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Boundary behavior for groups of subexponential growth

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Abstract

In this paper we introduce a method for partial description of the Poisson boundary for a certain class of groups acting on a segment. As an application we find among the groups of subexponential growth those that admit nonconstant bounded harmonic functions with respect to some symmetric (infinitely supported) measure μ of finite entropy $H(\mu)$. This implies that the entropy $h(\mu)$ of the corresponding random walk is (finite and) positive. As another application we exhibit certain discontinuity for the recurrence property of random walks. Finally, as a corollary of our results we get new estimates from below for the growth function of a certain class of Grigorchuk groups. In particular, we exhibit the first example of a group generated by a finite state automaton, such that the growth function is subexponential, but grows faster than $\exp(n^\alpha)$ for any $\alpha < 1$. We show that in some of our examples the growth function satisfies $\exp(\frac{n}{\ln^{2+\varepsilon}(n)}) \leq v_{G,S}(n) \leq \exp(\frac{n}{\ln^{1-\varepsilon}(n)})$ for any $\varepsilon > 0$ and any sufficiently large n.

1. Introduction

Let G be a finitely generated group and μ be a probability measure on G. Consider the random walk on G with transition probabilities $p(x|y) = \mu(x^{-1}y)$, starting at the identity. We say that the random walk is nondegenerate if μ generates G as a semigroup. In the sequel we assume, unless otherwise specified, that the random walk is nondegenerate.

The space of infinite trajectories G^∞ is equipped with the measure which is the image of the infinite product measure under the following map from G^∞ to G^∞:

$$(x_1, x_2, x_3 \dots) \to (x_1, x_1x_2, x_1x_2x_3 \dots).$$

Definition. *Exit boundary.* Let A_n^∞ be the σ-algebra of measurable subsets of the trajectory space G^∞ that are determined by the coordinates $y_n, y_{n+1}, \dots$ of the trajectory y. The intersection $A_\infty = \cap_n A_n^\infty$ is called the exit σ-algebra of the random walk. The corresponding G-space with measure is called *the exit boundary* of the random walk.

Equivalently, the exit boundary is the space of ergodic components of the time shift in the path space G^∞.

Recall that a real-valued function f on the group G is called μ-harmonic if $f(g) = \sum_x f(gx)\mu(x)$ for any $g \in G$.

It is known that the group admits nonconstant positive harmonic functions with respect to some nondegenerate measure μ if and only if the exit boundary of the corresponding random walk is nontrivial. The exit boundary can be defined in terms of bounded harmonic functions ([24]), and then it is called the Poisson (or Furstenberg) boundary.

There is a strong connection between amenability of the group and triviality of the Poisson boundary for random walks on it. Namely, any nondegenerate random walk on a nonamenable group has nontrivial Poisson boundary and any amenable group admits a symmetric measure with trivial boundary (see [24], [23] and [26]). First examples of symmetric random walks on amenable groups with nontrivial Poisson boundary were constructed in [24], where for some of the examples the corresponding measure has finite support.

Below we recall the definition of growth for groups.

Consider a finitely generated group G, let $S = (g_1, g_2, \dots, g_m)$ be a finite generating set of G, l_S and d_S be the word length and the word metric corresponding to S.

Recall that a growth function of G is

$$v_{G,S}(n) = \#\{g \in G : l_S(g) \le n\}.$$

Note that if S_1 and S_2 are two sets of generators of G, then there exist K_1, $K_2 > 0$ such that for any n, $v_{G,S_1}(n) \le v_{G,S_2}(K_2 n)$ and $v_{G,S_2}(n) \le v_{G,S_1}(K_1 n)$.

A group G is said to have polynomial growth if for some $A, d > 0$ and any positive integer n, $v_{G,S}(n) \le An^d$. A group G is said to have exponential growth if $v_{G,S}(n) \ge C^n$ for some $C > 1$. (Obviously, for any G, S $v_{G,S}(n) \le (2m-1)^n$ for any G, S.)

Clearly, the property of having exponential or polynomial growth does not depend on the set of generators chosen. The group is said to be of subexponential growth if it is not of exponential growth.

Recall that any group of subexponential growth is amenable. It is known (see Section 4) that the Poisson boundary is trivial for random walks on a group of subexponential growth if the corresponding measure μ has finite first moment (in particular, for any μ with finite support).

Moreover, any random walk on a finitely generated group of polynomial growth has trivial Poisson boundary. The aim of this paper is to show that this statement is not valid for subexponential growth. That is, for series of groups of intermediate growth we construct a random walk on them with nontrivial Poisson boundary. Some of our examples admit such random walks with a measure having finite entropy.

2. Grigorchuk groups G_w

It is known that a group has polynomial growth if and only if it is virtually nilpotent ([18]) and that any solvable or linear group has either polynomial or exponential growth (see [25] and [32] for solvable and [28] for linear case). The first examples of groups of intermediate (not polynomial and not exponential) growth were constructed by R. I. Grigorchuk in [13]. Below we recall one of his constructions from [13].

First we introduce the following notation. For any $i \geq 1$ fix a bijective map $m_i : (0,1] \to (0,1]$. Consider an element g that acts on $(0,1]$ as follows. On $(0, \frac{1}{2}]$ it acts as m_1 on $(0,1]$, on $(\frac{1}{2}, \frac{3}{4}]$ it acts as m_2 on $(0,1]$, on $(\frac{3}{4}, \frac{7}{8}]$ it acts as m_3 on $(0,1]$ and so on.

More precisely, take $r \geq 1$ and put

$$\Delta_r = \left(1 - \frac{1}{2^{(r-1)}}, 1 - \frac{1}{2^r}\right].$$

Consider the affine map α_r from Δ_r onto $(0,1]$. Note that $(0,1]$ is a disjoint union of Δ_r $(r \geq 1)$. The map $g : (0,1] \to (0,1]$ is defined by

$$g(x) = \alpha_r^{-1}(m_r(\alpha_r(x)))$$

for any $x \in \Delta_r$.

In this situation we write

$$g = m_1, m_2, m_3, \dots .$$

Let a be a cyclic permutation of the half-intervals of $(0,1]$. That is,

$$a(x) = x + \frac{1}{2} \text{ for } x \in (0, \tfrac{1}{2}] \text{ and } a(x) = x - \frac{1}{2} \text{ for } x \in (\tfrac{1}{2}, 1].$$

2.1. *Groups G_w.* Let $P = a$ and T be an identity map on $(0,1]$. We use here this notation as well as for b and d defined below following the original paper of Grigorchuk [13].

Consider any infinite sequence $w = PPTPTPTPPP\dots$ of symbols P and T such that each symbol P and T appears infinitely many times in w. We denote the set of such sequences by Ω^*. Let b act on $(0,1]$ as w, that is

$$b = P, P, T, P, T, P, T, P, P, P \dots$$

and d act on $(0,1]$ as

$$d = P, P, P, P, P, P, P, P, P, P \dots .$$

Let G_w be the group generated by a, b and d. For any $w \in \Omega^*$ the group G_w is of intermediate growth [13].

Remark 1. In the notation of [13], the G_w are the groups that correspond to sequences of 0 and 1 with infinite numbers of 0 and 1 (that is, from Ω_1 in the notation of [13]) In the papers of Grigorchuk the groups above are defined as groups acting on the segment $(0,1)$ with all dyadic points being removed. Then the action is continuous. We use other notation and do not remove dyadic points. Then the overall action is not continuous; however, it is continuous from the left.

In the sequel we use the following notation. If a and b are permutations on the segments of $[0,1]$ as above, or more generally for any a and b acting on $[0,1]$ we write $ab(x) = b(a(x))$ (not $a(b(x))$) for any $x \in [0,1]$.

3. Statement of the main result

Consider an action of a finitely generated group G on $(0,1]$. We assume that the action satisfies the following property (LN). For any $g \in G$, $x, y \in (0,1]$ such that $g(x) = y$ and any $\delta > 0$ there exist $\varepsilon > 0$ such that

$$g((x-\varepsilon, x]) \subset (y-\delta, y].$$

That is, g is continuous from the left and $g(y') < g(y)$ for each y and $y' < y$ close enough to y.

Definition. The action satisfies *the strong condition* $(*)$ if there exists a finite generating set S of G such that for any $g \in S$ and $x \in (0,1]$ satisfying $x \neq 1$ or $g(x) \neq 1$ there exist $a \in \mathbb{R}$ and $\varepsilon > 0$ such that for any $y \in (x-\varepsilon, x]$

$$g(y) = y + a.$$

Definition. The action satisfies *the weak condition* $(*)$ if there exists a finite generating set S of G such that for any $g \in S$ and $x \in (0,1]$ satisfying $x \neq 1$ there exist $a \in \mathbb{R}$ and $\varepsilon > 0$ such that for any $y \in (x-\varepsilon, x]$

$$g(y) = y + a.$$

For $g \in G$ define *the germ* $\mathrm{germ}(g)$ as the germ of the map $g(t) + 1 - g(1)$ in the left neighborhood of 1. More generally, for $g \in G$ and $y \in (0,1]$ define *the germ* $\mathrm{germ}_y(g)$ as the germ of the map $g(t+y-1) + 1 - g(y)$ in the left neighborhood of 1.

Below we introduce a notion of the group of germs $\mathrm{Germ}(G)$. We will need this notion for the description of the Poisson boundary.

Definition. Let G act on $(0,1]$ by LN maps. *The group of germs* $\mathrm{Germ}(G)$ of this action is the group generated by $\mathrm{germ}_y(g)$, where $g \in G$ and $y \in (0,1]$. Composition is the operation in $\mathrm{Germ}(G)$.

Remark 2. If G satisfies LN, then the group $\mathrm{Germ}(G)$ is well defined.

Proof. Note that for any $g \in G$ and $\delta > 0$ there exists $\varepsilon > 0$ such that

$$g((y-\varepsilon, y]) \subset (g(y)-\delta, g(y)].$$

Consequently,

$$(1-g(y)) + g(t+y-1) \subset (1-\delta, 1]$$

for any $t \in (1-\varepsilon, 1]$

Hence the composition of germs is well defined. □

Let $\mathrm{Germ}_1(G)$ be the subgroup of $\mathrm{Germ}(G)$ generated by $\mathrm{germ}_1(g) = \mathrm{germ}(g)$ for $g \in G$.

Remark 3. If the action of G on $(0,1]$ satisfies the weak condition $(*)$ then $\mathrm{Germ}(G) = \mathrm{Germ}_1(G)$.

Example 1. Let $G = G_w$ for some $w \in \Omega^*$. Put $c = bd$ and $S = a, b, c, d$. Then the action is by LN maps and satisfies the strong condition $(*)$. Moreover, $\mathrm{Germ}(G) = \mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$. Consider the subgroup $H = H_w$ of $G = G_w$ generated by ad. Clearly, $\mathrm{Germ}(H) = \mathbb{Z}/2\mathbb{Z}$.

The main result of this paper is the following theorem.

THEOREM 1. *Let G act on $(0,1]$ by* LN *maps and the action satisfy the strong condition $(*)$. Assume that there exists $g \in G$ such that $g^m(1) \neq 1$ for any $m \geq 1$ and that the subgroup generated by $\{\mathrm{germ}_y(g) | y \in (0,1]\}$ is not equal to* $\mathrm{Germ}(G)$. *Assume also that* $\mathrm{Germ}(G)$ *is finite. Let H be the subgroup of G generated by g. Then*

(1) *The group G admits a symmetric measure μ of finite entropy $H(\mu)$ such that the Poisson boundary is nontrivial.*

(2) *For any $0 < \varepsilon < 1$ the measure μ above can be chosen in such a way that its support* $\mathrm{supp}(\mu)$ *is equal to $H \cup K$ for some finite set K and there exists $C > 0$ such that for any $m \in \mathbb{Z}$*

$$\mu(g^m) = \frac{C}{|m|^{1+\varepsilon}}.$$

(3) *For any $p > 1$ the measure μ above can be chosen in such a way that its support* $\mathrm{supp}(\mu)$ *is equal to $H \cup K$ for some finite set K and there exists $C > 0$ such that for any $m \in \mathbb{Z}$*

$$\mu(g^m) = \frac{C \ln^p(|m|+1)}{|m|^2}.$$

Let $G = G_w$ and $H = H_w$ be as in Example 1. In Section 4 we will show that G, H satisfy the assumption the theorem above and hence G admits a symmetric measure of finite entropy with nontrivial Poisson boundary.

This shows that some groups of subexponential growth admit symmetric measures of finite entropy such that the Poisson boundary is nontrivial.

However, the entropic criterion for triviality of the boundary yields that any finitely supported measure (or, more generally, any measure having finite first moment) on a group of subexponetial growth has trivial boundary (see Section 4).

Let G be a finitely generated group, S be a symmetric finite generating set of G and H be a subgroup of G. Recall that the Schreier graph of G with respect to H is the graph whose vertexes are right cosets $H\backslash G$, that is, $\{Hg : g \in G\}$ and for any $s \in S$ and $g \in G$ there is an edge connecting $\{Hg\}$ and $\{Hgs\}$.

In Section 6 we will give a criterion for a graph being the Schreier graphs of $(G, \mathrm{Stab}(1))$ for groups G of intermediate growth acting on $(0, 1]$ with strong condition $(*)$. As a corollary of this criterion and our previous results we get the following example: there exist a finitely generated group A, a subgroup B of A, a finite set $K \subset A$ and a sequence of probability measures μ_i with the following properties. For any i the support of $\mu_i \subset K$. The sequence μ_i converges pointwise (on K) to a measure μ (clearly, μ is a probability measure and $\mathrm{supp}\mu \subset K$) and the subgroup B is a transient set for (A, μ); but for any i the subgroup B is recurrent for (A, μ).

In Section 6 as a corollary of Theorem 1 we get the following theorem.

Theorem 2. *Let G act on $(0, 1]$ by* LN *maps and the action satisfy the strong condition $(*)$. Assume that there exists $g \in G$ such that $g^m(1) \neq 1$ for any $m \geq 1$ and that the subgroup generated by $\{\mathrm{germ}_y(g) | y \in (0, 1]\}$ is not equal to* $\mathrm{Germ}(G)$. *Assume also that* $\mathrm{Germ}(G)$ *is finite. Then for any $\varepsilon > 0$ there exists N such that for any $n > N$*

$$v_{G,S}(n) \geq \exp\left(\frac{n}{\ln^{2+\varepsilon}(n)}\right).$$

This theorem can be applied in particular to any group G_w, $w \in \Omega^*$. Considering $w = PTPTPTPT\ldots$ and $G = G_w$ we obtain the first example of a (finite state) automatic group of intermediate growth for which $v_{G,S}(n)$ grows faster than $\exp(n^\alpha)$ for any $\alpha < 1$ (see Section 6).

In Subsection 6.1 we give an upper bound for the growth function of G_w (under some assumption on w). Combining this with Theorem 2 we obtain first examples of groups G with the growth function satisfying

$$\exp\left(\frac{n}{\ln^{2+\varepsilon}(n)}\right) \leq v_{G,S}(n) \leq \exp\left(\frac{n}{\ln^{1-\varepsilon}(n)}\right)$$

for any $\varepsilon > 0$ and any sufficiently large n.

For further applications of Theorem 1 to growth of groups see [10].

In the last section we discuss possible generalizations of Theorem 1. We obtain examples of groups with the growth function bounded from above by $\exp(n^\gamma)$ for some $\gamma < 1$ (and sufficiently large n) which admit symmetric measures with nontrivial Poisson boundary. (This is in contrast to Theorem 2.)

4. Proof of the main result

Recall that a Markov kernel ν on a countable set X is a set of probability measures on X $\nu_x(y) = \nu(x, y)$ ($x \in X$). A Markov kernel defines a Markov operator on X with transition probabilities

$$p(x|y) = \nu(x, y).$$

This operator acts on $l^2(X)$: if $f \in l^2(X)$, then

$$\nu f(x) = \sum_{x \in X} \nu(x, y) f(y).$$

A Markov kernel is called doubly stochastic if $\tilde{\nu}_x(y) = \nu(y, x)$ is also Markovian.

A weaker statement of the proposition below appears for the first time in [2].

Proposition 1 (Varopoulos, [29], [30]). *Let $\nu_1(x, y), \nu_2(x, y)$ be doubly stochastic kernels on a countable set X and assume that ν_1 is symmetric, that is, $\nu_1(x, y) = \nu_1(y, x)$. Suppose that there exists $k \geq 0$ such that*

$$\nu_1(x, y) \leq k\nu_2(x, y)$$

for any $x, y \in X$. Let ξ be a probability measure on $[0, 1]$ and $\xi_n = \int_0^1 \lambda^n d\xi(\lambda)$.

(1) *Then for any $0 \leq f \in l^2(X)$*

$$\sum_{n \geq 0} \xi_n \langle \nu_2^n f, f\rangle \leq k \sum_{n \geq 0} \xi_n \langle \nu_1^n f, f\rangle.$$

(2) *Let $p_n^i(x, x)$ ($i = 1$ or 2) be the n step transition probability for ν_i. Then*

$$\sum_{n \geq 0} \xi_n p_n^2(x, x) \leq k \sum_{n \geq 0} \xi_n p_n^2(x, x).$$

(This follows from (1) applied to a delta function f such that $f(x) = 1$.)

(3) *If ν_2 is recurrent, then ν_1 is also recurrent (following from (2) applied to a delta measure ξ such that $\xi(1) = 1$).*

We will mostly apply Proposition 1 for the case when both ν_1 and ν_2 are symmetric measures on the cosets $H\backslash G$ (for some group G and its subgroup H).

Proposition 2. *Let G act on $(0,1]$ by* LN *maps. Assume that the action satisfies the strong condition* $(*)$ *and that H is a subgroup of G. Assume also that* $\mathrm{Germ}(H) \neq \mathrm{Germ}(G)$, $\mathrm{Germ}(H)$ *is of finite index in* $\mathrm{Germ}(G)$ *and that μ is a probability measure on G such that* $\mathrm{Stab}_G(1)$ *is transient for* (G,μ). *Assume also that that* $\mathrm{supp}\mu \subset H \cup K$ *for some finite set $K \subset G$ and that the random walk is nondegenerate. Then the Poisson boundary of* (G,μ) *is nontrivial.*

Proof of Proposition 2. Consider the cosets

$$\Gamma = \mathrm{Germ}(G)/\mathrm{Germ}(H)$$

and a map $\pi_H : G \to \Gamma$ defined by

$$g \to \mathrm{germ}(g) \bmod \mathrm{Germ}(H).$$

Lemma 4.1. *With probability one,* $\pi_H(g)$ *stabilizes along an infinite trajectory of* (G,ν).

Proof. Consider an infinite trajectory

$$y_1, y_2, y_3, y_4, \dots$$

where $y_{i+1} = y_i g_{i+1}$, $g_{i+1} \in \mathrm{supp}(\nu)$.

Note that the weak condition $(*)$ for (G,S) implies that

$$\mathrm{germ}(gg') = \mathrm{germ}(g)$$

whenever $g(1) \neq 1$ and $g' \in S$.

Moreover, for any finite set $K \subset G$ there exists a finite set $\Sigma \subset [0,1]$ such that

$$\mathrm{germ}(gg') = \mathrm{germ}(g)$$

whenever $g(1) \notin \Sigma$ and $g' \in K$. Now, for any finite $K \subset G$ and any $k \in K$ fix a word u_k in the letters of the generating set S representing k in G; that is

$$k = u_k = s_1^k s_2^k s_3^k \dots s_{i_k}^k,$$

where $s_j^k \in S$ for any $1 \leq j \leq i_k$. Put

$$\tilde{K} = \{s_1^k s_2^k s_3^k \dots s_j^k : k \in K, 1 \leq j \leq i_k\}$$

and

$$\Sigma = \{\tilde{k}^{-1}(1), \tilde{k} \in \tilde{K}\}.$$

Note that if $g(1) \notin \Sigma$, then $(g\tilde{k})(1) = \tilde{k}(g(1)) \neq 1$ and hence $\mathrm{germ}(g\tilde{k}s) = \mathrm{germ}(g\tilde{k})$ for any $\tilde{k} \in \tilde{K}$ and $s \in S$. Arguing by induction on i_k we conclude that $\mathrm{germ}(gk) = \mathrm{germ}(g)$ for any $k \in K$.

Since Stab(1) is transient for (G,ν) and since Σ is a finite set, for almost all trajectories of this random walk there exists N such that $y_i(1) \notin \Sigma$ for any $i \geq N$.

Consider some $i > N$ and $y_{i+1} = y_i g_{i+1}$. We shall prove that $\pi_H(y_{i+1}) = \pi_H(y_i)$. Since $g_{i+1} \in \mathrm{supp}(\nu) \subset K \cup H$, either $g_{i+1} \in K$ or $g_{i+1} \in H$.

First case. $g_{i+1} \in K$. We know that $y_i(1) \notin \Sigma$, and hence

$$\mathrm{germ}(y_{i+1}) = \mathrm{germ}(y_i).$$

Consequently,

$$\pi_H(y_{i+1}) = \mathrm{germ}(y_{i+1}) \bmod \mathrm{Germ}(H) = \mathrm{germ}(y_i) \bmod \mathrm{Germ}(H) = \pi_H(y_i).$$

Second case. $g_{i+1} \in H$. Let $x = y_i(1)$. Note that

$$\mathrm{germ}_x(g_{i+1}) \in \mathrm{Germ}(H).$$

Consequently,

$$\mathrm{germ}_1(y_{i+1}) = \mathrm{germ}_1(y_i) \circ \mathrm{germ}_x(g_{i+1}) \equiv \mathrm{germ}_1(y_i) \bmod \mathrm{Germ}(H).$$

Thus

$$\pi_H(y_{i+1}) = \pi_H(y_i). \qquad \square$$

LEMMA 4.2. *For any* $\gamma \in \Gamma$

$$\Pr(\lim_{i\to\infty} \pi_h(y_i) = \gamma) \neq 0.$$

Proof. Recall that Γ is finite since $\mathrm{Germ}(H)$ is of finite index in $\mathrm{Germ}(G)$. Therefore,

$$\sum_{\gamma\in\Gamma} \Pr(\lim_{i\to\infty} \pi_H(y_i) = \gamma) = 1.$$

Consequently, there exists $\gamma_0 \in \Gamma$ such that

$$\Pr(\lim_{i\to\infty} \pi_H(y_i)) = \gamma_0 \neq 0.$$

Note that there exists $g \in G$ such that $\mathrm{germ}(g) \circ \gamma_0 = \gamma$.

There exist $s_1, s_2, \ldots, s_m \in S$ such that

$$g = s_1 s_2 \ldots s_m.$$

Consider an infinite trajectory $y_1, y_2, y_3, \ldots$ such that

$$\lim_{i\to\infty} \pi_H(y_i) = \gamma_0.$$

Consider now the trajectory $z = (z_1, z_2, z_3, \ldots)$ such that $z_1 = s_1$, $z_2 = s_1 s_2$, $z_3 = s_1 s_2 s_3$, $\ldots$ $z_m = s_1 s_2 s_3 \ldots s_m = g$ and $z_{m+k} = g y_k$ for any $k \geq 1$.

Note that

$$\lim_{j\to\infty} \pi_H(z_j) = \gamma.$$

Consequently,

$$\Pr(\lim_{i\to\infty} \pi_H(y_i) = \gamma) \geq \nu(s_1)\nu(s_2)\dots\nu(s_m)\Pr(\lim_{i\to\infty} \pi_H(y_i) = \gamma_0) > 0. \quad \square$$

Now we return to the proof of Proposition 2. Take $\gamma \in \Gamma$ and consider the set of trajectories A

$$y = (y_1, y_2, y_2, \dots)$$

such that

$$\lim_{i\to\infty} \pi_H(y_i) = \gamma.$$

Obviously, A is a measurable set in the set of infinite trajectories.

Since Γ contains at least two distinct elements, Lemma 4.2 and Lemma 4.3 imply that

$$0 < \nu^\infty(A) < 1.$$

It is clear that if two trajectories coincide after a finite number of steps and one of them belongs to A, then the other also belongs to A. Therefore A defines a subset $\tilde{A}$ in the exit boundary such that its measure in the boundary is equal to $\nu^\infty(A)$. And this implies that the exit boundary is nontrivial. $\square$

Remark 4. In Lemma 4.2 we used only that the action satisfies the weak condition $(*)$. For Lemma 4.3 the assumption that the action satisfies the strong condition $(*)$ is also not necessary. In fact, we used that the action satisfies the weak condition $(*)$ and that for any $g \in G$ there exists $\tilde{g} \in \mathrm{Stab}(1)$ such that $\mathrm{germ}(g) \equiv \mathrm{germ}(\tilde{g}) \bmod \mathrm{Germ}(H)$.

Remark 5. *The lamplighter boundary.* Under the assumptions of Proposition 2 (or more generally for any action satisfying the weak condition $(*)$, see Remark 4) we proved that $\mathrm{germ}_1(g) \bmod \mathrm{Germ}(H)$ stabilizes with probability 1 along infinite trajectories of the random walk.

In fact, in the same way we see that $\mathrm{germ}_y(g) \bmod \mathrm{Germ}(H)$ stabilizes for any $y \in [0,1]$.

(Note that this statement makes sense only if y belongs to the G-orbit of 1. Otherwise $\mathrm{germ}_y(g)$ is always trivial because of the weak condition $(*)$.)

Denote the G-orbit of 1 by Δ. To each $g \in G$ one can attach a map M_g from Δ to $\mathrm{Germ}(G) \bmod \mathrm{Germ}(H)$ and with probability 1 this map stabilizes pointwise along infinite trajectories of the random walk. (We know this for each point, and since Δ is countable it implies that this happens for all the points.) Note that G acts on the space of such maps M_g by 'taking the composition'. We call this space *the lamplighter boundary* of the action of G on $(0,1]$ with respect to the subgroup H. (For this definition we can consider arbitrary H, not necessarily as in Proposition 2. For example we can consider $H = \{e\}$.)

But under the assumptions of Proposition 2 the lamplighter boundary can be naturally endowed with a probability measure coming from the space of infinite trajectories G^∞. Hence we can identify it with some μ-boundary (that is, with a quotient of the Poisson boundary).

Definitions. Let X be a countable space with a discrete probability measure ν. *The entropy* of ν is defined as $H(\nu) = -\sum_x \nu(x)\ln(\nu(x))$.

The entropy of a random walk on (G, μ) (see [1]) is the limit

$$h(\mu) = \lim_{n\to\infty} H(\mu^{*n})/n.$$

The *drift* of the random walk (G, μ) is

$$l(\mu) = \lim_{n\to\infty} \frac{\mathrm{E}_{\mu^{*n}} l(g)}{n},$$

where l denotes the word length with respect to some finite generating set of G.

The exponential growth rate of G with respect to a finite generating set S is

$$v = v_{G,S} = \lim_{n\to\infty} \sqrt[n]{v_{G,S}(n)}.$$

It is not difficult to see that the limits in the three definitions above do exist (see [19]; for v see also e.g. [21]).

It is known that for any random walk on G, $h(\mu) \le \ln(v)l(\mu)$ ([19]). Consequently, any simple random walk (or, more generally, any random walk such that the transition measure μ has finite first moment) on a group of subexponential growth has zero entropy.

This is in contrast to the following result.

THEOREM 1. *Let G act on $(0,1]$ by* LN *maps and the action satisfies the strong condition* $(*)$. *Assume that there exists $g \in G$ such that $g^m(1) \neq 1$ for any $m \ge 1$ and that the subgroup generated by $\{\mathrm{germ}_y(g) | y \in (0,1]\}$ is not equal to* $\mathrm{Germ}(G)$. *Assume also that* $\mathrm{Germ}(G)$ *is finite. Let H be the subgroup of G generated by g. Then*

(1) *The group G admits a symmetric measure μ of finite entropy $H(\mu)$ such that the Poisson boundary is nontrivial.*

(2) *For any $0 < \varepsilon < 1$ the measure μ above can be chosen in such a way that its support* $\mathrm{supp}(\mu)$ *is equal to $H \cup K$ for some finite set K and there exists $C > 0$ such that for any $m \in \mathbb{Z}$*

$$\mu(g^m) = \frac{C}{|m|^{1+\varepsilon}}.$$

(3) *For any $p > 1$ the measure μ above can be chosen in such a way that its support* $\mathrm{supp}(\mu)$ *is equal to $H \cup K$ for some finite set K and there exists $C > 0$ such that for any $n \in \mathbb{Z}$*

$$\mu(g^n) = \frac{C \ln^p(|n| + 1)}{|n|^2}.$$

Proof of Theorem 1. Take the symmetric probability measure ν on $H = \mathbb{Z}$,

$$\nu((g)^N) = \frac{C}{|N|^{(1+\varepsilon)}}.$$

This measure is transient for any $0 < \varepsilon < 1$ [27]. The entropy of this measure

$$H(\nu) \sim \sum_N \frac{(1+\varepsilon)\ln(|N|)}{|N|^{(1+\varepsilon)}}$$

is obviously finite.

Note that $\mathrm{Stab}(1)$ is transient for (H, ν) since by the previous lemma we know that $H \cap \mathrm{Stab}_G(1) = e$.

Take any symmetric finite generating set S of G and consider the measure μ_2 equidistributed on S. Put $\mu = \frac{1}{2}(\nu + \mu_2)$. Obviously, μ is symmetric and $H(\mu) < \infty$).

From (3) of Proposition 1 we deduce that $\mathrm{Stab}(1)$ is transient for (G, μ). Hence we can apply Proposition 2 and get that the Poisson boundary of (G, μ) is nontrivial. So (1) and (2) of the theorem are proved.

Now we are going to prove (3). Consider the symmetric probability measure on H such that

$$\nu(g^n) = \frac{C_1 \ln^p(|n| + 1)}{|n|^2}.$$

We want to show that ν is transient. In fact, consider the real part of the Fourier transform of ν

$$\phi(t) = \sum_{n \in \mathbb{Z}} \cos(tn)\nu(n).$$

Note that for $1 \geq t \geq 0$

$$1 - \phi(t) = \sum_{n \in \mathbb{Z}} (1 - \cos(tn))\nu(n) \geq \sum_{n=0}^{[1/t]} (1 - \cos(tn))\nu(n).$$

Note also that there exists $A > 0$ such that $(1 - \cos(x)) \geq Ax^2$ for any $0 \leq x \leq 1$. Hence

$$1 - \phi(t) \geq A \sum_{n=0}^{[1/t]} (tn)^2 \frac{C_1 \ln^p(|n| + 1)}{|n|^2} = AC_1 t^2 \sum_{n=0}^{[1/t]} \ln^p(|n| + 1)$$

Now $\sum_{n=0}^{m} \ln^p(|n|+1) \geq A_2 m$ for some positive A_2 and any m large enough. Therefore,

$$1 - \phi(t) \geq A_3 t |\ln(t)|^p$$

for $0 < t < 1$ and some $A_3 > 0$. This implies that

$$\int_0^1 \frac{1}{1-\phi(t)} \geq \frac{1}{A_3} \int_0^1 \frac{1}{t|\ln(t)|^p} < \infty,$$

since $p > 1$. By the Recurrence Criterion (see e.g. [11]) this implies that ν is transient.

As before, we observe that then Stab(1) is transient for (H, ν). We take a symmetric nondegenerate finitely supported measure μ_2 and consider $\mu = \frac{1}{2}(\nu + \mu_2)$.

From (3) of Proposition 1 we deduce that Stab(1) is transient for (G, μ). Hence we can apply Proposition 2 and get that the Poisson boundary of (G, μ) is nontrivial. $\square$

Corollary 1. *For any $w \in \Omega$ the group G_w admits a symmetric measure μ such that $H(\mu) < \infty$, but the entropy of the random walk $h(\mu) > 0$.*

Proof. For the proof of the corollary it is sufficient to show that the group satisfies the assumption of Theorem 1. This is done in the following lemma, which statement is unexplicitly contained in [13, proof of Lemma 2.1].

Lemma 4.3. *For $G = G_w$ $(ad)^k \notin \mathrm{Stab}_G(1)$ for any $k \geq 1$.*

Proof of the lemma. Observe that $ad(0.5, 1] = (0, 0.5]$ and that $ad(0, 0.5] = (0.5, 1]$. Hence if k is odd then $(ad)^k(1) \in (0, 0.5]$. Consequently, if $(ad)^k \in \mathrm{Stab}_G(1)$ then k is even.

Let $k = 2l$. Note that $(ad)^2$ acts on $(0.5, 1]$ in the same way as (ad) acts on $(0, 1]$. If $(ad)^k(1) = 1$ then $(ad)^l(1) = 1$. Arguing by induction on k we come to the contradiction. $\square$

5. Applications to recurrence

The random walk on a finitely generated group G is called simple if the corresponding measure μ is equidistributed on some finite symmetric generating set of G. A random walk on a graph with finite valency of each vertex is called simple if from each vertex it walks with equal probability to one of its neighbors.

We say that a graph is recurrent if the simple random walk on it is recurrent. It is well known (and follows from (3) of Proposition 1) that the fact that the Schreier graph of G with respect to H is recurrent does not depend on the choice of the finite (symmetric) generating set of G (and more generally, the property of the graph to be recurrent is preserved by quasi-isometries).

Proposition 3. *Suppose that a group of intermediate growth G acts on $(0,1]$ by* LN *maps and that the action satisfies the strong condition* $(*)$. *Then the Schreier graph of* $(G, \mathrm{Stab}(1))$ *is recurrent. Moreover, for any finitely supported* (*not necessarily symmetric*) *measure μ on G such that* $\mathrm{supp}(\mu)$ *generates G as a semigroup the corresponding random walk on the Schreier graph of* $(G, \mathrm{Stab}(1))$ *is recurrent.*

Proof. Consider a finitely supported measure μ on G and assume that the corresponding random walk on the Schreier graph of $(G, \mathrm{Stab}(1))$ is transient.

Put $H = e$. Note that G, μ, H and $K = \mathrm{supp}(\mu)$ satisfy the assumption of Proposition 2. Consequently, (G, μ) has nontrivial Poisson boundary. But this is impossible since G has intermediate growth. This contradiction proves the proposition. □

Various examples of Schreier graphs of $(G, \mathrm{Stab}(1))$ are constructed in [3]. In that paper it was announced that in some examples the Schreier graphs have polynomial growth n^d for large d. By the proposition above all these graphs are recurrent, whenever G is of subexponential growth.

Discontinuity of recurrence. An example. Consider the group $G = G_w$ for some $w \in W^*$. As before, H is a subgroup of G generated by ad. Consider any measure μ such that $\mathrm{supp}\mu = \{ad, da\}$ and such that $\mu(ad) \neq \mu(da)$. Clearly, the random walk (H, μ) is transient. Since $H \cap \mathrm{Stab}(1) = e$ this implies that the random walk on the Schreier graph of $(G, \mathrm{Stab}(1))$ is transient. Now take a finite symmetric generating set K of G such that $ad, da \in K$. Take any sequence of measures μ_i such that $\mathrm{supp}(\mu_i) = K$ and the sequence μ_i tends pointwise to μ.

Since K generates G the proposition above implies that random walk on the Schreier graph of $(G, \mathrm{Stab}(1))$ is transient for any μ_i ($i \in \mathbb{N}$).

Note that a discontinuity as in the example above cannot happen for a symmetric measure μ, as follows from (3) of Proposition 1.

6. Applications to growth of groups

Theorem 2. *Let G act on $(0,1]$ by* LN *maps and let the action satisfy the strong condition* $(*)$. *Assume that there exists $g \in G$ such that $g^m(1) \neq 1$ for any $m \geq 1$ and that the subgroup generated by* $\{\mathrm{germ}_y(g) | y \in (0,1]\}$ *is not equal to* $\mathrm{Germ}(G)$. *Assume also that* $\mathrm{Germ}(G)$ *is finite. Then for any $\varepsilon > 0$ there exists N such that for any $n > N$*

$$v_{G,S}(n) \geq \exp\left(\frac{n}{\ln^{2+\varepsilon}(n)}\right).$$

Corollary 2. *For any* $w \in \Omega^*$ *and* $\varepsilon > 0$ *the growth function of* G_w *satisfies*

$$(\star) \qquad v_{G_w,S}(n) \geq \exp\left(\frac{n}{\ln^{2+\varepsilon}(n)}\right)$$

for any n *large enough* (*as already mentioned this group has subexponential growth*).

Proof. The corollary follows from Theorem 2 and Lemma 4.3. (Compare with Corollary 1.) □

In [13] it was shown that for any subexponential function f there exists a group G of intermediate growth such that (up to a natural equivalence relation) $v_{G,S}$ is asymptotically greater than f.

However, these examples from [13] are not generated by a finite state automaton. Moreover, for the known (finite state) automatic groups of subexponential growth (e.g. the first Grigorchuk group) there exists $\alpha < 1$ such that for any n large enough

$$v_{G,S}(n) \leq \exp(n^{\alpha}).$$

Now automatic groups satisfying ($\star$) can be constructed using Corollary 2. In fact, take

$$w = PTPTPTPTPTPT \ldots .$$

It is not difficult to see that G is generated by the finite state automaton shown in Figure 1.

The growth function can be defined for any finite state automata [16]. In that paper it is observed that this growth function is equal to the growth function of the semigroup generated by the automaton. The case when the automaton is invertible (that is, the corresponding semigroup is a group) is of particular interest.

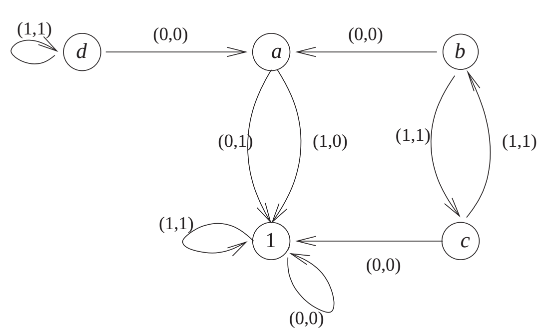

Figure 1.

In [13] it was shown that $G = G_w$ is commensurable with $G+G+G+G$. Moreover, it is possible to check that G is commensurable with $G+G$.

Let $B(e,r)$ denote the ball of radius r in the word metric, centered at e.

LEMMA 6.1. *Let μ be a probability measure on G such that*

$$\mu(G \setminus B(e,r)) \le C\frac{\ln^{\beta}(r+2)}{r}$$

for any r large enough and some $\beta > 1$, $C > 0$.

Then there exist $C', p > 0$ such that for any n large enough

$$\mu^{*n}(B(e, C'n\ln^{2\beta}(n)) > p.$$

(The initial form of this lemma was slightly changed after a talk with Th. Delzant.)

Proof of the lemma. Consider the measure ν on $\mathbb{Z}_+$ defined by $\nu(z) = \mu(S(e,z))$ for any positive integer z. Clearly, it suffices to prove the statement of the lemma for ν. We know that

$$\nu([r+1,\infty)) \le C\frac{\ln^{\beta}(r+2)}{r}.$$

Take R_0 such that $\frac{\ln^{\beta}(r+2)}{r}$ increases on $[R_0,\infty)$ and consider a measure ν_0 on $\mathbb{Z}_+$ such that

$$\nu_0([r+1,\infty)) = C\frac{\ln^{\beta}(r+2)}{r}$$

for any integer $r \ge R_0$. Put

$$m(n) = \frac{n}{\ln^{2\alpha}(n+2)}.$$

Since $m(n)$ increases on $\mathbb{Z}_+$, there exists $A_1, C_2 > 0$ such that

$$\begin{aligned} M = \sum_{n\ge 0}\frac{n}{\ln^{2\beta}(n+2)}\nu(n) &= \sum_{n\ge 0} m(n)\nu(n) = A_1 + \sum_{n\ge R_0} m(n)\nu(n)\\ &\le A_1 + \sum_{n\ge R_0} m(n)\nu_0(n) \le A_1 + C_2\sum_{n\ge R_0}\frac{1}{n\ln^{\beta}(n+2)} < \infty. \end{aligned}$$

The last inequality is due to the fact that $\beta > 1$.

Note that $m(n) = \frac{n}{\ln^{2\alpha}(n+2)}$ satisfies $m(a+b) \le m(a)+m(b)$ for any $a, b \ge 0$. Hence for any probability measures ν_1 and ν_2 on $\mathbb{Z}_+$

$$\sum_{n\ge 0} m(n)\nu_1 * \nu_2(n) \le \sum_{n\ge 0} m(n)\nu_1(n) + \sum_{n\ge 0} m(n)\nu_2(n).$$

Therefore,

$$\sum_{n\ge 0} m(n)\nu^{*k}(n) \le k\sum_{n\ge 0} m(n)\nu(n) = kM.$$

Consider $R = 3kM\ln^{2\beta}(k)$. Note that for k large enough $\frac{R}{\ln^{2\beta}(R+2)} \geq 2kM$. Hence

$$\mu^{*k}([0,R]) \geq 1/2. \qquad \square$$

Proof of Theorem 2. Let H be the subgroup of G generated by g. Take $\varepsilon > 0$. From 3 of Theorem 1 we know that there exists a symmetric measure μ on G such that $\mathrm{supp}(\mu) = H \cup K$, where K is some finite generating set of G,

$$\mu(g^n) = A_2 \frac{ln^{\beta}(|n|+1)}{n^2},$$

for some $1 < \beta < 1 + \varepsilon/2$ and $A_2 > 0$, and μ has nontrivial Poisson boundary.

Since the entropy of this measure is finite, the entropy on the random walk $h(\mu)$ is positive ([24]).

Put $C_3 = l(g)$. Note that $l(g^n) \geq C_3 n$. Consider $C_4 = \max l(k)$ for any $k \in K$. Note that for any $r > C_4$

$$\mu(G \setminus B(e,r)) \leq \sum_{i=r/C_3}^{\infty} \frac{\ln^{\beta}(|i|+1)}{i^2} \leq C\frac{\ln^{\beta}(r+2)}{r},$$

for some $C > 0$ (since $\beta > 1$).

Hence we can apply Lemma 6.1 and obtain that the convolution μ^{*n} is concentrated with positive probability on the ball $B(e, C'n\ln^{2\beta}(n))$. Since the entropy of the random walk is positive, Shannon's theorem [24] implies that the number of elements in this ball grow exponentially. That is, there exists some $c_2 > 0$ such that for any $n > N$

$$\#\left(B(e, C'n\ln^{2\beta}(n)\right) \geq \exp(c_2 n).$$

This inequality implies the statement of the theorem. $\square$

Remark 6. The same estimate as in Corollary 5 can be proved for the subgroup $G = \tilde{G}_w$ of G_w generated by ad and b. Let H be the subgroup of this subgroup generated by ad. Note that G and H do not satisfy the assumptions of Proposition 2, Theorem 1 and Theorem 2, but one can use Remark 4 instead.

6.1. *Estimates from above for the growth function.*

THEOREM 3. *Let $w = w_1, w_2, w_3 \ldots$ be the sequence of P and T satisfying the following property. There exists $M \geq 2$ such that for any $i \geq 1$ the elements $w_i, w_{i+1}, \ldots w_{i+M-1}$ are not all equal* (*that is, there are both P and T among them*). *Consider $G = G_w$. Then there exist $D > 0$ such that*

$$v_{G,S}(n) \leq \exp\left(\frac{D\ln(\ln(n))n}{\ln(n)}\right)$$

for any sufficiently large n.

Combining Theorem 3 above and Corollary 2 we obtain

Corollary 2′. *Let $w = PTPTPTPTPT\dots$ (or any other sequence satisfying the assumption of Theorem 3). Then*

$$\exp\left(\frac{n}{\ln^{2+\varepsilon}(n)}\right) \le v_{G_w,S}(n) \le \exp\left(\frac{n}{\ln^{1-\varepsilon}(n)}\right)$$

for any $\varepsilon > 0$ and any sufficient large n.

Proof of Theorem 3. The idea of the proof is similar to that in [13]. Take $w \in \Omega^*$. Let σ be the one-sided shift (that is, if $w = w_1, w_2, w_3, \dots$, then $\sigma(w) = w_2, w_3, \dots$). Let $H_r = H_{r,w}$ be the subgroup of G_w defined by

$$H_{r,w} = \{h \in G_w : h\left(\left(\frac{i}{2^r}\frac{i+1}{2^r}\right]\right) = \left(\frac{i}{2^r}\frac{i+1}{2^r}\right]$$

for any $0 \le i \le 2^r - 1$. Let β_i be the linear map from $\left(\frac{i}{2^r}, \frac{i+1}{2^r}\right]$ onto $(0,1]$. Note that for any $0 \le i \le 2^r - 1$ and any $g \in H_{r,w}$

$$\beta_i\left(g(\beta_i^{-1})\right) \in G_{\sigma^r(w)}.$$

This defines a map

$$\phi_r : H_{w,r} \to \underbrace{G_{\sigma^r(w)} + G_{\sigma^r(w)} + \dots + G_{\sigma^r(w)}}_{2^r}.$$

The group $G_{\sigma^r(w)}$ is generated by a, $b_{\sigma^r(w)}$, $c_{\sigma^r(w)}$ and $d_{\sigma^r(w)}$. In the product we consider the generating set which is the union of these generators of $G_{\sigma^r(w)}$. Below we always consider the word metric in this product which corresponds to this generating set.

For any $g \in G$ consider a shortest word u_g in the generating set $S = \{a, b, c, d\}$, representing g. Any such word clearly has the form $a * a * a \cdots * a$, $a * a * a \dots a*$, $*a * a \dots a*$ or $*a * a \cdots * a$, where $*$ stands for b, c or d. Let $\delta_b(u_g)$, $\delta_c(u_g)$ and $\delta_d(u_g)$ be the number of entries (in the word u_g) of b, c and d respectively.

Let $D_w^\varepsilon(n)$ consist of the elements g of G such that there exists a shortest word u_g of length n, representing g and satisfying

$$\delta_b(u_g) \le n(\frac{1}{2} - \varepsilon),\ \delta_c(u_g) \le n(\frac{1}{2} - \varepsilon)$$

and

$$\delta_d(u_g) \le n(\frac{1}{2} - \varepsilon).$$

The first part of the following lemma is proved in [13].

LEMMA 6.2. 1) *Let r be such that $w_1 = w_2 = \dots = w_r \neq w_{r+1}$. Then for any $g \in H_{w,r} \cap D_w^\varepsilon(n)$*

$$l(\phi_{r+1}(g)) \leq \left(1 - \frac{\varepsilon}{4}\right) n + 2^{r+1}$$

for any n and any $0 < \varepsilon < 1/4$.

2) *Let r be such that not all of the elements $w_1, w_2, \dots w_r$ are equal. Then for any $g \in H_{w,r} \cap D_w^\varepsilon(n)$*

$$l(\phi_r(g)) \leq \left(1 - \frac{\varepsilon}{4.1}\right) n$$

for any $\frac{1002^{r+1}}{n} < \varepsilon < 1/4$ and any n.

Proof. For (1) see [13, Lemma 6.3]. The second statement follows from 1) since for any r and any g

$$l(\phi_r(g)) \leq n + 2^r$$

(see [13]). □

Let $\gamma(n) = \gamma_{G,S}(n)$ be the number of elements of G of length n. By definition $v_{G,S}(n) = \gamma_{G,S}(1) + \gamma_{G,S}(2) + \dots + \gamma_{G,S}(n)$.

Note that the index of H_r in G is at most $C = 2^r!$. Hence there exists C_2, depending only on r, such that

$$v_{G,S}(n) \leq C|\{g \in H_{w,r} | l_{G,S}(g) \leq n + C_2\}|$$

for any $n \geq 0$. Let

$$\gamma_G^\varepsilon(r) = |(H_{w,r} \cap (S(e,r) D_w^\varepsilon(r))|,$$

where $S(e,r)$ is the sphere of radius r in the word metric of G, S, centered at e. Put $v_G^\varepsilon(r) = \gamma_G^\varepsilon(1) + \gamma_G^\varepsilon(2) + \dots + \gamma_G^\varepsilon(r)$.

Suppose that r and ε satisfy the assumption of the second part of the lemma. Then

$$v_{G,S}(n) \leq C \left(\sum_{\substack{n_1+n_2+\dots n_C \leq \\ (1-\varepsilon/4.1)(n+C_2)}} \prod_{i=1}^{C} v_{G_{\sigma^r(w)},S}(n_i) + v^\varepsilon_{G_{\sigma^r(w)},S(n+C_2)} \right)$$

for any $n > \tilde{N}$, for $\tilde{N}, C$ and C_2 depending only on r, as follows from the second part of the lemma.

There exists N, depending only on r such that

$$(1 - \varepsilon/4.1)(n + C_2) < (1 - \varepsilon/5)n$$

for any $\varepsilon > 40C_2$ and any $n > N$. Under this assumption

$$v_{G,S}(n) \leq C \left(\sum_{\substack{n_1+n_2+\dots n_C \leq \\ (1-\varepsilon/5)n}} \prod_{i=1}^{C} v_{G_{\sigma^r(w)},S}(n_i) + v^\varepsilon_{G_{\sigma^r(w)},S(n+C_2)} \right).$$

Lemma 6.3. *There exist $A'_1, C'_3, A_1, C_3 > 0$ and depending only on r such that for any $0 < \varepsilon < 1/4$ and any n*

$$\gamma^{\varepsilon}_G(n) \le C'_3 C_{n/2}^{A'_1 \varepsilon n} 2^{C'_3 \varepsilon n}$$

and

$$v^{\varepsilon}_G(n) \le C_3 C_{n/2}^{A_1 \varepsilon n} 2^{C_3 \varepsilon n}.$$

Proof. Note that

$$\delta_b(u_g) + \delta_c(u_g) + \delta_d(u_g) = [\frac{n}{2}] + x,$$

$x = 0, 1$ or -1, for any geodesic word u_g of length n. If $g \notin D^{\varepsilon}(n)$ then there exists a geodesic word u_g, representing g such that either

$$\delta_b(u_g) \ge (\frac{1}{2} - \varepsilon)n$$

or $\delta_c(u_g) \ge (\frac{1}{2} - \varepsilon)n$, or $\delta_d(u_g) \ge (\frac{1}{2} - \varepsilon)n$.

Hence

$$\gamma^{\varepsilon}_G(n) \le 3 \left(C^{\varepsilon n}_{[n/2]} + C^{\varepsilon n}_{[n/2]-1} + C^{\varepsilon n}_{[n/2]+1} + \right) 2^{\varepsilon n}.$$

This implies the first inequality in the statement of the lemma. Clearly, the second inequality follows from the first one. $\square$

Now consider $\varepsilon_n = 10C/\ln(n)$. Note that $40C_2/n < \varepsilon_n < 1/4$ and $1002^{r+1}/n < \varepsilon_n$ for any sufficiently large n.

Assume again that w, r satisfy the assumptions of the second part of Lemma 6.2. Lemma 6.3 implies that there exist $A, B, N > 0$, depending only on r such that

$$v_{G,S}(n) \le C \left(\sum_{\substack{n_1+n_2+\dots n_C \le \\ (1-\varepsilon_n/5)n}} \prod_{i=1}^{C} v_{G_{\sigma^r(w)},S}(n_i) + A C_{n/2}^{Bn/\ln(N)} \right)$$

for any $n > N$.

Note that Stirling's formula implies that

$$\ln\left(C_n^{L\ln(n)} \right) = O\left(\frac{\ln(\ln(n))n}{\ln(n)} \right)$$

and hence there exists $F > 0$ such that

$$v_{G,S}(n) \le C \left(\sum_{\substack{n_1+n_2+\dots n_C \le \\ (1-\varepsilon_n/5)n}} \prod_{i=1}^{C} v_{G_{\sigma^r(w)},S}(n_i) + \exp\left(\frac{F\ln(\ln(n))n}{\ln(n)} \right) \right).$$

Put

$$f_m(n) = \max v_{G_w,S}(n),$$

where the maximum is taken over all w satisfying the assumption of the theorem with a given constant m. (The function is well defined, since for $v_{G_w,S}(n) \leq 4^n$ for any w and any positive integer n.) Note that if w is as above, then $\sigma(w)$ satisfies the assumption of the theorem with the same constant m. Consider $r = m$. There exists N, depending on m, such that for some $C \geq 2$

$$f_m(n) \leq C\left(\sum_{\substack{n_1+n_2+\dots n_C\leq \\ (1-\varepsilon_n/5)n}} \prod_{i=1}^{C} f_m(n_i) + \exp\left(\frac{F\ln(\ln(n))n}{\ln(n)}\right)\right)$$

for any $n \geq N$.

LEMMA 6.4. *Let $f : \mathbb{N} \to \mathbb{R}_+$ be the function satisfying*

$$f(n) \leq C\left(\sum_{\substack{n_1+n_2+\dots n_C\leq \\ (1-2C/\ln(n))n}} \prod_{i=1}^{C} f(n_i) + \exp\left(\frac{F\ln(\ln(n))n}{\ln(n)}\right)\right)$$

for any $n \geq N$. Then there exists $D > 0$ such that

$$f(n) \leq \exp\left(\frac{D\ln(\ln(n))n}{\ln(n)}\right)$$

for any sufficiently large n.

Proof. Note that

$$\begin{aligned} f(n) &\leq C\left(n^C \max_{\substack{n_1+n_2+\dots n_C\leq \\ (1-2C/\ln(n))n}} \prod_{i=1}^{C} f(n_i) + \exp\left(\frac{F\ln(\ln(n))n}{\ln(n)}\right)\right) \\ &\leq n^{2C}\left(\max_{\substack{n_1+n_2+\dots n_C\leq \\ (1-2C/\ln(n))n}} \prod_{i=1}^{C} f(n_i) + \exp\left(\frac{F\ln(\ln(n))n}{\ln(n)}\right)\right) \end{aligned}$$

for any sufficiently large n.

Note that $\exp(x) + \exp(y) \leq 2\exp(\max(x, y))$, and hence

$$\begin{aligned} f(n) &\leq n^{2C} \max\left(\ln\left(\max_{\substack{n_1+n_2+\dots n_C\leq \\ (1-2C/\ln(n))n}} \prod_{i=1}^{C} f(n_i)\right), \frac{F\ln(\ln(n))n}{\ln(n)}\right) \\ &= n^{2C} \max\left(\left(\max_{\substack{n_1+n_2+\dots n_C\leq \\ (1-2C/\ln(n))n}} \sum_{i=1}^{C} \ln(f(n_i))\right), \frac{F\ln(\ln(n))n}{\ln(n)}\right). \end{aligned}$$

Put $g(n) = \ln(f(n))$. This function satisfies

$$g(n) \leq 2C\ln(n) + \max\left(\left(\max_{\substack{n_1+n_2+\dots n_C \leq \\ (1-2C/\ln(n))n}} \sum_{i=1}^{C} g(n_i)\right), \frac{F\ln(\ln(n))n}{\ln(n)}\right)$$

for any sufficiently large n.

Consider

$$g_0(n) = \frac{\ln(\ln(n+1000))(n+1000)}{\ln(n+1000)}.$$

Lemma 6.5. (1) *The function $g_0(n)$ is concave on $[0, \infty)$.*

(2) *There exists $N_1 > N$ such that for any $n > N_1$*

$$g_0\left(\frac{n(1-2C/\ln(n))}{C}\right) + 2C\ln(n) \leq g_0(n).$$

The proof of this lemma is omitted.

Now we return to the proof of Lemma 6.4. Take N_1 as in the second part of Lemma 6.5, such that

$$2C\ln(n) \leq Fg_0(n)$$

for any $n \geq N_1$.

Take $F_1 \geq 2F$ such that

$$g(n) \leq F_1 g_0(n)$$

for any $1 \leq n < N_1$. We are going to prove that then the inequality above holds for any positive integer n. The proof is by induction on n. Suppose that the inequality holds for any $n < n'$, $n' \geq N_1$. Note that

$$Fg_0(n') + 2C\ln(n') \leq 2Fg_0(n') \leq F_1 g_0(n')$$

and that, since g_0 is concave and since n' satisfies the assumption of the second part of Lemma 6.5

$$\begin{aligned} \max&\left(\max_{\substack{n_1+n_2+\dots n_C \leq \\ (1-2C/\ln(n'))n'}} \sum_{i=1}^{C} F_1 g_0(n_i)\right) + 2C\ln(n') \\ &\leq CF_1 g_0\left(\frac{(1-2C/\ln(n'))n'}{C}\right) + 2C\ln(n') \\ &\leq F_1\left(g_0\left(\frac{(1-2C/\ln(n'))n'}{C}\right) + 2C\ln(n')\right) \leq F_1 g_0(n'). \end{aligned}$$

This implies that $g(n') \leq F_1 g_0(n')$ and completes the proof of the lemma. □

Now we apply Lemma 6.4 to $f(n) = f_m(n)$ and this completes the proof of the theorem. □

7. Generalizations

In this section we weaken the assumptions of Theorem 1 and prove under these assumptions that the group admits a symmetric measure with nontrivial exit boundary. The difference between Theorem 1 and Theorem 4 below is that we do not assume in Theorem 4 that the subgroup H has an element of infinite order (and with infinite orbit of 1 with respect to the action on $(0,1]$). Thus the following theorem can be applied to torsion groups.

THEOREM 4. *Suppose that G acts on $(0,1]$ by* LN *maps and that the action satisfies the strong condition* $(*)$. *Suppose also that there exists a finitely generated subgroup H in G such that* $\mathrm{Germ}(H) \neq \mathrm{Germ}(G)$, $\mathrm{Germ}(H)$ *is of finite index in* $\mathrm{Germ}(G)$ *and the index*

$$[H : H \cap \mathrm{Stab}_G(1)] = \infty.$$

Then there exists a symmetric measure μ on G such that the Poisson boundary of (G, μ) is nontrivial.

COROLLARY 3. *As before, let a be a cyclic permutations of* $(0, 1/2]$ *and* $(1/2, 1]$. *Consider elements*

$$b_1 = PPTPPT\ PPTPPT\ PPTPPT \dots ,$$
$$b_2 = TPPTPP\ TPPTPP\ TPPTPP \dots ,$$

and

$$b_3 = PPTTPP\ PPTTPP\ PPTTPP \dots .$$

(b_1, b_2 *and* b_3 *are periodic with period* 6.) *Let G be the group generated by a, b_1, b_2 and b_3, and H be the subgroup of G generated by a, b_1, b_2.*

By construction, H is isomorphic to the first Grigorchuk group ([13]).

Note that $\mathrm{Germ}(G) = (\mathbb{Z}/2\mathbb{Z})^3 \neq \mathrm{Germ}(H) = (\mathbb{Z}/2\mathbb{Z})^2$. Note also that $\mathrm{Stab}(1)$ is of infinite index in H. Hence we can apply Theorem 3 and conclude that G admits a measure with nontrivial Poisson boundary.

Remark 7. Let G be as in the corollary above. Let H_6 be the subgroup of G such that for any $0 \le i < 2^6$

$$h\left((\frac{i}{2^6}, \frac{i+1}{2^6}] \right) = (\frac{i}{2^6}, \frac{i+1}{2^6}]$$

for any $h \in H_6$. Clearly, H_6 is of finite index in G.

Consider the system $S = \{a, b_1, b_2, b_3, ab_1b_2, b_2b_3, b_3, b_1, b_1b_2b_3\}$ of generators of G. Note that for any $b \neq a$, $b \in S$,

$$b = w_1, w_2, w_3, \dots$$

and for any i at least one of the elements $w_i, w_{i+1}, \dots w_{i+5}$ is equal to T.

Then similarly to the case of the first Grigorchuk group [13], one can check that there is an injective map

$$\psi : H \to \underbrace{G + G + \cdots + G}_{2^6}$$

and $\beta_1, \beta_2 > 0$ such that $\beta_1 < 1$ and for any $h \in H_6$

$$l(\psi(h)) \le \beta_1 l_{G,S}(h) + \beta_2.$$

(Here the word metric in the direct sum corresponds to the system of generators which is the union of generators of G.)

This implies that there exists $\alpha < 1$ for any n large enough and that the growth function of G satisfies

$$v_{G,S}(n) \le \exp(n^\alpha).$$

Before starting to prove Theorem 4, we prove the following lemma.

LEMMA 7.1. *Let A be a finitely generated group and B be a subgroup of infinite index in A. Then there exists a symmetric measure ν on A such that B is transient with respect to ν.*

Recall that B is transient with respect to ν if and only if the induced random walk $(A/B, \nu)$ on the cosets A/B is transient.

Proof of the lemma. Consider a symmetric measure μ with finite support on A containing l elements. Note that

$$\lim_{n\to\infty} \mu^{*n}(B) = 0.$$

In fact, consider the Schreier graph of (A, B). Since it is infinite, it contains an infinite array. For the proof of the formula it suffices to compare $(A/B, \nu)$ with the simple random walk on this array and to use (2) of Proposition 1 for the Lebesgue measure ξ on $[0, 1]$. In this case $\xi_n = n^{-1}$ and we see that

$$\sum_n \frac{1}{n} \mu^{*n}(B) < \infty.$$

Hence for some subsequence of $\mu^{*n}(B)$ tends to 0. But since $\mu^{*2n}(B)$ decreases in n (this follows from the spectral theorem) and since $\mu^{*2n}(B) \ge \frac{1}{l}\mu^{*(2n-1)}(B)$ this implies that

$$\lim_{n\to\infty} \mu^{*n}(B) = 0.$$

Now consider a sequence $n_i \in \mathbb{N}$ and $a_i \in \mathbb{R}$, $a_i \ge 0$, such that $\sum_{i=1}^n a_i = 1$. Put

$$\nu = \sum_{i=1}^n a_i \mu^{*n_i}.$$

It is clear that ν is a probability measure on A.

Note that

$$\nu^{*k} = \left(\sum_{i=1}^{n} a_i \mu^{*n_i}\right)^{*k} = \sum_{i_1,i_2,\dots,i_k \geq 0} a_{i_1} a_{i_2} \dots a_{i_k} \mu^{*(n_{i_1}+n_{i_2}+\cdots+n_{i_k})}.$$

Hence

$$\nu^{*k}(B) = \sum_{i_1,i_2,\dots,i_k \geq 0} a_{i_1} a_{i_2} \dots a_{i_k} \mu^{*(n_{i_1}+n_{i_2}+\cdots+n_{i_k})}(B).$$

Note that for any $d > 0$ there exists a decreasing sequence $a_i \geq 0$ and a constant $C > 0$ such that for any $k \in \mathbb{N}$

$$\left(\sum_{i=1}^{k} a_k\right)^k \leq \frac{C}{k^d}.$$

Take a_i as above and assume that the sequence n_i satisfies $\mu^{*m}(B) < 1/i^d$ for any $m \geq n_i$. Note that such sequences n_i do exist since $\mu^{*m}(B) \to 0$ as $m \to \infty$.

Also,

$$\begin{aligned}\nu^{*k}(B) &= \sum_{i_1,i_2,\dots,i_k \geq 0} a_{i_1} a_{i_2} \dots a_{i_k} \mu^{*(n_{i_1}+n_{i_2}+\cdots+n_{i_k})}(B)\\ &= \sum_{i_1,i_2,\dots,i_k \leq k} a_{i_1} a_{i_2} \dots a_{i_k} \mu^{*(n_{i_1}+n_{i_2}+\cdots+n_{i_k})}(B)\\ &\quad + \sum_{\substack{i_1,i_2,\dots,i_k:\\ \exists j: i_j > k}} a_{i_1} a_{i_2} \dots a_{i_k} \mu^{*(n_{i_1}+n_{i_2}+\cdots+n_{i_k})}(B).\end{aligned}$$

The first term is not greater than

$$\sum_{i_1,i_2,\dots,i_k \leq k} a_{i_1} a_{i_2} \dots a_{i_k} = (a_1 + a_2 + \cdots + a_k)^k \leq \frac{C}{k^d}.$$

Note that for each multi-index in the second term

$$(n_{i_1} + n_{i_2} + \cdots + n_{i_k}) \geq n_{i_j} \geq n_k.$$

Consequently the second term is at most

$$\begin{aligned}&\sum_{\substack{i_1,i_2,\dots,i_k:\\ \exists j: i_j > k}} a_{i_1} a_{i_2} \dots a_{i_k} \frac{1}{k^d}\\ &\qquad \leq \frac{1}{k^d} \sum_{i_1,i_2,\dots,i_k} a_{i_1} a_{i_2} \dots a_{i_k} \leq \frac{1}{k^d}(a_1 + a_2 + a_3 + \dots)^k = \frac{1}{k^d}.\end{aligned}$$

Consequently for any $k \geq 1$

$$\nu^{*k}(B) \leq \frac{C+1}{k^d}.$$

Hence taking $d > 1$ we get

$$\sum_{k=1}^{\infty} \nu^{*k}(B) < \infty.$$

This implies that B is transient. □

The assumption in the lemma above that A is finitely generated can be dropped. To see this, it suffices to consruct a measure μ on A such that

$$\mu^{*i}(B) \to 0 \text{ as } i \to \infty.$$

And then the previous argument applies.

Proof of Theorem 4. Take a finite symmetric set of generators S such that (G, S) satisfies the strong condition $(*)$. Consider a measure ν_1 on H such that $H \cap \mathrm{Stab}(1)$ is transient for (H, ν_1). This is possible due to the previous lemma since $H \cap \mathrm{Stab}(1)$ is of infinite index in H. Let ν_2 be the measure equidistributed on S.

Put

$$\nu = \frac{1}{2}(\nu_1 + \nu_2).$$

Let $k(\nu_1)$, $k(\nu_2)$ and $k(\nu)$ be the induced kernels on $G/\mathrm{Stab}(1)$. Clearly,

$$k(\nu)(x, y) = \frac{1}{2}(k(\nu_1)(x, y) + k(\nu_2)(x, y))$$

for any $x, y \in G/\mathrm{Stab}(1)$.

We know that $\mathrm{Stab}(1)$ is transient for (G, ν_1), and hence the random walk $(G/\mathrm{Stab}(1), k(\nu_1))$ is transient. Since $\nu_1 \leq 2\nu$, Proposition 1 implies that $(G/\mathrm{Stab}(1), k(\nu))$ is transient. That is $\mathrm{Stab}(1)$ is transient for (G, ν). Hence (G, ν) satisfies the assumption of Proposition 2 and, consequently, the Poisson boundary of (G, ν) is nontrivial. □

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