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The McKay conjecture and Galois automorphisms

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Abstract

The main problem of representation theory of finite groups is to find proofs of several conjectures stating that certain global invariants of a finite group G can be computed locally. The simplest of these conjectures is the "McKay conjecture" which asserts that the number of irreducible complex characters of G of degree not divisible by p is the same if computed in a p-Sylow normalizer of G. In this paper, we propose a much stronger version of this conjecture which deals with Galois automorphisms. In fact, the same idea can be applied to the celebrated Alperin and Dade conjectures.

1. Introduction

Much of the representation theory of finite groups these days is devoted to several conjectures which state that certain invariants of a finite group G can be computed locally. Perhaps, the most amazing (and the simplest) of these conjectures is the McKay conjecture which asserts that if G is a finite group, p is a prime number and $\mathrm{Irr}_{p'}(G)$ is the set of complex irreducible characters of G of degree not divisible by p, then

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|\,,$$

where P is a Sylow p-subgroup of G.

Why these two numbers coincide for every finite group is still a mystery for which no general explanation has been given. At the same time, this conjecture has been tested for so many classes of groups, that there is no reasonable doubt about its validity.

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The purpose of the present paper is to propose a much stronger form of the McKay conjecture which deals with Galois automorphisms of cyclotomic fields. As was our intention when we proposed the "congruence form" of the McKay conjecture in [11], it is our hope that these stronger conjectures will eventually lead us to understand what is really behind them.

Suppose that G is a finite group of order n. Richard Brauer's theorem that every irreducible complex character of G can be afforded by a representation with entries in the cyclotomic field $\mathbb{Q}_n$. Therefore, the Galois group $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ permutes the set $\mathrm{Irr}(G)$ of the irreducible complex characters of G. It is well-known that there cannot exist a bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ which commutes with *all* the elements of $\mathcal{G}$. (That would imply, for instance, that the number of rational characters in $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ is the same, and this is simply false as shown by the group $\mathrm{GL}(2,3)$ with $p = 3$, for instance.) We believe, however, that there should exist a bijection commuting with the elements of a very special subgroup of $\mathcal{G}$.

CONJECTURE A. *Let G be a finite group of order n and let p be a prime. Let e be a nonnegative integer and let $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ be any Galois automorphism sending every p'-root of unity ξ to ξ^{p^e}. Then σ fixes the same number of characters in $\mathrm{Irr}_{p'}(G)$ as it does in $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$.*

Of course, what Conjecture A is really proposing is that not only the sets $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ have the same cardinality, but that there is a connection between the character values of the elements in both sets.

If p is an odd prime dividing n and θ is any Galois automorphism fixing p'-roots of unity and having order $p-1$, we will show that it is a consequence of Conjecture A that a Sylow p-subgroup P of G is self-normalizing if and only if the principal character of G is the only irreducible character of G of p'-degree fixed by θ. (See Theorem (5.3) below.) If σ is the Galois automorphism which fixes 2-power roots of unity and sends every odd root of unity ξ to ξ^2, then it is a consequence of Conjecture A that a Sylow 2-subgroup of G is self-normalizing if and only if all irreducible characters of odd degree of G are σ-fixed. (See Theorem (5.2) below.) In particular, Conjecture A implies that we can read off from the character table of a finite group G if a Sylow p-subgroup of G is self-normalizing.

Our Conjecture A is saying something new, even in the classical case where a Sylow p-subgroup P of G is cyclic (a case that we shall prove in Section 3 below). For instance, if P is cyclic and self-centralizing, then it is true that $\chi(x) = \chi(x^p)$ whenever $\chi \in \mathrm{Irr}_{p'}(G)$ and $x \in G$ has order not divisible by p. (See Corollary (3.5) below.)

We give another consequence of Conjecture A: If p is an odd prime, then Conjecture A implies that the number of p-rational characters in $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ is the same. (See Theorem (5.4) below.)

When we stated the "congruence form" of the McKay conjecture in [11], we had noticed that there should be connections between the McKay conjecture and certain Galois automorphisms. We were assuming in [11], however, that those Galois automorphisms were fixing p'-roots of unity and had p-power order. These restrictions now seem unnecessary.

Now, what is the evidence for the validity in Conjecture A? When G is a group of odd order, M. Isaacs constructed in [9] a canonical bijection $^* : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$. In this case, * commutes with *every* Galois automorphism and therefore Conjecture A follows. More generally, when G is a solvable group E. C. Dade has recently checked our conjecture ([4]). This is already a surprisingly difficult theorem. If $G = S_n$ is a symmetric group, then all irreducible characters of G are rational. Our Conjecture A, predicts that all irreducible characters of degree not divisible by p of $\mathbf{N}_G(P)$ are σ-fixed (for the "right" σ, of course). This fact has been verified by P. Fong. Also, we have checked that Conjecture A is true for every sporadic group. This check, on which we will comment in Section 4, requires considerable work. In Section 3, and using the cyclic defect theory, we will show that the stronger block version of Conjecture A below is also true for blocks with cyclic defect group. In particular, this implies Conjecture A if a Sylow p-subgroup of G is cyclic.

The McKay conjecture was generalized to Brauer blocks by J. Alperin, and we do so for our Conjecture A.

CONJECTURE B. *Let G be a finite group of order n and let p be a prime. Let e be a nonnegative integer and let $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ be any Galois automorphism sending every p'-root of unity ξ to ξ^{p^e}. Let $B \in \mathrm{Bl}(G)$ be a p-block of G with defect group D, and let $b \in \mathrm{Bl}(\mathbf{N}_G(D))$ be the Brauer correspondent of B. Then σ fixes the same number of height zero ordinary irreducible characters in B as it does in b.*

Since the irreducible characters of G of p'-degree are exactly the height zero characters in blocks of full defect, it is clear that Conjecture B implies Conjecture A.

2. Galois automorphisms

Let $\xi \in \mathbb{C}$ be a primitive n-th root of unity and let $\mathbb{Q}_n = \mathbb{Q}(\xi)$ be the cyclotomic field. Let $R_n = \mathbb{Z}[\xi]$ be the Dedekind domain of algebraic integers in $\mathbb{Q}_n$. Now, the elements of $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ stabilize R_n.

We are interested in the subgroup $\mathcal{H}$ of $\mathcal{G}$ consisting of those elements $\sigma \in \mathcal{G}$ for which there is a nonnegative integer e such that $\sigma(\delta) = \delta^{p^e}$ whenever δ is a p'-root of unity in $\langle \xi \rangle$.

Let us write $n = p^a m$, where p does not divide m. Also, we write $\xi = \omega\delta$, where the order of ω is p^a and the order of δ is m. Hence, $\mathcal{G} = \mathcal{K} \times \mathcal{J}$, where

$\mathcal{K} = \{\tau \in \mathcal{G} | \tau(\delta) = \delta\}$ and $\mathcal{J} = \{\tau \in \mathcal{G} | \tau(\omega) = \omega\}$. Now, $\mathcal{K}$ is isomorphic to the group $\mathrm{Gal}(\mathbb{Q}_{p^a}/\mathbb{Q})$ of order $\varphi(p^a)$ and $\sigma \in \mathcal{J}$ is such that $\sigma(\delta) = \delta^p$. Also, the order t of σ is the order of p modulo m. We easily see that $\mathcal{H} = \mathcal{K} \times \langle \sigma \rangle$ is a group of order $\varphi(p^a)t$.

We shall need the following elementary result on algebraic number theory.

(2.1) Theorem. *With the previous notation, $\mathcal{H}$ is the subgroup of $\mathcal{G}$ which fixes every prime ideal $\mathcal{P}$ of R_n containing p.*

Proof. It is well-known that the prime ideals $\mathcal{P}$ of R_n containing p are $\mathcal{G}$-conjugate. Since $\mathcal{G}$ is an abelian group, it follows that $\tau \in \mathcal{G}$ fixes some prime ideal containing p if and only if τ fixes all of them. So let $\mathcal{P}$ be a prime ideal of R_n containing p and let $\mathcal{I}$ be the stabilizer of $\mathcal{P}$ in $\mathcal{G}$. Now, it is well-known that

$$pR_n = (\mathcal{P}_1 \cdots \mathcal{P}_r)^e \,,$$

where $e = \varphi(p^a)$, $\{\mathcal{P}_1, \ldots, \mathcal{P}_r\}$ is the set of all the different prime ideals of R_n containing p and $r = \varphi(m)/t$. Therefore, $|\mathcal{I}| = |\mathcal{G}|/r = \varphi(p^a)t = |\mathcal{H}|$. Hence, it suffices to show that the elements of $\mathcal{H}$ stabilize $\mathcal{P}$. First, notice that $\omega \equiv 1 \bmod \mathcal{P}$, since $R_n/\mathcal{P}$ is a field of characteristic p. Now, suppose that $\tau \in \mathcal{K}$ and let $f(\xi) \in \mathcal{P}$, where $f(x) = a_0 + a_1 x + \cdots + a_s x^s \in \mathbb{Z}[x]$. Then $\tau(\xi) = \omega^k \xi$ for some k. Next,

$$\tau(f(\xi)) = a_0 + a_1 \omega^k \xi + \cdots + a_s(\omega^k \xi)^s \equiv f(\xi) \bmod \mathcal{P} \,,$$

and therefore $\tau(f(\xi)) \in \mathcal{P}$. Finally, it suffices to show that $\sigma(f(\xi)) \in \mathcal{P}$. First, notice that $f(\xi)^p \equiv f(\xi^p) \bmod \mathcal{P}$. Hence, $f(\xi^p) \in \mathcal{P}$. Now, we can write $\sigma(\xi) = \omega^k \xi^p$ for some k. Then, $\sigma(f(\xi)) \equiv f(\xi^p) \bmod \mathcal{P}$, and we deduce that $\sigma(f(\xi)) \in \mathcal{P}$. $\square$

3. The cyclic case

In this section, we prove Conjecture B for blocks with a cyclic defect group. To do so, we adopt a clever argument by E. C. Dade and W. Feit which was pointed out to us by Dade.

(3.1) *Hypotheses and notation.* Suppose that G is a finite group of order n and let p be a prime number dividing n. Let $\xi \in \mathbb{C}$ be a primitive n-th root of unity and let $\mathbf{L} = \mathbb{Q}(\xi) = \mathbb{Q}_n$. Now, $R_n = \mathbb{Z}[\xi]$ is the ring of algebraic integers of $\mathbf{L}$, and a prime ideal $\mathcal{P}$ of R_n containing pR_n. Now, let $\mathbf{K} = \mathbf{L}_{\mathcal{P}}$ be the completion of $\mathbf{L}$ with respect to $\mathcal{P}$. Let R be the ring of $\mathcal{P}$-adic integers in $\mathbf{K}$ with unique maximal ideal (π). Let $\mathbf{F} = R/(\pi) \cong R_n/\mathcal{P}$, a field of characteristic p. Now $\mathbf{K}$ and $\mathbf{F}$ are splitting fields for G and for any of its subgroups. Also, $\mathbf{K} = \mathbf{Q}_p(\xi)$ is a finite extension of the p-adic field $\mathbf{Q}_p$. Now, let $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ and let $\mathcal{H}$ be as in Section 2. Then $\mathcal{H}$ fixes $\mathcal{P}$ by Theorem (2.1). Therefore, every $\tau \in \mathcal{H}$ extends to a unique automorphism of $\mathbf{K}$, say τ, such that $\tau(R) = R$. In particular, $\tau((\pi)) = (\pi)$ and therefore, τ defines an automorphism (of Frobenius type) on the field $\mathbf{F}$.

For blocks, we follow the notation in [13].

(3.2) LEMMA. *Let D be a p-subgroup of G and suppose that $U \triangleleft G$ is contained in D. Let $\bar{G} = G/U$. Let $b \in \mathrm{Bl}(\mathbf{N}_G(D))$ have defect group D and assume that $\bar{b}$ is a block of $\mathbf{N}_G(D)/U$ contained in b. Then b^G is the unique block of G containing the block $\bar{b}^{\bar{G}}$ of $\bar{G}$.*

Proof. Write $N = \mathbf{N}_G(D)$ and $C = \mathbf{C}_G(D)$. We claim that $\bar{b}$ has defect group D/U. Since $D/U \triangleleft N/U$, we have that D/U is contained in some defect group of $\bar{b}$. On the other hand, a defect group of $\bar{b}$ is contained in D/U (see Theorem (9.9.a) of [13]) and this proves the claim. In particular, $\bar{b}^{\bar{G}}$ is defined and has defect group D/U by the First Main Theorem.

Write $\bar{G} = G/U$. Let $g \in G$, $\bar{g} = gU$, $K = \mathrm{cl}(g)$ and let $\bar{K} = \mathrm{cl}(\bar{g})$. If $g \in \mathbf{C}_G(U)$ is a p'-element, by elementary group theory we have that $\mathbf{C}_{\bar{G}}(\bar{g}) = \mathbf{C}_G(g)/U$. In particular, if K is a class of p'-elements having defect group D, then $\bar{K}$ has defect group D/U. Suppose that K is such a class for the rest of this proof.

Let $\bar{B}$ be any block of $\bar{G} = G/U$ contained in B, a block of G. Let $\bar{\chi} \in \mathrm{Irr}(\bar{B})$ and let $\chi \in \mathrm{Irr}(G)$ be its lift. Then

$$\omega_\chi(\hat{K}) = |G : \mathbf{C}_G(g)|\chi(g)/\chi(1) = |\bar{G} : \mathbf{C}_{\bar{G}}(\bar{g})|\bar{\chi}(\bar{g})/\bar{\chi}(1) = \omega_{\bar{\chi}}(\bar{K})\,.$$

Hence, $\lambda_B(\hat{K}) = \lambda_{\bar{B}}(\hat{\bar{K}})$. Also, since $L = K \cap C$ is a conjugacy class of N with defect group D (see Lemma (4.16) of [13]), we have that $\lambda_b(\hat{L}) = \lambda_{\bar{b}}(\hat{\bar{L}})$.

Now, $\bar{K}$ is a conjugacy class of $\bar{G}$ with defect group $\bar{D}$ and again we have that $\bar{K} \cap \mathbf{C}_{\bar{G}}(\bar{D})$ is a conjugacy class of N/U. Since $\bar{L}$ is a conjugacy class of N/U contained in $\bar{K} \cap \mathbf{C}_{\bar{G}}(\bar{D})$, we have $\bar{L} = \bar{K} \cap \mathbf{C}_{\bar{G}}(\bar{D})$.

Now, set $\bar{B} = (\bar{b})^{\bar{G}}$. By the third paragraph of this proof,

$$\lambda_B(\hat{K}) = \lambda_{\bar{B}}(\hat{\bar{K}}) = \lambda_{\bar{b}^{\bar{G}}}(\hat{\bar{K}}) = \lambda_{\bar{b}}(\widehat{\bar{K} \cap \mathbf{C}_{\bar{G}}(\bar{D})}) = \lambda_{\bar{b}}(\hat{\bar{L}}) = \lambda_b(\hat{L}) = \lambda_{b^G}(\hat{K})\,.$$

If we choose a defect class K of b^G, we have that $\lambda_B(\hat{K}) \neq 0$. Hence, a defect group of B is contained in D. By Theorem (9.9.a) of [13], we conclude that B has defect group D and it follows that $B = b^G$. □

(3.3) LEMMA. *Suppose that B is a block of G with cyclic defect group D. Assume that $U < D$ is a normal subgroup of G and let $\bar{G} = G/U$. Then B contains a unique block $\bar{B}$ of $\bar{G}$. Also, $\bar{B}$ has defect group D/U.*

Proof. Suppose that $\bar{B}_1$ and $\bar{B}_2$ are blocks of $\bar{G}$ contained in B. By Theorem (9.9.a) of [13], we may find a defect group Δ_i/U of $\bar{B}_i$ contained in D/U. In particular, Δ_i/U is cyclic. Hence, all the characters in $\bar{B}_i$ have height zero (see Theorem VII.2.16) of [5]). Let $\bar{\chi}_i \in \mathrm{Irr}(\bar{B}_i)$ and let $\chi_i \in \mathrm{Irr}(B)$ be its lift. Then

$$|G : D|_p = \chi_i(1)_p = \bar{\chi}_i(1)_p = |G/U : \Delta_i/U|\,,$$

and we conclude that $\Delta_1 = D = \Delta_2$. To prove the lemma, it suffices to show that

$$\lambda_{\bar{B}_1}(\hat{\bar{K}}) = \lambda_{\bar{B}_2}(\hat{\bar{K}})$$

for p'-conjugacy classes $\bar{K}$ of $\bar{G}$ with defect group D/U (see, for instance, Problem (4.5) of [13]). Given such a $\bar{K}$ choose a p'-element $g \in G$ such that $\bar{g} \in \bar{K}$ and $\bar{g} \in \mathbf{C}_{\bar{G}}(\bar{D})$. Now, $[g, D] \leq U < D$. But $D = [D, g] \times \mathbf{C}_D(g)$ since g acts on D. Since D is cyclic, it follows that $g \in \mathbf{C}_G(D) \subseteq \mathbf{C}_G(U)$. Now, the lemma follows from the third paragraph in the proof of Lemma (3.2). □

For blocks with cyclic defect groups, we freely use the notation and the results in [2] and [5]. Suppose that B is a block of G with a cyclic defect group D with $|D| = p^a$. Let $N = \mathbf{N}_G(D)$ and let $C = \mathbf{C}_G(D)$. Let $b_0 \in \mathrm{Bl}(C)$ be a root of B. Then $(b_0)^G = B$ and $(b_0)^N = B_0$ is the Brauer correspondent of B. If E is the stabilizer of b_0 in N, then E/C is a cyclic group of order e dividing $p-1$. Also, E/C acts Frobeniusly on D (and therefore on the irreducible characters of $\mathrm{Irr}(D)$). Let Λ be a complete set of representatives of the action of E/C on $\mathrm{Irr}(D) - 1_D$. Then $|\Lambda| = (p^a - 1)/e$. We have that $\mathrm{IBr}(B) = \{\varphi_1, \ldots, \varphi_e\}$. Also, B has $e + |\Lambda|$ irreducible ordinary characters. If $|\Lambda| = 1$, then these will be denoted by $\chi_0, \chi_1, \ldots, \chi_e$. If $|\Lambda| > 1$, then $\mathrm{Irr}(B)$ is divided into two natural families. These are the *nonexceptional* characters $\mathrm{Irr}_{\mathrm{nex}}(B) = \{\chi_1, \ldots, \chi_e\}$ and the *exceptional* characters $\mathrm{Irr}_{\mathrm{ex}}(B) = \{\chi_\lambda | \lambda \in \Lambda\}$.

Assume (3.1) and let H be a subgroup of G. If $\tau \in \mathcal{G}$, then it is well-known that τ permutes the blocks of H. (This easily follows, for instance, from Theorem (3.19) of [13].) Now, by Theorem (2.1), it is easy to check that $\mathcal{H}$ acts naturally on $\mathrm{IBr}(H)$. Furthermore, $d_{\chi\varphi} = d_{\chi^\tau\varphi^\tau}$ for $\chi \in \mathrm{Irr}(b)$, $\varphi \in \mathrm{IBr}(H)$ and $\tau \in \mathcal{H}$. Hence, $\mathrm{Irr}(b^\tau) = \mathrm{Irr}(b)^\tau$ and $\mathrm{IBr}(b^\tau) = \mathrm{IBr}(b)^\tau$. Moreover, a defect group of b is one of b^τ. In fact, $\lambda_{b^\tau}(x) = \lambda_b(x)^\tau$ for $x \in \mathbf{ZF}H$. Thus

$$(b^\tau)^G = (b^G)^\tau\,.$$

So Brauer's First Main Theorem implies that τ fixes B if and only if τ fixes the Brauer correspondent of B.

(3.4) THEOREM. *Assume Hypotheses* (3.1). *Let B be a block of G with a cyclic defect group D and let B_0 be its Brauer correspondent. Let $\mathcal{H}_B$ be the subgroup of $\mathcal{H}$ fixing the block B.*

(a) *There exists a bijection* $F : \mathrm{Irr}(B) \to \mathrm{Irr}(B_0)$ *such that* $F(\chi^\tau) = F(\chi)^\tau$ *for all* $\chi \in \mathrm{Irr}(B)$ *and all* $\tau \in \mathcal{H}_B$. *Furthermore, if* $|\Lambda| > 1$, *then* F *sends the exceptional* (*nonexceptional*) *characters of* B *onto the exceptional* (*nonexceptional*) *characters of* B_0.

(b) *There exists a bijection* $S : \mathrm{IBr}(B) \to \mathrm{IBr}(B_0)$ *such that* $S(\varphi^\tau) = S(\varphi)^\tau$ *for all* $\varphi \in \mathrm{Irr}(B)$ *and all* $\tau \in \mathcal{H}_B$.

Proof. We prove the theorem by induction on $|G|$. We fix a root $b_0 \in \mathrm{Bl}(C)$ of B. Hence, $(b_0)^N = B_0$. Let E be the stabilizer of b_0 in N, so that $|E : C| = e$.

If $|\Lambda| > 1$, it easily follows from VII.6.3 of [5], that $\mathcal{H}_B$ permutes the exceptional (nonexceptional) characters among themselves. If $|\Lambda| = 1$, it follows from Theorem (2.4) and Lemma (3.1) of [6] that $\mathcal{H}_B$ fixes at least one character, say χ_0, of $\mathrm{Irr}(B)$.

Let $\tilde{D}$ be the unique subgroup of D of order p. Then $N \subseteq \tilde{N} = \mathbf{N}_G(\tilde{D})$. Also, $\tilde{B} = (b_0)^{\tilde{N}} = (B_0)^{\tilde{N}}$ is a block of defect group D which induces B and has Brauer correspondent B_0. Notice that $\tilde{B}$ is $\mathcal{H}_B$-invariant (by Proposition (3.9) of [3], for instance).

Next, we claim that parts (a) and (b) of this theorem are true if we replace B_0 by $\tilde{B}$. To prove the claim, we only have to check that Dade's bijections in Lemmas (4.8) through (4.10) in [3] are compatible with the action of $\mathcal{H}_B$. This is easily done by using the second paragraph of this proof and the fact that $\mathcal{H}_B$ commutes with the Green correspondence and induction of characters. Hence, by the inductive hypothesis, we may assume that $\tilde{D} \triangleleft G$.

Now, we freely use the notation and the results in [2]. Recall that D_i is the subgroup of D with $|D : D_i| = p^i$, $C_i = \mathbf{C}_G(D_i)$, $N_i = \mathbf{N}_G(D_i)$, so that $C_0 = C$, $N_0 = N$ and $D_{a-1} = \tilde{D}$. Hence $N_{a-1} = \tilde{N} = G$. We have that $C_i \subseteq C_{i+1}$, $N_i \subseteq N_{i+1}$, and $D_i \subseteq C_i \subseteq N_i$. Let $b_i = (b_0)^{C_i}$. For $i = 0, \ldots, a-1$, we have that $\mathrm{IBr}(b_i) = \{\varphi_i\}$. Also, $C_{a-1} = \mathbf{C}_G(\tilde{D}) \triangleleft G$.

If $\tau \in \mathcal{H}_B$, since B is τ-invariant, we have that $(b_0)^\tau$ is a root of B and it follows that $(b_0)^\tau = (b_0)^{n_\tau}$ for some $n_\tau \in N$. Also, since N normalizes C_i, $(b_0)^{C_i} = b_i$ and τ commutes with block induction, it follows that $(b_i)^\tau = (b_i)^{n_\tau}$.

Let us write $b = b_{a-1}$ and $\varphi = \varphi_{a-1}$. By Section 3 of [2], we have that $\mathrm{Irr}(b)$ consists of $|D|$ characters $\{\chi_{\lambda,b} \mid \lambda \in \mathrm{Irr}(D)\}$. The values of these characters on $x \in C_{a-1}$ are:

(1) $\chi_{\lambda,b}(x) = \lambda(x_p)\varphi(x_{p'})$, whenever $x_p \in D_{a-1}$.

(2) $\chi_{\lambda,b}(x) = (\delta_i/|C_i|) \sum_{z \in N_i \cap C_{a-1}} \lambda^z(x_p)(\varphi_i)^z(x_{p'})$ if x_p is conjugate in C_{a-1} to some y such that $y \in D_i - D_{i+1}$ for some $i = 0, \ldots, a-2$. Here, δ_i is a sign depending only on i.

(3) $\chi_{\lambda,b}(x) = 0$, otherwise.

If $n \in N$, by using these formulae and the fact that $(N_i \cap C_{a-1})^n = N_i \cap C_{a-1}$, we easily check that

$$(\chi_{\lambda,b})^n = \chi_{\lambda^n,b^n}\,.$$

Also,

$$(\chi_{\lambda,b})^\tau = \chi_{\lambda^\tau,b^\tau}$$

for $\tau \in \mathcal{H}_B$.

Now, the results in Section 4 of [2] tell us that $\mathrm{Irr}(G|\chi_{1_D,b}) = \{\chi_1, \dots, \chi_e\} = \mathrm{Irr}_{\mathrm{nex}}(B)$. Also, for $\lambda \neq 1$, we have that $(\chi_{\lambda,b})^G$ is irreducible, $(\chi_{\lambda,b})^G = (\chi_{\mu,b})^G$ if and only if $\lambda = \mu^z$ for some $z \in E$, and that $\mathrm{Irr}_{\mathrm{ex}}(B) = \{(\chi_{\lambda,b})^G | \lambda \in \Lambda\}$. Furthermore, it is straightforward that restriction to p'-elements defines a natural bijection from $\mathrm{Irr}_{\mathrm{nex}}(B)$ onto $\mathrm{IBr}(B)$.

Now, let $f = (b_0)^{\mathbf{N}_{C_{a-1}}(D)}$. By using the arguments in the previous paragraphs applied to the block B_0 in the group N, we have that the block f has p^a irreducible characters $\{\chi_{\lambda,f} \mid \lambda \in \mathrm{Irr}(D)\}$. Also, $\mathrm{Irr}(N|\chi_{1_D,f}) = \{\psi_1, \dots, \psi_e\} = \mathrm{Irr}_{\mathrm{nex}}(B_0)$, $(\chi_{\lambda,f})^N$ is irreducible for $\lambda \neq 1_D$, $(\chi_{\lambda,f})^N = (\chi_{\mu,f})^N$ if and only if $\lambda = \mu^z$ for some $z \in E$, and that $\{(\chi_{\lambda,f})^N | \lambda \in \Lambda\} = \mathrm{Irr}_{\mathrm{ex}}(B_0)$. Furthermore, we also have that restriction to p'-elements defines a natural bijection from $\mathrm{Irr}_{\mathrm{nex}}(B_0)$ onto $\mathrm{IBr}(B_0)$. Also $f^\tau = f^{n_\tau}$ for $\tau \in \mathcal{H}_B$.

We define a bijection $F : \mathrm{Irr}_{\mathrm{ex}}(B) \to \mathrm{Irr}_{\mathrm{ex}}(B_0)$. For $\lambda \neq 1_D$, we set

$$F((\chi_{\lambda,b})^G) = (\chi_{\lambda,f})^N\,.$$

Notice that F is well-defined since $(\chi_{\lambda,b})^G = (\chi_{\mu,b})^G$ if and only if $\lambda = \mu^z$ for some $z \in E$ if and only if $(\chi_{\lambda,f})^N = (\chi_{\mu,f})^N$. We claim that F commutes with $\tau \in \mathcal{H}_B$. If $\tau \in \mathcal{H}_B$, then

$$\begin{aligned}((\chi_{\lambda,b})^G)^\tau = ((\chi_{\lambda,b})^\tau)^G &= (\chi_{\lambda^\tau,b^\tau})^G = (\chi_{\lambda^\tau,b^{n_\tau}})^G \\ &= ((\chi_{\lambda^\tau,b^{n_\tau}})^{n_\tau^{-1}})^G = (\chi_{(\lambda^\tau)^{n_\tau^{-1}},b})^G\,,\end{aligned}$$

and by the same reasoning,

$$((\chi_{\lambda,f})^N)^\tau = (\chi_{(\lambda^\tau)^{n_\tau^{-1}},f})^N\,.$$

Now,

$$F(((\chi_{\lambda,b})^G)^\tau) = F((\chi_{(\lambda^\tau)^{n_\tau^{-1}},b})^G) = (\chi_{(\lambda^\tau)^{n_\tau^{-1}},f})^N = F((\chi_{\lambda,b})^G)^\tau\,,$$

as claimed.

Next, let $U = D_{a-1} \triangleleft G$ and $\bar{G} = G/U$. If $U = D$, then $N = G$, $B_0 = B$ and there is nothing to prove. So we may assume that $U < D$. Now, by Lemma (3.3), B contains a unique block $\bar{B}$ with cyclic defect group D/U. Thus $\mathrm{IBr}(\bar{B}) = \mathrm{IBr}(B)$. Also, B_0 contains a unique block $\bar{B}_0$ with defect group D/U and $\mathrm{IBr}(\bar{B}_0) = \mathrm{IBr}(B_0)$. By Lemma (3.2), we have that $\bar{B}_0$ is the Brauer correspondent of $\bar{B}$. By the inductive hypothesis, there is a bijection

$\mathrm{IBr}(\bar{B}) \to \mathrm{IBr}(\bar{B}_0)$ commuting with $\tau \in \mathcal{H}_B$. Hence, we have a bijection $S : \mathrm{IBr}(B) \to \mathrm{IBr}(B_0)$ commuting with every $\tau \in \mathcal{H}_B$. Since restriction to p-regular elements defines natural bijections of $\mathrm{Irr}_{\mathrm{nex}}(B)$ or $\mathrm{Irr}_{\mathrm{nex}}(B_0)$ onto $\mathrm{IBr}(B)$ or $\mathrm{IBr}(B_0)$, respectively, the proof of the theorem is complete. □

We mentioned the following consequence of Theorem (3.4) in the introduction.

(3.5) COROLLARY. *Let G be a finite group with a cyclic self-centralizing Sylow p-subgroup P. Let $\chi \in \mathrm{Irr}_{p'}(G)$ and let $x \in G$ be p-regular. Then $\chi(x) = \chi(x^p)$.*

Proof. Let σ be the Galois automorphism fixing p-power roots of unity and sending $\xi \mapsto \xi^p$ for p'-roots of unity ξ. Let $C = \mathbf{C}_G(P) = P$ and $N = \mathbf{N}_G(P)$. Suppose that $\chi \in \mathrm{Irr}(B)$, where B is a p-block of G. Since χ has p'-degree, P is a defect group of B. We claim that B is the principal block of G. By hypothesis, we have that $\mathbf{C}_G(P) = P$. Therefore, a root b_0 of B is the unique (principal) block of P. By the Third Main Theorem, $B = (b_0)^G$ is the principal block of G. Notice also, that $(b_0)^N$ is the unique block of N. In this case, N is the stabilizer of b_0 in N, and N/P is a cyclic group of order e dividing $p-1$. By Theorem (3.4), it suffices to check that all irreducible characters of N are σ-fixed. Let $\psi \in \mathrm{Irr}(N)$ and let $\lambda \in \mathrm{Irr}(P)$ be under ψ. If $\lambda \neq 1_P$, then $\psi = \lambda^N$ is σ-fixed. If $\lambda = 1_P$, then $\psi \in \mathrm{Irr}(N/P)$. Since $x^p = x$ for $x \in N/P$, it easily follows that ψ is σ-fixed. □

4. Sporadic groups

Suppose that G is an sporadic simple group and let P be a noncyclic Sylow p-subgroup of G. We find the character values of every $\chi \in \mathrm{Irr}_{p'}(G)$ in the ATLAS. In order to check our Conjecture A, we need therefore to know the character tables of the groups N/P', where $N = \mathbf{N}_G(P)$. Note that all but three of the groups N/P' are as described in [15]. (Those incorrectly described are Co_3 for $p = 3$, Fi_{23} for $p = 5$ and Fi_{24}' for $p = 3$.) Also, the character tables of the Sylow normalizers N of the sporadic groups (and therefore of the groups N/P') were calculated in his PhD thesis by Th. Ostermann ([14]), except for the seven groups Th, Fi_{22}, Fi_{23}, Fi_{24}', HN, B and M. (We should mention that the character tables of N/P' for J_4 and $p = 3$ and $G = Ly$ for $p = 2$ exhibited in [14] are not correct.) All these character tables calculated by Ostermann are now available in the GAP library ([7]). Also, the character tables of the seven groups not treated by Ostermann have been calculated by Thomas Breuer for the purposes of this paper and will appear in the library of the next version of GAP.

With this information, we have checked Conjecture A for every sporadic group.

5. Self-normalizing Sylow subgoups

We mentioned in the introduction that a consequence of Conjecture A that we can tell from the character table of G if a Sylow p-subgroup of G is self-normalizing. Let us start with the case $p = 2$. This is a consequence of the following elementary result.

(5.1) LEMMA. *Let G be a finite group and assume that x^2 is conjugate to x for every $x \in G$. Then $G = 1$.*

Proof. By hypothesis, we see that G has odd order. Now, suppose that $G > 1$ and let p be the smallest prime dividing $|G|$. Now, we let $x \in G$ be of order p. By hypothesis, there exists $g \in G$ such that $x^g = x^2$. Now, $\langle x \rangle^g = \langle x^2 \rangle = \langle x \rangle$, and therefore g normalizes $U = \langle x \rangle$. However, $\mathbf{N}_G(U)/\mathbf{C}_G(U)$ has order dividing $p-1$, and therefore $\mathbf{N}_G(U) = \mathbf{C}_G(U)$. Thus $x = x^g = x^2$ and $x = 1$, a contradiction. $\square$

(5.2) THEOREM. *Suppose that G is a finite group of order n, let $P \in \mathrm{Syl}_2(G)$ and let $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ be the Galois automorphism fixing 2-roots of unity and squaring $2'$-roots of unity. Assume Conjecture A. Then $P = \mathbf{N}_G(P)$ if and only if all irreducible characters of G of odd degree are σ-fixed.*

Proof. Assuming Conjecture A, we have to prove that a group G with a normal abelian Sylow 2-subgroup P is a 2-group if and only if all of its irreducible characters of G are σ-fixed. Of course, if G is a 2-group, this is obvious since σ fixes 2-roots of unity. Now, assume that all the irreducible characters of G are σ-fixed. In particular, all the irreducible characters of $H = G/P$ are σ-fixed. If $\psi \in \mathrm{Irr}(H)$ and $x \in H$, we have that $\psi(x)^\sigma = \psi(x^2)$. Therefore, x^2 and x are H-conjugate for every $x \in H$, and by Lemma (5.1) we deduce that $H = 1$. $\square$

P. Fong has shown that this consequence of Conjecture A is also true for the alternating groups or for $\mathrm{SL}(2, q)$. For solvable groups, R. Gow has a related result (when P/P' is elementary abelian, [8]). The obvious odd analog of Theorem (5.2) does not work. For instance, if $G = S_3$ and $p = 7$, then all irreducible characters of G are σ-fixed, while the Sylow 7-subgroup of G is normal!

(5.3) THEOREM. *Suppose that G is a finite group of order n, let p be an odd prime dividing n and let $P \in \mathrm{Syl}_p(G)$. Let $\theta \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ be fixing p'-roots of unity and having order $p-1$. Assume Conjecture A. Then $P = \mathbf{N}_G(P)$ if and only if the principal character of G is the only irreducible character of p'-degree of G which is fixed by θ.*

Proof. Write $N = \mathbf{N}_G(P)$. Assume first that $P < N$. Then $|\mathrm{Irr}(N/P)| \geq 2$. Since all the characters in N/P are θ-fixed, by Conjecture A we deduce that there are at least two irreducible characters of G of p'-degree that are θ-fixed. Assume now that $P = N$. By Conjecture A, it is enough to show that 1_P is the only character of P/P' which is θ-fixed. Suppose that $1_P \neq \lambda \in \mathrm{Irr}(P/P')$ is θ-fixed. Then, any power of λ is θ-fixed, and therefore, we may assume that λ has order p. In particular, $\lambda(x)$ is a primitive p-th root of unity for some $x \in G$. Now, $\theta(\lambda(x)) = \lambda(x)$, and therefore θ acts trivially on the field $\mathbb{Q}_p$. This is impossible since θ has order $p - 1$. □

Recall that $\chi \in \mathrm{Irr}(G)$ is p-rational if $\chi(g) \in \mathbb{Q}_m$ for every $g \in G$, where m is the p'-part of $|G|$.

(5.4) THEOREM. *Let G be a finite group, p an odd prime, and let $P \in \mathrm{Syl}_p(G)$. Assume Conjecture* A. *Then the number of p-rational characters in $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}(N/P')$ is the same.*

Proof. Write $|G| = p^a m$, where m is not divisible by p. Then $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_m) \cong \mathrm{Gal}(\mathbb{Q}_{p^a}/\mathbb{Q})$ is cyclic. Let $\tau \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_m)$ be any generator. Now, an irreducible character χ of G (or of $\mathbf{N}_G(P)$) is p-rational if and only if $\chi^\tau = \chi$. □

Finally, we think that there is no reason to restrict ourselves to the Alperin-McKay conjecture. We believe that the action of $\mathcal{H}$ is natural enough so that the obvious versions of the Alperin Weight conjecture, Dade's conjectures, or the Isaacs-Navarro conjecture should be true. It is perhaps also worth mentioning that a slightly stronger version of our Conjecture A seems to hold, namely, that the actions of $\mathcal{H}$ on the sets $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ are permutation isomorphic.

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