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A cornucopia of isospectral pairs of metrics on spheres with different local geometries

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Abstract

This article concludes the comprehensive study started in [Sz5], where the first nontrivial isospectral pairs of metrics are constructed on balls and spheres. These investigations incorporate four different cases since these balls and spheres are considered both on 2-step nilpotent Lie groups and on their solvable extensions. In [Sz5] the considerations are completely concluded in the ball-case and in the nilpotent-case. The other cases were mostly outlined. In this paper the isospectrality theorems are completely established on spheres. Also the important details required about the solvable extensions are concluded in this paper.

A new so called *anticommutator technique* is developed for these constructions. This tool is completely different from the other methods applied on the field so far. It brought a wide range of new isospectrality examples. Those constructed on geodesic spheres of certain harmonic manifolds are particularly striking. One of these spheres is homogeneous (since it is the geodesic sphere of a 2-point homogeneous space) while the other spheres, although isospectral to the previous one, are not even locally homogeneous. This shows that how little information is encoded about the geometry (for instance, about the isometries) in the spectrum of Laplacian acting on functions.

Research in spectral geometry started out in the early 60's. This field might as well be called audible versus nonaudible geometry. This designation much more readily suggests the fundamental question of the field: To what extent is the geometry of compact Riemann manifolds encoded in the spectrum of the Laplacian acting on functions?

It started booming in the 80's, however, all the isospectral metrics constructed until the early 90's had the same local geometry and they differed from each other only by their global invariants, such as fundamental groups.

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Then, in 1993, the first examples of isospectral pairs of metrics with different local geometries were constructed both on closed manifolds [G1] and on manifolds with boundaries [Sz3], [Sz4]. Gordon established her examples on closed nil-manifolds (which were diffeomorphic to tori) while this author performed his constructions on topologically trivial principal torus bundles over balls, i.e., on $B^m \times T^3$. The boundaries of the latter manifolds are the torus bundles $S^{m-1} \times T^3$. The isospectrality proofs are completely different in these two cases. On manifolds with boundaries the proof was based on an explicit computation of the spectrum.The main tool in these computations was the Fourier-Weierstrass decomposition of the L^2-function space on the torus fibres T^3_p.

The results of this author were first announced during the San Antonio AMS Meeting, which was held January 13–16, 1993 (cf. *Notices of AMS*, Dec. 1992, vol 39(10), p. 1245) and, thereafter, in several seminar talks given at the University of Pennsylvania, Rutgers University and at the Spectral Geometry Festival held at MSRI(Berkeley), in November, 1993. It was circulated in preprint form but it was published much later [Sz4]. The later publication includes new materials, such as establishment of the isospectrality theorem on the boundaries $S^{m-1} \times T^3$ of the considered manifolds as well.

The author's construction strongly related to the Lichnerowicz conjecture (1946) concerning harmonic manifolds. This connection is strongly present also in this paper since the striking examples offered below also relate to the conjecture.

A *Riemann manifold* is said to be *harmonic* if its harmonic functions yield the classical mean value theorem. One can easily establish this harmonicity on two-point homogeneous manifolds. The conjecture claims this statement also in the opposite direction: The harmonic manifolds are exactly the two-point homogeneous spaces.

The conjecture was established on compact, simply connected manifolds by this author [Sz1], in 1990. Then, in 1991, Damek and Ricci [DR] found infinitely many counterexamples for the conjecture in the noncompact case by proving that the natural left-invariant metrics on the solvable extensions of Heisenberg-type groups are harmonic. The Heisenberg-type groups are particular 2-step nilpotent groups attached to Clifford modules (i.e., to representations of Clifford algebras) [K]. Among them are the groups $H_3^{(a,b)}$ defined by imaginary quaternionic numbers (cf. (2.13) and below).

In constructing the isospectrality examples described in [Sz3], [Sz4], the center $\mathbf{R}^3$ of these groups was factorized by a full lattice Γ to obtain the torus $T^3 = \Gamma\backslash\mathbf{R}^3$ and the torus bundle $\mathbf{R}^{4(a+b)} \times T^3 = \Gamma\backslash H_3^{(a,b)}$. Then this torus bundle was restricted onto a ball $B \subset \mathbf{R}^{4(a+b)}$ and both the Dirichlet and Neumann spectrum of the bundle $B \times T^3$ (topological product) was computed. It turned out that both spectra depended only on the value $(a+b)$, proving the desired isospectrality theorem for the ball×torus-type domains of the metric groups $H_3^{(a,b)}$ having the same $(a+b)$.

Gordon and Wilson [GW3] generalized the isospectrality result of [Sz3], [Sz4] to the ball×torus-type domains of general 2-step nilpotent Lie groups. Such a Lie group is uniquely determined by picking a linear space, E, of skew endomorphisms acting on a Euclidean space $\mathbf{R}^m$ (cf. formula (0.1)). Two endomorphism spaces are said to be *spectrally equivalent* if there exists an orthogonal transformation between them which corresponds isospectral (conjugate) endomorphisms to each other. (This basic concept of the field was introduced in [GW3]. Note that the endomorphism spaces belonging to the Heisenberg type groups $H_3^{(a,b)}$ satisfying $(a+b) = constant$ are spectrally equivalent.)

By the first main theorem of [GW3], *the corresponding ball×torus domains are both Dirichlet and Neumann isospectral on* 2*-step nilpotent Lie groups which are defined by spectrally equivalent endomorphism spaces.* Then this general theorem is used for constructing continuous families of isospectral metrics on $B^m \times T^2$ such that the distinct family members have different local geometries.

It turned out too that these metrics induce nontrivial isospectral metrics also on the boundaries, $S^{m-1} \times T$, of these manifolds. This statement was independently established both with respect to the [GW3]-examples (in [GGSWW]) and the [Sz3]-examples (in [Sz4]). Each of these examples has its own interesting new features. Article [GGSWW] provides the first continuous families of isospectral metrics on closed manifolds such that the distinct family members have different local geometries. In [Sz3] one has only a discrete family $g_3^{(a,b)}$ of isospectral metrics on $S^{4(a+b)} \times T^3$ (such a family is defined by the constant $a+b$). The surprising new feature is that the metric $g_3^{(a+b,0)}$ is homogeneous while the metrics $g_3^{(a,b)}$ satisfying $ab \neq 0$ are locally inhomogeneous.

At this point of the development no nontrivial isospectral metrics constructed on simply connected manifolds were known in the literature. The first such examples were constructed by Schueth [Sch1]. The main idea of her construction is the following: She enlarged the torus T^2 of the torus bundle $S^{m-1} \times T^2$ considered in [GGSWW] into a compact simply connected Lie group S such that T^2 is a maximal torus in S. Then the isospectral metrics were constructed on the enlarged manifold $S^{m-1} \times S$. Also this enlarged manifold is a T^2-bundle with respect to the left action of T^2 on the second factor. The original bundle $S^{m-1} \times T^2$ is a sub-bundle in this enlarged bundle. Then the parametric families of isospectral metrics introduced in [GGSWW] on manifolds $S^{m-1} \times T^2$ are extended such that they provide isospectral metrics also on the enlarged manifold. In special cases she obtained examples on the product of spheres. The metrics with the lowest dimension were constructed on $S^4 \times S^3 \times S^3$.

In [Sch2] this technique is reformulated in a more general form such that certain principal torus bundles are considered with a fixed metric on the base space and with the natural flat metric on the torus T. (Important basic concepts of this general theory are abstracted from works [G2], [GW3].) The isospectral metrics are constructed on the total space such that they have the following three properties: (1) The elements of the structure group T act as isometries. (2) The torus fibers have the prescribed natural flat metric. (3) The projection onto the base space is a Riemannian submersion.

One can define such a Riemannian metric just by choosing a connection on this principal torus bundle for defining the orthogonal complement to the torus fibers. Then the isospectral metrics with different local geometries are found by appropriate changing (deforming) of these connections. This combination of extension- and connection-techniques is a key feature of Schueth's constructions, which provided new surprising examples including isospectral pairs of metrics with different local geometries with the lowest known dimension on $S^2 \times T^2$.

Let us mention that in each of the papers [G1], [G2], [GW3], [GGSWW], [Sch1] the general torus bundles involved in the constructions have total geodesic torus fibres. This assumption is not used in establishing the isospectrality theorem on the special torus bundle considered [Sz3], [Sz4]. This assumption is removed and the torus bundle technique is formulated in a very general form in [GSz]. Though this form of the general isospectrality theorem opens up new directions, yet examples constructed on balls or on spheres were still out of touch by this technique, since no ball or sphere can be considered as the total space of a torus bundle, where $\dim(T) \geq 2$.

The first examples of isospectral metrics on balls and spheres have been constructed most recently by this author [Sz4] and, very soon thereafter, by Gordon [G3] independently. The techniques applied in these two constructions are completely different, providing completely different examples of isospectral metrics. Actually none of these examples can be constructed by the technique used for constructing the other type of examples.

First we describe Gordon's examples. The crucial new idea in her construction is a generalization of the torus bundle technique such that, instead of a principal torus bundle, just a torus action is considered which is not required to be free anymore. Yet this generalization is benefited by the results and methods of the bundle technique (for instance, by the Fourier-Weierstrass decomposition of function spaces on the torus fibres for establishing the isospectrality theorem) since they are still applicable on the everywhere-dense open subset covered by the maximal dimensional principal torus-orbits. This idea really gives the chance for constructing appropriate isospectral metrics on balls and sphere, since these manifolds admit such nonfree torus actions.

In her construction Gordon uses the metrics defined on $B\times T^l$ resp. $S\times T^l$ introduced in [GW3] resp. [GGSWW]. First, she represents the torus $T^l = \mathbf{Z}^l\backslash\mathbf{R}^l$ in $\mathrm{SO}(2l)$ by using the natural identification $T^l = \times_l \mathrm{SO}(2)$. By this representation she gets an enlarged bundle with the base space $\mathbf{v} = \mathbf{R}^k$ and with the total space $\mathbf{R}^{k+2l}$ such that the torus is nonfreely acting on the total space. Then a metric is defined on the total space. This metric inherits the Euclidean metric of the torus orbits and its projection onto the base space is the original Euclidean metric. Therefore, only the horizontal subspaces (which are perpendicular to the orbits) should be defined. They are introduced by the alternating bilinear form $B : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^l$, where $\langle B(X,Y),Z\rangle = \langle J_Z(X),Y\rangle$. Her final conclusion is as follows:

If the one parametric family g_t, considered in the first step on the manifolds $B\times T^l$, or, on $S\times T^l$, consists of isospectral metrics then also the above constructed metrics $\widetilde{g}_t$ are isospectral on the Euclidean balls and spheres of the total space $\mathbf{R}^{k+2l}$.

This construction provides locally inhomogeneous metrics because the torus actions involved have degenerated orbits. In the concrete examples, since the metrics g_t constructed in [GW3] and [GGSWW] are used, the torus T is 2-dimensional. In another theorem Gordon proves that the metrics $\widetilde{g}_t$ can be arbitrarily close to the standard metrics of Euclidean balls and spheres.

Constructing by the anticommutator technique. The Lie algebra of a 2-step nilpotent metric Lie group is described by a system $\{\mathbf{n} = \mathbf{v}\oplus\mathbf{z}, \langle,\rangle, J_Z\}$, where the Euclidean vector space $\mathbf{n}$, with the inner product $\langle,\rangle$, is decomposed into the indicated orthogonal direct sum. Furthermore, J_Z is a skew endomorphism acting on $\mathbf{v}$ for all $Z\in\mathbf{z}$ such that the map $J:\mathbf{z}\to \mathrm{End}(\mathbf{v})$ is linear and one-to-one. The linear space of endomorphisms J_Z is denoted by $J_{\mathbf{z}}$. Then the nilpotent Lie algebra with the center $\mathbf{z}$ is defined by

$$(0.1)\qquad \langle [X,Y],Z\rangle = \langle J_Z(X),Y\rangle\,;\ [X,Y] = \sum_\alpha \langle J_{Z_\alpha}(X),Y\rangle Z_\alpha,$$

where $X,Y\in\mathbf{v}$; $Z\in\mathbf{z}$ and $\{Z_1,\dots,Z_l\}$ is an orthonormal basis on $\mathbf{z}$.

Note that such a Lie algebra is uniquely determined by a linear space, $J_{\mathbf{z}}$, of skew endomorphisms acting on a Euclidean vector space $\mathbf{v}$. The natural Euclidean norm is defined by $||Z||^2 = -\mathrm{Tr}(J_Z^2)$ on $\mathbf{z}$. The constructions below admit arbitrary other Euclidean norms on $\mathbf{z}$.

The Lie group defined by this Lie algebra is denoted by G. The Riemann metric, g, is defined by the left invariant extension of the above Euclidean inner product introduced on the tangent space $T_0(G) = \mathbf{n}$ at the origin 0. The exponential map identifies the Lie algebra $\mathbf{n}$ with the vector space $\mathbf{v}\oplus\mathbf{z}$. Explicit formulas for geometric objects such as the invariant vector fields $(\mathbf{X}_i,\mathbf{Z}_\alpha)$, Laplacian, etc. are described in (1.1)–(1.6).

The particular Heisenberg-type nilpotent groups are defined by special endomorphism spaces satisfying $\mathbf{J}_Z^2 = -|Z|^2\mathrm{id}$, for all $Z \in \mathbf{z}$ [K]. If $l = \dim(\mathbf{z}) = 3\mathrm{mod}4$, then there exist (up to equivalence) exactly two Heisenberg-type endomorphism spaces, $J_l^{(1,0)}$ and $J_l^{(0,1)}$, acting irreducibly on $\mathbf{v} = \mathbf{R}^{n_l}$ (see the explanations at (2.6)). The reducible endomorphism spaces can be described by an appropriate Cartesian product in the form $J_l^{(a,b)}$ (see more about this notation below (2.14)). When quaternionic- resp. Cayley-numbers are used for constructions, the corresponding endomorphism spaces are denoted by $J_3^{(a,b)}$ resp. $J_7^{(a,b)}$. The family $J_l^{(a,b)}$, defined by fixed values of l and $(a+b)$, consists of spectrally equivalent endomorphism spaces.

Any 2-step nilpotent Lie group N extends to a solvable group SN defined on the half space $\mathbf{n}\times\mathbf{R}_+$ (cf. (1.8) and (1.9)). The first spectral investigations on these solvable extensions are established in [GSz].

The ball×torus-type domains, sketchily introduced above, are defined by the factor manifold $\Gamma_Z\setminus\mathbf{n}$, where Γ_Z is a full lattice on the Z-space $\mathbf{z}$ such that this principal torus bundle is considered over a Euclidean ball B_δ of radius δ around the origin of the X-space $\mathbf{v}$. The boundary of this manifold is the principal torus bundle (S_δ, T).

The main tool in proving the isospectrality theorem on such domains is the Fourier-Weierstrass decomposition $W = \oplus_\alpha W^\alpha$ of the L^2 function space on the group G, where, in the nilpotent case, the W^α is spanned by the functions of the form $F(X,Z) = f(X)e^{-2\pi\sqrt{-1}\langle Z_\alpha,Z\rangle}$. It turns out that each W^α is invariant under the action of the Laplacian, $(\Delta_G F)(X,Z) = \Box_\alpha(f)(X)e^{-2\pi\sqrt{-1}\langle Z_\alpha,Z\rangle}$, such that $\Box_\alpha$ depends, besides some universal terms and Δ_X, only on J_{Z_α} and it does not depend on the other endomorphisms. Since J_{Z_α} and $J_{Z'_\alpha}$ are isospectral, one can intertwine the Laplacian on the subspaces W^α separately by the orthogonal transformation conjugating J_{Z_α} to $J_{Z'_\alpha}$. This tool extends not only to the general ball×torus-cases considered in [GSz] but also to the torus-action-cases considered in [G3].

The simplicity of the isospectrality proofs by the above described Z-Fourier transform is due to the fact that, on an invariant subspace W^α, one should deal only with one endomorphism, J_α, while the others are eliminated.

New, so called *ball-type domains* were introduced in [Sz5] whose spectral investigation has no prior history. These domains are diffeomorphic to Euclidean balls whose smooth boundaries are described as level sets by equations of the form $\varphi(|X|,Z) = 0$, resp. $\varphi(|X|,Z,t) = 0$, according to the nilpotent, resp. solvable, cases. The boundaries of these domains are diffeomorphic to Euclidean spheres which are called sphere-type manifolds, or, sphere-type hypersurfaces.

The technique of the Z-Fourier transform breaks down on these domains and hypersurfaces, since the functions gotten by this transform do not satisfy the required boundary conditions. The Fourier-Weierstrass decomposition does not apply on the sphere-type hypersurfaces either. The difficulties in proving the isospectrality on these domains originate from the fact that no such Laplacian-invariant decomposition of the corresponding L^2 function spaces is known which keeps, on an invariant subspace, only one of the endomorphisms active while it gets rid of the other endomorphisms. The isospectrality proofs on these manifolds require a new technique whose brief description follows.

Let us mention first that a wide range of spectrally equivalent endomorphism spaces were introduced in [Sz5] by means of the so called *σ-deformations*. These deformations are defined by an involutive orthogonal transformation σ on $\mathbf{v}$ which commutes with all of the endomorphisms from $J_{\mathbf{z}}$. The σ-deformed endomorphism space, $J^\sigma_{\mathbf{z}}$, consists of endomorphisms of the form σJ_Z. This new endomorphism space is clearly spectrally equivalent to the old one. Note that no restriction on $\dim(\mathbf{z})$ is imposed in this case. These deformations are of discrete type, however, which can be considered as the generalizations of deformations considered on the endomorphism spaces $J_l^{(a,b)}$ in [Sz3], [Sz4]. These deformations provide isospectral metrics on the ball$\times$torus-type domains by the Gordon-Wilson theorem.

The new so-called *anticommutator technique*, developed for establishing the spectral investigations on ball- and sphere-type manifolds, does not apply for all the σ-deformations. We can accomplish the isospectrality theorems by this technique only for those particular endomorphism spaces which include nontrivial *anticommutators*.

A nondegenerated endomorphism $A \in J_{\mathbf{z}}$ is an anticommutator if and only if $A \circ B = -B \circ A$ holds for all $B \in J_{\mathbf{A}^\perp}$. If an endomorphism space $J_{\mathbf{z}}$ contains an anticommutator A, then, by the Reduction Theorem 4.1 of [Sz5], a σ-deformation is equivalent to the simpler deformation where one performs σ-deformation only on the anticommutator A. That is, only A is switched to $A^\sigma = \sigma \circ A$ and one keeps the orthogonal complement $J_{\mathbf{A}^\perp}$ unchanged. In [Sz5] and in this paper the isospectrality theorems are established for such, so called, σ_A-deformations.

The constructions concern four different cases, since we perform them on the ball- and sphere-type domains both of 2-step nilpotent Lie groups and their solvable extensions. The details are shared between these two papers. Roughly speaking, the proofs are completely established in [Sz5] on the ball-type domains and all the technical details are complete on 2-step nilpotent groups. Though the other cases were outlined to some extent, the important details concerning the sphere-type domains and the solvable extensions are left to this paper.

We start with a review of the solvable extensions of 2-step nilpotent groups. Then, in Proposition 2.1, we describe all the endomorphism spaces having an anticommutator A (alias ESW$_A$'s) in a representation theorem, where the Pauli matrices play a very crucial role. The basic examples of

ESW_A's are the endomorphism spaces $J_l^{(a,b)}$ belonging to Clifford modules. In this case each endomorphism is an anticommutator. The representation theorem describes a great abundance of other examples.

In Section 2 the so called $\mathrm{unit}_{\mathrm{endo}}$-deformations are introduced just by choosing two different unit anticommutators A_0 and B_0 to a fixed endomorphism space $\mathbf{F}$ (the corresponding ESW_A's are $\mathbf{R}A_0 \oplus \mathbf{F}$ and $\mathbf{R}B_0 \oplus \mathbf{F}$). Also these deformations can be used for isospectrality constructions. By clarifying a strong connection between $\mathrm{unit}_{\mathrm{endo}}$- and σ_A-deformations (cf. Theorem 2.2) we point out that the anticommutator technique is a discrete isospectral construction technique. In fact, we prove that continuous $\mathrm{unit}_{\mathrm{endo}}$-deformations provide conjugate ESW_A's and therefore the corresponding metrics are isometric.

The main isospectrality theorems are stated in the following form in this paper.

Main Theorems 3.2 and 3.4. *Let $J_{\mathbf{z}} = J_A \oplus J_{A^\perp}$ and $J_{\mathbf{z}'} = J_{A'} \oplus J_{A'^\perp}$ be endomorphism spaces acting on the same space $\mathbf{v}$ such that $J_{A^\perp} = J_{A'^\perp}$; furthermore, the anticommutators J_A and $J_{A'}$ are either unit endomorphisms (i.e., $A^2 = (A')^2 = -\mathrm{id}$) or they are σ-equivalent. Then the map $\partial\kappa = T' \circ \partial\kappa^* T^{-1}$ intertwines the corresponding Laplacians on the sphere-type boundary ∂B of any ball-type domain, both on the nilpotent groups N_J and $N_{J'}$ and/or on their solvable extensions SN_J and $SN_{J'}$. Therefore the corresponding metrics are isospectral on these sphere-type manifolds.*

In [Sz5], the corresponding theorem is established only for balls and for σ_A deformations. The investigations on spheres are just outlined and even these sketchy details concentrate mostly on the striking examples.

The constructions of the intertwining operators κ and $\partial\kappa$ require an appropriate decomposition of the function spaces. This decomposition is, however, completely different from the Fourier-Weierstrass decomposition applied in the torus-bundle cases since this decomposition is performed on the L^2-function space of the X-space. The details are as follows.

The crucial terms in the Laplacian acting on the X-space are the Euclidean Laplacian Δ_X and the operators $D_A\bullet, D_F\bullet$ derived from the endomorphisms (cf. (1.5), (1.12), (3,7), (3.33)). The latter operators commute with Δ_X. In the first step only the operators Δ_X and $D_A\bullet$ are considered and a common eigensubspace decomposition of the corresponding L^2 function space is established. This decomposition results in a refined decomposition of the spherical harmonics on the spheres of the X-space. Then the operators $\kappa, \partial\kappa$ are defined such that they preserve this decomposition. Though one cannot get rid of the other operators $D_F\bullet$ by this decomposition, the anticommutativity of A by the perpendicular endomorphisms F ensures that also the terms containing the operators $D_F\bullet$ in the Laplacian are intertwined by κ and $\partial\kappa$.

By proving also the appropriate nonisometry theorems, these examples provide a wide range of isospectral pairs of metrics constructed on spheres with different local geometries. These nonisometry proofs are achieved by an independent *Extension Theorem* asserting that an isometry between two sphere-type manifolds extends to an isometry between the ambient manifolds. (In order to avoid an even more complicated proof, the theorem is established for sphere-type manifolds described by equations of the form $\varphi(|X|,|Z|) = 0$ resp. $\varphi(|X|,|Z|,t) = 0$. It is highly probable that one can establish this extension in the most general cases by extending the method applied here.) This theorem traces back the problem of nonisometry to the ambient manifolds, where the nonisometry was thoroughly investigated in [Sz5]. The extension can be used also for determining the isometries of a sphere-type manifold by the isometries acting on the ambient manifold.

The abundance of the isospectral pairs of metrics constructed by the anticommutator technique on spheres with different local geometries is exhibited in *Cornucopia Theorem* 4.9, which is the combination of the isospectrality theorems and of the nonisometry theorems.

These isospectral pairs include the so called *striking examples* constructed on the geodesic spheres of the solvable groups $\mathrm{SH}_3^{(a,b)}$. (These examples are outlined in [Sz5] with fairly complete details, yet some of these details are left to this paper.) These spheres are homogeneous on the 2-point homogeneous space $\mathrm{SH}_3^{(a+b,0)}$ while the other spheres on $\mathrm{SH}_3^{(a,b)}$ are locally inhomogeneous. These examples demonstrate the surprising fact that no information about the isometries is encoded in the spectrum of Laplacian acting on functions.

1. Two-step nilpotent Lie algebras and their solvable extensions

A metric 2-step nilpotent Lie algebra is described by the system

$$\mathfrak{n} = \{\mathbf{n} = \mathbf{v} \oplus \mathbf{z}, \langle,\rangle, J_Z\}, \tag{1.1}$$

where $\langle,\rangle$ is an inner product defined on the algebra $\mathbf{n}$ and the space $\mathbf{z} = [\mathbf{n},\mathbf{n}]$ is the center of $\mathbf{n}$; furthermore $\mathbf{v}$ is the orthogonal complement to $\mathbf{z}$. The map $J : \mathbf{z} \to \mathrm{SkewEndo}(\mathbf{v})$ is defined by $\langle J_Z(X), Y\rangle = \langle Z, [X,Y]\rangle$.

The vector spaces $\mathbf{v}$ and $\mathbf{z}$ are called X-space and Z-space respectively.

Such a Lie algebra is well defined by the endomorphisms J_Z. The linear space of these endomorphisms is denoted by $J_{\mathbf{z}}$. For a fixed X-vector $X \in \mathbf{v}$, the subspace spanned by the X-vectors $J_Z(X)$ (for all $Z \in \mathbf{z}$) is denoted by $J_{\mathbf{z}}(X)$.

Consider the orthonormal bases $\{E_1;\dots;E_k\}$ and $\{e_1;\dots;e_l\}$ on the X- and the Z-spaces respectively. The corresponding coordinate systems defined by these bases are denoted by $\{x^1;\dots;x^k\}$ and $\{z^1;\dots;z^l\}$. According to [Sz5] the left-invariant extensions of the vectors $E_i; e_\alpha$ are the vector fields

(1.2)
$$\begin{aligned}\mathbf{X}_i &= \partial_i + \frac{1}{2}\sum_{\alpha=1}^{l}\langle [X,E_i],e_\alpha\rangle\partial_\alpha \\ &= \partial_i + \frac{1}{2}\sum_{\alpha=1}^{l}\langle J_\alpha(X),E_i\rangle\partial_\alpha \quad ; \quad \mathbf{Z}_\alpha = \partial_\alpha,\end{aligned}$$

where $\partial_i = \partial/\partial x^i$, $\partial_\alpha = \partial/\partial z^\alpha$ and $J_\alpha = J_{e_\alpha}$.

The covariant derivative acting on invariant vector fields is described as follows.

(1.3)
$$\nabla_X X^* = \frac{1}{2}[X,X^*]\,;\ \nabla_X Z = \nabla_Z X = -\frac{1}{2}J_Z(X)\,;\ \nabla_Z Z^* = 0.$$

The Laplacian, Δ, acting on functions can be explicitly established by substituting (1.2) and (1.3) into the following well-known formula

(1.4)
$$\Delta = \sum_{i=1}^{k}\left(\mathbf{X}_i^2 - \nabla_{\mathbf{X}_i}\mathbf{X}_i\right) + \sum_{\alpha=1}^{l}\left(\mathbf{Z}_\alpha^2 - \nabla_{\mathbf{Z}_\alpha}\mathbf{Z}_\alpha\right).$$

Then we obtain

(1.5)
$$\Delta = \Delta_X + \Delta_Z + \frac{1}{4}\sum_{\alpha,\beta=1}^{l}\langle J_\alpha(X),J_\beta(X)\rangle\partial^2_{\alpha\beta} + \sum_{\alpha=1}^{l}\partial_\alpha D_\alpha\bullet,$$

where $D_\alpha\bullet$ means differentiation (directional derivative) with respect to the vector field

(1.6)
$$D_\alpha : X \to J_\alpha(X)$$

tangent to the X-space; furthermore $\partial_{\alpha\beta} = \partial^2/\partial z^\alpha\partial z^\beta$.

Some other basic objects (such as Riemannian curvature, Ricci curvature, d- and δ-operators acting on forms) are also explicitly established in [Sz5]. Finally, we mention a theorem describing the isometries on 2-step nilpotent Lie groups.

Proposition 1.1 ([K], [E], [GW3], [W]). *The 2-step nilpotent metric Lie groups (N,g) and (N',g') are isometric if and only if there exist orthogonal transformations $A:\mathbf{v}\to\mathbf{v}'$ and $C:\mathbf{z}\to\mathbf{z}'$ such that*

(1.7)
$$AJ_ZA^{-1} = J'_{C(Z)}$$

holds for any $Z\in\mathbf{z}$.

Any 2-step nilpotent Lie group, N, extends to a solvable group, SN, defined on the half-space $\mathbf{n}\times\mathbf{R}_+$ with multiplication given by

(1.8)
$$(X,Z,t)(X',Z',t') = \left(X + t^{\frac{1}{2}}X', Z + tZ' + \frac{1}{2}t^{\frac{1}{2}}[X,X'], tt'\right).$$

This formula provides the multiplication also on the nilpotent group N, since the latter is a subgroup determined by $t = 1$.

The Lie algebra of this solvable group is $\mathbf{s} = \mathbf{n} \oplus \mathbf{t}$. The Lie bracket is completely determined by the formulas

$$(1.9) \qquad [\partial_t, X] = \frac{1}{2}X \quad ; \quad [\partial_t, Z] = Z \quad ; \quad [\mathbf{n}, \mathbf{n}]_{/SN} = [\mathbf{n}, \mathbf{n}]_{/N},$$

where $X \in \mathbf{v}$ and $Z \in \mathbf{z}$.

In [GSz], a scaled inner product $\langle\, ,\, \rangle_c$ with scaling factor $c > 0$ is introduced on $\mathbf{s}$ defined by the rescaling $|\partial_t| = c^{-1}$ and by keeping the inner product on $\mathbf{n}$ as well as keeping the relation $\partial_t \perp \mathbf{n}$. The left invariant extension of this inner product is denoted by g_c.

The left-invariant extensions $\mathbf{Y}_i, \mathbf{V}_\alpha, \mathbf{T}$ of the unit vectors

$$E_i = \partial_i \quad , \quad e_\alpha = \partial_\alpha \quad , \quad \varepsilon = c\partial_t$$

at the origin are

$$(1.10) \qquad \mathbf{Y}_i = t^{\frac{1}{2}}\mathbf{X}_i \quad ; \quad \mathbf{V}_\alpha = t\mathbf{Z}_\alpha \quad ; \quad \mathbf{T} = ct\partial_t,$$

where $\mathbf{X}_i$ and $\mathbf{Z}_\alpha$ are the invariant vector fields on N (cf. (1.2)).

One can establish these latter formulas by the following standard computations. Consider the vectors $\partial_i, \partial_\alpha$ and ∂_t at the origin $(0, 0, 1)$ such that they are the tangent vectors of the curves $c_A(s) = (0, 0, 1) + s\partial_A$, where $A = i, \alpha, t$. Then transform these curves to an arbitrary point by left multiplications described in (1.8). Then the tangent of the transformed curve gives the desired left invariant vector at an arbitrary point.

The covariant derivative can be computed by the well known formula

$$\langle \nabla_P Q, R\rangle = \frac{1}{2}\{\langle P, [R, Q]\rangle + \langle Q, [R, P]\rangle + \langle [P, Q], R\rangle\},$$

where P, Q, R are invariant vector fields. Then we get

$$(1.11) \quad \nabla_{X+Z}(X^* + Z^*) = \nabla^N_{X+Z}(X^* + Z^*) + c\left(\frac{1}{2}\langle X, X^*\rangle + \langle Z, Z^*\rangle\right)\mathbf{T};$$

$$\nabla_X \mathbf{T} = -\frac{c}{2}X \quad ; \quad \nabla_Z \mathbf{T} = -cZ \quad ; \quad \nabla_T X = \nabla_T Z = \nabla_T T = 0,$$

where ∇^N is the covariant derivative on N (cf. (1.3)) and

$$X, X^* \in \mathbf{v}; \qquad Z, Z^* \in \mathbf{z}; \qquad T \in \mathbf{t}.$$

The Laplacian on these solvable groups can be established by the same computation performed on N. Then we get

$$(1.12) \qquad \Delta = t\Delta_X + t^{\frac{1}{2}}\Delta_Z + \frac{1}{4}t\sum_{\alpha;\beta=1}^{l}\langle J_\alpha(X), J_\beta(X)\rangle\partial^2_{\alpha\beta}$$

$$+t\sum_{\alpha=1}^{l}\partial_\alpha D_\alpha \bullet + c^2t^2\partial_t^2 + c^2\left(1 - \frac{k}{2} - l\right)t\partial_t.$$

Also the Riemannian curvature can be computed straightforwardly such that formulas (1.11) are substituted into the standard formula of the Riemannian curvature. Then we get

$$(1.13)\qquad R_c(X^*\wedge X) = R(X^*\wedge X) - \frac{c}{2}[X^*,X]\wedge \mathbf{T} + \frac{c^2}{4}X^*\wedge X;$$

$$R_c(X\wedge Z) = R(X\wedge Z) - \frac{c}{4}J_Z(X)\wedge \mathbf{T} + \frac{c^2}{2}X\wedge Z;$$

$$R_c(Z^*\wedge Z) = R(Z^*\wedge Z) + c^2 Z^*\wedge Z;$$

$$R_c((X+Z),\mathbf{T})(.) = c\nabla_{\frac{1}{2}X+Z}(.);$$

$$R_c((X+Z)\wedge \mathbf{T}) = \frac{1}{2}c\left(\sum_\alpha J_\alpha(X)\wedge e_\alpha - J^*_Z\right) + c^2\left(\frac{1}{4}X+Z\right)\wedge \mathbf{T},$$

where J^*_Z is the 2-vector dual to the 2-form $\langle J_Z(X_1),X_2\rangle$ and R is the Riemannian curvature on N, described by

$$(1.14)\quad R(X,Y)X^* = \frac{1}{2}J_{[X,Y]}(X^*) - \frac{1}{4}J_{[Y,X^*]}(X) + \frac{1}{4}J_{[X,X^*]}(Y);$$

$$R(X,Y)Z = -\frac{1}{4}[X,J_Z(Y)] + \frac{1}{4}[Y,J_Z(X)] \quad ; \quad R(Z_1,Z_2)Z_3 = 0;$$

$$R(X,Z)Y = -\frac{1}{4}[X,J_Z(Y)] \quad ; \quad R(X,Z)Z^* = -\frac{1}{4}J_Z J_{Z^*}(X);$$

$$R(Z,Z^*)X = -\frac{1}{4}J_{Z^*}J_Z(X) + \frac{1}{4}J_Z J_{Z^*}(X),$$

where $X;X^*;Y\in\mathbf{v}$ and $Z;Z^*;Z_1;Z_2;Z_3\in\mathbf{z}$ are elements of the Lie algebra (see [E]).

By introducing $H(X,X^*,Z,Z^*) := \langle J_Z(X),J_{Z^*}(X^*)\rangle$, for the Ricci curvature we have

$$(1.15)\quad \mathrm{Ricc}_c(X) = \mathrm{Ricc}(X) - c^2\left(\frac{k}{4}+\frac{l}{2}\right)X;$$

$$\mathrm{Ricc}_c(Z) = \mathrm{Ricc}(Z) - c^2\left(\frac{k}{2}+l\right)Z\ ;\quad \mathrm{Ricc}_c(T) = -c^2\left(\frac{k}{4}+l\right)T,$$

where the Ricci tensor *Ricc* on N is described by formulas

$$(1.16)\qquad \mathrm{Ricc}(X,X^*) = -\frac{1}{2}\sum_{\alpha=1}^{l}H(X,X^*,e_\alpha,e_\alpha) = -\frac{1}{2}H_{\mathbf{v}}(X,X^*);$$

$$\mathrm{Ricc}(Z,Z^*) = \frac{1}{4}\sum_{i=1}^{k}H(E_i,E_i,Z,Z^*) = \frac{1}{4}H_{\mathbf{z}}(Z,Z^*)$$

and by $\mathrm{Ricc}(X,Z)=0$.

By (1.9) we get that the subspaces $\mathbf{v}, \mathbf{z}$ and $\mathbf{t}$ are eigensubspaces of the Ricci curvature operator and, except for finitely many scaling factors c, the eigenvalue on $\mathbf{t}$ is different from the other eigenvalues. For these scaling factors c, an isometry $\alpha : SN_c \to SN'_c$ maps $\mathbf{T}$ to $\mathbf{T}'$ and for a fixed t, the hypersurface (N, t) is mapped to the hypersurface (N', t'), where t' is an appropriate fixed parameter. By using left-products on SN', we may suppose that the α maps the origin $(0, 1)$ to the origin of SN' and therefore $\alpha(N, 1) = (N', 1)$. By (1.2) and (1.10), the restrictions of the metric tensors g_c and g'_c onto the hyper-surfaces $(N, 1)$ and $(N', 1)$ are nothing but the metric tensors g resp. g' on the nilpotent groups. Thus the α defines an isometry between (N, g) and (N', g') and so we have:

Proposition 1.2. *Except for finitely many scaling factors c, the solvable extensions (SN, g_c) and (SN', g'_c) are locally isometric if and only if the nilpotent metric groups (N, g) and (N', g') are locally isometric.*

It should be mentioned that the above assumption about the scaling factor can be dropped. This stronger theorem is proved in [GSz, Prop. 2.13] by a completely different (much more elaborate) technique.

We conclude this section by considering the spectrum of the curvature operator acting as a symmetric endomorphism on the 2-vectors. These considerations can be used to establish the nonisometry proofs. These nonisometry proofs will be established in many different ways, however, in order to get a deeper insight into the realm of nonaudible geometry. Even though the next theorem is an interesting contribution to this geometry, the understanding of the main thesis of this paper is not disturbed by continuing the study in the next section.

Two symmetric operators are said to be *isotonal* if the elements of their spectra are the same but the multiplicities may be different. This property is accomplished for the curvature operators of $\sigma^{(a+b)}$-equivalent nilpotent groups in [Sz5, Prop. 5.4]. Now we establish this statement also on the solvable extensions of these groups. The technical definition of the groups $N_J^{(a,b)}$ and the $\sigma^{(a+b)}$-deformations can be found both in [Sz5] and in formulas (2.12)–(2.14) of this paper.

In the nilpotent case we used the following decomposition, which technique extends also to the solvable case.

First decompose the X-space of the considered nilpotent Lie-algebras in the form $\mathbf{v} = \mathbf{v}^{(a)} \oplus \mathbf{v}^{(b)}$ such that the involution $\sigma^{(a,b)}$ acts on $\mathbf{v}^{(a)} = \mathbf{R}^{na}$ by id and also on the subspace $\mathbf{v}^{(b)} = \mathbf{R}^{nb}$ by $-$id. Then the subspaces

$$\mathbf{D} = (\mathbf{v}^{(a)} \wedge \mathbf{v}^{(a)}) \oplus (\mathbf{v}^{(b)} \wedge \mathbf{v}^{(b)}) \oplus (\mathbf{z} \wedge \mathbf{z}); \qquad (1.17)$$
$$\mathbf{F} = \mathbf{v}^{(a)} \wedge \mathbf{v}^{(b)} \quad ; \quad \mathbf{G} = \mathbf{v} \wedge \mathbf{z},$$

in $\mathbf{n}\wedge\mathbf{n}$, are invariant under the action of the curvature operator $R^{ij}{}_{kl}$. The space $\mathbf{F}$ is further decomposed into the mixed boxes $\mathbf{F}_{rs}=\mathbf{v}_r^{(a)}\wedge\mathbf{v}_s^{(b)}$, where $\mathbf{v}_r^{(a)}$ is the r^{th} component subspace, $\mathbf{R}^n$, in the Cartesian product $\mathbf{v}^{(a)}=\times\mathbf{R}^n$. Then one can prove that the spectrum on such a mixed box is the same on $\sigma^{(a+b)}$-equivalent spaces and, furthermore, it is the negative of the spectrum on a mixed box $\mathbf{v}_p^{(a)}\wedge\mathbf{v}_q^{(a)};\mathbf{v}_p^{(b)}\wedge\mathbf{v}_q^{(b)}\subseteq\mathbf{D}$, where $p\neq q$. These latter 2-vectors span the complement space, $\mathbf{Dg}^\perp$, to the diagonal space

$$\mathbf{Dg}=\left(\sum_r\mathbf{v}_r^{(a)}\wedge\mathbf{v}_r^{(a)}\right)\oplus\left(\sum_r\mathbf{v}_r^{(b)}\wedge\mathbf{v}_r^{(b)}\right)\oplus(\mathbf{z}\wedge\mathbf{z}).$$

On the invariant space $\mathbf{Dg}\oplus\mathbf{G}$ one can prove that the spectra of the considered operators are the same, since they are isospectral to the curvature operator on the group $N_{\mathbf{z}}^{(a+b,0)}$. Therefore, comparing the two spectra, we get that only the multiplicities of eigenvalues belonging to the mixed boxes of the invariant spaces $\mathbf{F}$ resp. $\mathbf{Dg}^\perp$ are different, while the elements of the spectra are the same. These multiplicities depend on the number of the mixed boxes, i.e., on ab. This proves that the curvature operator on $N_{\mathbf{z}}^{(a+b,0)}$ is subtonal to the operator on $N_{\mathbf{z}}^{(a,b)}$ and the curvature operators on $N_{\mathbf{z}}^{(a,b)}$ and $N_{\mathbf{z}}^{(a',b')}$, where $a+b=a'+b'$ and $aba'b'\neq 0$, are isotonal.

On the solvable extensions, $SN_{\mathbf{z}}^{(a,b)}$, the corresponding invariant subspaces are the following ones:

$$\mathbf{F}\quad,\quad\mathbf{Dg}^\perp\quad;\quad\mathbf{G}\quad;\quad\mathbf{Dg}\oplus(\mathbf{n}\wedge\mathbf{t}).\tag{1.18}$$

First consider the last subspace. From (1.13) we get that the map τ, defined by $\tau=-\mathrm{id}$ on the space $(\mathbf{v}^{(b)}\wedge\mathbf{v}^{(b)})\oplus(\mathbf{v}^{(b)}\wedge\mathbf{t})$ and by $\tau=\mathrm{id}$ on the orthogonal complement, intertwines the curvature operators of the spaces $SN_{\mathbf{z}}^{(a,b)}$ and $SN_{\mathbf{z}}^{(a+b,0)}$ on this subspace. Actually, this statement is true on the direct sum of $\mathbf{G}$ and the subspace listed in the last place of (1.18). Furthermore, the spectrum $\{\nu_i\}$ on a mixed box $\mathbf{F}_{pq}$ is the same on $\sigma^{(a+b)}$-equivalent spaces which can be expressed with the help of the corresponding spectrum $\{\lambda_i\}$ on the nilpotent group in the form $\nu_i=-Q^2+\lambda_i$. Then the spectrum on a mixed box of $\mathbf{Dg}^\perp$ has the form $\{-Q^2-\lambda_i\}$. We get again that only the multiplicities corresponding to these eigenvalues are different with respect to the two spectra, since these multiplicities depend on the number of the mixed boxes (i.e., on ab). Thus we have

Proposition 1.3. *The curvature operators on the σ-equivalent metric Lie groups $SN_{\mathbf{z}}^{(a,b)}$ and $SN_{\mathbf{z}}^{(a',b')}$ with $aba'b'\neq 0$ are isotonal.*

In many cases they are isotonal yet nonisospectral. This is the case, for instance, on the groups $S\mathbf{H}_3^{(a,b)}$ with the same $a+b$ and $ab\neq 0$, where the curvature operators are isotonal yet nonisopectral unless $(a,b)=(a',b')$ up to an order.

A general criterion can be formulated as follows: The Riemannian curvatures on the spaces $SN_{\mathbf{z}}^{(a,b)}$ *and* $SN_{\mathbf{z}}^{(a',b')}$ *with* $(a+b)=(a'+b')$ *and* $0\neq ab\neq a'b'\neq 0$ *are strictly isotonal if and only if the spectrum of the curvature of the corresponding nilpotent group changes on the mixed boxes* $\mathbf{F}_{pq}$ *when it is multiplied by* -1.

The curvature of $SN_{\mathbf{z}}^{(a+b,0)}$ *is just subtonal (i.e., the tonal spectrum is a proper subset of the other tonal spectrum) to the curvatures of the manifolds* $SN_{\mathbf{z}}^{(a,b)}$ *with* $ab\neq 0$.

2. Endomorphism spaces with anticommutators (alias ESW_A)

For the isospectrality examples a new, so called, anticommutator technique is developed in [Sz5]. A nondegenerated endomorphism $A=J_Z$ is called an *anticommutator* in $J_{\mathbf{z}}$ if $A\circ B=-B\circ A$ holds for all $B\in J_{Z^\perp}$. That is, the endomorphism A anticommutes with each endomorphism orthogonal to A.

An anticommutator satisfying $A^2=-\mathrm{id}$ is said to be a *unit anticommutator*. Any anticommutator can be rescaled to a unit anticommutator, since it can be written in the form $A=S\circ A_0$, where the symmetric "scaling" operator S is one of the square-roots of the operator $-A^2$, furthermore, A_0 is a unit anticommutator. Then the operator S is commuting with all elements of the endomorphism space.

The isospectrality examples are accomplished by certain deformations performed on ESW_A's. By these deformations only the A is deformed to a new anticommutator A' which is isospectral (conjugate) to A. The orthogonal endomorphisms are kept unchanged (i.e., $A^\perp=A'^\perp$) and for a general endomorphism the deformation is defined according to the direct sum $A\oplus A^\perp$. Such deformations are, for example, the σ_A-deformations introduced in [Sz5] (see the definition also in this paper at (2.12)–(2.14)). Another obvious example is when both A and A' are unit endomorphisms anticommuting with the endomorphisms of a given endomorphism space $A^\perp=A'^\perp$. We call these deformations $\mathrm{unit}_{\mathrm{endo}}$-*deformations*. In this paper we consider only these two sorts of deformations; the general isospectral deformations of an anticommutator will be studied elsewhere.

A brief outline of this section is as follows.

First we explicitly describe all the ESW_A's in a representation theorem, where the endomorphisms are represented as matrices of Pauli matrices. (In [Sz5] only particular ESW_A's were constructed to show the wide range of examples covered by this concept.)

Then the explicit description of the $\mathrm{unit}_{\mathrm{endo}}$-deformations follows. We also prove that the endomorphism spaces ESW_A and $\mathrm{ESW}_{A'}$ are conjugate if A and A' can be connected by a continuous curve passing through unit anticommutators. This statement shows that the anticommutator technique developed in these papers is a discrete construction technique since the corresponding metrics constructed by continuous $\text{unit}_{\text{endo}}$-deformations are isometric.

This section is concluded by describing those σ_A- or $\text{unit}_{\text{endo}}$-deformations which provide nonconjugate endomorphism spaces and therefore also the corresponding metrics are locally nonisometric.

The Jordan form of an ESW_A. First, we explicitly describe a general ESW_A by a matrix-representation. Then more specific endomorphism spaces such as quaternionic ESW_A's (alias $\mathbf{H}\text{ESW}_A$) and Heisenberg-type ESW_A's will be considered.

(A) In the following matrix-representation of an ESW_A the endomorphisms are represented as block-matrices; more precisely, they are the matrices of the following 2×2 matrices (blocks).

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{j} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

The matrix product with these matrices are described as follows.

$$\mathbf{i}^2 = -\mathbf{1}\,, \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{1}\,, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}\,, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}\,, \mathbf{kj} = -\mathbf{jk} = \mathbf{i}. \tag{2.2}$$

The second and the last group of these equations show that (2.1) is not a representation of the quaternionic numbers. Note that the matrices

$$\sigma_x = \mathbf{k} \quad , \quad \sigma_y = -\sqrt{-1}\,\mathbf{i} \quad , \quad \sigma_z = -\mathbf{j} \tag{2.3}$$

are the so called Pauli spin matrices.

From the above relations the following *observation* follows immediately: *a* 2×2-*matrix*, Y, *anticommutes with* $\mathbf{i}$ *if and only if it has the form* $Y = y_2\mathbf{j} + y_3\mathbf{k}$.

In the following we describe the whole space of skew endomorphisms anticommuting with a fixed skew endomorphism A. The endomorphisms are considered to be represented in matrix form such that the matrix of A is a diagonal Jordan matrix. One can establish this representation of an ESW_A by an orthonormal Jordan basis corresponding to the anticommutator A.

We consider the eigenvalues of the symmetric endomorphism A^2 arranged in the form $-a_1^2 < \cdots < -a_s^2 \leq 0$. The corresponding multiplicities are denoted by $m_1, \ldots, m_s$. First suppose that A is nondegenerated and therefore the main diagonal of its Jordan matrix is built up by 2×2 matrices in the block-form

$$(|a_1|\mathbf{i}, \ldots, |a_1|\mathbf{i}, \ldots\ldots, |a_s|\mathbf{i}, \ldots, |a_s|\mathbf{i}). \tag{2.4}$$

The $m_c = 2n_c$-dimensional eigensubspace belonging to the eigenvalue $-a_c^2$ is denoted by $\mathbf{B}_c^{m_c}$.

We seek also the anticommuting matrices in the above block-form, i.e., we consider them as matrices of 2×2-matrices. Any matrix, F, anticommuting with A leaves the eigensubspaces $\mathbf{B}_c^{m_c}$ invariant. Therefore it can be written in the form $F = \oplus F_c$, where F_c operates on $\mathbf{B}_c^{m_c}$. By the above observation we get that F is anticommuting with A if and only if the matrix of F_c, considered as the matrix of 2×2 matrices, has the block-entries of the form $F_{cml} = j_{cml}\mathbf{j} + k_{cml}\mathbf{k}$. Since the matrices $\mathbf{j}$ and $\mathbf{k}$ are symmetric, the main diagonal is trivial ($F_{cll} = \mathbf{0}$); furthermore, $j_{cml} = -j_{clm}$; $k_{clm} = -k_{cml}$ hold. That is, an endomorphism F anticommutes with A if and only if the real matrices j_c and k_c are skew symmetric.

If A is degenerated, then $a_s = 0$ and its action is trivial on the maximal eigensubspace $\mathbf{B}_s^{m_s}$. In this case the block F_s can be an arbitrary real skew-matrix.

An *irreducible block-decomposition* of an ESW_A is defined as follows. First we decompose the eigensubspaces $\mathbf{B}_c^{m_c}$ into orthogonal subspaces $\mathbf{B}_{ci}^{m_{ci}}$ such that the endomorphisms leave them invariant and act on them irreducibly. Then we consider a basis whose elements are in these irreducible spaces. With respect to such a basis, all the endomorphisms appear in the form $F_c = \bigoplus F_{ci}$. This irreducible decomposition of the X-space is the most refined one such that the endomorphisms F still can be represented in the form $F_{ci} = \{j_{cikl}\mathbf{j}+k_{cikl}\mathbf{k}\}$. The entry $a_{ci}\mathbf{i}$ is constant with multiplicity m_{ci}.

These statements completely describe the space of skew endomorphisms anticommuting with A. If A is nondegenerated, the dimension of this space is $\sum_c n_c(n_c - 1)$. If A is degenerated, the last term in this sum should be changed to $(1/2)m_s(m_s - 1)$. A general ESW_A is an A-including subspace of this maximal space.

By summing up we have

PROPOSITION 2.1. *Let $\bigoplus B_c$ be the above described Jordan decomposition of the* X-*space with respect to an anticommutator A such that A^2 has the constant eigenvalue $-a_c^2$ on B_c. Then all the endomorphisms from* ESW_A *leave these Jordan subspaces invariant and, in case $a_c \neq 0$, an $F \in A^\perp$ can be represented as a matrix of 2×2 matrices in the form $F_c = (F_{cml} = j_{cml}\mathbf{j} + k_{cml}\mathbf{k})$, where j_c and k_c are real skew matrices. If $a_c = 0$, the matrix representation of F_c can be an arbitrary real skew matrix.*

This Jordan decomposition, $\mathrm{ESW}_A = \bigoplus \mathrm{ESW}_{A_c}$, *can be refined by decomposing a subspace B_c into irreducible subspaces $\mathbf{B}_{ci}^{m_{ci}}$. Then any F_c can be represented in the form $F_c = \bigoplus F_{ci}$ such that the components, F_{ci}, still have the above described form.*

For a fixed anticommutator A, the dimension of the maximal ESW_A *consisting of all the skew endomorphisms anticommuting with A is $\sum_{ci} m_{ci}(m_{ci}-1)$. A general* ESW_A *is an A-including subspace of this maximal endomorphism space.*

(B) Particular, so called *quaternionic endomorphism spaces with anticommutators* (alias $\mathbf{H}\mathrm{ESW}_A$) can be introduced by using matrices with quaternionic entries. In this case the X-space is the n-dimensional quaternionic vector space identified with $\mathbf{R}^{4n}$. We suppose that the entries of an $n \times n$ quaternionic matrix A acts by left side products on the component of the quaternionic n-vectors. Such a matrix defines a skew symmetric endomorphism on $\mathbf{R}^{4n}$ if and only if it is a Hermitian skew matrix, i.e., $a_{ij} = -\overline{a}_{ji}$ holds for the entries.

Notice that in this case the entries in the main diagonal are imaginary quaternions. Furthermore A^2 is a Hermitian symmetric matrix and therefore the entries in the main diagonal of the matrices $(A^2)^k$ are real numbers.

There is atypical example of an $\mathbf{H}\mathrm{ESW}_A$ when A is a diagonal matrix having the same imaginary quaternion (say $\mathbf{I}$) in the main diagonal and the anticommuting matrices are symmetric matrices with entries of the form $y_2\mathbf{J}+y_3\mathbf{K}$. If the action of endomorphisms is irreducible and we build up diagonal block matrices by using such blocks, we get the quaternionic version of the above Proposition 2.1.

Notice that the matrices in a general ESW_A cannot be represented as such quaternionic matrices in general. In fact, the endomorphisms restricted to a subspace $\mathbf{B}_{ci}^{m_{ci}}$ can be commonly transformed into quaternionic matrices if and only if the multiplicities m_{ci} are multiples of 4 ($m_{ci} = 4k_{ci}$) and the matrices are matrices of such 4×4 blocks which are the linear combinations of matrices of the form

$$\mathbf{I} = \begin{pmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{pmatrix}, \ \mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{j} \\ -\mathbf{j} & \mathbf{0} \end{pmatrix}, \ \mathbf{K} = \begin{pmatrix} \mathbf{0} & \mathbf{k} \\ -\mathbf{k} & \mathbf{0} \end{pmatrix}. \tag{2.5}$$

Note that in this quaternionic matrix form, two anticommuting matrices can be pure diagonal matrices.

(C) Other specific *endomorphism spaces* are those *where all the endomorphisms are anticommutators.*

Since on a Heisenberg-type group the equation

$$J_Z J_{Z^*} + J_{Z^*} J_Z = -2\langle Z, Z^*\rangle \mathrm{id}$$

holds (cf. (1.4) in [CDKR], where this statement is proved by polarizing $J_Z^2 = -|Z|^2\,\mathrm{id}$), all the endomorphisms are anticommutators in the endomorphism space $J_{\mathbf{z}}$ of these groups.

The endomorphism spaces belonging to Heisenberg-type groups are attached to Clifford modules (which are representations of Clifford algebras) [K]. Therefore we call them Heisenberg-type, or, Cliffordian endomorphism spaces. The classification of Clifford modules is well known, providing classification also for the Cliffordian endomorphism spaces. Next we briefly summarize some of the main results of this theory (cf. [L]).

If $l = \dim(J_{\mathbf{z}}) \neq 3 (\mathrm{mod}\ 4)$ then there exists (up to equivalence) exactly one irreducible H-type endomorphism space acting on $\mathbf{R}^{n_l}$, where the dimension n_l, depending on l, is described below. This endomorphism space is denoted by $J_l^{(1)}$. If $l = 3 (\mathrm{mod}\ 4)$, then there exist (up to equivalence) exactly two nonequivalent irreducible H-type endomorphism spaces acting on $\mathbf{R}^{n_l}$ which are denoted by $J_l^{(1,0)}$ and $J_l^{(0,1)}$ separately. The values n_l corresponding to $l = 8p, 8p+1, \dots, 8p+7$ are

$$n_l = 2^{4p}\,,\ 2^{4p+1}\,,\ 2^{4p+2}\,,\ 2^{4p+2}\,,\ 2^{4p+3}\,,\ 2^{4p+3}\,,\ 2^{4p+3}\,,\ 2^{4p+3}. \tag{2.6}$$

The reducible Cliffordian endomorphism spaces can be built up by these irreducible ones. They are denoted by $J_l^{(a)}$ resp. $J_l^{(a,b)}$, corresponding to the definition of $J_3^{(a,b)}$ and $J_7^{(a,b)}$. (See an explanation about this notation after formula (2.14).)

Riehm [R] described these endomorphism spaces explicitly and used his description to determine the isometries on Heisenberg-type metric groups.

From our point of view particularly important examples are the groups $H_3^{(a,b)}$. The endomorphism space $J_3^{(a,b)}$ of these groups, defined by appropriate multiplications with imaginary quaternions, are thoroughly described in [Sz5]. Another interesting case is $H_7^{(a,b)}$, where the imaginary Cayley numbers are used for constructions. A brief description of this Cayley-case is as follows.

We identify the space of imaginary Cayley numbers with $\mathbf{R}^7$ and we introduce also the maps $\Phi : \mathbf{R}^7 \to \mathbf{H} = \mathbf{R}^4$ and $\Psi : \mathbf{R}^7 \to \mathbf{H}$ defined by $(Z_1, \dots, Z_7) \to Z_1\mathbf{i} + Z_2\mathbf{j} + Z_3\mathbf{k}$ and $(Z_1, \dots, Z_7) \to Z_4 + Z_5\mathbf{i} + Z_6\mathbf{j} + Z_7\mathbf{k}$ respectively. That is, if we consider the natural decomposition $\mathbf{Ca} = \mathbf{H}^2$ on the space $\mathbf{Ca}$ of Cayley numbers, then the above maps are the corresponding projections onto the factor spaces. Then the right product R_Z by an imaginary Cayley number $Z \in \mathbf{R}^7$ is described by the following formula:

$$R_Z(v_1, v_2) = (v_1\Phi(Z), -v_2\Phi(Z)) + (\Psi(Z)v_2, -\overline{\Psi}(Z)v_1),$$

where (v_1, v_2) corresponds to the decomposition $\mathbf{Ca} = \mathbf{H}^2$.

The R_Z is a skew symmetric endomorphism satisfying the property $R_Z^2 = -|Z|^2\mathrm{id}$ and the whole space of these endomorphisms defines the Heisenberg type Lie algebra $\mathfrak{n}_7^1$. Then the Lie algebras $\mathfrak{n}_7^{(a,b)}$ can be similarly defined, and then the algebras $\mathfrak{n}_3^{(a,b)}$. (See more about this notation below (2.14).)

The $\mathrm{unit}_{\mathrm{endo}}$*-deformations* $\delta_{\mathbf{F}} : A_0 \to B_0$. So far only σ_A-deformations of an ESW_A have been considered. Seemingly a new type of deformations can be introduced as follows.

Consider an endomorphism space $\mathbf{F}$ spanned by the orthonormal basis $\{, F^{(1)}, \dots, F^{(s)}\}$ and let A_0 and B_0 be unit endomorphisms ($A_0^2 = B_0^2 = -\mathrm{id}$) such that both anticommute with the elements of $\mathbf{F}$. Then the linear map defined by $A_0 \to B_0$ and $F^{(i)} \to F^{(i)}$ between $\mathrm{ESW}_{A_0} = \mathbf{R}A_0 \oplus \mathbf{F}$ and $\mathrm{ESW}_{B_0} = \mathbf{R}B_0 \oplus \mathbf{F}$ is an orthogonal transformation relating isospectral endomorphisms to each other. The latter statement immediately follows with $(F + cA_0)^2 = (F + cB_0)^2 = F^2 - c^2\mathrm{id}$.

These transformations are called $\text{unit}_{\text{endo}}$-*deformations* and are denoted by $\delta_{\mathbf{F}} : A_0 \to B_0$. Since the isospectrality theorem extends to these deformations, it is important to compare them with the σ_A-deformations. This problem is completely answered by the following theorem.

THEOREM 2.2. *Let A_0 resp. B_0 unit anticommutators with respect to the same system $\mathbf{F} = \mathrm{Span}\{F^{(1)}, \dots, F^{(s)}\}$. Then the orthogonal endomorphism $\sqrt{D}$, where the D is derived from $A_0 B_0^{-1}$ by* 2.7, *conjugates B_0 to an anticommutator of $\mathbf{F}$ such that it is a σ-deformation of A. Thus any nontrivial* $\text{unit}_{\text{endo}}$-*deformation, $\delta_{\mathbf{F}} : A_0 \to B_0$, is equivalent to a σ_{A_0}-deformation.*

A continuous family of $\text{unit}_{\text{endo}}$-*deformations is always trivial. That is, it is the family of conjugate endomorphism spaces; therefore the corresponding metric groups are isometric.*

Proof. In this proof we seek an orthogonal transformation conjugating B_0 to an endomorphism of the form $B_0' = \hat{S} \circ A_0$, where $\hat{S}$ is a symmetric endomorphism satisfying $\hat{S}^2 = \mathrm{id}$, such that the conjugation fixes, meanwhile, all the endomorphisms from $\mathbf{F}$. Then one can easily establish that B_0' is the $\sigma = \hat{S}$-deformation of A_0.

The endomorphism $E = A_0 \circ B_0^{-1} = -A_0 \circ B_0$ commutes with the F's since they anticommute both with A_0 and B_0. Decompose E into the form

$$E = A_0 \circ B_0^{-1} = -A_0 \circ B_0 = S + S^* \circ C = \widetilde{S} \circ D, \tag{2.7}$$

where $S = (1/2)(A_0 \circ B_0^{-1} + B_0^{-1} \circ A_0)$ is the symmetric part, the endomorphism $S^* \circ C = (1/2)(A_0 \circ B_0^{-1} - B_0^{-1} \circ A_0)$ is the skew-symmetric part written in scaled form ($C^2 = -\mathrm{id}$), and the orthogonal endomorphism D (commuting with all endomorphisms $\{F^{(1)}, \dots, F^{(s)}\}$) is constructed as follows.

Notice that S and $S^* \circ C$ commute and therefore a common Jordan decomposition can be established such that the matrix of E appears as a diagonal matrix of 2×2 matrices of the form

$$E_a = \begin{pmatrix} S_a & S_a^* \\ -S_a^* & S_a \end{pmatrix} = \widetilde{S}_a \begin{pmatrix} \cos\alpha_a & \sin\alpha_a \\ -\sin\alpha_a & \cos\alpha_a \end{pmatrix}, \tag{2.8}$$

where $\widetilde{S}_a = (S_a^2 + S_a^{*2})^{-1/2}$. This Jordan decomposition can be described also in the following more precise form.

The skew endomorphism $[A_0, B_0]$ vanishes exactly on the subspace K, where A_0 and B_0 commute and therefore we should deal only with these endomorphisms on the orthogonal complement $K^\perp$. On this space the nondegenerated operators $[A_0, B_0]$ and B_0 anticommute, generating the quaternionic numbers and both can be represented as diagonal quaternionic matrices such that $C_a = [B_0, A_0]_{0a} = \mathbf{I}$ and $B_{0a} = \mathbf{J}$ (cf. Lemma 2.4). Then a 4×4 quaternionic block E_a of E appears in the form $E_a = \widetilde{S}_a(\cos\alpha_a\, \mathbf{1} + \sin\alpha_a\, \mathbf{I})$.

On the subspace K (which can be considered as a complex space with the complex structure B_0), the 2×2 Jordan blocks introduced in (2.8) are $E_a = S_a \mathbf{1}$ and $B_{0a} = \mathbf{i}$.

The endomorphism B_0 commutes with the symmetric part D_+ of D and it is anticommuting with the skew part D_- of D. The same statement is true with respect to the square root operator $\sqrt{D}$, which has the Jordan blocks

$$\begin{pmatrix} \cos(\alpha_a/2) & \sin(\alpha_a/2) \\ -\sin(\alpha_a/2) & \cos(\alpha_a/2) \end{pmatrix}. \tag{2.9}$$

Therefore $\sqrt{D}B_0 = B_0\sqrt{D}^T = B_0\sqrt{D}^{-1}$ and thus

$$\sqrt{D} \circ B_0 \circ \sqrt{D}^{-1} = D \circ B_0 = \hat{S} \circ A_0, \tag{2.10}$$

where $\hat{S}$ is a symmetric, while $\hat{S}A_0$ is a skew-symmetric unit endomorphism. Thus $\hat{S}A_0 = A_0\hat{S}$, $\hat{S}^2 = \mathrm{id}$ and $B'_0 = \hat{S}A_0$ commutes with A_0.

The operator D commutes with each of the operators $\{F^{(1)}, \dots, F^{(s)}\}$; therefore the matrices of the F's are symmetric quaternionic block matrices with entries of the form $f_{ij}\mathbf{I}$ such that the blocks F_{ck} corresponding to an irreducible subspace B_{ck} are included in the blocks determined by those maximal eigensubspaces where the values $\widetilde{S}_a$ are constant. Therefore also $\sqrt{D}$ commutes with these operators.

Thus the endomorphisms from $\mathbf{F}$ anticommute with $B'_0 = \hat{S}A_0$ and commute with the orthogonal transformation $\hat{S}$. It follows that B'_0 is an anticommutator with respect to the system $\mathbf{F}$ such that it is the $\sigma = \hat{S}$-deformation of A_0.

The second part of the theorem obviously follows from the first one. Thus the proof is concluded. □

The above theorem proves that one cannot construct nontrivial continuous families of isospectral metrics by the $\text{unit}_{\text{endo}}$-deformations. The following theorem establishes a similar statement corresponding to the 2-dimensional ESW_A's.

THEOREM 2.3. *On a* 2-*dimensional* ESW_A *any* σ_A-*deformation* (*or unit-endo-deformation*) *is trivial*, *resulting in conjugate endomorphism spaces.*

Proof. This theorem is established by the following:

LEMMA 2.4. *Let A and F be anticommuting endomorphisms. If both are nondegenerated*, *they generate the quaternionic numbers and both can be represented as a diagonal quaternionic matrix such that there are* $\mathbf{I}$'*s on the diagonal of A and there are* $\mathbf{J}$'*s on the diagonal of F.*

This lemma easily settles the statement.

In fact, if the anticommuting endomorphisms A and F form a basis in the ESW_A such that both are nondegenerate, then they are represented in the above described diagonal quaternionic matrix form. Since the irreducible subspaces B_{ck} are nothing but the 4-dimensional quaternionic spaces $\mathbf{H} = \mathbf{R}^4$, a σ_A-deformation should operate such that some of the matrices $\mathbf{I}$ are switched to $-\mathbf{I}$ at some entries on the diagonal. This operation results in the new endomorphism A'. Let d_- be the set of positions where these switchings are done. Since $\mathbf{J}^{-1}\mathbf{I}\mathbf{J} = -\mathbf{I}$ and $\mathbf{J}^{-1}\mathbf{J}\mathbf{J} = \mathbf{J}$, the quaternionic diagonal matrix, having the entry $\mathbf{J}$ at a position listed in d_- and the entry $\mathbf{1}$ at the other positions, conjugates A to A' while this conjugation fixes F.

If one of the endomorphisms, say F, is degenerate on a maximal subspace K, then A leaves this space invariant. If A is nondegenerate on K, then it defines a complex structure on it. The conjugation by the reflection in a real subspace takes $-A_{/K}$ to $A_{/K}$.

The problem of conjugation is trivial on the maximal subspace L where both endomorphisms are degenerate. This proves the statement completely.

The proof is concluded by *proving Lemma* 2.4.

Represent the nondegenerate endomorphisms A and F in the scaled form $A = S_A A_0\,;\, F = S_F F_0$. Consider also the skew endomorphism $E = AF = S_E E_0$, where $S_E = S_A S_F$ and $E_0 = A_0 F_0$. It anticommutes with the endomorphisms A and F. Then the endomorphisms

$$A_0 = J_{\mathbf{i}} \quad , \quad F_0 = J_{\mathbf{j}} \quad , \quad E_0 = J_{\mathbf{k}} \tag{2.11}$$

define a quaternionic structure on the X-space and the symmetric endomorphisms S_A , S_F , S_E commute with each other as well as with the skew endomorphisms listed above.

Because of these commutativities, the X-space can be decomposed into a Cartesian product $\mathbf{v} = \oplus \mathbf{H}_i$ of pairwise perpendicular 4-dimensional quaternionic spaces such that all the above endomorphisms can be represented as diagonal quaternionic matrices. In this matrix form the entries of the matrices corresponding to the symmetric endomorphisms S are real numbers which are nothing but the eigenvalues of these matrices. From the quaternionic representation we get that each of these eigenvalue-entries has multiplicity 4.

This completes the proof both of the lemma and the theorem. □

Remark 2.5. The isospectrality theorem in [Sz5] states that σ_A-deformations provide pairs of endomorphism spaces such that the ball-type domains with the same radius-function are isospectral on the corresponding nilpotent groups as well as on their solvable extensions.

We would like to modify Remark 4.4 of [Sz5], where the extension of the above isospectrality theorem to arbitrary isospectral deformations of an anticommutator is suggested. The spectral investigation of these general deformations appears to be a far more difficult problem than it seemed to be earlier. In this paper we give only a weaker version of this generalization, where A is supposed to be a unit anticommutator.

This weaker generalization immediately follows from Theorem 2.2.

THEOREM 2.6. *The* ESW_A*-extensions of a fixed endomorphism space* $\mathbf{F} = \mathrm{Span}\{F^{(1)}, \dots F^{(k)}\}$ *by unit anticommutators A define nilpotent groups* (*and solvable extensions*) *such that for any two of these metric groups the ball-type domains with the same* X*-radius function are isospectral.*

An ESW_A-extension of the above fixed set means adding such a skew endomorphism A to the system which anticommutes with the endomorphisms $F^{(i)}$.

σ_A-deformations providing nonconjugate ESW_A*'s.* The precise forms of theorems quoted below require the precise forms of definitions given for $\sigma-$, σ_A-, $\sigma^{(a,b)}$- and $\sigma_A^{(a,b)}$-deformations, performed on an endomorphism space. These concepts were introduced in [Sz5] as follows.

Let σ be an involutive orthogonal transformation commuting with the endomorphisms of an $\mathrm{ESW}_A = \mathbf{A} \oplus \mathbf{A}^\perp$, where $\mathbf{A} = \mathbf{R}A$. Then the σ_A-deformation of the endomorphism space is defined by deforming A to $A' = \sigma \circ A$ while keeping the orthogonal endomorphisms unchanged. The deformation of a general element is defined according to the direct sum $\mathrm{ESW}_A = \mathbf{A} \oplus \mathbf{A}^\perp$.

These deformations provide spectrally equivalent endomorphism spaces, since

$$(A^\perp + \sigma A)^2 = (A^\perp)^2 + A^2 = (A^\perp + A)^2. \tag{2.12}$$

Therefore, there exists an orthogonal transformation between ESW_A and $\mathrm{ESW}_{A'}$ such that the corresponding endomorphisms are isospectral

Variants of these deformations are the so called $\sigma_A^{(a,b)}$-deformations defined as follows.

Consider an $\mathrm{ESW}_A = J_{\mathbf{z}} = \mathbf{A} \oplus \mathbf{A}^\perp$ such that the endomorphisms act on $\mathbf{R}^n$. For a pair (a, b) of natural numbers the endomorphism space $\mathrm{ESW}_A^{(a,b)} = J_{\mathbf{z}}^{(a,b)}$ is defined by a new representation, $B^{(a,b)} = J_B^{(a,b)}$, of the endomorphisms $B \in \mathrm{ESW}_A$ on the new X-space $\mathbf{v} = \mathbf{R}^n \times \cdots \times \mathbf{R}^n$ (the Cartesian product is taken $(a + b)$-times) such that the endomorphisms $A^{(a,b)}$ and $F^{(a,b)}$, where $F \in A^\perp$, are defined by

$$\begin{aligned} A^{(a,b)}(X) &= (A(X_1), \dots, A(X_a), -A(X_{a+1}), \dots, -A(X_{a+b})), \\ F^{(a,b)}(X) &= (F(X_1), \dots, F(X_{a+b})). \end{aligned} \tag{2.13}$$

If $\sigma^{(a,b)}$ is the involutive orthogonal transformation defined on $\mathbf{v}$ by

$$\sigma^{(a,b)}(X) = (X_1, \dots, X_a, -X_{a+1}, \dots, -X_{a+b}), \tag{2.14}$$

then the $\sigma_A^{(a,b)}$-deformation sends $\mathrm{ESW}_A^{(a,b)}$ and $\mathrm{ESW}_A^{(a+b,0)}$ to each other.

In [Sz5] also another type of deformation, called a σ-deformation, was introduced. It is defined for general endomorphism spaces such that the deformation $\sigma \circ B$ is performed on all elements of the endomorphism space. (Also in this case the σ is an involutive orthogonal transformation commuting with all the elements of the endomorphism space.)

Though they seem to be completely different deformations, Reduction Theorem 4.1 in [Sz5] asserts that, on ESW_A's, σ_A-deformations are equivalent to σ-deformations. In this spectral investigation we prefer the σ_A deformations to the σ deformations of an ESW_A because of the simplicity offered by considering only the deformation of a single endomorphism.

The 2-step nilpotent Lie algebras (resp. Lie groups) corresponding to $\mathrm{ESW}_A^{(a,b)} = J_{\mathbf{z}}^{(a,b)}$ is denoted by $\mathfrak{n}_J^{(a,b)}$ (resp. $N_J^{(a,b)}$).

This notation is consistent with the notation of $\mathfrak{h}_3^{(a,b)}$. In this case the space $\mathbf{z} = \mathbf{R}^3$ is identified with the space of the imaginary quaternions, and the skew endomorphisms $J_Z = L_Z$ acting on $\mathbf{R}^4 = \mathbf{H}$ are defined by left products with Z. Notice that in this case $J_{\mathbf{z}} \simeq so(3) \subset so(4)$ hold and this endomorphism space is closed with respect to the Lie bracket.

In the case of Cayley numbers the Z-space $\mathbf{z} = \mathbf{R}^7$ is identified with the space of imaginary Cayley numbers and the endomorphism space $J_{\mathbf{z}} = R_{\mathbf{z}}$ is defined by the right product described below formula (2.6) (the left products result in equivalent endomorphism spaces). Notice that this endomorphism space is not closed with respect to the Lie bracket. The corresponding Lie algebra is denoted by $\mathfrak{h}_7^{(a,b)}$.

Note that σ-deformations provide pairs of endomorphism spaces such that the metrics on the corresponding groups have different local geometries in general. Nonisometry Theorem 2.1 in [Sz5] asserts that *for endomorphism spaces $J_{\mathbf{z}}$ which are either non-Abelian Lie algebras or, more generally, contain non-Abelian Lie subalgebras, the metric on $N_J^{(a,b)}$ is locally nonisometric to the metric on $N_J^{(a',b')}$ unless $(a,b) = (a',b')$ up to an order. Yet the Ball× Torus-type domains are both Dirichlet and Neumann isospectral on these locally different spaces.*

The key idea of this theorem's proof is that $\sigma^{(a,b)}$-deformations impose changing on the algebraic structure of the endomorphism spaces and that is why they cannot be conjugate.

This general theorem proves the nonisometry with respect to the groups $H_3^{(a,b)}$, however, it does not prove it with respect to the $H_7^{(a,b)}$'s, or, for the other Cliffordian endomorphism spaces. Fortunately enough, the nonisometry statement in the latter case is well known (described at (2.6)) and can be established exactly for those Heisenberg-type groups, $H_l^{(a,b)}$, where $l = 3\mathrm{mod}(4)$.

The nonisometry proofs on the solvable extensions are traced back to the nilpotent subgroups and on the sphere-type domains they are traced back to the ambient space. That is, the question of nonisometry is always traced back to the question of nonconjugacy of the corresponding endomorphism spaces and, therefore, to the above theorem.

3. Isospectrality theorems on sphere-type manifolds

In [Sz5], the isospectrality theorems are completely established on ball- and ball×torus-type manifolds; however, the proofs are only outlined on the boundary, i.e., on sphere- and sphere×torus-type manifolds. Even these sketchy details concentrate mostly on the striking examples. In this chapter the isospectrality theorems are completely established also on these boundary manifolds.

There are three sections ahead. In the first two sections the nilpotent case is considered where, after establishing an explicit formula for the Laplacian on the boundary manifolds, the isospectrality theorems are accomplished by constructing intertwining operators. In the third section these considerations are settled on the solvable extensions.

Normal vector field and Laplacian on the boundary manifolds. We start by a brief description of the ball×torus- and ball-type domains in the nilpotent case.

(1) Let Γ be a full lattice on the Z-space spanned by a basis $\{e_1, \dots, e_l\}$. For an l-tuple $\alpha = (\alpha_1, \dots, \alpha_l)$ of integers the corresponding lattice point is $Z_\alpha = \alpha_1 e_1 + \cdots + \alpha_l e_l$. Since Γ is a discrete subgroup, one can consider the factor manifold $\Gamma\backslash N$ with the factor metric. This factor manifold is a principal fibre bundle with the base space $\mathbf{v}$ and with the fibre T_X at a point $X \in \mathbf{v}$. Each fibre T_X is naturally identified with the torus $T = \Gamma\backslash\mathbf{z}$. The projection $\pi : \Gamma\backslash N \to \mathbf{v}$ defined by $\pi : T_X \to X$ projects the inner product from the horizontal subspace (defined by the orthogonal complement of the fibres) to the Euclidean inner product $\langle\, , \rangle$ on the X-space.

Consider also a Euclidean ball B_δ of radius δ around the origin of the X-space and restrict the fibre bundle onto B_δ. Then the fibre bundle (B_δ, T) has the boundary (S_δ, T), which is also a principal fibre bundle over the sphere S_δ.

Prior to paper [Sz5] only these manifolds were involved to constructions of isospectral metrics with different local geometries.

(2) In these papers we consider also such domains around the origin which are homeomorphic to a $(k + l)$-dimensional ball and their smooth boundaries can be described as level surfaces by equations of the form $f(|X|, Z) = 0$. The boundaries of these domains are homeomorphic to the sphere S^{k+l-1} such that the boundary points form a Euclidean sphere of radius $\delta(Z)$, for any fixed Z. That is, the boundary can be described by the equation $|X|^2 - \delta^2(Z) = 0$. We call these cases *Ball-cases* resp. *Sphere-cases.*

In this section we provide explicit formulas for the normal vector field and for the Laplacian on the sphere-type manifolds only. However, these formulas establish the corresponding formulas also in the sphere×torus-cases, such that one substitutes the constant radius R for the function $\delta(Z)$ in order to have the formula also on the latter manifolds.

First the normal vector $\boldsymbol{\mu}$ is computed. From the equation

$$\nabla f = \operatorname{grad} f = \sum_i \mathbf{X}_i(f)\mathbf{X}_i + \sum_\alpha \mathbf{Z}_\alpha(f)\mathbf{Z}_\alpha$$

we get (by using the special function $f(|X|, Z) = |X|^2 - \delta^2(Z)$) that this unit normal vector at a point (X, Z) is

$$\boldsymbol{\mu} = \left(4|X|^2 + \frac{1}{4}|J_{\nabla\delta^2}(X)|^2 + |\nabla\delta^2|^2\right)^{-\frac{1}{2}} \left(2X - \frac{1}{2}\mathbf{J}_{\nabla\delta^2}(X) - \nabla\delta^2\right), \tag{3.1}$$

where $\boldsymbol{\mu}$ is considered as an element of the Lie algebra. Notice that in the sphere×torus case the $\boldsymbol{\mu}$ has the simple form $\boldsymbol{\mu} = X/|X|$.

By (1.2), this normal vector can be written also in the following regular vector form:

(3.2)

$$\boldsymbol{\mu} = C\left(2|X|E_0 - \sum_{\alpha=1}^{l}(\partial_\alpha\delta^2)\left(\frac{1}{2}|X|E_\alpha + \sum_{\beta=1}^{l}\left(1 + \frac{1}{4}\langle J_\alpha(X), J_\beta(X)\rangle\right)e_\beta\right)\right),$$

where $\{e_1, \dots, e_l\}$ is an orthonormal basis of $\mathbf{z}$ and $E_\alpha = J_\alpha(E_0); E_0 = X/|X|$; furthermore,

$$C = (4|X|^2 + \frac{1}{4}|J_{\nabla\delta^2}(X)|^2 + |\nabla\delta^2|^2)^{-\frac{1}{2}}. \tag{3.3}$$

In the following we always make it clear which representation of a particular vector is considered.

Over a fixed point Z, the X-cross section with the boundary ∂D is the sphere $S_X(Z)$ with radius $\delta(Z)$. The corresponding Z-cross section over a fixed point X is denoted by $S_Z(X)$. Notice that these latter manifolds are only homeomorphic to Euclidean spheres in general and they are Euclidean spheres for all points X if and only if the function δ depends only on $|Z|$. The Euclidean(!) normal vector $\boldsymbol{\mu}_Z$ to $S_Z(X)$ is

$$\boldsymbol{\mu}_Z = \left(\sum_\alpha(\partial_\alpha\delta)^2\right)^{-1/2}\sum_\beta \partial_\beta(\delta)e_\beta = \sum \boldsymbol{\mu}_{Z\beta}, \tag{3.4}$$

which is different from the orthogonal projection of $\boldsymbol{\mu}$ onto the Z-space.

Let $\tilde\nabla$ (resp. $\tilde\Delta$) be the covariant derivative (resp. the Laplace operator) on the boundary ∂D. The second fundamental form and the Minkowski curvature are denoted by $M(V, W)$ and $\mathbf{M}$. Then the formula

$$\nabla^2 f(V, V) = V \cdot V(f) - \nabla_V V \cdot (f) = \tilde\nabla^2 f(V, V) + M(V, V)f' \tag{3.5}$$

($f' := \boldsymbol{\mu} \cdot (f)$) holds for any function f defined on the ambient space and for any vector field V tangent to ∂D. Thus

$$\tilde{\Delta} f = \Delta f - f'' - \mathbf{M} f'. \tag{3.6}$$

Choose such functions f around the boundary ∂D which are constant with respect to the normal direction (i.e. $f' = f'' = 0$). Then check the formula

$$\begin{aligned} \tilde{\Delta} = \Delta_{S_X(Z)} + \Delta_{S_Z(X)} + \sum_{\alpha=1}^{l} (\partial_\alpha - \boldsymbol{\mu}_{Z\alpha}) D_\alpha \bullet \\ + \frac{1}{4} \sum_{\alpha\beta=1}^{l} \langle J_\alpha(X), J_\beta(X) \rangle (\partial_\alpha - \boldsymbol{\mu}_{Z\alpha})(\partial_\beta - \boldsymbol{\mu}_{Z\beta}). \end{aligned} \tag{3.7}$$

This formula is simpler on the level surfaces described by equations of the form $|X| = \delta(|Z|)$ (cf. Chapter 4). On the sphere×torus-type manifolds we get the Laplacian by performing the simple modification $\boldsymbol{\mu}_{Z\alpha} = 0$ in the above formula.

Isospectrality theorems on sphere-type manifolds. In order to establish the isospectrality theorems on sphere-type manifolds, one should appropriately modify the technique developed for the ambient ball-type manifolds in [Sz5]. The main tool of this technique is the following:

Harmonic analysis developed for a unit anticommutator J_A. A brief account of this analysis is as follows.

As indicated, the anticommutator A is a unit anticommutator. By the normalization described at the beginning of Section 2, each nondegenerate anticommutator can be rescaled to a unit anticommutator.

First notice that the Euclidean Laplacian Δ_S (defined on the unit sphere S around the origin of the X-space) and the differential operator $\mathbf{D}_A \bullet$ commute since the vector field $J_A(X_u)$, where $X_u \in S$, is an infinitesimal generator of isometries on S. Therefore a common eigensubspace decomposition of the L^2 function space exists which can be established as follows.

The eigenfunctions of Δ_S are the well known spherical harmonics which are the restrictions of the homogeneous harmonic polynomials of the ambient X-space onto the sphere S. The space of the q^{th}-order spherical harmonic polynomials is denoted by $\mathbf{H}^{(q)}$. In the following we describe the eigenfunctions of the operator $\mathbf{D}_A \bullet$.

For a fixed X-vector Q, we define the complex valued function

$$\Theta_Q(X) = \langle Q + \mathbf{i} J_A(Q), X \rangle = \langle \mathbf{Q}, X \rangle, \tag{3.8}$$

where $\mathbf{Q} = Q + \mathbf{i} J_A(Q)$. Then the polynomials of the form

$$\Phi_{(Q_i, p_i, Q_j^*, p_j^*)}(X) = \Pi_{i=1}^{r} \Theta_{Q_i}^{p_i}(X) \Pi_{j=1}^{r^*} \overline{\Theta}_{Q_j^*}^{p_j^*}(X) \tag{3.9}$$

are eigenfunctions of the operator $\mathbf{D}_A\bullet$ with the eigenvalue $(2s-q)\mathbf{i}$, where $q=\sum_i p_i+\sum_j p_j^*=s+(q-s)$.

Notice, that the functions of the pure form

$$\Phi_{(Q_i,p_i)}(X)=\Pi_{i=1}^r\Theta_{Q_i}^{p_i}(X)\quad;\quad\overline{\Phi}_{(Q_i,p_i)}(X)=\Pi_{i=1}^r\overline{\Theta}_{Q_i}^{p_i}(X)\tag{3.10}$$

are harmonic with respect to the Euclidean Laplacian Δ_X on the X-space. In fact, on the Euclidean Kähler manifold $\{\mathbf{v},\langle,\rangle,A\}$ these functions correspond to the holomorphic resp. anti-holomorphic polynomials. One can directly check this property also by $\langle\mathbf{Q}_i,\mathbf{Q}_j\rangle=\langle\overline{\mathbf{Q}_i},\overline{\mathbf{Q}_j}\rangle=0$.

However, the polynomials of the mixed form are not harmonic, since

$$\Delta_X\Theta_Q(X)\overline{Q}_{Q^*}(X)=2\langle Q,Q^*\rangle+2\mathbf{i}\langle J_A(Q),Q^*\rangle.\tag{3.11}$$

Let us note that the whole space $\mathbf{H}^{(q)}$ of q^{th} order eigenfunctions is not spanned by the above polynomials of pure form. The "missing" functions can be furnished by the orthogonal projection of the q^{th} order mixed polynomials $\Phi_{(Q_i,p_i,Q_i^*,p_i^*)}$ onto the function space $\mathbf{H}^{(q)}$. The range of this projection is denoted by $\mathbf{H}^{(s,q-s)}$. Since the operators Δ_S and $D_A\bullet$ commute, the space $\mathbf{H}^{(q)}$ is invariant under the action of $D_A\bullet$ and the subspace $\mathbf{H}^{(s,q-s)}\subset\mathbf{H}^{(q)}$ is an eigensubspace of this operator with the eigenvalue $(2s-q)\mathbf{i}$. Thus the decomposition $\mathbf{H}^{(q)}=\oplus_{s=0}^q\mathbf{H}^{(s,q-s)}$ is an orthogonal direct sum corresponding to the common eigensubspace decomposition of the two commuting differential operators Δ_S and $D_A\bullet$.

A more accurate description of the above mentioned projections can be given by the kernel functions $H_{(q)}(Q_u,Q_u^*)$ introduced for the subspaces $\mathbf{H}^{(q)}$ by

$$H_{(q)}(Q_u,Q_u^*)=\sum_{j=1}^{N_q}\eta_j^{(q)}(Q_u)\eta_j^{(q)}(Q_u^*),\tag{3.12}$$

where $\{\eta_1^{(q)},\dots,\eta_{N_q}^{(q)}\}$ is an orthonormal basis on the subspace $\mathbf{H}^{(q)}$. In [Be] it is proved (cf. Lemma 6.94) that the eigenfunction $H_{(q)}(Q_u,.)$ is radial for any fixed Q_u. (That is, it has the form $C_q\langle Q_u,.\rangle^q+\dots+C_1\langle Q_u,.\rangle+C_0$ with $H_{(q)}(Q_u,Q_u)=1$.) Furthermore, for any function $\Psi\in L^2(S)$ we have

$$\Psi(Q_u)=\sum_{q=0}^\infty\int H_{(q)}(Q_u,Q_u^*)\Psi(Q_u^*)dQ_u^*=\sum h_{(q)}(\Psi)_{/Q_u},\tag{3.13}$$

which is called the *spherical decomposition of* Ψ *by the spherical harmonics.* The operators $h_{(q)}:L^2(S)\to\mathbf{H}^{(q)}$ project the L^2 function space to the corresponding eigensubspace of the Laplacian.

For an r^{th} order polynomial $\Pi_i\Theta_{Q_i}^{p_i}\overline{\Theta}_{Q_i}^{r_i-p_i}$, $\sum r_i=r$, the projection $h_{(r)}$ can be computed also by the formula

$$h_{(r)}(\Pi_i\Theta_{Q_i}^{p_i}\overline{\Theta}_{Q_i}^{r_i-p_i})=\sum_s B_s\langle X,X\rangle^s\Delta_X^s\Pi_i\Theta_{Q_i}^{p_i}\overline{\Theta}_{Q_i}^{r_i-p_i},\tag{3.14}$$

where $B_0 = 1$ and the other coefficients can be determined by the recursive formula

$$2s(2(s+r)-1)B_s + B_{s-1} = 0.$$

These formulas are established by the fact that the function on the right side of (3.14) is a homogeneous harmonic polynomial exactly for these coefficients.

One of the most important properties of these operators is that they commute with the differential operators $D_\alpha \bullet$. This statement immediately follows from the fact that the vector fields $J_\alpha(X)$ are infinitesimal generators of one-parametric families of isometries on the Euclidean sphere S and the projections $h_{(q)}$ are invariant with respect to these isometries. One can imply this commutativity also by (3.14) and by the commutativity of the operators Δ_X and $D_\alpha \bullet$. By the spherical decomposition theorem the kernel of this projection on the r^{th} order polynomial space $\mathbf{P}^{(r)}$ consists of those (nonproperly represented) polynomials which are products of radial and lower order polynomials. They are properly represented on the lower level.

By substituting $Q_u = 1/2(\mathbf{Q}_u + \overline{\mathbf{Q}}_u)$, where $\mathbf{Q} = Q + \mathbf{i}J_A(Q)$, into the above expression of the radial kernel $H_{(q)}(Q_u, .)$, we get

Proposition 3.1 ([Sz5]). *The polynomial space $\mathbf{P}^{(r)}$ is the direct sum of the subspaces $\mathbf{P}^{(p,r-p)}$ spanned by the polynomials of the form $\Theta_Q^p \overline{\Theta}_Q^{r-p}$, where $Q \in \mathbf{v}$ and $0 \leq p \leq r$. The space $\mathbf{P}^{(p,r-p)}$ consists of the r^{th} order eigen polynomials of the differential operator $D_A \bullet$ with eigenvalue $(r-p)\mathbf{i}$.*

The projection $h_{(r)}$ is surjective establishing a one to one map, $h_{/(r)}$, between a complement $\mathbf{P}_{/S}^{(p,r-p)}$ of the kernel and $\mathbf{H}^{(p,r-p)}$. The direct sum $T = \oplus_r h_{/(r)}$ of these maps defines an invertible operator on the whole function space L_S^2. This operator T commutes with the differential operators $D_\alpha \bullet$.

Above, $\mathbf{P}_S^{(p,r-p)}$ denotes the space of the corresponding restricted functions onto the sphere S. In the following we represent the functions $\psi \in \mathbf{H}^{(r)}$ in the form $\psi = h_{/(r)}(\psi^*)$, where $\psi^* \in \mathbf{P}_S^{(r)}$. On the whole ambient space $\mathbf{n}$, the function space is spanned by the functions of the form

$$F(X,Z) = \varphi(|X|, Z) h_{(r)}(\Theta_{Q_u}^p \overline{\Theta}_{Q_u}^{r-p})(X_u), \tag{3.15}$$

where $X_u = X/|X|$ is a unit vector. The decomposition with respect to these functions is called *spherical decomposition*.

Constructing the intertwining operator. Let $J_{\mathbf{z}} = J_{\mathbf{A}} \oplus J_{\mathbf{A}^\perp}$ and $J_{\mathbf{z}'} = J_{\mathbf{A}'} \oplus J_{\mathbf{A}'^\perp}$ be endomorphism spaces with the unit anticommutators J_A and $J_{A'}$ such that the endomorphisms are acting on the same space and, even more, $J_{\mathbf{A}^\perp} = J_{\mathbf{A}'^\perp}$ holds. In [Sz5] we proved that the ball- and ball×torus-type domains with the same radius-function are both Dirichlet and Neumann isospectral on the metric groups $N = N_{J_{\mathbf{z}}}$ and $N' = N_{J_{\mathbf{z}'}}$ constructed by these

ESW$_A$'s. This isospectrality theorem is established for σ_A-deformation where A is not necessarily a unit anticommutator.

For the proof of this theorem we constructed the intertwining operator

$$\kappa : L^2(N,\mathbf{C}) \to L^2(N',\mathbf{C}),\qquad \kappa : F(X,Z) \to F'(X,Z) = \varphi(|X|,Z)h'_{(r)}(\Theta'^p_{Q_u}\overline{\Theta'}^{r-p}_{Q_u})(X_u), \tag{3.16}$$

with F as introduced in (3.15). The functions Θ' are defined on N' by means of $J_{A'}$.

The function space $\mathbf{H}^{(r)}$ maps onto $\mathbf{H}'^{(r)}$ and is defined by means of the map

$$\kappa^* : \mathbf{P}^{(r)} \to \mathbf{P}'^{(r)} \quad , \quad \kappa^* : \Theta^p_Q\overline{\Theta}^{r-p}_Q \to \Theta'^p_Q\overline{\Theta}'^{r-p}_Q \tag{3.17}$$

and of the projection $h_{(r)}$. Unlike κ, the map κ^* can be easily handled. If $\{E_1,\dots,E_K\}$ is a basis on the X-space then $\kappa^*(\Pi\Theta_{E_{i_r}}\Pi\overline{\Theta}_{E_{j_r}}) = \Pi\Theta'_{E_{i_r}}\Pi\overline{\Theta}'_{E_{j_r}}$. In general, for arbitrary vectors Q_m, we get

$$\kappa^*(\Pi\Theta_{Q_{i_r}}\Pi\overline{\Theta}_{Q_{j_r}}) = \Pi\Theta'_{Q_{i_r}}\Pi\overline{\Theta}'_{Q_{j_r}}. \tag{3.18}$$

The operators κ and κ^* are connected by the equation $\kappa = T'\circ\kappa^*\circ T^{-1}$. Since the kernels of the projections involved are corresponded by κ^*, the κ is independent from the particular choice of the complements $\mathbf{P}^{(r)}_{/S}$. In [Sz5], the intertwining property of the map κ is proved.

The intertwining operator $\partial\kappa$ on the sphere-type boundary is constructed by an appropriate restriction of κ onto the boundary. Also in this case, we first suppose that J_A is a unit anticommutator and in the end we make the necessary modifications in order to establish the intertwining for general σ_A-deformations.

The function space $L^2(\partial B)$ is spanned by the functions of the form

$$F(\partial X,\partial Z) = \varphi(|\partial X|,\partial Z)h_{(r)}(\Theta^p_{Q_u}\overline{\Theta}^{r-p}_{Q_u})(X_u), \tag{3.19}$$

where $(\partial X,\partial Z)\in\partial B$. Then the operators $\partial\kappa,\partial\kappa^* L^2(\partial B)\to L^2(\partial B)$ are defined again by the formulas (3.16)–(3.17), which are used for defining κ and κ^* on the ambient space. However, in this case, the function φ depends on variables $(|\partial X|,\partial Z)$. Then the intertwining property can be proved by the following steps.

Consider the explicit expression (1.6) of the Laplacian $\tilde{\Delta}$ on ∂B. Since $\partial\kappa : \mathbf{H}^{(q)} \to \mathbf{H}'^{(q)} = \mathbf{H}^{(q)}$, the terms $\Delta_{S_X(Z)} + \Delta_{S_Z(X)}$ and $\Delta'_{S_X(Z)} + \Delta'_{S_Z(X)}$ are clearly intertwined by this map.

For the next step choose an orthonormal basis $\{e_0,e_1,\dots,e_{l-1}\}$ on the Z-space such that $e_0 = A$. The Greek characters are used for the indices $\{0,1,\dots,l-1\}$ and the Latin characters are used for the indices $\{1,\dots,l-1\}$. Then $D_c\bullet = D'_c\bullet$; furthermore,

$$D_0\bullet\Theta_Q(X) = \mathbf{i}\Theta_Q(X) \quad , \quad D_0\bullet\overline{\Theta}_Q(X) = -\mathbf{i}\overline{\Theta}_Q(X), \tag{3.20}$$

and

$$D_c \bullet \Theta_Q = \langle \mathbf{Q}, J_c(X)\rangle = -\overline{\Theta}_{J_c(Q)} \quad , \quad D_c \bullet \overline{\Theta}_Q = -\Theta_{J_c(Q)}. \tag{3.21}$$

(Let us mention that the switching of the conjugation in (3.20) is due to the equation $J_0J_c = -J_cJ_0$.) Therefore $\partial\kappa^* D_\alpha \bullet (\psi^*) = D'_\alpha \bullet \partial\kappa^*(\psi^*)$ for any $\psi^* \in \mathbf{P}_S^{(r)}$. Since $\partial\kappa = T' \circ \partial\kappa^* \circ T^{-1}$, we also have $\partial\kappa D_\alpha \bullet (\psi) = D'_\alpha \bullet \partial\kappa(\psi)$ and thus the terms $\sum(\partial_\alpha - \boldsymbol{\mu}_{Z\alpha})D_\alpha\bullet$ and $\sum(\partial_\alpha - \boldsymbol{\mu}_{Z\alpha})D'_\alpha\bullet$ in (1.6) are intertwined by the map $\partial\kappa$.

Only the term $(1/4)\sum\langle J_\alpha(X), J_\beta(X)\rangle(\partial_\alpha - \boldsymbol{\mu}_{Z\alpha})(\partial_\beta - \boldsymbol{\mu}_{Z\beta})$ should be considered yet.

First notice that on the Heisenberg-type groups $H_l^{(a,b)}$, this operator is nothing but $(1/4)|X|^2\Delta_{S_Z(X)}$. Therefore it is intertwined by the $\partial\kappa$ and the proof is completely established on this rather wide range of manifolds defined by Cliffordian endomorphism spaces $J_l^{(a,b)}$.

Now consider this operator in general cases. Since $J_0 \circ J_c$ is a skew symmetric endomorphism, $\langle J_0(X), J_c(X)\rangle = 0$. Thus this operator is the same one on the considered spaces N and N'. Yet, for the sake of completeness, we should prove that the $\partial\kappa$ maps a function of the form

$$\langle J_c(X), J_d(X)\rangle h_{(r)}(\Theta^p\overline{\Theta}^{r-p}) := J_{cd}(X)h_{(r)}(\Theta^p\overline{\Theta}^{r-p}) \tag{3.22}$$

to a function of the very same form on N'. In [Sz5] this problem is settled (cf. formulas (4.13)–(4.17)) by proving, first, that in the spherical decomposition of the function $J_{cd}\Theta_Q^p\overline{\Theta}^{r-p}$ the component-spherical-harmonics are linear combinations of the functions of the form

$$\Theta_P\overline{\Theta}_R\Theta_{Q_u}^{p-s}\overline{\Theta}_{Q_u}^{r-p-s} \quad , \quad \Theta_{Q_u}^{p-v}\overline{\Theta}_{Q_u}^{r-p-v}$$

such that the combinational coefficients depend only on the constants

$$r\,,\ p\,,\ s\,,\ v\,,\ \mathrm{Tr}(J_c \circ J_d)\,,\ \langle J_c(Q_u), J_d(Q_u)\rangle.$$

Then the same statement is established for the preimages (with respect to the map T) of these functions. Since these terms do not depend on J_A, the function $J_{cd}h_{(r)}\Theta_Q^p\overline{\Theta}_Q^{r-p}$ is intertwined with the corresponding function $J_{cd}h_{(r)}\Theta_Q'^p\overline{\Theta}_Q'^{r-p}$ by the map κ.

This argument settles the proof of intertwining also for $\partial\kappa$, yet we would like to give a simpler and more comprehensive proof for this part.

The radial spherical harmonics span the eigensubspaces $\mathbf{H}^{(r)}$ (cf. (3.13)). Therefore, it is enough to consider the real functions of the form $K_{(r)}(X) = J_{cd}(X)h_{(r)}\langle Q, X\rangle^r$. In this case the function $h_{/(r)}\langle Q, X\rangle^r$ is nothing but a constant multiple of $H_{(r)}(Q, X)$. By (3.14), the spherical decomposition of $K_{(r)}$ is $K_{(r)} = \sum C_p\langle X, X\rangle^p L_p(X)$, where the functions $L_p(X)$ are spherical harmonics built up by the functions of the form

$$\langle Q,X\rangle^p\,,\ J_{cd}(X)\langle Q,X\rangle^q\,,\qquad (3.23)$$

$$\langle (J_cJ_d+J_dJ_c)(Q),X\rangle^s\langle Q,X\rangle^v=\langle Q_{cd},X\rangle^s\langle Q,X\rangle^v,$$

such that the combinational coefficients depend only on the constants r,p,s,v, $\mathrm{Tr}J_c\circ J_d$ and on $\langle J_c(Q),J_d(Q)\rangle$. More precisely, the function L_p is of the form $L_p=h_{(p)}U_p$, where U_p is one of the functions from the set (3.23).

The function $J_{cd}(X)$ can be written in the form

(3.24)

$$J_{cd}(X)=\sum_{i=1}^{K}\frac{1}{4}\{(\langle X,\mathbf{Q}_{ci}\rangle\langle X,\overline{\mathbf{Q}}_{di}\rangle+\langle X,\overline{\mathbf{Q}}_{ci}\rangle\langle X,\mathbf{Q}_{di}\rangle)$$

$$+(\langle X,\mathbf{Q}_{ci}\rangle\langle X,\mathbf{Q}_{di}\rangle+\langle X,\overline{\mathbf{Q}}_{ci}\rangle\langle X,\overline{\mathbf{Q}}_{di}\rangle)\}=J^{(1)}_{cd}(X)+J^{(2)}_{cd}(X),$$

where $E_1,\dots,E_K$ is an orthonormal basis on the X-space ($K=k(a+b)$) and $\mathbf{Q}_{ei}=\mathbf{J}_e(\mathbf{E}_i)$. The proof of this formula immediately follows from

$$J_{cd}(X)=\sum_i\langle J_c(X),E_i\rangle\langle J_d(X),E_i\rangle\quad,\quad E_i=\frac{1}{2}(\mathbf{E}_i+\overline{\mathbf{E}}_i).\qquad (3.25)$$

Notice that the second function of (3.24) is vanishing. This statement immediately follows from equations $\langle\mathbf{Q}_1,\mathbf{Q}_2\rangle=\langle\overline{\mathbf{Q}}_1,\overline{\mathbf{Q}}_2\rangle=0$.

By the substitution $Q=(1/2)(\mathbf{Q}+\overline{\mathbf{Q}})$ we get that the spherical harmonics L_p are linear combinations of functions of the form

$$\Theta_Q^{p-s}\overline{\Theta}_Q^{r-p-s}\,,\ J_{cd}\Theta_Q^{p-s}\overline{\Theta}_Q^{r-p-s}\,,\ (\Theta_{Q_{cd}}+\overline{\Theta}_{Q_{cd}})\Theta_Q^{p-v}\overline{\Theta}_Q^{r-p-v}\qquad (3.26)$$

such that the combinational coefficients depends only on the constants r,p,s,v, $\mathrm{Tr}J_c\circ J_d$ and $\langle J_c(Q),J_d(Q)\rangle$.

The considered problem can be settled by representing J_{cd} in the form (3.24) in formula (3.22). In fact, the coefficients discussed above are the same on both spaces (since they do not depend on the unit anticommutator J_0). By (3.14) we get that the preimages (with respect to the map T) of spherical-harmonics L_p are combinations of appropriate functions which have the same form (3.26) on the other manifold and the coefficients do not depend on J_0. This completely proves that $\partial\kappa$ maps a function $K_{(r)}$ to an appropriate function desired in this problem.

The above constructions and proofs can be easily extended to the cases, when J_A is just a nondegenerated anticommutator and the anticommutators A and A' are σ_A-related.

In this case, first, the unit anticommutator A_0 should be introduced by an appropriate rescaling of A. This A_0 may not be in the endomorphism space; however, it commutes with A and it anticommutes with the elements of $J_{\mathbf{A}^\perp}$. The operator $\partial\kappa$ should be established by means of A_0.

Since σ-deformations do not change the maximal eigensubspaces of the operators involved, the operators $D_\alpha \bullet$ and $D'_\alpha \bullet$ are intertwined by$\partial\kappa$. By the very same reason also the operators $\langle J_A(X), J_A(X)\rangle(\partial_0 - \boldsymbol{\mu}_{Z0})^2$ and $\langle J_{A'}(X), J_{A'}(X)\rangle(\partial_0 - \boldsymbol{\mu}_{Z0})^2$ are intertwined by the $\partial\kappa$. Since these are the only terms in the Laplacian which depend on the eigenvalues of the endomorphism J_A, the intertwining property is established in the considered case. Thus we have.

Main Theorem 3.2. *Let $J_{\mathbf{z}} = J_{\mathbf{A}} \oplus J_{\mathbf{A}^\perp}$ and $J_{\mathbf{z}'} = J_{\mathbf{A}'} \oplus J_{\mathbf{A}'^\perp}$ be endomorphism spaces acting on the same space such that $J_{\mathbf{A}^\perp} = J_{\mathbf{A}'^\perp}$. Furthermore, the anticommutators J_A and $J_{A'}$ are either unit endomorphisms or they are σ-related. Then the map $\partial\kappa = T' \circ \partial\kappa^* T^{-1}$ intertwines the corresponding Laplacians on the sphere-type boundary ∂B of any ball-type domain on the metric groups N_J and $N_{J'}$. Therefore the corresponding metrics on these sphere-type manifolds are isospectral.*

Remark 3.3. The map $\partial\kappa$ establishes the isospectrality theorem also on the sphere$\times$torus-type boundaries of the ball$\times$torus-domains in the considered cases, offering a completely new proof for the theorem.

Intertwining operators on the solvable extensions. The above isospectrality theorem extends to the solvable extensions of nilpotent groups. In this case one should consider the following domains.

The solvable ball $\times$ torus cases. A group SN can be considered as a principal fibre bundle (vector bundle) over the (X,t)-space, fibrated by the Z-spaces. Let Γ be again a full lattice on the Z-space, and $D_{R(t)}$ be a domain on the (X,t)-space such that it is diffeomorphic to a $(k+1)$-dimensional ball whose smooth boundary can be described by an equation of the form $|X| = R(t)$. Then consider the torus bundle $(\Gamma/\mathbf{z}, D_{R(t)})$ over $D_{R(t)}$. The normal vector $\boldsymbol{\mu}$ at a boundary point (X,Z,t) is of the form $\boldsymbol{\mu} = A(t)X + B(t)\partial_t$, where $A(t)$ and $B(t)$ are determined by $R(t)$.

The solvable ball case. In this case we consider a domain D diffeomorphic to a $(k+l+1)$-dimensional ball whose smooth boundary (diffeomorphic to S^{k+l}) can be described as a levelset in the form $|X| = \delta(Z,t)$.

The normal vector $\boldsymbol{\mu}$ at a boundary point (X,Z,t) can be similarly computed as in the nilpotent case. Then, by (1.10) and (3.1), we get:

$$\boldsymbol{\mu} = F(|X|,Z,t,c)t^{\frac{1}{2}}\left(2X - \frac{1}{2}\mathbf{J}_{\mathrm{grad}_Z\delta^2}(X)\right) - t(\mathrm{grad}_Z\delta^2 + c\partial_t(\delta^2)\mathbf{T}), \tag{3.27}$$

where $\boldsymbol{\mu}$ is considered as an element of the Lie algebra and

$$F = \left(4t|X|^2 + \frac{1}{4}t|J_{\mathrm{grad}_Z\delta^2}(X)|^2 + t^2(|\mathrm{grad}_Z\delta^2|^2 + c^2(\partial_t\delta^2)^2)\right)^{-\frac{1}{2}}. \tag{3.28}$$

Therefore this vector can be written in the following regular vector form

$$\boldsymbol{\mu} = F_0(|X|, Z, t, c)E_0 + C(Z, t, c)\partial_t + \sum_{i=1}^{l}(F_i(|X|, Z, t, c)E_i + L_i(|X|, Z, t, c)e_i), \tag{3.29}$$

where the functions F_α , L_i and C are determined by $\delta(Z, t)$ and c.

The Laplacian can be established by formulas (1.12),(3.5) and (3.6). Then we get

$$\begin{aligned}\tilde{\Delta} = {} & t\Delta_{S_X} + t^{\frac{1}{2}}\Delta_{S_Z} \\ & + \frac{1}{4}t\sum_{\alpha;\beta=1}^{l}\langle J_\alpha(X), J_\beta(X)\rangle(\partial_\alpha - \boldsymbol{\mu}_{Z\alpha})(\partial_\beta - \boldsymbol{\mu}_{Z\beta}) \\ & + t\sum_{\alpha=1}^{l}(\partial_\alpha - \boldsymbol{\mu}_{Z\alpha})D_\alpha \bullet + c^2t^2(\partial_t - \boldsymbol{\mu}_t)^2 + c^2\left(1 - \frac{k}{2} - l\right)t(\partial_t - \boldsymbol{\mu}_t).\end{aligned} \tag{3.30}$$

By repeating the very same arguments used in the nilpotent cases (only the function φ in (3.19) should be of the form $\varphi(|\partial X|, \partial Z, t)$), we get

Main Theorem 3.4. *Let $J_{\mathbf{z}}$ and $J_{\mathbf{z}'}$ be endomorphism spaces described in Main Theorem* 3.2. *Then the corresponding metrics on the sphere-type surfaces having the same radius-function δ are isospectral on the solvable groups SN_J and $SN_{J'}$.*

4. Extension and nonisometry theorems on sphere-type manifolds

The nonisometry theorems are established by an independent statement asserting that an isometry between two sphere-type manifolds extends to an isometry between the corresponding ambient manifolds. Therefore the nonisometry on the sphere-type boundary manifolds can be checked by checking the nonisometry on the ambient manifolds. Since the nonisometry proofs on the solvable ambient manifolds are traced back to the nilpotent cases, where the question of nonisometry is equivalent to the nonconjugacy of the endomorphism spaces involved, one can always check on the nonisometry simply by checking the nonconjugacy of the corresponding endomorphism spaces. These kinds of theorems, concerning the nonconjugacy of spectrally equivalent endomorphism spaces, are established in [Sz5] and are reviewed in the last part (cf. below formula) of Section 2 in this paper.

The proof of this *extension theorem* is rather complicated, due to the circumstances that no general technique has been found covering the diverse isospectrality examples constructed in this paper. It requires different techniques depending on the sphere-type manifolds. On the largest class of examples the scalar curvature is used such that the extension of an isometry from a sphere-type boundary to the ambient space is settled for those manifolds where the gradient of the scalar curvature is nonvanishing almost everywhere (this assumption is formulated in a more precise form later). However, this proof does not cover the important case of the striking examples, since the considered geodesic spheres have constant scalar curvature. In this case the Ricci curvature should be involved to establish the desired extension of the isometry onto the ambient space. On this example we establish more nonisometry proofs, revealing surprising spectrally undetermined objects. The most surprising revelation is that the spectrum of the Laplacian acting on functions may give no information about the isometries.

The proofs are described in a hierarchic order. First in the nilpotent- and then in the solvable-case such sphere-type manifolds are considered which satisfy the above mentioned condition concerning the scalar curvature. The striking examples are considered in the third part. There are hierarchies also within these groups of considerations. For instance, in the first big group of the proofs, first the Heisenberg-type nilpotent groups are considered, since they provide a simple situation. Yet this proof clearly points into the direction of a general solution.

The nilpotent case.

Technicalities on sphere-type manifolds. In order to avoid long technical computations, we give detailed extension and nonisometry proofs on the particular sphere-type domains which can be described as level sets by equations of the form $\varphi(|X|,|Z|)=0$. (For local description we use the explicit function of the form $|X|=\delta(|Z|)$ or an appropriate variant of this function.) For such level sets both the X-cross sections, $S_X(Z)$, over a point Z and the Z-cross sections, $S_Z(X)$, over a point X are Euclidean spheres in the corresponding Euclidean spaces.

Without proof let us mention that the geodesic spheres around the origin of a Heisenberg-type group belong to this category. (We do not use this statement in the following considerations. Later we independently prove that the geodesic spheres around the origin $(0,1)$ of the solvable extension SH of a Heisenberg-type group H are level sets of the form $\varphi(|X|,|Z|,t)=0$.)

The normal vector $\boldsymbol{\mu}$ can be computed by means of formulas (1.2). By these formulas, regular X- and Z-vectors tangent to the sphere-type manifold can be expressed as Lie algebra elements. After such a computation we get that the perpendicular normal vector has the form:

$$\boldsymbol{\mu}=C(2X-D'\mathbf{J}_Z(X))-2CD'Z=\boldsymbol{\mu}_X+\boldsymbol{\mu}_Z, \tag{4.1}$$

where $\boldsymbol{\mu}$ is considered as an element of the Lie algebra. The function D is defined by $D(|Z|^2)=\delta^2(|Z|)$; furthermore,

$$C=(4|X|^2+(D')^2(|\mathbf{J}_Z(X)|^2+4|Z|^2))^{-\frac{1}{2}}. \tag{4.2}$$

(The prime in formula D' means differentiation with respect to the argument $\tau=|Z|^2$.)

The Weingarten map $B(\tilde{U})=\nabla_{\tilde{U}}\boldsymbol{\mu}$ or the second fundamental form $M(\tilde{U},\tilde{U}^*)=g(\tilde{U},B(\tilde{U}^*))$, where $\tilde{U}$ and $\tilde{U}^*$ are tangent to the hyper-surface, can be computed by the decompositions $X=\sum x^i\mathbf{X}_i$ and $Z=\sum z^\alpha\mathbf{Z}_\alpha$. Then, by (1.2) and (1.3), we get:

$$M(\widetilde{X}_1,\widetilde{X}_2)=C(2\langle\widetilde{X}_1,\widetilde{X}_2\rangle-\sum_\beta d_\beta\langle J_\beta(X),\widetilde{X}_1\rangle\langle J_\beta(X),\widetilde{X}_2\rangle); \tag{4.3}$$

$$M(\widetilde{Z}_1,\widetilde{Z}_2)=-2CD'\langle\widetilde{Z}_1,\widetilde{Z}_2\rangle;$$

$$M(\widetilde{X},\widetilde{Z})=M(\widetilde{Z},\widetilde{X})=-\frac{1}{2}\langle J_{\widetilde{Z}}(\boldsymbol{\mu}_X+2CD'X),\widetilde{X}\rangle,$$

where $d_0=\frac{1}{2}D'+|Z|^2D''$ and $d_i=\frac{1}{2}D',\forall\ 0<i\leq(l-1)$. In the first formula an orthonormal basis $e_0,e_1,\dots,e_{l-1}$ is considered on the Z-space such that $e_0=Z/|Z|$.

The Riemannian curvature of the considered hypersurfaces can be computed by the Gauss equation:

$$\begin{aligned}\widetilde{R}(\widetilde{V},\widetilde{Y})\widetilde{W}=&R(\widetilde{V},\widetilde{Y})\widetilde{W}-\langle R(\widetilde{V},\widetilde{Y})\widetilde{W},\boldsymbol{\mu}\rangle\boldsymbol{\mu}\\&+M(\widetilde{Y},\widetilde{W})B(\widetilde{V})-M(\widetilde{V},\widetilde{W})B(\widetilde{Y}).\end{aligned} \tag{4.4}$$

In the following we compute also the Ricci curvature $\widetilde{r}(\widetilde{U},\widetilde{V})$ and the scalar curvature $\widetilde{\kappa}=Tr(\widetilde{r})$ on these hypersurfaces. From the Gauss equation and from (1.14) we get:

$$\begin{aligned}\widetilde{r}(\widetilde{Y},\widetilde{W})=&r(\widetilde{Y},\widetilde{W})-\langle R(\boldsymbol{\mu},\widetilde{Y})\widetilde{W},\boldsymbol{\mu}\rangle\\&+\big\langle\widetilde{Y},\big((\mathrm{Tr}\,B)B-B^2\big)(\widetilde{W})\big\rangle;\end{aligned} \tag{4.5}$$

$$\begin{aligned}\langle R(\boldsymbol{\mu},\widetilde{X}_1),\widetilde{X}_2,\boldsymbol{\mu}\rangle=&-\frac{3}{4}\sum_\alpha\langle\widetilde{X}_1,J_\alpha(\boldsymbol{\mu}_X)\rangle\langle\widetilde{X}_2,J_\alpha(\boldsymbol{\mu}_X)\rangle\\&+\frac{1}{4}\langle J_{\boldsymbol{\mu}_z}(\widetilde{X}_1),J_{\boldsymbol{\mu}_z}(\widetilde{X}_2)\rangle;\end{aligned}$$

$$\langle R(\boldsymbol{\mu},\widetilde{Z})\widetilde{X},\boldsymbol{\mu}\rangle=\frac{1}{2}\langle(J_{\boldsymbol{\mu}_z}J_{\widetilde{Z}}-\frac{1}{2}J_{\widetilde{Z}}J_{\boldsymbol{\mu}_z})(\widetilde{X}),\boldsymbol{\mu}_X\rangle;$$

$$\langle R(\boldsymbol{\mu},\widetilde{Z}_1),\widetilde{Z}_2,\boldsymbol{\mu}\rangle=\frac{1}{4}\langle J_{\widetilde{Z}_1}(\boldsymbol{\mu}_X),J_{\widetilde{Z}_2}(\boldsymbol{\mu}_X)\rangle.$$

In the following the scalar curvature $\widetilde{\kappa}$ is used to establish the extension theorem. In general, this scalar curvature is a complicated expression depending on the functions

$$(4.6) \qquad |Z|\,,\ |J_Z(X)|\,,\ |J_Z^2(X)|\,,\ \sum_i |J_i(X)|^2\,,\ \sum_{i,j}\langle J_i(X), J_j(X)\rangle,$$

$$\sum_i \langle J_i(X), J_Z(X)\rangle\,,\ \sum_i \langle J_i J_Z(X), J_i J_Z(X)\rangle,$$

where the index 0 concerns the unit vector Z_0 and the indices $i, j > 0$ concern an orthonormal system $\{Z_1, \dots, Z_{l-1}\}$ of vectors perpendicular to Z_0. For a fixed X, the latter vectors can be chosen such that they are eigenvectors of the bilinear form $\langle J_Z(X), J_Z(X)\rangle$, restricted to the space $Z_0^\perp$. By using this basis, the fourth term can be reduced to the third one in the first line of (4.6).

The above formulas concern groups defined by general endomorphism spaces. In this paper we deal with groups defined by ESW_A's and we should explicitly compute the scalar curvature $\widetilde{\kappa}$ at points (X, Z), where Z is an anticommutator. The endomorphisms defined by the elements of the above introduced orthonormal basis $e_0, e_1, \dots, e_{l-1}$ on the Z-space ($e_0 = Z/|Z|$) are denoted by J_i. Then the operators $L_0 = J_0^2$ and $L_\perp = \sum_{i=1}^{l-1} J_i^2$ commute and a common eigensubspace decomposition can be established for them. In the following the scalar curvature $\widetilde{\kappa}$ is explicitly computed at a particular point (X, Z) where X is in the common eigensubspace of L_0 and $L_\perp$. The corresponding eigenvalues are denoted by λ_0 and $\lambda_\perp$.

To be more precise, the scalar curvature will be computed at the points of a 2-dimensional surface, called a *Hopf hull*, which are included in a higher dimension, so-called, X-*hulls*. These hulls are constructed as follows.

For the unit vector Z_0, consider the one-parametric family $S_X(sZ_0)$ of the X-cross sections describing the so-called X-hull around Z_0. This X-hull is denoted by $\mathrm{Hull}_X(sZ_0)$. We can construct it by an appropriate rotation of the graph of the function $|X| = \delta(|s|)$. This construction shows that an X-hull is a k-dimensional manifold diffeomorphic to a sphere and also the s-parameter lines on this surface are well defined by the rotated graph. The point on the hull satisfying $|X| = \delta(|S|) = 0$ is the so called *vertex* of the hull. The sphere $S_X(0)$ in the middle is the *eye* of this manifold. This eye is shared by all of the X-hulls. It is a total-geodesic submanifold since it is fixed by the isometry $(X, Z) \to (X, -Z)$. The vertexes of the hulls form the so called *rim* of the sphere-type manifold. This rim is nothing but the Z-sphere $S_Z(0)$ over the origin of the X-space.

The Hopf hulls are sub-hulls of X-hulls, constructed as follows.

On the X-hull consider the vector field $J_Z(X)$ tangent to the X-spheres. The integral curves of this vector field are Euclidean circles (called *Hopf circles*) through an X if and only if X is an eigenvector of J_Z^2 with eigenvalue, say λ_0.

For other X-vectors these curves may not be closed or they become proper Euclidean ellipses. If we fix a Hopf circle HC(0) on the sphere $S_X(0)$ at the origin and we consider the s-parameter lines only through this circle, we get a 2-dimensional, so called *Hopf hull*, $\mathrm{HHull}_{\mathrm{C}}(sZ_0)$, which is built up by a 1-parametric family of parallel Hopf circles HC(s).

One can get this Hopf hull by cutting it out from the ambient X-hull by the 3-dimensional space T_X spanned by the vectors $\{X\,,\,J_Z(X)\,,\,Z\}$. From (1.3) and

$$[X, J_Z(X)] = \sum_\alpha \langle J_Z(X), J_\alpha(X)\rangle Z_\alpha \tag{4.7}$$

we get that T_X is a total-geodesic manifold on the ambient space, for any X, if and only if the J_Z is an anticommutator. Then the T_X is a scaled metric Heisenberg group such that $|J_Z(X)| = \sqrt{-\lambda_0}|Z||X|$ holds.

Thus we get

Lemma 4.1. *A Hopf hull,* $\mathrm{HHull}_{\mathrm{C}}(sZ_0)$, *is totally geodesic on a sphere-type manifold ∂D for any Hopf-circle* C *if and only if Z_0 is an anticommutator. These Hopf hulls are intersections of ∂D by the total-geodesic scaled Heisenberg groups T_X.*

The scalar curvature $\widetilde{\kappa}$ (cf. above (4.5)) is computed on such a Hopf hull by formula

$$\widetilde{\kappa} = \kappa - 2\mathrm{Ricc}(\boldsymbol{\mu},\boldsymbol{\mu}) + (\mathrm{Tr}B)^2 - \mathrm{Tr}(B^2). \tag{4.8}$$

We use the new parametrization $\tau = |Z|^2 = s^2$ on the parameter lines. Then D' means differentiation with respect to this variable. By (1.16) and (4.1)–(4.5) one gets, by a lengthy but straightforward computation, that the scalar curvature has the rational form

$$\widetilde{\kappa} = \frac{\mathrm{Pol}(\tau, D(\tau), D', D'')}{(4-\lambda_0 t(D')^2)^2(4D - \lambda_0\tau D(D')^2 + 4\tau)}, \tag{4.9}$$

where the coefficients of the polynomial Pol (depending on λ_0, $\lambda_\perp$ and on constants such as k and l) can be computed by the following formulas:

$$\begin{aligned} \kappa &= \frac{1}{4}(\lambda_0 + \mathrm{Tr}\,L_\perp), \\ -2\mathrm{Ricc}(\boldsymbol{\mu},\boldsymbol{\mu}) &= C^2(-4D\lambda_\perp \\ &\quad +\lambda_0\tau(-4D + (D')^2(2 + D(\lambda_\perp + \lambda_0)))), \\ \mathrm{Tr}\,B &= C\Bigg(2(k-1-D'(l-1)) \\ &\quad + D\left(\frac{1}{2}\lambda_\perp D' + \lambda_0\left(\frac{1}{2}D' + \tau D''\right)\Omega\right)\Bigg), \end{aligned} \tag{4.10}$$

$$
\begin{aligned}
-\mathrm{Tr}\, B^2 = & -C^2 \Big(4(k-1+(D')^2(l-1)) \\
& +\lambda_\perp D(2(D'-(1+D')^2)+\frac{1}{2}\lambda_0\tau\Big) \\
& +4\lambda_0 D\left(\frac{1}{2}D'+tD''\right)\Omega+\left(\lambda_0 D\left(\frac{1}{2}D'+\tau D''\right)\Omega\right)^2, \\
\Omega = & \, 4(4-\lambda_0\tau(D')^2)^{-1}, \; C^2=(4D-\lambda_0\tau D(D')^2+4\tau)^{-1}.
\end{aligned}
$$

The explicit computation of the coefficients of the polynomial Pol in formula (4.9) requires further tedious computations even in the simple cases when the sphere-type domain is nothing but the Euclidean sphere described by the equation $|X|^2+|Z|^2=R^2$. In these cases $D(\tau)=R^2-\tau$, $D'=-1, D''=0$ hold and Pol is a fourth order polynomial of t. If $D(\tau)$ is a higher order polynomial of t, then also $\mathrm{Pol}(\tau)$ is a higher order polynomial and, except only one polynomial, the $\widetilde{\kappa}(\tau)$ is a nonconstant rational function of τ and $\widetilde{\kappa}'$ has only finitely many zero places. These examples show the wide range of the sphere-type domains for which $\widetilde{\kappa}'\neq 0$ almost everywhere.

These formulas allow us to compare the inner scalar curvature $\widetilde{\kappa}_H$ of a Hopf hull with the scalar curvature $\widetilde{\kappa}$ of the ambient sphere-type manifold. The above formulas can be applied also for computing $\widetilde{\kappa}_H$ by the substitutions $L_\perp=0, \lambda_\perp=0, k=2, l=1$. Then the $\widetilde{\kappa}_H$ can be expressed by means of the functions $\tau, D(\tau), D', D''$ and by λ_0. The *Hopf curvature* $\widetilde{\kappa}_{HD}(\tau)$ of a function $D(\tau)$ defining a sphere-type domain is defined by the scalar curvature $\widetilde{\kappa}_H$ such that $\lambda_0=-1$. That is, it is the scalar curvature of the sphere-type domain defined by $D(\tau)$ on the standard 3-dimensional Heisenberg group.

The scalar curvature has a simple form on groups with Heisenberg-type endomorphism spaces. (Reminder: An endomorphism space $J_{\mathbf{z}}$ is said to be Heisenberg-type if $J_Z^2=-|Z|^2 id$, $\forall Z\in\mathbf{z}$.) In fact, by (4.6), the $\widetilde{\kappa}$ depends only on the functions τ, D, D' in these cases. From the above arguments we get that $\widetilde{\kappa}'$ is nonvanishing almost everywhere if and only if $\widetilde{\kappa}'_{HD}$ is nonvanishing almost everywhere.

The Extension Theorem. The main result of this section is:

THEOREM 4.2. *Let $\widetilde{\Phi}$ be an isometry between two sphere-type hypersurfaces defined by the same function $D(|Z|^2)$ on the Heisenberg-type groups N and N^* such that the derivative $\widetilde{\kappa}'_{HD}$ is nonvanishing almost everywhere.* (*The abundance of such manifolds is described below formula* (4.10).) *Then the $\widetilde{\Phi}$ extends into an isometry of the form $\Phi=(\Phi_{/X},\Phi_{/Z})$ between the ambient spaces, where the component maps are appropriate orthogonal transformations on the* X- *resp.* Z-*spaces. Therefore the metrics $\widetilde{g}$ and $\widetilde{g}^*$ are isometric if and only if the ambient spaces are isometric.*

We consider this problem first on groups defined by Heisenberg-type ESW_A's. This simplification provides good ideas for proving the theorem in the much more complicated general cases.

The Extension Theorem on Heisenberg-type groups. In this case we consider a whole X-hull along with the restriction of the scalar curvature $\widetilde{\kappa}$ of the sphere-type hypersurface onto it. Since this scalar curvature depends only on the functions $|Z|$, D, D', the vectors grad $\widetilde{\kappa}$ always point in the directions of the s-parameter lines. By the above remark, made about comparing $\widetilde{\kappa}'$ and $\widetilde{\kappa}'_{HD}$, we get also that $\widetilde{\kappa}'$ is nonvanishing almost everywhere. Therefore grad$(\widetilde{\kappa})$ is nonvanishing almost everywhere on every X-hull. Since this vector field is invariant by isometries, the isometries must also keep the parameter lines. The vertex points are intersections of parameter-lines; therefore X-hulls are mapped to X-hulls such that vertex is mapped to vertex and eye is mapped to eye. That is, the isometries must keep the X-cross sections $S_X(Z)$ as well as the Z-cross sections $S_Z(X)$.

From the construction of the X-hull it is clear that the s-parameter lines identify the X-cross sections $S_X(sZ_0)$ and, with respect to this identification, an isometry $\widetilde{\Phi}$ defines the same map, $\widetilde{\Phi}_{/X}(Z_0)$, on the distinct X-spheres since all these maps are identified with the one defined on $S_X(Z_0)$. In the following we show that the $\widetilde{\Phi}_{/X}(Z_0)$ is the restriction of an orthogonal transformation (defined on the ambient space) to the considered sphere.

To prove this statement, we explicitly compute the metric tensor

$$g_{ij} = g(\partial_i, \partial_j) \quad , \quad g_{i\alpha} = g(\partial_i, \partial_\alpha) \quad , \quad g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$$

on the ambient space. From (1.2) we get

$$\begin{aligned} g_{ij} &= \delta_{ij} + \frac{1}{4}\langle [X, \partial_i], [X, \partial_j]\rangle \\ &= \delta_{ij} + \frac{1}{4} D \sum_{\alpha=1}^{l} \langle \mathbf{J}_\alpha(X_0), \partial_i\rangle \langle \mathbf{J}_\alpha(X_0), \partial_j\rangle; \\ g_{i\alpha} &= -\frac{1}{2}\langle \mathbf{J}_\alpha(X), \partial_i\rangle \quad ; \quad g_{\alpha\beta} = \delta_{\alpha\beta}. \end{aligned} \tag{4.11}$$

The metric tensor δ_{ij} (in the above formula concerning g_{ij}) defines the standard round metric on the above considered X-spheres. Notice too that, because of the function D in the second term, the metrics on the X-spheres with different radius are nonhomotetic. Since the $\widetilde{\Phi}_{/X}(Z_0)$ keeps all of these different metrics, it must keep them separately. That is, it keeps $\widetilde{\delta}_{ij}$ as well as $\sum \langle \mathbf{J}_\alpha(X_0), \widetilde{\partial}_i\rangle \langle \mathbf{J}_\alpha(X_0), \widetilde{\partial}_j\rangle$. Therefore it is derived from an orthogonal transformation $\Phi_{/X}(Z_0)$ on the ambient X-space. Actually, the transformations $\widetilde{\Phi}_{/X}(Z_0)$ and $\Phi_{/X}(Z_0)$ do not depend on Z_0, since all the X-hulls share the X-sphere $S_X(0)$ over $Z = 0$ and isometries must keep this X-sphere at the origin. Thus they can be simply denoted by $\widetilde{\Phi}_{/X}$ resp. $\Phi_{/X}$.

In the following we describe the isometry $\widetilde{\Phi}_Z(X)$ induced on the Z-sphere $S_Z(X)$ over X. By the last formula of (4.11), each map $\widetilde{\Phi}_{/Z}(X) : S_Z(X) \to S_{Z^*}(X^*)$ is the restriction of the orthogonal map $\Phi_{/Z}(X) : Z_X \to Z^*_{X^*}$, where Z_X denotes the Z-space over X.

On the other hand, from the formula given for $g_{i\alpha}$ in (4.11) we get

$$\langle J_{Z^*}(X^*), Y^* \rangle = \langle J_Z(X), Y \rangle \quad , \quad J_Z = \Phi^{-1}_{/X} J_{Z^*} \Phi_{/X}; \tag{4.12}$$

i.e. the orthogonal transformations $\Phi_{/Z}(X)$ do not depend on X and all are equal to the orthogonal transformation defined by the conjugation $J_{Z^*} = \Phi_{/X} J_Z \Phi^{-1}_{/X}$.

This proves the desired extension theorem on Heisenberg-type groups completely.

The extension in the general cases. Next we prove the extension theorem on spaces which are defined by general ESW_A's. The key idea of this general proof is the same as before; first we show that any isometry, $\widetilde{\Phi}$, between two sphere-type domains defined by the same function $D(|Z|^2)$ leaves the X-space invariant such that it is the restriction of a uniquely determined orthogonal transformation $\Phi_{/X} : \mathbf{v} \to \mathbf{v}^*$. This is the significant part of the proof since the construction of the appropriate orthogonal transformation $\Phi_{/Z} : \mathbf{z} \to \mathbf{z}^*$ can be completed in the same way shown above.

The technique used for constructing $\Phi_{/X}$ on Heisenberg-type groups cannot be directly applied in the general cases because the hull should be around an anticommutator and even on such an X-hull the vector field grad$(\widetilde{\kappa})$ is not tangent to the s-parameter lines in general. The latter tangent property is valid only on the much thinner Hopf hulls which are in a common eigensubspace of the commuting operators L_0 and $L_\perp$. On these Hopf hulls both grad$(\widetilde{\kappa})$ and grad$(\widetilde{\kappa}_H)$ are tangent to the s-parameter lines. (By the argument explained at establishing the Hopf curvature $\widetilde{\kappa}_{HD}$ we get that these gradients are nonvanishing almost everywhere if and only if $\widetilde{\kappa}'_{HD} \neq 0$ almost everywhere.) On the Hopf hulls whose Hopf circles are still in an eigensubspace of L_0 but are not in an eigensubspace of $L_\perp$, only grad$(\widetilde{\kappa}_H)$ is tangent to the parameter lines. If a parameter line is not inside an eigensubspace of L_0 then none of the above gradients is tangent to it.

We can deal with these difficulties in the following way.

Consider an X-hull around an anticommutator Z. By Lemma 4.1 and by the technique developed for Heisenberg type groups we get that the vertex Z is mapped by the isometry $\widetilde{\Phi}$ to an anticommutator Z^* such that the Hopf hulls, which are inside a common eigensubspace parametrized by the pair $(\lambda_0, \lambda_\perp)$ of eigenvalues, are mapped to Hopf hulls which are inside a common eigensubspace parametrized by the same pair of eigenvalues. On these Hopf hulls also the parameter lines (together with parametrization) are kept by the isometry. Using all the total geodesic Hopf hulls whose Hopf circles are in an eigensubspace E_{λ_0} of L_0, we can establish the above statement for those parameter lines which are in $E_{\lambda_0} \oplus Z$. This means that the sub-eye $E_{\lambda_0} \cap S_X(0)$ is mapped to the sub-eye $E^*_{\lambda_0} \cap S^*_{X^*}(0)$. It is clear too that the isometry restricted to $(E_{\lambda_0} \cap S_X(0)) \oplus Z$ extends to the sub-ambient space $E_{\lambda_0} \oplus Z$ and thus defines a uniquely determined orthogonal transformation $\Phi_{\lambda_0} : E_{\lambda_0} \to E^*_{\lambda_0}$. Our goal is to show that the orthogonal transformation $\Phi = \bigoplus \Phi_{\lambda_0}$, defined on the whole X-space, is the desired one to the problem. To establish this statement it is enough to prove that a general parameter line is mapped to parameter line such that the parametrization also is kept on it.

The above defined orthogonal transformation Φ defines a correspondence among the parameter lines of the two manifolds. Next we show that the $\widetilde{\Phi}$ maps the corresponding parameter lines to each other.

A parameter line $p(s)$ can be represented in the form $\delta X_0 + sZ_0$. By (1.3), (4.1) and (4.3), the curvature vector $\widetilde{\nabla}_P P$, where $P(\tau) = p'(\tau)$ is the tangent vector, has the form

$$\widetilde{\nabla}_P P = (\delta''/\delta')(P - Z_0) - \delta' J_{Z_0}(X_0).$$

This means that the parameter lines corresponding to each other are the solutions of the same second order differential equation. This equation is invariant under the action of isometries; therefore parameter lines are mapped to parameter lines.

Since the complete eye is a total geodesic submanifold, it is mapped to the eye $S^*_{X^*}(0)$. Furthermore, the vertex-to-vertex property is also satisfied. Therefore a general parameter line is mapped to a parameter line such that the parametrization is also kept on it.

It follows that isometries keep the X-spheres. More precisely, if an X-sphere $S_X(\widetilde{Z})$ is mapped to an X^*-sphere over $\widetilde{Z}^*$ then the isometry can be described by the pair $(\widetilde{\Phi}_{/X}(\widetilde{X}) = \widetilde{X}^*, \widetilde{\Phi}_{/Z}(\widetilde{Z}) = \widetilde{Z}^*)$ such that both are the restrictions of the corresponding orthogonal transformations from the pair $(\Phi_{/X}, \Phi_{/Z})$. By the same argument applied on Heisenberg-type groups we get that $J_{Z^*} = \Phi_{/X} J_Z \Phi^{-1}_{/X}$, which proves the desired extension and the theorem in the general cases completely.

The extension theorem in the solvable cases. In the solvable cases we prove the nonisometry on such sphere-type hypersurfaces which can be described by an equation of the form $|X|^2 = D(|Z|^2, t) = D(\tau, t)$. By (3.24), the normal unit vector of such a surface is

$$\boldsymbol{\mu}^S = Ct^{\frac{1}{2}}(2X - D_\tau \mathbf{J}_Z(X)) - 2CtD_\tau Z - cCtD_t\mathbf{T} = \boldsymbol{\mu}^S_X + \boldsymbol{\mu}^S_Z + \boldsymbol{\mu}^S_T, \tag{4.13}$$

where $\boldsymbol{\mu}$ is considered to be an element of the Lie algebra; furthermore,

$$C = \big(t(4|X|^2 + (D_\tau)^2|\mathbf{J}_Z(X)|^2) + t^2(4\tau(D_\tau)^2 + c^2D_t^2)\big)^{-\frac{1}{2}}. \tag{4.14}$$

The unit vector $\mathbf{t}$ parallel to ∂_t is perpendicular to $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_Z$; thus

$$\mathbf{t} = (4D + (ctD_t)^2)^{-\frac{1}{2}}(ctD_tX_0 + 2D^{\frac{1}{2}}\mathbf{T}) = \mathbf{t}_X + \mathbf{t}_T. \tag{4.15}$$

Also this vector is considered to be a Lie algebra element.

These formulas, together with (1.10), (1.11) and with the formulas established on the nilpotent groups, can be used for computing the corresponding formulas also on the solvable groups. Then we get:

$$\begin{aligned}
M_S(\widetilde{X}_1,\widetilde{X}_2) &= tM(\widetilde{X}_1,\widetilde{X}_2) + \frac{1}{2}c^2CtD_t\langle\widetilde{X}_1,\widetilde{X}_2\rangle\,; \\
M_S(\widetilde{X},\widetilde{Z}) &= M_S(\widetilde{Z},\widetilde{X}) = t^{\frac{1}{2}}M(\widetilde{X},\widetilde{Z})\,; \\
M_S(\widetilde{X},\mathbf{t}) &= M_S(\mathbf{t},\widetilde{X}) = P\Bigg(tD_t\left(1+\frac{1}{4}c^2D_t\right)\langle X_0,\widetilde{X}\rangle \\
&\qquad -D\tau^{\frac{1}{2}}D_{t\tau}\langle J_{Z_0}(X_0),\widetilde{X}\rangle\Bigg)\,; \\
P &= 2cCt(4D+(ctD_t)^2)^{-\frac{1}{2}}\,; \\
M_S(\widetilde{Z}_1,\widetilde{Z}_2) &= Ct(c^2-2D_\tau)\langle\widetilde{Z}_1,\widetilde{Z}_2\rangle\,; \\
M_S(\widetilde{Z},\mathbf{t}) &= M_S(\mathbf{t},\widetilde{Z}) = P^*\langle J_{Z_0}(X_0),J_{\widetilde{Z}}(X_0)\rangle\,; \\
P^* &= \frac{1}{2}cC(4D+(ctD_t)^2)^{-\frac{1}{2}}t^{\frac{3}{2}}\tau^{\frac{1}{2}}D^{\frac{1}{2}}D_\tau D_t\,; \\
M_S(\mathbf{t},\mathbf{t}) &= C\Bigg(2t\left(1+\frac{c^2}{4}D_t\right)|\mathbf{t}_X|^2 - cDt^{\frac{1}{2}}|\mathbf{t}_X||\mathbf{t}_Z| \\
&\qquad -c^2(tD_t+t^2D_{tt})|\mathbf{t}_T|^2\Bigg).
\end{aligned} \tag{4.16}$$

(Also in this case D_τ means differentiation with respect to the argument $\tau = |Z|^2$.)

If J_Z is an anticommutator, the formulas for $M_S(\widetilde{X},\mathbf{t})$ and $M_S(\widetilde{Z},\mathbf{t})$ can be considerably simplified. In fact, the latter expression vanishes and the first expression vanishes on the tangent vectors of the form $\widetilde{X} = J_{\widetilde{Z}}(X)$ (these vectors are tangent to the surface, since they are perpendicular to $\boldsymbol{\mu}_X^S$). Therefore it is nontrivial only on the vectors $\widetilde{X} = J_Z(\boldsymbol{\mu}_X)$; furthermore, the plane spanned by this vector and by $\mathbf{t}$ is invariant by the action of the Weingarten map B_S.

The proofs of the extension theorem can be straightforwardly adopted from the nilpotent case to this solvable case.

First choose a unit vector Z_0 and consider the half-plane $\mathbf{Z}_0 \oplus \mathbf{R}_+$ parametrized by (s,t). Then the X-hull, $\mathrm{Hull}_X(sZ_0,t)$, is a 2-parametric family of the X-spheres $S_X(sZ_0,t)$ defined on a closed domain diffeomorphic to a closed disk. The boundary of this domain (which is diffeomorphic to a circle) is the so called (s,t)-rim. The point $(s_v,1)$ resp. $(0,t_v)$, where $D=|X|^2=0$, is called Z-vertex- resp. t-vertex-parameters. The corresponding points on the surface are the corresponding vertexes. Also the Hopf hulls, $\mathrm{HHull}_{\mathrm{C}}(sZ_0,t)$, are 2-parametric families, $\mathrm{HC}(s,t)$, of corresponding Hopf circles. One can get such a Hopf hull by cutting it out from the ambient X-hull by the 4-dimensional space T_S spanned by the vectors $\{X\,,\,J_Z(X)\,,\,Z\,,T\}$. From (1.11) and (4.7) we get that the space T_S is total-geodesic in the ambient space if and only if the J_Z is an anticommutator.

The scalar curvature $\widetilde{\kappa}_S$ depends on the functions listed in (4.6) and the parameter t. Thus, by the very same arguments applied in the nilpotent case we get

Lemma 4.3. (A) *A* $\mathrm{HHull}_{\mathrm{C}}(sZ_0,t)$ *is totally geodesic on a Sphere-type manifold ∂D if and only if Z_0 is an anticommutator. These Hopf hulls are the intersections of the sphere-type manifold by the total geodesic submanifolds T_S, which are the solvable extensions of the corresponding Heisenberg subgroups, T, introduced in the nilpotent case. That is, the metrics on these submanifolds are complex hyperbolic metrics of constant holomorphic sectional curvature.*

(B) *If Z_0 is an anticommutator then* $\mathrm{grad}\,\widetilde{\kappa}_S(\neq 0)$ *is tangent to the (s,t)-parameter plane on a Hopf-hull,* $\mathrm{HHull}_{\mathrm{C}}(sZ_0,t)$, *if and only if the Hopf circles* $\mathrm{HC}(s,t)$ *are in a common eigensubspace of the commuting operators $L_0=J_0^2$ and $L_\perp=\sum_{i=1}^{l-1}J_i^2$.*

The explicit computation of the scalar curvature can be performed on a $\mathrm{HHull}_{\mathrm{C}}$ by using (4.8),(1.15),(1.16) and (4.10). These lengthy computations are relatively simple when D is a polynomial of τ and t. (For instance, for Euclidean spheres with center $(0,0,t_0)$ and radius R, this function is $D=R^2-\tau-(t-t_0)^2$.) In these cases the $\widetilde{\kappa}_S$ is a rational function of τ and t.

The Hopf curvature $\widetilde{\kappa}_{HD}(s,t)$ is defined by the scalar curvature of the sphere-type manifold defined by D on the standard complex hyperbolic space of -1 holomorphic sectional curvature.

As in the nilpotent case we get

Theorem 4.4. *Any isometry $\widetilde{\Phi}$ between two sphere-type manifolds, defined by the same function $D(s,t)$ such that* $\mathrm{grad}(\widetilde{\kappa}_{HD})\neq 0$ *almost everywhere, extends to an isometry Φ between the ambient spaces SN and SN^*. That is, $\widetilde{g}_c$ and $\widetilde{g}_c^*$ are isometric if and only if the ambient groups are isometric.*

Extension- and nonisometry-theorems on the striking examples. The above proof of the extension theorem breaks down on important hypersurfaces such as the geodesic spheres on the solvable groups $SH_3^{(a,b)}$. In [Sz5] we pointed out that the most striking examples can be constructed exactly on these geodesic spheres. In fact, these geodesic spheres with the same radius are isospectral on spaces with the same $a+b$, yet the spheres belonging to the 2-point homogeneous space $SH_3^{(a+b,0)}$ is homogeneous while the others are locally inhomogeneous.

Next we establish the extension theorem along with other nonisometry theorems also on the geodesic spheres. The key idea is an explicit computation of the eigensubspaces of the Ricci curvature. The invariance of these eigensubspace-distributions guarantees that both the X-spaces and Z-spaces are invariant under the actions of isometries and they extend into an isometry between the ambient spaces. Also in this section the nilpotent and the solvable cases are considered separately

Extensions from the geodesic spheres of $H_3^{(a,b)}$ We use the notation introduced in (2.12)–(2.14). In this case $\{e_1 = \mathbf{i}\,, e_2 = \mathbf{j}\,, e_3 = \mathbf{k}\}$ is a basis in the space $\mathbf{R}^3$ of the imaginary quaternions and $J_c = J_{e_c}$; $\mathbf{R}^4 \to \mathbf{R}^4$ is defined by the appropriate left product on the space $\mathbf{R}^4$ of quaternionic numbers. The endomorphism $J_c^{(a,b)}$ acting on $\mathbf{R}^{4(a+b)}$ is introduced in (2.14). The endomorphism spaces $J_{\mathbf{z}}^{(a,0)}$ and $J_{\mathbf{z}}^{(0,b)}$ are used accordingly. Let us note that $J_{\mathbf{z}}^{(a,0)}$ and $J_{\mathbf{z}}^{(0,a)}$ are equivalent endomorphism spaces (in the sense of (1.7)) and they correspond to the left- resp. to the right-representation of $so(3)$ on $\mathbf{R}^{4a}$.

We introduce also the distribution $\rho^{(a,b)}$ tangent to the X-spheres $S_X(Z)$ of the spaces $H_3^{(a,b)}$, spanned by the vectors $J_{\mathbf{z}}^{(a,b)}(X)$ at a vector X. Let us point out again that this distribution is considered as a regular X-distribution and the spanning vectors are regular X-vectors. Therefore the ρ is not perpendicular to the distribution $\widetilde{z}$ defined by the Z-vectors tangent to the considered surface at a point. We introduce also the distribution $K^{(a,b)}$ consisting of vectors perpendicular to $\rho^{(a,b)} \oplus \widetilde{z}$. From (1.2) we immediately get that this latter distribution is spanned by regular X-vectors also; i.e., $\rho^{(a,b)} \oplus K^{(a,b)}$ is an orthogonal direct sum decomposition of the tangent space of the Euclidean X-spheres around the origin.

On the space $H_3^{(a,0)}$ (resp. on $H_3^{(0,b)}$) the distribution $\rho^{(a,0)}$ (resp. $\rho^{(0,b)}$) is integrable and the 3-dimensional integral manifolds are the fibres of a principal fibre bundle with the structure group SO(3). This fibration is nothing but the quaternionic Hopf fibration and the factor space is the 2-point homogeneous quaternionic projective space [Be]. If $a > 1$ (resp. $b > 1$), the distribution $K^{(a,0)}$ (resp. $K^{(0,b)}$) is an irreducible connection on this bundle with an irreducible curvature form $\omega(X,Y) = [X,Y]_\rho; X,Y \in K$. This proves that $[K^{(a,b)}, K^{(a,b)}]_\rho = \rho^{(a,b)}$. Thus we have

LEMMA 4.5. *If $a, b > 1$, then $[K^{(a,b)}, K^{(a,b)}]_\rho = \rho^{(a,b)}$ and therefore the K generates the whole tangent space on a sphere $S_X(Z)$ by Lie brackets.*

In the following step we compute the Ricci curvature on ∂D and it turns out that both $\rho \oplus \widetilde{z}$ and K are eigensubspaces of this Ricci operator with a completely different set of eigenvalues. This observation offers more options for establishing the extension theorem.

We use a special basis to compute the matrix of the Ricci curvature. At a fixed point (X,Z) on the hypersurface the unit normal vector Z_0 is denoted by $\mathbf{i}$, furthermore, the last two vectors from the right handed orthonormal system $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ are chosen such that they are tangents to the hypersurface. The unit vectors

$$\widetilde{E}_{\mathbf{i}} = J_{\mathbf{i}}(\boldsymbol{\mu}_{X0}) = -(4+|Z|^2(D')^2)^{-\frac{1}{2}}(D'|Z|X_0 + 2J_{\mathbf{i}}(X_0)),$$
$$\widetilde{E}_{\mathbf{j}} = J_{\mathbf{j}}(X_0) \quad , \quad \widetilde{E}_{\mathbf{k}} = J_{\mathbf{k}}(X_0) \tag{4.17}$$

(considered as Lie algebra elements) are tangent to the hypersurface, standing perpendicular to the Z-space and to the distribution $\rho^{(a,b)}$. Notice that $\widetilde{E}_{\mathbf{j}}$ and $\widetilde{E}_{\mathbf{k}}$ are tangent to $\rho^{(a,b)}$ while the vector $\widetilde{E}_{\mathbf{i}}$ is not tangent to this distribution, expressing the fact that ρ and $\widetilde{z}$ are not perpendicular in general. We consider an orthonormal basis $\{\widetilde{K}_1,\dots,\widetilde{K}_{k-3}\}$ also on $K^{(a,b)}$ and the matrix of the Ricci operator $\widetilde{r}$ is computed with respect to the basis $\{\widetilde{K}_1,\dots,\widetilde{K}_{k-3},\widetilde{E}_{\mathbf{i}},\widetilde{E}_{\mathbf{j}},\widetilde{E}_{\mathbf{k}},\mathbf{j},\mathbf{k}\}$. For the computations we use the following formulas:

$$\begin{aligned} J_{\mathbf{i}}(\boldsymbol{\mu}_X) &= D^{\frac{1}{2}}C(4+|Z|^2(D')^2)^{\frac{1}{2}}\widetilde{E}_{\mathbf{i}},\\ J_{\mathbf{j}}(\boldsymbol{\mu}_X) &= D^{\frac{1}{2}}C(2\widetilde{E}_{\mathbf{j}} + D'|Z|\widetilde{E}_{\mathbf{k}}),\\ J_{\mathbf{k}}(\boldsymbol{\mu}_X) &= D^{\frac{1}{2}}C(2\widetilde{E}_{\mathbf{k}} - D'|Z|\widetilde{E}_{\mathbf{j}}). \end{aligned} \tag{4.18}$$

Then by (4.1)–(4.5) we get that this matrix is of the form

$$\widetilde{r} = \begin{pmatrix} \varepsilon I_K & 0 & 0 & 0\\ 0 & (\varepsilon+E_{ll})I_l & 0 & 0\\ 0 & 0 & (\varepsilon+E_{LL})I_L & E_{L\widetilde{z}}\\ 0 & 0 & E_{\widetilde{z}L} & (\varepsilon+E_{\widetilde{z}\widetilde{z}})I_{\widetilde{z}} \end{pmatrix}, \tag{4.19}$$

where I_K, $I_{\widetilde{z}}$ and I_L (resp. I_l) are unit matrices on the spaces K, $\widetilde{z}$ and on the space L spanned by the vectors $\mathbf{j}$ and $\mathbf{k}$ (resp. on the 1-dimensional space l spanned by $\mathbf{i}$). Furthermore,

$$\begin{aligned} \varepsilon &= -\frac{3}{2} + 2C\,Tr(B) - 4C^2 = -\frac{3}{2} + 2C^2(\mathrm{Tr}(b)-2)\\ &= -\frac{3}{2} + 2C^2\big(2(k-2) - DD' - d_0D\Omega\big);\\ E_{ll} &= C^2\big(4 + (3D-4)\Omega^{-1}\\ &\quad -d_0D(2(k-3) + D(d_0-D'))\Omega + 2(d_0D)^2\Omega^2\big);\\ E_{LL} &= C^2\left(4 + (6D-4)\Omega^{-1} - \frac{1}{2}DD'\left(2(k-3) - \frac{1}{2}DD'\right)\right.\\ &\quad \left. + \frac{1}{2}D^2D'd_0\Omega\right); \end{aligned} \tag{4.20}$$

$$E_{\widetilde{z}\widetilde{z}} = \frac{k}{4} + \frac{3}{2} - 2C^2\left(\frac{1}{2}D\Omega^{-1} - (1+D')(2(k-2) - D'(D-2)) - D(D'+1)d_0\Omega)\right),$$

where $\Omega = 4(4 + |Z|^2(D'))^{-1}$ is as introduced in (4.12) and $d_0 = \frac{1}{2}D' + D''$ is as introduced in (4.3). The 2×2 matrices $E_{L\widetilde{z}} = E_{\widetilde{z}L}$ have the following form:

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

where the functions A and B are

$$(4.20') \qquad A = C^2 D^{\frac{1}{2}}\left(\frac{1}{8}(3C^{-1} - D^{\frac{1}{2}})\Omega^{-1} + 6C^{-1} + D(1+D')d_0\Omega - (2D^{\frac{1}{2}}D'(D'+2) + (1+D')(2(k-1) - DD'))\right);$$

$$B = C^2 D^{\frac{1}{2}} C^{-1} D'|Z|\left(k + 2 - \frac{1}{2}DD' - \frac{1}{2}Dd_0\Omega\right).$$

From these formulas and from the characteristic equation $\det(\widetilde{r} - \lambda I) = 0$ we get that the subspaces K and $\rho \oplus \widetilde{z}$ are eigensubspaces of the Ricci operator and the eigenvalue ε on K is different from the other eigenvalues if and only if the following determinant

$$(4.21) \qquad \det\begin{pmatrix} E_{LL}I_L & E_{L\widetilde{z}} \\ E_{\widetilde{z}L} & E_{\widetilde{z}\widetilde{z}}I_{\widetilde{z}} \end{pmatrix} = (A^2 + B^2 - E_{LL}E_{\widetilde{z}\widetilde{z}})^2$$

is nonzero. Since this determinant is zero on an open set only in the case when the function D satisfies a certain differential equation on some open intervals, it is clear that this last assumption is satisfied on an everywhere dense open set for the most general sphere-type manifolds in $H_3^{(a,b)}$. Then on this set the Ricci tensor has distinct eigenvalues on the invariant subspaces K and $\rho \oplus \widetilde{z}$. Since we concentrate on the geodesic spheres in the next section, we would like to give more details about certain particular cases in order to prepare the next section.

Later we will see that the geodesic spheres on the solvable extensions intersect the nilpotent level sets in Sphere-type surfaces described by functions of the form $D(\tau) = \sqrt{Q - \tau} + Q^*$, where $\tau = |Z|^2$ and Q, Q^* are constants. In this case we introduce the new variable $u = \sqrt{Q - \tau}$. Then all the functions D, D', d_0, C^2, Ω are rational functions of u, while the functions $D^{\frac{1}{2}} = \sqrt{u + Q^*}$ and $C^{-1}(u) = \frac{1}{2u}G(u) = \frac{1}{2u}H^{\frac{1}{2}}(u)$, where

$$(4.22) \qquad H(u) = -16u^4 + 15u^3 + (16Q + 15Q^*)u^2 - Qu - QQ^*$$

are nonrational functions of u. (One can prove, by an elementary computation, that the polynomial $H(u)$ can be written in the quadratic form $H(u) = (q_1u^2 + q_2u + q_3)^2$ only for particular constants Q and Q^*, and even in this particular case the coefficients q_s are pure imaginary numbers ($q_1 = \pm 4\mathbf{i}$ can be checked immediately)). This argument proves that the function $B^2 - E_{LL}E_{\widetilde{z}\widetilde{z}}$ is a rational function of u; however, the function A^2 has the following nonrational form

$$A^2(u) = R_1(u) + R_2(u)D^{\frac{1}{2}}(u) + R_3(u)H^{\frac{1}{2}}(u), \tag{4.23}$$

where the functions $R_i(u)$ are nontrivial rational functions. This proves that the determinant in (4.17) is nonvanishing almost everywhere since the nonrational terms cannot be canceled out from this function either.

The proof of the following extension and nonisometry theorem is completely prepared by the above considerations.

Theorem 4.6. *Suppose that the function defined in* (4.21) *is nonzero on an everywhere dense open set* (*this assumption is most widely satisfied*, *including the manifolds described above around formula* (4.22)). *Then the isometries* $\widetilde{\Phi} : \partial D \to \partial D^*$ *keep both the* X-*spaces and the* Z-*spaces and they extend to isometries* Φ *acting between the ambient spaces. Therefore the metrics* $\widetilde{g}$ *and* $\widetilde{g}^*$ *are isometric if and only if the ambient metrics* g *and* g^* *are isometric.*

Proof. First we suppose that $\dim(K) > 0$. Since $[K,K]_\rho = \rho$ and the $\widetilde{\Phi}$ keeps the distribution K by the above arguments, the $\widetilde{\Phi}$ keeps the tangent spaces of the X-spheres S_X. Therefore the image of $S_X(Z)$ is an X*-sphere $S_{X^*}(Z^*)$. Thus the $\widetilde{\Phi}$ defines an orthogonal transformation between these 2-spheres, transforming the vector field $J_Z(X)$ to $J_{Z^*}(X^*)$. (The last statement follows from (4.11) by the same arguments used there, since also in this case the $\widetilde{\Phi}$ keeps the tensors δ_{ij} and $\sum_\alpha \langle J_\alpha(X), \widetilde{\partial}_i\rangle\langle J_\alpha(X), \widetilde{\partial}_j\rangle$ together with the tensor $g_{ij} = -1/2\langle J_j(X), \widetilde{\partial}_i\rangle$ separately. Therefore it is derived from an orthogonal transformation on the ambient space such that the form $\langle J_0(X), \widetilde{\partial}_i\rangle$ is also preserved. This latter statement can also be proved by using the invariant property of the eigensubspace l of the Ricci operator $\widetilde{r}$.)

Let $\widetilde{\rho}$ be the subspace in $\rho \oplus \widetilde{t}$ perpendicular to $\widetilde{z}$. Then the $\widetilde{\Phi}$ keeps $\widetilde{\rho}$ by (4.13) and by the previous argument. Consequently, it also keeps $\widetilde{z}$. That is, a Z-sphere $S_Z(X)$ is mapped to the Z^*-sphere $S_{Z^*}(X^*)$ and the $\widetilde{\Phi}$ extends to an orthogonal transformation between the ambient spaces z_X and z_{X^*}. Since the ambient spaces are isometric if and only if $(a,b) = (a^*,b^*)$ up to an order, this extension theorem also proves the nonisometry completely. □

Remark 4.7. The nonisometry can be established otherwise as follows.

(A) For the vector fields U and V tangent to $\rho^{(a,b)} \oplus t^2$, let $L(U,V)$ be the orthogonal projection of $[U,V]$ onto $K^{(a,b)}$. Then L is obviously a tensor field of type (2,1) on ∂D such that it is invariant with respect to the isometries of the space. It turns out too that the L vanishes exactly at the points of the form $(X^{(a)}, Z)$ or $(X^{(b)}, Z)$. That is, the induced metrics on the hypersurface ∂D of the spaces $H_3^{(a,b)}$ and $H_3^{(a^*,b^*)}$ cannot be isometric unless $(a,b) = (a^*, b^*)$ up to an order.

(B) One can demonstrate the nonisometry also by determining the isometries on the considered hypersurfaces. It turns out that the group of isometries is $\{O(\mathbf{H}^a) \times O(\mathbf{H}^b)\}SO(3)$, where $O(\mathbf{H}^c)$ is the quaternionic orthogonal group acting on $\mathbf{H}^c$. This also proves the above local nonisometry on ∂D. This proof is not independent of the above extension theorem, since the above isometry group is determined by the fact that the isometries on ∂D are the restrictions of those isometries on the ambient space which fix the origin.

(C) One can establish a general(!) nonisometry proof (not just on $H_3^{(a,b)}$) also by [Sz5, Prop. 5.4], where the isotonal property of the corresponding curvature operators is established (cf. also Proposition 1.3 in this paper). The complete proof on sphere-type manifolds needs the extension theorem also in this case.

The nonisometry proofs on the geodesic spheres of $SH_3^{(a,b)}$. In order to establish these results on the solvable extension SN, first we explicitly compute the equation of a geodesic sphere around the origin $(0,0,1)$. This computation can be carried out by using the generalized Cayley transform constructed on the solvable extensions of Heisenberg-type groups [CDKR]. This transform maps the unit ball

$$B = \{(X, Z, t) \in \mathbf{n} \oplus \mathbf{a} \,|\, |X|^2 + |Z|^2 + t^2 = r^2 < 1\} \tag{4.24}$$

onto SN, by the formula

$$C(X, Z, t) = ((1-t)^2 + |Z|^2)^{-1}\big(2(1 - t + J_Z)(X)\,,\, 2Z\,,\, 1 - r^2\big). \tag{4.25}$$

By pulling back we get the ball-representation of the metric, having the property that the geodesics through the origin are nothing but the rays $(\tanh(s)/r)(X, Z, t)$ such that they are parametrized by the arc-length s starting at the origin. That is, the considered geodesic spheres with radius s match the Euclidean spheres with radius $\tanh(s)$ on the Ball-model. Then, by computing the inverse Cayley map (the reader can consult for more details [CDKR] (pages 14–15)), by a routine computation we get

Lemma 4.8. *The equation of a geodesic sphere of radius s around the origin* $(0,0,1)$ *of SN is*

$$|X|^2 = 4((e^s + e^{-s} + 2)t - |Z|^2)^{\frac{1}{2}} - 4(t+1). \tag{4.26}$$

Notice that on groups $SH_3^{(a,b)}$ with the same $a+b$, the geodesic spheres with the same radius R around the origin are the same level sets, described by the same equation.

In the following step we compute the Ricci curvature $\widetilde{r}_S$ on the geodesic spheres with respect to the basis $\{\widetilde{K}_1,\dots,\widetilde{K}_{k-3},\widetilde{E}_{\mathbf{i}},\widetilde{E}_{\mathbf{j}},\widetilde{E}_{\mathbf{k}},\mathbf{j},\mathbf{k},\mathbf{t}\}$. By (4.16), (1.13), (4.1), (4.5), (4.13) and by the fact that

$$\begin{aligned} R(\boldsymbol{\mu}^S,\widetilde{U},\widetilde{V},\boldsymbol{\mu}^S) = {} & R(\boldsymbol{\mu}^S_X+\boldsymbol{\mu}^S_Z,\widetilde{U},\widetilde{V},\boldsymbol{\mu}^S_X+\boldsymbol{\mu}^S_Z)+R(\boldsymbol{\mu}^S_T,\widetilde{U},\widetilde{V},\boldsymbol{\mu}^S_T) \\ & +R(\boldsymbol{\mu}^S_T,\widetilde{U},\widetilde{V},\boldsymbol{\mu}^S_X+\boldsymbol{\mu}^S_Z)+R(\boldsymbol{\mu}^S_X+\boldsymbol{\mu}^S_Z,\widetilde{U},\widetilde{V},\boldsymbol{\mu}^S_T), \end{aligned}$$

we get

$$\widetilde{r}_S=\begin{pmatrix} \sigma I_K & 0 & 0 & 0 & 0 \\ 0 & (\sigma+S_{ll}) & 0 & 0 & S_{l\mathbf{t}} \\ 0 & 0 & (\sigma+S_{LL})I_L & t^{\frac{1}{2}}S_{L\widetilde{z}} & 0 \\ 0 & 0 & t^{\frac{1}{2}}S_{\widetilde{z}L} & (\sigma+S_{\widetilde{z}\widetilde{z}})I_{\widetilde{z}} & 0 \\ 0 & S_{\mathbf{t}l} & 0 & 0 & \sigma+S_{\mathbf{tt}} \end{pmatrix}. \tag{4.27}$$

By the very same proof given in the nilpotent case we get that the eigenvalue σ on the eigenspace K is different from the other eigenvalues on an everywhere dense open set and therefore this distribution is invariant by the actions of the isometries.

In fact, the determinant corresponding to (4.21) is nonvanishing also in this case, since the corresponding term A^2 now has the form

$$A^2(u,t)=R_1(u,t)+R_2(u,t)D^{\frac{1}{2}}(u,t)+R_3(u,t)H^{\frac{1}{2}}(u,t), \tag{4.28}$$

where $u=((e^R+e^{-R}+2)t-\tau)^{\frac{1}{2}}$, while the other terms are rational functions. A straitforward computation shows also that $S_{ll}S_{\mathbf{tt}}-S^2_{l\mathbf{t}}$ is nonvanishing on an everywhere dense open set, which proves the above statement concerning the distinctness of σ from the other eigenvalues.

The extension- and nonisometry-proofs can be similarly established as in the nilpotent case.

As in the nilpotent case, the invariant tensor field $L(\widetilde{U},\widetilde{V})$ defined on the distribution $\rho\oplus\widetilde{z}\oplus\mathbf{t}$ can be used to establish the nonisometry. This tensor field vanishes exactly at the X-vectors of the form $(X^{(a)},0)$ or $(0,X^{(b)})$, proving the nonisometry of the considered manifolds.

If $ab\neq 0$, the group of isometries is the nontransitive group $\{\mathrm{O}(\mathbf{H}^a)\times \mathrm{O}(\mathbf{H}^b)\}\mathrm{SO}(3)$ on the geodesic spheres, while the geodesic spheres of the 2-point homogeneous spaces $SH_3^{(a+b,0)}$ have transitive groups of isometries whose unit component is isomorphic to $\mathbf{Sp}((a+b+1))\mathbf{Sp}(1)$. This is the third proof of the nonisometry. This demonstration is not independent of the extension theorem, since the isometry group is determined by the fact that the isometries on the geodesic sphere are restrictions of those isometries on the ambient space which fix the center of the sphere.

A different type of the nonisometry proofs can be established by Propsition 1.3, though also this proof involves the extension theorem.

By summing up, both the isospectrality and the nonisometry theorems, we have

CORNUCOPIA THEOREM 4.9. (A) *Let $J_{\mathbf{z}}$ be an endomorphism space with anticommutator (i.e., an ESW_A) such that it either contains a nonAbelian Lie subalgebra or it is one of the irreducible Cliffordian endomorphism spaces $J_{4k+3}^{(1,0)} \simeq J_{4k+3}^{(0,1)}$. Consider the family, determined by the constant $(a+b)$, of 2-step nilpotent metric Lie groups $(N_J^{(a,b)}, g^{(a,b)})$ (resp. the family $(SN_J^{(a,b)}, g^{(a,b)})$ of solvable extensions) defined by the endomorphism spaces $J_{\mathbf{z}}^{(a,b)}$ (cf. (2.12)–(2.14)). Such a family is represented on the same manifold $M = \mathbf{R}^{k(a+b)+l}$ (resp. on $M = \mathbf{R}^{k(a+b)+l} \times \mathbf{R}_+$ in the solvable case).*

While the induced metrics $\widetilde{g}^{(a,b)}$ and $\widetilde{g}^{(b,a)}$ are isometric, the other metrics from the family have local geometries different from $\widetilde{g}^{(a,b)}$, on any sphere-type hypersurface $\partial D \subset M$ defined by the same function $\varphi(|X|,|Z|) = 0$ (resp. $\varphi(|X|,|Z|,t) = 0$) where the condition $\mathrm{grad}(\widetilde{\kappa}_{HD}) \neq 0$ *is satisfied almost everywhere on the corresponding Hopf-hull. Yet the metrics $\widetilde{g}^{(a,b)}$ on a ∂D, belonging to a family, are isospectral.*

(B) *The above statement is established also on the geodesic spheres of the solvable extensions $SH_3^{(a,b)}$ (the technique applied for proving Theorem* (A) *breaks down in this case). That is, the geodesic spheres having the same radius and belonging to the same family have different local geometries unless $(a,b) = (a^*, b^*)$ up to an order. Yet, the induced metrics are isospectral also in this case.*

The geodesic spheres on $SH_3^{(a+b,0)}$ are homogeneous, while the geodesic spheres on the other manifolds $SH_3^{(a,b)}$ are locally inhomogeneous. This demonstrates the fact: The group of isometries, even the local homogeneity property, is lost to the nonaudible in the debate of audible versus nonaudible geometry.

The abundance of the isospectrality examples constructed in these papers is due to the abundance of the ESW_A's, described in Section 2, and to the great variety of the sphere-type manifolds which can be chosen for a fixed family of endomorphism spaces, both on the nilpotent groups and the solvable extensions. Let us mention again that the isospectrality theorem is established for any sphere-type manifold and the nonisometry theorems are established for the particular manifolds defined by equations of the form $\varphi(|X|,|Z|) = 0$ (resp. $\varphi(|X|,|Z|,t) = 0$) only because we have not wanted to make the proofs even more complicated than they are in this simplified situation. It is highly likely that the extension and nonisometry proofs can be extended to sphere-type manifolds defined by general functions of the form $\varphi(|X|,Z)$ (resp. $\varphi(|X|,Z,t)$).

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