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The space of embedded minimal surfaces of fixed genus in a 3-manifold III; Planar domains

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0. Introduction

This paper is the third in a series where we describe the space of all embedded minimal surfaces of fixed genus in a fixed (but arbitrary) closed 3-manifold. In [CM3]–[CM5] we describe the case where the surfaces are topologically disks on any fixed small scale. Although the focus of this paper, general planar domains, is more in line with [CM6], we will prove a result here (namely, Corollary III.3.5 below) which is needed in [CM5] even for the case of disks. Roughly speaking, there are two main themes in this paper. The first is that stability leads to improved curvature estimates. This allows us to find large graphical regions. These graphical regions lead to two possibilities:

- Either they "close up" to form a graph,
- Or a multi-valued graph forms.

The second theme is that in certain important cases we can rule out the formation of multi-valued graphs, i.e., we can show that only the first possibility can arise. The techniques that we develop here apply both to general planar domains and to certain topological annuli in an embedded minimal disk; the latter is used in [CM5]. The current paper is third in the series since the techniques here are needed for our main results on disks.

The above hopefully gives a rough idea of the present paper. To describe these results more precisely and explain in more detail why and how they are needed for our results on disks, we will need to briefly outline those arguments. There are two local models for embedded minimal disks (by an embedded disk, we mean a smooth injective map from the closed unit ball in $\mathbf{R}^2$

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into $\mathbf{R}^3$). One model is the plane (or, more generally, a minimal graph), the other is a piece of a helicoid. In the first four papers of this series, we will show that every embedded minimal disk is either a graph of a function or is a double spiral staircase where each staircase is a multi-valued graph. This will be done by showing that if the curvature is large at some point (and hence the surface is not a graph), then it is a double spiral staircase. To prove that such a disk is a double spiral staircase, we will first prove that it can be decomposed into N-valued graphs where N is a fixed number. This was initiated in [CM3] and a version of it was completed in [CM4]. *To get the version needed in* [CM5], *we need one result that will be proved here*, *namely Corollary* III.3.5. This result asserts that in an embedded minimal disk, then above and below any given multi-valued graph, there are points of large curvature and thus, by the results of [CM3], [CM4], there are other multi-valued graphs both above and below the given one. Iterating this gives the decomposition of such a disk into multi-valued graphs. The fourth paper of this series will deal with how the multi-valued graphs fit together and, in particular, prove regularity of the set of points of large curvature – the axis of the double spiral staircase.

To describe general planar domains (in [CM6]) we need in addition to the results of [CM3]–[CM5] a key estimate for embedded *stable* annuli which is the main result of this paper (see Theorem 0.3 below). This estimate asserts that such an annulus is a graph away from its boundary if it has only one interior boundary component and if this component lies in a small (extrinsic) ball.

Planar domains arise when one studies convergence of embedded minimal surfaces of a fixed genus in a fixed 3-manifold. This is due to the next theorem which loosely speaking asserts that any sequence of embedded minimal surfaces of fixed genus has a subsequence which consists of uniformly planar domains away from finitely many points. (In fact, this describes only "(1)" and "(2)" of Theorem 0.1. Case "(3)" is self explanatory and "(4)" very roughly corresponds to whether the surface locally "looks like" the genus one helicoid; cf. [HoKrWe], or has "more than one end.")

Before stating the next theorem about embedded minimal surfaces of a given fixed genus, it may be in order to recall what the genus is for a surface with boundary. Given a surface Σ with boundary $\partial\Sigma$, the *genus* of Σ ($\text{gen}(\Sigma)$) is the genus of the closed surface $\hat{\Sigma}$ obtained by adding a disk to each boundary circle. The genus of a union of disjoint surfaces is the sum of the genuses. Therefore, a surface with boundary has nonnegative genus; the genus is zero if and only if it is a planar domain. For example, the disk and the annulus are both genus zero; on the other hand, a closed surface of genus g with k disks removed has genus g.

In the next theorem, M^3 will be a closed 3-manifold and Σ_i^2 a sequence of closed embedded oriented minimal surfaces in M with fixed genus g.

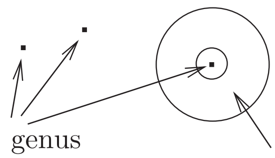


Figure 1: (1) and (2) of Theorem 0.1: Any sequence of genus g surfaces has a subsequence for which the genus concentrates at at most g points. Away from these points, the surfaces are locally planar domains.

THEOREM 0.1 (see Figure 1). *There exist $x_1, \ldots, x_m \in M$ with $m \leq g$ and a subsequence Σ_j so that the following hold*:

(1) *For $x \in M \setminus \{x_1, \ldots, x_m\}$, there are $j_x, r_x > 0$ so that for $j > j_x$,*

$$\text{gen}(B_{r_x}(x) \cap \Sigma_j) = 0 \, .$$

(2) *For each x_k, there are $\ell_k, r_k > 0, r_k > r_{k,j} \to 0$ so that for all j there are components $\{\Sigma^\ell_{k,j}\}_{\ell \leq \ell_k}$ of $B_{r_k}(x_k) \cap \Sigma_j$ with*

$$\text{gen}(B_{r_k}(x_k) \cap \Sigma_j) = \sum_{\ell \leq \ell_k} \text{gen}(\Sigma^\ell_{k,j}) \leq g \, ,$$
$$\text{gen}(B_{r_{k,j}}(x_k) \cap \Sigma^\ell_{k,j}) = \text{gen}(\Sigma^\ell_{k,j}) \textit{ for } \ell \leq \ell_k \, .$$

(3) *For every k, ℓ, j, there is only one component $\tilde{\Sigma}^\ell_{k,j}$ of $B_{r_{k,j}}(x_k) \cap \Sigma^\ell_{k,j}$ with genus > 0.*

(4) *For each k, ℓ, either $\partial \Sigma^\ell_{k,j}$ is connected or a component of $\partial \tilde{\Sigma}^\ell_{k,j}$ separates two components of $\partial \Sigma^\ell_{k,j}$.*

To explain why the next two theorems are crucial for what we call "the pairs of pants decomposition" of embedded minimal planar domains, recall the following prime examples of such domains: Minimal graphs (over disks), a helicoid, a catenoid or one of the Riemann examples. (Note that the first two are topologically disks and the others are disks with one or more subdisks removed.) Let us describe the nonsimply connected examples in a little more detail. The catenoid (see Figure 2) is the (topological) annulus

$$(\cosh s \, \cos t, \cosh s \, \sin t, s) \tag{0.2}$$

where $s, t \in \mathbf{R}$. To describe the Riemann examples, think of a catenoid as roughly being obtained by connecting two parallel planes by a neck. Loosely speaking (see Figure 3), the Riemann examples are given by connecting (infinitely many) parallel planes by necks; each adjacent pair of planes is connected by exactly one neck. In addition, all of the necks are lined up along an

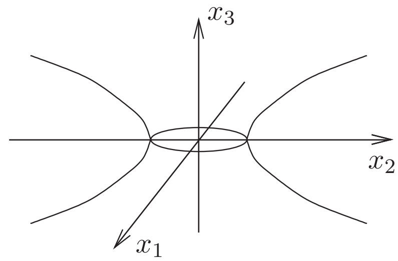


Figure 2: The catenoid given by revolving $x_1 = \cosh x_3$ around the x_3-axis.

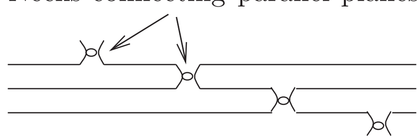


Figure 3: The Riemann examples: Parallel planes connected by necks.

axis and the separation between each pair of adjacent ends is constant (in fact the surfaces are periodic). Locally, one can imagine connecting $\ell - 1$ planes by $\ell - 2$ necks and add half of a catenoid to each of the two outermost planes, possibly with some restriction on how the necks line up and on the separation of the planes; see [FrMe], [Ka], [LoRo].

To illustrate how Theorem 0.3 below will be used in [CM6] where we give the actual "pair of pants decomposition" observe that the catenoid can be decomposed into two minimal annuli each with one exterior convex boundary and one interior boundary which is a short simple closed geodesic. (See also [CM9] for the "pair of pants decomposition" in the special case of annuli.) In the case of the Riemann examples (see Figure 4), there will be a number of "pairs of pants", that is, topological disks with two subdisks removed. Metrically these "pairs of pants" have one convex outer boundary and two interior boundaries each of which is a simple closed geodesic. Note also that this decomposition can be made by putting in minimal graphical annuli in the complement of the domains (in $\mathbf{R}^3$) which separate each of the pieces; cf. Corollary 0.4 below. Moreover, after the decomposition is made then every intersection of one of the "pairs of pants" with an extrinsic ball away from the interior boundaries is simply connected and hence the results of [CM3]–[CM5] apply there.

The next theorem is a kind of effective removable singularity theorem for embedded stable minimal surfaces with small interior boundaries. It asserts that embedded stable minimal surfaces with small interior boundaries are graphical away from the boundary. Here small means contained in a small ball

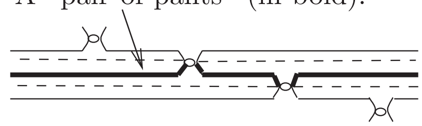


Figure 4: Decomposing the Riemann examples into "pair of pants" by cutting along small curves; these curves bound minimal graphical annuli separating the ends.

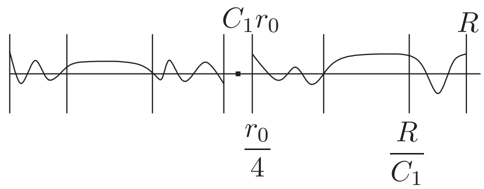


Figure 5: Theorem 0.3: Embedded stable annuli with small interior boundary are graphical away from their boundary.

in $\mathbf{R}^3$ (and not that the interior boundary has small length). This distinction is important; in particular if one had a bound for the area of a tubular neighborhood of the interior boundary, then Theorem 0.3 would follow easily; see Corollary II.1.34 and cf. [Fi].

THEOREM 0.3 (see Figure 5). *Given $\tau > 0$, there exists $C_1 > 1$, so that if $\Gamma \subset B_R \subset \mathbf{R}^3$ is an embedded stable minimal annulus with $\partial\Gamma \subset \partial B_R \cup B_{r_0/4}$ (for $C_1^2\, r_0 < R$) and $B_{r_0} \cap \partial\Gamma$ is connected, then each component of $B_{R/C_1} \cap \Gamma \setminus B_{C_1\, r_0}$ is a graph with gradient $\leq \tau$.*

Many of the results of this paper will involve either graphs or multi-valued graphs. Graphs will always be assumed to be single-valued over a domain in the plane (as is the case in Theorem 0.3).

Combining Theorem 0.3 with the solution of a Plateau problem of Meeks-Yau (proven initially for convex domains in Theorem 5 of [MeYa1] and extended to mean convex domains in [MeYa2]), we get (the result of Meeks-Yau gives the existence of Γ below):

COROLLARY 0.4 (see Figure 6). *Given $\tau > 0$, there exists $C_1 > 1$, so that the following holds*:

Let $\Sigma \subset B_R \subset \mathbf{R}^3$ with $\partial\Sigma \subset \partial B_R$ be an embedded minimal surface with $\text{gen}(\Sigma) = \text{gen}(B_{r_1} \cap \Sigma)$ *and let Ω be a component of $B_R \setminus \Sigma$.*

If $\gamma \subset B_{r_0} \cap \Sigma \setminus B_{r_1}$ is noncontractible and homologous in $\Sigma \setminus B_{r_1}$ to a component of $\partial\Sigma$ and $r_0 > r_1$, then a component $\hat{\Sigma}$ of $\Sigma \setminus \gamma$ is an annulus and there is a stable embedded minimal annulus $\Gamma \subset \Omega$ with $\partial\Gamma = \partial\hat{\Sigma}$.

Moreover, each component of $(B_{R/C_1} \setminus B_{C_1 r_0}) \cap \Gamma$ is a graph with gradient $\leq \tau$.

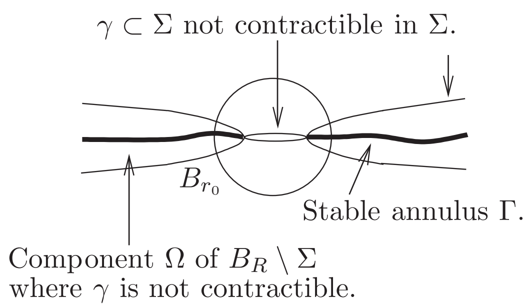


Figure 6: Corollary 0.4: Solving a Plateau problem gives a stable graphical annulus separating the boundary components of an embedded minimal annulus.

Stability of Γ in Theorem 0.3 is used in two ways: To get a pointwise curvature bound on Γ and to show that certain sectors have small curvature. In Section 2 of [CM4], we showed that a pointwise curvature bound allows us to decompose an embedded minimal surface into a set of bounded area and a collection of (almost stable) sectors with small curvature. Using this, we see that the proof of Theorem 0.3 will also give (if $0 \in \Sigma$, then $\Sigma_{0,t}$ denotes the component of $B_t \cap \Sigma$ containing 0):

Theorem 0.5. *Given C, there exist $C_2, C_3 > 1$, so that the following holds*:

Let $0 \in \Sigma \subset B_R \subset \mathbf{R}^3$ be an embedded minimal surface with connected $\partial\Sigma \subset \partial B_R$. If $\text{gen}(\Sigma_{0,r_0}) = \text{gen}(\Sigma)$, $r_0 \leq R/C_2$, *and*

$$\sup_{\Sigma \setminus B_{r_0}} |x|^2 \, |A|^2(x) \leq C \, , \tag{0.6}$$

then

$$\text{Area}(\Sigma_{0,r_0}) \leq C_3 \, r_0^2 \, .$$

The examples constructed in [CM13] show that the quadratic curvature bound (0.6) is necessary to get the area bound in Theorem 0.5.

In [CM5] a strengthening of Theorem 0.5 (this strengthening is Theorem III.3.1 below) will be used to show that, for limits of a degenerating sequence of embedded minimal disks, points where the curvatures blow up are not isolated. This will eventually give (Theorem 0.1 of [CM5]) that for a subsequence such points form a Lipschitz curve which is infinite in two directions and transversal to the limit leaves; compare with the example given by a sequence of rescaled helicoids where the singular set is a single vertical line perpendicular to the horizontal limit foliation.

To describe a neighborhood of each of the finitely many points, coming from Theorem 0.1, where the genus concentrates (specifically to describe when there is one component $\tilde{\Sigma}^{\ell}_{k,j}$ of genus > 0 in "(3)" of Theorem 0.1), we will need in [CM6]:

Corollary 0.7. *Given C, g, there exist C_4, C_5 so that the following holds:*
Let $0 \in \Sigma \subset B_R \subset \mathbf{R}^3$ be an embedded minimal surface with connected $\partial\Sigma \subset \partial B_R$, $r_0 < R/C_4$, and $\text{gen}(\Sigma_{0,r_0}) = \text{gen}(\Sigma) \leq g$. If

$$\sup_{\Sigma \setminus B_{r_0}} |x|^2 \, |A|^2(x) \leq C \,, \tag{0.8}$$

then

$$\Sigma \textit{ is a disk and } \Sigma_{0,R/C_5} \textit{ is a graph with gradient } \leq 1.$$

This corollary follows directly by combining Theorem 0.5 and theorem 1.22 of [CM4]. That is, we note first that for $r_0 \leq s \leq R$, it follows from the maximum principle (since Σ is minimal) and Corollary I.0.11 that $\partial\Sigma_{0,s}$ is connected and $\Sigma \setminus \Sigma_{0,s}$ is an annulus. Second, theorem 0.5 bounds $\text{Area}(\Sigma_{0,R/C_2})$ and Theorem 1.22 of [CM4] then gives the corollary.

Theorems 0.3, 0.5 and Corollary 0.7 are local and are for simplicity stated and proved only in $\mathbf{R}^3$ although they can with only very minor changes easily be seen to hold for minimal planar domains in a sufficiently small ball in any given fixed Riemannian 3-manifold.

Throughout Σ, $\Gamma \subset M^3$ will denote complete minimal surfaces possibly with boundary, sectional curvatures K_Σ, K_Γ, and second fundamental forms A_Σ, A_Γ. Also, Γ will be assumed to be stable and have trivial normal bundle. Given $x \in M$, $B_s(x)$ will be the usual ball in $\mathbf{R}^3$ with radius s and center x. Likewise, if $x \in \Sigma$, then $\mathcal{B}_s(x)$ is the intrinsic ball in Σ. Given $S \subset \Sigma$ and $t > 0$, let $\mathcal{T}_t(S, \Sigma) \subset \Sigma$ be the intrinsic tubular neighborhood of S in Σ with radius t and set

$$\mathcal{T}_{s,t}(S, \Sigma) = \mathcal{T}_t(S, \Sigma) \setminus \mathcal{T}_s(S, \Sigma) \,.$$

Unless explicitly stated otherwise, all geodesics will be parametrized by arclength.

We will often consider the intersections of various curves and surfaces with extrinsic balls. We will always assume that these intersections are transverse since this can be achieved by an arbitrarily small perturbation of the radius.

I. Topological decomposition of surfaces

In this part we will first collect some simple facts and results about planar domains and domains that are planar outside a small ball. These results will then be used to show Theorem 0.1. First we recall an elementary lemma:

LEMMA I.0.9 (see Figure 7). *Let Σ be a closed oriented surface (i.e., $\partial\Sigma = \emptyset$) with genus g. There are transverse simple closed curves $\eta_1, \dots, \eta_{2g} \subset \Sigma$ so that for $i < j$*

$$\#\{p \,|\, p \in \eta_i \cap \eta_j\} = \delta_{i+g,j}\,. \tag{I.0.10}$$

Furthermore, for any such $\{\eta_i\}$, if $\eta \subset \Sigma \setminus \cup_i \eta_i$ is a closed curve, then η divides Σ.

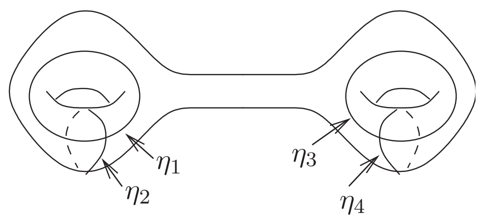


Figure 7: Lemma I.0.9: A basis for homology on a surface of genus g.

Recall that if $\partial\Sigma \neq \emptyset$, then $\hat{\Sigma}$ is the surface obtained by replacing each circle in $\partial\Sigma$ with a disk. Note that a closed curve $\eta \subset \Sigma$ divides Σ if and only if η is homologically trivial in $\hat{\Sigma}$.

COROLLARY I.0.11. *If $\Sigma_1 \subset \Sigma$ and* $\text{gen}(\Sigma_1) = \text{gen}(\Sigma)$, *then each simple closed curve $\eta \subset \Sigma \setminus \Sigma_1$ divides Σ.*

Proof. Since Σ_1 has genus $g = \text{gen}(\Sigma)$, Lemma I.0.9 gives transverse simple closed curves $\eta_1, \dots, \eta_{2g} \subset \Sigma_1$ satisfying (I.0.10). However, since η does not intersect any of the η_i's, Lemma I.0.9 implies that η divides Σ. □

COROLLARY I.0.12. *If Σ has a decomposition $\Sigma = \cup_{\beta=1}^{\ell} \Sigma_\beta$ where the union is taken over the boundaries and each Σ_β is a surface with boundary consisting of a number of disjoint circles, then*

$$\sum_{\beta=1}^{\ell} \text{gen}(\Sigma_\beta) \leq \text{gen}(\Sigma)\,. \tag{I.0.13}$$

Proof. Set $g_\beta = \text{gen}(\Sigma_\beta)$. Lemma I.0.9, gives transverse simple closed curves

$$\eta_1^\beta, \dots, \eta_{2g_\beta}^\beta \subset \Sigma_\beta$$

satisfying (I.0.10). Since $\Sigma_{\beta_1} \cap \Sigma_{\beta_2} = \emptyset$ for $\beta_1 \neq \beta_2$, this implies that the rank of the intersection form on the first homology (mod 2) of $\hat{\Sigma}$ is $\geq 2\sum_{\beta=1}^{\ell} g_\beta$. In particular, we get (I.0.13). □

In the next lemma, M^3 will be a closed 3-manifold and Σ_i^2 a sequence of closed embedded oriented minimal surfaces in M with fixed genus g.

LEMMA I.0.14. *There exist $x_1, \dots, x_m \in M$ with $m \leq g$ and a subsequence Σ_j so that the following hold*:

- *For $x \in M \setminus \{x_1, \dots, x_m\}$, there exist $j_x, r_x > 0$ so that* $\text{gen}(B_{r_x}(x) \cap \Sigma_j) = 0$ *for $j > j_x$.*
- *For each x_k, there exist $R_k, g_k > 0, R_k > R_{k,j} \to 0$ so that $\sum_{k=1}^m g_k \leq g$ and for all j,*

$$\text{gen}(B_{R_k}(x_k) \cap \Sigma_j) = g_k = \text{gen}(B_{R_{k,j}}(x_k) \cap \Sigma_j)\,.$$

Proof. Suppose that for some $x_1 \in M$ and any $R_1 > 0$ we have infinitely many i's where

$$\text{gen}(B_{R_1}(x_1) \cap \Sigma_i) = g_{1,i} > 0\,.$$

By Corollary I.0.12, we have $g_{1,i} \leq g$ and hence there is a subsequence Σ_j and a sequence $R_{1,j} \to 0$ so that for all j

$$\text{gen}(B_{R_{1,j}}(x_1) \cap \Sigma_j) = g_1 > 0\,. \tag{I.0.15}$$

By repeating this construction, we can suppose that there are disjoint points $x_1, \dots, x_m \in M$ and $R_{k,j} > 0$ so that for any k we have $R_{k,j} \to 0$ and

$$\text{gen}(B_{R_{k,j}}(x_k) \cap \Sigma_j) = g_k > 0\,.$$

However, Corollary I.0.12 implies that for j sufficiently large

(I.0.16)

$$0 \leq \text{gen}(\Sigma_j \setminus \cup_k B_{R_{k,j}}(x_k)) \leq \text{gen}(\Sigma_j) - \sum_{k=1}^m \text{gen}(B_{R_{k,j}}(x_k) \cap \Sigma_j) \leq g - \sum_{k=1}^m g_k\,.$$

In particular, $\sum_{k=1}^m g_k \leq g$ and we can therefore assume that $\sum_{k=1}^m g_k$ is maximal. This has two consequences:

- First, given $x \in M \setminus \{x_1, \dots, x_m\}$, there exist $r_x > 0$ and j_x so that $\text{gen}(B_{r_x}(x) \cap \Sigma_j) = 0$ for $j > j_x$.
- Second, for each x_k, there exist $R_k > 0$ and j_k so that $\text{gen}(B_{R_k}(x_k) \cap \Sigma_j) = g_k$ for $j > j_k$.

The lemma now follows easily. □

By Corollary I.0.12, each $R_k, R_{k,j}$ from Lemma I.0.14 can (after going to a further subsequence) be replaced by any $R'_k, R'_{k,j}$ with $R'_k \leq R_k$ and $R'_{k,j} \geq R_{k,j}$. Similarly, each r_x can be replaced by any $r'_x \leq r_x$. This will be used freely in the proof of Theorem 0.1 below.

Proof of Theorem 0.1. Let $x_k, g_k, R_k, R_{k,j}$ and r_x be from Lemma I.0.14. We can assume that each $R_k > 0$ is sufficiently small so that $B_{R_k}(x_k)$ is essentially Euclidean (e.g., $R_k < \min\{i_0/4, \pi/(4k^{1/2})\}$). Part (1) follows directly from Lemma I.0.14.

For each x_k, we can assume that there are ℓ_k and $n_{\ell,k}$ so that:

- $B_{R_k}(x_k) \cap \Sigma_j$ has components $\{\Sigma^\ell_{k,j}\}_{1 \leq \ell \leq \ell_k}$ with genus > 0.
- $B_{R_{k,j}}(x_k) \cap \Sigma^\ell_{k,j}$ has $n_{\ell,k}$ components with genus > 0.

We will use repeatedly that, by (1) and Corollary I.0.12, $n_{\ell,k}$ is nonincreasing if either $R_{k,j}$ increases or R_k decreases. For each ℓ, k with $n_{\ell,k} > 1$, set

$$\rho^\ell_{k,j} = \inf\{\rho > R_{k,j} \,|\, \#\{\text{components of } B_\rho(x_k) \cap \Sigma^\ell_{k,j}\} < n_{\ell,k}\}\,. \tag{I.0.17}$$

There are two cases. If $\liminf_{j\to\infty} \rho^\ell_{k,j} = 0$, then choose a subsequence Σ_j with $\rho^\ell_{k,j} \to 0$; $n_{\ell,k}$ decreases if we replace $R_{k,j}$ with any $R'_{k,j} > \rho^\ell_{k,j}$. Otherwise, set $2\,\rho^\ell_k = \liminf_{j\to\infty} \rho^\ell_{k,j} > 0$ and choose a subsequence Σ_j so that $\rho^\ell_{k,j} < \rho^\ell_k$; ℓ_k increases if we replace R_k with any $R'_k \leq \rho^\ell_k$. In either case, $\sum_{\ell,k}(n_{\ell,k} - 1)$ decreases. Since $\sum_{\ell,k} n_{\ell,k} \leq g$ (by Corollary I.0.12), repeating this $\leq g$ times gives

$$0 < R'_k \leq R_k \text{ and } R_{k,j} \leq R'_{k,j} \to 0 \text{ (as } j \to \infty)$$

as well as a subsequence so that only one component $\tilde{\Sigma}^\ell_{k,j}$ of $B_{R'_{k,j}}(x_k) \cap \Sigma^\ell_{k,j}$ has genus > 0 (i.e., each new $n_{\ell,k} = 1$). By Corollary I.0.12 (and (1)) and the remarks before the proof, Parts (1), (2), and (3) now hold for any $r_k \leq R'_k$ and $R'_{k,j} \leq r_{k,j} \to 0$.

Suppose that for some k, ℓ there exists $j_{k,\ell}$ so that $\partial \Sigma^\ell_{k,j}$ has at least two components for all $j > j_{k,\ell}$. For $R'_{k,j} \leq t \leq R'_k$, let $\Sigma^\ell_{k,j}(t)$ be the component of $B_t(x_k) \cap \Sigma$ containing $\tilde{\Sigma}^\ell_{k,j}$. Set

$$r^\ell_{k,j} = \inf\{t > R_{k,j} \,|\, \#\{\text{components of } \partial \Sigma^\ell_{k,j}(t)\} > 1\}\,. \tag{I.0.18}$$

There are two cases:

- If $\liminf_{j\to\infty} r^\ell_{k,j} = 0$, then choose a subsequence Σ_j with $r^\ell_{k,j} \to 0$. By the maximum principle (since Σ is minimal) and Corollary I.0.11, a component of (the new) $\partial \tilde{\Sigma}^\ell_{k,j}$ separates two components of $\partial \Sigma^\ell_{k,j}$ for any $r_{k,j} \to 0$ with $r_{k,j} > r^\ell_{k,j}$.
- On the other hand, if $\liminf_{j\to\infty} r^\ell_{k,j} = 2r^\ell_k > 0$, then choose a subsequence so that (the new) $\partial \Sigma^\ell_{k,j}$ is connected for any $r_k \leq r^\ell_k$.

After repeating this $\leq g$ times (each time either increasing $R'_{k,j}$ or decreasing R'_k), Part (4) also holds. □

In [CM6] we will need the following (here, and elsewhere, if $0 \in \Sigma \subset \mathbf{R}^3$, then $\Sigma_{0,t}$ denotes the component of $B_t \cap \Sigma$ containing 0):

Proposition I.0.19. *Let $0 \in \Sigma_i \subset B_{S_i} \subset \mathbf{R}^3$ with $\partial \Sigma_i \subset \partial B_{S_i}$ be a sequence of embedded minimal surfaces with genus $\leq g < \infty$ and $S_i \to \infty$. After going to a subsequence, Σ_j, and possibly replacing S_j by R_j and Σ_j by Σ_{0,j,R_j} where $R_0 \leq R_j \leq S_j$ and $R_j \to \infty$, then*

$$\mathrm{gen}(\Sigma_{j,0,R_0}) = \mathrm{gen}(\Sigma_j) \leq g$$

and either (a) *or* (b) *holds*:

(a) *$\partial \Sigma_{j,0,t}$ is connected for all $R_0 \leq t \leq R_j$.*

(b) *$\partial \Sigma_{j,0,R_0}$ is disconnected.*

Proof. We will first show that there exists $R_0 > 0$, a subsequence Σ_j, and a sequence $R_j \to \infty$ with $R \leq R_j \leq S_j$, such that (after replacing Σ_j by $\Sigma_{j,0,R_j}$)

$$\mathrm{gen}(\Sigma_{j,0,R_0}) = \mathrm{gen}(\Sigma_j) \leq g\,.$$

Suppose not; it follows easily from the monotonicity of the genus (i.e., Corollary I.0.12) that there exists a subsequence Σ_j and a sequence $G_k \to \infty$ such that for all k there exists a j_k so that for $j \geq j_k$

$$g \geq \mathrm{gen}(\Sigma_{j,0,G_{k+1}}) > \mathrm{gen}(\Sigma_{j,0,G_k})\,, \tag{I.0.20}$$

which is a contradiction.

For each j, let $R_{0,j}$ be the infimum of R with $R_0 \leq R \leq R_j$ where $\partial \Sigma_{j,0,R}$ is disconnected; set $R_{0,j} = R_j$ if no such exists. There are now two cases:

- If $\liminf R_{0,j} < \infty$, then, after going to a subsequence and replacing R_0 by $\liminf R_{0,j}+1$, we are in (b) by the maximum principle.
- If $\liminf R_{0,j} = \infty$, then we are in (a) after replacing R_j by $R_{0,j}$. □

II. Estimates for stable minimal surfaces with small interior boundaries

In this part we prove Theorem 0.3. That is, we will show that all embedded stable minimal surfaces with small interior boundaries are graphical away from the boundary. Here *small* means contained in a small ball in $\mathbf{R}^3$ (and not that the interior boundary has small length).

II.1. Long stable sectors contain multi-valued graphs

In [CM3], [CM4] we proved estimates for the total curvature and area of stable sectors. A stable sector in the sense of [CM3], [CM4] is a stable subset of a minimal surface given as half of a normal tubular neighborhood (in the surface) of a strictly convex curve. For instance, a curve lying in the boundary of an intrinsic ball is strictly convex. In this section we give similar estimates for half of normal tubular neighborhoods of curves lying in the intersection of the surface and the boundary of an extrinsic ball. These domains arise naturally in our main result and are unfortunately somewhat more complicated to deal with due to the lack of convexity of the curves.

In this section, the surfaces Σ and Γ will be planar domains and, hence, simple closed curves will divide the surface into two planar (sub)domains.

We will need some notation for multi-valued graphs. Let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbf{C} \setminus \{0\}$ with global (polar) coordinates (ρ, θ) and set

$$S_{r,s}^{\theta_1,\theta_2} = \{r \leq \rho \leq s \, , \, \theta_1 \leq \theta \leq \theta_2\} \, .$$

An N-valued graph Σ of a function u over the annulus $D_s \setminus D_r$ (see Figure 8) is a (single-valued) graph (of u) over $S_{r,s}^{-N\,\pi,N\,\pi}$ ($\Sigma_{r,s}^{\theta_1,\theta_2}$ will denote the subgraph of Σ over $S_{r,s}^{\theta_1,\theta_2}$). The separation $w(\rho,\theta)$ between consecutive sheets is (see Figure 8)

$$w(\rho,\theta) = u(\rho,\theta + 2\pi) - u(\rho,\theta) \, . \tag{II.1.1}$$

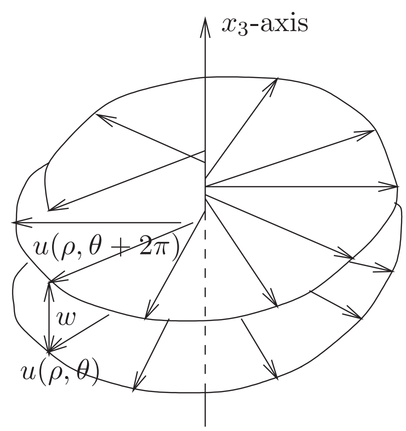


Figure 8: The separation w for a multi-valued graph in (II.1.1).

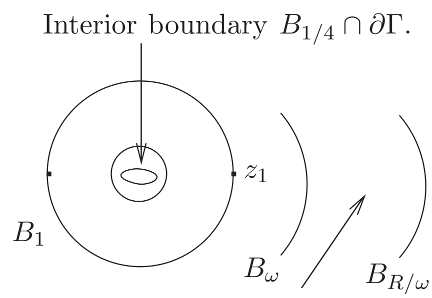
The main result of the next two sections is the following theorem ($\Gamma_1(\partial)$ is the component of $B_1 \cap \Gamma$ containing $B_1 \cap \partial\Gamma$):

THEOREM II.1.2 (see Figure 9). *Given $N, \tau > 0$, there exist $\omega > 1$, d_0 so that the following holds*:

Let Γ be a stable embedded minimal annulus with $\partial\Gamma \subset B_{1/4} \cup \partial B_R$, $B_{1/4} \cap \partial\Gamma$ connected, and $R > \omega^2$. Given a point $z_1 \in \partial B_1 \cap \partial\Gamma_1(\partial)$, then (*after a rotation of* $\mathbf{R}^3$) *either* (1) *or* (2) *below holds*:

(1) *Each component of $B_{R/\omega} \cap \Gamma \setminus B_\omega$ is a graph with gradient $\leq \tau$.*

(2) *Γ contains a graph $\Gamma^{-N\pi,N\pi}_{\omega,R/\omega}$ with gradient $\leq \tau$ and* $\mathrm{dist}_{\Gamma\setminus\Gamma_1(\partial)}(z_1, \Gamma^{0,0}_{\omega,\omega}) < d_0$.



Γ contains a large "flat region" between B_ω and $B_{R/\omega}$. Since Γ is embedded, this either (1) closes up to give a graphical annulus or (2) spirals to give an N-valued graph.

Figure 9: Theorem II.1.2: Embedded stable annuli with small interior boundary contain either: (1) a graphical annulus, or (2) an N-valued graph away from its boundary.

Note that if Γ is as in Theorem II.1.2 and one component of $B_{R/\omega} \cap \Gamma \setminus B_\omega$ contains a graph over $D_{R/(2\omega)} \setminus D_{2\omega}$ with gradient ≤ 1, then every component of

$$B_{R/(C\omega)} \cap \Gamma \setminus B_{C\omega}$$

is a graph for some $C > 1$. Namely, embeddedness and the gradient estimate (which applies because of stability) would force any nongraphical component to spiral indefinitely, contradicting that Γ is compact. Thus it is enough to find one component that is a graph. This will be used below.

We will eventually show in Section II.3 that (2) in Theorem II.1.2 does not happen; thus every component is a (single-valued) graph. This will easily give Theorem 0.3.

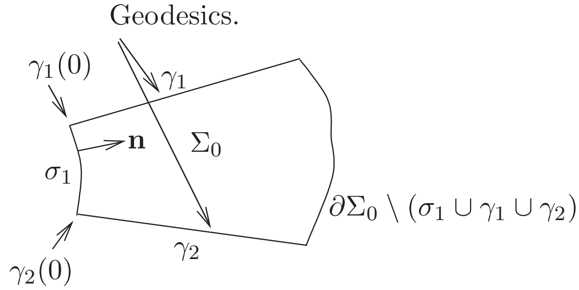


Figure 10: The subdomain $\Sigma_0 \subset \Sigma$ in Lemma II.1.3 and below.

See Figure 10. Throughout this section (except in Corollary II.1.34):

- $\Sigma \subset \mathbf{R}^3$ will be an embedded minimal planar domain (if the domain is stable, then we use Γ instead of Σ).
- $\Sigma_0 \subset \Sigma$ will be a subdomain.
- γ_1, γ_2, $\sigma_1 \subset \partial\Sigma_0$ will be curves (γ_1, γ_2 geodesics) so that $\gamma_1 \cup \gamma_2 \cup \sigma_1$ is a simple curve and $\gamma_i(0) \in \sigma_1$.

(By a geodesic we will mean a curve with zero geodesic curvature. This definition of geodesic is needed when the curve intersects the boundary of the surface.) Below we will sometimes require one or more of the following properties:

(A) $\text{dist}_\Sigma(\gamma_i(t), \sigma_1) \geq t - C_0$ for $0 \leq t \leq \text{Length}(\gamma_i)$.

(B) $\partial_{\mathbf{n}}|x| \geq 0$ along σ_1 (where $\mathbf{n}$ is the inward normal to $\partial\Sigma_0$).

(C) $\gamma_1 \perp \sigma_1$, $\gamma_2 \perp \sigma_1$ (i.e., angle $\pi/2$).

(D) $\text{dist}_{\Sigma_0}(\sigma_1, \partial\Sigma_0 \setminus (\sigma_1 \cup \gamma_1 \cup \gamma_2)) \geq \ell$ (thus $\ell \leq \text{Length}(\gamma_i)$).

Note that if $\sigma_1 \subset \partial B_1$ (and Σ_0 is leaving B_1 along σ_1), then (B) is automatically satisfied.

The main component of the proof of Theorem II.1.2 is Proposition II.1.20 below which shows that certain stable sectors have subsectors with small total curvature. To show this, we will use an argument in the spirit of [CM2], [CM4] to get good curvature estimates for our nonstandard stable domains. As in [CM2], [CM4], to estimate the total curvature we show first an area bound. That is, we being with the following lemma (here k_g is the geodesic curvature of σ_1):

Lemma II.1.3. *Let $\Gamma_0 = \Gamma \subset \mathbf{R}^3$ be stable and satisfy* (A) *for $C_0 = 0$,* (C), (D). *If $0 \leq \chi \leq 1$ is a function on Γ_0 which vanishes on each γ_i, then for $1 < R < \ell$*

$$\text{(II.1.4)} \qquad \operatorname{Area}(\mathcal{T}_R(\sigma_1,\Gamma_0)) \leq C\,R^2 \int_{\sigma_1} |k_g| + C\,R\,\operatorname{Length}(\sigma_1) + C\,R^2 \left(\int_{\mathcal{T}_1(\sigma_1,\Gamma_0)} (1+|A|^2) + \int_{\mathcal{T}_R(\sigma_1,\Gamma_0)} |\nabla\chi|^2 + \int_{\mathcal{T}_R(\sigma_1,\Gamma_0)\cap\{\chi<1\}} |A|^2 \right) .$$

Proof. Set $\mathcal{T}_{s,t} = \mathcal{T}_{s,t}(\sigma_1,\Gamma_0)$ and $\mathrm{r} = \operatorname{dist}_\Gamma(\sigma_1,\cdot)$. Define a (radial) cut-off function ϕ by

$$\text{(II.1.5)} \qquad \phi = \begin{cases} \mathrm{r} & \text{on } \mathcal{T}_1\,, \\ (R-\mathrm{r})/(R-1) & \text{on } \mathcal{T}_{1,R}\,, \\ 0 & \text{otherwise}\ . \end{cases}$$

By the stability inequality applied to $\phi\,\chi$ and the inequality, $2ab \leq a^2+b^2$,

$$\text{(II.1.6)} \qquad \int_{\mathcal{T}_{1,R}} |A|^2\,[(R-\mathrm{r})/(R-1)]^2 \leq \int |A|^2\,\phi^2 \leq 2\int |\nabla\phi|^2 + 2\int_{\mathcal{T}_R} |\nabla\chi|^2 + \int_{\mathcal{T}_R\cap\{\chi<1\}} |A|^2$$
$$\leq 2\operatorname{Area}(\mathcal{T}_1) + 2(R-1)^{-2}\operatorname{Area}(\mathcal{T}_{1,R}) + 2\int_{\mathcal{T}_R} |\nabla\chi|^2 + \int_{\mathcal{T}_R\cap\{\chi<1\}} |A|^2 .$$

Set $K(s) = \int_{\mathcal{T}_{1,s}} |A|^2$. By the co-area formula and integrating (II.1.6) by parts twice, we get

$$\text{(II.1.7)} \quad 2\,(R-1)^{-2} \int_1^R \int_1^t K(s)ds\,dt \leq 2/(R-1) \int_1^R K(s)(R-s)/(R-1)ds$$
$$\leq \int_1^R K'(s)\,((R-s)/(R-1))^2 ds$$
$$\leq 2\operatorname{Area}(\mathcal{T}_1) + 2(R-1)^{-2}\operatorname{Area}(\mathcal{T}_{1,R}) + 2\int_{\mathcal{T}_R} |\nabla\chi|^2 + \int_{\{\chi<1\}} |A|^2 .$$

Given $y \in \sigma_1$, let $\gamma_y : [0,r_y] \to \Gamma$ be the (inward from $\partial\Gamma$) normal geodesic up to the cut-locus of σ_1 (so $\operatorname{dist}_\Gamma(\sigma_1,\gamma_y(r_y)) = r_y$) and J_y the corresponding Jacobi field with $J_y(0) = 1$ and $J_y'(0) = k_g(y)$. Set $R_y = \min\{r_y,R\}$. By the Jacobi equation,

$$\text{(II.1.8)} \qquad \int_0^{R_y} J_y(s)\,ds = R_y^2\,k_g(y)/2 + R_y - \int_0^{R_y} \int_0^t \int_0^s \mathrm{K}_\Gamma(\gamma_y(\tau))\,J_y(\tau)\,d\tau\,ds\,dt\,.$$

If $R_y < R$, then we extend $J_y(\tau)$, $\mathrm{K}_y(\tau) = \mathrm{K}_\Gamma(\gamma_y(\tau))$ to functions $\tilde{J}_y$, $\tilde{\mathrm{K}}_y$ on $[0, R]$ by setting

$$\tilde{J}_y \text{ and } \tilde{\mathrm{K}}_y = \begin{cases} J_y \text{ and } \tilde{\mathrm{K}}_y & \text{on } [0, R_y]\,, \\ 0 & \text{otherwise}\ . \end{cases}$$

(Obviously, if $R_y = R$, then $\tilde{J}_y = J_y$ and $\tilde{\mathrm{K}}_y = \mathrm{K}_y$.) Since $\mathrm{K}_\Gamma = -|A|^2/2$ (in particular, is ≤ 0), by (II.1.8)

$$\text{(II.1.9)} \quad \int_0^{R_y} J_y(s)\, ds \leq R^2\, |k_g(y)|/2 + R - \int_0^R \int_0^t \int_0^s \tilde{\mathrm{K}}_y(\tau)\, \tilde{J}_y(\tau)\, d\tau\, ds\, dt\,.$$

Since $K(s) = -2 \int_{\sigma_1} \int_1^s \tilde{\mathrm{K}}_y(\tau)\, \tilde{J}_y(\tau)\, d\tau\, dy$ (this uses (C)), integrating (II.1.9) over σ_1 gives

$$\text{(II.1.10)} \qquad \begin{aligned} \mathrm{Area}\,(\mathcal{T}_R) \leq & \frac{R^2}{2} \int_{\sigma_1} |k_g| + R\,\mathrm{Length}(\sigma_1) \\ & + \int_1^R \int_1^t \frac{K(s)}{2}\, ds\, dt + \frac{R^2}{2} \int_{\mathcal{T}_1} |A|^2\,. \end{aligned}$$

(Here we also used $\int_0^R \int_0^t f(s)\, ds\, dt \leq \int_1^R \int_1^t [f(s) - f(1)]\, ds\, dt + R^2\, f(1)$ for the nondecreasing function $f(t) = \int_{\mathcal{T}_t} |A|^2 \geq 0$.) Combining (II.1.7) and (II.1.10) gives (II.1.4). □

To apply Lemma II.1.3, we will need to replace a given curve, in a minimal disk, by a curve lying within a fixed tubular neighborhood of it and with length and total geodesic curvature bounded in terms of the area of the tubular neighborhood as in the following lemma:

LEMMA II.1.11 (see Figure 11). *If $\Sigma \subset \mathbf{R}^3$ is an immersed minimal disk, $\partial\Sigma = \gamma_1 \cup \gamma_2 \cup \sigma_1 \cup \sigma_2$, the γ_i's are geodesics with*

$$2 \leq \mathrm{Length}(\gamma_i) = \mathrm{dist}_\Sigma(\sigma_2 \cap \gamma_i, \sigma_1) \text{ and } 1 \leq \mathrm{dist}_\Sigma(\sigma_1, \sigma_2)\,,$$

then there exists a simple curve $\check{\sigma}_1 \subset \mathcal{T}_{1/64,1/4}(\sigma_1)$ connecting γ_1 to γ_2 and with

$$\text{(II.1.12)} \qquad \mathrm{Length}(\check{\sigma}_1) + \int_{\check{\sigma}_1} |k_g| \leq C_1\, (1 + \mathrm{Area}\,(\mathcal{T}_{1/4}(\sigma_1)))\,.$$

Moreover, $\check{\sigma}_1$ can be chosen to intersect γ_i orthogonally so that $\mathrm{Length}(\check{\gamma}_i) = \mathrm{dist}_\Sigma(\sigma_2 \cap \gamma_i, \check{\sigma}_1)$, *where $\check{\gamma}_i$ denotes the component of $\gamma_i \setminus \check{\sigma}_1$ which intersects σ_2.*

Proof. We will do this in three steps. First, we use the co-area formula to find a level set of the distance function with bounded length. Local replacement then gives a broken geodesic with the same length bound and a bound on the number of breaks. Third, we find a simple subcurve and use the Gauss-Bonnet theorem to control the number of breaks.

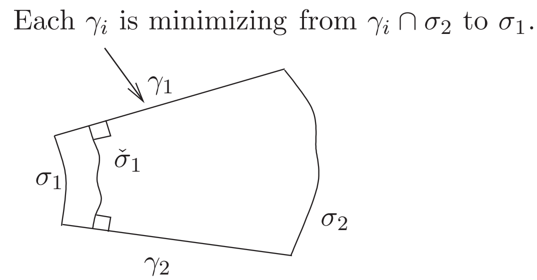


Figure 11: Lemma II.1.11: Connecting γ_1 and γ_2 by a curve $\breve{\sigma}_1$ with length and total curvature bounded.

Set $\mathrm{r}(\cdot) = \mathrm{dist}_\Sigma(\sigma_1, \cdot)$. By the co-area formula applied to (a regularization of) r, there exists d_0 between $1/16$ and $3/32$ with

$$\mathrm{Length}(\{\mathrm{r} = d_0\}) \leq 32\,\mathrm{Area}(\mathcal{T}_{1/8}(\sigma_1))$$

and so that $\{\mathrm{r} = d_0\}$ is transverse. Since the level set $\{\mathrm{r} = d_0\}$ separates σ_1 and σ_2, a component $\tilde{\sigma}$ of $\{\mathrm{r} = d_0\}$ goes from γ_1 to γ_2.

Parametrize $\tilde{\sigma}$ by arclength and let

$$0 = t_0 < \cdots < t_n = \mathrm{Length}(\tilde{\sigma})$$

be a subdivision with $t_{i+1} - t_i \leq 1/32$ and $n \leq 32\,\mathrm{Length}(\tilde{\sigma}) + 1$. Since $\mathcal{B}_{1/32}(y)$ is a disk for all $y \in \tilde{\sigma}$, it follows that we can replace $\tilde{\sigma}$ with a broken geodesic $\tilde{\sigma}_1$ (with breaks at $\tilde{\sigma}(t_i) = \tilde{\sigma}_1(t_i)$) which is homotopic to $\tilde{\sigma}$ in $\mathcal{T}_{1/32}(\tilde{\sigma})$. We can assume that $\tilde{\sigma}_1$ intersects the γ_i's only at its endpoints.

Let $[a, b]$ be a maximal interval so that $\tilde{\sigma}_1|_{[a,b]}$ is simple. We are done if $\tilde{\sigma}_1|_{[a,b]} = \tilde{\sigma}_1$. Otherwise, $\tilde{\sigma}_1|_{[a,b]}$ bounds a disk in Σ and the Gauss-Bonnet theorem implies that $\tilde{\sigma}_1|_{(a,b)}$ contains a break. Hence, replacing $\tilde{\sigma}_1$ by $\tilde{\sigma}_1 \setminus \tilde{\sigma}_1|_{(a,b)}$ gives a subcurve from γ_1 to γ_2 but does not increase the number of breaks. Repeating this eventually gives a simple subcurve with the same bounds for the length and the number of breaks. Smoothing this at the breaks gives the desired $\breve{\sigma}_1$.

Finally, since γ_i minimizes distance from $\gamma_i \cap \sigma_2$ to σ_1, it follows easily by adding segments in γ_1, γ_2 to $\breve{\sigma}_1$ and then perturbing infinitesimally near γ_1, γ_2 that we can choose $\breve{\sigma}_1$ to intersect γ_i orthogonally and so each $\breve{\gamma}_i$ minimizes distance back to $\breve{\sigma}_1$; this gives at most a bounded contribution to the length and total curvature. □

We will also need a version of Lemma II.1.11 where σ is a noncontractible curve (cf. Lemma 1.21 in [CM4]). This version is the following lemma:

Lemma II.1.13. *Let $\Sigma \subset \mathbf{R}^3$ be an immersed minimal planar domain and $\sigma = B_1 \cap \partial\Sigma$ a simple closed curve with*

$$\mathrm{dist}_\Sigma(\sigma, \partial\Sigma \setminus \sigma) > 1\,.$$

Then there exists a simple noncontractible curve $\breve{\sigma} \subset \mathcal{T}_{1/32,1/4}(\sigma)$ *with*

$$\text{Length}(\breve{\sigma}) + \int_{\breve{\sigma}} |k_g| \leq C_1 \left(1 + \text{Area}\left(\mathcal{T}_{1/4}(\sigma)\right)\right). \tag{II.1.14}$$

Proof. Following the first two steps of the proof of Lemma II.1.11 (with the obvious modifications), we get a simple closed broken geodesic $\tilde{\sigma}_1$ which is noncontractible with length and the number of breaks $\leq C\,\text{Area}\,(\mathcal{T}_{1/4}(\sigma))$.

As in the third step of the proof of Lemma II.1.11, let $\tilde{\sigma}_1|_{[a,b]}$ be a maximal simple subcurve. It follows that $\tilde{\sigma}_1|_{[a,b]}$ is closed (and has at most one more break than $\tilde{\sigma}_1$). If $\tilde{\sigma}_1|_{[a,b]}$ is noncontractible, then we are done. Otherwise, if $\tilde{\sigma}_1|_{[a,b]}$ bounds a disk, then we apply the Gauss-Bonnet theorem to see that $\tilde{\sigma}_1|_{(a,b)}$ contains a break and proceed as in the proof of Lemma II.1.11. □

In Proposition II.1.20 below, we will also need a lower bound for the area growth of tubular neighborhoods of a curve. To get such a bound, it is necessary that the curve not be completely "crumpled up." This will follow when

$$(t + C_0)\,(t+1) \leq \delta\,\text{Area}(\mathcal{T}_1(\sigma_1))\,.$$

The lower bound for the area growth of tubular neighborhoods needed in Proposition II.1.20 is the following:

LEMMA II.1.15. *Let* $\Sigma_0 = \Sigma$ *satisfy* (A), (B) *and* (D). *If* $\sigma_1 \subset B_1$, $1 \leq s < t \leq \ell$ *and*

$$(t + C_0)\,(t+1) \leq \delta\,\text{Area}(\mathcal{T}_1(\sigma_1))\,,$$

then

$$(t+1)^{2\delta - 2}\,\text{Area}(\mathcal{T}_t(\sigma_1)) \geq (s+1)^{2\delta-2}\,\text{Area}\left(\mathcal{T}_s(\sigma_1)\right). \tag{II.1.16}$$

Proof. Set $\mathcal{T}_t = \mathcal{T}_t(\sigma_1)$ and define the "length function" $L(s)$ by

$$L(s) = \int_{\partial \mathcal{T}_s \setminus \partial \Sigma} 1\,.$$

By minimality, Stokes' theorem, (A), (B) and $\text{dist}_\Sigma(\sigma_1, x) + 1 \geq |x|$, we get that

$$4\,\text{Area}\left(\mathcal{T}_s\right) = \int_{\mathcal{T}_s} \Delta\,|x|^2 \leq 2\,(s+1)\,L(s) + 4\,(s + C_0)\,(s+1)\,. \tag{II.1.17}$$

By the co-area formula, $(\text{Area}\,(\mathcal{T}_s))' = L(s)$ for almost every s. Hence, for almost every s with $\text{dist}_\Sigma(\sigma_1, \sigma_2) \geq s \geq 1$,

$$(\log\,\text{Area}\,(\mathcal{T}_s))' \geq \frac{2}{s+1} - \frac{2\,(s+C_0)}{\text{Area}(\mathcal{T}_s)} \geq \frac{2\,(1-\delta)}{s+1}\,. \tag{II.1.18}$$

Since $\text{Area}(\mathcal{T}_s)$ is a monotonic function of s, a standard argument then gives (II.1.16). □

Remark II.1.19. In the special case of Lemma II.1.15 where Σ is an annulus with $\partial\Sigma = \sigma_1 \cup \sigma_2$, i.e., where $\gamma_i = \emptyset$ and σ_1, σ_2 are closed, the proof simplifies in an obvious way and δ can be chosen to be zero.

We are now ready to apply Lemma II.1.3 and to use the logarithmic cut-off trick to show that certain stable sectors have small curvature. This is the following proposition:

PROPOSITION II.1.20. *Let* $\Gamma_0 \subset \Gamma \subset \mathbf{R}^3$ *satisfy* (A) (*with* $C_0 = 0$), (B), (D), *and*

$$\mathrm{dist}_\Gamma(\Gamma_0, \partial\Gamma) > 1/4\,.$$

Suppose that Γ *is stable,* $\omega > 2$, $\ell > R_0 > \omega^2$, *and* $\sigma_1 \subset B_1$. *If* Γ_0 *is a disk and*

$$4\,R_0^2\,(R_0+1) \leq \mathrm{Area}(\mathcal{T}_1(\sigma_1, \Gamma_0))\,,$$

then for $\omega^2 \leq t \leq R_0$,

(II.1.21)

$$\mathrm{Area}\,(\mathcal{T}_2(\sigma_1,\Gamma_0))\,t^2/C \leq \mathrm{Area}(\mathcal{T}_{\omega,t}(\sigma_1,\Gamma_0)) \leq C\,\mathrm{Area}\,(\mathcal{T}_2(\sigma_1,\Gamma_0))\,t^2\,,$$

$$\text{(II.1.22)} \qquad \int_{\mathcal{T}_{\omega,R_0/\omega}(\sigma_1,\Gamma_0)} |A|^2 \leq C\,R_0 + \frac{C}{\log\omega}\mathrm{Area}\,(\mathcal{T}_2(\sigma_1,\Gamma_0))\,.$$

Proof. Define a function χ on Γ_0 by

$$\text{(II.1.23)} \qquad \chi = \begin{cases} 2\,\mathrm{dist}_\Gamma(\gamma_1\cup\gamma_2,\cdot) & \text{on } \mathcal{T}_{1/2}(\gamma_1\cup\gamma_2)\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

We will use χ to cut-off on the sides γ_1, γ_2. Using the estimates for stable surfaces of [Sc], [CM2], and $\mathrm{dist}_\Gamma(\Gamma_0, \partial\Gamma) > 1/4$, we get

$$\text{(II.1.24)} \qquad \int_{\mathcal{T}_2(\sigma_1,\Gamma_0)} (1+|A|^2) \leq C_1\,\mathrm{Area}(\mathcal{T}_2(\sigma_1,\Gamma_0))\,,$$

$$\text{(II.1.25)} \qquad 2\int_{\mathcal{T}_{R_0}(\sigma_1,\Gamma_0)} |\nabla\chi|^2 + \int_{\mathcal{T}_{R_0}(\sigma_1,\Gamma_0)\cap\{\chi<1\}} |A|^2 \leq C_1\,R_0$$
$$\leq C_1\,\mathrm{Area}(\mathcal{T}_1(\sigma_1,\Gamma_0))\,.$$

Since $\sigma_1 \subset \partial\Gamma_0$ satisfies (A) with $C_0 = 0$ and (D), Lemma II.1.11 gives a simple curve $\check{\sigma}_1$ (and $\check{\gamma}_1$, $\check{\gamma}_2$) satisfying (A) with $C_0 = 0$, (C), (D), and (II.1.12); let $\check{\Gamma}_0 \subset \Gamma_0$ be the component of $\Gamma_0 \setminus \check{\sigma}_1$ containing σ_2. By the triangle inequality, we have

$$\text{(II.1.26)} \qquad \mathcal{T}_t(\check{\sigma}_1,\Gamma_0) \subset \mathcal{T}_{t+1/4}(\sigma_1,\Gamma_0) \subset \mathcal{T}_{t+1/4}(\check{\sigma}_1,\check{\Gamma}_0) \cup (\Gamma_0 \setminus \check{\Gamma}_0)\,.$$

Note that $\Gamma_0 \setminus \check{\Gamma}_0$ is a disk with boundary

$$\sigma_1 \cup \check{\sigma}_1 \cup (\gamma_1 \setminus \check{\gamma}_1) \cup (\gamma_2 \setminus \check{\gamma}_2)\,.$$

Hence, by minimality, Stokes' theorem, (B), $|x| \leq 5/4$ on $\partial(\Gamma_0 \setminus \check{\Gamma}_0)$, and (II.1.12), we get

$$\text{(II.1.27)} \qquad \begin{aligned} 4\,\text{Area}(\Gamma_0 \setminus \check{\Gamma}_0) &= \int_{\Gamma_0 \setminus \check{\Gamma}_0} \Delta |x|^2 \leq 2 \int_{\check{\sigma}_1 \cup (\gamma_1 \setminus \check{\gamma}_1) \cup (\gamma_2 \setminus \check{\gamma}_2)} |x| \\ &\leq C_1'\,\text{Area}(\mathcal{T}_1(\sigma_1, \Gamma_0))\,. \end{aligned}$$

Inserting (II.1.24), (II.1.25) into Lemma II.1.3 applied to $\check{\sigma}_1$ and using (II.1.12), (II.1.26), (II.1.27), for $2 \leq t \leq R_0$, we get

$$\text{(II.1.28)} \qquad \text{Area}(\mathcal{T}_t(\sigma_1, \Gamma_0)) \leq C_2\,\text{Area}\,(\mathcal{T}_2(\sigma_1, \Gamma_0))\, t^2\,,$$

which gives the second inequality in (II.1.21). Set $\mathcal{T}_t = \mathcal{T}_t(\sigma_1, \Gamma_0)$ (define $\mathcal{T}_{s,t}$ similarly) and set $L(t) = \int_{\partial \mathcal{T}_t \setminus \partial \Gamma_0} 1$. By (II.1.28), the co-area formula, and integration by parts, we get

$$\text{(II.1.29)} \qquad \begin{aligned} \int_{R_0/\omega}^{R_0} L(t)\, t^{-2} dt &= \left[\text{Area}(\mathcal{T}_{R_0/\omega, t})\, t^{-2}\right]_{R_0/\omega}^{R_0} \\ &\quad + 2 \int_{R_0/\omega}^{R_0} \text{Area}(\mathcal{T}_{R_0/\omega, t})\, t^{-3}\, dt \\ &\leq C_2\,(1 + 2 \log\, \omega)\,\text{Area}\,(\mathcal{T}_2) \leq C_3\, \log \omega\,\text{Area}\,(\mathcal{T}_2)\,, \end{aligned}$$

$$\text{(II.1.30)} \qquad \begin{aligned} \int_1^{\omega} L(t)\, t^{-2} dt &\leq \text{Area}(\mathcal{T}_{1,\omega})\, \omega^{-2} + 2 \int_1^{\omega} \text{Area}(\mathcal{T}_{1,t})\, t^{-3}\, dt \\ &\leq C_3\, \log\, \omega\,\text{Area}\,(\mathcal{T}_2)\,. \end{aligned}$$

Define a (radial) cut-off function η by

$$\text{(II.1.31)} \qquad \eta = \begin{cases} \log \text{dist}_{\Gamma_0}(\sigma_1, \cdot) / \log \omega & \text{on } \mathcal{T}_{1,\omega}\,, \\ 1 & \text{on } \mathcal{T}_{\omega, R_0/\omega}\,, \\ \left[\log R_0 - \log \text{dist}_{\Gamma_0}(\sigma_1, \cdot)\right] / \log \omega & \text{on } \mathcal{T}_{R_0/\omega, R_0}\,. \end{cases}$$

Using the bounds (II.1.29) and (II.1.30), we get

$$\text{(II.1.32)} \qquad \begin{aligned} \int |\nabla \eta|^2 &= \int_{\mathcal{T}_{1,\omega}} |\nabla \eta|^2 + \int_{\mathcal{T}_{R_0/\omega, R_0}} |\nabla \eta|^2 \\ &\leq \frac{1}{(\log \omega)^2} \int_1^{\omega} \frac{L(t)}{t^2} dt + \frac{1}{(\log \omega)^2} \int_{R_0/\omega}^{R_0} \frac{L(t)}{t^2} dt \\ &\leq \frac{C_3\,\text{Area}\,(\mathcal{T}_2)}{\log \omega}\,. \end{aligned}$$

Substituting $\eta\,\chi$ into the stability inequality, we get using (II.1.25) and (II.1.32) that

$$\text{(II.1.33)} \qquad \int_{\mathcal{T}_{\omega,R_0/\omega}} |A|^2 \leq \int_{\mathcal{T}_{R_0}\cap\{\chi<1\}} |A|^2 + 2\int_{\mathcal{T}_{R_0}} |\nabla\chi|^2 + 2\int |\nabla\eta|^2 \leq C_1\,R_0 + \frac{2\,C_3\,\text{Area}\,(\mathcal{T}_2)}{\log\omega}\,.$$

Finally, Lemma II.1.15 (and (II.1.28) for $t = \omega$) gives the first inequality in (II.1.21). □

We will prove Theorem II.1.2 by considering two separate cases depending on the area of $\mathcal{T}_1(\sigma)$:

- When Area($\mathcal{T}_1(\sigma)$) is small, the next corollary will show that (1) of Theorem II.1.2 holds.
- When Area($\mathcal{T}_1(\sigma)$) is large, we will show in the next section, using Corollary II.1.45 below, that (2) of Theorem II.1.2 holds.

COROLLARY II.1.34. *Given C_a, there exists $\Omega_a > 4$ so that the following holds*:

Let $\Gamma \subset \mathbf{R}^3$ be a stable embedded minimal planar domain, $\sigma = B_1 \cap \partial\Gamma$ connected, and $\text{dist}_\Gamma(\sigma, \partial\Gamma \setminus \sigma) > R$. If $R > \Omega_a^2$ and

$$\text{Area}(\mathcal{T}_1(\sigma)) \leq C_a\,,$$

then Γ contains a graph Γ_g (after a rotation) over $D_{R/\Omega_a} \setminus D_{\Omega_a}$ with gradient ≤ 1 and $\text{dist}_\Gamma(\sigma, \Gamma_g) \leq 2\,\Omega_a$.

Proof. Lemma II.1.13 gives a simple closed noncontractible curve $\breve{\sigma} \subset \mathcal{T}_{1/32,1/4}(\sigma)$ with

$$\text{Length}(\breve{\sigma}) + \int_{\breve{\sigma}} |k_g| \leq C_1\,[\text{Area}\,(\mathcal{T}_1(\sigma)) + 1]\,.$$

Since Γ is a planar domain, $\breve{\sigma}$ separates in Γ; let $\check{\Gamma}$ be the component of $\Gamma \setminus \breve{\sigma}$ which does not contain σ. By Lemma II.1.3 (which applies with $\chi \equiv 1$ since $\gamma_1 = \gamma_2 = \emptyset$), we get for $1 \leq t \leq R$

$$\text{(II.1.35)} \qquad \text{Area}(\mathcal{T}_t(\breve{\sigma}, \check{\Gamma})) \leq C\,(C_a + 1)\,t^2\,.$$

Given $\Omega > 4$, by (II.1.35) and the logarithmic cut-off trick in the stability inequality (cf. (II.1.33)), we get that

$$\int_{\mathcal{T}_{\Omega/2,2R/\Omega}(\breve{\sigma},\check{\Gamma})} |A|^2 \leq C_2\,(C_a + 1)/\log\Omega\,.$$

Combining this with (II.1.35) and the Cauchy-Schwarz inequality give for $\Omega/2 \leq t \leq R/\Omega$

$$\text{(II.1.36)} \qquad \int_{\mathcal{T}_{t,2t}(\check{\sigma},\check{\Gamma})} |A| \leq \left(\text{Area}(\mathcal{T}_{2t}(\check{\sigma},\check{\Gamma})) \int_{\mathcal{T}_{\Omega/2,2R/\Omega}(\check{\sigma},\check{\Gamma})} |A|^2 \right)^{1/2} \leq \frac{C_3\,(C_a+1)\,t}{(\log \Omega)^{1/2}} \,.$$

Applying the co-area formula on $\mathcal{T}_{t,2t}$ for $t = \Omega/2, R/\Omega$, we see that (II.1.36) gives a (possibly disconnected) planar domain

$$\Gamma_0 \subset \mathcal{T}_{\Omega/2,2R/\Omega}(\check{\sigma},\check{\Gamma})$$

with $\mathcal{T}_{\Omega,R/\Omega}(\check{\sigma},\check{\Gamma}) \subset \Gamma_0$, $\partial \Gamma_0 = \cup_{i=1}^n \sigma_i$, and

$$\text{(II.1.37)} \qquad \sum_{i=1}^n \int_{\sigma_i} |A| \leq \frac{C_3\,(C_a+1)}{(\log \Omega)^{1/2}} \,.$$

We now fix a large constant $\Omega = \Omega(C_a) > 4$ so that

$$C_2\,(C_a+1)/\log \Omega < \pi \,,$$
$$C_3\,(C_a+1)\,(\log \Omega)^{-1/2} < 1/4 \,.$$

Since the Gauss map is conformal, the L^2 curvature bound on Γ_0 and the L^1 bound on $\partial \Gamma_0$ imply that the unit normal $\mathbf{n}_\Gamma$ is almost constant on each component of Γ_0. To be precise, proposition 1.12 of [CM7] implies that on each component Γ_0^k of Γ_0 we get

$$\mathbf{n}_\Gamma(\Gamma_0^k) \subset \mathcal{B}_{1/2}(a_k) \,,$$

where each a_k is a point in the unit sphere. In particular, the unit normal to each component of Γ_0 is almost constant and, hence, Γ_0 is a either a graph or a multi-valued graph. Since Γ is embedded, the corollary now follows easily (cf. lemma 1.10 in [CM4]). □

We construct next from curves σ_1, γ_1, γ_2 in a stable surface the desired multi-valued graph. (The existence of the curves σ_1, γ_1, γ_2 will be established in the next section.) First we need two lemmas. The first of these is the following:

Lemma II.1.38. *Given $C_1, \varepsilon_0 > 0$, there exists $\varepsilon_1 > 0$ so that if $\mathcal{B}_1 \subset \Sigma$ is minimal with*

$$\sup_{\mathcal{B}_{1/2}} |A|^2 \leq \varepsilon_1 \text{ and } \sup_{\mathcal{B}_1} |A|^2 \leq C_1 \,,$$

then

$$\sup_{\mathcal{B}_{3/4}} |A|^2 \leq \varepsilon_0 \,.$$

Proof. Suppose not; it follows that there is a sequence Σ_j of minimal surfaces with

$$\sup_{\mathcal{B}_{1/2}} |A|^2 \leq 1/j\,,$$
$$\sup_{\mathcal{B}_1} |A|^2 \leq C_1\,,$$
$$\sup_{\mathcal{B}_{3/4}} |A|^2 > \varepsilon_0 > 0\,.$$

The uniform bound $\sup_{\mathcal{B}_1} |A|^2 \leq C_1$ (and standard elliptic estimates) gives a subsequence which converges in $C^{2,\alpha}$ to a limit Σ_∞. It follows that Σ_∞ is minimal, $|A|^2 = 0$ on $\mathcal{B}_{1/2}$, and

$$\sup_{\mathcal{B}_{3/4}} |A|^2 \geq \varepsilon_0 > 0\,.$$

By unique continuation, Σ_∞ is flat contradicting that $\sup_{\mathcal{B}_{3/4}} |A|^2 \geq \varepsilon_0 > 0$. □

The next lemma will be applied both when Γ is an annulus and when Γ has boundary on the sides. When Γ is an annulus, the condition (II.1.40) will be trivially satisfied and it will be possible for Γ to contain a graph instead of a multi-valued graph.

LEMMA II.1.39. *Given $N, S_0 > 4$, $\varepsilon > 0$, there exist $C_b > 1$, $\delta > 0$ so that the following holds*:

Let $\Gamma \subset \mathbf{R}^3$ be a stable embedded minimal surface and $\sigma = B_1 \cap \partial\Gamma$. If $\gamma : [0, S_0] \to \Gamma$ is a geodesic so that for $0 \leq t \leq S_0$ we have

$$\text{(II.1.40)} \qquad \operatorname{dist}_\Gamma(\gamma(t), \sigma) = t\,,$$
$$\sup_{\mathcal{B}_{S_0/16}(\gamma(S_0))} |A|^2 \leq \delta\, S_0^{-2}\,,$$
$$\operatorname{dist}_{\Gamma \setminus \mathcal{T}_{t/8}(\sigma)}(\gamma(t), \partial\Gamma) \geq C_b\, t\,,$$

then (*after a rotation of* $\mathbf{R}^3$) Γ *contains either*

- *An N-valued graph $\Gamma_{2,S_0/2}^{-N\pi,N\pi}$ with $\gamma(4) \in \Gamma_{2,5}^{-\pi,\pi}$ or*
- *A graph $\Gamma_{2,S_0/2}$ with $\gamma(4) \in \Gamma_{2,5}$.*

In either case, *the graph has gradient $\leq \varepsilon$ and $|A| \leq \varepsilon/r$.*

Proof. Combining estimates for stable surfaces of [Sc], [CM2] and (II.1.40), gives for $0 \leq t \leq S_0$

$$\text{(II.1.41)} \qquad \sup_{\mathcal{B}_{t/2}(\gamma(t))} |A| \leq C_0\, t^{-1}\,.$$

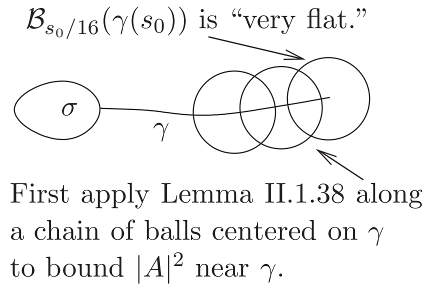


Figure 12: The proof of Lemma II.1.39: Repeatedly applying Lemma II.1.38 along chains of balls builds out a "flat" region in Γ.

Fix $\delta_0 > 0$ to be chosen small depending on S_0. Using (II.1.41) and repeatedly applying Lemma II.1.38 along a chain of balls with centers in γ, see Figure 12, there exists

$$\delta_1 = \delta_1(S_0, \delta_0, C_0) > 0$$

so that if $\delta \leq \delta_1$, then for $1 \leq t \leq S_0$

$$\sup_{\mathcal{B}_{t/32}(\gamma(t))} |A| \leq \delta_0 \, t^{-1} . \tag{II.1.42}$$

Since γ is a geodesic in Γ, (II.1.42) gives the bound

$$k_g^{\mathbf{R}^3}(t) \leq \delta_0 \, t^{-1}$$

for the geodesic curvature of γ in $\mathbf{R}^3$. It follows that for $1 \leq t \leq S_0$

$$|\mathbf{n}_\Gamma(\gamma(t)) - \mathbf{n}_\Gamma(\gamma(1))| + |\gamma'(t) - \gamma'(1)| \leq 2\delta_0 \int_1^{S_0} \frac{ds}{s} \leq 2\delta_0 \, \log S_0 \, ; \tag{II.1.43}$$

i.e., γ is C^1-close to a straight line segment in $\mathbf{R}^3$ and $\mathbf{n}_\Gamma$ is almost constant on γ. Rotate so that $\gamma'(1) = (1, 0, 0)$ (i.e., so that $\gamma'(1)$ points in the x_1-direction). For $\delta_0 > 0$ small, (II.1.43) (and $\gamma(0) \in B_1$) implies that for $1 \leq t \leq S_0$

$$3t/4 - 2 \leq x_1(\gamma(t)) \leq 1 + t \, . \tag{II.1.44}$$

We will now argue as in (II.1.41) and (II.1.42) to extend the region where Γ is graphical, this time using balls centered on cylinders (i.e., building out the multi-valued graph in the θ direction). Suppose now that $4 \leq s \leq S_0/2$ and

$$y_{0,s} = \{x_1^2 + x_2^2 = s^2\} \cap \gamma \, .$$

Using (II.1.42), we see that $\mathcal{B}_{C_2 s}(y_{0,s})$ is a graph with gradient $\leq C_2' \, \delta_0$ over $\mathbf{n}_\Gamma(y_{0,s})$. In particular, also using (II.1.43), $\partial \mathcal{B}_{C_2 s}(y_{0,s})$ contains a point

$$y_{1,s} \in \{x_1^2 + x_2^2 = s^2\} \, .$$

Using Lemma II.1.38, we can therefore repeat this to find $y_{2,s}$, etc. It follows from (II.1.40) that we can continue this until Γ either closes up (giving a graph) or we have the desired N-valued graph $\Gamma_{2,S_0/2}^{-N\pi,N\pi}$, with gradient $\leq \varepsilon$, $|A| \leq \varepsilon/r$, which contains $\gamma(4)$. $\square$

In the next corollary $\Gamma \subset B_{2R} \subset \mathbf{R}^3$ will be a stable embedded minimal annulus with

$$\partial\Gamma \subset B_{1/4} \cup \partial B_{2R}$$

where $B_1 \cap \partial\Gamma$ is connected and suppose $\Gamma_0 \subset \Gamma$ is a disk satisfying (A) for $C_0 = 0$, (B), (D). Let $\sigma = B_1 \cap \partial\Gamma$ so that $\sigma_1 \subset \sigma$ and σ is a simple closed curve. Assume also that the following strengthening of (A) holds:

(A′) $\text{dist}_\Gamma(\gamma_i(t), \sigma) = t$ for $0 \leq t \leq \text{Length}(\gamma_i)$.

Corollary II.1.45 (see Figure 13). *Given $N, \varepsilon > 0$, there exist ω_0, $R_0 > 1$ so that if Γ and Γ_0 are as above, and*

$$\text{Area}(\mathcal{T}_1(\sigma_1)) \geq 4\, R_0^2\, (R_0 + 1)\,,$$

then (after a rotation of $\mathbf{R}^3$) Γ contains an N-valued graph $\Gamma_{\omega_0,R/\omega_0}^{-N\pi,N\pi}$ with gradient $\leq \varepsilon$, $|A| \leq \varepsilon/r$, and

$$\text{dist}_\Gamma(z_1, \Gamma_{\omega_0,\omega_0}^{0,0}) < 2\,\omega_0 + C_1\, \text{Area}(\mathcal{T}_1(\sigma_1, \Gamma))\,. \tag{II.1.46}$$

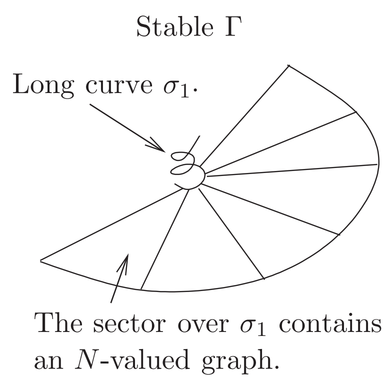


Figure 13: Corollary II.1.45: A stable tubular neighborhood of a long curve σ_1 contains an N-valued graph $\Gamma_{\omega_0,R/\omega_0}^{-N\pi,N\pi}$.

Proof. Proposition II.1.20 gives C so that for $\omega \leq t \leq R_0/\omega$ (where $\omega > 4$ and $R_0^2 > 2\omega^2$)

$$\text{Area}\,(\mathcal{T}_2(\sigma_1, \Gamma_0))\, t^2/C \leq \text{Area}(\mathcal{T}_{\omega,t}(\sigma_1, \Gamma_0))\,, \tag{II.1.47}$$

$$\int_{\mathcal{T}_{\omega,R_0/\omega}(\sigma_1,\Gamma_0)} |A|^2 \leq \frac{C}{\log \omega} \text{Area}\,(\mathcal{T}_2(\sigma_1, \Gamma_0))\,. \tag{II.1.48}$$

(Here we also used $\text{Area}(\mathcal{T}_1(\sigma_1)) \geq 4\,R_0^2\,(R_0+1)$ in (II.1.48).) Set $S = \omega$. Choose a maximal disjoint collection of balls $\mathcal{B}_{S/4}(y_1), \dots, \mathcal{B}_{S/4}(y_n)$ with centers in $\mathcal{T}_{S,2S}(\sigma_1, \Gamma_0)$. Since Γ is an annulus without boundary on the sides and $R_0 > 5S/2$, it follows from (A$'$) that

$$\mathcal{B}_{S/2}(y_j) \cap \partial\Gamma = \emptyset\,.$$

We will use this twice below. First, since $\mathcal{T}_{S,2S}(\sigma_1, \Gamma_0)$ is contained in the union of the double balls and, by stability (see [CM2]),

$$\pi (S/4)^2 \leq \text{Area}(\mathcal{B}_{S/4}(y_j)) \leq \text{Area}(\mathcal{B}_{S/2}(y_j)) \leq C\pi\, S^2\,,$$

we have $n \geq C\,S^{-2}\,\text{Area}\,(\mathcal{T}_{S,2S}(\sigma_1, \Gamma_0))$. Second, again by stability, [CM2], we have

$$\int_{\mathcal{T}_{S/4}(\gamma_1 \cup \gamma_2) \cap \mathcal{T}_{S/2,3S}(\sigma_1)} |A|^2 \leq C\,.$$

Combining this with (II.1.47) and (II.1.48), we can find j so that

$$\int_{\mathcal{B}_{S/4}(y_j)} |A|^2 < C/\log\omega\,.$$

Therefore, by the mean value inequality, we have

$$\sup_{\mathcal{B}_{S/8}(y_j)} |A|^2 < C\,S^{-2}/\log\omega\,.$$

Let $\gamma : [0, \ell] \to \Gamma$ be a minimal geodesic from y_j to σ_1; note that $S \leq \ell \leq 2S$. Since the sides γ_1, γ_2 are minimizing (i.e., (A$'$)), it follows that $\gamma \subset \Gamma_0$. Furthermore, since Γ is an annulus, (A$'$) implies that

$$\text{dist}_{\Gamma\setminus\mathcal{T}_1(\sigma)}(\gamma(\ell), \partial\Gamma) \geq R/2\,. \tag{II.1.49}$$

In particular, given $\omega_1, N_1 > 1$ and and $\varepsilon_1 > 0$, there exists ω (and hence R_0) large so that we can apply Lemma II.1.39 to get either a graph $\Gamma_{S/\omega_1,S/2}$ or an initial multi-valued graph $\Gamma^{-N_1\pi,N_1\pi}_{S/\omega_1,S/2}$ with gradient $\leq \varepsilon_1$, $|A| \leq \varepsilon_1/r$, and

$$\gamma(4S/\omega_1) \in \Gamma^{-\pi,\pi}_{2S/\omega_1,5S/\omega_1}\,.$$

However, since

$$\text{Area}(\mathcal{T}_1(\sigma_1)) \geq 4\,R_0^2\,(R_0+1)\,,$$

Γ cannot contain a graph $\Gamma_{S/\omega_1,S/2}$.

Using Theorem II.0.21 of [CM3], we will next extend $\Gamma^{-N_1\pi,N_1\pi}_{S/\omega_1,S/2}$ to the desired N-valued graph $\Gamma^{-N\pi,N\pi}_{\omega_0,R/\omega_0}$. Namely, let P be the vertical plane

$$P = \{x_1 = 2S/\omega_1\}\,.$$

We claim first that

$$\text{each component of } P \cap \Gamma \text{ goes off to } \partial B_{2R}\,.$$

To see this, note that by the maximum principle, any closed curve in $P \cap \Gamma$ would be homologous to the interior boundary of Γ and together these two curves would span an annulus in Γ violating the convex hull property (using the multi-valued graph in Γ to connect this annulus to $\{x_1 = -S/\omega_1\}$). It follows that two of these nodal curves connect the multi-valued graph out to ∂B_{2R}, giving a curve η in Γ with both endpoints in ∂B_{2R}. One component of $\Gamma \setminus \eta$ is a stable disk which is forced to spiral initially. Therefore, by theorem II.0.21 of [CM3], this extends to the desired multi-valued graph. □

II.2. The minimizing geodesics and the proof of Theorem II.1.2

In Proposition II.2.9 and Corollary II.2.10 below, we will construct the minimizing geodesics γ_1 and γ_2 needed for Corollary II.1.45. To do this we will first need the following lemmas and corollaries (here $\mathcal{T}_t$ is the closed tubular neighborhood and $\mathcal{T}_t^\circ$ is the open):

The first lemma finds the disk Σ_4 and the curve σ_4 in Figure 15.

Lemma II.2.1 (see Figures 14 and 15). *Let Σ be an annulus with $\partial\Sigma = \sigma_1 \cup \sigma_2$, where $\text{dist}_\Sigma(\sigma_1, \sigma_2) > \ell + \varepsilon$ for ℓ, $\varepsilon > 0$ and let E be the connected component of $\Sigma \setminus \mathcal{T}_\ell(\sigma_1)$ containing σ_2. Let γ_1 and γ_2 be geodesics with*

$$\text{(II.2.2)} \qquad \begin{aligned} \gamma_i : [0, \ell] &\to \Sigma\,, \\ \text{dist}_\Sigma(\gamma_i(t), \sigma_1) &= t \textit{ for } 0 \le t \le \ell\,, \\ \gamma_i(\ell) &\in \overline{E}\,. \end{aligned}$$

If $\sigma_3 \subset \sigma_1$ is a segment connecting $\gamma_1(0)$ and $\gamma_2(0)$, then there exists a curve

$$\sigma_4 \subset \mathcal{T}_\varepsilon^\circ(E) \cap \mathcal{T}_\varepsilon^\circ(\Sigma \setminus E)$$

connecting $\gamma_1(\ell)$ and $\gamma_2(\ell)$ and so $\sigma_3 \cup \sigma_4 \cup \gamma_1 \cup \gamma_2$ bounds a disk Σ_4. Moreover, $\sigma_4 \subset \mathcal{T}_{\ell+\varepsilon}(\sigma_1) \setminus \mathcal{T}_{\ell-\varepsilon}(\sigma_1)$.

Proof. First, note that $\gamma_1(\ell), \gamma_2(\ell) \in \overline{E} \cap \overline{\Sigma \setminus E}$ and by definition E, hence $\mathcal{T}_\varepsilon^\circ(E)$, is connected. Moreover, if $x \in \Sigma \setminus E$ and $\gamma : [0, \ell_\gamma] \to \Sigma$ is a geodesic with $\gamma(\ell_\gamma) = x$ and $\text{dist}_\Sigma(\gamma(t), \sigma_1) = t$ for $0 \le t \le \ell_\gamma$, then $\gamma \cap E = \emptyset$. Hence, also $\Sigma \setminus E$ and $\mathcal{T}_\varepsilon^\circ(\Sigma \setminus E)$ are connected. Since $\sigma_1 \subset \mathcal{T}_\varepsilon^\circ(\Sigma \setminus E)$ and $\sigma_2 \subset \mathcal{T}_\varepsilon^\circ(E)$, applying van Kampen's theorem to

$$\Sigma = \mathcal{T}_\varepsilon^\circ(\Sigma \setminus E) \cup \mathcal{T}_\varepsilon^\circ(E)$$

gives that $\mathcal{T}_\varepsilon^\circ(E) \cap \mathcal{T}_\varepsilon^\circ(\Sigma \setminus E)$ is path-connected and has fundamental group $\mathbf{Z}$ which injects into $\pi_1(\Sigma)$. In particular, we get simple curves

$$\sigma_{4,1}, \sigma_{4,2} \subset \mathcal{T}_\varepsilon^\circ(E) \cap \mathcal{T}_\varepsilon^\circ(\Sigma \setminus E)$$

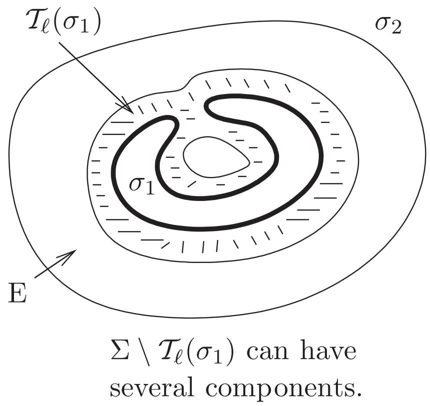


Figure 14: The set E in Lemma II.2.1.

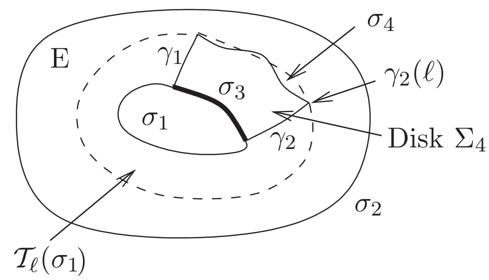


Figure 15: In an annulus Σ with $\partial\Sigma = \sigma_1 \cup \sigma_2$, given geodesics γ_1, γ_2 and a curve $\sigma_3 \subset \sigma_1$ connecting $\gamma_1(0)$ and $\gamma_2(0)$, Lemma II.2.1 finds a disk Σ_4 with $\partial\Sigma_4 = \sigma_3 \cup \sigma_4 \cup \gamma_1 \cup \gamma_2$ where each point in σ_4 is almost distance ℓ from σ_1.

connecting $\gamma_1(\ell)$ to $\gamma_2(\ell)$ so that $\sigma_{4,1} \cup \sigma_{4,2}$ is homologous to σ_1. Fix $\Sigma_0 \subset \Sigma$ with

$$\partial\Sigma_0 = \sigma_1 \cup (\sigma_{4,1} \cup \sigma_{4,2}) .$$

The curve $\sigma_3 \cup \gamma_1 \cup \gamma_2$ divides Σ_0 into two components, one of which is a disk with σ_3, γ_1, γ_2, and either $\sigma_{4,1}$ or $\sigma_{4,2}$ in its boundary.

Finally, since $\sigma_4 \subset \mathcal{T}^\circ_\varepsilon(E)$ it follows that $\sigma_4 \subset \Sigma \setminus \mathcal{T}_{\ell-\varepsilon}(\sigma_1)$. Likewise it follows from the fact that

$$\sigma_4 \subset \mathcal{T}^\circ_\varepsilon(E) \cap \mathcal{T}^\circ_\varepsilon(\Sigma \setminus E)$$

and the triangle inequality that $\sigma_4 \subset \mathcal{T}_{\ell+\varepsilon}(\sigma_1)$. □

The next corollary finds the geodesic γ_3 between γ_1 and γ_2 in Figure 16.

Corollary II.2.3 (see Figure 16). *Let* Σ, E, σ_1, σ_2, σ_3, γ_1, γ_2 *be as in Lemma* II.2.1.

If $\gamma_1(\ell) \neq \gamma_2(\ell)$, *then there exists a geodesic* γ_3 *different from* γ_1, γ_2, *intersecting* σ_3, *and satisfying* (II.2.2).

Proof. Let $\eta \subset \Sigma \setminus (\gamma_1 \cup \gamma_2)$ be a simple curve from σ_3 to σ_2 so that $\eta \cap \sigma_1 \subset \eta \cap \sigma_3$ is one point. Fix $\mu > 0$ with $3\mu < \text{dist}_\Sigma(\gamma_1 \cup \gamma_2, \eta)$. For $\varepsilon > 0$ small (in particular, $\varepsilon < \text{dist}_\Sigma(\sigma_1, \sigma_2) - \ell$), let $\sigma_{\varepsilon,4}$, $\Sigma_{\varepsilon,4}$ be given by Lemma II.2.1. Let η_ε be the component of $\eta \cap \Sigma_{\varepsilon,4}$ intersecting σ_3 and let

$$\gamma_{\varepsilon,3} : [0, \ell_\varepsilon] \to \Sigma$$

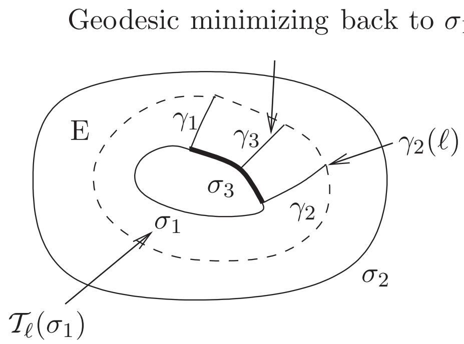


Figure 16: Corollary II.2.3: Finding a geodesic γ_3 satisfying (II.2.2) between two other geodesics γ_1, γ_2.

be a geodesic with $\gamma_{\varepsilon,3}(\ell_\varepsilon) = \partial\eta_\varepsilon \setminus \sigma_1$ and $\text{dist}_\Sigma(\gamma_{\varepsilon,3}(t), \sigma_1) = t$ for $0 \leq t \leq \ell_\varepsilon$. Since $\sigma_{\varepsilon,4} \subset \mathcal{T}_{\ell+\varepsilon}(\sigma_1) \setminus \mathcal{T}_{\ell-\varepsilon}(\sigma_1)$, we see that

$$\ell - \varepsilon < \ell_\varepsilon \leq \ell + \varepsilon\,.$$

Moreover, by the triangle inequality, if $\varepsilon < \mu$, then $\mathcal{B}_\mu(\gamma_k(\ell)) \cap \gamma_{\varepsilon,3} = \emptyset$ for $k = 1, 2$, hence

$$\mathcal{B}_\mu(\gamma_k(\ell)) \cap (\eta \cup \gamma_{\varepsilon,3}) = \emptyset \text{ for } k = 1, 2 \text{ and } \varepsilon < \mu\,.$$

We claim that

$$\text{(II.2.4)} \qquad \eta_\varepsilon \cup \gamma_{\varepsilon,3} \subset \mathcal{T}_\delta(\Sigma \setminus E) \text{ where } \delta \to 0 \text{ as } \varepsilon \to 0\,.$$

Suppose that (II.2.4) fails; it follows that there exists a sequence $\varepsilon_i \to 0$ and $x_i \in \eta_{\varepsilon_i}$ with $x_i \to x$, where

$$x_i, x \in \Sigma \setminus \mathcal{T}^\circ_\delta(\Sigma \setminus E) \subset E$$

for some $\delta > 0$. Since E is open and connected, there exists a curve $\nu \subset E$ from x to σ_2 and so $\nu \subset E \setminus \mathcal{T}_{\delta_0}(\Sigma \setminus E)$ for some $\delta_0 = \delta_0(x) > 0$. For i sufficiently large, we can extend ν to a curve

$$\nu_i \subset E \setminus \mathcal{T}_{\delta_0/2}(\Sigma \setminus E)$$

from x_i to σ_2. However, the curve

$$\gamma_1 \cup \gamma_2 \cup \sigma_{\varepsilon_i,4} \subset \mathcal{T}_{\varepsilon_i}(\Sigma \setminus E)$$

separates x_i from σ_2 which is a contradiction for i sufficiently large. Hence, (II.2.4) holds.

Pick a sequence $\varepsilon_i > 0$ with $\varepsilon_i \to 0$. After passing to a subsequence, we can assume that $\gamma_{\varepsilon_j,3} \to \gamma_3$. It is clear that $\gamma_3 : [0, \ell] \to \Sigma$ is a geodesic with $\gamma_3(0) \in \sigma_1 \setminus \{\gamma_1(0), \gamma_2(0)\}$, $\text{dist}_\Sigma(\gamma_3(t), \sigma_1) = t$ for $0 \leq t \leq \ell$, and $\gamma_3(\ell) \in \overline{E}$. It remains to show that $\gamma_3(0) \in \sigma_3$.

If $\gamma_3(0) \notin \sigma_3$, then $\text{dist}_\Sigma(\gamma_3(0), \sigma_3) > 0$ (since $\gamma_3(0) \in \sigma_1 \setminus \{\gamma_1(0), \gamma_2(0)\}$) and therefore for j large we have

$$\text{dist}_\Sigma(\gamma_{\varepsilon_j,3}(0), \sigma_3) > 0\,.$$

It follows that $\eta_{\varepsilon_j} \cup \gamma_{\varepsilon_j,3}$ divides Σ into two components $\Sigma_{\varepsilon_j,1}$, $\Sigma_{\varepsilon_j,2}$ with

$$\gamma_1(\ell) \in \Sigma_{\varepsilon_j,1} \text{ and } \gamma_2(\ell) \in \Sigma_{\varepsilon_j,2}\,.$$

(That $\gamma_1(\ell)$, $\gamma_2(\ell)$ are in different components follows from $\gamma_{\varepsilon,3} \cap \gamma_1 = \emptyset = \gamma_{\varepsilon,3} \cap \gamma_2$ by the triangle inequality.) After possibly switching γ_1 and γ_2 (and going to a subsequence), we can assume that $\sigma_2 \subset \Sigma_{\varepsilon_j,2}$. Note that

$$\mathcal{B}_\mu(\gamma_1(\ell)) \subset \Sigma_{\varepsilon_j,1}$$

since we showed above that $\mathcal{B}_\mu(\gamma_1(\ell)) \cap (\eta \cup \gamma_{\varepsilon,3}) = \emptyset$. We will use this to contradict that $\gamma_1(\ell) \in \overline{E}$. Namely, choose

$$x \in \mathcal{B}_{\mu/2}(\gamma_1(\ell)) \cap E$$

(note that such an x exists since $\gamma_1(\ell) \in \overline{E}$). Since E is open and connected, there exists a curve

$$\nu \subset E \setminus \mathcal{T}_{\delta_0}(\Sigma \setminus E)$$

for some sufficiently small $\delta_0 = \delta_0(x) > 0$ which connect x and σ_2. This contradicts (II.2.4) for j sufficiently large since $\eta_{\varepsilon_j} \cup \gamma_{\varepsilon_j,3}$ separate $\Sigma_{\varepsilon_j,1}$ and σ_2. □

The next lemma bounds the area of a minimal surface with two sides and an interior boundary in a small ball in terms of the length of the sides, provided that the surface "initially leaves" the small ball.

LEMMA II.2.5. *If $\Sigma \subset \mathbf{R}^3$ is an immersed minimal surface with $\partial\Sigma = \gamma_1 \cup \gamma_2 \cup \sigma$ where $\sigma \subset B_1$, $\partial_{\mathbf{n}}|x| \geq 0$ on σ ($\mathbf{n}$ is the inward normal to $\partial\Sigma$), and γ_1, γ_2 have length $\leq \ell$, then*

$$\text{Area}(\mathcal{T}_1(\sigma)) \leq 4\,\ell\,(\ell+1)\,. \tag{II.2.6}$$

Proof. By minimality, Stokes' theorem, $\partial_{\mathbf{n}}|x| \geq 0$ on σ, and $|x| \leq \ell + 1$ on γ_i,

$$4\,\text{Area}(\Sigma) = \int_\Sigma \Delta|x|^2 \leq 2\int_{\gamma_1\cup\gamma_2} |x|\,|\partial_{\mathbf{n}}|x||\leq 4\,(\ell+1)\,\ell\,. \tag{II.2.7}$$ □

In what follows, if $\sigma \subset \partial\Sigma$ is a simple curve, $\mathbf{n}$ is the inward normal to σ, $\tilde{\sigma}$ is a segment of σ, then (see Figure 17)

$$\mathcal{T}_s(\tilde{\sigma}, \mathbf{n}) = \{\exp_{\tilde{\sigma}(t)}(\tau\,\mathbf{n}(t))\,|\,\text{dist}_\Sigma(\exp_{\tilde{\sigma}(t)}(\tau\,\mathbf{n}(t)), \sigma) = \tau \leq s\}\,. \tag{II.2.8}$$

In the next proposition and Corollary II.2.10, we will construct the minimizing geodesics γ_1, γ_2 needed for Corollary II.1.45.

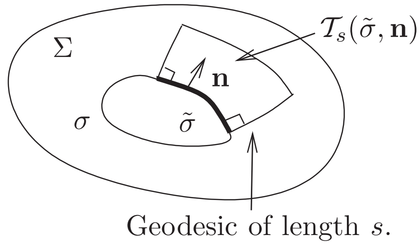


Figure 17: The region $\mathcal{T}_s(\tilde\sigma,\mathbf{n})$ in (II.2.8).

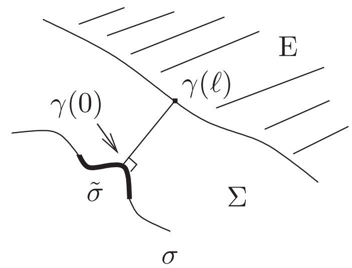


Figure 18: Proposition II.2.9: Finding a geodesic $\gamma \subset \Sigma$ which minimizes back to the curve σ.

PROPOSITION II.2.9 (see Figure 18). *Let $\Sigma \subset \mathbf{R}^3$ be an immersed minimal annulus, $\sigma \subset B_1 \cap \partial\Sigma$ a simple closed curve with* $\text{dist}_\Sigma(\sigma, \partial\Sigma \setminus \sigma) > \ell$, $\partial_{\mathbf{n}}|x| \geq 0$ *on σ, and let E be as in Lemma* II.2.1. *If $\tilde\sigma$ is a segment of σ and*

$$\text{Area}(\mathcal{T}_1(\tilde\sigma,\mathbf{n})) > 4\,\ell\,(\ell+1)\,,$$

then there exists a geodesic $\gamma : [0,\ell] \to \Sigma$ with $\text{dist}_\Sigma(\gamma(t),\sigma) = t$ *for* $0 \leq t \leq \ell$ *and $\gamma(0) \in \tilde\sigma$, $\gamma(\ell) \in \overline{E}$.*

Proof. Suppose that there is no such geodesic γ. Let B be the set of geodesics satisfying (II.2.2) for $\sigma_1 = \sigma$. It follows easily that

$$A = \{\gamma_0(0) \,|\; \gamma_0 \in B\}$$

is a closed subset of $\sigma \setminus \tilde\sigma$ containing more than two points. Let $\hat\sigma$ be the connected component of $\sigma \setminus A$ containing $\tilde\sigma$ (note that $\hat\sigma$ is open) and let $\partial\hat\sigma = \{\gamma_1(0), \gamma_2(0)\}$ where γ_1, γ_2 are the corresponding minimizing geodesics of lengths ℓ.

By Corollary II.2.3, $\gamma_1(\ell) = \gamma_2(\ell)$. In fact, there exists a subset $\hat\Sigma$ of Σ with $\partial\hat\Sigma = \gamma_1 \cup \gamma_2 \cup \hat\sigma$. Since

$$\text{Area}(\mathcal{T}_1(\hat\sigma,\tilde\Sigma)) \geq \text{Area}(\mathcal{T}_1(\tilde\sigma,\tilde\Sigma)) = \text{Area}(\mathcal{T}_1(\tilde\sigma,\mathbf{n})) > 4\,\ell\,(\ell+1)\,,$$

it follows from Lemma II.2.5 that $A \cap \tilde\sigma \neq \emptyset$ which is the desired contradiction; the proposition follows. □

COROLLARY II.2.10. *Let $\Sigma \subset \mathbf{R}^3$ be an immersed minimal annulus, $\sigma \subset B_1 \cap \partial\Sigma$ be a simple closed curve with* $\text{dist}_\Sigma(\sigma, \partial\Sigma \setminus \sigma) > \ell \geq 1$, $\partial_{\mathbf{n}}|x| \geq 0$ *on σ, and*

$$\text{Area}(\mathcal{T}_1(\sigma,\Sigma)) > 12\,\ell^2\,(\ell+1)\,.$$

For each $z_1 \in \sigma$ there is a segment $\sigma_1 \subset \sigma$ with $z_1 \in \sigma_1$ and geodesics γ_1, $\gamma_2 : [0,\ell] \to \Sigma$ with $\{\gamma_1(0), \gamma_2(0)\} = \partial\sigma_1$,

$$\text{dist}_\Sigma(\gamma_i(t),\sigma) = t \text{ for } 0 \leq t \leq \ell\,. \tag{II.2.11}$$

Moreover, for all $\varepsilon > 0$ *a disk* $\Sigma_0 \subset \Sigma$ *has* σ_1, $\gamma_i \subset \partial\Sigma_0$, $\text{dist}_{\Sigma_0}(\partial\Sigma_0 \setminus \sigma_1 \cup \gamma_1 \cup \gamma_2) > \ell - \varepsilon$,

$$15\,\ell^2\,(\ell+1) > \text{Area}(\mathcal{T}_1(\sigma_1, \Sigma_0)) > 4\,\ell^2\,(\ell+1)\,. \tag{II.2.12}$$

Proof. Let $\sigma^1_{z_1}$, $\sigma^2_{z_1}$, $\sigma^3_{z_1}$ be three consecutive (disjoint) subsegments of σ with $z_1 \in \sigma^2_{z_1}$ being the "middle one" so that for each i

$$5\,\ell^2\,(\ell+1) > \text{Area}(\mathcal{T}_1(\sigma^i_{z_1}, \mathbf{n})) > 4\,\ell^2\,(\ell+1)\,. \tag{II.2.13}$$

By Proposition II.2.9 applied to both $\sigma^1_{z_1}$ and $\sigma^3_{z_1}$, we get geodesics γ_1, $\gamma_2 : [0,\ell] \to \Sigma$ satisfying (II.2.11) and with

$$\gamma_1(0) \in \sigma^1_{z_1},\ \gamma_2(0) \in \sigma^3_{z_1},\ \text{and } \gamma_i(\ell) \in \overline{E}$$

(where E is the connected component of $\Sigma \setminus \mathcal{T}_\ell(\sigma)$ containing σ_2). Let σ_1 be the segment of σ between $\gamma_1(0)$ and $\gamma_2(0)$ containing $\sigma^2_{z_1}$. By Lemma II.2.1 there is a disk $\Sigma_0 \subset \Sigma$ with σ_1, γ_1, $\gamma_2 \subset \partial\Sigma_0$, and

$$\text{dist}_{\Sigma_0}(\partial\Sigma_0 \setminus \sigma_1 \cup \gamma_1 \cup \gamma_2) > \ell - \varepsilon\,.$$

We need to show (II.2.12). Since $\sigma^2_{z_1} \subset \sigma_1$, the lower bound in (II.2.12) follows easily from (II.2.13). To see the upper bound, observe that if $x \in \mathcal{T}_1(\sigma_1, \Sigma_0)$, then clearly

$$\text{dist}_\Sigma(x, \sigma) = \text{dist}_{\Sigma_0}(x, \sigma_1)$$

and hence

$$\mathcal{T}_1(\sigma_1, \Sigma_0) \subset \cup_{i=1,2,3} \mathcal{T}_1(\sigma^i_{z_1}, \mathbf{n})\,. \tag{II.2.14}$$

From (II.2.13) and (II.2.14), the upper bound in (II.2.12) follows. □

Proof of Theorem II.1.2. Given N, ε, let ω_0, R_0 be given by Corollary II.1.45. Set

$$\sigma = \partial B_1 \cap \partial\Gamma_1(\partial)$$

and note that $\partial_{\mathbf{n}}|x| \geq 0$ and $B_1 \cap \partial\Gamma$ is connected since Γ is an annulus. Suppose that $\text{dist}_\Sigma(\sigma, \partial\Gamma \setminus \sigma) > R_0$. By Corollary II.1.34, (1) holds if

$$\text{Area}(\mathcal{T}_1(\sigma)) \leq 12\,R_0^2\,(R_0+1)\,.$$

(Recall that if one component of $B_{R/\omega} \cap \Gamma \setminus B_\omega$ contains a graph over $D_{R/(2\omega)} \setminus D_{2\omega}$ with gradient ≤ 1, then every component of $B_{R/(C\omega)} \cap \Gamma \setminus B_{C\omega}$ is a graph for some $C > 1$.)

On the other hand if

$$\text{Area}(\mathcal{T}_1(\sigma)) > 12\,R_0^2\,(R_0+1)\,,$$

then it follows from Corollary II.1.45 together with Corollary II.2.10 that (2) holds. □

Using the fact that the curvature of a 2-valued embedded minimal graph decays faster than quadratically (this was shown in [CM8]), we show next (this will be needed in the next section) that such 2-valued graphs contain minimal geodesics close to the radial curve $\theta = 0$. (In particular, there is such a geodesic which does not spiral.) In this corollary, $\Gamma_{\lambda\omega}(\partial)$ denotes the component of $B_{\lambda\omega} \cap \Gamma$ containing $B_{\lambda\omega} \cap \partial\Gamma$.

COROLLARY II.2.15. *There exists* $\lambda > 1$ *so that the following holds*:
If Γ *is as in Theorem* II.1.2, $\Gamma^{-3\pi,3\pi}_{\omega,R/\omega}$ *is as in* (2) *of that theorem* (*with* $N \geq 3$, $\tau \leq 1$), *and* $R > \lambda\omega^2$, *then there exists a geodesic* $\gamma : [0,\ell] \to \Gamma^{-\pi,\pi}_{\omega,R^{1/2}}$ *with* $\gamma(0) \in \partial B_{\lambda\omega}$, $\ell \geq R^{1/2}/4$, *and*

$$\mathrm{dist}_{\Gamma}(\gamma(t), \Gamma_{\lambda\omega}(\partial)) = t\,.$$

Proof. Fix $\lambda > 1$ large to be chosen. Set

$$\mathrm{r} = \mathrm{dist}_{\Gamma^{-2\pi,2\pi}_{\omega,R^{1/2}}}(\Gamma^{-2\pi,2\pi}_{\omega,\omega}, \cdot)$$

and let $\Gamma^{-3\pi,3\pi}_{\omega,R/\omega}$ be the graph of u. By Corollary 1.14 of [CM8], on $S^{-2\pi,2\pi}_{\omega,R^{1/2}}$ we have

$$\rho\,|\mathrm{Hess}_u| \leq C'\,(\rho/\omega)^{-5/12}\,.$$

Hence, on $\Gamma^{-2\pi,2\pi}_{\omega,R^{1/2}}$ we have

$$\mathrm{r}\,|A| \leq C\,\omega^{5/12}\,\mathrm{r}^{-5/12}\,. \tag{II.2.16}$$

Let $\gamma : [0,\ell] \to \Gamma$ be a minimizing geodesic in Γ from (the point) $\Gamma^{0,0}_{R^{1/2}/3,R^{1/2}/3}$ to $\Gamma_{\lambda\omega}(\partial)$, so that for $0 \leq t \leq \ell$ we get

$$\mathrm{dist}_{\Gamma}(\gamma(t), \Gamma_{\lambda\omega}(\partial)) = t\,.$$

In particular, $\gamma(0) \in \partial\Gamma_{\lambda\omega}(\partial)$ and $\gamma(\ell) = \Gamma^{0,0}_{R^{1/2}/3,R^{1/2}/3}$. Using the radial curve $\Gamma^{0,0}_{\omega,R^{1/2}/3}$ as a comparison (and $\tau \leq 1$), we see that

$$\mathrm{Length}(\gamma) = \ell \leq R^{1/2}/2\,.$$

Let $\tilde{\gamma}$ be the maximal segment of γ in $\Gamma^{-\pi,\pi}_{\omega,R^{1/2}}$ containing $\gamma(\ell)$. Since $\tilde{\gamma}$ is a geodesic in Γ, (II.2.16) gives the bound

$$k_g^{\mathbf{R}^3}(t) \leq C\,\omega^{5/12}\,t^{-1-5/12}$$

for the geodesic curvature of $\tilde{\gamma}$ in $\mathbf{R}^3$. It follows that for $\lambda\omega \leq t \leq \ell$

$$|\tilde{\gamma}'(t) - \gamma'(\ell)| \leq C\,\omega^{5/12} \int_{\lambda\omega}^{\infty} s^{-17/12}\,ds \leq 12\,C\,\lambda^{-5/12}/5\,; \tag{II.2.17}$$

i.e., $\tilde{\gamma}$ is C^1-close to a straight line segment in $\mathbf{R}^3$. For λ large, (II.2.17) implies that either

$$\tilde{\gamma} \subset \Gamma^{-3\pi/4,3\pi/4}_{\omega,R^{1/2}/2} \text{ or } \gamma \text{ leaves } B_{R^{1/2}}\,.$$

The latter is impossible since $\text{Length}(\gamma) \leq R^{1/2}/2$. We conclude that $\tilde{\gamma} \subset \Gamma_{\omega,R^{1/2}/2}^{-3\pi/4,3\pi/4}$. In particular, $\tilde{\gamma} = \gamma$ and the corollary follows. □

II.3. Area growth of stable sectors and the proof of Theorem 0.3

In this section, we show that case (2) in Theorem II.1.2 does not happen and thus Theorem 0.3 follows easily. To do that, we first prove upper and lower bounds for the area of a stable sector over a curve σ_1 if the sides γ_1, γ_2 of the sector are contained in multi-valued graphs Σ_1, Σ_2. By [CM8], the number of sheets of each Σ_i grows at least like $\log^2 \rho$, giving the lower area bound

$$\text{Area} \geq \rho^2 \, \log^2 \rho$$

when the Σ_i's are disjoint. We use this growing number of sheets to construct a function χ, with small energy, which vanishes on the sides γ_1, γ_2. Inserting χ in Lemma II.1.3 gives the upper area bound

$$\text{Area} \leq \rho^2 (C + \log\log\rho)$$

(where $C = C(\sigma_1)$). If ρ is large depending on C, then these bounds are contradictory and hence the Σ_i's cannot be disjoint.

We will use several times the fact that, given $\alpha > 0$, Proposition II.2.12 of [CM3] gives $N_g > 0$ so that if u satisfies the minimal surface equation on

$$S_{\mathrm{e}^{-N_g},\mathrm{e}^{N_g} R}^{-N_g,2\pi+N_g}$$

with $|\nabla u| \leq 1$, and $w < 0$ (where w is the separation), then on $S_{1,R}^{0,2\pi}$,

$$\rho \, |\text{Hess}_u| + \rho \, |\nabla w|/|w| \leq \alpha \, .$$

Theorem 3.36 of [CM7] then yields

$$|\nabla u - \nabla u(1,0)| \leq C\alpha \, .$$

We can therefore assume (after rotating so that $\nabla u(1,0) = 0$) that

$$\text{(II.3.1)} \qquad |\nabla u| + \rho \, |\text{Hess}_u| + 4 \, \rho \, |\nabla w|/|w| + \rho^2 \, |\text{Hess}_w|/|w| \leq \varepsilon < 1/(2\pi) \, .$$

The bound on $|\text{Hess}_w|$ follows from the other bounds and standard elliptic theory.

The next lemma shows that an embedded multi-valued minimal graph in a concave cone (intersected with cylindrical shells; see Figure 19)

$$\text{(II.3.2)} \qquad \mathcal{C}_{\Lambda,R}(h) = \{x \,|\, (x_3 - h)^2 \leq \Lambda^2 \, (x_1^2 + x_2^2) \, , \, 1/4 \leq x_1^2 + x_2^2 \leq R^2\}$$

has at least $\log^2 \rho$ many sheets. Note that the axis of the cone $\mathcal{C}_{\Lambda,R}(h)$ is the x_3-axis and the vertex is $(0,0,h)$. We will only need $\log \rho$ sheets for most of what follows, except for the lower bound for the area given in Corollary II.3.16 below.

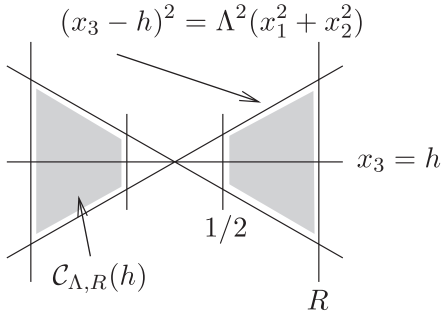


Figure 19: The truncated cone $\mathcal{C}_{\Lambda,R}(h)$.

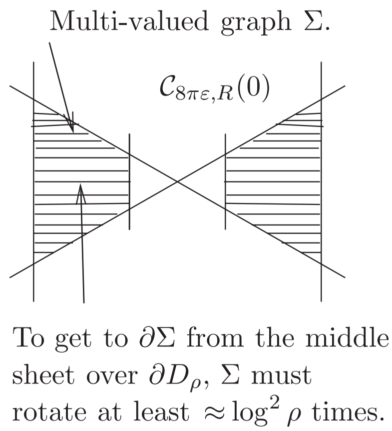


Figure 20: Lemma II.3.3: It takes at least $\approx \log^2 \rho$ rotations for a multi-valued graph to spiral out of the cone $\mathcal{C}_{8\,\pi\,\varepsilon,R}(0)$.

Lemma II.3.3 (see Figure 20). *Given $\varepsilon > 0$, there exist $0 < C_1 < 1$ and C_2 so that the following holds*:

Let $\Sigma \subset \mathcal{C}_{8\,\pi\,\varepsilon,R}(0)$ with $\partial\Sigma \subset \partial\mathcal{C}_{8\,\pi\,\varepsilon,R}(0)$ be a minimal multi-valued graph of u with $w < 0$ and $u(1,0) = 0$. If the domain of u contains $S_{1/2,R}^{-2\pi,2\pi}$ and u satisfies (II.3.1), *then Σ contains a* (*multi-valued*) *graph over*

$$\{(\rho,\theta)\,|\,|\theta| \leq C_1 \log^2 \rho + \pi,\ 1 \leq \rho \leq R^{3/4}/2\} \tag{II.3.4}$$

with

$$\rho^2\,|A|^2 \leq C_2\,\rho^{-5/18}\,. \tag{II.3.5}$$

Proof. Corollary 1.14 of [CM8] gives on $S_{1,R^{3/4}}^{-\pi,\pi}$ that

$$\rho^2\,|\mathrm{Hess}_u(\rho,\theta)|^2 \leq C\,\rho^{-5/18}\,, \tag{II.3.6}$$

directly giving (II.3.5) for $|\theta| \leq \pi$. By Corollary 5.7 of [CM8], Σ contains a (multi-valued) graph over

$$\{(\rho,\theta)\,|\,c^2\,|\theta| \leq \log^2 \rho,\ 1 \leq \rho \leq R^{3/4}\}$$

so that if $n \in \mathbf{Z}$ satisfies $2\,\pi\,c^2\,|n| \leq \log^2 \rho$, then

$$|u(\rho,2\pi n) - u(\rho,0)| \leq \rho^\varepsilon\,.$$

Applying the Harnack inequality and elliptic estimates to the function

$$w_n(\rho,\theta) = u(\rho,2\pi n + \theta) - u(\rho,\theta)$$

(cf. (1.17) of [CM8]), we get

(II.3.7) $$\rho\,|\nabla u(\rho,2\pi n)-\nabla u(\rho,0)|+\rho^2\,|\mathrm{Hess}_u(\rho,2\pi n)-\mathrm{Hess}_u(\rho,0)|\leq C'\,\rho^{\varepsilon}\,.$$

Combining (II.3.6) and (II.3.7) then easily gives (II.3.5) in general. □

We first define a function $0\leq\chi\leq 1$ on $\mathcal{P}$ (the universal cover of $\mathbf{C}\setminus\{0\}$) which is

- 0 on $S^{-\pi,\pi}_{3/4,\infty}$,
- 1 on $\{\rho<R^{3/4}/2\}\setminus(S^{-2\pi,2\pi}_{1/2,R}\cup$ (II.3.4)), and
- so that $|\nabla_{\mathcal{P}}\chi|^2$ is of the order $(\rho\log\rho)^{-2}$ for ρ large.

Namely, set

(II.3.8)
$$\chi(\rho,\theta)=\begin{cases}3-4\rho & \text{for } 1/2\leq\rho<3/4,\ |\theta|\leq\pi\,,\\ 1-(C_1-|\theta|+\pi)(4\rho-2)/C_1 & \text{for } 1/2\leq\rho<3/4,\ \pi\leq|\theta|\leq C_1+\pi\,,\\ 0 & \text{for } |\theta|\leq\pi,\ 3/4\leq\rho\,,\\ (|\theta|-\pi)/C_1 & \text{for } 3/4\leq\rho<\mathrm{e},\ \pi\leq|\theta|\leq C_1+\pi\,,\\ (|\theta|-\pi)/(C_1\,\log\rho) & \text{for } \mathrm{e}\leq\rho,\ \pi\leq|\theta|\leq C_1\,\log\rho+\pi\,,\\ 1 & \text{otherwise}\,.\end{cases}$$

Using (II.3.8), define χ on a (multi-valued) graph over a domain containing

$$S^{-2\pi,2\pi}_{1/2,R}\cup\ \text{(II.3.4)}$$

in the obvious way. Note that if Σ is as in Lemma II.3.3, then $1-\chi$ is one on the central sheet $\Sigma^{-\pi,\pi}_{3/4,R}$ and vanishes before Σ leaves the cone on the top, bottom, or inside.

Corollary II.3.9. *Given $\varepsilon>0$, there exists C_3 so that if Σ and u are as in Lemma* II.3.3, *then*

- $\chi=0$ *over* $S^{-\pi,\pi}_{3/4,R}$,
- $\chi=1$ *on* $\{x_1^2+x_2^2\leq R^{3/2}/4\}\cap\partial\Sigma$,

and for $\mathrm{e}<t\leq R^{3/4}/2$,

(II.3.10) $$\int_{\{\chi<1,\,x_1^2+x_2^2\leq t^2\}}|A|^2+\int_{\{x_1^2+x_2^2\leq t^2\}\cap\Sigma}|\nabla\chi|^2\leq C_3\,(1+\log\log t)\,.$$

Proof. Clearly, $\chi=0$ over $S^{-\pi,\pi}_{3/4,R}$. By Lemma II.3.3, $\chi=1$ on $\{x_1^2+x_2^2\leq R^{3/2}/4\}\cap\partial\Sigma$. To get (II.3.10), first consider χ as a function downstairs on $\mathcal{P}$. On $\{\rho\leq\mathrm{e}\}$,

$$|\nabla_{\mathcal{P}}\chi|\leq C_0 \text{ and } \{|\nabla_{\mathcal{P}}\chi|\neq 0\}\subset\{|\theta|\leq C_1+\pi,\ 1/2\leq\rho\}\,.$$

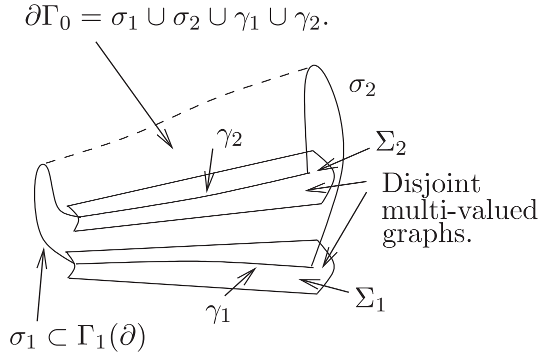


Figure 21: A stable Γ satisfying i)–iii): $\Gamma_0 \subset \Gamma$ is a disk with geodesics $\gamma_1, \gamma_2 \subset \partial\Gamma_0$ which are in the middle sheets of multi-valued graphs Σ_1, Σ_2.

Similarly, on $\{\mathrm{e} \leq \rho\}$,

$$|\partial_\theta \chi|/\rho \leq 1/(C_1 \rho \log \rho) \text{ and } |\partial_\rho \chi| \leq 1/(\rho \log \rho)\,,$$

so that

$$|\nabla_{\mathcal{P}} \chi|^2 \leq 2\,(C_1 \rho \log \rho)^{-2} \text{ and } \{|\nabla_{\mathcal{P}} \chi| \neq 0\} \subset \{\pi \leq |\theta| \leq C_1 \log \rho + \pi\}\,.$$

Therefore, since Σ is a graph with gradient ≤ 1, it follows easily that

$$\text{(II.3.11)} \qquad \int_{\{x_1^2+x_2^2 \leq t^2\} \cap \Sigma} |\nabla \chi|^2 \leq C_0' + \frac{12}{C_1} \int_{\mathrm{e}}^{t} \frac{ds}{s \log s} = C_0' + \frac{12 \log \log t}{C_1}\,.$$

Similarly, using (II.3.5) gives

$$\text{(II.3.12)} \quad \int_{\{\chi<1,\, x_1^2+x_2^2 \leq t^2\}} |A|^2 \leq C + 4\,C_2 \int_{\mathrm{e}}^{\infty} (\pi + C_1 \log s) s^{-23/18}\, ds \leq C'\,.$$

Finally, combining (II.3.11) and (II.3.12) gives (II.3.10). □

The next corollary gives upper and lower bounds for the areas of tubular neighborhoods in a Γ which satisfies i)–iii) below; see Figure 21. ($\Gamma_t(\partial)$ is the component of $B_t \cap \Gamma$ containing $B_t \cap \partial\Gamma$.)

i) $\Gamma \subset B_{2R}$ is a stable embedded minimal surface, $\partial\Gamma \subset B_{1/4} \cup \partial B_{2R}$, $B_{1/4} \cap \partial\Gamma$ is connected, and $\Gamma_0 \subset \Gamma$ is a disk with

$$\partial\Gamma_0 = \gamma_1 \cup \gamma_2 \cup \sigma_1 \cup \sigma_2\,,$$

where $\gamma_i : [0, \ell_i] \to \Sigma$ is a geodesic, $\gamma_i(0) \in \sigma_1 \subset \Gamma_1(\partial)$, and $\gamma_i \perp \sigma_1$.

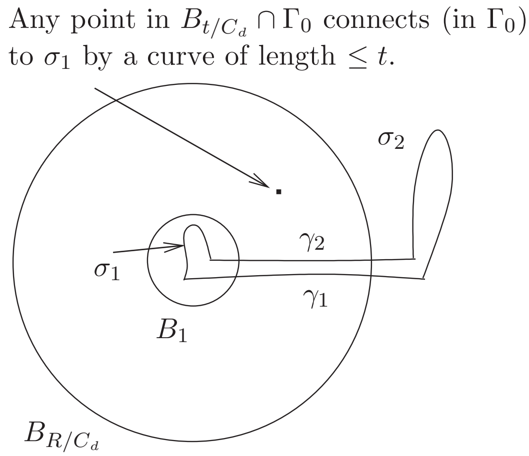


Figure 22: Lemma II.3.13: A chord-arc property for a stable Γ satisfying i)–iii).

ii) $\Sigma_1, \Sigma_2 \subset \Gamma$ are disjoint (multi-valued) graphs over domains containing $S^{-2\pi,2\pi}_{1/2,R}$ of functions u_1, u_2 satisfying (II.3.1), $w_i < 0$,

$$\Sigma_i \subset \mathcal{C}_{8\,\pi\,\varepsilon,R}(u_i(1,0))\,,$$
$$\partial\Sigma_i \subset \partial\mathcal{C}_{8\,\pi\,\varepsilon,R}(u_i(1,0))\,,$$
$$\gamma_i \subset (\Sigma_i)^{-\pi,\pi}_{3/4,R}\,.$$

iii) $\text{dist}_\Gamma(\gamma_i(t), \Gamma_1(\partial)) = t$ for $0 \leq t \leq \ell_i$, $\ell_i \geq R-1$, and $\text{dist}_\Gamma(\sigma_2, \Gamma_1(\partial)) \geq R-1$.

We first show that intrinsic and extrinsic distances to σ_1 are roughly equivalent (see Figure 22) in the following lemma:

LEMMA II.3.13. *There exists $C_d > 1$ so that if* i)–iii) *hold and $R > C_d$, then $B_{R/C_d} \cap \sigma_2 = \emptyset$ and $B_{t/C_d} \cap \Gamma_0 \subset \mathcal{T}_t(\sigma_1, \Gamma_0)$ for $C_d < t < R$.*

Proof. Both of these assertions follow easily from stability together with the assumption that Γ contains multi-valued graphs. That is, suppose that either one failed. It follows easily that there exists a point in Γ which is extrinsically much closer to the origin than its intrinsic distance to the inner boundary of Γ. This easily implies by stability that Γ contains a large almost flat graph over a disk centered at the origin which easily contradicts that Γ contains multi-valued graphs since these would be forced to spiral into the almost flat graph. We will now make this argument precise.

Fix $C_d > 1$ to be chosen. We show first for $1 < t < R/C_d$ that

$$B_t \cap \Gamma \subset \mathcal{T}_{C_d t}(B_{1/4} \cap \partial\Gamma)\,.$$

Suppose that $y \in B_{R/C_d} \cap \Gamma$. Fix $C > 2$ and $\delta > 0$ to be chosen. Since Γ is stable, the estimates of [Sc] and [CM2] give a constant $C'_d = C'_d(C, \delta)$ so that:

> If $\text{dist}_\Gamma(y, \partial\Gamma) > C'_d\,(1 + |y|)$, then $\mathcal{B}_{C'_d\,(1+|y|)}(y)$ contains a graph Γ_y with gradient $\leq \delta$ over a disk $B_{C\,(1+|y|)}(y) \cap P_y$, where $P_y \subset \mathbf{R}^3$ is the plane tangent to Γ at y.

Since Γ is embedded (and since Γ contains a multi-valued graph Σ_1 around γ_1 with $\gamma_1(0) \in B_1$), we can choose C, δ so that Γ would then be forced to spiral into Γ_y. This is impossible since Γ is compact. Since $\partial\Gamma \subset \Gamma_1(\partial) \cup \partial B_{2R}$, it follows that

$$B_t \cap \Gamma \subset \mathcal{T}_{2C'_d t}(B_{1/4} \cap \partial\Gamma)$$

for $1 < t < R/C_d$. Combining this and iii) gives $B_{(R-1)/(2C'_d)} \cap \sigma_2 = \emptyset$.

Suppose that $y \in B_{R/C_d} \cap \Gamma_0$ so that (by the first part) $y' \in \partial\Gamma_0$ with

$$\text{dist}_{\Gamma_0}(y, y') + \text{dist}_\Gamma(y', B_{1/4} \cap \partial\Gamma) \leq C'_d\,(1 + |y|) < R\,. \tag{II.3.14}$$

In particular, $y' \in \sigma_1 \cup \gamma_1 \cup \gamma_2$. Since $\text{dist}_\Gamma(\gamma_i(t), \Gamma_1(\partial)) = t$,

$$\text{dist}_{\sigma_1 \cup \gamma_i}(y', \sigma_1) \leq C'_d\,(1 + |y|)\,, \tag{II.3.15}$$

so that $\text{dist}_{\Gamma_0}(y, \sigma_1) \leq 2\,C'_d\,(1 + |y|)$. The lemma follows. □

The next corollary gives upper and lower bounds for the areas of tubular neighborhoods in a Γ which satisfies i)–iii).

Corollary II.3.16. *Given* $\varepsilon, C_I > 0$, *there exists* $C_4 > 0$ *so that if* i)–iii) *hold and* $R^{3/4} > 12C_d$, *then for* $\mathrm{e} < t \leq R^{3/4}/4 - 1$,

$$\begin{aligned} &C_4 \log^2 t \leq t^{-2}\text{Area}(\mathcal{T}_t(\sigma_1, \Gamma_0)) \\ &\qquad \leq \frac{1}{C_4}\left(1 + \int_{\mathcal{T}_{C_I}(\sigma_1,\Gamma_0)} (1 + |A|^2) + \int_{\sigma_1} (1 + |k_g|) + \log\log t\right). \end{aligned} \tag{II.3.17}$$

Proof. Since $\sigma_1 \subset \Gamma_1(\partial)$, i) and iii) imply (A) with $C_0 = 0$, (C), and (D) with $\ell = R - 1$. Using Corollary II.3.9 on Σ_1, Σ_2, we can define χ on $\{x_1^2 + x_2^2 \leq R^{3/2}/4\} \cap \Gamma$ which

- vanishes on γ_1, γ_2,
- is one on $\{x_1^2 + x_2^2 \leq R^{3/2}/4\} \cap \Gamma \setminus (\Sigma_1 \cup \Sigma_2)$, and
- satisfies (II.3.10) (with double the constant).

Since $\mathcal{T}_t(\sigma_1, \Gamma_0) \subset \{x_1^2 + x_2^2 \leq R^{3/2}/4\}$, inserting (II.3.10) into Lemma II.1.3 (and scaling so $C_I \to 1$) gives the second inequality in (II.3.17).

By Lemma II.3.3, we have that Σ_1 and Σ_2 each contain a (multi-valued) graph over (II.3.4). Suppose now that

$$\mathrm{e} < t < R^{3/4}/(4C_d)\,.$$

By Lemma II.3.13, we have

$$\{1 < x_1^2 + x_2^2 \leq t^2\} \cap \Gamma \subset B_{2t} \cap \Gamma \subset \mathcal{T}_{2C_d t}(\sigma_1, \Gamma)$$

and $B_{2t} \cap \sigma_2 = \emptyset$ (by iii)). Since $\sigma_1 \subset B_1$, $\gamma_i \subset (\Sigma_i)^{-\pi,\pi}_{3/4,R}$, and $\Sigma_1 \cap \Sigma_2 = \emptyset$, it then follows easily that $\mathcal{T}_{2C_d t}(\sigma_1, \Gamma_0)$ contains one component of

$$\{1 < x_1^2 + x_2^2 \leq t^2\} \cap \Sigma_1 \setminus (\Sigma_1)^{-\pi,\pi}_{1,t}\,.$$

The first inequality in (II.3.17) follows immediately (after possibly decreasing $C_4 > 0$). □

We are now finally ready to prove Theorem 0.3. That is, we will show that all embedded stable minimal surfaces with small interior boundaries are graphical away from the boundary.

Proof of Theorem 0.3. Rescale so that $r_0 = 1$. Set $\hat{\Gamma} = \Gamma \setminus \Gamma_1(\partial)$ so (since Γ is topologically an annulus) $\partial\hat{\Gamma} = \sigma \cup \hat{\sigma}$ where $\sigma \subset \partial B_1$, $\hat{\sigma} \subset \partial B_R$ are the two connected components of $\partial\hat{\Gamma}$, and $\partial_{\mathbf{n}}|x| \geq 0$ on σ (where $\mathbf{n}$ is the inward normal to $\partial\hat{\Gamma}$).

By Theorem II.1.2 we need only prove that (2) does not happen for $\hat{\Gamma}$. Suppose it does; we will obtain a contradiction. The key point will be to find two oppositely oriented multi-valued graphs in Γ which have fixed bounded distance between them and then apply Corollary II.3.16 for t sufficiently large to get a contradiction.

Fix (ordered) points $z_1, \ldots, z_m \in \sigma$ so that $\sigma \setminus \{z_1, \ldots, z_m\}$ has components $\{\sigma_{z_1}, \ldots, \sigma_{z_m}\}$ where $\partial\sigma_{z_i} = \{z_i, z_{i+1}\}$ (set $z_{m+1} = z_1$) and $\mathrm{Length}(\sigma_{z_i}) \leq 1$. By Theorem II.1.2 (and the discussion surrounding (II.3.1)), we have that Γ contains 3-valued graphs Σ_{z_i} of u_{z_i} satisfying (II.3.1) over $D_{R/\omega} \setminus D_\omega$ (after a rotation of $\mathbf{R}^3$; *a priori* this rotation may depend on z_i) and with

$$\mathrm{dist}_{\hat{\Gamma}}(z_i, (\Sigma_{z_i})^{0,0}_{\omega,\omega}) < d_0\,.$$

Combining this with Corollary II.2.15, we get 3-valued graphs $\{\Sigma_{z_i}\}$, geodesics

$$\gamma_{z_i} : [0, \ell_{z_i}] \to (\Sigma_{z_i})^{-\pi,\pi}_{\omega,R^{1/2}}$$

with $\gamma_{z_i}(0) \in \partial B_{\lambda\omega}$, $\mathrm{dist}_\Gamma(\gamma_{z_i}(t), \Gamma_{\lambda\omega}(\partial)) = t$ for $0 \leq t \leq \ell_{z_i}$, and $\gamma_i(\ell_{z_i}) \subset \Gamma \setminus B_{R^{1/2}/3}$. After possibly increasing λ, we can assume that $\lambda\omega > 2d_0 + 2$. Hence, the curves in $\hat{\Gamma}$ from z_i to $(\Sigma_{z_i})^{0,0}_{\omega,\omega}$ given by Theorem II.1.2 are contained in $B_{\lambda\omega/2}$. Therefore, since $(\Sigma_{z_i})^{-3\pi,3\pi}_{\omega,\lambda\omega}$ is a graph, we can choose curves $\eta_{z_i} \subset$

$\Gamma_{\lambda\omega}(\partial)$ from $\gamma_{z_i}(0)$ to z_i with length $\leq 2\lambda\omega + 4\pi\omega$ and so $\eta_{z_i} \setminus B_{\lambda\omega/2}$ is simple with

$$\int_{\eta_{z_i} \setminus B_{\lambda\omega/2}} |k_g| \leq C\,.$$

It follows immediately from embeddedness that the Σ_{z_i}'s are graphs over a common plane. From the gradient estimate (which applies because of estimates for stable surfaces of [Sc], [CM2]), each component of Γ intersected with a concave cone is also a multi-valued graph. Since $\partial B_{\lambda\omega} \cap \partial \Gamma_{\lambda\omega}(\partial)$ is a closed curve, it must pass between the sheets of each Σ_{z_i}. It is now easy to see that each Σ_{z_i} contains an oppositely oriented multi-valued graph $\hat{\Sigma}_{z_i}$ between its sheets (i.e., $\mathbf{n}_\Gamma$ points in almost opposite directions on Σ_{z_i} and $\hat{\Sigma}_{z_i}$). Furthermore, since Lemma II.3.13 bounds the distance in $\hat{\Gamma}$ from $\hat{\Sigma}_{z_i}$ to σ, we can assume that two of the Σ_{z_i}'s are oppositely oriented. We can therefore choose two consecutive 3-valued graphs, Σ_{z_j}, $\Sigma_{z_{j+1}}$, which are oppositely oriented; rename these Σ_1, Σ_2 (and similarly the corresponding γ_1, γ_2, ℓ_1, ℓ_2).

By replacing $B_{\lambda\omega/2} \cap (\sigma_{z_j} \cup \eta_{z_j} \cup \eta_{z_{j+1}})$ with a broken geodesic and finding a simple subcurve as in Lemma II.1.11, we get a simple curve

$$\sigma_1 \subset \Gamma_{\lambda\omega}(\partial) \setminus \Gamma_{7/8}(\partial)$$

from $\gamma_1(0)$ to $\gamma_2(0)$ with

$$\int_{\sigma_1} (1 + |k_g|) \leq C_a\,. \tag{II.3.18}$$

Furthermore, since $\sigma_1 \subset \Gamma_{\lambda\omega}(\partial))$, we get for $0 \leq t \leq \ell_i$ that

$$\text{dist}_\Gamma(\gamma_i(t), \sigma_1) = t\,.$$

Let Γ_0 be the component of

$$\Gamma_{R^{1/2}/3}(\partial) \setminus (\sigma_1 \cup \gamma_1 \cup \gamma_2)$$

which does not contain $\Gamma_{7/8}(\partial)$; set

$$\sigma_2 = \partial\Gamma_0 \setminus (\sigma_1 \cup \gamma_1 \cup \gamma_2)\,.$$

It follows that Γ_0 is a disk and $\text{dist}_\Gamma(\Gamma_0, \partial\Gamma) \geq 5/8$. Since $(\Sigma_{z_i})^{-3\pi,3\pi}_{\omega,\lambda\omega}$ is a graph, we can perturb σ_1 near $\gamma_1(0), \gamma_2(0)$ to arrange that $\sigma_1 \perp \gamma_1$ and $\sigma_1 \perp \gamma_2$ and so σ_1 still satisfies (II.3.18) with a slightly larger constant C_a. Combining (II.3.18) and estimates for stable surfaces of [Sc], [CM2], we get

$$\int_{\mathcal{T}_{1/8}(\sigma_1,\Gamma_0)} (1 + |A|^2) + \int_{\sigma_1} (1 + |k_g|) \leq C_b\,. \tag{II.3.19}$$

Hence (after rescaling), $\Gamma_0, \Gamma, \Sigma_1, \Sigma_2, \gamma_1, \gamma_2, \sigma_1$ satisfy i) and iii). To get ii), we use [Sc], [CM2] and the gradient estimate to extend Σ_1, Σ_2 as multi-valued graphs inside the cones

$$\mathcal{C}_{8\,\pi\,\varepsilon, R^{1/2}/4}(u_i(1,0))\,;$$

the opposite orientation guarantees that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Corollary II.3.16 and (II.3.19) give for $C_5 < t < R^{3/8}/C_5$

$$C_4 \log^2 t \leq t^{-2} \text{Area}(\mathcal{T}_t(\sigma_1, \Gamma_0)) \leq (1 + C_b + \log \log t) / C_4 \,. \tag{II.3.20}$$

This gives the desired contradiction for R large, completing the proof. □

III. Nearby points with large curvature

In this part, we extend Theorem 0.3 (proven for stable surfaces) to surfaces with *extrinsic* quadratic curvature decay

$$|A|^2 \leq C \, |x|^{-2} \,.$$

As mentioned in the introduction, this extension is needed in both [CM5] and [CM6]. In [CM5] it is used for disks to get points of large curvature nearby and on each side of a given point with large curvature (in particular it is used to show that such points are not extrinsically isolated).

Stability was used in the proof of Theorem 0.3 for two purposes:

(a) To show *intrinsic* quadratic curvature decay.

(b) To bound the total curvature using the stability inequality.

To get the extension to the extrinsic quadratic curvature decay case, we will deal with (a) and (b) separately in the next two sections. To get (a), we relate extrinsic and intrinsic distances (i.e., we show a "chord-arc" property). For (b), we follow Section 2 of [CM4] to decompose a surface with quadatric curvature decay into disjoint almost stable subdomains and a "remainder" with quadratic area growth.

For applications of the results of this part in [CM5], Σ will be a disk and hence $\partial \Sigma_{0,t}$ is connected for all t (here, and elsewhere, if $0 \in \Sigma$, then $\Sigma_{0,t}$ denotes the component of $B_t \cap \Sigma$ containing 0). However, in [CM6], when we apply the results here to deal with the first possibility in (4) of Theorem 0.1 (i.e., the analog of the genus one helicoid), Σ is no longer a disk but $\partial \Sigma$ is still connected (which is assumed in many of the results below).

III.1. Relating intrinsic and extrinsic distances

In this section, $0 \in \Sigma \subset B_R$ is an embedded minimal surface with $\partial \Sigma \subset \partial B_R$ satisfying:

- $|A|^2 \leq C_1^2 |x|^{-2}$ on $\Sigma \setminus B_1$.
- $\partial \Sigma_{0,t}$ is connected for $1 \leq t \leq R$.

The next lemma shows that only one component of $B_{C_b} \cap \Sigma$ intersects B_2. The second lemma bounds the radius of the intrinsic tubular neighborhood of $B_2 \cap \Sigma$ containing this component. Combining these iteratively (on decreasing scales) in Corollary III.1.5 gives the "chord-arc" property needed to establish (a).

LEMMA III.1.1. *Given C_1, there exists C_b so that if $\Sigma_{0,1}$ is not a graph, then*

$$B_2 \cap \Sigma \subset \Sigma_{0,C_b} .$$

Proof. Suppose that Σ_1, Σ_2 are disjoint components of $B_{C_b} \cap \Sigma$ with $B_2 \cap \Sigma_i \neq \emptyset$. It follows that there is a component Ω of $B_{C_b} \setminus \Sigma$ and a segment $\eta \subset B_2 \setminus \Sigma$ so that $\partial \Sigma_{0,C_b}$ is linked with η in Ω (cf. Lemma 2.1 in [CM9]). Since Ω is mean convex, we can solve the Plateau problem as in [MeYa2] to get a stable minimal surface $\Gamma \subset \Omega$ with $\partial \Gamma = \partial \Sigma_{0,C_b}$. The linking implies that $B_2 \cap \Gamma \neq \emptyset$. The curvature estimates of [Sc], [CM2] then give a graph $\Gamma_0 \subset \Gamma$ of a function u_0 over $D_{C_b/C}$ (after a rotation) with

$$|u_0(z)| \leq |z| .$$

By Corollary 1.14 of [CM8] (applied with $w = 0$), we can assume that on $D_{C_b^{1/2}/C}$

$$|\nabla u_0|(z) \leq C' \, |z|^{-5/12} . \tag{III.1.2}$$

In particular, Γ_0 is close to a horizontal plane. The lemma now follows from an argument used in [CM9] (see also [CM10]) which we now outline: Σ intersects a narrow cone about Γ_0, then contains a long chain of graphical balls (by the gradient estimate), and must then either spiral indefinitely or close up as a graph. Namely, for $t < C_b^{1/2}/C$, the surface $\Sigma_{0,t}$ sits on one side of Γ_0. However, by Lemma 2.4 of [CM9] (for $t > C'$), we have that $\partial \Sigma_{0,t}$ contains a "low point," i.e., a point y_0 with

$$|x_3(y_0)| \leq \delta \, t$$

with $\delta > 0$ small. The gradient estimate (since $|A|^2 \leq C_1^2 |x|^{-2}$ on $\Sigma \setminus B_1$) gives a long chain of balls $\mathcal{B}_{c\,t}(y_i)$ with

$$y_i \in \partial \Sigma_{0,t} \cap \{|x_3| \leq C' \, \delta \, t\}$$

which is a (possibly multi-valued) graph. Since $\partial \Sigma_{0,t}$ cannot spiral forever, this graph closes up. By Rado's theorem (note that no assumption on the topology is needed for this application of Rado's theorem; cf. the proof of theorem 1.22 in [CM4]), $\Sigma_{0,t}$ is itself a graph, giving the lemma. $\square$

The next lemma bounds the radius of the intrinsic tubular neighborhood of $B_2 \cap \Sigma$ containing the only component of $B_{C_b} \cap \Sigma$ intersecting B_2.

Lemma III.1.3. *Given C_1, C_b, there exists C_c so that if $R > C_c$, then for all $y \in \Sigma_{0,C_b}$*

$$\text{(III.1.4)} \qquad \text{dist}_{\Sigma}(y, B_1 \cap \Sigma) \leq C_c \, .$$

Proof. Let $\tilde{\Sigma}$ be the universal cover of Σ and $\tilde{\Pi} : \tilde{\Sigma} \to \Sigma$ the covering map. With the definition of δ-stable as in Section 2 of [CM4], the argument of [CM2] (i.e., curvature estimates for 1/2-stable surfaces) gives $C > 10$ so that if $\mathcal{B}_{CC_b/2}(\tilde{z}) \subset \tilde{\Sigma}$ is 1/2-stable and $\tilde{\Pi}(\tilde{z}) = z$, then

$$\tilde{\Pi} : \mathcal{B}_{5C_b}(\tilde{z}) \to \mathcal{B}_{5C_b}(z)$$

is one-to-one and $\mathcal{B}_{5C_b}(z)$ is a graph with $B_{4C_b}(z) \cap \partial \mathcal{B}_{5C_b}(z) = \emptyset$. Corollary 2.13 in [CM4] gives $\varepsilon = \varepsilon(C, C_1, C_b) > 0$ so that if $|z_1 - z_2| < \varepsilon$ and $|A|^2 \leq C_1^2$ on (the disjoint balls) $\mathcal{B}_{CC_b}(z_i)$, then each $\mathcal{B}_{CC_b/2}(\tilde{z}_i) \subset \tilde{\Sigma}$ is 1/2-stable where $\tilde{\Pi}(\tilde{z}_i) = z_i$.

We claim that there exists n so that

$$B_1 \cap \mathcal{B}_{(2n+1)\, CC_b}(y) \neq \emptyset \, .$$

Suppose not; we get a curve

$$\sigma \subset \Sigma_{0,C_b} \setminus \mathcal{T}_{CC_b}(B_1 \cap \Sigma)$$

from y to $\partial \mathcal{B}_{2n\, CC_b}(y)$. For $i = 1, \ldots, n$, fix points $z_i \in \partial \mathcal{B}_{2i\, CC_b}(y) \cap \sigma$. The intrinsic balls $\mathcal{B}_{CC_b}(z_i) \subset \Sigma \setminus B_1$ are disjoint, have centers in $B_{C_b} \subset \mathbf{R}^3$, and

$$\sup_{\mathcal{B}_{CC_b}(z_i)} |A|^2 \leq C_1^2 \, .$$

Hence, there exist i_1 and i_2 with

$$0 < |z_{i_1} - z_{i_2}| < C' \, C_b \, n^{-1/3} < \varepsilon \, ,$$

and, by Corollary 2.13 in [CM4], each $\mathcal{B}_{CC_b/2}(\tilde{z}_{i_j}) \subset \tilde{\Sigma}$ is 1/2-stable where $\tilde{\Pi}(\tilde{z}_{i_j}) = z_{i_j}$. By [CM2], each $\mathcal{B}_{5C_b}(z_{i_j})$ is a graph with $B_{4C_b}(z_{i_j}) \cap \partial \mathcal{B}_{5C_b}(z_{i_j}) = \emptyset$. In particular,

$$B_{C_b} \cap \partial \mathcal{B}_{5C_b}(z_{i_j}) = \emptyset \, .$$

This contradicts the fact that $\sigma \subset B_{C_b}$ connects z_{i_j} to $\partial \mathcal{B}_{CC_b}(z_{i_j})$. □

The next corollary combines the two previous lemmas to get the "chord-arc" property needed to establish (a).

Corollary III.1.5. *Given C_1, there exists C_c so that if $\Sigma_{0,1}$ is not a graph and $y \in B_{R/C_c} \cap \Sigma$, then*

$$\text{(III.1.6)} \qquad \text{dist}_{\Sigma}(y, B_1 \cap \Sigma) \leq 2\, C_c \, |y| \, .$$

Proof. Suppose $y \in B_{2^n} \setminus B_{2^{n-1}}$. By Lemma III.1.1, we have $y \in \Sigma_{0,C_b 2^{n-1}}$ where $C_b = C_b(C_1)$. Set $y_n = y$. Lemma III.1.3 gives $y_{n-1} \in B_{2^{n-1}} \cap \Sigma$ with

$$\text{dist}_\Sigma(y_n, y_{n-1}) \leq C_c\, 2^{n-1}\,.$$

We can now repeat the argument. Namely, by Lemma III.1.1, we have $y_{n-1} \in \Sigma_{0,C_b 2^{n-2}}$ and then Lemma III.1.3 gives $y_{n-2} \in B_{2^{n-2}} \cap \Sigma$ with

$$\text{dist}_\Sigma(y_{n-1}, y_{n-2}) \leq C_c\, 2^{n-2}\,.$$

After n steps, we get $y_0 \in B_1 \cap \Sigma$ with

$$\text{dist}_\Sigma(y, y_0) \leq \sum_{i=1}^{n} \text{dist}_\Sigma(y_i, y_{i-1}) \leq \sum_{i=1}^{n} C_c\, 2^{i-1} \leq 2\, C_c\, |y|\,. \tag{III.1.7}$$ □

III.2. A decomposition from [CM4]

In Lemma 2.15 of [CM4], we decomposed an embedded minimal surface in a ball with bounded curvature into disjoint, almost stable subdomains and a remainder with bounded area. The same argument gives the following lemma:

LEMMA III.2.1. *Given C_1, there exists C_d so that the following holds*:
If $\Sigma \subset B_{2R}$ is an embedded minimal surface with $\partial\Sigma \subset \partial B_{2R} \cup B_{1/2}$, and

$$|A|^2 \leq C_1^2\, |x|^{-2}\,,$$

then there exist disjoint $1/2$-stable subdomains $\Omega_j \subset \Sigma$ and a function $0 \leq \psi \leq 1$ on Σ which vanishes on $(B_R \setminus B_1) \cap \Sigma \setminus (\cup_j \Omega_j)$ so that

$$\text{Area}(\{x \in (B_R \setminus B_1) \cap \Sigma \,|\, \psi(x) < 1\}) \leq C_d\, R^2\,, \tag{III.2.2}$$

$$\int_{B_R \cap \Sigma} |\nabla \psi|^2 \leq C_d\, \log R\,. \tag{III.2.3}$$

In the proof of Theorem 0.5 in the next section, Lemma III.2.1 will be used to extend the area bounds for stable surfaces proved in Sections II.1 and II.3 (specifically those in Lemma II.1.3, Proposition II.1.20, and Corollary II.3.16) to minimal surfaces with $|A|^2 \leq C_1^2\, |x|^{-2}$. This is very similar to how Lemma 2.15 of [CM4] was used in Lemma 3.1 of [CM4].

By Lemma III.2.1, we have that

$$\int_{B_R \cap \Sigma} |\nabla \psi|^2 + \int_{B_R \cap \{\psi < 1\}} |A|^2$$

grows (in R) at most like $\log R$. We use this below in the 1/2-stability inequality to get the total curvature bound needed for (b). This is used in the proof of Theorem III.3.1.

III.3. Theorem 0.5 and a generalization

As already mentioned, stability was used in the proof of Theorem 0.3 to establish (a) and (b) in the introduction to Part III; these were extended in the two previous sections to surfaces with a quadratic curvature bound. In [CM5] we will need the contrapositive of Theorem 0.5, i.e., we will need to find points where the quadratic bound fails. In fact, what we will really need is to find points on "each side" of a multi-valued graph where this fails; this is the following theorem:

(Here $u_1(r_0, 2\pi) < u_2(r_0, 0) < u_1(r_0, 0)$ just says that the two graphs

THEOREM III.3.1 (see Figure 23). *Given C_1, there exists C_2 so that the following holds*:

Let $0 \in \Sigma \subset B_{2C_2\, r_0}$ be an embedded minimal surface with connected $\partial \Sigma \subset \partial B_{2C_2\, r_0}$ and $\text{gen}(\Sigma_{0,r_0}) = \text{gen}(\Sigma)$. *Suppose that*

$$\Sigma_1 \text{ and } \Sigma_2 \subset \Sigma \cap \{x_3^2 \leq (x_1^2 + x_2^2)\}$$

are (multi-valued) graphs of functions u_i satisfying (II.3.1) *on $S^{-2\pi,2\pi}_{r_0, C_2 r_0}$ with*

$$u_1(r_0, 2\pi) < u_2(r_0, 0) < u_1(r_0, 0)\,,$$

and $\nu \subset \partial \Sigma_{0,2r_0}$ is a curve from Σ_1 to Σ_2. If Σ_0 is the component of

$$\Sigma_{0,C_2 r_0} \setminus (\Sigma_1 \cup \Sigma_2 \cup \nu)$$

which does not contain Σ_{0,r_0}, then

$$\sup_{x \in \Sigma_0 \setminus B_{4r_0}} |x|^2\, |A|^2(x) \geq 4\, C_1^2\,. \tag{III.3.2}$$

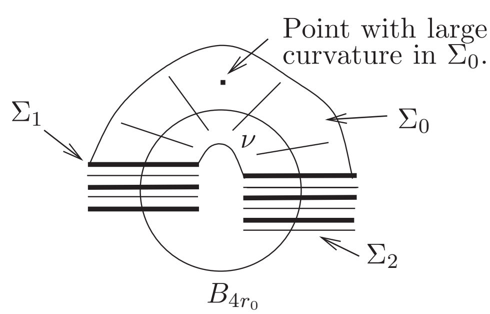


Figure 23: Theorem III.3.1 and Corollary III.3.5 — existence of nearby points with large curvature.

Proof. Suppose that (III.3.2) fails for some C_1; as in the proof of Theorem 0.3, we will show contradictory upper and lower bounds for the area growth for C_2 sufficiently large.

Note that for $r_0 \leq s \leq 2C_2 r_0$, it follows from the maximum principle (since Σ is minimal) and Corollary I.0.11 that $\partial \Sigma_{0,s}$ is connected and $\Sigma \setminus \Sigma_{0,s}$ is an annulus.

Note also that the gradient estimate (which applies because of the curvature bound) allows us to extend each Σ_i (inside Σ_0) as a graph of u_i over ∂D_ρ as long as

$$|u_i(\rho,\theta) - u_i(\rho,[\theta])| \leq C_g\, \rho\,,$$

where $\theta - [\theta] \in 2\pi\mathbf{Z}$ and $0 \leq [\theta] \leq 2\pi$. By Corollary 1.14 of [CM8], the curvature of Σ_i decays faster than quadratically. Combining these (and increasing the inner radius), we can assume that each Σ_i extends (inside Σ_0) as a graph until it leaves a cone $\{x_3^2 \leq \Lambda^2(x_1^2 + x_2^2)\}$ for some small $\Lambda > 0$. Moreover, these extended multi-valued graphs must stay disjoint since

$$u_1(r_0, 2\pi) < u_2(r_0, 0) < u_1(r_0, 0)\,.$$

We next choose the inner boundary curve where we argue as in Theorem 0.3. By Lemma III.1.1, we have

$$B_{4r_0} \cap \Sigma \subset \Sigma_{0,2C_b r_0}\,.$$

In particular, $\partial\Sigma_{0,2C_b r_0}$ separates $B_{4r_0} \cap \Sigma$ from $\partial\Sigma$. We can therefore replace ν with a segment of $\partial\Sigma_{0,2C_b r_0}$ from Σ_1 to Σ_2 so (for the new Σ_0)

$$\sup_{x\in\Sigma_0} |x|^2\, |A|^2(x) \leq 4\,\bar{C}_1^2\,. \tag{III.3.3}$$

By Corollary III.1.5 (the "chord-arc" property), intrinsic and extrinsic distances to $B_{4r_0} \cap \Sigma$ are compatible. Hence, we get

$$\sup_{x\in\Sigma_0} \operatorname{dist}^2_\Sigma(x, B_{4r_0} \cap \Sigma)\, |A|^2(x) \leq C_3\,. \tag{III.3.4}$$

The proof of Theorem 0.3 now applies with two changes (and the minor modifications which result):

(a′) The curvature estimates for stable surfaces of [Sc], [CM2] are replaced with (III.3.4).

(b′) The total curvature bound from the stability inequality in (II.1.6) is replaced with the bound using Lemma III.2.1 and the 1/2-stability inequality (cf. Lemma 3.1 of [CM4]).

Namely, using (a′) and (b′), the proof of Theorem II.1.2 extends from stable surfaces to surfaces satisfying (III.3.4) (with (b′) being used in Lemma II.1.3 and Proposition II.1.20 exactly as in [CM4]). It follows that each z in (the new) ν is a fixed bounded distance from a multi-valued graph (either Σ_1, Σ_2 or a new multi-valued graph in between). Hence, as in the proof of Theorem 0.3, we can choose two consecutive multi-valued graphs which are oppositely oriented; let σ_1 be the curve connecting these. Next, (b′) contributes a new

$$C_4\, t^2 \, \log t$$

term to the upper bound for the area of a sector $\mathcal{T}_t(\sigma_1)$ in the upper bound for the area in Corollary II.3.16 where C_4 does not depend on σ_1 (see the last paragraph of Section III.2). However, since the lower bound for the area is on the order of

$$t^2 \, \log^2 t \, ,$$

we get the desired contradiction as before. □

In [CM5], we will use the special case of Theorem III.3.1 where Σ is a disk:

Corollary III.3.5 (see Figure 23). *Given C_1, there exists C_2 so that the following holds*:

Let $0 \in \Sigma \subset B_{2C_2\, r_0}$ be an embedded minimal disk. Suppose that

$$\Sigma_1 \text{and} \Sigma_2 \subset \Sigma \cap \{x_3^2 \leq (x_1^2 + x_2^2)\}$$

are graphs of functions u_i satisfying (II.3.1) *on $S^{-2\pi,2\pi}_{r_0,C_2 r_0}$ with*

$$u_1(r_0, 2\pi) < u_2(r_0, 0) < u_1(r_0, 0) \, ,$$

and $\nu \subset \partial \Sigma_{0,2r_0}$ is a curve from Σ_1 to Σ_2. Let Σ_0 be the component of

$$\Sigma_{0,C_2 r_0} \setminus (\Sigma_1 \cup \Sigma_2 \cup \nu)$$

which does not contain Σ_{0,r_0}.

If either:

- $\partial \Sigma \subset \partial B_{2C_2\, r_0}$, *or*
- Σ *is stable and* Σ_0 *does not intersect* $\partial \Sigma$,

then

$$\sup_{x \in \Sigma_0 \setminus B_{4r_0}} |x|^2 \, |A|^2(x) \geq 4\, C_1^2 \, . \qquad \text{(III.3.6)}$$

Proof. Since Σ is a disk, $\partial \Sigma$ is connected and

$$\text{gen}(\Sigma_{0,r_0}) = \text{gen}(\Sigma) = 0 \, .$$

Hence, Theorem III.3.1 gives the corollary when $\partial \Sigma \subset \partial B_{2C_2\, r_0}$.

When Σ is stable and Σ_0 does not intersect $\partial \Sigma$, then Σ_1, Σ_2 each extend inside cones in at least one direction as multi-valued graphs. This gives essentially half of the multi-valued graphs Σ_1, Σ_2 used in Section II.3 which is all that is needed in the proof of Theorem 0.3. The corollary now follows easily from the proof of Theorem 0.3 (with Σ_1, Σ_2 causing the same modifications as in Theorem III.3.1). □

Note that if C_1 is large, then (III.3.6) would contradict the curvature estimate for stable surfaces of [Sc], [CM2]. In [CM5], we will apply Corollary III.3.5 in this way, showing that such a stable Σ does not exist.

In [CM5], we will also use the other case of Corollary III.3.5, where Σ is not assumed to be stable, to get points of large curvature "metrically" on each side of the multi-valued graph Σ_1. Namely, note first that the curve $\partial \Sigma_{0,2r_0} \setminus \nu$ in Corollary III.3.5 has the same properties as ν. In [CM5], ν (and hence also Σ_0) will be on one side of Σ_1, Σ_2 while $\partial \Sigma_{0,2r_0} \setminus \nu$ is on the other. Applying Corollary III.3.5 to each of these will give points of large curvature "topologically " on each side of Σ_1, Σ_2.

In fact, we will see in [CM5] that if an embedded minimal disk Σ contains one multi-valued graph Σ_1, then it will contain a second multi-valued graph Σ_2 which spirals together with Σ_1 ("the other half"). We will also see there that

$$\partial \Sigma_{0,Cr_0} \setminus (\Sigma_1 \cup \Sigma_2)$$

has exactly two components $\nu_\pm$; it follows easily that we can assume ν_+ is above and ν_- is below Σ_1. Applying Corollary III.3.5 to both $\nu_\pm$ will give points of large curvature "metrically" on each side of Σ_1.

Proof of Theorem 0.5. It suffices to show that if $\text{Area}(\Sigma_{0,r_0}) > C_3\, r_0^2$, then (0.6) fails.

Note that for $r_0 \leq s \leq R$, it follows from the maximum principle (since Σ is minimal) and Corollary I.0.11 that $\partial \Sigma_{0,s}$ is connected and $\Sigma \setminus \Sigma_{0,s}$ is an annulus.

The proof is now virtually identical to the proof of Theorem III.3.1 except that it simplifies since we no longer keep track of the two sides and (1) in (an analog of) Theorem II.1.2 becomes $\text{Area}(\Sigma_{0,r_0}) \leq C_3'\, r_0^2$. □

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References

[CM1] T. H. Colding and W. P. Minicozzi II, *Minimal Surfaces*, *Courant Lecture Notes in Math.* **4**, New York University Press, Courant Institute of Math. Sciences, New York (1999).

[CM2] ———, Estimates for parametric elliptic integrands, *Internat. Math. Res. Not.* **6** (2002), 291–297.

[CM3] ———, The space of embedded minimal surfaces of fixed genus in a 3-manifold I; Estimates off the axis for disks, *Ann. of Math.* **160** (2004), 27–68; math.AP/0210106.

[CM4] T. H. Colding and W. P. Minicozzi II, The space of embedded minimal surfaces of fixed genus in a 3-manifold II; Multi-valued graphs in disks, *Ann. of Math.* **160** (2004), 69–92; math.AP/0210086.

[CM5] ———, The space of embedded minimal surfaces of fixed genus in a 3-manifold IV; Locally simply connected, *Ann. of Math.* **160** (2004), 573–615; math.AP/0210119.

[CM6] ———, The space of embedded minimal surfaces of fixed genus in a 3-manifold V; Fixed genus, in preparation.

[CM7] ———, Minimal annuli with and without slits, *J. Symplectic Geom.* **1** (2001), 47–61.

[CM8] ———, Multi-valued minimal graphs and properness of disks, *Internat. Math. Res. Not.* **21** (2002), 1111–1127.

[CM9] ———, On the structure of embedded minimal annuli, *Internat. Math. Res. Not.* **29** (2002), 1539–1552.

[CM10] ———, Complete properly embedded minimal surfaces in $\mathbf{R}^3$, *Duke Math. J.* **107** (2001), 421–426.

[CM11] ———, Embedded minimal disks, in *Minimal surfaces* (*MSRI*, 2001), *Clay Mathematics Proceedings*, 405–438, D. Hoffman, ed., AMS, Providence, RI, 2004; math.DG/0206146.

[CM12] ———, Disks that are double spiral staircases, *Notices Amer. Math. Soc.* **50** (2003), 327–339.

[CM13] ———, Embedded minimal disks: Proper versus nonproper — global versus local, *Trans. Amer. Math. Soc.* **356** (2004), 283–289.

[Fi] D. Fischer-Colbrie, On complete minimal surfaces with finite Morse index in 3-manifolds, *Invent. Math.* **82** (1985), 121–132.

[FrMe] C. Frohman and W. H. Meeks, The topological uniqueness of complete one-ended minimal surfaces and Heegaard surfaces in $\mathbf{R}^3$, *J. Amer. Math. Soc.* **10** (1997), 495–512.

[HoKrWe] D. Hoffman, H. Karcher, and F. Wei, Adding handles to the helicoid, *Bull. Amer. Math. Soc.* **29** (1993), 77–84.

[Ka] N. Kapouleas, On desingularizing the intersections of minimal surfaces, *Proc. 4th Internat. Congress of Geometry* (Thessaloniki, 1996), 34–41, Giachoudis-Giapoulis, Thessaloniki, 1997.

[LoRo] F. Lopez and A. Ros, On embedded complete minimal surfaces of genus zero, *J. Differential Geom.* **33** (1991), 293–300.

[MeYa1] W. Meeks III and S.-T. Yau, The classical Plateau problem and the topology of three-dimensional manifolds. The embedding of the solution given by Douglas-Morrey and an analytic proof of Dehn's lemma, *Topology* **21** (1982), 409–442.

[MeYa2] ———, The existence of embedded minimal surfaces and the problem of uniqueness, *Math. Z.* **179** (1982), 151–168.

[Sc] R. Schoen, Estimates for stable minimal surfaces in three-dimensional manifolds, in *Seminar on Minimal Submanifolds*, *Ann. of Math. Studies* **103**, 1111–126, Princeton Univ. Press, Princeton, NJ (1983).

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