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On the periods of motives with complex multiplication and a conjecture of Gross-Deligne

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Abstract

We prove that the existence of an automorphism of finite order on a $\overline{\mathbf{Q}}$-variety X implies the existence of algebraic linear relations between the logarithm of certain periods of X and the logarithm of special values of the Γ-function. This implies that a slight variation of results by Anderson, Colmez and Gross on the periods of CM abelian varieties is valid for a larger class of CM motives. In particular, we prove a weak form of the period conjecture of Gross-Deligne [11, p. 205][1]. Our proof relies on the arithmetic fixed-point formula (equivariant arithmetic Riemann-Roch theorem) proved by K. Köhler and the second author in [13] and the vanishing of the equivariant analytic torsion for the de Rham complex.

1. Introduction

In the following article, we shall be concerned with the computation of periods in a very general setting. Recall that a period of an algebraic variety defined by polynomial equations with algebraic coefficients is the integral of an algebraic differential against a rational homology cycle. In his article [16, formule 26, p. 303] Lerch proved (see also [3]) that the abelian integrals that arise as periods of elliptic curves with complex multiplication (i.e. whose rational endomorphism ring is an imaginary quadratic field) can be related to special values of the Γ-function. A special case of his result is the following identity (already known to Legendre [15, 1-ère partie, no. 146, 147, p. 209])

$$\int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2(t)}} = \frac{2^{\frac{2}{3}}3^{\frac{1}{4}}}{8\pi}\Gamma^3(\frac{1}{3}),$$

where $k=\sin(\frac{\pi}{12})$, which is associated to an elliptic curve whose rational endomorphism ring is isomorphic to $\mathbf{Q}(\sqrt{-3})$. The formula of Lerch (now known

[1] This should not be confused with the conjecture by Deligne relating periods and values of L-functions.

as the Chowla-Selberg formula) has been generalised to higher dimensional abelian varieties in the work of several people (precise references are given below), including Anderson and Colmez. They show that the abelian integrals arising as periods of abelian varieties of dimension d with complex multiplication by a a CM field (i.e. a totally complex number field endowed with an involution which becomes complex conjugation in any complex embedding) whose Galois group over $\mathbf{Q}$ is abelian of order $2d$, are related to special values of the Γ-function.

Consider now any algebraic variety X defined over the algebraic numbers. The transcendence properties of the periods of X are influenced by the algebraic subvarieties of X; a subvariety of X has a cycle class in the dual of a rational homology space of X and the duals of these cycle classes span a subspace of homology, which might be large. Up to normalisation, the integral of an algebraic differential against a cycle class will be an algebraic number. The celebrated Hodge conjecture describes the space spanned by the classes of the algebraic cycles in terms of the decomposition of complex cohomology in bidegrees (the Hodge decomposition) and its underlying rational structure. This set of data is called a Hodge structure. The Hodge conjecture implies that the periods of X depend only on the Hodge structure of its complex cohomology and thus any algebraic variety whose cohomology contains a Hodge structure related to a Hodge structure appearing in the cohomology of an abelian variety with complex multiplication as above should have periods that are related to the special values of the Γ-function. This leads to the conjecture of Gross-Deligne, which is described precisely in the last section of this paper.

The main contribution of this paper is the proof of a (slight variant of) the conjecture of Gross-Deligne, in the situation where the Hodge structure with complex multiplication arises has the direct sum of the nontrivial eigenspaces of an automorphism of finite prime order acting on the algebraic variety. We use techniques of higher-dimensional Arakelov theory to do so. Arakelov theory is an extension of Grothendieck style algebraic geometry, where the algebraic properties of polynomial equations with algebraic coefficients and the differential-geometric properties of their complex solutions are systematically studied in a common framework.

Many theorems of Grothendieck algebraic geometry have been extended to Arakelov theory, in particular there is an intersection theory, a Riemann-Roch theorem ([9]) and a fixed-point formula of Lefschetz type ([13]). Our proof of the particular case of the Gross-Deligne conjecture described above relies on this last theorem; we write out the fixed-point formula for the de Rham complex and obtain a first formula (11) which involves differential-geometric invariants (in particular, the equivariant Ray-Singer analytic torsion); these invariants are shown to vanish and we are left with an identity (12) which involves only the topological and algebraic structure. More work implies that this is a rewording of a part of the conjecture of Gross-Deligne. Our proof is thus an instance of a collapse of structure, where fine differential-geometric quantities are ultimately shown to depend on less structure than they appear to.

In the rest of this introduction, we shall give a precise description of our results and conjectures.

So let $\mathcal{M}$ be a (homological Grothendieck) motive defined over Q_0, where Q_0 is an algebraic extension of $\mathbf{Q}$ embedded in $\mathbf{C}$. We shall use the properties of the category of motives over a field which are listed at the beginning of [5]. The complex singular cohomology $H(\mathcal{M}, \mathbf{C})$ of the manifold of complex points of $\mathcal{M}$ is then endowed with two natural Q_0-structures. The first one is induced by the standard Betti $\mathbf{Q}$-structure $H(\mathcal{M}, \mathbf{Q})$ via the identifications $H(\mathcal{M}, Q_0) = H(\mathcal{M}, \mathbf{Q}) \otimes_{\mathbf{Q}} Q_0$ and $H(\mathcal{M}, \mathbf{C}) = H(\mathcal{M}, Q_0) \otimes_{Q_0} \mathbf{C}$ and will be referred to as the Betti (or singular) Q_0-structure on $H(\mathcal{M}, \mathbf{C})$. The second one arises from the comparison isomorphism between $H(\mathcal{M}, \mathbf{C})$ and the de Rham cohomology of $\mathcal{M}$ (tensored with $\mathbf{C}$ over Q_0) and will be referred to as the de Rham Q_0-structure.

Let Q be a finite (algebraic) extension of $\mathbf{Q}$ and suppose that the image of any embedding of Q into $\mathbf{C}$ lies inside Q_0. Furthermore, suppose that $\mathcal{M}$ is endowed with a Q-motive structure (over Q_0). A Q-motive is also called a motive with coefficients in Q (see [5, Par. 2]). The Q-motive structure of $\mathcal{M}$ induces a direct sum decomposition

$$H(\mathcal{M}, \mathbf{C}) = \bigoplus_{\sigma \in \mathrm{Hom}(Q, \mathbf{C})} H(\mathcal{M}, \mathbf{C})_\sigma$$

which respects both Q_0-structures. The notation $H(\mathcal{M}, \mathbf{C})_\sigma$ refers to the complex vector subspace of $H(\mathcal{M}, \mathbf{C})$ where Q acts via $\sigma \in \mathrm{Hom}(Q, \mathbf{C})$. The determinant $\det_{\mathbf{C}}(H(\mathcal{M}, \mathbf{C})_\sigma)$ thus has two Q_0-structures. Let v_{sing} (resp. v_{dR}) be a nonvanishing element of $\det_{\mathbf{C}}(H(\mathcal{M}, \mathbf{C})_\sigma)$ defined over Q_0 for the singular (resp. for the de Rham) Q_0-structure. We write $P_\sigma(\mathcal{M})$ for the (uniquely defined and independent of the choices made) image in $\mathbf{C}^\times / Q_0^\times$ of the complex number λ such that $v_{\mathrm{dR}} = \lambda \cdot v_{\mathrm{sing}}$.

Let χ be an odd simple Artin character of Q and suppose at this point that $\mathcal{M}$ is homogeneous of degree k (in particular, its cohomological realisations are homogeneous of degree k). Consider the following conjecture:

Conjecture A$(\mathcal{M}, \chi)$. *The equality of complex numbers*

$$\sum_{\sigma \in \mathrm{Hom}(Q, \mathbf{C})} \log |P_\sigma(\mathcal{M})| \chi(\sigma) = \frac{L'(\chi, 0)}{L(\chi, 0)} \sum_{\sigma \in \mathrm{Hom}(Q, \mathbf{C})} \sum_{p+q=k} p \cdot \mathrm{rk}(H^{p,q}(\mathcal{M}, \mathbf{C})_\sigma) \chi(\sigma)$$

is verified, up to addition of a term of the form $\sum_{\sigma \in \mathrm{Hom}(Q, \mathbf{C})} \log |\alpha_\sigma| \chi(\sigma)$, *where* $\alpha_\sigma \in Q_0^\times$.

Recall that an Artin character of Q is a character of a finite dimensional complex representation of the automorphism group of the normalisation $\widetilde{Q}$ of Q over $\mathbf{Q}$, which is trivial on all the automorphisms of $\widetilde{Q}$ whose restriction to Q is the identity. The normalisation $\widetilde{Q}$ may be embedded in Q_0 and in order for the equality of Conjecture A to make sense, one has to choose such an embedding; it is a part of the conjecture that the equality holds whatever the choice.

Conjecture A is a slight strengthening of the case $n = 1$, $Y = \operatorname{Spec} Q_0$ of the statement in [17, Conj. 3.1]. Notice that this conjecture has both a "motivic" and an "arithmetic" content. More precisely, *if the Hodge conjecture holds* and $Q_0 = \overline{\mathbf{Q}}$, this conjecture can be reduced to the case where $\mathcal{M}$ is a submotive of an abelian variety with complex multiplication by Q. Indeed, assuming the Hodge conjecture, one can show by examining its associated Hodge structures that some exterior power of $\mathcal{M}$ (taken over Q) is isomorphic to a motive over $\overline{\mathbf{Q}}$ lying in the tannakian category generated by abelian varieties with maximal complex multiplication by Q. In this latter case, the Conjecture A is contained in a conjecture of Colmez [4]. Performing this reduction to CM abelian varieties or circumventing it is the "motivic" aspect of the conjecture.

However, even in the case of CM abelian varieties, the conjecture seems far from proof: as far as the authors know, only the case of Dirichlet characters has been tackled up to now; obtaining a proof of Conjecture A *for nonabelian Artin characters* (i.e. for abelian varieties with complex multiplication by a field whose Galois group over $\mathbf{Q}$ is nonabelian) is the "arithmetic" aspect alluded to above.

In this text we shall be concerned with both aspects, but our original contribution concerns the "motivic" aspect, more precisely, in finding a way to circumvent the Hodge conjecture.

We now state a weaker form of Conjecture A. Let χ be a simple odd Artin character of Q as before, and N be a subring of $\overline{\mathbf{Q}}$. Let $\mathcal{M}_0$ be a motive over Q_0 (not necessarily homogeneous) and suppose that $\mathcal{M}_0$ is endowed with a Q-motive structure (over Q_0). Let $\mathcal{M}_0^k$ ($k \geqslant 0$) be the motive corresponding to the k^{th} cohomology group of $\mathcal{M}_0$.

Conjecture B($\mathcal{M}_0, N, \chi$). *The equality of complex numbers*

$$\sum_{k\geqslant 0}(-1)^k \sum_{\sigma\in \operatorname{Hom}(Q,\mathbf{C})} \log|P_\sigma(\mathcal{M}_0^k)|\chi(\sigma)$$
$$= \sum_{k\geqslant 0}(-1)^k \frac{L'(\chi,0)}{L(\chi,0)} \sum_{\sigma\in \operatorname{Hom}(Q,\mathbf{C})} \sum_{p+q=k} p\cdot \operatorname{rk}(H^{p,q}(\mathcal{M},\mathbf{C})_\sigma)\chi(\sigma)$$

is verified, up to addition of a term of the form

$$\sum_{\sigma\in \operatorname{Hom}(Q,\mathbf{C})} \sum_i (b_{i,\sigma}\log|\alpha_{i,\sigma}|)\chi(\sigma),$$

where $\alpha_{i,\sigma} \in Q_0^\times$, $b_{i,\sigma} \in N$ *and* i *runs over a finite set of indices.*

Note that Conjecture A (resp. B) only depends on the vector space $H(\mathcal{M}, \mathbf{C})$ (resp. $H(\mathcal{M}_0, \mathbf{C})$), together with its Hodge structure (over $\mathbf{Q}$), its de Rham Q_0-structure and its additional Q-structure. If V is a $\mathbf{Q}$-vector space together with the just described structures on $V \otimes_{\mathbf{Q}} \mathbf{C}$ (all of them satisfying the obvious compatibility relations), we shall accordingly write $\mathrm{A}(V, \chi)$ (resp. $\mathrm{B}(V, N, \chi)$) for the corresponding statement, even if V possibly does not arise from a motive.

In this article we shall prove Conjecture B (and to a lesser extent, part of Conjecture A) for a large class of motives, which include abelian varieties with complex multiplication by an abelian extension of $\mathbf{Q}$, without assuming the Hodge conjecture (or any other conjecture about motives). Even in the case of abelian varieties, our method of proof is completly different from the existing ones.

A consequence of our results is that on any $\overline{\mathbf{Q}}$-variety X, the existence of a finite group action implies the existence of nontrivial algebraic linear relations between the logarithm of the periods of the eigendifferentials of X (for the action of the group) and the logarithm of special values of the Γ-function (recall that they are related to the logarithmic derivatives of Dirichlet L-functions at 0 via the Hurwitz formula). More precisely, our results are the following:

Let X be a smooth and projective variety together with an automorphism $g : X \to X$ of order n, with everything defined over a number field Q_0. Let us denote by $\mu_n(\mathbf{C})$ (resp. $\mu_n(\mathbf{C})^\times$) the group of n^{th} roots of unity (resp. the set of primitive n^{th} roots of unity) in $\mathbf{C}$. Suppose that Q_0 is chosen large enough so that it contains $\mathbf{Q}(\mu_n)$; and let $P_n(T) \in \mathbf{Q}[T]$ be the polynomial

$$P_n(T) = \sum_{\zeta \in \mu_n(\mathbf{C})^\times} \prod_{\xi \in \mu_n(\mathbf{C}) \setminus \{\zeta\}} \frac{T - \xi}{\zeta - \xi}.$$

The submotive $\mathcal{X}(g) = \mathcal{X}(X, g)$ cut out in X by the projector $P_n(g)$ is endowed by construction with a natural $Q := \mathbf{Q}(\mu_n)$-motive structure.

Theorem 1. *For all the odd primitive Dirichlet characters χ of $\mathbf{Q}(\mu_n)$, Conjecture* $\mathrm{B}(\mathcal{X}(g), \mathbf{Q}(\mu_n), \chi)$ *holds.*

Let now Q be a finite abelian extension of $\mathbf{Q}$ with conductor f_Q and let $\mathcal{M}_0$ be the motive associated to an abelian variety defined over Q_0 with (not necessarily maximal) complex multiplication by $\mathcal{O}_Q$. We suppose that the action of $\mathcal{O}_Q$ is defined over Q_0 and that $\mathbf{Q}(\mu_{f_Q}) \subseteq Q_0$.

Theorem 2. *For all the odd Dirichlet characters χ of Q, Conjecture* $\mathrm{B}(\mathcal{M}_0^1, \mathbf{Q}(\mu_{f_Q}), \chi)$ *holds.*

As a consequence of the existence of the Picard variety and of Theorems 1 and 2, we get:

Corollary. *Let the hypotheses of Theorem* 1 *hold and suppose also that* X *is a surface. For all the odd primitive Dirichlet characters* χ *of* $\mathbf{Q}(\mu_n)$, *the conjecture* $\mathrm{B}(H^2(\mathcal{X}(X,g)),\mathbf{Q}(\mu_n),\chi)$ *holds.*

Our method of proof relies heavily on the arithmetic fixed-point formula (equivariant arithmetic Riemann-Roch theorem) proved by K. Köhler and the second author in [13]. More precisely, we write down the fixed-point formula as applied to the de Rham complex of a variety equipped with the action of a finite group. This yields a formula for some linear combinations of logarithms of periods of the variety in terms of derivatives of (partial) Lerch ζ-functions. Using the Hurwitz formula and some combinatorics, we can translate this into Theorems 1 and 2. In general the fixed-point formula of [13], like the arithmetic Riemann-Roch theorem, contains an anomalous term, given by the equivariant Ray-Singer analytic torsion, which has proved to be difficult to compute explicitly. In the case of the de Rham complex, this anomalous term vanishes for simple symmetry reasons. It is this fact that permits us to conclude.

When $Q_0=\overline{\mathbf{Q}}$, Q is an abelian extension of $\mathbf{Q}$ and $\mathcal{M}_0$ is an abelian variety with maximal complex multiplication by Q, the assertion $\mathrm{A}(\mathcal{M}_0^1,\chi)$ was proved by Anderson in [1], whereas the statement $\mathrm{A}(\mathcal{M}_0^1,\chi)$ had already been proved by Gross [11, Th. 3, Par. 3, p. 204] in the case where Q is an imaginary quadratic extension of $\mathbf{Q}$, $Q_0=\overline{\mathbf{Q}}$ and $\mathcal{M}_0$ is an abelian variety with (not necessarily maximal) complex multiplication by Q. One could probably derive Theorem 2 from the results of Anderson, using the result of Deligne on absolute Hodge cycles on abelian varieties [7] (proved after the theorem of Gross and inspired by it), which can be used as a substitute of the Hodge conjecture in this context. In the case where $\mathcal{M}_0$ is an abelian variety with maximal complex multiplication and Q is an abelian extension of $\mathbf{Q}$, Colmez [4] proves a much more precise version of $\mathrm{A}(\mathcal{M}_0^1,\chi)$. He uses the Néron model of the abelian variety to normalise the periods so as to eliminate all the indeterminacy and proves an equation similar to Theorem 2 for those periods. A slightly weaker form of his result (but still much more precise than Theorem 2) can also be obtained from the arithmetic fixed-point formula, when applied to the Néron models. This is carried out in [14]. Finally, when $\mathcal{M}_0$ is the motive of a CM elliptic curve, Theorem 2 is just a weak form of the Chowla-Selberg formula [3]. For a historical introduction to those results, see [19, p. 123–125].

In the last section of the paper, we compare Conjecture A with the period conjecture of Gross-Deligne [11, Sec. 4, p. 205]. This conjecture is a translation into the language of Hodge structures of a special case of Conjecture A, with Q an abelian extension of $\mathbf{Q}$. For example, we show the following: Theorem 1 implies that if S is a surface defined over $\overline{\mathbf{Q}}$ and if S is endowed with an action of an automorphism g of finite prime order p, then the natural embedding of the Hodge structure $\det_{\mathbf{Q}(\mu_p)}(H^2(\mathcal{X}(S,g),\mathbf{Q}))$ into $H(\times_{r=1}^d S,\mathbf{Q})$, where $d=\dim_{\mathbf{Q}(\mu_p)}H^2(\mathcal{X}(S,g),\mathbf{Q})$, satisfies a weak form of the period conjecture.

In light of the application of the arithmetic fixed-point formula to Conjectures A and B, it would be interesting to investigate whether this formula is related to the construction of the cycles whose existence (postulated by the Hodge conjecture) would be necessary to reduce the Conjecture A to abelian varieties.

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2. Preliminaries

2.1. *Invariance properties of the conjectures.* Let Q_0 and Q be number fields taken as in the introduction, and let H be a (homogeneous) Hodge structure (over $\mathbf{Q}$). The $\mathbf{C}$-vector space $H_{\mathbf{C}} := H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$ comes with a natural Q_0-structure given by $H_{\mathbf{Q}} \otimes_{\mathbf{Q}} Q_0$. Suppose that $H_{\mathbf{C}}$ is endowed with another Q_0-structure. The first of these two Q_0-structures will be referred to as the Betti (or singular) one, and the second as the de Rham Q_0-structure on $H_{\mathbf{C}}$. Suppose furthermore that $H_{\mathbf{C}}$ is endowed with an additional Q-vector space structure compatible with both the Hodge structure and the (Betti and de Rham) Q_0-structures. This Q-structure induces an inner direct sum of $\mathbf{C}$-vector spaces $H_{\mathbf{C}} := \oplus_{\sigma\in \mathrm{Hom}(Q,\mathbf{C})} H_\sigma$. Let $V := \oplus_{\sigma\in \mathrm{Hom}(Q,\mathbf{C})} \det_{\mathbf{C}}(H_\sigma)$ and let $m := \dim_Q(H)$. There is an embedding $\iota : V \hookrightarrow \otimes_{k=1}^m H_{\mathbf{C}}$ given by $\iota(\oplus_\sigma v_1^\sigma \wedge \cdots \wedge v_m^\sigma) := \sum_\sigma \mathrm{Alt}(v_1^\sigma \otimes \cdots \otimes v_m^\sigma)$. Recall that Alt is the alternation map, described by the formula $\mathrm{Alt}(x_1 \otimes \cdots \otimes x_m) := \frac{1}{m!}\sum_{\pi\in\mathfrak{S}_m} \mathrm{sign}(\pi)\pi(x_1 \otimes \cdots \otimes x_m)$; here $\mathfrak{S}_m$ is the permutation group on m elements and π acts on $\otimes_{k=1}^m H_{\mathbf{C}}$ by permutation of the factors.

LEMMA 2.1. *The space V inherits the Hodge structure as well as the Betti and de Rham Q_0-structures of $\otimes_{k=1}^m H_{\mathbf{C}}$ via the map ι.*

Proof. The bigrading of $H_{\mathbf{C}}$ is described by the weight of H and by an action $\upsilon : \mathbf{C}^\times \to \mathrm{End}_{\mathbf{C}}(H_{\mathbf{C}})$ of the complex torus $\mathbf{C}^\times$, which commutes with complex conjugation. The bigrading of $\otimes_{k=1}^m H_{\mathbf{C}}$ is described by the weight $m \cdot \mathrm{weight}(H)$ and the tensor product action $\upsilon^{\otimes m} : \mathbf{C}^\times \to \mathrm{End}_{\mathbf{C}}(\otimes_{k=1}^m H_{\mathbf{C}})$. On the other hand we can describe a bigrading on each $\det_{\mathbf{C}}(H_\sigma)$ by the weight $m \cdot \mathrm{weight}(H)$ and by the exterior product action. The map ι commutes with both actions by construction.

To prove that V inherits the Hodge $\mathbf{Q}$-structure, consider that there is an action by $\mathbf{Q}$-vector space automorphisms of $\mathrm{Aut}(\mathbf{C})$ on $\otimes_{k=1}^m H_{\mathbf{C}}$ given by

$a((h_1 \otimes z_1) \otimes \cdots \otimes (h_m \otimes z_m)) := (h_1 \otimes a(z_1)) \otimes \cdots \otimes (h_m \otimes a(z_m))$. An element t of $\otimes_{k=1}^m H_{\mathbf{C}}$ is defined over $\mathbf{Q}$ (for the Hodge $\mathbf{Q}$-structure) if and only if $a(t) = t$ for all $a \in \mathrm{Aut}(\mathbf{C})$. For each $\sigma \in \mathrm{Hom}(Q, \mathbf{C})$, let $b_1^\sigma, \dots, b_m^\sigma$ be a basis of H_σ, which is defined over $\sigma(Q)$ such that $a(b_i^\sigma) = b_i^{a(\sigma)}$ for all $a \in \mathrm{Aut}(\mathbf{C})$. This can be achieved by taking the conjugates under the action of $\mathrm{Aut}(\mathbf{C})$ of a given basis. Now choose a basis $c_1, \dots, c_{d_Q}$ of Q over $\mathbf{Q}$ and let $e_i := \sum_\sigma \sigma(c_i) b_1^\sigma \wedge \cdots \wedge b_m^\sigma$. By construction, the elements $\iota(e_1), \dots, \iota(e_{d_Q})$ are invariant under $\mathrm{Aut}(\mathbf{C})$ and they are linearly independent over $\mathbf{C}$, because the determinant of the transformation matrix from the basis $\{b_1^\sigma \wedge \cdots \wedge b_m^\sigma\}_\sigma$ to the basis formed by the e_i is the discriminant of the basis e_i over $\mathbf{Q}$. They thus define over V a $\mathbf{Q}$-structure $V_{\mathbf{Q}}$ which is compatible with the Hodge $\mathbf{Q}$-structure of $\otimes_{k=1}^m H_{\mathbf{C}}$. The Betti Q_0-structure on V is then just taken to be $V_{\mathbf{Q}} \otimes_{\mathbf{Q}} Q_0$.

To show that V inherits the de Rham Q_0-structure of $\otimes_{k=1}^m H_{\mathbf{C}}$, just notice that for each $\sigma \in \mathrm{Hom}(Q, \mathbf{C})$, the space H_σ is a basis $\alpha_1^\sigma, \dots, \alpha_m^\sigma$ defined over the de Rham Q_0-structure of $H_{\mathbf{C}}$. The elements $\alpha_1^\sigma \wedge \cdots \wedge \alpha_m^\sigma$ form a basis of V and $\iota(\alpha_1^\sigma \wedge \cdots \wedge \alpha_m^\sigma)$ is by construction defined over Q_0. □

In view of the last lemma the complex vector space V arises from a (homogeneous) Hodge structure over $\mathbf{Q}$ that we shall denote by $\det_Q(H)$. The embedding ι arises from an embedding of Hodge structures $\det_Q(H) \hookrightarrow \otimes_{k=1}^m H$ and $\det_Q(H)$ inherits a Betti and a de Rham Q_0-structure from this embedding. If $H' = \oplus_{w \in \mathbf{Z}} H_w$ is a direct sum of homogeneous Hodge structures (graded by the weight), each of them satisfying the hypotheses of Lemma 2.1, we extend the previous definition to H' by letting $\det_Q(H') := \oplus_{w \in \mathbf{Z}} \det_Q(H_w)$.

Proposition 2.2. *The assertion* $\mathrm{A}(\mathcal{M}, \chi)$ (*resp.* $\mathrm{B}(\mathcal{M}_0, N, \chi)$) *is equivalent to the assertion* $\mathrm{A}(\det_Q(H(\mathcal{M}, \mathbf{Q})), \chi)$ (*resp.* $\mathrm{B}(\det_Q(H(\mathcal{M}_0, \mathbf{Q})), N, \chi)$).

Proof. We examine both sides of the equality in the assertion $\mathrm{A}(H(\mathcal{M}, \mathbf{Q}), \chi)$, when $H(\mathcal{M}, \mathbf{Q})$ is replaced by $\det_Q(H(\mathcal{M}, \mathbf{Q}))$. From the definition of $\det_Q(H(\mathcal{M}, \mathbf{Q}))$, we see that the left-hand side is unchanged. As to the right-hand side, it is sufficient to show that

$$\sum_{p+q=k} p \cdot \mathrm{rk}(H_\sigma^{p,q}) = \sum_{p+q=r\cdot k} p \cdot \mathrm{rk}(\det_{\mathbf{C}}(H_\sigma)^{p,q}),$$

where $r := \mathrm{rk}(H_\sigma)$, k is the weight of $\mathcal{M}$ and $H := H(\mathcal{M}, \mathbf{Q})$. To prove it, we let $v_1, \dots, v_r$ be a basis of H_σ, which is homogeneous for the grading. The last equality follows from the equality

$$\sum_{j=1}^r p_{\mathcal{H}}(v_j) = p_{\mathcal{H}}(v_1 \wedge \cdots \wedge v_r)$$

(where $p_{\mathcal{H}}$ stands for the Hodge p-type) which holds from the definitions. The proof of the second equivalence runs along the same lines. □

Let $\mathcal{M}$ be a Q-motive (over Q_0) and let E be a Q-vector space. We denote by $\mathcal{M}\otimes_Q E$ the motive such that $\mathrm{Hom}_Q(\mathcal{M}', \mathcal{M}\otimes_Q E) = \mathrm{Hom}_Q(\mathcal{M}', \mathcal{M})\otimes_Q E$ for any Q-motive $\mathcal{M}'$. If χ is a character of Q, recall that $\mathrm{Ind}_Q^E(\chi)$ is the character on E (the induced character) defined by the formula $\mathrm{Ind}_Q^E(\chi)(\sigma_E) := \chi(\sigma_E|_Q)$.

Proposition 2.3. *Let E be a finite extension of Q, such that the image of all the embeddings of E in $\mathbf{C}$ are contained in Q_0. The statement* $\mathrm{A}(\mathcal{M}\otimes_Q E, \mathrm{Ind}_Q^E(\chi))$ *(resp.* $\mathrm{B}(\mathcal{M}_0\otimes_Q E, N, \mathrm{Ind}_Q^E(\chi))$*) holds if and only if* $\mathrm{A}(\mathcal{M}, \chi)$ *(resp.* $\mathrm{B}(\mathcal{M}_0, N, \chi)$*) holds.*

Proof. Let r be the dimension of E over Q. The choice of a basis $x_1, \ldots, x_r$ of E as a Q-vector space induces an isomorphism of Q-motives $\mathcal{M}\otimes_Q E \simeq \oplus_{j=1}^r \mathcal{M}$ and thus an isomorphism of $\mathbf{C}$-vector spaces

$$\bigoplus_{j=1}^{r} H(\mathcal{M}, \mathbf{C}) \simeq H((\mathcal{M}\otimes_Q E), \mathbf{C})$$

which respects the Hodge structure and both Q_0-structures. Under this isomorphism, we also have a decomposition

$$\bigoplus_{j=1}^{r} H(\mathcal{M}, \mathbf{C})_{\sigma_Q} \simeq \bigoplus_{\sigma_E|\sigma_Q} H((\mathcal{M}\otimes_Q E), \mathbf{C})_{\sigma_E}$$

where $\sigma_Q \in \mathrm{Hom}(Q, \mathbf{C})$ and the $\sigma_E \in \mathrm{Hom}(E, \mathbf{C})$ restrict to σ_Q. This decomposition again respects the Hodge structure and both Q_0-structures. We now compute the left-hand side of the equality predicted by $\mathrm{A}(\mathcal{M}_E := \mathcal{M}\otimes_Q E, \mathrm{Ind}_Q^E(\chi))$:

$$\begin{aligned}\sum_{\sigma_E} &\log|P_{\sigma_E}(\mathcal{M}_E)|\,\mathrm{Ind}_Q^E(\chi)(\sigma_E)\\ &= \sum_{\sigma_Q}\chi(\sigma_Q)\sum_{\sigma_E|\sigma_Q}\log|P_{\sigma_E}(\mathcal{M}_E)|\\ &= \sum_{\sigma_Q}\chi(\sigma_Q)\sum_{j=1}^{r}\log|P_{\sigma_Q}(\mathcal{M})| = r\cdot\sum_{\sigma_Q}\log|P_{\sigma_Q}(\mathcal{M})|\,\chi(\sigma_Q).\end{aligned}$$

As for the right-hand side, we compute

$$\begin{aligned}\sum_{\sigma_E}\sum_{p,q} &p\cdot \mathrm{rk}(H^{p,q}(\mathcal{M}_E, \mathbf{C})_{\sigma_E})\mathrm{Ind}_Q^E(\chi)(\sigma_E)\\ &= \sum_{\sigma_Q}\sum_{p,q} p\cdot\chi(\sigma_Q)\,\mathrm{rk}(\oplus_{\sigma_E|\sigma_Q}H^{p,q}(\mathcal{M}_E, \mathbf{C})_{\sigma_E})\\ &= \sum_{\sigma_Q}\sum_{p,q} p\cdot\chi(\sigma_Q)\cdot r\cdot \mathrm{rk}(H^{p,q}(\mathcal{M}, \mathbf{C})_{\sigma_Q});\end{aligned}$$

dividing both sides by r, we are reduced to the conjecture $\mathrm{A}(\mathcal{M}, \chi)$. The proof of the second equivalence is similar. □

2.2. *The arithmetic fixed-point formula.* For the sake of completness and in order to fix notation, we shall review in this section the arithmetic fixed-point formula proved by K. Köhler and the second author in [13]. Many results will be stated without proof; we refer to [13, Sec. 4] for more details and further references to the literature.

Let D be a regular arithmetic ring, i.e. a regular, excellent, Noetherian integral ring, together with a finite set $\mathcal{S}$ of injective ring homomorphisms of $D \hookrightarrow \mathbf{C}$, which is invariant under complex conjugation. Let μ_n be the diagonalisable group scheme over D associated to the group $\mathbf{Z}/n$. An *equivariant arithmetic variety* $f : Y \to \operatorname{Spec} D$ is a regular integral scheme, endowed with a μ_n-action over $\operatorname{Spec} D$, such that there exists a μ_n-equivariant ample line bundle on Y. We write $Y(\mathbf{C})$ for the complex manifold $\coprod_{\sigma\in\mathcal{S}} Y \otimes_{\sigma(D)} \mathbf{C}$. The group $\mu_n(\mathbf{C})$ acts on $Y(\mathbf{C})$ by holomorphic automorphisms and we shall write g for the automorphism corresponding to a fixed primitive n^{th} root of unity $\zeta = \zeta(g)$. The subfunctor of fixed points of the functor associated to Y is representable and we call the representing scheme the *fixed-point scheme* and denote it by Y_{μ_n}. It is regular and there are natural isomorphisms of complex manifolds $Y_{\mu_n}(\mathbf{C}) \simeq Y(\mathbf{C})_g$, where $Y(\mathbf{C})_g$ is the set of fixed points of Y under the action of g. We write f^{μ_n} for the map $Y_{\mu_n} \to \operatorname{Spec} D$ induced by f. Complex conjugation of coefficients induces an antiholomorphic automorphism of $Y(\mathbf{C})$ and $Y_{\mu_n}(\mathbf{C})$, both of which we denote by F_∞. We write $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ for $\widetilde{\mathfrak{A}}(Y(\mathbf{C})_g) := \bigoplus_{p\geqslant 0}(\mathfrak{A}^{p,p}(Y(\mathbf{C})_g)/(\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}))$, where $\mathfrak{A}^{p,p}(\cdot)$ denotes the set of smooth complex differential forms ω of type (p,p) such that $F_\infty^*\omega = (-1)^p\omega$.

A hermitian equivariant sheaf (resp. vector bundle) on Y is a coherent sheaf (resp. a vector bundle) E on Y, assumed locally free on $Y(\mathbf{C})$, endowed with a μ_n-action which lifts the action of μ_n on Y and a hermitian metric h on $E_{\mathbf{C}}$, the bundle associated to E on the complex points, which is invariant under F_∞ and μ_n. We shall write (E,h) or $\overline{E}$ for a hermitian equivariant sheaf (resp. vector bundle). There is a natural $(\mathbf{Z}/n)$-grading $E|_{Y_{\mu_n}} \simeq \oplus_{k\in\mathbf{Z}/n} E_k$ on the restriction of E to Y_{μ_n}, whose terms are orthogonal, because of the invariance of the metric. We write $\overline{E}_k$ for the k^{th} term ($k \in \mathbf{Z}/n$), endowed with the induced metric. We shall also write $\overline{E}_{\neq 0}$ for $\oplus_{k\in(\mathbf{Z}/n)\setminus\{0\}}\overline{E}_k$.

We write $\operatorname{ch}_g(\overline{E}) := \sum_{k\in\mathbf{Z}/n} \zeta(g)^k \operatorname{ch}(\overline{E}_k)$ for the equivariant Chern character form $\operatorname{ch}_g(E_{\mathbf{C}}, h)$ associated to the restriction of $(E_{\mathbf{C}}, h)$ to $Y_{\mu_n}(\mathbf{C})$. Recall also that $\operatorname{Td}_g(\overline{E})$ is the differential form $\operatorname{Td}(\overline{E}_0)\Big(\sum_{i\geqslant 0}(-1)^i \operatorname{ch}_g(\Lambda^i(\overline{E}_{\neq 0}))\Big)^{-1}$. If $\mathcal{E} : 0 \to E' \to E \to E'' \to 0$ is an exact sequence of equivariant sheaves (resp. vector bundles), we shall write $\overline{\mathcal{E}}$ for the sequence $\mathcal{E}$ together with $\mu_n(\mathbf{C})$ and F_∞-invariant hermitian metrics on $E'_{\mathbf{C}}$, $E_{\mathbf{C}}$ and $E''_{\mathbf{C}}$. To $\overline{\mathcal{E}}$ and ch_g is associated an equivariant Bott-Chern secondary class $\widetilde{\operatorname{ch}}_g(\overline{\mathcal{E}}) \in \widetilde{\mathfrak{A}}(Y_{\mu_n})$, which satisfies the equation $\frac{\overline{\partial}\partial}{2\pi i}\widetilde{\operatorname{ch}}_g(\overline{\mathcal{E}}) = \operatorname{ch}_g(\overline{E}') + \operatorname{ch}_g(\overline{E}'') - \operatorname{ch}_g(\overline{E})$.

Definition 2.4. The *arithmetic equivariant Grothendieck group* $\widehat{K}_0^{\mu_n\prime}(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$) of Y is the free abelian group generated by the elements of $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ and by the equivariant isometry classes of hermitian equivariant sheaves (resp. vector bundles), together with the relations

(i) for every exact sequence $\overline{\mathcal{E}}$ as above, $\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}}) = \overline{E}' - \overline{E} + \overline{E}''$;

(ii) if $\eta \in \widetilde{\mathfrak{A}}(Y_{\mu_n})$ is the sum in $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ of two elements η' and η'', then $\eta = \eta' + \eta''$ in $\widehat{K}_0^{\mu_n\prime}(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$).

We shall now define a product on $\widehat{K}_0^{\mu_n\prime}(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$). Let $\overline{V}$, $\overline{V}'$ be hermitian equivariant sheaves (resp. vector bundles) and let η, η' be elements of $\widetilde{\mathfrak{A}}(Y_{\mu_n})$. We define a product $\cdot$ on the generators of $\widehat{K}_0^{\mu_n\prime}(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$) by the rules $\overline{V} \cdot \overline{V}' := \overline{V} \otimes \overline{V}'$, $\overline{V} \cdot \eta = \eta \cdot \overline{V} := \mathrm{ch}_g(\overline{V}) \wedge \eta$ and $\eta \cdot \eta' := \frac{\overline{\partial}\partial}{2\pi i}\eta \wedge \eta'$ and we extend it by linearity. This product is compatible with the relations defining $\widehat{K}_0^{\mu_n\prime}(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$) and defines a commutative ring structure on $\widehat{K}_0^{\mu_n\prime}(Y)$ (resp. $\widehat{K}_0^{\mu_n}(Y)$).

Suppose now that f is projective. Fix an F_∞-invariant Kähler metric on $Y(\mathbf{C})$, with Kähler form ω_Y and suppose that $\mu_n(\mathbf{C})$ acts by isometries with respect to this Kähler metric. Let $\overline{E} := (E, h)$ be an equivariant hermitian sheaf on Y. We write $T_g(\overline{E})$ for the equivariant analytic torsion $T_g(E_{\mathbf{C}}, h) \in \mathbf{C}$ of $(E_{\mathbf{C}}, h)$ over $Y(\mathbf{C})$; see [12, Sec. 2] or subsection 2.3 for the definition. Let $f : Y \to \operatorname{Spec} D$ be the structure morphism. We let $R^i f_* \overline{E}$ be the i^{th} direct image sheaf of E endowed with its natural equivariant structure and L^2-metric. Let $d_Y := \dim(Y(\mathbf{C}))$. The L^2-metric on $R^i f_* E_{\mathbf{C}} \simeq \coprod_{\sigma \in S} H^i_{\overline{\partial}}(Y \times_{\sigma(D)} \mathbf{C}, E_{\sigma,\mathbf{C}})$ is defined by the formula

$$\frac{1}{(2\pi)^{d_Y}} \int_{Y(\mathbf{C})} (s, t)\omega_Y^{d_Y} \tag{1}$$

where s and t are harmonic (i.e. in the kernel of the Kodaira Laplacian $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$) sections of $\Lambda^i(T^{*(0,1)}Y(\mathbf{C})) \otimes E_{\mathbf{C}}$. The pairing $(\cdot,\cdot)$ is the natural metric on $\Lambda^i(T^{*(0,1)}Y(\mathbf{C})) \otimes E_{\mathbf{C}}$. This definition is meaningful because by Hodge theory there is exactly one harmonic representative in each cohomology class. We also write $\overline{H^i(Y, E)}$ for $R^i f_* \overline{E}$ and $R^{\cdot} f_* \overline{E}$ for the linear combination $\sum_{i \geqslant 0} (-1)^i R^i f_* \overline{E}$. Let $\eta \in \widetilde{\mathfrak{A}}(Y_{\mu_n})$ and consider the rule which associates the element $R^{\cdot} f_* \overline{E} - T_g(\overline{E})$ of $\widehat{K}_0^{\mu_n\prime}(D)$ to $\overline{E}$ and the element $\int_{Y(\mathbf{C})_g} \mathrm{Td}_g(\overline{TY})\eta \in \widehat{K}_0^{\mu_n\prime}(D)$ to η.

Proposition 2.5. *The above rule induces a well defined group homomorphism* $f_* : \widehat{K}_0^{\mu_n\prime}(Y) \to \widehat{K}_0^{\mu_n\prime}(D)$.

One can show that $\widehat{K}_0^{\mu_n}(D)$ is isomorphic to $\widehat{K}_0^{\mu_n\prime}(D)$ via the natural map so that by composition the last proposition yields a map $\widehat{K}_0^{\mu_n}(Y) \to \widehat{K}_0^{\mu_n}(D)$, which we shall also call f_*.

Finally, to formulate the fixed point theorem, we define the homomorphism $\rho : \widehat{K}_0^{\mu_n}(Y) \to \widehat{K}_0^{\mu_n}(Y_{\mu_n})$, which is obtained by restricting all the involved objects from Y to Y_{μ_n}. If $\overline{E}$ is a hermitian vector bundle on Y, we write $\lambda_{-1}(\overline{E}) := \sum_{k=0}^{\mathrm{rk}(E)}(-1)^k\Lambda^k(\overline{E}) \in \widehat{K}_0^{\mu_n}(Y)$, where $\Lambda^k(\overline{E})$ is the k^{th} exterior power of $\overline{E}$, endowed with its natural hermitian and equivariant structure. If $\overline{E}$ is the orthogonal direct sum of two hermitian equivariant vector bundles $\overline{E}'$ and $\overline{E}''$, then $\lambda_{-1}(\overline{E}) = \lambda_{-1}(\overline{E}')\cdot\lambda_{-1}(\overline{E}'')$. Let $R(\mu_n)$ be the Grothendieck group of finitely generated projective μ_n-comodules over D. There are natural isomorphisms $R(\mu_n) \simeq K_0(D)[\mathbf{Z}/n] \simeq K_0(D)[T]/(1-T^n)$. Let $\overline{I}$ be the μ_n-comodule whose term of homogeneous degree $1 \in \mathbf{Z}/n$ is D endowed with the trivial metric and whose other terms are 0. We make $\widehat{K}_0^{\mu_n}(D)$ an $R(\mu_n)$-algebra under the ring morphism which sends T to $\overline{I}$. In the next theorem, which is the arithmetic fixed-point formula, let $\mathcal{R}$ be any $R(\mu_n)$-algebra such that the elements $1-T^k$ $(k = 1,\ldots,n-1)$ are invertible in $\mathcal{R}$.

Let now $\theta \in \mathbf{R}$. For all $s \in \mathbf{C}$ such that $\Re(s) > 1$ we define Lerch's partial ζ-functions $\zeta(\theta, s) := \sum_{n\geqslant 1}\frac{\cos(n\theta)}{n^s}$ and $\eta(\theta, s) := \sum_{n\geqslant 1}\frac{\sin(n\theta)}{n^s}$, and using analytic continuation, we extend them to meromorphic functions of s over $\mathbf{C}$. Let $R(\theta, t)$ be the formal power series

$$\sum_{n\geqslant 1,\ n\text{ odd}}(2\zeta'(\theta,-n)+\sum_{j=1}^{n}\frac{\zeta(\theta,-n)}{j})\frac{t^n}{n!} + i\sum_{n\geqslant 0,\ n\text{ even}}(2\eta'(\theta,-n)+\sum_{j=1}^{n}\frac{\eta(\theta,-n)}{j})\frac{t^n}{n!}.$$

We shall need $R(\theta,(\cdot))$, which is by definition the unique additive characteristic class on holomorphic vector bundles such that $R(\theta, L) = R(\theta, c_1(L))$ for each line bundle L. Let V be a μ_n-equivariant vector bundle on Y; we define

$$R_g(V) := \sum_{k=1}^{\mathrm{rk}(V)} R(\arg(\zeta(g)^k), V_k).$$

Choose any μ_n-invariant hermitian metric on $V_{\mathbf{C}}$; this hermitian metric induces a connection of type $(1,0)$ on each $V_{\mathbf{C},k}$; using this connection, we may compute a differential form representative of $R(\arg(\zeta(g)^k), V_k)$ in complex de Rham cohomology; this representative is a sum of differential forms of type (p,p) $(p \geq 0)$, which is both $\partial-$ and $\overline{\partial}-$closed. In the next theorem, we may thus consider that the values of $R_g(\cdot)$ lie in $\widetilde{\mathfrak{A}}(Y_{\mu_n})$.

THEOREM 2.6. *Let $\overline{N}_{Y/Y_{\mu_n}}$ be the normal bundle of Y_{μ_n} in Y, endowed with its quotient equivariant structure and quotient metric structure* (*which is F_∞-invariant*).

(i) *The element $\Lambda := \lambda_{-1}(\overline{N}^{\vee}_{Y/Y_{\mu_n}})$ has an inverse in $\widehat{K}_0^{\mu_n}(Y_{\mu_n}) \otimes_{R(\mu_n)} \mathcal{R}$.*

(ii) *When* $\Lambda_R := \Lambda \cdot (1 + R_g(N_{Y/Y_{\mu_n}}))$, *the diagram*[2]

$$\begin{array}{ccc} \widehat{K}_0^{\mu_n}(Y) & \xrightarrow{\Lambda_R^{-1}\cdot\rho} & \widehat{K}_0^{\mu_n}(Y_{\mu_n}) \otimes_{R(\mu_n)} \mathcal{R} \\ \downarrow\ f_* & & \downarrow\ f_*^{\mu_n} \\ \widehat{K}_0^{\mu_n}(D) & \xrightarrow{\mathrm{Id}\otimes 1} & \widehat{K}_0^{\mu_n}(D) \otimes_{R(\mu_n)} \mathcal{R} \end{array}$$

commutes.

The proof of this theorem is the object of [13], it combines the deformation to the normal cone technique with deep results of Bismut on the behaviour of equivariant analytic torsion under immersions [2].

2.3. *The equivariant analytic torsion and the L^2-metric of the de Rham complex.* In this subsection, we shall prove the vanishing of the equivariant analytic torsion for the de Rham complex. Before doing so, we shall review some results on the polarisation induced by an ample line bundle on the singular cohomology of a complex manifold.

Let M be a complex projective manifold of dimension d and L be an ample line bundle on M. Let us denote by $\omega \in H^2(M, \mathbf{Q})$ the first Chern class of L and for $k \leqslant d$, let $P^k(M, \mathbf{C}) \subseteq H^k(M, \mathbf{C})$ be the primitive cohomology associated to ω; this is a Hodge substructure of $H^k(M, \mathbf{C})$. Recall that for any $k \geqslant 0$, the primitive decomposition theorem establishes an isomorphism

$$H^k(M, \mathbf{C}) \simeq \oplus_{r \geqslant \max(k-d,0)}\, \omega^r \wedge P^{k-2r}(M, \mathbf{C}).$$

Define the cohomological star operator $* : H^k(M, \mathbf{C}) \to H^{2d-k}(M, \mathbf{C})$ by the rule $*\,\omega^r \wedge \phi := i^{p-q}(-1)^{(p+q)(p+q+1)/2} \frac{r!}{(d-p-q-r)!} \omega^{d-p-q-r} \wedge \phi$ if ϕ is a primitive element of pure Hodge type (p, q) and extend it by additivity. We can now define a hermitian metric on $H^k(M, \mathbf{C})$ by the formula

$$(\nu, \eta)_L := \frac{1}{(2\pi)^d} \int_M \nu \wedge * \overline{\eta}$$

for any $\nu, \eta \in H^k(M, \mathbf{C})$. This metric is sometimes called the Hodge metric. The next lemma follows from the definition of the L^2-metric, Hodge's theorem on the representability of cohomology classes by harmonic forms and the Hodge-Kähler identities.

LEMMA 2.7. *Endow M with a Kähler metric whose Kähler form ω_M represents the cohomology class of ω in Betti cohomology and equip the bundles Ω_M^p with the corresponding metrics. Endow $\oplus_{p+q=k} H^q(M, \Omega_M^p)$ with the L^2-metric and $H^k(M, \mathbf{C})$ with the Hodge metric. The Hodge-de Rham isomorphism $H^k(M, \mathbf{C}) \simeq \oplus_{p+q=k} H^q(M, \Omega_M^p)$ is an isometry.*

[2]Note that a misprint found its way into the statement [13, Th. 4.4] (= Th. 2.6). In [13, Th. 4.4] the term $\Lambda \cdot (1 - R_g(N_{Y/Y_{\mu_n}}))$ has to be replaced by $\Lambda \cdot (1 + R_g(N_{Y/Y_{\mu_n}}))$ in the expression for Λ_R.

Proof. Let $*'$ be the Hodge star operator of differential geometry. By the definitions of the L^2-metric and of the operator $*'$, we have the formula

$$\frac{1}{(2\pi)^d}\int_M \nu\wedge *'\overline{\eta}$$

for the L^2-hermitian product of two harmonic representatives of $H^q(M,\Omega^p_M)$. Furthermore, the operator $\omega_M\wedge(\cdot)$ sends harmonic forms to harmonic forms and the identity $*'\,\omega^r_M\wedge\phi := i^{p-q}(-1)^{(p+q)(p+q+1)/2}\frac{r!}{(d-p-q-r)!}\omega_M^{d-p-q-r}\wedge\phi$ is verified if ϕ is a primitive harmonic form of pure Hodge type (p,q) (see [21]). This implies the result. □

Suppose now that M is a complex compact Kähler manifold endowed with a unitary automorphism g, and let $\overline{E}$ be a hermitian holomorphic vector bundle on M which is equipped with a unitary lifting of the action of g. Let $\Box^E_q$ be the differential operator $(\overline{\partial}+\overline{\partial}^*)^2$ acting on the C^∞-sections of the bundle $\Lambda^q T^{*(0,1)}M\otimes E$. This space of sections is equipped with the L^2-metric and the operator $\Box^E_q$ is symmetric for that metric; we let $\mathrm{Sp}(\Box^E_q)\subseteq\mathbf{R}$ be the set of eigenvalues of $\Box^E_q$ (which is discrete and bounded from below) and we let $\mathrm{Eig}^E_q(\lambda)$ be the eigenspace associated to an eigenvalue λ (which is finite-dimensional). Define

$$Z(\overline{E},g,s):=\sum_{q\geqslant 1}(-1)^{q+1}q\sum_{\lambda\in\mathrm{Sp}(\Box^E_q)\setminus\{0\}}\mathrm{Tr}(g^*|_{\mathrm{Eig}^E_q(\lambda)})\lambda^{-s}$$

for $\Re(s)$ sufficiently large. The function $Z(\overline{E},g,s)$ has a meromorphic continuation to the whole plane, which is holomorphic around 0 (see [12]). By definition, the equivariant analytic torsion of $\overline{E}$ is given by $T_g(\overline{E}):=Z'(\overline{E},g,0)$.

The nonequivariant analog of the following lemma (the proof of which is similar) can be found in [18].

LEMMA 2.8. *Let M be a complex compact Kähler manifold and let g be a unitary automorphism of M. The identity*

$$\sum_{p\geqslant 0}(-1)^p T_g(\Lambda^p(\overline{\Omega}_M))=0$$

holds.

Proof. Recall the Hodge decomposition (see [21, Chap. IV, no. 3, Cor. 2])

$$A^{p,q}(M)=\mathcal{H}^{p,q}(M)\oplus\partial(A^{p-1,q}(M))\oplus\partial^*(A^{p+1,q}(M))$$

where $\mathcal{H}^{p,q}(M)$ are the harmonic forms for the usual Kodaira-Laplace operator $\Box_q=(\partial+\partial^*)^2=(\overline{\partial}+\overline{\partial}^*)^2$ and $A^{p,q}(M)$ is the space of C^∞-differential forms of type (p,q) on M. Let us write $A^{p,q}_1(M)$ for $\partial(A^{p-1,q}(M))$ and $A^{p,q}_2(M)$ for $\partial^*(A^{p+1,q}(M))$. The map $\partial|_{A^{p,q}_2(M))}$ is an injection and its image is $A^{p+1,q}_1(M)$.

Notice also that the operator $\Box_q$ commutes with ∂ and ∂^*. Notice as well that the C^∞-sections of $\Lambda^q T^{*(0,1)}M \otimes \Lambda^p(\Omega_M)$ correspond to the space $A^{p,q}(M)$ and that $\Box_q^{\Lambda^p(\Omega)} = \Box_q|_{A^{p,q}}$. For $\lambda \in \mathbf{R}^\times$, we write $L_\lambda^{p,q} = \mathrm{Ker}(\Box_q^{\Lambda^p(\Omega)} - \lambda)$, $L_{\lambda,1}^{p,q} = L_\lambda^{p,q} \cap A_1^{p,q}$ and $L_{\lambda,2}^{p,q} = L_\lambda^{p,q} \cap A_2^{p,q}$. We compute

$$\sum_{p\geqslant 0}(-1)^p \mathrm{Tr}(g^*|_{L_\lambda^{p,q}}) = \sum_{p\geqslant 0}(-1)^p[\mathrm{Tr}(g^*|_{L_{\lambda,1}^{p,q}}) + \mathrm{Tr}\,(g^*|_{L_{\lambda,2}^{p,q}})] = 0$$

and from this, we conclude that $\sum_{p\geqslant 0}(-1)^p Z(\Lambda^p(\overline{\Omega}_M), g, s) \equiv 0$. $\Box$

2.4. *An invariant of equivariant arithmetic K_0-theory.* From now on, we restrict ourselves to the case $D = Q_0$, where $Q_0 \overset{\iota_0}{\hookrightarrow} \mathbf{C}$ is a number field embedded in $\mathbf{C}$, and we fix a primitive n^{th} root of unity $\zeta := e^{2\pi i/n}$. We use this choice to identify the set $\mu_n(\mathbf{C})^\times$ of primitive n^{th} roots of unity with the Galois group $G := \mathrm{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q}) = \mathrm{Hom}(\mathbf{Q}(\mu_n), \mathbf{C})$. The ring morphism $R(\mu_n) \to \mathbf{Q}(\mu_n)$ which sends the generator T on ζ makes $\mathbf{Q}(\mu_n)$ an $R(\mu_n)$-algebra and allows us to take $\mathcal{R} := \mathbf{Q}(\mu_n)$. We let $\widehat{\mathrm{CH}}(Q_0)$ be the arithmetic Chow ring of Q_0 with the set of embeddings $\mathcal{S} := \{\iota_0, \overline{\iota_0}\}$, in the sense of Gillet-Soulé (see [8]). There is a natural isomorphism $\widehat{\mathrm{CH}}(Q_0) \simeq \mathbf{Z} \oplus \mathbf{R}/\log|Q_0^\times|$ and a ring isomorphism $\widehat{K}_0(Q_0) \simeq \widehat{\mathrm{CH}}(Q_0)$ given by the arithmetic Chern character $\widehat{\mathrm{ch}}$ (see [8]), the ring $\widehat{K}_0(Q_0)$ being defined similarly to the ring $\widehat{K}_0^{\mu_1}(Q_0)$, with $\mathfrak{A}^{p,p}(\cdot)$ replaced by the space $\mathfrak{A}_{\mathbf{R}}^{p,p}(\cdot)$ of real (not complex) differential forms of type (p,p). The ring structure on $\mathbf{Z} \oplus \mathbf{R}/\log|Q_0^\times|$ is given by the formula $(r \oplus x)\cdot(r' \oplus x') := (r\cdot r', r\cdot x' + r'\cdot x)$. On generators of $\widehat{K}_0(Q_0)$, the arithmetic Chern character is defined as follows: For $\overline{V}$ a hermitian vector bundle on $\mathrm{Spec}\, Q_0$, the arithmetic Chern character $\widehat{\mathrm{ch}}(\overline{V})$ is the element $\mathrm{rk}(V) \oplus (-\log||s||)$, where s is a nonvanishing section of $\det(V)$ and $||\cdot||$ is the norm on $\det(V)_{\mathbf{C}}$ induced by the metric on $V_{\mathbf{C}}$. For an element $\eta \in \mathfrak{A}_{\mathbf{R}}(\mathrm{Spec}\, Q_0) \simeq \mathbf{R}$, the arithmetic Chern character $\widehat{\mathrm{ch}}(\eta)$ is the element $0 \oplus \frac{1}{2}\eta$.

Let now $\aleph_0$ be the additive subgroup of $\mathbf{C}$ generated by the elements $z\cdot\log|q_0|$ where $q_0 \in Q_0^\times$ and $z \in \mathbf{Q}(\mu_n)$. We define $\widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0) := \mathbf{Q}(\mu_n) \oplus \mathbf{C}/\aleph_0$ and we define a ring structure on $\widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$ by the rule $(z,x)\cdot(z',x') := (z\cdot z', z\cdot x' + z'\cdot x)$. Notice that there is a natural ring morphism $\psi : \widehat{\mathrm{CH}}(Q_0) \to \widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$ and that there is a natural $\mathbf{Q}(\mu_n)$-module structure on $\widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$. Define a rule which associates elements of $\widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$ to generators of $\widehat{K}_0^{\mu_n}(Q_0)$ as follows. Associate the element $\zeta^k\cdot\psi(\widehat{\mathrm{ch}}(\overline{V}))$ to a μ_n-equivariant hermitian vector bundle $\overline{V}$ of pure degree k (for the natural $(\mathbf{Z}/n)$-grading) on $\mathrm{Spec}\, Q_0$; furthermore associate the element $0 \oplus \frac{1}{2}\eta$ to $\eta \in \widetilde{\mathfrak{A}}(\mathrm{Spec}\, Q_0) \simeq \mathbf{C}$.

Lemma 2.9. *The above rule induces a morphism of $R(\mu_n)$-modules $\widehat{\mathrm{ch}}_{\mu_n} : \widehat{K}_0^{\mu_n}(Q_0) \to \widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$.*

Proof. Let

$$\mathcal{V}:\ 0\to V'\to V\to V''\to 0$$

be a an exact sequence of μ_n-equivariant vector bundles (μ_n-comodules) over Q_0. We endow the members of $\mathcal{V}$ with (conjugation invariant) hermitian metrics h', h and h'' respectively, such that the pieces of the various gradings are orthogonal. The equality

$$\widetilde{\mathrm{ch}}_\zeta(\overline{\mathcal{V}})=\sum_{k\in\mathbf{Z}/n}\zeta^k\widetilde{\mathrm{ch}}(\overline{\mathcal{V}}_k)$$

holds (see [13, Th. 3.4, Par. 3.3]). From this and the well-defined quality of the arithmetic Chern character, the result follows. □

We shall write $\widehat{c}^1_{\mu_n}$ for the second component of $\widehat{\mathrm{ch}}_{\mu_n}$, i.e. the component lying in $\mathbf{C}/\aleph_0$. If $\overline{V}$ is a hermitian μ_n-equivariant vector bundle on $\operatorname{Spec}Q_0$ with a trivial μ_n-action, we shall write $\widehat{\mathrm{c}}_1(\overline{V})$ for $\widehat{c}^1_{\mu_n}(\overline{V})$.

Lemma 2.10. *Let $\overline{V}$ be a hermitian μ_n-equivariant vector bundle on* $\operatorname{Spec}Q_0$*. The equation*

$$\prod_{l\in\mathbf{Z}/n}(1-\zeta^l)^{-\mathrm{rk}(V_l)}\widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V}))=1\oplus\Big(-\sum_{l\in\mathbf{Z}/n}\frac{\zeta^l}{1-\zeta^l}\widehat{\mathrm{c}}_1(\overline{V}_l)\Big)\tag{2}$$

holds.

Proof. We shall make use of the canonical isomorphism

$$\det(\Lambda^k(W))\simeq\det(W)^{\otimes\frac{(r-1)!}{(r-k)!(k-1)!}},$$

valid for any vector space W of rank r over a field and any $1\leqslant k\leqslant r$, and constructed as follows: For any basis $b_1,\dots,b_r$ of W, the element

$$\bigwedge_{1\leqslant i_1<\cdots<i_k\leqslant r}(b_{i_1}\wedge\cdots\wedge b_{i_k})$$

of $\det(\Lambda^k(W))$ is sent to the element

$$\bigotimes_{j=1}^{\frac{(r-1)!}{(r-k)!(k-1)!}}(b_1\wedge\cdots\wedge b_r)$$

of $\det(W)^{\otimes\frac{(r-1)!}{(r-k)!(k-1)!}}$. This isomorphism is by construction invariant under base change to a field extension. Furthermore, if one applies the above description to the orthonormal basis of a vector space over $\mathbf{C}$ endowed with a hermitian metric, one find that this isomorphism is also an isometry for the natural metrics on both sides. Thus, for any hermitian vector bundle $\overline{W}$ over $\operatorname{Spec} Q_0$, the isomorphism of vector bundles

$$\det(\Lambda^k(W)) \simeq \det(W)^{\otimes \frac{(r-1)!}{(r-k)!(k-1)!}}$$

is an isometry. Using the definition of the ring structure of $\widehat{\mathrm{CH}}_{\mathbf{Q}(\mu_n)}(Q_0)$ and the fact that $\widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V} \oplus \overline{V}')) = \widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V})) \cdot \widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V}'))$ for any two hermitian equivariant vector bundles over Q_0, we see that as functions of $\overline{V}$, both sides of the equality in (2) are multiplicative for direct sums of hermitian vector bundles. We are thus reduced to proving the equality

$$(1-\zeta^l)^{-\mathrm{rk}(V_l)}\widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V}_l)) = 1 \oplus \frac{-\zeta^l}{1-\zeta^l}\,\widehat{\mathrm{c}}_1(\overline{V}_l)$$

for all $l \in \mathbf{Z}/n$. Let $r_l := \mathrm{rk}(V_l)$; we compute

$$\begin{aligned}\widehat{\mathrm{ch}}_{\mu_n}(\lambda_{-1}(\overline{V}_l)) &= \sum_{k=0}^{r_l}(-1)^k\zeta^{lk}\widehat{\mathrm{ch}}(\Lambda^k(\overline{V}_l))\\ &= \sum_{k=0}^{r_l}(-1)^k\zeta^{lk}\frac{r_l!}{k!(r_l-k)!} \oplus (\sum_{k=1}^{r_l}(-1)^k\zeta^{lk}\frac{(r_l-1)!}{(k-1)!(r_l-k)!})\widehat{c}_1(\overline{V}_l).\end{aligned}$$

Using the binomial formula, we see that the last expression can be rewritten as

$$(1-\zeta^l)^{r_l} \oplus (-\zeta^l(1-\zeta^l)^{r_l-1})\widehat{c}_1(\overline{V}_l)$$

and the result follows. □

3. Proof of Theorems 1 and 2

3.1. *Two lemmas.* For z belonging to the unit circle $\mathcal{S}_1$, we define Lerch's ζ-function $\zeta_L(z,s) := \sum_{k\geqslant 1}\frac{z^k}{k^s}$ for $s \in \mathbf{C}$ such that $\Re(s) > 1$, and using analytic continuation, we extend it to a meromorphic function of s over $\mathbf{C}$.

LEMMA 3.1. *Let Y be a scheme, smooth over $\mathbf{C}$, and let E be a vector bundle on Y together with an automorphism $g : E \to E$ of finite order* (*acting fiberwise*). *Let κ be the class*

$$\kappa := \mathrm{Td}(E_0)\frac{\sum_{p\geqslant 0}(-1)^p p\cdot \mathrm{ch}_g(\Lambda^p(E^\vee))}{\sum_{p\geqslant 0}(-1)^p \mathrm{ch}_g(\Lambda^p(E^\vee_{\neq 0}))}.$$

The equality

$$\kappa^{[l+\mathrm{rk}(E_0)]} = -c^{\mathrm{top}}(E_0)\sum_{z\in\mathcal{S}_1}\zeta_{\mathrm{L}}(z,-l)\,\mathrm{ch}^{[l]}(E^\vee_z)$$

holds.

Proof. According to the splitting principle, we may suppose that E is an equivariant direct sum of $r := \mathrm{rk}(E)$ line bundles F_i on which g acts by multiplication with the eigenvalue $\alpha_i^{-1} \in \mathcal{S}_1$. If we set $\gamma_i := c_1(F_i^\vee)$ for all i, we can write

$$\sum_{p\geqslant 0}(-1)^p p\cdot \mathrm{ch}_g(\wedge^p(E^\vee)) = \sum_{p\geqslant 0}(-1)^p p \sum_{1\leqslant i_1<\cdots<i_p\leqslant r} \alpha_{i_1}\cdots\alpha_{i_p} e^{\gamma_{i_1}+\cdots+\gamma_{i_p}}. \tag{3}$$

If we take the formal derivative (with respect to t) of the identity

$$\prod_{i=1}^r (1-\alpha_i e^{\gamma_i} t) = \sum_{p\geqslant 0}(-1)^p \left[\sum_{1\leqslant i_1<\cdots<i_p\leqslant r} \alpha_{i_1}\cdots\alpha_{i_p} e^{\gamma_{i_1}+\cdots+\gamma_{i_p}}\right] t^p$$

set $t=1$ and apply (3), we obtain

$$\sum_{p\geqslant 0}(-1)^p p\cdot \mathrm{ch}_g(\wedge^p(E^\vee)) = -\prod_{i=1}^r (1-\alpha_i e^{\gamma_i}) \sum_{j=1}^r \frac{\alpha_j e^{\gamma_j}}{1-\alpha_j e^{\gamma_j}}\,. \tag{4}$$

Notice now that we can write

$$\begin{aligned}\prod_{i=1}^r (1-\alpha_i e^{\gamma_i}) &= \prod_{\alpha_i\neq 1}(1-\alpha_i e^{\gamma_i}) \prod_{\alpha_i=1}(1-e^{\gamma_i}) \\ &= \sum_{p\geqslant 0}(-1)^p\,\mathrm{ch}_g(\wedge^p(E_{\neq 0}^\vee))\,\frac{c^{\mathrm{top}}(E_0)}{\mathrm{Td}(E_0)}\,;\end{aligned}$$

this together with (4) shows that

$$\kappa = -c^{\mathrm{top}}(E_0)\sum_{j=1}^r \frac{\alpha_j e^{\gamma_j}}{1-\alpha_j e^{\gamma_j}}\,. \tag{5}$$

Furthermore, notice that for any smooth function $f(x)$ and any $k\in\mathbb{N}$,

$$\frac{d^k}{dt^k} f(\alpha e^t) = \left(\left[x\frac{d}{dx}\right]^k f\right)(\alpha e^t)$$

and (for $x\neq 1$)

$$\zeta_{\mathrm{L}}(x,-k) = \left[x\frac{d}{dx}\right]^k \zeta_{\mathrm{L}}(x,0)$$

and also,

$$\zeta_{\mathrm{L}}(x,0) = \frac{x}{1-x}.$$

We deduce that (for $\alpha\neq 1$)

$$\frac{\alpha e^t}{1-\alpha e^t} = \sum_{p\geqslant 0}\zeta_{\mathrm{L}}(\alpha,-p)\frac{t^p}{p!}\,. \tag{6}$$

When $\alpha = 1$, we have the the classical expansion

$$\frac{e^t}{1-e^t} = -\frac{1}{t} + \sum_{p\geqslant 0} \zeta_{\mathbb{Q}}(-p)\frac{t^p}{p!}\,.$$

Using in (5) the formula just above or the formula (6) according to whether α_j is equal to 1 or not, we find that for all $l \geqslant 0$

$$\kappa^{[l+\mathrm{rk}(E_0)]} = -c^{\mathrm{top}}(E_0)\Big[\sum_{z\in\mathcal{S}_1\setminus\{1\}} \zeta_{\mathrm{L}}(z,-l)\,\mathrm{ch}^{[l]}(E_z^\vee) + \zeta_{\mathbb{Q}}(-l)\,\mathrm{ch}^{[l]}(E_0^\vee)\Big]$$

which, noticing that $\zeta_{\mathrm{L}}(1,-l) = \zeta_{\mathbb{Q}}(-l)$, concludes the proof. □

Caution. In what follows, in contradiction to classical usage and to the introduction to this article, the notation $L(\chi, s)$ will always refer to the *nonprimitive* L-function associated with a Dirichlet character χ. We shall write χ_{prim} for the primitive character associated with χ and accordingly write $L(\chi_{\mathrm{prim}}, s)$ for the associated primitive L-function.

Recall that the relationship between primitive and nonprimitive Dirichlet L-functions is given by the equality

$$L(\chi,s) = L(\chi_{\mathrm{prim}},s)\prod_{p|n}(1-\chi_{\mathrm{prim}}(p)p^{-s}) \tag{7}$$

where χ is a character of $\mathrm{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q})$. This implies in particular the formula

$$\frac{L'(\chi,s)}{L(\chi,s)} = \frac{L'(\chi_{\mathrm{prim}},s)}{L(\chi_{\mathrm{prim}},s)} + \sum_{p|n}\frac{\chi_{\mathrm{prim}}(p)p^{-s}}{1-\chi_{\mathrm{prim}}(p)p^{-s}}\log(p) \tag{8}$$

obtained by taking the logarithmic derivative of both sides of (7).

The following lemma, proved in [14, Lemma 5.2, Sec. 5], establishes the link between Lerch ζ-functions and Dirichlet L-functions. It follows from the functional equation of Dirichlet L-functions when the character is primitive. Recall that by definition (see before Theorem 2.6), the following identity relates $\zeta_L(z,s)$, $\zeta(\arg(z),s)$ and $\eta(\arg(z),s)$:

$$\zeta_L(z,s) = \zeta(\arg(z),s) + i\cdot\eta(\arg(z),s)$$

where $s\in\mathbf{C}$ and $z\in\mathbf{C}$, $|z|=1$.

LEMMA 3.2. *Let χ be an odd character of $G = \mathrm{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q})$. The equality*

$$\sum_{\sigma\in G}\eta(\arg(\sigma(\zeta)),s)\,\overline{\chi(\sigma)} = n^{1-s}\frac{\Gamma(1-s/2)}{\Gamma((s+1)/2)}\pi^{s-1/2}L(\overline{\chi},1-s)$$

holds for all $s\in\mathbf{C}$.

If χ is a character of G, we shall write $\tau(\chi) := \sum_{\sigma\in G}\sigma(\zeta)\chi(\sigma)$ for the Gauss sum associated to χ. Recall that if χ is primitive (i.e. not induced from a subfield $\mathbf{Q}(\mu_m)$ with $m<n$) the following equation holds (see

[20, Lemma 4.7, p. 36])

$$\sum_{\sigma\in G}\sigma(\zeta^l)\chi(\sigma)=\tau(\chi)\overline{\chi(l)} \tag{9}$$

where we used the identification $G\simeq(\mathbf{Z}/n)^\times$ to give meaning to $\chi(l)$.

If one combines the preceding lemma with the functional equation of primitive L-functions, one obtains the equation

$$\sum_{\sigma\in G}\eta(\arg(\sigma(\zeta)),s)\,\chi(\sigma)=-i\cdot\tau(\chi)L(\overline{\chi},s) \tag{10}$$

for all $s\in\mathbf{C}$, if χ is a *primitive* and *odd* Dirichlet character. Another way to prove this equality is to apply (9) to the definition of the function η.

3.2. *The proofs.* The notations of Sections 1 and 2, and the conventions of subsection 2.4 are still in force. If $N^\times$ is a subgroup of $\mathbf{C}^\times$ and $z,z'\in\mathbf{C}$, we shall write $z\sim_{N^\times}z'$ if $z=\lambda\cdot z'$ with $\lambda\in N^\times$. Recall that $f:X\to\operatorname{Spec}Q_0$ is a smooth and projective variety acted upon by g, an automorphism of order n (defined over Q_0). Suppose that Q_0 contains $\mathbf{Q}(\mu_n)$. Endow $X(\mathbf{C})$ with a g-invariant Kähler metric. We will denote by $\overline{\Omega}$ the sheaf of relative differentials of f equipped with the induced metric.

We shall now prove Theorems 1 and 2. To do so, we first apply the arithmetic fixed-point formula (Theorem 2.6) to the de Rham complex $\lambda_{-1}(\overline{\Omega})$.

$$\begin{aligned}
f_*(\lambda_{-1}(\overline{\Omega}))=&\,T_g(\lambda_{-1}(\overline{\Omega}))\\
&-\int_{X_{\mu_n}(\mathbf{C})}R_g(TX)\mathrm{Td}_g(TX)\mathrm{ch}_g(\lambda_{-1}(\Omega))\\
&+f_*^{\mu_n}(\lambda_{-1}^{-1}(\overline{N}^\vee_{X/X_{\mu_n}})\lambda_{-1}(\rho(\overline{\Omega})))\\
=&\,T_g(\lambda_{-1}(\overline{\Omega}))-\int_{X(\mathbf{C})_g}R_g(TX)\mathrm{Td}(TX_g)\mathrm{ch}(\lambda_{-1}(TX_g^\vee))\\
&+f_*^{\mu_n}(\lambda_{-1}(\overline{\Omega}(f^{\mu_n})))\\
=&\,T_g(\lambda_{-1}(\overline{\Omega}))-\int_{X(\mathbf{C})_g}R_g(TX)c^{\mathrm{top}}(TX_g)+f_*^{\mu_n}\big(\lambda_{-1}(\overline{\Omega}(f^{\mu_n}))\big).
\end{aligned}$$

Applying $\widehat{c}^{\,1}_{\mu_n}(\cdot)$ to both sides of the last equality, we obtain

$$\begin{aligned}
\widehat{c}^{\,1}_{\mu_n}(f_*(\lambda_{-1}(\overline{\Omega})))=&\,\frac{1}{2}T_g(\lambda_{-1}(\overline{\Omega}))\\
&-\frac{1}{2}\int_{X(\mathbf{C})_g}R_g(TX)c^{\mathrm{top}}(TX_g)+\widehat{\mathrm{c}}_1(f_*^{\mu_n}\big(\lambda_{-1}(\overline{\Omega}(f^{\mu_n}))\big)).
\end{aligned} \tag{11}$$

We deduce from Lemma 2.8 that $T_g(\lambda_{-1}(\overline{\Omega}))=0$. The following lemma shows that the third term in (11) likewise vanishes:

LEMMA 3.3. *The equality* $\widehat{\mathrm{c}}_1(f_*^{\mu_n}(\lambda_{-1}(\overline{\Omega}(f^{\mu_n})))) = 0$ *holds.*

Proof. Let d_μ be the relative dimension of X_{μ_n} over $\operatorname{Spec} Q_0$. The expression $\widehat{\mathrm{c}}_1(f_*^{\mu_n}(\lambda_{-1}(\overline{\Omega}(f^{\mu_n}))))$ can be subdivided into a linear combination of terms of the following kind:

$$\widehat{\mathrm{c}}_1(R^q f_*^{\mu_n}(\Lambda^p(\overline{\Omega}(f^{\mu_n})))) + \widehat{\mathrm{c}}_1(R^{d_\mu - q} f_*^{\mu_n}(\Lambda^{d_\mu - p}(\overline{\Omega}(f^{\mu_n})))).$$

By Serre duality, the spaces $R^q f_*^{\mu_n}(\Lambda^p(\Omega(f^{\mu_n})))$ and $R^{d_\mu - q} f_*^{\mu_n}(\Lambda^{d_\mu - p}(\Omega(f^{\mu_n})))$ are dual to each other, and even more, this duality is a duality of hermitian vector bundles (for the last statement, see [9]). Hence, from the definition of $\widehat{\mathrm{c}}_1$, it follows that $\widehat{\mathrm{c}}_1(R^q f_*^{\mu_n}(\Lambda^p(\overline{\Omega}(f^{\mu_n})))) = -\widehat{\mathrm{c}}_1(R^{d_\mu - q} f_*^{\mu_n}(\Lambda^{d_\mu - p}(\overline{\Omega}(f^{\mu_n}))))$ which ends the proof. □

We shall write $H^k_{\mathrm{Dlb}}(X) := \bigoplus_{p+q=k} R^q f_*(\Lambda^p(\Omega(f)))$ and $H_{\mathrm{Dlb}}(X)$ for the direct sum of the all the $H^k_{\mathrm{Dlb}}(X)$. Furthermore, we shall write $H_{\mathrm{Dlb}}(\overline{X})$ for $H_{\mathrm{Dlb}}(X)$ equipped with its natural L^2-metric. From the preceding discussion and (11), there exists the equality

$$\sum_{k\geqslant 0}(-1)^k \widehat{c}^{\,1}_{\mu_n}(H^k_{\mathrm{Dlb}}(\overline{X})) = -\frac{1}{2}\int_{X(\mathbf{C})_g} R_g(TX)c^{\mathrm{top}}(TX_g). \tag{12}$$

To show that (12) implies Theorems 1 and 2, we will use Lemma 3.2 to express derivatives of Lerch ζ-functions occurring in $R_g(TX)$ in terms of derivatives of Dirichlet L-functions, and then Lemma 3.1 to give a global (cohomological) expression for the right side of (12).

Proof of Theorem 1. Let L be a g-equivariant ample line bundle over X and suppose now that $X(\mathbf{C})$ is endowed with a Kähler metric whose Kähler form represents the first Chern class of L in Betti cohomology. We compute

$$\sum_{k\geqslant 0}(-1)^k \widehat{c}^{\,1}_{\mu_n}(H^k_{\mathrm{Dlb}}(\overline{X})) = \sum_{k\geqslant 0}(-1)^k \sum_{l\geqslant 0}\zeta^l\,\widehat{\mathrm{c}}_1(H^k_{\mathrm{Dlb}}(\overline{X})_l)$$

and

$$\begin{aligned}
&\sum_{\sigma\in G}\sum_{k\geqslant 0}(-1)^k \sum_{l\geqslant 0}\sigma(\zeta)^l\,\widehat{\mathrm{c}}_1(H^k_{\mathrm{Dlb}}(\overline{X})_l)\chi(\sigma)\\
&\quad= \tau(\chi)\sum_{k\geqslant 0}(-1)^k\sum_{l\geqslant 0}\overline{\chi(l)}\widehat{\mathrm{c}}_1(H^k_{\mathrm{Dlb}}(\overline{X})_l)\\
&\quad= -\tau(\chi)\int_{X(\mathbf{C})_g} L'(\overline{\chi},0)\sum_{l}\overline{\chi}(l)\,\mathrm{rk}(TX_l)c^{\mathrm{top}}(TX_g)\\
&\quad= -\tau(\chi)\int_{X(\mathbf{C})_g}\frac{L'(\overline{\chi},0)}{L(\overline{\chi},0)\tau(\chi)}\sum_{\sigma\in G}\chi(\sigma)\mathrm{Td}(TX_g)\frac{\sum_{p\geqslant 0}(-1)^p p\cdot \mathrm{ch}_{\sigma(\zeta)}(\Lambda^p(\Omega))}{\sum_{p\geqslant 0}(-1)^p\mathrm{ch}_{\sigma(\zeta)}(\Lambda^p(N^\vee))}\\
&\quad= -\tau(\chi)\frac{L'(\overline{\chi},0)}{L(\overline{\chi},0)}\sum_{\sigma\in G}\sum_{p,q}(-1)^{p+q}p\cdot\mathrm{rk}(H^{p,q}(X(\mathbf{C}))_\sigma)\chi(\sigma)
\end{aligned}$$

which shows the result. For the first equality in the last string of equalities, we have used (9); for the second one, we have used (12) and (10); for the third one, we used Lemma 3.1 and the fact that $L(\overline{\chi},0)\tau(\chi) = L(1,\chi)\frac{i\cdot n}{\pi} \neq 0$ (see [20, p. 36, after Cor. 4.6] and [20, Cor. 4.4]); for the last equality, we have applied the holomorphic Lefschetz trace formula [10, 3.4, p. 422] to the virtual vector bundle $1 - \Omega + 2\cdot\Lambda^2(\Omega) - \cdots + (-1)^{\dim(X)}\dim(X)\cdot\Lambda^{\dim(X)}(\Omega)$. The proof now follows from the coming lemma, the definition of $\widehat{c}^1_{\mu_n}(\cdot)$ and the fact that the Fourier transform of a constant function vanishes on odd characters. To formulate it, let σ be any element of G and let l be the corresponding element in $(\mathbf{Z}/n)^\times$. Let $k \geq 0$ and let $\omega^1_\sigma, \ldots, \omega^t_\sigma$ be a basis of $H^k_{\mathrm{Dlb}}(X)_l$ (as a Q_0-vector space), which is homogeneous for the decomposition in bidegrees. For the time of the lemma, endow $\det_{\mathbf{C}}(H^k_{\mathrm{Dlb}}(X)_{l,\mathbf{C}})$ with the exterior power metric induced by the L^2-metric on $H^k_{\mathrm{Dlb}}(X)_{l,\mathbf{C}}$ and denote the resulting norm by $|\cdot|_{L^2}$ as well. Let $\mathcal{M} := \mathcal{X}(X,g)$.

LEMMA 3.4. *The relation* $|P_\sigma(\mathcal{M}^k)| \sim_{|Q_0^\times|} (2\pi)^{\frac{\dim(X(\mathbf{C}))t}{2}}|\omega^1_\sigma \wedge \cdots \wedge \omega^t_\sigma|_{L^2}$ *holds.*

Proof. Let $\omega_\sigma := \omega^1_\sigma \wedge \cdots \wedge \omega^t_\sigma$. We know that the Hodge filtration on $H^k(X(\mathbf{C}),\mathbf{C})$ is defined over Q_0. Since the Hodge to de Rham spectral sequence degenerates, we also know that the successive quotients of this filtration are isomorphic to the spaces $H^q(X(\mathbf{C}),\Omega^p)$, where $p+q=k$, via the canonical embedding $H^q(X(\mathbf{C}),\Omega^p) \hookrightarrow H^k(X(\mathbf{C}),\mathbf{C})$. Furthermore, these isomorphisms are compatible with the Q_0-structure of $H^q(X(\mathbf{C}),\Omega^p)$ and with the Q_0-structure of the quotients of the filtration which arise from the de Rham structure of $H^k(X(\mathbf{C}),\mathbf{C})$. These facts implie that $\omega_\sigma \in \det_{\mathbf{C}}H(\mathcal{M}^k,\mathbf{C})_\sigma \simeq \det_{\mathbf{C}}(H^k_{\mathrm{Dlb}}(X)_{l,\mathbf{C}})$ is rational for the de Rham structure. Let z be a complex number such that $z\cdot\omega_\sigma$ is defined over the singular Q_0-structure of $\det_{\mathbf{C}}H(\mathcal{M}^k,\mathbf{C})_\sigma$. Since $z\cdot\omega_\sigma$ is of pure Hodge type, the construction of the Hodge metric and Lemma 2.7 show that the number $(2\pi)^{\dim(X(\mathbf{C}))t}|z\cdot\omega_\sigma|_{L^2}$ lies in $|Q_0^\times|$ and from this the result follows. □

Proof of Theorem 2. By class-field theory and Proposition 2.3, we are reduced to the case $Q = \mathbf{Q}(\mu_n)$. We may thus suppose that X is an abelian variety A with (not necessarily maximal) complex multiplication by $\mathcal{O}_{\mathbf{Q}(\mu_n)}$. In this case the sheaf of differentials Ω is equivariantly isomorphic to $f^*(H^0(A,\Omega))$. Let L be a g-equivariant ample line bundle over A and let ω_L be a real, translation-invariant $(1,1)$-form which represents the first Chern class of L in Betti cohomology. Let $\omega_A := \lambda\cdot\omega_L$, where $\lambda \in \mathbf{R}$ is chosen so that $\frac{1}{(2\pi)^{\dim(A(\mathbf{C}))}}\int_{A(\mathbf{C})}\omega_A^{\dim(A(\mathbf{C}))} = 1$. We endow $A(\mathbf{C})$ with the Kähler metric ω_A. Recall that the natural map of Q_0-algebras $\Lambda^k(H^1_{\mathrm{Dlb}}(A)) \to H^k_{\mathrm{Dlb}}(A)$ is an isomorphism. The definition of ω_Y, the definition of the L^2-metric and the fact that the harmonic forms on $A(\mathbf{C})$ are precisely the translation invariant ones, imply that this map is an isometry. Let $\overline{V} := H^1_{\mathrm{Dlb}}(\overline{A})$. By the usual Lefschetz trace formula, the number of fixed points on $A(\mathbf{C})$ of the automorphism corresponding to $\zeta \in \mathcal{O}_{\mathbf{Q}(\mu_n)}$ is $\prod_{l\in \mathbf{Z}/n}(1-\zeta^l)^{\mathrm{rk}(V_l)}$. Thus we get, by (12) and Lemma 2.10

$$-\sum_l \frac{\zeta^l}{1-\zeta^l}\widehat{\mathrm{c}}_1(\overline{V}_l) = -\frac{1}{2}R_g(\Omega^\vee) = -\frac{1}{2}\sum_l 2i(\partial/\partial s)\eta(\arg(\zeta^l),0)\mathrm{rk}((\Omega^\vee)_l).$$

Now notice that $\mathrm{Im}\frac{\zeta^l}{1-\zeta^l} = \eta(\arg(\zeta^l),0)$ and that $(\Omega^\vee)_l = (\Omega_{-l})^\vee$ and thus

$$\sum_l \eta(\arg(\zeta^l),0)\widehat{\mathrm{c}}_1(\overline{V}_l) = \sum_l (\partial/\partial s)\eta(\arg(\zeta^l),0)\mathrm{rk}(\Omega_{-l}).$$

Next we take the Fourier transform of both sides of the last equality for the action of $G = \mathrm{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q})$ and change variables from l to $-l$ on the right side of the last equality. We get:

$$\sum_l [\sum_{\sigma\in G}\eta(\arg(\sigma(\zeta)^l),0)\chi(\sigma)]\widehat{\mathrm{c}}_1(\overline{V}_l)$$
$$= -\sum_l [\sum_{\sigma\in G}(\partial/\partial s)\eta(\arg(\sigma(\zeta)^l),0)\chi(\sigma)]\mathrm{rk}(\Omega_l)$$

and by changing variables

$$-[\sum_{\sigma\in G}\eta(\arg(\sigma(\zeta)),0)\chi(\sigma)]\sum_l \overline{\chi}(l)\widehat{\mathrm{c}}_1(\overline{V}_l)$$
$$= [\sum_{\sigma\in G}(\partial/\partial s)\eta(\arg(\sigma(\zeta)),0)\chi(\sigma)]\sum_l \overline{\chi}(l)\mathrm{rk}(\Omega_l).$$

We now calculate, using Lemma 3.2 and (8),

$$\frac{[\sum_{\sigma\in G}(\partial/\partial s)\eta(\arg(\sigma(\zeta)),0)\chi(\sigma)]}{[\sum_{\sigma\in G}\eta(\arg(\sigma(\zeta)),0)\chi(\sigma)]}$$
$$= -\log(n) - \frac{1}{2}(\frac{\Gamma'(1)}{\Gamma(1)} + \frac{\Gamma'(1/2)}{\Gamma(1/2)}) + \log(\pi) - \frac{L'(\chi,1)}{L(\chi,1)}$$
$$= -\log(n) - \frac{1}{2}(\frac{\Gamma'(1)}{\Gamma(1)} + \frac{\Gamma'(1/2)}{\Gamma(1/2)}) + \log(\pi)$$
$$-\frac{L'(\chi_{\mathrm{prim}},1)}{L(\chi_{\mathrm{prim}},1)} - \sum_{p|n}\frac{\chi_{\mathrm{prim}}(p)}{p-\chi_{\mathrm{prim}}(p)}\log(p)$$
$$= -\log(n) - \frac{1}{2}(\frac{\Gamma'(1)}{\Gamma(1)} + \frac{\Gamma'(1/2)}{\Gamma(1/2)}) + \log(\pi) - \log(\frac{2\pi}{f_\chi})$$
$$+\frac{\Gamma'(1)}{\Gamma(1)} + \frac{L'(\overline{\chi}_{\mathrm{prim}},0)}{L(\overline{\chi}_{\mathrm{prim}},0)} - \sum_{p|n}\frac{\chi_{\mathrm{prim}}(p)}{p-\chi_{\mathrm{prim}}(p)}\log(p)$$
$$= \log(\frac{f_\chi}{n}) + \frac{L'(\overline{\chi}_{\mathrm{prim}},0)}{L(\overline{\chi}_{\mathrm{prim}},0)} - \sum_{p|n}\frac{\chi_{\mathrm{prim}}(p)}{p-\chi_{\mathrm{prim}}(p)}\log(p),$$

where we used the functional equation of primitive Dirichlet L-functions for the third equality. Now notice that with our choice of metric and by Lemma 3.4, we have $\widehat{c}_1(\overline{V}_l) := -\log|P_\sigma(\mathcal{M}_0^1)| + c$ for each $l \in (\mathbf{Z}/n)^\times$ and its corresponding σ, where c is independent of l. Since the Fourier transform of any constant function on $(\mathbf{Z}/n)^\times$ vanishes on any odd character, we have proved Theorem 2. □

4. The period conjecture of Gross-Deligne

In this section, we shall indicate the consequences of Theorem 1 and Theorem 2 for the period conjecture of Gross-Deligne [11, Sec. 4, p. 205]. We first recall the latter conjecture. Let Q be a finite abelian extension of $\mathbf{Q}$ and let H be a rational and homogeneous Hodge structure of dimension $[Q : \mathbf{Q}]$ and homogeneous degree r. Suppose that there is a morphism of rings $\iota : Q \hookrightarrow \mathrm{End}(H)$ (in other words, H has maximal complex multiplication by Q). Suppose also that H is embedded in the singular cohomology $H(X, \mathbf{Q})$ of a variety X defined over $\overline{\mathbf{Q}}$. We let f_Q be the conductor of Q and we choose an embedding of Q in $\mathbf{Q}(\mu_{f_Q})$ (this is possible by class-field theory). Choose an embedding $\varphi : Q \hookrightarrow \mathbf{C}$ and an isomorphism $\mathrm{Gal}(\mathbf{Q}(\mu_{f_Q})/\mathbf{Q}) \simeq (\mathbf{Z}/f_Q)^\times$. There is a natural map $\mathrm{Gal}(\mathbf{Q}(\mu_{f_Q})/\mathbf{Q}) \to \mathrm{Hom}(Q, \mathbf{C})$ given for each $\sigma \in \mathrm{Gal}(\mathbf{Q}(\mu_{f_Q})/\mathbf{Q})$ by $\varphi \circ \sigma|_Q$, and we thus obtain a map $(\mathbf{Z}/f_Q)^\times \to \mathrm{Hom}(Q, \mathbf{C})$. For each $u \in (\mathbf{Z}/f_Q)^\times$ let $\omega_u^H \in H \otimes_{\mathbf{Q}} \mathbf{C}$ be a nonvanishing element affording the embedding corresponding to u and defined over $\overline{\mathbf{Q}}$ for the de Rham $\overline{\mathbf{Q}}$-structure of $H(X, \mathbf{C})$ (such an element is well-defined up to multiplication by a nonzero algebraic number). We attach to ω_u^H a period $\mathrm{Per}(\omega_u^H) := v(\omega_u^H)$ where $v \in H^\vee$ is any (nonzero) element of the dual of the $\mathbf{Q}$-vector space H. The number $\mathrm{Per}(\omega_u^H)$ is independent of the choices of v and ω_u^H, up to multiplication by a nonzero algebraic number, and only depends on u and H. In the notation at the beginning of the introduction, with $Q_0 = \overline{\mathbf{Q}}$, we have $\mathrm{Per}(\omega_u^H) = P_u(H)$ (where we identify u with the corresponding embedding of Q after the equality sign). Let $(p(u), q(u))$ be the Hodge type of ω_u^H. By [6, Lemme 6.12], there exists a (nonunique) function $\varepsilon^H : \mathbf{Z}/f_Q \to \mathbf{Q}$ which satisfies the equation

$$p(u) = \sum_{a \in \mathbf{Z}/f_Q} \varepsilon^H(a)[u \cdot a/f_Q]$$

for all $u \in (\mathbf{Z}/f_Q)^\times$. Here $[\cdot]$ takes the fractional part. The following conjecture is formulated by Gross in [11, p. 205]; he indicates that the precise form of it was suggested to him by Deligne. This conjecture is related by Deligne to his conjecture on motives of rank 1 in [5, 8.9, p. 338].

Period conjecture. Let $u \in (\mathbf{Z}/f_Q)^\times$. The relations

$$\text{(i)} \qquad |\mathrm{Per}(\omega_u^H)| \sim_{\overline{\mathbf{Q}}^\times} \prod_{a \in \mathbf{Z}/f_Q} \Gamma(1 - \frac{a}{f_Q})^{\varepsilon^H(a/u)}$$

$$\text{(ii)} \qquad \mathrm{Per}(\omega_u^H) \sim_{\overline{\mathbf{Q}}^\times} |\mathrm{Per}(\omega_u^H)|$$

hold.

The relations (i) and (ii) can of course be condensed in the single relation $\mathrm{Per}(\omega_u^H) \sim_{\overline{\mathbf{Q}}^\times} \prod_{a \in \mathbf{Z}/f_Q} \Gamma(1 - \frac{a}{f_Q})^{\varepsilon^H(a/u)}$.

One can show that the period conjecture is independent of the choice of the function ε^H (see the appendix by Koblitz and Ogus to [5]).
In particular, the numbers

$$\prod_{a \in \mathbf{Z}/f_Q} \Gamma(1 - \frac{a}{f_Q})^{\gamma(a/u)},$$

where γ runs over all the functions $\mathbf{Z}/f_Q \to \mathbf{Q}$ satisfying the equation

$$\sum_{a \in \mathbf{Z}/f_Q} \gamma(a)[u \cdot a/f_Q] = 0$$

for all $u \in (\mathbf{Z}/f_Q)^\times$, span an algebraic extension Γ_{f_Q} of the rationals.

Suppose now that $Q = \mathbf{Q}(\mu_p)$ for a prime number p and let $Q_0 \subseteq \overline{\mathbf{Q}}$ be a field of definition of X containing Q and Γ_p. Suppose that H inherits the de Rham Q_0-structure of $H(X, \mathbf{C})$ and that its Q-vector space structure is compatible with that Q_0-structure. Suppose furthermore that all the ω_u^H are defined over Q_0 (for the de Rham Q_0-structure) and that $\mathrm{Per}(\omega_u^H) \sim_{Q_0^\times} \frac{(2\pi i)^r}{\mathrm{Per}(\omega_{-u}^H)}$ for all u.

If χ is an Artin character, we define $\mathbf{Q}(\chi)$ as the field generated over $\mathbf{Q}$ by the values of χ. We denote by E the compositum in $\mathbf{C}$ of the field $\mathbf{Q}(\mu_p)$ and of the fields $\mathbf{Q}(\chi)$ for all the odd (simple) Artin characters χ of $\mathbf{Q}(\mu_p)$, and we let $E^+ := E \cap \mathbf{R}$.

LEMMA 4.1. *If the conjecture* $\mathrm{B}(H, E, \chi)$ *holds for all the odd simple Artin characters* χ *of* $Q = \mathbf{Q}(\mu_p)$, *then the identity*

$$\log|\mathrm{Per}(\omega_u^H)| = \log|\prod_{a \in \mathbf{Z}/p} \Gamma(1 - \frac{a}{p})^{\varepsilon^H(a/u)}| + \sum_i b_{u,i} \log|a_{u,i}|$$

holds for all u. *Here* $b_{u,i} \in E^+$, $a_{u,i} \in Q_0$ *and* i *runs over a finite set of indices.*

Proof. We can plainly assume that $\varepsilon^H(0) = 0$; set $\varepsilon := \varepsilon^H$. The statement of the lemma is equivalent to the set of equalities

$$\sum_u (\log|\mathrm{Per}(\omega_u^H)| - \sum_i b_{u,i}\log|a_{u,i}|)\chi(u) = \sum_u \chi(u)\sum_a \log(\Gamma(1-a/p))\varepsilon(a/u)$$
$$= \sum_a \log(\Gamma(1-a/p))\sum_u \chi(u)\varepsilon(a/u) = \sum_a \log(\Gamma(1-a/p))\chi(a)\sum_u \chi(u)\varepsilon(1/u)$$

where χ runs over all the odd or trivial characters of the group $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ (notice that with our conventions $\chi(0) = 0$ for all characters including the trivial one). Now by the definition of ε,

$$\sum_u \chi(u)p(u) = \sum_a [a/p]\sum_u \chi(u)\varepsilon(a/u) = (\sum_a \chi(a)[a/p])(\sum_u \chi(u)\varepsilon(1/u))$$

and our set of equalities becomes

$$\sum_u (\log|\mathrm{Per}(\omega_u^H)| - \sum_i b_{u,i}\log|a_{u,i}|)\chi(u) = (\sum_a \log(\Gamma(1-a/p))\chi(a))\frac{\sum_u \chi(u)p(u)}{\sum_a \chi(a)[a/p]}. \tag{13}$$

If χ is odd, Hurwitz's formula implies that the right-hand side of (13) is equal to

$$(\frac{L'(\chi,0)}{L(\chi,0)} + \log(p))\sum_u \chi(u)p(u)$$

and in that case (13) is a consequence of $B(H,E,\chi)$. If χ is trivial, standard identities satisfied by the Γ-function and the relation $p(-u) = r - p(u)$ imply that the right-hand side of (13) is equal to

$$(\frac{p-1}{2}\log(2\pi) - \frac{1}{2}\log(p))r.$$

If we use the identity $\mathrm{Per}(\omega_u^H) \sim_{Q_0^\times} \frac{(2\pi i)^r}{\mathrm{Per}(\omega_{-u}^H)}$ to compute the sum on the left-hand side of (13), we can conclude this last case and finish the proof. □

Suppose moreover that X is acted upon by an automorphism g (defined over Q_0) of order p, and let $d_k := \dim_Q(H^k(\mathcal{X}(g), \mathbf{Q}))$ for any $k \geqslant 0$. Arising from the Künneth isomorphism $H(\times_{j=1}^{d_k} X, \mathbf{Q}) \simeq \otimes_{j=1}^{d_k} H(X, \mathbf{Q})$ and Lemma 2.1, there is a natural embedding of Hodge structures $\det_Q(H^k(\mathcal{X}(g), \mathbf{Q})) \hookrightarrow H(\times_{j=1}^{d_k} X, \mathbf{Q})$ which respects the de Rham Q_0-structures.

The next two results follow from Theorem 1 and Lemma 4.1.

COROLLARY 4.2. *Suppose that there is at most one k such that $0 \leqslant k \leqslant \dim(X)$ and $d_k \neq 0$. The Hodge structure* $\det_Q(H^k(\mathcal{X}(g), \mathbf{Q})) \subseteq H(\times_{j=1}^{d_k} X, \mathbf{Q})$

satisfies the hypothesis of the period conjecture and the identity

$$\log|\mathrm{Per}(\omega_u^{\det_Q(H^k(\mathcal{X}(g),\mathbf{Q}))})| = \log|\prod_{a\in\mathbf{Z}/p}\Gamma(1-\frac{a}{p})^{\varepsilon^{\det_Q(H^k(\mathcal{X}(g),\mathbf{Q}))}(a/u)}| + \sum_i b_{u,i}\log|a_{u,i}|$$

holds for all u, *where* $b_{u,i} \in E^+$, $a_{u,i} \in Q_0$ *and* i *runs over a finite set of indices.*

Notice that the weak Lefschetz theorem implies that the last corollary applies if X is a hypersurface of a projective space.

COROLLARY 4.3. *If* X *is a surface, the Hodge structure* $\det_Q(H^2(\mathcal{X}(g),\mathbf{Q})) \subseteq H(\times_{j=1}^{d_2}X,\mathbf{Q})$ *satisfies the hypothesis of the period conjecture and the identity*

$$\log|\mathrm{Per}(\omega_u^{\det_Q(H^2(\mathcal{X}(g),\mathbf{Q}))})| = \log|\prod_{a\in\mathbf{Z}/p}\Gamma(1-\frac{a}{p})^{\varepsilon^{\det_Q(H^2(\mathcal{X}(g),\mathbf{Q}))}(a/u)}| + \sum_i b_{u,i}\log|a_{u,i}|$$

holds for all u, *where* $b_{u,i} \in E^+$, $a_{u,i} \in Q_0$ *and* i *runs over a finite set of indices.*

Proof. The conjecture $\mathrm{B}(H^1(\mathcal{X}(g),\mathbf{Q}),E,\chi)$ is verified for all the odd simple Artin characters χ of $\mathbf{Q}(\mu_p)$ because of the existence of the Picard variety and Theorem 2. Theorem 1 and Lemma 4.1 now imply that the conjecture is verified for $H^2(\mathcal{X}(g),\mathbf{Q})$. □

Notice that when $p = 3$ in either of the last corollaries, the assertion (i) in the period conjecture holds for the Hodge structures under consideration (for all $u \in (\mathbf{Z}/p)^\times$), since $E^+ = \mathbf{Q}$ in that case.

Remark. The assertion (ii) in the period conjecture seems to be out of the reach of the techniques developed in the present paper.

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