponding to χ. Recall that in §2(c) we have associated with each P_N a probability measure $\underline{P}_N$ on Ω, which is the pushforward of P_N under an embedding of $\mathbb{GT}_N$ into Ω. According to Theorem 2.2, the measures $\underline{P}_N$ weakly converge to P as $N \to \infty$. Let us form the probability measures

$$\underline{\mathcal{P}}^{(N)} = \iota(\underline{P}_N),$$

which are point processes on $\mathfrak{X}$. Theorem 2.2 suggests the idea that the process $\mathcal{P}$ should be a limit of the processes $\underline{\mathcal{P}}^{(N)}$ as $N \to \infty$. For instance, the correlation measures of $\mathcal{P}$ should be limits of the respective correlation measures of $\underline{\mathcal{P}}^{(N)}$'s. Then this would give us a possibility of finding the correlation measures of $\mathcal{P}$ through the limit transition in the correlation measures of the processes $\underline{\mathcal{P}}^{(N)}$ as $N \to \infty$. The goal of this section is to provide a general justification of such a limit transition.

First of all, let us give a slightly different (but equivalent) definition of the processes $\underline{\mathcal{P}}^{(N)}$. Recall that in Section 4, we have associated to each P_N a point process $\mathcal{P}^{(N)}$ on the lattice $\mathfrak{X}^{(N)}$. Consider the map

$$\mathfrak{X}^{(N)} \longrightarrow \underline{\mathfrak{X}}^{(N)} = \tfrac{1}{N}\,\mathfrak{X}^{(N)} \subset \mathfrak{X}, \qquad x \mapsto \frac{x}{N}\,. \tag{9.2}$$

Then the process $\underline{\mathcal{P}}^{(N)}$ can be identified with the pushforward of the process $\mathcal{P}^{(N)}$ under this map of the phase spaces. Hence, denoting by $\rho_k^{(N)}$ and $\underline{\rho}_k^{(N)}$ the k^{th} correlation measure of $\mathcal{P}^{(N)}$ and $\underline{\mathcal{P}}^{(N)}$, respectively, we see that $\underline{\rho}_k^{(N)}$ is the pushforward of $\rho_k^{(N)}$ under the map (9.2).

Let us assume that $\mathcal{P}^{(N)}$ is a determinantal process with a kernel $K^{(N)}(x,y)$ on $\mathfrak{X}^{(N)} \times \mathfrak{X}^{(N)}$ (which actually holds in our concrete situation). Then $\underline{\mathcal{P}}^{(N)}$ is a determinantal process, too. It is convenient to take as the reference measure on the lattice $\underline{\mathfrak{X}}^{(N)}$ not the counting measure but the measure $\underline{\mu}^{(N)}$ such that $\underline{\mu}^{(N)}(\{x\}) = \frac{1}{N}$ for any $x \in \underline{\mathfrak{X}}^{(N)}$. The reason is that, as $N \to \infty$, the measures $\underline{\mu}^{(N)}$ approach the Lebesgue measure on $\mathfrak{X}$. Taking account of the factor $\frac{1}{N}$ we see that the kernel

$$\underline{K}^{(N)}(x,y) = N \cdot K^{(N)}(Nx, Ny), \qquad x, y \in \mathfrak{X}^{(N)}, \tag{9.3}$$

is a correlation kernel for $\underline{\mathcal{P}}^{(N)}$.

We need one more bit of notation: given $x \in \mathfrak{X}$, let x_N be the node of the lattice $\mathfrak{X}^{(N)}$ which is closest to Nx (any of two if Nx fits exactly at the middle between two nodes).

The main result of this section is as follows.

THEOREM 9.2. *Let χ be a character of* $\mathrm{U}(\infty)$ *and let P, $\{P_N\}$, $\mathfrak{X}$, $\mathcal{P}$, $\mathcal{P}^{(N)}$ be as above. Assume that each $\mathcal{P}^{(N)}$ is a determinantal process on $\mathfrak{X}^{(N)}$ with a correlation kernel $K^{(N)}(x,y)$ whose restriction both to $\mathfrak{X}^{(N)}_{\text{in}} \times \mathfrak{X}^{(N)}_{\text{in}}$ and $\mathfrak{X}^{(N)}_{\text{out}} \times \mathfrak{X}^{(N)}_{\text{out}}$ is Hermitian symmetric. Further, assume that*

$$\lim_{N\to\infty} N \cdot K^{(N)}(x_N, y_N) = K(x,y), \qquad x, y \in \mathfrak{X},$$

uniformly on compact subsets of $\mathfrak{X} \times \mathfrak{X}$, where $K(x,y)$ is a continuous function on $\mathfrak{X} \times \mathfrak{X}$.

Then $\mathcal{P}$ is a determinantal point process and $K(x,y)$ is its correlation kernel.

The proof will be given after preparatory work. First, we review a few necessary definitions and facts from [Ol3].

A *path* in the Gelfand-Tsetlin graph $\mathbb{GT}$ is an infinite sequence $t = (t_1, t_2, \dots)$ such that $t_N \in \mathbb{GT}_N$ and $t_N \prec t_{N+1}$ for any $N = 1, 2, \dots$. The set of the paths will be denoted by $\mathcal{T}$.

Consider the natural embedding $\mathcal{T} \subset \prod\limits_N \mathbb{GT}_N$. We equip $\prod\limits_N \mathbb{GT}_N$ with the product topology (the sets $\mathbb{GT}_N$ are viewed as discrete spaces). The set $\mathcal{T}$ is closed in this product space. We equip $\mathcal{T}$ with the induced topology. Then $\mathcal{T}$ turns into a totally disconnected topological space. Let $\tau = (\tau_1, \dots, \tau_N)$ be an arbitrary *finite* path in the graph $\mathbb{GT}$, i.e., $\tau_1 \in \mathbb{GT}_1, \dots, \tau_N \in \mathbb{GT}_N$ and

$\tau_1 \prec \cdots \prec \tau_N$. The cylinder sets of the form

$$C_\tau = \{t \in \mathcal{T} \mid t_1 = \tau_1, \ldots, t_N = \tau_N\}$$

form a base of topology in $\mathcal{T}$.

Consider an arbitrary signature $\lambda \in \mathbb{GT}_N$. The set of finite paths $\tau = (\tau_1 \prec \cdots \prec \tau_N)$ ending at λ has cardinality equal to $\operatorname{Dim}_N \lambda = \chi^\lambda(e)$. The cylinder sets C_τ corresponding to these finite paths τ are pairwise disjoint, and their union coincides with the set of infinite paths t passing through λ.

A *central measure* is any probability Borel measure on $\mathcal{T}$ such that the mass of any cylinder set C_τ depends only on its endpoint λ. These definitions are inspired by [VK1].

There exists a natural bijective correspondence $M \longleftrightarrow \{P_N\}$ between central measures M and coherent systems $\{P_N\}$, defined by the relations

$$\operatorname{Dim}_N \lambda \cdot M(C_\tau) = P_N(\lambda),$$

where $N = 1, 2, \ldots$, $\lambda \in \mathbb{GT}_N$, and τ is an arbitrary finite path ending at λ. In other words, for any N, the pushforward of M under the natural projection

$$\prod_{N=1}^{\infty} \mathbb{GT}_N \supset \mathcal{T} \to \mathbb{GT}_N \tag{9.4}$$

coincides with P_N.

By Proposition 2.1 we get a bijective correspondence between central measures and characters of $\mathrm{U}(\infty)$. This correspondence is an isomorphism of convex sets. So, extreme central measures exactly correspond to extreme characters.

On the other hand, by virtue of Theorem 1.2, we get a bijection $M \longleftrightarrow P$ between central measures on $\mathcal{T}$ and probability measures on Ω. In more detail, the correspondence $M \to P$ has the form

$$M \to \{P_N\} \to \chi \to P.$$

Given a path t, let $\widetilde{p}_i^{\pm}(N,t)$ and $\widetilde{q}_i^{\pm}(N,t)$ denote the modified Frobenius coordinates of the Young diagram $(t_N)^{\pm}$. We say that t is a *regular path* if there exist limits

$$\lim_{N\to\infty} \frac{\widetilde{p}_i^{\pm}(N,t)}{N} = \alpha_i^{\pm}, \quad \lim_{N\to\infty} \frac{\widetilde{q}_i^{\pm}(N,t)}{N} = \beta_i^{\pm}, \quad \lim_{N\to\infty} \frac{|(t_N))^{\pm}|}{N} = \delta^{\pm},$$

where $i = 1, 2, \ldots$. Then the limit values are the coordinates of a point $\omega \in \Omega$, and we say that ω is the *end* of the regular path t or that t *ends* at ω.

Let $\mathcal{T}_{\mathrm{reg}} \subset \mathcal{T}$ be the subset of regular paths. This is a Borel set. Let $\mathcal{T}_{\mathrm{reg}} \to \Omega$ be the projection assigning to a regular path its end. This is a Borel map.

Theorem 9.3. *Let M be a central measure on $\mathcal{T}$. Then M is supported by the Borel set $\mathcal{T}_{\mathrm{reg}}$ and hence can be viewed as a probability measure on $\mathcal{T}_{\mathrm{reg}}$. The pushforward of M under the projection $\mathcal{T}_{\mathrm{reg}} \to \Omega$ coincides with the spectral measure P that appears in the correspondence $M \to P$ defined above.*

Proof. See [Ol3, Th. 10.7]. □

Corollary 9.4. *Both the measure P and all the measures P_N can be represented as pushforwards of the measure M, with respect to the maps $\mathcal{T}_{\mathrm{reg}} \to \Omega$ and $\mathcal{T}_{\mathrm{reg}} \to \mathbb{GT}_N$, respectively, where the latter map is given by restricting (9.4) to $\mathcal{T}_{\mathrm{reg}} \subset \mathcal{T}$.*

Proof. The claim concerning P is exactly Theorem 9.3. The claim concerning P_N follows from the discussion above. □

For any compact set $A \subset \mathfrak{X}$, let $\mathcal{N}_{A,N}$ denote the random variable $\mathcal{N}_A$ associated with the process $\underline{\mathcal{P}}^{(N)}$. Here it is convenient to consider as the phase space of $\underline{\mathcal{P}}^{(N)}$ not the lattice $\underline{\mathfrak{X}}^{(N)}$ but the ambient continuous space $\mathfrak{X}$. Recall that by $\underline{\rho}_k^{(N)}$ we have denoted the k^{th} correlation measure of $\underline{\mathcal{P}}^{(N)}$. The first step towards Theorem 9.2 is the following

Proposition 9.5. *Assume that for any compact set $A \subset \mathfrak{X}$ there exist uniform in N bounds for the moments of $\mathcal{N}_{A,N}$,*

$$\mathbb{E}[(\mathcal{N}_{A,N})^l] \le C_l, \qquad l = 1, 2, \dots,$$

where the symbol $\mathbb{E}$ means expectation.

Then for any $k = 1, 2, \dots,$ the k^{th} correlation measure ρ_k of the process $\mathcal{P}$ exists. Moreover, for any continuous compactly supported function F on $\mathfrak{X}^k = \mathfrak{X} \times \cdots \times \mathfrak{X}$,

$$\lim_{N\to\infty} \langle F, \underline{\rho}_k^{(N)} \rangle = \langle F, \rho_k \rangle.$$

Proof. We argue as in the proof of [BO4, Lemma 6.2]. First, using Corollary 9.4, we put all the processes on one and the same probability space, $(\mathcal{T}_{\mathrm{reg}}, M)$.

Given a regular path $t = (t_N)_{N=1,2,\dots}$, we assign to it point configurations

$$\mathcal{C}(t) \in \mathrm{Conf}(\mathfrak{X}), \quad \mathcal{C}_N(t) \in \mathrm{Conf}(\mathfrak{X}), \qquad N = 1, 2, \dots,$$

as follows.

Let

$$\omega(t) = (\alpha^+(t), \beta^+(t); \alpha^-(t), \beta^-(t); \delta^+(t), \delta^-(t)) \in \Omega$$

be the end of t. Then we set

$$\mathcal{C}(t) = \{\alpha_i^+(t) + \tfrac{1}{2}\} \sqcup \{\tfrac{1}{2} - \beta_i^+(t)\} \sqcup \{-\alpha_j^-(t) - \tfrac{1}{2}\} \sqcup \{-\tfrac{1}{2} + \beta_j^-(t)\}$$

(cf. (9.1)). Here, as in (9.1), we omit possible 0's in $\alpha^+(t)$, $\beta^+(t)$, $\alpha^-(t)$, $\beta^-(t)$, and also possible 1's in $\beta^+(t)$ or $\beta^-(t)$. Equivalently,

$$\mathcal{C}(t) = \mathcal{C}(\omega(t)) = \iota(\omega(t)).$$

Likewise, for any $N = 1, 2, \dots$, let $\widetilde{p}_i^{\pm}(N,t)$ and $\widetilde{q}_i^{\pm}(N,t)$ denote the modified Frobenius coordinates of $(t_N)^{\pm}$, and let

$$a_i^{\pm}(N,t) = \frac{\widetilde{p}_i^{\pm}(N,t)}{N}, \quad b_i^{\pm}(N,t) = \frac{\widetilde{q}_i^{\pm}(N,t)}{N}, \qquad i = 1, 2, \dots .$$

We set

$$\mathcal{C}_N(t) = \{a_i^+(N,t) + \tfrac{1}{2}\} \sqcup \{\tfrac{1}{2} - b_i^+(N,t)\} \sqcup \{-a_j^-(N,t) - \tfrac{1}{2}\} \sqcup \{-\tfrac{1}{2} + b_j^-(N,t)\}.$$

Equivalently, $\mathcal{C}_N(t)$ is the image of t_N under the composite map $\mathbb{GT}_N \to \Omega \to \mathrm{Conf}(\mathfrak{X})$.

We view $\mathcal{C}(t)$ and $\mathcal{C}_N(t)$ (for any $N = 1, 2, \dots$) as random configurations defined on the common probability space $(\mathcal{T}_{\mathrm{reg}}, M)$. By Corollary 9.4, these are exactly the same as the random configurations corresponding to the point processes $\mathcal{P}$ and $\underline{\mathcal{P}}^{(N)}$, respectively.

From now on all the random variables will be referred to the probability space $(\mathcal{T}_{\mathrm{reg}}, M)$. Fix a continuous compactly supported function F on $\mathfrak{X}^k$. It will be convenient to assume that F is nonnegative (this does not mean any loss of generality). Introduce random variables f and f_N as follows:

(9.5)
$$f(t) = \sum_{x_1,\dots,x_k \in \mathcal{C}(t)} F(x_1, \dots, x_k), \qquad f_N(t) = \sum_{x_1,\dots,x_k \in \mathcal{C}_N(t)} F(x_1, \dots, x_k),$$

summed over ordered k-tuples of points with pairwise distinct labels. Any such sum is actually finite because F is compactly supported and the point configurations are locally finite.

By the definition of the correlation measures,

$$\langle F, \rho_k \rangle = \mathbb{E}[f], \qquad \langle F, \underline{\rho}_k^{(N)} \rangle = \mathbb{E}[f_N],$$

and the very existence of ρ_k is guaranteed if $\mathbb{E}[f] < \infty$ for any F as above. So, we have to prove that $\mathbb{E}[f_N] \to \mathbb{E}[f] < \infty$ as $N \to \infty$. By a general theorem (see [Sh, Ch. II, §6, Th. 4]), it suffices to check the following two conditions:

Condition 1. $f_N(t) \to f(t)$ for any $t \in \mathcal{T}_{\mathrm{reg}}$.

Condition 2. The random variables f_N are uniformly integrable, that is,

$$\sup_N \int_{\{t \mid f_N(t) \geq c\}} f_N(t) M(dt) \to 0, \quad \text{as } c \to +\infty.$$

Let us check Condition 1. This condition does not depend on M, it is a simple consequence of the regularity property. Indeed, for $\varepsilon > 0$ let $\mathfrak{X}_\varepsilon$ be obtained from $\mathfrak{X}$ by removing the ε-neighborhoods of the points $\pm\frac{1}{2}$,

$$\mathfrak{X}_\varepsilon = \mathfrak{X} \setminus ((-\tfrac{1}{2} - \varepsilon, -\tfrac{1}{2} + \varepsilon) \cup (\tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon)).$$

Choose ε so small that the function F is supported by $\mathfrak{X}_\varepsilon^k$. Fix l so large that $\alpha_j^\pm < \varepsilon$, $\beta_j^\pm < \varepsilon$ for all indices $j \ge l$. By the definition of $\mathfrak{X}_\varepsilon$, if a point

$$x = \begin{cases} \frac{1}{2} + \alpha_i^+(t) \\ \frac{1}{2} - \beta_i^+(t) \\ -\frac{1}{2} - \alpha_i^-(t) \\ -\frac{1}{2} + \beta_i^-(t) \end{cases} \tag{9.6}$$

lies in $\mathfrak{X}_\varepsilon$ then $i < l$.

By the definition of regular paths, for any index i,

$$a_i^\pm(N,t) \to \alpha_i^\pm(t), \quad b_i^\pm(N,t) \to \beta_i^\pm(t), \qquad N \to \infty. \tag{9.7}$$

Therefore, we have $a_l^\pm(N,t) < \varepsilon$ and $b_l^\pm(N,t) < \varepsilon$ for all N large enough. By monotonicity, the same inequality holds for the indices $l+1, l+2, \dots$ as well. This means that if N is large enough and a point

$$x = \begin{cases} \frac{1}{2} + a_i^+(N,t) \\ \frac{1}{2} - b_i^+(N,t) \\ -\frac{1}{2} - a_i^-(N,t) \\ -\frac{1}{2} + b_i^-(N,t) \end{cases} \tag{9.8}$$

lies in $\mathfrak{X}_\varepsilon$ then $i < l$.

It follows that in the sums (9.5), only the points (9.6) or (9.8) with indices $i = 1, \dots, l-1$ may really contribute. By (9.7) and continuity of F we conclude that $f_N(t) \to f(t)$.

Let us check Condition 2. Choose a compact set A such that F is supported by A^k. Denoting by C the supremum norm of F, we have the bound

$$f_N(t) \le C \cdot N_{A,N}(t)(\mathcal{N}_{A,N}(t) - 1) \dots (\mathcal{N}_{A,N}(t) - k + 1) \le C \cdot (\mathcal{N}_{A,N}(t))^k.$$

Therefore the random variables f_N are uniformly integrable provided that this is true for the random variables $(\mathcal{N}_{A,N})^k$ for any fixed k. But the latter fact follows from the assumption of the proposition and Chebyshev's inequality. □

To apply Proposition 9.5 we must check the required uniform bound for the moments. By the assumption of Theorem 9.2, each point process $\underline{\mathcal{P}}_N$ is a determinantal process on $\underline{\mathfrak{X}}^{(N)}$ such that its kernel, restricted to $\underline{\mathfrak{X}}_{\rm in}^{(N)} \times \underline{\mathfrak{X}}_{\rm in}^{(N)}$ and to $\underline{\mathfrak{X}}_{\rm out}^{(N)} \times \underline{\mathfrak{X}}_{\rm out}^{(N)}$, is Hermitian symmetric. Here we set $\underline{\mathfrak{X}}_{\rm in}^{(N)} = \underline{\mathfrak{X}}^{(N)} \cap \mathfrak{X}_{\rm in}$, $\underline{\mathfrak{X}}_{\rm out}^{(N)} = \underline{\mathfrak{X}}^{(N)} \cap \mathfrak{X}_{\rm out}$. For a compact set $A \subset \mathfrak{X}$ we denote by $\underline{K}_A^{(N)}$ the restriction of the kernel to $A\cap\underline{\mathfrak{X}}^{(N)}$. If A is entirely contained in $\mathfrak{X}_{\rm in}$ or $\mathfrak{X}_{\rm out}$ then $\underline{K}_A^{(N)}$ is a finite-dimensional *nonnegative* operator.

Proposition 9.6. *Assume that for any compact set A, which is entirely contained in $\mathfrak{X}_{\rm in}$ or $\mathfrak{X}_{\rm out}$, there is a bound of the form* $\operatorname{tr}\underline{K}_A^{(N)}\le\text{const}$, *where the constant does not depend on N. Then the assumption of Proposition* 9.5 *is satisfied.*

Proof. Let $A\subset\mathfrak{X}$ be an arbitrary compact set. Then $A=A_{\rm in}\cup A_{\rm out}$, where $A_{\rm in}\subset\mathfrak{X}_{\rm in}$ and $A_{\rm out}\subset\mathfrak{X}_{\rm out}$ are compact sets. We have $\mathcal{N}_{A,N}=\mathcal{N}_{A_{\rm out},N}+\mathcal{N}_{A_{\rm in},N}$. If we have uniform bounds for the moments of $\mathcal{N}_{A_{\rm out},N}$ and $\mathcal{N}_{A_{\rm in},N}$ then, by the Cauchy-Schwarz-Bunyakovskii inequality, we get such bounds for $\mathcal{N}_{A,N}$, too. Consequently, we can assume that A is entirely contained in $\mathfrak{X}_{\rm out}$ or $\mathfrak{X}_{\rm in}$, so that $\underline{K}_A^{(N)}$ is nonnegative.

Instead of ordinary moments we can deal with factorial moments. Given $l=1,2,\dots$, the l-th factorial moment of $\mathcal{N}_{A,N}$ is equal to

$$\underline{\rho}_l^{(N)}(A^l)=\int_{A^l}\det[\underline{K}_A^{(N)}(x_i,x_j)]_{1\le i,j\le l}\,dx_1\dots dx_l=l!\operatorname{tr}(\wedge^l\underline{K}_A^{(N)}).$$

Since $\underline{K}_A^{(N)}$ is nonnegative, we have

$$\operatorname{tr}(\wedge^l\underline{K}_A^{(N)})\le\operatorname{tr}(\otimes^l\underline{K}_A^{(N)})=(\operatorname{tr}(\underline{K}_A^{(N)}))^l.$$

This concludes the proof, because we have a uniform bound for the traces by assumption. □

Proof of Theorem 9.2. We shall approximate the process $\mathcal{P}$ by the processes $\underline{\mathcal{P}}^{(N)}$. Recall that the process $\underline{\mathcal{P}}^{(N)}$ is a scaled version of the process $\mathcal{P}^{(N)}$, and their correlation kernels, $\underline{K}^{(N)}$ and $K^{(N)}$, are related as follows

$$\underline{K}^{(N)}(x,y)=N\cdot K^{(N)}(Nx,Ny),\qquad x,y\in\underline{\mathfrak{X}}^{(N)}=\tfrac{1}{N}\mathfrak{X}^{(N)}.$$

Let us check that the assumption of Proposition 9.6 is satisfied. Without loss of generality we may assume that A is a closed interval $[a,b]$, contained either in $\mathfrak{X}_{\rm in}$ or $\mathfrak{X}_{\rm out}$. We have

$$\operatorname{tr}\underline{K}_A^{(N)}=\frac{1}{N}\sum_{x\in A\cap\underline{\mathfrak{X}}^n}\underline{K}^{(N)}(x,x),$$

where the factor $\frac{1}{N}$ comes from the reference measure $\mu^{(N)}$ on the lattice $\mathfrak{X}^{(N)}$ (recall that $\mu^{(N)}$ assigns weight $\frac{1}{N}$ to each node). By the relation between the two kernels, this can be rewritten as follows

$$\operatorname{tr}\underline{K}_A^{(N)}=\frac{1}{N}\sum_{x\in A\cap\underline{\mathfrak{X}}^n}N\cdot K^{(N)}(Nx,Nx).$$

By the assumption of Theorem 9.2, the right-hand side tends, as $N \to \infty$, to the integral $\int_a^b K(x,x)dx$; hence the traces above are uniformly bounded, as required.

Proposition 9.6 makes it possible to apply Proposition 9.5. It shows that the process $\mathcal{P}$ possesses correlation measures ρ_k, which are weak limits of the measures $\underline{\rho}_k^{(N)}$.

On the other hand, we know that for any k, the correlation measure $\underline{\rho}_k^{(N)}$ is expressed, by a determinantal formula, through the correlation kernel $\underline{K}^{(N)}$. Further, by the assumption of Theorem 9.2, this kernel tends to the kernel K as $N \to \infty$. This implies that the limit measure ρ_k is expressed, by the same determinantal formula, through the limit kernel K. □

Now we return to the discussion of the correspondence $P \mapsto \mathcal{P}$. Recall that it is based on the map $\iota : \Omega \to \operatorname{Conf}(\mathfrak{X})$, sending a point $\omega = (\alpha^+, \beta^+; \alpha^-, \beta^-; \delta^+, \delta^-)$ to a configuration $\mathcal{C}(\omega)$. This is a continuous analog of the map $\mathbb{GT}_N \to \operatorname{Conf}(\mathfrak{X}^{(N)})$ sending a signature λ to a point configuration $X(\lambda)$; see (4.1). The correspondence $\lambda \mapsto X(\lambda)$ is reversible while the map $\omega \mapsto \mathcal{C}(\omega)$ is not. This is caused by three factors listed below.

(i) The coordinates β_i^+ and β_j^- become indistinguishable: given a point $x \in \mathcal{C}(\omega) \cap \mathfrak{X}_{\text{in}}$, there is no way to decide whether it comes from a coordinate $\frac{1}{2} - x$ of β^+ or from a coordinate $x + \frac{1}{2}$ of β^-. Note that in the discrete case such a problem does not arise. Indeed, if d^+ and d^- stand for the numbers of points $x \in X(\lambda)$ that are on the right and on the left of $\mathfrak{X}_{\text{in}}$, respectively, then $X(\lambda)$ has exactly $d^+ + d^-$ points in $\mathfrak{X}_{\text{in}}$ of which the leftmost d^- points come from λ^- while the remaining d^+ points come from λ^+. But in the continuous case, such an argument fails, because the total number of points is, generally speaking, infinite.

(ii) The map ι ignores the coordinates $\delta^\pm$.

(iii) The map ι ignores possible 1's in $\beta^\pm$.

Let us discuss the significance of these factors in succession.

Factor (i). Note that exactly the same effect of mixture of the plus and minus β-coordinates arises when an extreme character $\chi^{(\omega)}$ is restricted from the group $\mathrm{U}(\infty)$ to the subgroup $\mathrm{SU}(\infty)$; see [Ol3, Remark 1.7]. Hence, if one agrees to view characters that coincide on $\mathrm{SU}(\infty)$ as equivalent ones, then the factor in question becomes not too important.

Factor (ii). We conjecture that in our concrete situation (i.e., for the characters $\chi_{z,z',w,w'}$) the spectral measure P is concentrated on the subset of ω's with $\gamma^\pm = 0$ (see the definition of $\gamma^\pm$ in §1). If this is true then $\delta^\pm$ is almost surely equal to $\sum(\alpha_i^\pm + \beta_i^\pm)$. The conjecture is supported by the fact that the vanishing of the gamma parameters was proved in similar situations; see [P.I, Th. 6.1] and [BO4, Th. 7.3]. The method of [BO4] makes it possible to reduce the conjecture to the following claim concerning the first correlation measures of the processes $\underline{\mathcal{P}}^{(N)}$:

$$\lim_{\varepsilon\to 0}\int_{\frac12-\varepsilon}^{\frac12+\varepsilon}|x-\tfrac12|\,\underline{\rho}_1^{(N)}(dx)=0,\qquad \lim_{\varepsilon\to 0}\int_{-\frac12-\varepsilon}^{-\frac12+\varepsilon}|x+\tfrac12|\,\underline{\rho}_1^{(N)}(dx)=0$$

uniformly on N.

Factor (iii). Again, we remark that possible 1's in $\beta^{\pm}$ play no role when characters are restricted to the subgroup $\mathrm{SU}(\infty)$. On the other hand, one can argue that that in concrete situations the 1's do not appear almost surely.

The above arguments (though not rigorous) present a justification of the passage $P\mapsto\mathcal{P}$.

We conclude the section with one more general result which will be used in Section 10.

Observe that the set of characters of $\mathrm{U}(\infty)$ is stable under the operation of pointwise multiplication by $\det(\,\cdot\,)^k$, where $k\in\mathbb{Z}$.

PROPOSITION 9.7. *Let χ be a character of $\mathrm{U}(\infty)$, P be its spectral measure on Ω, and $\mathcal{P}$ be the corresponding point process on $\mathfrak{X}=\mathbb{R}\setminus\{\pm\frac12\}$. Then $\mathcal{P}$ does not change under $\chi\mapsto\chi\det(\,\cdot\,)^k$, $k\in\mathbb{Z}$.*

Proof. It suffices to prove this for $k=1$. Assume first that χ is extreme, so that $\chi=\chi^{(\omega)}$, where $\omega=(\alpha^+,\beta^+;\alpha^-,\beta^-;\delta^+,\delta^-)\in\Omega$, and P shrinks to the Dirac mass at $\{\omega\}$. From the explicit expression for $\chi^{(\omega)}$, see (1.2), it follows that $\chi^{(\omega)}\det(\cdot)$ is an extreme character, too. Moreover, the corresponding element $\bar\omega\in\Omega$ looks as follows: the parameters $\alpha^{\pm}$ and $\delta^{\pm}$ do not change, while

$$\beta^+\mapsto(1-\beta_1^-,\beta_1^+,\beta_2^+,\dots),\qquad \beta^-\mapsto(\beta_2^-,\beta_3^-,\dots).$$

On the other hand, from the definition of the projection $\omega\mapsto\mathcal{C}(\omega)$, see (9.1), it follows that the change $\omega\mapsto\bar\omega$ does not affect the configuration $\mathcal{C}(\omega)$. This proves the needed claim for extreme χ.

Now let χ be arbitrary. By the very definition of spectral measures (see Theorem 1.2), the spectral measure of the character $\chi\det(\,\cdot\,)$ coincides with the pushforward of the spectral measure P under the map $\omega\mapsto\bar\omega$ of Ω. We have just seen that this map does not affect the projection $\omega\mapsto\mathcal{C}(\omega)$. Since $\mathcal{P}$ is the image of P under this projection, we conclude that $\mathcal{P}$ remains unchanged. □

10. The correlation kernel of the process $\mathcal{P}$

Our goal in this section is to compute the correlation functions of the process $\mathcal{P}$ associated to the spectral measure for zw-measures; see the beginning of Section 9 for the definitions. Theorem 10.1 below is the main result of the paper.

In the formulas below we use the *Gauss hypergeometric function*. Recall that this is a function in one complex variable (say, u) defined inside the unit circle by the series

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix} \,\Big|\, u\right] = \sum_{k\geq 0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k}\, u^k. $$

Here a, b, c are complex parameters, $c \notin \{0, -1, \dots\}$.

This function can be analytically continued to the domain $u \in \mathbb{C} \setminus [1, +\infty)$; see, e.g., [Er, Vol. 1, Ch. 2]. We will need the fact that for any fixed u in $\mathbb{C} \setminus [1, +\infty)$, the expression

$$ \frac{1}{\Gamma(c)}\, {}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix} \,\Big|\, u\right] $$

defines an entire function in $(a, b, c) \in \mathbb{C}^3$. This follows, e.g., from [Er, 2.1.3(15)].

THEOREM 10.1. *Let $(z, z', w, w') \in \mathcal{D}_{\mathrm{adm}}$ and $\mathcal{P}$ be the corresponding point process on $\mathfrak{X} = \mathbb{R} \setminus \{\pm\frac{1}{2}\}$ defined in §9.*

For any $n = 1, 2, \dots$ and $x_1, \dots, x_n \in \mathfrak{X}$, the n^{th} correlation function $\rho_n(x_1, \dots, x_n)$ of the process $\mathcal{P}$ has the determinantal form

$$ \rho_n(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^{n}. $$

The kernel $K(x, y)$ with respect to the splitting $\mathfrak{X} = \mathfrak{X}_{\mathrm{out}} \sqcup \mathfrak{X}_{\mathrm{in}}$ has the following form

$$ \begin{aligned} K_{\mathrm{out,out}}(x, y) &= \sqrt{\psi_{\mathrm{out}}(x)\psi_{\mathrm{out}}(y)}\, \frac{R_{\mathrm{out}}(x)S_{\mathrm{out}}(y) - S_{\mathrm{out}}(x)R_{\mathrm{out}}(y)}{x - y}, \\ K_{\mathrm{out,in}}(x, y) &= \sqrt{\psi_{\mathrm{out}}(x)\psi_{\mathrm{in}}(y)}\, \frac{R_{\mathrm{out}}(x)R_{\mathrm{in}}(y) - S_{\mathrm{out}}(x)S_{\mathrm{in}}(y)}{x - y}, \\ K_{\mathrm{in,out}}(x, y) &= \sqrt{\psi_{\mathrm{in}}(x)\psi_{\mathrm{out}}(y)}\, \frac{R_{\mathrm{in}}(x)R_{\mathrm{out}}(y) - S_{\mathrm{in}}(x)S_{\mathrm{out}}(y)}{x - y}, \\ K_{\mathrm{in,in}}(x, y) &= \sqrt{\psi_{\mathrm{in}}(x)\psi_{\mathrm{in}}(y)}\, \frac{R_{\mathrm{in}}(x)S_{\mathrm{in}}(y) - S_{\mathrm{in}}(x)R_{\mathrm{in}}(y)}{x - y}, \end{aligned} $$

where

$$\psi_{\text{out}}(x)=\begin{cases} C(z,z')\cdot\left(x-\frac12\right)^{-z-z'}\left(x+\frac12\right)^{-w-w'}, & x>\frac12\,, \\ C(w,w')\cdot\left(-x-\frac12\right)^{-w-w'}\left(-x+\frac12\right)^{-z-z'}, & x<-\frac12\,, \end{cases}$$

$$\psi_{\text{in}}(x)=\left(\tfrac12-x\right)^{z+z'}\left(\tfrac12+x\right)^{w+w'},\quad -\tfrac12<x<\tfrac12,$$

$$C(z,z')=\frac{\sin\pi z\sin\pi z'}{\pi^2},\quad C(w,w')=\frac{\sin\pi w\sin\pi w'}{\pi^2},$$

and

$$R_{\text{out}}(x)=\left(\frac{x+\frac12}{x-\frac12}\right)^{w'}{}_2F_1\left[\left.\begin{matrix} z+w',\ z'+w' \\ z+z'+w+w'\end{matrix}\right|\frac{1}{\frac12-x}\right],$$

$$S_{\text{out}}(x)=\Gamma\left[\begin{matrix} z+w+1,\ z+w'+1,\ z'+w+1,\ z'+w'+1 \\ z+z'+w+w'+1,\ z+z'+w+w'+2\end{matrix}\right]$$

$$\times\frac{1}{x-\frac12}\left(\frac{x+\frac12}{x-\frac12}\right)^{w'}{}_2F_1\left[\left.\begin{matrix} z+w'+1,\ z'+w'+1 \\ z+z'+w+w'+2\end{matrix}\right|\frac{1}{\frac12-x}\right],$$

$$R_{\text{in}}(x)=-\frac{\sin\pi z}{\pi}\Gamma\left[\begin{matrix} z'-z,\ z+w+1,\ z+w'+1 \\ z+w+z'+w'+1\end{matrix}\right]$$

$$\times\left(\frac12+x\right)^{-w}\left(\frac12-x\right)^{-z'}{}_2F_1\left[\left.\begin{matrix} z+w'+1,\ -z'-w \\ z-z'+1\end{matrix}\right|\frac12-x\right]$$

$$-\frac{\sin\pi z'}{\pi}\Gamma\left[\begin{matrix} z-z',\ z'+w+1,\ z'+w'+1 \\ z+w+z'+w'+1\end{matrix}\right]$$

$$\times\left(\frac12+x\right)^{-w}\left(\frac12-x\right)^{-z}{}_2F_1\left[\left.\begin{matrix} z'+w'+1,\ -z-w \\ z'-z+1\end{matrix}\right|\frac12-x\right],$$

$$S_{\text{in}}(x)=-\frac{\sin\pi z}{\pi}\Gamma\left[\begin{matrix} z'-z,\ z+z'+w+w' \\ z'+w,\ z'+w'\end{matrix}\right]$$

$$\times\left(\frac12+x\right)^{-w}\left(\frac12-x\right)^{-z'}{}_2F_1\left[\left.\begin{matrix} z+w',\ -z'-w+1 \\ z-z'+1\end{matrix}\right|\frac12-x\right]$$

$$-\frac{\sin\pi z'}{\pi}\Gamma\left[\begin{matrix} z-z',\ z+z'+w+w' \\ z+w,\ z+w'\end{matrix}\right]$$

$$\times\left(\frac12+x\right)^{-w}\left(\frac12-x\right)^{-z}{}_2F_1\left[\left.\begin{matrix} z'+w',\ -z-w+1 \\ z'-z+1\end{matrix}\right|\frac12-x\right].$$

The indeterminacy on the diagonal $x=y$ *is resolved by L'Hospital's rule:*

$$K_{\text{out,out}}(x,x)=\psi_{\text{out}}(x)\left(R'_{\text{out}}(x)S_{\text{out}}(x)-S'_{\text{out}}(x)R_{\text{out}}(x)\right),$$
$$K_{\text{in,in}}(x,x)=\psi_{\text{in}}(x)\left(R'_{\text{in}}(x)S_{\text{in}}(x)-S'_{\text{in}}(x)R_{\text{in}}(x)\right).$$

Singularities. The formulas for the function $R_{\rm out}, S_{\rm out}, R_{\rm in}, S_{\rm in}$ above have no singularities for

$$(z,z',w,w')\in\mathcal{D}_0\setminus\big(\{\Sigma=0\}\cup\{z-z'\in\mathbb{Z}\}\big).$$

Moreover, we will prove that the value of the kernel $K(x,y)$ can be extended to a continuous function on $\mathcal{D}_{\rm adm}$ for any fixed $x,y\in\mathfrak{X}$. (Recall that the process $\mathcal{P}$ is defined for $(z,z',w,w')\in\mathcal{D}_{\rm adm}$.)

Vanishing of the kernel. Note that if $(z,z')\in\mathcal{Z}_{\rm degen}$ (see §3 for the definition of $\mathcal{Z}_{\rm degen}$) then the function $\psi_{\rm out}$ vanishes on $(\frac12,+\infty)$, because $C(z,z')=0$. This implies that $K(x,y)=0$ whenever x or y is greater than $\frac12$. It follows that the configurations of the process $\mathcal{P}$ do not have points in $(\frac12,+\infty)$.

Likewise, if $(w,w')\in\mathcal{Z}_{\rm degen}$ then the configurations of the process do not intersect $(-\infty,-\frac12)$.

Proof. First of all, observe that it suffices to prove the theorem when $(z,z',w,w')\in\mathcal{D}'_{\rm adm}$. Indeed, if $(z,z',w,w')\in\mathcal{D}_{\rm adm}\setminus\mathcal{D}'_{\rm adm}$ then (z,z',w,w') can be moved to $\mathcal{D}'_{\rm adm}$ by an appropriate shift of the parameters, which is equivalent to multiplying the initial character χ by $\det(\,\cdot\,)^k$ with a certain $k\in\mathbb{Z}$; see Remarks 6.4 and 3.7. Next, according to Proposition 9.7, multiplication by $\det(\,\cdot\,)^k$ does not affect the point process $\mathcal{P}$.

To carry out the desired reduction we have to check that the formulas for the functions ρ_n given in Theorem 10.1 are also invariant under any shift of the parameters of the form

$$(z,z',w,w')\mapsto(z+k,z'+k,w-k,w'-k),\quad k\in\mathbb{Z}.$$

Note that the kernel $K(x,y)$ is not invariant under such a shift. To see what happens with the kernel we observe that

$$\psi_{\rm out}(x)\to\left(\frac{x+\frac12}{x-\frac12}\right)^{2k}\psi_{\rm out}(x),\quad \psi_{\rm in}(x)\to\left(\frac{x+\frac12}{-x+\frac12}\right)^{-2k}\psi_{\rm in}(x),$$
$$R_{\rm out}(x)\to\left(\frac{x+\frac12}{x-\frac12}\right)^{-k}R_{\rm out}(x),\quad S_{\rm out}(x)\to\left(\frac{x+\frac12}{x-\frac12}\right)^{-k}S_{\rm out}(x),$$
$$R_{\rm in}(x)\to(-1)^k\left(\frac{x+\frac12}{-x+\frac12}\right)^{k}R_{\rm in}(x),\quad S_{\rm in}(x)\to(-1)^k\left(\frac{x+\frac12}{-x+\frac12}\right)^{k}S_{\rm in}(x).$$

It follows that

$$K(x,y)\to\phi(x)K(x,y)(\phi(y))^{-1},$$

where

$$\phi(x)=\begin{cases}1, & x\in\mathfrak{X}_{\rm out},\\ (-1)^k, & x\in\mathfrak{X}_{\rm in}.\end{cases}$$

But such a transformation of the kernel does not affect the determinantal formula for the correlation functions.

From now on we will assume that $(z, z', w, w') \in \mathcal{D}'_{\text{adm}}$, as in Theorem 8.7. At this moment we impose additional restrictions $\Sigma \neq 0$ and $z - z' \notin \mathbb{Z}$. We will need these restrictions in Propositions 10.3–10.4 below. After that they will be removed.

For any $x \in \mathfrak{X} = \mathbb{R} \setminus \{\pm\frac{1}{2}\}$, let x_N denote the point of the lattice $\mathfrak{X}^{(N)} = \mathbb{Z} + \frac{N-1}{2}$ which is closest to Nx (if there are two such points then we choose either of them). By Theorems 8.7 and 9.2, it suffices to prove that

$$\lim_{N\to\infty} N \cdot K^{(N)}(x_N, y_N) = K(x, y), \tag{10.1}$$

uniformly on compact sets of $\mathfrak{X} \times \mathfrak{X}$. Here $K^{(N)}$ is the kernel of Theorem 8.7.

To do this we will establish the uniform convergence of all six functions $\psi^{(N)}_{\text{out}}$, $\psi^{(N)}_{\text{in}}$, $R^{(N)}_{\text{out}}$, $S^{(N)}_{\text{out}}$, $R^{(N)}_{\text{in}}$, $S^{(N)}_{\text{in}}$ of Theorem 8.7 to the respective functions of Theorem 10.1. In order to overcome the difficulty arising from vanishing of the denominators at $x = y$ we will establish the convergence of $R^{(N)}_{\text{out}}$, $S^{(N)}_{\text{out}}$, $R^{(N)}_{\text{in}}$, $S^{(N)}_{\text{in}}$ in a complex region containing $\mathfrak{X}$.

The needed convergence (10.1) follows from Propositions 10.2–10.4 below.

Proposition 10.2. *There exist limits*

$$\lim_{N\to\infty} N^{\Sigma}\psi^{(N)}_{\text{out}}(x_N) = \psi_{\text{out}}(x), \qquad \lim_{N\to\infty} N^{-\Sigma}\psi^{(N)}_{\text{in}}(x_N) = \psi_{\text{in}}(x),$$

as $N \to \infty$, where the functions $\psi^{(N)}_{\text{in}}$ and $\psi^{(N)}_{\text{out}}$ are as defined in (6.2), (6.3). The convergence is uniform on the compact subsets of $\mathfrak{X}_{\text{out}} = \mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]$ and $\mathfrak{X}_{\text{in}} = (-\frac{1}{2}, \frac{1}{2})$, respectively.

Proposition 10.3. *Let I be a compact subset of $\mathfrak{X}_{\text{out}}$. Set*

$$I_\varepsilon = \{\zeta \in \mathbb{C} \,|\, \Re\zeta \in I,\, |\Im\zeta| < \varepsilon\}.$$

Then for $\varepsilon > 0$ small enough, for any $\zeta \in I_\varepsilon$,

$$\lim_{N\to\infty} R^{(N)}_{\text{out}}(N\zeta) = R_{\text{out}}(\zeta), \quad \lim_{N\to\infty} N^{-\Sigma} S^{(N)}_{\text{out}}(N\zeta) = S_{\text{out}}(\zeta), \tag{10.2}$$

and the convergence is uniform on I_ε.

Proposition 10.4. *Let J be a compact subset of $\mathfrak{X}_{\text{in}}$. Set*

$$J_\varepsilon = \{\zeta \in \mathbb{C} \,|\, \Re\zeta \in J,\, |\Im\zeta| < \varepsilon\}.$$

Then for $\varepsilon > 0$ small enough, for any $\zeta \in J_\varepsilon$,

$$\lim_{N\to\infty} R^{(N)}_{\text{in}}(N\zeta) = R_{\text{in}}(\zeta), \quad \lim_{N\to\infty} N^{\Sigma} S^{(N)}_{\text{in}}(N\zeta) = S_{\text{in}}(\zeta), \tag{10.3}$$

and the convergence is uniform on J_ε.

Proof of Proposition 10.2. This follows from the following uniform estimates.

For $x \in \mathfrak{X}_{\text{in}}$ we have

$$\frac{\Gamma(-x_N+u+\frac{N+1}{2})}{\Gamma(-x_N+\frac{N+1}{2})} = N^u \left(-x+\frac{1}{2}\right)^u (1+\mathrm{O}(N^{-1})), \quad u = z \text{ or } z',$$

$$\frac{\Gamma(x_N+v+\frac{N+1}{2})}{\Gamma(x_N+\frac{N+1}{2})} = N^v \left(x+\frac{1}{2}\right)^v (1+\mathrm{O}(N^{-1})), \quad v = w \text{ or } w',$$

as $N \to \infty$.

For $x \in \mathfrak{X}_{\text{out}}$ and $x > \frac{1}{2}$ we use the formulas

$$\frac{1}{\Gamma(-x_N+u+\frac{N+1}{2})} = \frac{\sin\left(\pi(-x_N+u+\frac{N+1}{2})\right)}{\pi}\,\Gamma\left(x_N-u-\frac{N-1}{2}\right)$$
$$= (-1)^{-x_N+\frac{N+1}{2}}\,\frac{\sin(\pi u)}{\pi}\,\Gamma\left(x_N-u-\frac{N-1}{2}\right), \quad u = z \text{ or } z',$$

and the asymptotic relations

$$\frac{\Gamma(x_N-u-\frac{N-1}{2})}{\Gamma(x_N-\frac{N-1}{2})} = N^{-u} \left(x-\frac{1}{2}\right)^{-u} (1+\mathrm{O}(N^{-1})), \quad u = z \text{ or } z',$$

$$\frac{\Gamma(x_N+\frac{N+1}{2})}{\Gamma(x_N+v+\frac{N+1}{2})} = N^{-v} \left(x+\frac{1}{2}\right)^{-v} (1+\mathrm{O}(N^{-1})), \quad v = w \text{ or } w',$$

as $N \to \infty$.

For $x \in \mathfrak{X}_{\text{out}}$ and $x < -\frac{1}{2}$ we use the formulas

$$\frac{1}{\Gamma(x_N+v+\frac{N+1}{2})} = \frac{\sin\left(\pi(x_N+v+\frac{N+1}{2})\right)}{\pi}\,\Gamma\left(-x_N-v-\frac{N-1}{2}\right)$$
$$= (-1)^{x_N+\frac{N+1}{2}}\,\frac{\sin(\pi v)}{\pi}\,\Gamma\left(-x_N-v-\frac{N-1}{2}\right), \quad v = w \text{ or } w',$$

and the asymptotic relations

$$\frac{\Gamma(-x_N+\frac{N+1}{2})}{\Gamma(-x_N+u+\frac{N+1}{2})} = N^{-u} \left(-x+\frac{1}{2}\right)^{-u} (1+\mathrm{O}(N^{-1})), \quad u = z \text{ or } z',$$

$$\frac{\Gamma(-x_N-v-\frac{N-1}{2})}{\Gamma(-x_N-\frac{N-1}{2})} = N^{-v} \left(-x-\frac{1}{2}\right)^{-v} (1+\mathrm{O}(N^{-1})), \quad v = w \text{ or } w',$$

as $N \to \infty$. □

Proof of Proposition 10.3. We will employ the formulas (8.1) and (8.2). This gives

$$R_{\text{out}}^{(N)}(N\zeta) = \Gamma\left[\begin{array}{c} N\zeta-\frac{N-1}{2},\ N\zeta+w'+\frac{N+1}{2} \\ N\zeta+\frac{N+1}{2},\ N\zeta+w'-\frac{N-1}{2} \end{array}\right] {}_3F_2\left[\left.\begin{array}{c} -N,\ z+w',\ z'+w' \\ \Sigma,\ N\zeta+w'-\frac{N-1}{2} \end{array}\right| 1\right].$$

Assume $\Re\zeta < -\frac{1}{2}$. Handling the gamma factors is easy:

$$\Gamma\left[\begin{array}{c} N\zeta - \frac{N-1}{2},\, N\zeta + w' + \frac{N+1}{2} \\ N\zeta + \frac{N+1}{2},\, N\zeta + w' - \frac{N-1}{2} \end{array}\right]$$

$$= \frac{\Gamma(-N\zeta - \frac{N-1}{2})}{\Gamma(-N\zeta - w' - \frac{N-1}{2})} \cdot \frac{\Gamma(-N\zeta - w' + \frac{N+1}{2})}{\Gamma(-N\zeta + \frac{N+1}{2})}$$

$$= \left(-\zeta - \frac{1}{2}\right)^{w'} \left(-\zeta + \frac{1}{2}\right)^{-w'} (1 + \mathrm{O}(N^{-1})) = \left(\frac{\zeta + \frac{1}{2}}{\zeta - \frac{1}{2}}\right)^{w'} (1 + \mathrm{O}(N^{-1})),$$

as $N \to \infty$, and the estimate is uniform on a small complex neighborhood of any compact subset $I \subset (-\infty, -\frac{1}{2})$.

To complete the proof of the first limit relation (10.1) we need to have the equality

$$\lim_{N\to\infty} {}_3F_2\left[\left.\begin{array}{c} -N,\, z + w',\, z' + w' \\ z + z' + w + w',\, N\zeta + w' - \frac{N-1}{2} \end{array}\right| 1\right] = {}_2F_1\left[\left.\begin{array}{c} z + w',\, z' + w' \\ z + z' + w + w' \end{array}\right| \frac{1}{\frac{1}{2} - \zeta}\right]$$

with uniform convergence. This limit transition is justified by the following lemma.

Lemma 10.5. *Let A, B, C be complex numbers, $C \neq 0, -1, -2, \ldots$, $\{D_N\}_{N=1}^{\infty}$ be a sequence of complex numbers, $D_N \neq 0, -1, \ldots$ for all N. Assume that*

$$\lim_{N\to\infty} \frac{-N}{D_N} = q \in (0, 1).$$

Then

$$\lim_{N\to\infty} {}_3F_2\left[\left.\begin{array}{c} -N,\, A,\, B \\ C,\, D_N \end{array}\right| 1\right] = {}_2F_1\left[\left.\begin{array}{c} A,\, B \\ C \end{array}\right| q\right].$$

The convergence is uniform on any set of sequences $\{D_N\}_{N=1}^{\infty}$ such that $\{\frac{N}{D_N} - \lim \frac{N}{D_N}\}$ uniformly converges to 0 as $N \to \infty$ and $\lim \frac{N}{D_N}$ is uniformly bounded from -1 on this set.

Proof. We have

$${}_3F_2\left[\left.\begin{array}{c} -N,\, A,\, B \\ C,\, D_N \end{array}\right| 1\right] = \sum_{k=0}^{N} \frac{(-N)_k (A)_k (B)_k}{k! (C)_k (D_N)_k}.$$

Let us show that these sums converge uniformly in $N > N_0$ for some $N_0 > 0$. Thanks to our hypothesis, for large enough N we can pick $q_0 \in (q, 1)$ and a positive number d_N such that for all our sequences $\{D_N\}$, $-\Re D_N > d_N > N q_0^{-1}$. Then (note that $k \leq N$)

$$|(D_N)_k| = |(-D_N)(-D_N - 1) \ldots (-D_N - k + 1)| \geq d_N (d_N - 1) \cdots (d_N - k + 1).$$

Hence,

$$\left|\frac{(-N)_k}{(D_N)_k}\right| \le \frac{N(N-1)\cdots(N-k+1)}{d_N(d_N-1)\cdots(d_N-k+1)} \le \left(\frac{N}{d_N}\right)^k < q_0^k.$$

Thus, the sums above for large enough N are majorized by the convergent series

$$\sum_{k=0}^{\infty}\left|\frac{(A)_k(B)_k}{k!(C)_k}\right| q_0^k$$

and, therefore, converge uniformly. This means that to compute the limit as $N\to\infty$ we can pass to the limit $N\to\infty$ in every term of the sum. Since for any fixed k

$$\lim_{N\to\infty}\frac{(-N)_k}{(D_N)_k} = q^k,$$

this yields

$$\lim_{N\to\infty} {}_3F_2\left[\begin{matrix}-N,\,A,\,B\\ C,\,D_N\end{matrix}\,\middle|\,1\right] = \sum_{k\ge 0}\frac{(A)_k(B)_k}{k!(C)_k}\,q^k = {}_2F_1\left[\begin{matrix}A,\,B\\ C\end{matrix}\,\middle|\,q\right].$$

The fact that we majorized the series by the same convergent series for all our sequences $\{D_N\}$, and the uniform convergence of the terms of the series guarantee the needed uniform convergence on the set of sequences. □

To prove the first limit relation for $I\subset(\frac12,+\infty)$ we just note that by uniqueness of monic orthogonal polynomials with a fixed weight, $R^{(N)}_{\mathrm{out}}(x)$ is invariant with respect to the substitution

$$x\mapsto -x,\qquad (z,z')\longleftrightarrow(w',w);$$

cf. Lemmas 8.4, 8.5, and so is $R_{\mathrm{out}}(x)$, because of the transformation formula

$${}_2F_1\left[\begin{matrix}A,\,B\\ C\end{matrix}\,\middle|\,\zeta\right] = (1-\zeta)^{-A}{}_2F_1\left[\begin{matrix}A,\,C-B\\ C\end{matrix}\,\middle|\,\frac{\zeta}{\zeta-1}\right].$$

The proof of the second relation (10.1) is similar. Note that both $S^{(N)}_{\mathrm{out}}(x)$ and $S_{\mathrm{out}}(x)$ are skew-symmetric with respect to the substitution above.

The proof of Proposition 10.3 is complete. □

Proof of Proposition 10.4. The argument is quite similar to the proof of Proposition 10.3 above. Let us evaluate the asymptotics of the right-hand side of (8.4): We look at the first term. Clearly, the argument for the second term will be just the same. Gamma factors give (here $\Re z\in J\subset(-\frac12,\frac12)$)

$$\Gamma\left[\begin{matrix}N\zeta+\frac{N+1}{2},\,-N\zeta+\frac{N+1}{2},\,N+1+\Sigma\\ -N\zeta+z'+\frac{N+1}{2},\,N\zeta+w+\frac{N+1}{2},\,N+1+z+w'\end{matrix}\right]$$
$$= \left(\zeta+\frac12\right)^{-w}\left(\zeta-\frac12\right)^{-z'}(1+\mathrm{O}(N^{-1})),$$

as $N \to \infty$, uniformly on a neighborhood of J. To complete the proof of the first relation (10.2) we need to show that

$$\lim_{N\to\infty} {}_3F_2\left[\begin{matrix} z+w'+1,\ -z'-w,\ -N\zeta+z+\frac{N+1}{2} \\ z-z'+1,\ N+1+z+w' \end{matrix}\,\middle|\, 1\right] = {}_2F_1\left[\begin{matrix} z+w'+1,\ -z'-w \\ z-z'+1 \end{matrix}\,\middle|\, \frac{1}{2}-\zeta\right]$$

uniformly in ζ. This is achieved by the following lemma.

LEMMA 10.6. *Let A, B, C, δ be complex numbers, $C \neq 0, -1, -2, \dots$, $\{D_N\}_{N=1}^{\infty}$ be a sequence of complex numbers. Assume that*

$$\lim_{N\to\infty} \frac{D_N}{N} = q \in (0,1).$$

Then

$$\lim_{N\to\infty} {}_3F_2\left[\begin{matrix} A,\ B,\ D_N \\ N+\delta,\ C \end{matrix}\,\middle|\, 1\right] = {}_2F_1\left[\begin{matrix} A,\ B \\ C \end{matrix}\,\middle|\, q\right].$$

The convergence is uniform on any set of sequences $\{D_N\}_{N=1}^{\infty}$ such that $\{\frac{D_N}{N} - \lim \frac{D_N}{N}\}$ uniformly converges to 0 as $N \to \infty$ and $\lim \frac{D_N}{N}$ is uniformly bounded from 1 on this set.

Proof. We have

$${}_3F_2\left[\begin{matrix} A,\ B,\ D_N \\ N+\delta,\ C \end{matrix}\,\middle|\, 1\right] = \sum_{k=0}^{\infty} \frac{(A)_k (B)_k (D_N)_k}{k!(N+\delta)_k (C)_k}.$$

Let us show that these sums converge uniformly in $N > N_0$ for some $N_0 > 0$.

Let d_N be the smallest integer greater than $\sup |D_N|$, where the supremum is taken over all our sequences $\{D_N\}$. Let l be the largest integer less than $\Re\delta$. Then for large enough N our series is majorized by the series

$$\sum_{k=0}^{\infty} \left| \frac{(A)_k (B)_k (d_N)_k}{k!(N+l)_k (C)_k} \right|.$$

Using the hypothesis we may assume that for large enough N, $d_N < q_0(N+l)$ for some $q_0 \in (0,1)$. In particular, $d_N < N+l$. If $k \le N+l-d_N$ then

$$\frac{(d_N)_k}{(N+l)_k} = \frac{d_N(d_N+1)\cdots(d_N+k-1)}{(N+l)(N+l+1)\cdots(N+l+k-1)} \le \left(\frac{d_N+k-1}{N+l+k-1}\right)^k$$
$$\le \left(\frac{d_N+(N+l-d_N)}{N+l+(N+l-d_N)}\right)^k = \left(\frac{N+l}{2N-d_N+2l}\right)^k \le (2-q_0)^{-k}$$

for large enough N. If $k \geq N + l - d_N$ then

$$\begin{aligned}\frac{(d_N)_k}{(N+l)_k} &= \frac{d_N(d_N+1)\cdots(d_N+k-1)}{(N+l)(N+l+1)\cdots(N+l+k-1)} \\ &= \frac{d_N(d_N+1)\cdots(N+l-1)}{(d_N+k)(d_N+k+1)\cdots(N+l+k-1)} \leq \left(\frac{N+l}{N+l+k}\right)^{N+l-1-d_N} \\ &\leq \left(1+\frac{k}{N+l}\right)^{-(N+l-1-d_N)} \leq \left(1+\frac{k}{N+l}\right)^{-(1-q_0)N}\end{aligned}$$

for large enough N. The last expression is a decreasing function in N (with $N + l > 0$). Hence, for $N > N_0$,

$$\frac{(d_N)_k}{(N+l)_k} \leq \left(1+\frac{k}{N_0+l}\right)^{-(1-q_0)(N_0+l)}.$$

Thus, we have proved that $(d_N)_k/(N+l)_k$ does not exceed the maximum of the k^{th} member of a geometric progression with ratio $(2-q_0)^{-1} < 1$ and the inverse of the value at the point k of a polynomial of arbitrarily large (equal to $[(1-q_0)(N_0+l)]$) degree. Since $(A)_k(B)_k/(k!(C)_k)$ has polynomial behavior in k for large k, this means that we majorized the series

$${}_3F_2\left[\begin{matrix}A,\, B,\, D_N \\ N+\delta,\, C\end{matrix}\,\middle|\, 1\right] = \sum_{k=0}^{\infty} \frac{(A)_k(B)_k(D_N)_k}{k!(N+\delta)_k(C)_k}$$

by a convergent series with terms not depending on N. Therefore, to compute the limit of this series as $N \to \infty$, we can take the limit term by term. Since for any fixed $k \geq 0$

$$\lim_{N\to\infty} \frac{(D_N)_k}{(N+\delta)_k} = q^k,$$

we get

$$\lim_{N\to\infty} \sum_{k=0}^{\infty} \frac{(A)_k(B)_k(D_N)_k}{k!(N+\delta)_k(C)_k} = \sum_{k=0}^{\infty} \frac{(A)_k(B)_k}{k!(C)_k}\, q^k = {}_2F_1\left[\begin{matrix}A,\, B \\ C\end{matrix}\,\middle|\, q\right].$$

As in the proof of Lemma 10.5, the fact that we majorized the series by the same convergent series for all our sequences $\{D_N\}$, and the uniform convergence of the terms of the series guarantee the needed uniform convergence on the set of sequences. □

The proof of Proposition 10.4 is complete. □

To conclude the proof of Theorem 10.1 we need to get rid of the extra restrictions $\Sigma \neq 0$ and $z - z' \notin \mathbb{Z}$ imposed in the beginning of the proof.

Define a function $\Psi : \mathfrak{X} \to \mathbb{C}$, which is similar to the function $\Psi^{(N)}$ introduced in (8.10), by

$$\Psi(x) = \begin{cases} \psi_{\text{out}}(x), & x \in \mathfrak{X}_{\text{out}}, \\ \dfrac{1}{\psi_{\text{in}}(x)}, & x \in \mathfrak{X}_{\text{in}}. \end{cases}$$

Note that for any $x \in \mathfrak{X}$, $\Psi(x)$ is an entire function in (z, z', w, w').

LEMMA 10.7. *Let $(z, z', w, w') \in \mathcal{D}'_{\mathrm{adm}}$. The kernel $K(x, y)$ can be written in the form*

$$K(x,y) = \sqrt{\Psi(x)\Psi(y)}\,\overset{\circ}{K}(x,y),$$

where $\overset{\circ}{K}(x,y)$ admits a holomorphic continuation in the parameters to the domain $\mathcal{D}'_0 \supset \mathcal{D}'_{\mathrm{adm}}$. Moreover, for any $(z, z', w, w') \in \mathcal{D}'_0$,

$$\overset{\circ}{K}(x,y) = \lim_{N\to\infty} N^{1-\Sigma} \overset{\circ}{K}{}^{(N)}(x_N, y_N)$$

uniformly on compact subsets of $\mathfrak{X} \times \mathfrak{X}$.

Recall that the kernel $\overset{\circ}{K}{}^{(N)}$ was defined in Lemma 8.8 and $\mathcal{D}'$ was defined just before this lemma.

Proof of the lemma. It will be convenient to use more detailed notation for the kernels in question. So, we will use the notation $\overset{\circ}{K}{}^{(N)}(x, y \mid z, z', w, w')$ instead of $\overset{\circ}{K}{}^{(N)}(x, y)$. Next, we define the kernel $\overset{\circ}{K}(x, y \mid z, z', w, w')$: in the block form,

$$\begin{aligned}
\overset{\circ}{K}_{\mathrm{out,out}}(x, y \mid z, z', w, w') &= \frac{R_{\mathrm{out}}(x)S_{\mathrm{out}}(y) - S_{\mathrm{out}}(x)R_{\mathrm{out}}(y)}{x-y}\,,\\
\overset{\circ}{K}_{\mathrm{out,in}}(x, y \mid z, z', w, w') &= \Psi(y)\,\frac{R_{\mathrm{out}}(x)R_{\mathrm{in}}(y) - S_{\mathrm{out}}(x)S_{\mathrm{in}}(y)}{x-y}\,,\\
\overset{\circ}{K}_{\mathrm{in,out}}(x, y \mid z, z', w, w') &= \Psi(x)\,\frac{R_{\mathrm{in}}(x)R_{\mathrm{out}}(y) - S_{\mathrm{in}}(x)S_{\mathrm{out}}(y)}{x-y}\,,\\
\overset{\circ}{K}_{\mathrm{in,in}}(x, y \mid z, z', w, w') &= \Psi(x)\Psi(y)\,\frac{R_{\mathrm{in}}(x)S_{\mathrm{in}}(y) - S_{\mathrm{in}}(x)R_{\mathrm{in}}(y)}{x-y}\,.
\end{aligned}$$

These expressions are well defined if (z, z', w, w') is in the subdomain

$$\mathcal{D}''_0 = \{(z, z', w, w') \in \mathcal{D}'_0 \mid \Sigma \neq 0,\ z - z' \notin \mathbb{Z}\}.$$

By virtue of Propositions 10.2, 10.3, and 10.4,

$$\lim_{N\to\infty} N^{1-\Sigma} \overset{\circ}{K}{}^{(N)}(x_N, y_N \mid z, z', w, w') = \overset{\circ}{K}(x, y \mid z, z', w, w') \tag{10.4}$$

for any fixed $(z, z', w, w') \in \mathcal{D}''_0$, uniformly on compact subsets of $\mathfrak{X} \times \mathfrak{X}$. Moreover, one can verify that the estimates of Propositions 10.3 and 10.4 are uniform in (z, z', w, w') varying on any compact subset of the domain $\mathcal{D}''_0$. Thus, the limit relation (10.4) holds uniformly on compact subsets of $\mathfrak{X} \times \mathfrak{X} \times \mathcal{D}''_0$.

On the other hand, we know that the kernel $\overset{\circ}{K}{}^{(N)}$ is holomorphic in (z, z', w, w') on the larger domain $\mathcal{D}_0' \supset \mathcal{D}_0''$. It follows that the additional restrictions $\Sigma \neq 0$ and $z - z' \notin \mathbb{Z}$ can be removed. Specifically, the right-hand side of (10.4) can be extended to the domain $\mathcal{D}_0'$ and the limit relation (10.4) holds on $\mathfrak{X} \times \mathfrak{X} \times \mathcal{D}_0'$. Indeed, we can avoid the hyperplanes $\Sigma = 0$ or $z - z' = k$, where $k \in \mathbb{Z}$, by making use of Cauchy's integral over a small circle in the z-plane.

This completes the proof of Lemma 10.7. □

Now we can complete the proof of the relation (10.1). Proposition 10.2 implies that

$$\lim_{N\to\infty} N^{\Sigma}\, \Psi^{(N)}(x_N) = \Psi(x) \tag{10.5}$$

for any $(z, z', w, w') \in \mathcal{D}_0'$, uniformly on compact subsets of $\mathfrak{X}$. Indeed, as is seen from the proof of Proposition 10.2, it does not use the additional restrictions and holds for any $(z, z', w, w') \in \mathcal{D}_0$. To pass from the functions $\psi_{\text{out}}^{(N)}$, $\psi_{\text{in}}^{(N)}$, and ψ_{out}, ψ_{in} to the functions $\Psi^{(N)}$ and Ψ, we use the assumption $(z, z', w, w') \in \mathcal{D}_0'$ which makes it possible to invert the 'inner' functions for all values of parameters. (Note that if $(z, z') \in \mathcal{Z}_{\text{degen}}$ then for $x > \frac{1}{2}$ and large enough N both $\Psi^{(N)}(x_N)$ and $\Psi(x)$ vanish. Similarly, if $(w, w') \in \mathcal{Z}_{\text{degen}}$ then the vanishing happens for $x < -\frac{1}{2}$.)

Since

$$\begin{aligned} K(x,y) &= \overset{\circ}{K}(x,y)\sqrt{\Psi(x)\Psi(y)},\\ K^{(N)}(x,y) &= \overset{\circ}{K}{}^{(N)}(x,y)\sqrt{\Psi^{(N)}(x)\Psi^{(N)}(y)}, \end{aligned}$$

(10.1) follows from (10.4) and (10.5). This completes the proof of Theorem 10.1. □

We conclude this section by a list of properties (without proofs) of the correlation kernel K and functions R_{out}, S_{out}, R_{in}, S_{in}. The proofs can be found in [BD].

All the results below should be compared with similar results for $K^{(N)}$ and $R_{\text{out}}^{(N)}$, $S_{\text{out}}^{(N)}$, $R_{\text{in}}^{(N)}$, $S_{\text{in}}^{(N)}$, which were proved in the previous sections.

Symmetries. All four functions R_{out}, S_{out}, R_{in}, S_{in} are invariant with respect to the transpositions $z \leftrightarrow z'$ and $w \leftrightarrow w'$.

Further, let us denote by $\mathcal{S}$ the following familiar change of parameters and the variable: $(z, z', w, w', x) \longleftrightarrow (w, w', z, z', -x)$. Then

$$\begin{aligned} &\mathcal{S}(\psi_{\text{out}}) = \psi_{\text{out}}, \quad \mathcal{S}(\psi_{\text{in}}) = \psi_{\text{in}},\\ &\mathcal{S}(R_{\text{out}}) = R_{\text{out}}, \quad \mathcal{S}(S_{\text{out}}) = -S_{\text{out}}, \quad \mathcal{S}(R_{\text{in}}) = R_{\text{in}}, \quad \mathcal{S}(S_{\text{in}}) = -S_{\text{in}}. \end{aligned}$$

The functions $R_{\rm out}$, $S_{\rm out}$, $R_{\rm in}$, $S_{\rm in}$ and the kernel K for admissible values of parameters take real values on $\mathfrak{X}$. Moreover, the kernel $K(x,y)$ is J-symmetric; see §5(f). That is,

$$K_{\rm out,out}(x,y)=K_{\rm out,out}(y,x),\quad K_{\rm in,in}(x,y)=K_{\rm in,in}(y,x),$$
$$K_{\rm in,out}(x,y)=-K_{\rm out,in}(y,x).$$

Branching of analytic continuations. The formulas for $R_{\rm out}$, $S_{\rm out}$, $R_{\rm in}$, $S_{\rm in}$ above provide analytic continuations of these functions. We can view $R_{\rm out}$ and $S_{\rm out}$ as functions which are analytic and single-valued on $\mathbb{C}\setminus\mathfrak{X}_{\rm in}$, and $R_{\rm in}$ and $S_{\rm in}$ as functions analytic and single-valued on $\mathbb{C}\setminus\mathfrak{X}_{\rm out}$. (Recall that the Gauss hypergeometric function can be viewed as an analytic and single valued function on $\mathbb{C}\setminus[1,+\infty)$.)

For a function $F(\zeta)$ defined on $\mathbb{C}\setminus\mathbb{R}$ we will denote by F^+ and F^- its boundary values:

$$F^+(x)=F(x+i0),\qquad F^-(x)=F(x-i0).$$

Then we have

$$\text{on } \mathfrak{X}_{\rm in}\qquad \frac{1}{\psi_{\rm in}}\,\frac{S^-_{\rm out}-S^+_{\rm out}}{2\pi i}=R_{\rm in}\,,\quad \frac{1}{\psi_{\rm in}}\,\frac{R^-_{\rm out}-R^+_{\rm out}}{2\pi i}=S_{\rm in}\,,$$
$$\text{on } \mathfrak{X}_{\rm out}\qquad \frac{1}{\psi_{\rm out}}\,\frac{S^-_{\rm in}-S^+_{\rm in}}{2\pi i}=R_{\rm out}\,,\quad \frac{1}{\psi_{\rm out}}\,\frac{R^-_{\rm in}-R^+_{\rm in}}{2\pi i}=S_{\rm out}\,.$$

This can also be restated as follows. Let us form a matrix

$$m=\begin{bmatrix} R_{\rm out} & -S_{\rm in}\\ -S_{\rm out} & R_{\rm in}\end{bmatrix}.$$

Then the matrix m satisfies the jump relation $m_+=m_-v$ on $\mathfrak{X}$, where the jump matrix equals

$$v(x)=\begin{cases}\begin{pmatrix}1 & 2\pi i\,\psi_{\rm out}(x)\\ 0 & 1\end{pmatrix}, & x\in\mathfrak{X}_{\rm out},\\[2mm] \begin{pmatrix}1 & 0\\ 2\pi i\,\psi_{\rm in}(x) & 1\end{pmatrix}, & x\in\mathfrak{X}_{\rm in}.\end{cases}$$

Furthermore, if $\Sigma>0$ then $m(\zeta)\sim I$ as $\zeta\to\infty$.

Differential equations. We use Riemann's notation

$$P\begin{pmatrix} t_1 & t_2 & t_3 & \\ a & b & c & \zeta\\ a' & b' & c' & \end{pmatrix}$$

to denote the two-dimensional space of solutions to the second order Fuchs' equation with singular points t_1,t_2,t_3 and exponents a,a'; b,b'; c,c'; see, e.g., [Er, Vol. 1, 2.6].

We have

$$R_{\text{out}}(x) \in P\begin{pmatrix} -\frac12 & \infty & \frac12 \\ w & 0 & z\;\; x \\ w' & 1-\Sigma & z' \end{pmatrix}, \quad S_{\text{out}}(x) \in P\begin{pmatrix} -\frac12 & \infty & \frac12 \\ w & 1 & z\;\; x \\ w' & -\Sigma & z' \end{pmatrix},$$

$$R_{\text{in}}(x) \in P\begin{pmatrix} -\frac12 & \infty & \frac12 \\ -w' & 0 & -z'\;\; x \\ -w & 1+\Sigma & -z \end{pmatrix}, \quad S_{\text{in}} \in P\begin{pmatrix} -\frac12 & \infty & \frac12 \\ -w' & 1 & -z'\;\; x \\ -w & \Sigma & -z \end{pmatrix}.$$

The resolvent kernel. There exists a limit

$$L(x,y) = \lim_{N\to\infty} N\cdot L^{(N)}(x_N, y_N)\,, \quad x,y\in\mathfrak{X}.$$

In the block form corresponding to the splitting $\mathfrak{X} = \mathfrak{X}_{\text{out}} \sqcup \mathfrak{X}_{\text{in}}$, the kernel $L(x,y)$ looks as follows:

$$L = \begin{bmatrix} 0 & \mathcal{A} \\ -\mathcal{A}^* & 0 \end{bmatrix},$$

where $\mathcal{A}$ is a kernel on $\mathfrak{X}_{\text{out}} \times \mathfrak{X}_{\text{in}}$ of the form

$$\mathcal{A}(x,y) = \frac{\sqrt{\psi_{\text{out}}(x)\psi_{\text{in}}(y)}}{x-y}.$$

This kernel defines a bounded operator in $L^2(\mathfrak{X}, dx)$ if and only if $|z+z'|<1$ and $|w+w'|<1$. If, in addition, we know that $\Sigma>0$ then we can prove that $L = K/(1-K)$ or $K = L/(1+L)$ as bounded operators in $L^2(\mathfrak{X}, dx)$.

11. Integral parameters z and w

If one of the parameters z, z' and one of the parameters w, w' are integral then the measure P_N defined in Section 3 is concentrated on a finite set of signatures, and there is a somewhat simpler way to compute the correlation kernel of $\mathcal{P}$.

Let us assume that $z=k$ and $w=l$, where $k,l\in\mathbb{Z}$, $k+l\ge 1$. Then (z,z',w,w') forms an admissible quadruple of parameters (see Definition 3.4) if z' and w' are real and $z'-k>-1$, $w'-l>-1$. We excluded the case $k+l=0$ from our consideration because in this case the measure P_N is concentrated on one signature.

It is easily seen from the definition of P_N that the measure P_N is now concentrated on the signatures $\lambda\in\mathbb{GT}_N$ such that

$$k \ge \lambda_1 \ge \cdots \ge \lambda_N \ge -l.$$

Note that it may happen that this set does not include the zero signature because k and $-l$ can be of the same sign.

From now on in this section we will assume that λ satisfies the inequalities above. Denote

$$\mathfrak{X}_{k,l}^{(N)} = \left\{-\frac{N-1}{2} - l, \dots, \frac{N-1}{2} + k\right\}.$$

Then

$$\mathcal{L}(\lambda) = \left\{\lambda_1 - 1 + \frac{N+1}{2}, \dots, \lambda_N - N + \frac{N+1}{2}\right\} \subset \mathfrak{X}_{k,l}^{(N)}.$$

Let us associate to λ a point configuration $Y(\lambda)$ in $\mathfrak{X}^{(N)}$ as follows

$$Y(\lambda) = \mathfrak{X}_{k,l}^{(N)} \setminus \mathcal{L}(\lambda).$$

Note that $Y(\lambda)$ defines λ uniquely. Since $|\mathcal{L}(\lambda)| = N$, we have $|Y(\lambda)| = k + l$. Let

$$Y(\lambda) = \{y_1, \dots, y_{k+l}\}.$$

The configuration $Y(\lambda)$ coincides with the configuration $X(\lambda) = \mathcal{L}(\lambda)^{\Delta}$ from (4.1) on the set $\mathfrak{X}_{\text{in}}^{(N)} \cap \mathfrak{X}_{k,l}^{(N)}$.

Proposition 11.1. *Let $z = k$, $w = l$ be integers, $k+l \geq 1$, and $z' > k-1$, $w' > l-1$ be real numbers. Then*

$$P_N(\lambda) = \text{const} \prod_{i=1}^{k+l} \frac{\Gamma(-y_i + z' + \frac{N+1}{2})\Gamma(y_i + w' + \frac{N+1}{2})}{\Gamma(-y_i + k + \frac{N+1}{2})\Gamma(y_i + l + \frac{N+1}{2})} \prod_{1 \leq i < j \leq k+l} (y_i - y_j)^2.$$

Remark 11.2. Note that for any integer n the shift

$$k \mapsto k + n, \quad l \mapsto l - n, \quad z' \mapsto z' + n, \quad w' \mapsto w' - n, \quad y \mapsto y + n$$

leaves the measure P_N invariant; cf. Remark 3.7. This means that essentially the measure depends on three, not four, parameters. If we now set $l = 0$ then λ can be viewed as a Young diagram. Then one can show that $Y(\lambda) = \{\frac{N-1}{2} - \lambda'_j + j\}_{j=1}^{k}$, where λ' is the transposed diagram.

Proof of Proposition 11.1. Set $x_i = \lambda_i - i + \frac{N+1}{2}$. Then by Proposition 6.1

$$P_N(\lambda) = \text{const} \prod_{i=1}^{N} f(x_i) \prod_{1 \leq i < j \leq N} (x_i - x_j)^2.$$

Now, since $\{y_i\}_{i=1}^{k+l} = \mathfrak{X}_{k,l}^{(N)} \setminus \{x_i\}_{i=1}^{N}$, similarly to Proposition 5.7 we get

$$P_N(\lambda) = \text{const} \prod_{i=1}^{N} h(y_i) \prod_{1 \leq i < j \leq k+l} (y_i - y_j)^2,$$

where

$$h(y) = \frac{1}{f(y) \prod\limits_{x \in \mathfrak{X}_{k,l}^{(N)} \setminus y} (y-x)^2} .$$

Substituting

$$\prod_{x \in \mathfrak{X}_{k,l}^{(N)} \setminus y} (y-x)^2 = \Gamma^2\left(-y+k+\frac{N+1}{2}\right)\Gamma^2\left(y+l+\frac{N+1}{2}\right)$$

and $f(x)$ from (6.1) we see that

$$h(y) = \frac{\Gamma(-y+z'+\frac{N+1}{2})\Gamma(y+w'+\frac{N+1}{2})}{\Gamma(-y+k+\frac{N+1}{2})\Gamma(y+l+\frac{N+1}{2})} . \qquad \square$$

Denote by $\mathcal{P}_{k,l}^{(N)}$ the point process consisting of the measure $P_N(\lambda)$ on point configurations $Y(\lambda)$.

Below we will be using Hahn polynomials. These are classical orthogonal polynomials on a finite set, and we will follow the notation of [NSU].

Proposition 11.3. *For any $n = 1, 2, \ldots$, the n^{th} correlation function of the process $\mathcal{P}_{k,l}^{(N)}$ has the form*

$$\rho_n^{(N)}(y_1, \ldots, y_n) = \det[K_{k,l}^{(N)}(y_i, y_j)]_{i,j=1}^n .$$

$K_{k,l}^{(N)}$ is the normalized Christoffel-Darboux kernel for shifted Hahn polynomials defined as follows:

$$K_{k,l}^{(N)}(x,y) = \frac{A_{m-1}}{A_m H_{m-1}} \frac{\mathfrak{P}_m(x)\mathfrak{P}_{m-1}(y) - \mathfrak{P}_{m-1}(x)\mathfrak{P}_m(y)}{x-y} \sqrt{h(x)h(y)} ,$$

where $m = k+l$, $h(x)$ is as above,

$$\mathfrak{P}_m(x) = h_m^{(z'-k, w'-l)}\left(x+l+\frac{N-1}{2}, m+N\right),$$
$$\mathfrak{P}_{m-1}(x) = h_{m-1}^{(z'-k, w'-l)}\left(x+l+\frac{N-1}{2}, m+N\right)$$

are Hahn polynomials,

$$H_{m-1} = \left\| h_{m-1}^{(z'-k,w'-l)}(x, m+N) \right\|^2 ,$$

and the numbers A_{m-1}, A_m are the leading coefficients of $h_{m-1}^{(z'-k,w'-l)}(x, m+N)$ and $h_m^{(z'-k,w'-l)}(x, m+N)$.

Proof. Note that if we shift our phase space $\mathfrak{X}_{k,l}^{(N)}$ by $l+\frac{N-1}{2}$ then the weight function turns into the function

$$h\left(y-l-\frac{N-1}{2}\right)=\frac{\Gamma(-y+l+z'+N)\Gamma(y-l+w'+1)}{\Gamma(-y+k+l+N)\Gamma(y+1)}$$

on the space

$$\mathfrak{X}_{k,l}^{(N)}+l+\frac{N-1}{2}=\{0,1,\dots,k+l+N-1\}.$$

But this is exactly the weight function for the Hahn polynomials

$$h_n^{(z'-k,w'-l)}(y,k+l+N),\quad n=0,1,2,\dots;$$

see [NSU, 2.4]. Then the claim follows from Proposition 5.1. □

Explicit formulas for the Hahn polynomials and their data can be found in [NSU].

We know that the processes $\mathcal{P}^{(N)}$ and $\mathcal{P}_{k,l}^{(N)}$ restricted to the set $\mathfrak{X}_{\text{in}}^{(N)}\cap\mathfrak{X}_{k,l}^{(N)}$ coincide by construction. The same is true for the correlation kernels, but it is not obvious (recall that the correlation kernel of a determinantal point process is not defined uniquely see Section 5(b)).

PROPOSITION 11.4. *For any $x,y\in\mathfrak{X}_{\text{in}}^{(N)}\cap\mathfrak{X}_{k,l}^{(N)}$,*

$$K_{k,l}^{(N)}(x,y)=K_{\text{in,in}}^{(N)}(x,y).$$

Proof. This follows from the relations (here $x\in\mathfrak{X}_{k,l}^{(N)}$)

$$\begin{aligned}\mathfrak{P}_m(x)\sqrt{h(x)}&=(-1)^{x-k-\frac{N-1}{2}}\mathfrak{p}_{N-1}(x)\sqrt{f(x)},\\ \mathfrak{P}_{m-1}(x)\sqrt{h(x)}&=(-1)^{x-k-\frac{N-1}{2}}\mathfrak{p}_{N}(x)\sqrt{f(x)},\\ A_{m-1}=h_N,\quad A_m&=h_{N-1},\quad H_{m-1}=h_N.\end{aligned}$$

(The polynomials $\mathfrak{p}_{N-1}(x),\mathfrak{p}_N(x)$ were introduced in (7.1).)

These relations can be proved either by a direct verification using explicit formulas (which is rather tedious), or they can be deduced from the following general fact.

LEMMA 11.5 ([B5]). *Let $\mathcal{X}=\{x_0,x_1,\dots,x_M\}$ be a finite set of distinct points on the real line, $u(x)$ and $v(x)$ be two positive functions on $\mathcal{X}$ such that*

$$u(x_k)v(x_k)=\frac{1}{\prod_{i\neq k}(x_k-x_i)^2},\quad k=0,1,\dots,M,$$

and $P_0,P_1,\dots,P_M$ and $Q_0,Q_1,\dots,Q_M$ be the systems of orthogonal polynomials on $\mathcal{X}$ with respect to the weights $u(x)$ and $v(x)$, respectively,

$$\begin{aligned}\deg P_i=\deg Q_i=i,\quad \|P_i\|^2=p_i,\quad \|Q_i\|^2=q_i,\\ P_i=a_ix^i+\text{lower terms},\quad Q_i=b_ix^i+\text{lower terms}.\end{aligned}$$

Assume that the polynomials are normalized so that $p_i = q_{M-i}$ *for all* $i = 0, 1, \dots, M$.

Then

$$P_i(x)\sqrt{u(x)} = \epsilon(x) Q_{M-i}(x)\sqrt{v(x)}, \quad x \in \mathcal{X},$$
$$a_i b_{M-i} = p_i = q_{M-i}, \quad i = 0, 1, \dots, M,$$

where

$$\epsilon(x_k) = \operatorname{sgn} \prod_{i \neq k} (x_k - x_i), \quad k = 0, 1, \dots, M.$$

Taking

$$M = N + m - 1, \quad \mathcal{X} = \mathfrak{X}_{k,l}^{(N)}, \quad u(x) = f(x), \quad v(x) = h(x)$$

we get the needed formulas. The proof of Proposition 11.4 is complete. □

THEOREM 11.6. *Assume* $z = k$ *and* $w = l$ *are integers,* $k + l \geq 1$, z' *and* w' *are real numbers such that* $z' - k > -1$, $w' - l > -1$. *Then the correlation kernel of the process* $\mathcal{P}$ *vanishes if at least one of the arguments is in* $\mathfrak{X}_{\mathrm{out}}$, *and on* $\mathfrak{X}_{\mathrm{in}} \times \mathfrak{X}_{\mathrm{in}}$ *it is equal to the normalized* $(k+l)^{\mathrm{th}}$ *Christoffel-Darboux kernel for the Jacobi polynomials on* $\left(-\frac{1}{2}, \frac{1}{2}\right)$ *with the weight function* $(\frac{1}{2} - x)^{z'-k}(\frac{1}{2} + x)^{w'-l}$.

Proof. One way to prove this statement is to substitute integrals z and w into the formulas of Theorem 10.1. A simpler way, however, is to use the asymptotic relation

$$\frac{1}{M^n}\, h_n^{\alpha,\beta}\left(\left[\frac{M(1+s)}{2}\right], M\right) = P_n^{(\alpha,\beta)}(s) + \mathrm{O}(M^{-1}), \quad M \to \infty,$$

where $P_n^{(\alpha,\beta)}$ is the n^{th} Jacobi polynomial with parameters (α, β); see, e.g., [NSU, (2.6.3)]. The estimate is uniform in s belonging to any compact set inside $(-1, 1)$. It is not hard to see that the weight function $h(y)$ as well as the constants H_{m-1}, A_{m-1}, A_m, see above, converge to the weight function and the corresponding constants for the Jacobi polynomials. Then Theorem 9.2 and Proposition 11.3 imply the claim. □

Appendix

The hypergeometric series ${}_3F_2$ evaluated at the unity viewed as a function of parameters has a large number of two and three-term relations. A lot of them were discovered by J. Thomae back in 1879. In 1923, F. J. W. Whipple introduced a notation which provided a clue to the numerous formulas obtained by Thomae. An excellent exposition of Whipple's work was given by W. N. Bailey in [Ba, Ch. 3]. We will be using the notation of [Ba] below.

Proof of (7.6). The formula (7.6) coincides with the relation

$$Fp(0;4,5)=Fp(0;1,5);$$

see [Ba, 3.5, 3.6].

Proof of Lemma 8.2. The right-hand side of (8.6) contains two ${}_3F_2$'s. We will use appropriate transformation formulas to rewrite both of them.

For the first one we employ the relation

$$\frac{\sin\pi\beta_{14}}{\pi\Gamma(\alpha_{014})}\,Fp(0)=\frac{Fn(1)}{\Gamma(\alpha_{234})\Gamma(\alpha_{245})\Gamma(\alpha_{345})}-\frac{Fn(4)}{\Gamma(\alpha_{123})\Gamma(\alpha_{125})\Gamma(\alpha_{135})}\,, \tag{A.1}$$

which is [Ba, 3.7(1)] with the indices 1 and 2 interchanged, and 3 and 4 interchanged.

By [Ba, 3.5, 3.6], we have

$$\begin{aligned} Fp(0)&=Fp(0;4,5)\\ &=\frac{1}{\Gamma(\alpha_{123})\Gamma(\beta_{40})\Gamma(\beta_{50})}\,{}_3F_2\left[\begin{matrix}\alpha_{145},\,\alpha_{245},\,\alpha_{345};\\ \beta_{40},\,\beta_{50}\end{matrix}\right]\\ &=\frac{1}{\Gamma(s)\Gamma(e)\Gamma(f)}\,{}_3F_2\left[\begin{matrix}a,\,b,\,c;\\ e,\,f\end{matrix}\right],\end{aligned}$$

$$\begin{aligned} Fn(1)&=Fn(1;2,4)=\frac{1}{\Gamma(\alpha_{124})\Gamma(\beta_{12})\Gamma(\beta_{14})}\,{}_3F_2\left[\begin{matrix}\alpha_{135},\,\alpha_{013},\,\alpha_{015};\\ \beta_{12},\,\beta_{14}\end{matrix}\right]\\ &=\frac{1}{\Gamma(e-c)\Gamma(1+a-b)\Gamma(1-b-c+f)}\,{}_3F_2\left[\begin{matrix}f-b,\,1-b,\,1-e+a;\\ 1+a-b,\,1-b-c+f\end{matrix}\right],\end{aligned}$$

$$\begin{aligned} Fn(4)&=Fp(4;1,2)=\frac{1}{\Gamma(\alpha_{124})\Gamma(\beta_{41})\Gamma(\beta_{42})}\,{}_3F_2\left[\begin{matrix}\alpha_{034},\,\alpha_{045},\,\alpha_{345};\\ \beta_{41},\,\beta_{42}\end{matrix}\right]\\ &=\frac{1}{\Gamma(e-c)\Gamma(1+b+c-f)\Gamma(1+a+c-f)}\,{}_3F_2\left[\begin{matrix}1-f+c,\,1-s,\,c;\\ 1+b+c-f,\,1+a+c-f\end{matrix}\right],\end{aligned}$$

where $s=e+f-a-b-c$. Thus, (A.1) takes the form

$$\begin{aligned} &{}_3F_2\left[\begin{matrix}a,\,b,\,c;\\ e,\,f\end{matrix}\right]\\ &\quad=\Gamma\left[\begin{matrix}1-f+a,\,s,\,e,\,f,\,b+c-f\\ e-a,\,b,\,c,\,e-c,\,1+a-b\end{matrix}\right]{}_3F_2\left[\begin{matrix}f-b,\,1-b,\,1-e+a;\\ 1+a-b,\,1-b-c+f\end{matrix}\right]\\ &\qquad+\Gamma\left[\begin{matrix}1-f+a,\,e,\,f,\,-b-c+f\\ f-c,\,f-b,\,e-c,\,1+a+c-f\end{matrix}\right]{}_3F_2\left[\begin{matrix}1-f+c,\,1-s,\,c;\\ 1+b+c-f,\,1+a+c-f\end{matrix}\right],\end{aligned}$$

where we used the identity

$$\frac{\pi}{\sin\pi(1-b-c+f)}=\Gamma(1-b-c+f)\Gamma(b+c-f)=-\Gamma(-b-c+f)\Gamma(1+b+c-f).$$

Now set

$$a=N,\ b=-z-w',\ c=-z-w,\ e=u-z+\frac{N+1}{2},\ f=-z-z'-w-w',$$

$$s=e+f-a-b-c=u-z'-\frac{N-1}{2}.$$

Also, recall the notation $\Sigma=z+z'+w+w'$. Multiplying the last relation by

$$\Gamma\left[\begin{array}{c} u+\frac{N+1}{2},\ u-z-\frac{N-1}{2} \\ u-\frac{N-1}{2},\ u-z+\frac{N+1}{2} \end{array}\right]$$

we get

(A.2)

$$\Gamma\left[\begin{array}{c} u+\frac{N+1}{2},\ u-z-\frac{N-1}{2} \\ u-\frac{N-1}{2},\ u-z+\frac{N+1}{2} \end{array}\right] {}_3F_2\left[\begin{array}{c} N,\ -z-w',\ -z-w; \\ u-z+\frac{N+1}{2},\ -\Sigma \end{array}\right]$$

$$=\Gamma\left[\begin{array}{c} u+\frac{N+1}{2},\ 1+N+\Sigma,\ u-z'-\frac{N-1}{2},\ -\Sigma,\ -z+z' \\ u-\frac{N-1}{2},\ -z-w',\ -z-w,\ u+w+\frac{N+1}{2},\ 1+N+z+w' \end{array}\right]$$

$$\times {}_3F_2\left[\begin{array}{c} -z'-w,\ z+w'+1,\ -u+z+\frac{N+1}{2}; \\ 1+N+z+w',\ 1+z-z' \end{array}\right]$$

$$+\big\{\text{a similar expression with } z \text{ and } z' \text{ interchanged}\big\}.$$

This is the transformation for the first term in (8.6).

As for the second term, we use [Ba, 3.2(2)] which reads

$${}_3F_2\left[\begin{array}{c} a,\ b,\ c; \\ e,\ f \end{array}\right]=\Gamma\left[\begin{array}{c} 1-a,\ e,\ f,\ c-b \\ e-b,\ f-b,\ 1+b-a,\ c \end{array}\right] {}_3F_2\left[\begin{array}{c} b,\ b-e+1,\ b-f+1; \\ 1+b-c,\ 1+b-a \end{array}\right]$$

$$+\big\{\text{a similar expression with } b \text{ and } c \text{ interchanged}\big\}.$$

Set

$$a=-N+1,\ b=z+w'+1,\ c=z'+w'+1,\ e=\Sigma+2,\ f=u+w'-\frac{N-3}{2}.$$

We get

$${}_3F_2\left[\begin{array}{c} -N+1,\ z+w'+1,\ z'+w'+1; \\ \Sigma+2,\ u+w'-\frac{N-3}{2} \end{array}\right]$$

$$=\Gamma\left[\begin{array}{c} N,\ \Sigma+2,\ u+w'-\frac{N-3}{2},\ z'-z \\ z'+w+1,\ u-z-\frac{N-1}{2},\ N+1+z+w',\ z'+w'+1 \end{array}\right]$$

$$\times {}_3F_2\left[\begin{array}{c} z+w'+1,\ -z'-w,\ -u+z+\frac{N+1}{2}; \\ 1+z-z',\ 1+N+z+w' \end{array}\right]$$

$$+\big\{\text{a similar expression with } z \text{ and } z' \text{ interchanged}\big\}.$$

Let us multiply this by the prefactor of the hypergeometric function in the second term of (8.6). Recalling the formula for $h_{N-1}=h(N-1,z,z',w,w')$, see (7.2), and canceling some gamma factors we find that the second term of (8.6) equals

(A.3)

$$\Gamma\left[\begin{matrix} u+\frac{N+1}{2},\, -u+\frac{N+1}{2},\, 1+N+\Sigma,\, z+w+1,\, z+w'+1,\, z'-z \\ -u+z'+\frac{N+1}{2},\, u+w+\frac{N+1}{2},\, \Sigma+1,\, 1+N+z+w' \end{matrix}\right]$$

$$\times\frac{\sin\pi(u-z-\frac{N-1}{2})}{\pi}\, F(u)\, {}_3F_2\left[\begin{matrix} z+w'+1,\, -z'-w,\, -u+z+\frac{N+1}{2}; \\ 1+z-z',\, 1+N+z+w' \end{matrix}\right]$$

$$+\big\{\text{a similar expression with } z \text{ and } z' \text{ interchanged}\big\},$$

where we also used the identity

$$\Gamma(u-z-\tfrac{N-1}{2})\Gamma(-u+z+\tfrac{N+1}{2}) = \frac{\pi}{\sin\pi(u-z-\frac{N-1}{2})}.$$

(Recall that $F(u)$ was defined right before Lemma 8.2.)

Now, in order to get (8.6) we have to add (A.2) and (A.3). Since both expressions have two parts with the second parts different from the first parts by switching z and z', it suffices to transform the sum of the first parts. We immediately see that the hypergeometric functions entering the first parts of (A.2) and (A.3) are identical. By factoring out the ${}_3F_2$'s and some of the gamma factors, and using the identity $\Gamma(\tau)\Gamma(1-\tau)=\pi/\sin\pi\tau$ several times, we see that the sum of the first parts of (A.2) and (A.3) equals

$$\Gamma\left[\begin{matrix} u+\frac{N+1}{2},\, -u+\frac{N+1}{2},\, 1+N+\Sigma,\, z+w+1,\, z+w'+1,\, z'-z \\ -u+z'+\frac{N+1}{2},\, u+w+\frac{N+1}{2},\, \Sigma+1,\, 1+N+z+w' \end{matrix}\right]$$

$$\times\, {}_3F_2\left[\begin{matrix} z+w'+1,\, -z'-w,\, -u+z+\frac{N+1}{2}; \\ 1+z-z',\, 1+N+z+w' \end{matrix}\right]$$

multiplied by

(A.4)

$$\frac{1}{\pi}\left(-\frac{\sin\pi(u-\frac{N-1}{2})\sin\pi(z+w')\sin\pi(z+w)}{\sin\pi(u-z'-\frac{N-1}{2})\sin\pi\Sigma}+\sin\pi\left(u-z-\tfrac{N-1}{2}\right)\,F(u)\right).$$

Observe that $F(u)$ as a function in u is a linear combination of $1/\sin\pi(-u+z+\frac{N+1}{2})$ and $1/\sin\pi(-u+z'+\frac{N+1}{2})$. Thus, (A.4) is a meromorphic function. It is easily verified that all the singularities of (A.4) are removable and (A.4) is an entire function. Moreover, since the ratios $\sin(u+\alpha)/\sin(u+\beta)$ are periodic with period 2π and are bounded as $\Im u\to\pm\infty$ (for arbitrary $\alpha,\beta\in\mathbb{C}$), $F(u)$ is bounded on the entire complex plane. By

Liouville's theorem, $F(u)$ does not depend on u. Substituting $u=\frac{N-1}{2}$ we see that (A.4) is equal to $-\sin\pi z/\pi$.

This immediately implies that the sum of (A.2) and (A.3) is equal to the right-hand side of (8.4), and the first part of Lemma 8.2 is proved.

Now let us look at the formula (8.7). Note that the hypergeometric functions in (8.7) can be obtained from those in (8.6) by the following shift:

$$N\mapsto N+1,\quad z\mapsto z-\frac{1}{2},\quad z'\mapsto z'-\frac{1}{2},\quad w\mapsto w-\frac{1}{2},\quad w'\mapsto w'-\frac{1}{2}.$$

We use for them exactly the same transformation formulas used for (8.6). By computations very similar to the above, we find that the first term of (8.7) is equal to

(A.5)

$$-\Gamma\begin{bmatrix} u+\frac{N+1}{2},\, -u+\frac{N+1}{2},\, N+1,\, \Sigma,\, z'-z \\ -u+z'+\frac{N-1}{2},\, u+w+\frac{N+1}{2},\, N+1+z+w',\, z'+w,\, z'+w' \end{bmatrix}$$

$$\times\frac{\sin\pi(u-\frac{N-1}{2})\sin\pi(z+w')\sin\pi(z+w)}{\pi\sin\pi(u-z'-\frac{N-1}{2})\sin\pi\Sigma}$$

$$\times{}_3F_2\begin{bmatrix} -z'-w+1,\, z+w',\, -u+z+\frac{N+1}{2}; \\ N+1+z+w',\, 1+z-z' \end{bmatrix}$$

$$-\big\{\text{a similar expression with } z \text{ and } z' \text{ interchanged}\big\},$$

while the second term of (8.7) is equal to

(A.6)

$$\Gamma\begin{bmatrix} u+\frac{N+1}{2},\, -u+\frac{N+1}{2},\, N+1,\, \Sigma,\, z'-z \\ -u+z'+\frac{N-1}{2},\, u+w+\frac{N+1}{2},\, N+1+z+w',\, z'+w,\, z'+w' \end{bmatrix}$$

$$\times\frac{\sin\pi(u-z-\frac{N-1}{2})}{\pi}\,F(u)\,{}_3F_2\begin{bmatrix} -z'-w+1,\, z+w',\, -u+z+\frac{N+1}{2}; \\ N+1+z+w',\, 1+z-z' \end{bmatrix}$$

$$+\big\{\text{a similar expression with } z \text{ and } z' \text{ interchanged}\big\},$$

Adding (A.5) and (A.6) and using the fact that (A.4) is equal to $-\sin\pi z/\pi$, we arrive at the right-hand side of (8.5). □

Proof of Lemma 8.4. We start by deriving a convenient transformation formula for ${}_3F_2$. [Ba, 3.7(6)] with indices 4 and 5 interchanged reads

$$\text{(A.7)}\quad \frac{\sin\pi\beta_{40}Fp(0)}{\pi\Gamma(\alpha_{045})\Gamma(\alpha_{034})\Gamma(\alpha_{024})\Gamma(\alpha_{014})}$$

$$=-\frac{Fn(0)}{\Gamma(\alpha_{345})\Gamma(\alpha_{245})\Gamma(\alpha_{145})\Gamma(\alpha_{134})\Gamma(\alpha_{234})\Gamma(\alpha_{124})}+K_0Fn(4),$$

where

$$K_0 = \frac{\sin\pi\alpha_{145}\sin\pi\alpha_{245}\sin\pi\alpha_{345} + \sin\pi\alpha_{123}\sin\pi\beta_{40}\sin\pi\beta_{50}}{\pi^3}.$$

By [Ba, 3.6] we have

$$\begin{aligned}
Fp(0) = Fp(0;4,5) &= \frac{1}{\Gamma(\alpha_{123})\Gamma(\beta_{40})\Gamma(\beta_{50})}\,{}_3F_2\left[\begin{matrix}\alpha_{145},\ \alpha_{245},\ \alpha_{345};\\ \beta_{40},\ \beta_{50}\end{matrix}\right]\\
&= \frac{1}{\Gamma(s)\Gamma(e)\Gamma(f)}\,{}_3F_2\left[\begin{matrix}a,\ b,\ c;\\ e,\ f\end{matrix}\right],\\
Fn(0) = Fn(0;4,5) &= \frac{1}{\Gamma(\alpha_{045})\Gamma(\beta_{04})\Gamma(\beta_{05})}\,{}_3F_2\left[\begin{matrix}\alpha_{023},\ \alpha_{013},\ \alpha_{012};\\ \beta_{04},\ \beta_{05}\end{matrix}\right]\\
&= \frac{1}{\Gamma(1-s)\Gamma(2-e)\Gamma(2-f)}\,{}_3F_2\left[\begin{matrix}1-a,\ 1-b,\ 1-c;\\ 2-e,\ 2-f\end{matrix}\right],\\
Fn(4) = Fn(4;0,3) &= \frac{1}{\Gamma(\alpha_{034})\Gamma(\beta_{40})\Gamma(\beta_{43})}\,{}_3F_2\left[\begin{matrix}\alpha_{124},\ \alpha_{145},\ \alpha_{245};\\ \beta_{40},\ \beta_{43}\end{matrix}\right]\\
&= \frac{1}{\Gamma(1-f+c)\Gamma(e)\Gamma(1+a+b-f)}\,{}_3F_2\left[\begin{matrix}e-c,\ a,\ b;\\ e,\ 1+a+b-f\end{matrix}\right],\\
K_0 &= \frac{\sin\pi a\sin\pi b\sin\pi c + \sin\pi s\sin\pi e\sin\pi f}{\pi^3},
\end{aligned}$$

where $s = e+f-a-b-c$. Then (A.7) takes the form

(A.8)

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}a,\ b,\ c;\\ e,\ f\end{matrix}\right]\\
&= -\frac{\pi}{\sin\pi e}\,\Gamma\left[\begin{matrix}1-f+c,\ 1-f+b,\ 1-f+a,\ s,\ e,\ f\\ c,\ b,\ a,\ e-b,\ e-a,\ e-c,\ 2-e,\ 2-f\end{matrix}\right]{}_3F_2\\
&\qquad\times\left[\begin{matrix}1-a,\ 1-b,\ 1-c;\\ 2-e,\ 2-f\end{matrix}\right]\\
&\quad+\frac{\pi K_0}{\sin\pi e}\,\Gamma\left[\begin{matrix}1-s,\ 1-f+b,\ 1-f+a,\ s,\ f\\ 1+a+b-f\end{matrix}\right]{}_3F_2\left[\begin{matrix}e-c,\ a,\ b;\\ e,\ 1+a+b-f\end{matrix}\right],
\end{aligned}$$

with K_0 as above.

Let us apply (A.8) to both hypergeometric functions in (8.6). For the first one we set

$$a = N,\quad b = -z-w',\quad c = -z-w,\quad e = -\Sigma,\quad f = u - z + \frac{N+1}{2},$$
$$s = e+f-a-b-c = u - z' - \frac{N-1}{2}.$$

Then $K_0 = -\sin\pi\Sigma \sin\pi(u-z'-\frac{N-1}{2})\sin\pi(u-z+\frac{N+1}{2})/\pi^3$,[11] and rewriting the sines in K_0 as products of gamma functions, we have

(A.9)

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} N,\, -z-w',\, -z-w; \\ u-z+\frac{N+1}{2},\, -\Sigma \end{matrix}\right] \\
&= \Gamma\left[\begin{matrix} -u-w-\frac{N-1}{2},\, -u-w'-\frac{N-1}{2},\, -u+z+\frac{N+1}{2},\, u-z'-\frac{N-1}{2},\, -\Sigma,\, u-z+\frac{N+1}{2} \\ -z-w,\, -z-w',\, N,\, -z'-w,\, -\Sigma-N,\, -z'-w',\, 2+\Sigma,\, -u+z-\frac{N-3}{2} \end{matrix}\right] \\
&\times \frac{\pi}{\sin\pi\Sigma}\, {}_3F_2\left[\begin{matrix} -N+1,\, z+w'+1,\, z+w+1; \\ \Sigma+2,\, -u+z-\frac{N-3}{2} \end{matrix}\right] \\
&+\Gamma\left[\begin{matrix} -u-w'-\frac{N-1}{2},\, -u+z+\frac{N+1}{2} \\ -u-w'+\frac{N+1}{2},\, -u+z-\frac{N-1}{2} \end{matrix}\right] {}_3F_2\left[\begin{matrix} -z'-w',\, N,\, -z-w'; \\ -\Sigma,\, -u-w'+\frac{N+1}{2} \end{matrix}\right].
\end{aligned}
$$

For the second one we set

$$
\begin{aligned}
&a=-N+1, \quad b=z+w'+1, \quad c=z'+w'+1, \quad e=\Sigma+2, \\
&f=u+w'-\frac{N-3}{2}, s=e+f-a-b-c=u+w+\frac{N+1}{2}.
\end{aligned}
$$

Then $K_0 = \sin\pi\Sigma\sin\pi(u+w+\frac{N+1}{2})\sin\pi(u+w'-\frac{N-3}{2})/\pi^3$. Observe that the first term on the right-hand side of (A.8) will now vanish thanks to the $\Gamma(a)$ in the denominator (remember that $N\in\{1,2,\dots\}$). Rewriting sines as products of gamma functions again, we get

(A.10)
$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} -N+1,\, z+w'+1,\, z'+w'+1; \\ \Sigma+2,\, u+w'-\frac{N-3}{2} \end{matrix}\right]
&= \Gamma\left[\begin{matrix} -u+z+\frac{N+1}{2},\, -u-w'-\frac{N-1}{2} \\ -u-w'+\frac{N-1}{2},\, -u+z-\frac{N-3}{2} \end{matrix}\right] \\
&\times {}_3F_2\left[\begin{matrix} z+w+1,\, -N+1,\, z+w'+1; \\ \Sigma+2,\, -u+z-\frac{N-3}{2} \end{matrix}\right].
\end{aligned}
$$

Now let us substitute (A.9) and (A.10) into (8.6). Observe that the two ${}_3F_2$'s from the right-hand sides of (A.9) and (A.10) are obtained from the ${}_3F_2$'s from (8.6) by the change

(A.11)
$$u\mapsto -u, \quad (z,z',w,w')\mapsto (w',w,z',z).$$

Our goal is to show that the prefactors of these ${}_3F_2$'s after substitution will be symmetric to the prefactors of ${}_3F_2$'s in (8.6) with respect to (A.11).

One prefactor is easy to handle. The coefficient of

$$
{}_3F_2\left[\begin{matrix} -z'-w',\, N,\, -z-w'; \\ -\Sigma,\, -u-w'+\frac{N+1}{2} \end{matrix}\right]
$$

[11] Here we used the facts that $N\in\mathbb{Z}$ and $\sin\pi a = \sin\pi N = 0$.

after the substitution of (A.9) and (A.10) into (8.6) equals

$$\Gamma\left[\begin{array}{c} u+\frac{N+1}{2},\, u-z-\frac{N-1}{2} \\ u-\frac{N-1}{2},\, u-z+\frac{N+1}{2}\end{array}\right]\Gamma\left[\begin{array}{c} -u-w'-\frac{N-1}{2},\, -u+z+\frac{N+1}{2} \\ -u-w'+\frac{N+1}{2},\, -u+z-\frac{N-1}{2}\end{array}\right]$$
$$=\Gamma\left[\begin{array}{c} -u+\frac{N+1}{2},\, -u+w'-\frac{N-1}{2} \\ -u-\frac{N-1}{2},\, -u+w'+\frac{N+1}{2}\end{array}\right],$$

which is symmetric to $\Gamma\left[\begin{array}{c} u+\frac{N+1}{2},\, u-z-\frac{N-1}{2} \\ u-\frac{N-1}{2},\, u-z+\frac{N+1}{2}\end{array}\right]$ with respect to (A.11).

As for the other prefactor, the verification is more involved. Namely, we need to prove the following equality:

$$\Gamma\left[\begin{array}{c} -u-w-\frac{N-1}{2},\, -u-w'-\frac{N-1}{2},\, -u+z+\frac{N+1}{2},\, u-z'-\frac{N-1}{2},\, -\Sigma,\, u-z+\frac{N+1}{2} \\ -z-w,\, -z-w',\, N,\, -z'-w,\, -\Sigma-N,\, -z'-w',\, 2+\Sigma,\, -u+z-\frac{N-3}{2}\end{array}\right]$$
$$\times\Gamma\left[\begin{array}{c} u+\frac{N+1}{2},\, u-z-\frac{N-1}{2} \\ u-\frac{N-1}{2},\, u-z+\frac{N+1}{2}\end{array}\right]\frac{\pi}{\sin\pi\Sigma}$$
$$+\Gamma\left[\begin{array}{c} u+\frac{N+1}{2},\, -u+\frac{N+1}{2} \\ -u+z+\frac{N+1}{2},\, -u+z'+\frac{N+1}{2},\, u+w+\frac{N+1}{2},\, u+w'-\frac{N-3}{2}\end{array}\right]$$
$$\times\Gamma\left[\begin{array}{c} -u+z+\frac{N+1}{2},\, -u-w'-\frac{N-1}{2} \\ -u-w'+\frac{N-1}{2},\, -u+z-\frac{N-3}{2}\end{array}\right]\frac{1}{h(N-1,z,z',w,w')}\,\frac{1}{\sin\pi\Sigma\,\sin\pi(z'-z)}$$
$$\times\left(\frac{\sin\pi(z+w)\sin\pi(z+w')\sin\pi z'}{\sin\pi(-u+z'+\frac{N+1}{2})}-\frac{\sin\pi(z'+w)\sin\pi(z'+w')\sin\pi z}{\sin\pi(-u+z+\frac{N+1}{2})}\right)$$
$$=\Gamma\left[\begin{array}{c} -u+\frac{N+1}{2},\, u+\frac{N+1}{2} \\ u+w'+\frac{N+1}{2},\, u+w+\frac{N+1}{2},\, -u+z'+\frac{N+1}{2},\, -u+z-\frac{N-3}{2}\end{array}\right]$$
$$\times\frac{1}{h(N-1,w',w,z',z)}\,\frac{1}{\sin\pi\Sigma\,\sin\pi(w-w')}$$
$$\times\left(\frac{\sin\pi(z'+w')\sin\pi(z+w')\sin\pi w}{\sin\pi(u+w+\frac{N+1}{2})}-\frac{\sin\pi(z'+w)\sin\pi(z+w)\sin\pi w'}{\sin\pi(u+w'+\frac{N+1}{2})}\right).$$

After massive cancellations[12] this equality is reduced to the following trigonometric identity (here $y=u-\frac{N-1}{2}$):

$$\frac{\sin\pi y\sin\pi(z+w)\sin\pi(z'+w)\sin\pi(z+w')\sin\pi(z'+w')}{\sin\pi(y-z)\sin\pi(y-z')\sin\pi(y+w)\sin\pi(y+w')}$$
$$=\frac{\sin\pi(z'+w)\sin\pi(z'+w')\sin\pi z}{\sin\pi(z-z')\sin\pi(y-z)}+\frac{\sin\pi(z+w)\sin\pi(z+w')\sin\pi z'}{\sin\pi(z'-z)\sin\pi(y-z')}$$
$$+\frac{\sin\pi(z+w')\sin\pi(z'+w')\sin\pi w}{\sin\pi(w-w')\sin\pi(y+w)}+\frac{\sin\pi(z+w)\sin\pi(z'+w)\sin\pi w'}{\sin\pi(w'-w)\sin\pi(y+w')}.$$

[12]The cancellations also rely on the fact that $N\in\mathbb{Z}$; cf. the previous footnote.

One way to prove this identity is to view both sides as meromorphic functions in y. Then it is easily verified that the difference of the left-hand side and the right-hand side is an entire function. Moreover, both sides are periodic with period 2π and bounded for $|\Im y|$ large enough. This implies that both sides are identically equal. The proof of Lemma 8.4 is complete.

On the proof of Lemma 8.5. This proof is very similar to that of Lemma 8.4 above. The needed transformation formulas for the ${}_3F_2$'s are obtained from (A.9) and (A.10) by the shift $(N, z, z', w, w') \mapsto (N+1, z-\frac{1}{2}, z'-\frac{1}{2}, w-\frac{1}{2}, w'-\frac{1}{2})$. After substituting the resulting expressions into (8.7) we collect the coefficients of ${}_3F_2$'s and compare them with what we want. As in the proof of Lemma 8.4, one of the desired equalities follows immediately, while the other is reduced to the same trigonometric identity.

On analytic continuation of the series ${}_3F_2(1)$. Here we prove that the function

$$\frac{1}{\Gamma(e)\Gamma(f)\Gamma(e+f-a-b-c)}\,{}_3F_2\left[\begin{matrix}a,\, b,\, c\\ e,\, f\end{matrix}\,\middle|\,1\right]$$

can be analytically continued to an entire function in five complex variables a, b, c, e, f. We stated this claim in the beginning of Section 7 and used it in Section 8.

Apply the transformation formula

$${}_3F_2\left[\begin{matrix}a,\, b,\, c\\ e,\, f\end{matrix}\,\middle|\,1\right] = \Gamma\left[\begin{matrix}e,\, f,\, s\\ a,\, s+b,\, s+c\end{matrix}\right]{}_3F_2\left[\begin{matrix}e-a,\, f-a,\, s\\ s+b,\, s+c\end{matrix}\,\middle|\,1\right],$$

where $s = e+f-a-b-c$. This allows us to conclude that the function in question continues to the domain $\Re(a) > 0$ (other parameters being arbitrary).

Likewise, we can continue to the domain $\Re(b) > 0$ and also to the domain $\Re(c) > 0$. Then one can apply a general theorem about 'forced' analytic continuation of holomorphic functions on tube domains: see, e.g., [H, Th. 2.5.10].

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