

Geometry of the uniform spanning forest: Transitions in dimensions 4, 8, 12, . . .

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Abstract

The uniform spanning forest (USF) in \mathbb{Z}^d is the weak limit of random, uniformly chosen, spanning trees in $[-n, n]^d$. Pemantle [11] proved that the USF consists a.s. of a single tree if and only if $d \leq 4$. We prove that any two components of the USF in \mathbb{Z}^d are adjacent a.s. if $5 \leq d \leq 8$, but not if $d \geq 9$. More generally, let $N(x, y)$ be the minimum number of edges outside the USF in a path joining x and y in \mathbb{Z}^d . Then

$$\max\{N(x, y) : x, y \in \mathbb{Z}^d\} = \lfloor (d-1)/4 \rfloor \text{ a.s.}$$

The notion of stochastic dimension for random relations in the lattice is introduced and used in the proof.

1. Introduction

A *uniform spanning tree* (UST) in a finite graph is a subgraph chosen uniformly at random among all spanning trees. (A spanning tree is a subgraph such that every pair of vertices in the original graph are joined by a unique simple path in the subgraph.) The *uniform spanning forest* (USF) in \mathbb{Z}^d is a random subgraph of \mathbb{Z}^d , that was defined by Pemantle [11] (following a suggestion of R. Lyons), as follows: The USF is the weak limit of uniform spanning trees in larger and larger finite boxes. Pemantle showed that the limit exists, that it does not depend on the sequence of boxes, and that every connected component of the USF is an infinite tree. See Benjamini, Lyons, Peres and Schramm [2] (denoted BLPS below) for a thorough study of the construction and properties of the USF, as well as references to other works on the subject. Let $T(x)$ denote the tree in the USF which contains the vertex x .

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Also define

$$N(x, y) = \min\{\text{number of edges outside the USF in a path from } x \text{ to } y\}$$

(the minimum here is over all paths in \mathbb{Z}^d from x to y).

Pemantle [11] proved that for $d \leq 4$, almost surely $T(x) = T(y)$ for all $x, y \in \mathbb{Z}^d$, and for $d > 4$, almost surely $\max_{x,y} N(x, y) > 0$. The following theorem shows that a.s. $\max N(x, y) \leq 1$ for $d = 5, 6, 7, 8$, and that $\max N(x, y)$ increases by 1 whenever the dimension d increases by 4.

THEOREM 1.1.

$$\max\{N(x, y) : x, y \in \mathbb{Z}^d\} = \left\lfloor \frac{d-1}{4} \right\rfloor \quad \text{a.s.}$$

Moreover, a.s. on the event $\{T(x) \neq T(y)\}$, there exist infinitely many disjoint simple paths in \mathbb{Z}^d which connect $T(x)$ and $T(y)$ and which contain at most $\lfloor (d-1)/4 \rfloor$ edges outside the USF.

It is also natural to study

$$D(x, y) := \liminf_{n \rightarrow \infty} \{ |u - v| : u \in T(x), v \in T(y), |u|, |v| \geq n \},$$

where $|u| = \|u\|_1$ is the l^1 norm of u . The following result is a consequence of Pemantle [11] and our proof of Theorem 1.1.

THEOREM 1.2. *Almost surely, for all $x, y \in \mathbb{Z}^d$,*

$$D(x, y) = \begin{cases} 0 & \text{if } d \leq 4, \\ 1 & \text{if } 5 \leq d \leq 8 \text{ and } T(x) \neq T(y), \\ \infty & \text{if } d \geq 9 \text{ and } T(x) \neq T(y). \end{cases}$$

When $5 \leq d \leq 8$, this provides a natural example of a translation invariant random partition of \mathbb{Z}^d , into infinitely many components, each pair of which comes infinitely often within unit distance from each other.

The lower bounds on $N(x, y)$ follow readily from standard random walk estimates (see Section 5), so the bulk of our work will be devoted to the upper bounds.

Part of our motivation comes from the conjecture of Newman and Stein [10] that invasion percolation clusters in \mathbb{Z}^d , $d \geq 6$, are in some sense 4-dimensional and that two such clusters, which are formed by starting at two different vertices, will intersect with probability 1 if $d < 8$, but not if $d > 8$. A similar phenomenon is expected for minimal spanning trees on the points of a homogeneous Poisson process in \mathbb{R}^d . These problems are still open, as the tools presently available to analyze invasion percolation and minimal

spanning forests are not as sharp as those available for the uniform spanning forest.

In the next section the notion of stochastic dimension is introduced. A random relation $\mathcal{R} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ has *stochastic dimension* $d - \alpha$, if there is some constant $c > 0$ such that for all $x \neq z$ in \mathbb{Z}^d ,

$$c^{-1}|x - z|^{-\alpha} \leq \mathbf{P}[x\mathcal{R}z] \leq c|x - z|^{-\alpha},$$

and if a natural correlation inequality (2.2) (an upper bound for $\mathbf{P}[x\mathcal{R}z, y\mathcal{R}w]$) holds. The results regarding stochastic dimension are formulated and proved in this generality, to allow for future applications.

The bulk of the paper is devoted to the proof of the upper bound on $\max N(x, y)$ in (1.1). We now present an overview of this proof. Let $\mathcal{U}^{(n)}$ be the relation $N(x, y) \leq n - 1$. Then $x\mathcal{U}^{(1)}y$ means that x and y are in the same USF tree, and $x\mathcal{U}^{(2)}y$ means that $T(x)$ is equal to or adjacent to $T(y)$. We show that $\mathcal{U}^{(1)}$ has stochastic dimension 4 when $d \geq 4$. When $\mathcal{R}, \mathcal{L} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ are independent relations with stochastic dimensions $\dim_S(\mathcal{R})$ and $\dim_S(\mathcal{L})$, respectively, it is proven that the composition $\mathcal{L}\mathcal{R}$ (defined by $x\mathcal{L}\mathcal{R}y$ if and only if there is a z such that $x\mathcal{L}z$ and $z\mathcal{R}y$) has stochastic dimension $\min\{\dim_S(\mathcal{R}) + \dim_S(\mathcal{L}), d\}$. It follows that the composition of $m + 1$ independent copies of $\mathcal{U}^{(1)}$ has stochastic dimension d , where m is equal to the right hand of (1.1). By proving that $\mathcal{U}^{(m+1)}$ stochastically dominates the composition of $m + 1$ independent copies of $\mathcal{U}^{(1)}$, we conclude that $\dim_S(\mathcal{U}^{(m+1)}) = d$, which implies $\inf_{x, y \in \mathbb{Z}^d} \mathbf{P}[N(x, y) \leq m] > 0$. Nonobvious tail-triviality arguments then give $\mathbf{P}[N(x, y) \leq m] = 1$ for every x and y in \mathbb{Z}^d , which proves the required upper bound.

In Section 4 we present the relevant USF properties needed; in particular, we obtain a tight upper bound, Theorem 4.3, for the probability that a finite set of vertices in \mathbb{Z}^d is contained in one USF component. Fundamental for these results is a method from BLPS [2] for generating the USF in any transient graph, which is based on an algorithm by Wilson [15] for sampling uniformly from the spanning trees in finite graphs. (We recall this method in Section 4.)

Our main results are established in Section 5. Section 6 describes several examples of relations having a stochastic dimension, including long-range percolation, and suggests some conjectures. We note that proving $D(x, y) \in \{0, 1\}$ for $5 \leq d \leq 8$, is easier than the higher dimensional result. (The full power of Theorem 2.4 is not needed; Corollary 2.9 suffices.)

2. Stochastic dimension and compositions

Definition 2.1. When $x, y \in \mathbb{Z}^d$, we write $\langle xy \rangle := 1 + |x - y|$, where $|x - y| = \|x - y\|_1$ is the distance from x to y in the graph metric on \mathbb{Z}^d . Suppose that $W \subset \mathbb{Z}^d$ is finite, and τ is a tree on the vertex set W (τ need not be a

subgraph of \mathbb{Z}^d). Then let $\langle \tau \rangle := \prod_{\{x,y\} \in \tau} \langle xy \rangle$ denote the product of $\langle xy \rangle$ over all undirected edges $\{x, y\}$ in τ . Define the *spread* of W by $\langle W \rangle := \min_{\tau} \langle \tau \rangle$, where τ ranges over all trees on the vertex set W .

For three vertices, $\langle xyz \rangle = \min\{\langle xy \rangle \langle yz \rangle, \langle yz \rangle \langle zx \rangle, \langle zx \rangle \langle xy \rangle\}$. More generally, for n vertices, $\langle x_1 \dots x_n \rangle$ is a minimum of n^{n-2} products (since this is the number of trees on n labeled vertices); see Remark 2.7 for a simpler equivalent expression.

Definition 2.2 (Stochastic dimension). Let \mathcal{R} be a random subset of $\mathbb{Z}^d \times \mathbb{Z}^d$. We think of \mathcal{R} as a relation, and usually write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$. Let $\alpha \in [0, d)$. We say that \mathcal{R} has *stochastic dimension* $d - \alpha$, and write $\dim_S(\mathcal{R}) = d - \alpha$, if there is a constant $C = C(\mathcal{R}) < \infty$ such that

$$(2.1) \quad \mathbf{P}[x\mathcal{R}z] \geq \langle xz \rangle^{-\alpha},$$

and

$$(2.2) \quad \mathbf{P}[x\mathcal{R}z, y\mathcal{R}w] \leq C \langle xz \rangle^{-\alpha} \langle yw \rangle^{-\alpha} + C \langle xzyw \rangle^{-\alpha},$$

hold for all $x, y, z, w \in \mathbb{Z}^d$.

Observe that (2.2) implies

$$(2.3) \quad \mathbf{P}[x\mathcal{R}z] \leq 2C \langle xz \rangle^{-\alpha},$$

since we may take $x = y$ and $z = w$. Also, note that $\dim_S(\mathcal{R}) = d$ if and only if $\inf_{x,z \in \mathbb{Z}^d} \mathbf{P}[x\mathcal{R}z] > 0$.

To motivate (2.2), focus on the special case in which \mathcal{R} is a random equivalence relation. Then heuristically, the first summand in (2.2) represents an upper bound for the probability that x, z are in one equivalence class and y, w are in another, while the second summand, $C \langle xzyw \rangle^{-\alpha}$, represents an upper bound for the probability that x, z, y, w are all in the same class. Indeed, when the equivalence classes are the components of the USF, we will make this heuristic precise in Section 4.

Several examples of random relations that have a stochastic dimension are described in Section 6. The main result of Section 4, Theorem 4.2, asserts that the relation determined by the components of the USF has stochastic dimension 4.

Definition 2.3 (Composition). Let $\mathcal{L}, \mathcal{R} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ be random relations. The *composition* $\mathcal{L}\mathcal{R}$ of \mathcal{L} and \mathcal{R} is the set of all $(x, z) \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that there is some $y \in \mathbb{Z}^d$ with $x\mathcal{L}y$ and $y\mathcal{R}z$.

Composition is clearly an associative operation, that is, $(\mathcal{L}\mathcal{R})\mathcal{Q} = \mathcal{L}(\mathcal{R}\mathcal{Q})$. Our main goal in this section is to prove,

THEOREM 2.4. *Let $\mathcal{L}, \mathcal{R} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ be independent random relations. Then*

$$\dim_S(\mathcal{LR}) = \min\left\{\dim_S(\mathcal{L}) + \dim_S(\mathcal{R}), d\right\},$$

when $\dim_S(\mathcal{L})$ and $\dim_S(\mathcal{R})$ exist.

Notation. We write $\phi \preceq \psi$ (or equivalently, $\psi \succeq \phi$), if $\phi \leq C\psi$ for some constant $C > 0$, which may depend on the laws of the relations considered. We write $\phi \asymp \psi$ if $\phi \preceq \psi$ and $\phi \succeq \psi$. For $v \in \mathbb{Z}^d$ and $0 \leq n < N$, define the *dyadic shells*

$$H_n^N(v) := \{x \in \mathbb{Z}^d : 2^n \leq \langle vx \rangle < 2^{n+1}\}.$$

Remark 2.5. As the proof will show, the composition rule of Theorem 2.4 for random relations in \mathbb{Z}^d is valid for any graph where the shells $H_k^{k+1}(v)$ satisfy $|H_k^{k+1}(v)| \asymp 2^{dk}$.

For sets $V, W \subset \mathbb{Z}^d$ let

$$\rho(V, W) := \min\{\langle vw \rangle : v \in V, w \in W\}.$$

In particular, if V and W have nonempty intersection, then $\rho(V, W) = 1$. We write $\langle Wx \rangle$ as an abbreviation for $\langle W \cup \{x\} \rangle$.

LEMMA 2.6. *For every $M > 0$, every $x \in \mathbb{Z}^d$ and every $W \subset \mathbb{Z}^d$ with $|W| \leq M$,*

$$\langle Wx \rangle \leq \langle W \rangle \rho(x, W) \preceq \langle Wx \rangle,$$

where the constant implicit in the \preceq notation depends only on M .

Proof. We may assume without loss of generality that $x \notin W$. The inequality $\langle Wx \rangle \leq \langle W \rangle \rho(x, W)$ holds because given a tree on W we may obtain a tree on $W \cup \{x\}$ by adding an edge connecting x to the closest vertex in W . For the second inequality, consider some tree τ with vertices $W \cup \{x\}$. Let W' denote the neighbors of x in τ , and let $u \in W'$ be such that $\langle xu \rangle = \rho(x, W')$. Let τ' be the tree on W obtained from τ by replacing each edge $\{w', x\}$ where $w' \in W' \setminus \{u\}$, by the edge $\{w', u\}$. (See Figure 2.1.) It is easy to verify that τ' is a tree. For each $w' \in W'$, we have $\langle uw' \rangle \leq \langle ux \rangle + \langle xw' \rangle \leq 2\langle xw' \rangle$. Hence $\langle \tau' \rangle \langle ux \rangle \leq 2^M \langle \tau \rangle$, and the inequality $\langle W \rangle \rho(x, W) \leq 2^M \langle Wx \rangle$ follows. \square

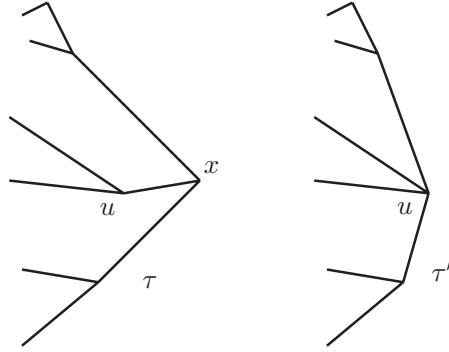


Figure 2.1: The trees τ and τ' .

Remark 2.7. Repeated application of Lemma 2.6 yields that for any set $\{x_1, \dots, x_n\}$ of n vertices in \mathbb{Z}^d ,

$$(2.4) \quad \langle x_1 \dots x_n \rangle \asymp \prod_{i=1}^{n-1} \rho(x_i, \{x_{i+1}, \dots, x_n\}),$$

where the implied constants depend only on n .

Our next goal in the proof of Theorem 2.4 is to establish (2.1) for the composition \mathcal{LR} . For this, the following lemma will be essential.

LEMMA 2.8. *Let \mathcal{L} and \mathcal{R} be independent random relations in \mathbb{Z}^d . Suppose that $\dim_S(\mathcal{L}) = d - \alpha$ and $\dim_S(\mathcal{R}) = d - \beta$ exist, and denote $\gamma := \alpha + \beta - d$. For $u, z \in \mathbb{Z}^d$ and $1 \leq n \leq N$, let*

$$S_{uz} = S_{uz}(n, N) := \sum_{x \in H_n^{N+1}(u)} \mathbf{1}_{u\mathcal{L}x} \mathbf{1}_{x\mathcal{R}z}.$$

If $\langle uz \rangle < 2^{n-1}$ and $N \geq n$, then

$$(2.5) \quad \mathbf{P}[S_{uz} > 0] \gtrsim \frac{\sum_{k=n}^N 2^{-k\gamma}}{\sum_{k=0}^N 2^{-k\gamma}}.$$

Proof. For $k \geq n$ and $x \in H_k^{k+1}(u)$, we have $\mathbf{P}[u\mathcal{L}x] \asymp 2^{-k\alpha}$ (use (2.1) and (2.3)). Also, for $x \in H_k^{k+1}(u)$, $k \geq n$,

$$\frac{1}{2} \langle xu \rangle \leq \langle xu \rangle - \langle zu \rangle \leq \langle xz \rangle \leq \langle xu \rangle + \langle zu \rangle \leq 2 \langle xu \rangle$$

and $\mathbf{P}[x\mathcal{R}z] \asymp 2^{-k\beta}$.

Because \mathcal{L} and \mathcal{R} are independent and $|H_k^{k+1}(u)| \asymp 2^{dk}$, we have

$$(2.6) \quad \mathbf{E}[S_{uz}] \asymp \sum_{k=n}^N 2^{kd} 2^{-k\alpha} 2^{-k\beta} = \sum_{k=n}^N 2^{-k\gamma}.$$

To estimate the second moment, observe that if $2\langle uz \rangle \leq \langle ux \rangle \leq \langle uy \rangle$, then

$$\langle uy \rangle \leq \langle uz \rangle + \langle zy \rangle \leq \frac{1}{2}\langle uy \rangle + \langle zy \rangle,$$

so that $\langle xy \rangle \leq \langle xu \rangle + \langle uy \rangle \leq 2\langle uy \rangle \leq 4\langle zy \rangle$. Applying the first two inequalities here, (2.2) for \mathcal{L} and Lemma 2.6, we obtain that

$$\begin{aligned} \mathbf{P}[u\mathcal{L}x, u\mathcal{L}y] &\preceq \langle ux \rangle^{-\alpha} \langle uy \rangle^{-\alpha} + \langle uxy \rangle^{-\alpha} \\ &\preceq \langle ux \rangle^{-\alpha} (\langle uy \rangle^{-\alpha} + \langle xy \rangle^{-\alpha}) \preceq \langle ux \rangle^{-\alpha} \langle xy \rangle^{-\alpha}. \end{aligned}$$

Similarly, from Lemma 2.6 and the inequality $\langle xy \rangle \leq 4\langle yz \rangle$ above, it follows that

$$\mathbf{P}[x\mathcal{R}z, y\mathcal{R}z] \preceq \langle xz \rangle^{-\beta} (\langle xy \rangle^{-\beta} + \langle yz \rangle^{-\beta}) \preceq \langle xz \rangle^{-\beta} \langle xy \rangle^{-\beta}.$$

Since

$$\mathbf{E}[S_{uz}^2] \leq 2 \sum_{x \in H_n^{N+1}(u)} \sum_{y \in H_n^{N+1}(u)} \mathbf{1}_{\{\langle uy \rangle \geq \langle ux \rangle\}} \mathbf{P}[u\mathcal{L}x, u\mathcal{L}y] \mathbf{P}[x\mathcal{R}z, y\mathcal{R}z],$$

we deduce by breaking the inner sum up into sums over $y \in H_j^{j+1}(x)$ for various j that

(2.7)

$$\mathbf{E}[S_{uz}^2] \preceq \sum_{x \in H_n^{N+1}(u)} \langle ux \rangle^{-\alpha} \langle xz \rangle^{-\beta} \sum_{y \in H_0^{N+2}(x)} \langle xy \rangle^{-\alpha-\beta} \preceq \mathbf{E}[S_{uz}] \sum_{j=0}^N 2^{j(d-\alpha-\beta)}.$$

Finally, recall that $\gamma = \alpha + \beta - d$, and apply the Cauchy-Schwarz inequality in the form

$$\mathbf{P}[S_{uz} > 0] \geq \frac{\mathbf{E}[S_{uz}]^2}{\mathbf{E}[S_{uz}^2]}.$$

The estimates (2.6) and (2.7) then yield the assertion of the lemma. \square

COROLLARY 2.9. *Under the assumptions of Theorem 2.4, $\mathbf{P}[u\mathcal{L}\mathcal{R}z] \succcurlyeq \langle uz \rangle^{-\gamma'}$ for all $u, z \in \mathbb{Z}^d$, where $\gamma' := \max\{0, d - \dim_S(\mathcal{L}) - \dim_S(\mathcal{R})\}$.*

Proof. Let $n := \lceil \log_2 \langle uz \rangle \rceil + 2$ and $\gamma := \alpha + \beta - d$ with $\alpha := d - \dim_S(\mathcal{L})$, $\beta := d - \dim_S(\mathcal{R})$. Apply the lemma with $N := n$ if $\gamma \neq 0$ and $N := 2n$ if $\gamma = 0$. \square

The proof of (2.2) for the composition $\mathcal{L}\mathcal{R}$ requires some further preparation.

LEMMA 2.10. *Let $\alpha, \beta \in [0, d)$ satisfy $\alpha + \beta > d$. Let $\gamma := \alpha + \beta - d$. Then for $v, w \in \mathbb{Z}^d$,*

$$\sum_{x \in \mathbb{Z}^d} \langle vx \rangle^{-\alpha} \langle xw \rangle^{-\beta} \asymp \langle vw \rangle^{-\gamma}.$$

Proof. Suppose that $2^N \leq \langle vw \rangle \leq 2^{N+1}$. By symmetry, it suffices to sum over vertices x such that $\langle xv \rangle \leq \langle xw \rangle$. For such x in the shell $H_n^{n+1}(v)$, we have by the triangle inequality

$$\langle xv \rangle^{-\alpha} \langle xw \rangle^{-\beta} \asymp 2^{-n\alpha} 2^{-\max\{n, N\}\beta}.$$

Multiplying by 2^{dn} (for the volume of the shell) and summing over all n prove the lemma. \square

LEMMA 2.11. *Let $M > 0$ be finite and let $V, W \subset \mathbb{Z}^d$ satisfy $|V|, |W| \leq M$. Let $\alpha, \beta \in [0, d]$ satisfy $\alpha + \beta > d$. Denote $\gamma := \alpha + \beta - d$. Then*

$$(2.8) \quad \sum_{x \in \mathbb{Z}^d} \langle Vx \rangle^{-\alpha} \langle Wx \rangle^{-\beta} \preceq \langle V \rangle^{-\alpha} \rho(V, W)^{-\gamma} \langle W \rangle^{-\beta} \leq \langle VW \rangle^{-\gamma},$$

where the constant implicit in the \preceq relation may depend only on M .

Proof. Using Lemma 2.6 we see that

$$\langle Vx \rangle^{-\alpha} \preceq \langle V \rangle^{-\alpha} \rho(x, V)^{-\alpha} \leq \langle V \rangle^{-\alpha} \sum_{v \in V} \langle vx \rangle^{-\alpha}$$

and similarly $\langle Wx \rangle^{-\beta} \preceq \langle W \rangle^{-\beta} \sum_{w \in W} \langle wx \rangle^{-\beta}$. Therefore, by Lemma 2.10,

$$\begin{aligned} \sum_{x \in \mathbb{Z}^d} \langle Vx \rangle^{-\alpha} \langle Wx \rangle^{-\beta} &\preceq \langle V \rangle^{-\alpha} \langle W \rangle^{-\beta} \sum_{v \in V} \sum_{w \in W} \langle vw \rangle^{-\gamma} \\ &\leq M^2 \langle V \rangle^{-\alpha} \langle W \rangle^{-\beta} \rho(V, W)^{-\gamma}. \end{aligned}$$

The rightmost inequality in (2.8) holds because $\gamma \leq \min\{\alpha, \beta\}$ and given trees on V and on W , a tree on $V \cup W$ can be obtained by adding an edge $\{v, w\}$ with $v \in V, w \in W$ and $\langle vw \rangle = \rho(V, W)$ (unless $V \cap W \neq \emptyset$, in which case $|V \cap W| - 1$ edges have to be deleted to obtain a tree on $V \cup W$). \square

LEMMA 2.12. *Let $\alpha, \beta \in [0, d]$ satisfy $\gamma := \alpha + \beta - d > 0$. Then*

$$\sum_{a \in \mathbb{Z}^d} \langle ax \rangle^{-\alpha} \langle ay \rangle^{-\beta} \langle az \rangle^{-\gamma} \preceq \langle xyz \rangle^{-\gamma}$$

holds for $x, y, z \in \mathbb{Z}^d$.

Proof. Without loss of generality, we may assume that $\langle zx \rangle \leq \langle zy \rangle$. Consider separately the sum over $A := \{a : \langle az \rangle \leq \frac{1}{2} \langle zx \rangle\}$ and its complement. For $a \in A$ we have $\langle ax \rangle \geq \frac{1}{2} \langle zx \rangle$ and $\langle ay \rangle \geq \frac{1}{2} \langle zy \rangle$. Therefore,

$$\begin{aligned} \sum_{a \in A} \langle ax \rangle^{-\alpha} \langle ay \rangle^{-\beta} \langle az \rangle^{-\gamma} &\preceq \langle zx \rangle^{-\alpha} \langle zy \rangle^{-\beta} \sum_{a \in A} \langle az \rangle^{-\gamma} \\ &\preceq \langle zx \rangle^{-\alpha} \langle zy \rangle^{-\beta} \langle zx \rangle^{d-\gamma} \\ &= \langle zx \rangle^{-\gamma} \langle zy \rangle^{-\gamma} \frac{\langle zx \rangle^{d-\alpha}}{\langle zy \rangle^{d-\alpha}} \leq \langle xyz \rangle^{-\gamma}, \end{aligned}$$

because of the assumption $\langle zx \rangle \leq \langle zy \rangle$ and $\gamma > 0$. Passing to the complement of A , we have,

$$\begin{aligned} \sum_{a \in \mathbb{Z}^d \setminus A} \langle az \rangle^{-\gamma} \langle ax \rangle^{-\alpha} \langle ay \rangle^{-\beta} &\preceq \langle zx \rangle^{-\gamma} \sum_{a \in \mathbb{Z}^d} \langle ax \rangle^{-\alpha} \langle ay \rangle^{-\beta} \\ &\preceq \langle zx \rangle^{-\gamma} \langle xy \rangle^{-\gamma} \leq \langle xyz \rangle^{-\gamma}, \end{aligned}$$

where Lemma 2.10 was used in the next to last inequality. Combining these two estimates completes the proof of the lemma. \square

The following slight extension of Lemma 2.12 will also be needed. Under the same assumptions on α, β, γ ,

$$(2.9) \quad \sum_{a \in \mathbb{Z}^d} \langle axw \rangle^{-\alpha} \langle ay \rangle^{-\beta} \langle az \rangle^{-\gamma} \preceq \langle xwyz \rangle^{-\gamma}.$$

This is obtained from Lemma 2.12 by using $\langle axw \rangle \succcurlyeq \langle xw \rangle \min\{\langle ax \rangle, \langle aw \rangle\}$, which is an application of Lemma 2.6, and $\langle xw \rangle \langle wyz \rangle \geq \langle xwyz \rangle$, which holds by the definition of the spread.

Proof of Theorem 2.4. If $\dim_S(\mathcal{L}) + \dim_S(\mathcal{R}) \geq d$, then Corollary 2.9 shows that

$$\inf_{x, y \in \mathbb{Z}^d} \mathbf{P}[x\mathcal{L}\mathcal{R}y] > 0,$$

which is equivalent to $\dim_S(\mathcal{L}\mathcal{R}) = d$. Therefore, assume that $\dim_S(\mathcal{L}) + \dim_S(\mathcal{R}) < d$. Let $\alpha := d - \dim_S(\mathcal{L})$, $\beta := d - \dim_S(\mathcal{R})$ and $\gamma := \alpha + \beta - d$. Since Corollary 2.9 verifies (2.1) for the composition $\mathcal{L}\mathcal{R}$, it suffices to prove (2.2) for $\mathcal{L}\mathcal{R}$ with γ in place of α . Independence of the relations \mathcal{L} and \mathcal{R} , together with (2.2) for \mathcal{L} and for \mathcal{R} with β in place of α imply

$$\begin{aligned} (2.10) \quad \mathbf{P}[x\mathcal{L}\mathcal{R}z, y\mathcal{L}\mathcal{R}w] &\leq \sum_{a, b \in \mathbb{Z}^d} \mathbf{P}[x\mathcal{L}a, y\mathcal{L}b] \mathbf{P}[a\mathcal{R}z, b\mathcal{R}w] \\ &\preceq \sum_{a, b \in \mathbb{Z}^d} \left(\langle xa \rangle^{-\alpha} \langle yb \rangle^{-\alpha} + \langle xayb \rangle^{-\alpha} \right) \left(\langle az \rangle^{-\beta} \langle bw \rangle^{-\beta} + \langle azbw \rangle^{-\beta} \right). \end{aligned}$$

Opening the parentheses gives four sums, which we deal with separately. First,

$$(2.11) \quad \sum_{a, b \in \mathbb{Z}^d} \langle xa \rangle^{-\alpha} \langle yb \rangle^{-\alpha} \langle az \rangle^{-\beta} \langle bw \rangle^{-\beta} \preceq \langle xz \rangle^{-\gamma} \langle yw \rangle^{-\gamma},$$

by Lemma 2.10 applied twice. Second, by two applications of Lemma 2.11,

$$\begin{aligned} (2.12) \quad &\sum_{a \in \mathbb{Z}^d} \sum_{b \in \mathbb{Z}^d} \langle xayb \rangle^{-\alpha} \langle azbw \rangle^{-\beta} \\ &\preceq \sum_{a \in \mathbb{Z}^d} \rho(\{x, a, y\}, \{z, a, w\})^{-\gamma} \langle xay \rangle^{-\alpha} \langle zaw \rangle^{-\beta} \\ &= \sum_{a \in \mathbb{Z}^d} \langle xay \rangle^{-\alpha} \langle zaw \rangle^{-\beta} \preceq \langle xyzw \rangle^{-\gamma}. \end{aligned}$$

Third, by Lemma 2.11 and (2.9),

$$\begin{aligned}
 (2.13) \quad & \sum_{a \in \mathbb{Z}^d} \sum_{b \in \mathbb{Z}^d} \langle xayb \rangle^{-\alpha} \langle az \rangle^{-\beta} \langle bw \rangle^{-\beta} \\
 & \preceq \sum_{a \in \mathbb{Z}^d} \rho(w, \{x, y, a\})^{-\gamma} \langle az \rangle^{-\beta} \langle xay \rangle^{-\alpha} \\
 & \leq \sum_{a \in \mathbb{Z}^d} \langle wa \rangle^{-\gamma} \langle az \rangle^{-\beta} \langle xay \rangle^{-\alpha} + \rho(w, \{x, y\})^{-\gamma} \sum_{a \in \mathbb{Z}^d} \langle az \rangle^{-\beta} \langle xay \rangle^{-\alpha} \\
 & \preceq \langle wzxy \rangle^{-\gamma} + \rho(w, \{x, y\})^{-\gamma} \langle zxy \rangle^{-\gamma} \leq 2 \langle wzxy \rangle^{-\gamma}.
 \end{aligned}$$

By symmetry, we also have

$$\sum_{a, b \in \mathbb{Z}^d} \langle xa \rangle^{-\alpha} \langle yb \rangle^{-\alpha} \langle azbw \rangle^{-\beta} \preceq \langle xywz \rangle^{-\gamma}.$$

This, together with (2.10), (2.11), (2.12) and (2.13), implies that \mathcal{LR} satisfies the correlation inequality (2.2) with γ in place of α , and completes the proof. \square

3. Tail triviality

Consider the USF on \mathbb{Z}^d . Given $n = 1, 2, \dots$, let \mathcal{R}_n be the relation consisting of all pairs $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that y may be reached from x by a path which uses no more than $n - 1$ edges outside of the USF. We show below that \mathcal{R}_1 has stochastic dimension 4 (Theorem 4.2) and that \mathcal{R}_n stochastically dominates the composition of n independent copies of \mathcal{R}_1 (Theorem 4.1). By Theorem 2.4, \mathcal{R}_n dominates a relation with stochastic dimension $\min\{4n, d\}$. When $4n \geq d$, this says that $\inf_{x, y \in \mathbb{Z}^d} \mathbf{P}[x\mathcal{R}_ny] > 0$. For Theorem 1.1, the stronger statement that $\inf_{x, y \in \mathbb{Z}^d} \mathbf{P}[x\mathcal{R}_ny] = 1$ is required. For this purpose, tail triviality needs to be discussed.

Definition 3.1 (Tail triviality). Let $\mathcal{R} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ be a random relation with law \mathbf{P} . For a set $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}^d$, let \mathfrak{F}_Λ be the σ -field generated by the events $x\mathcal{R}y, (x, y) \in \Lambda$. Let the *left tail field* $\mathfrak{F}_L(v)$ corresponding to a vertex v be the intersection of all $\mathfrak{F}_{\{v\} \times K}$ where $K \subset \mathbb{Z}^d$ ranges over all subsets such that $\mathbb{Z}^d \setminus K$ is finite. Let the *right tail field* $\mathfrak{F}_R(v)$ be the intersection of all $\mathfrak{F}_{K \times \{v\}}$ where $K \subset \mathbb{Z}^d$ ranges over all subsets such that $\mathbb{Z}^d \setminus K$ is finite. Let the *remote tail field* \mathfrak{F}_W be the intersection of all $\mathfrak{F}_{K_1 \times K_2}$, where $K_1, K_2 \subset \mathbb{Z}^d$ range over all subsets of \mathbb{Z}^d such that $\mathbb{Z}^d \setminus K_1$ and $\mathbb{Z}^d \setminus K_2$ are finite. The random relation \mathcal{R} with law \mathbf{P} is said to be *left tail trivial* if $\mathbf{P}[A] \in \{0, 1\}$ for every $A \in \mathfrak{F}_L(v)$ and every $v \in \mathbb{Z}^d$. Analogously, define *right tail triviality* and *remote tail triviality*.

We will need the following known lemma, which is a corollary of the main result of von Weizsäcker (1983). Its proof is included for the reader's convenience.

LEMMA 3.2. *Let $\{\mathfrak{F}_n\}$ and $\{\mathfrak{G}_n\}$ be two decreasing sequences of complete σ -fields in a probability space (X, \mathfrak{F}, μ) , with \mathfrak{G}_1 independent of \mathfrak{F}_1 , and let \mathfrak{T} denote the trivial σ -field, consisting of events with probability 0 or 1. If $\bigcap_{n \geq 1} \mathfrak{F}_n = \mathfrak{T} = \bigcap_{n \geq 1} \mathfrak{G}_n$, then $\bigcap_{n \geq 1} (\mathfrak{F}_n \vee \mathfrak{G}_n) = \mathfrak{T}$ as well.*

Proof. Let $h \in \bigcap_{n=1}^\infty L^2(\mathfrak{F}_n \vee \mathfrak{G}_n)$. It suffices to show that h is constant. Suppose that $f_n \in L^2(\mathfrak{F}_n)$, $g_n \in L^\infty(\mathfrak{G}_n)$ and $f \in L^\infty(\mathfrak{F}_1)$. Then

$$\mathbf{E}[f_n g_n f | \mathfrak{G}_1] = g_n \mathbf{E}[f_n f | \mathfrak{G}_1] = g_n \mathbf{E}[f_n f] \quad \text{a.s.}$$

As the linear span of such products $f_n g_n$ is dense in $L^2(\mathfrak{F}_n \vee \mathfrak{G}_n)$, it follows that

$$(3.1) \quad \mathbf{E}[h f | \mathfrak{G}_1] \in L^2(\mathfrak{G}_n).$$

By our assumption $\mathfrak{T} = \bigcap_{n \geq 1} \mathfrak{G}_n$, we infer from (3.1) that

$$(3.2) \quad \mathbf{E}[h f | \mathfrak{G}_1] = \mathbf{E}[h f] \quad \text{for } f \in L^\infty(\mathfrak{F}_1).$$

In particular, $\mathbf{E}[h | \mathfrak{G}_1] = \mathbf{E}[h]$. By symmetry, $\mathbf{E}[h | \mathfrak{F}_1] = \mathbf{E}[h]$, whence

$$\forall f \in L^\infty(\mathfrak{F}_1), \quad \mathbf{E}[h f] = \mathbf{E}[f \mathbf{E}[h | \mathfrak{F}_1]] = \mathbf{E}[h] \mathbf{E}[f].$$

Inserting this into (3.2), we conclude that for $f \in L^\infty(\mathfrak{F}_1)$ and $g \in L^\infty(\mathfrak{G}_1)$ (so that f and g are independent),

$$\mathbf{E}[h f g] = \mathbf{E}[g \mathbf{E}[h f | \mathfrak{G}_1]] = \mathbf{E}[h] \mathbf{E}[f] \mathbf{E}[g] = \mathbf{E}[h] \mathbf{E}[f g].$$

Thus $h - \mathbf{E}[h]$ is orthogonal to all such products $f g$, so it must vanish a.s. \square

Suppose that $\mathcal{R}, \mathcal{L} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ are independent random relations which have trivial remote tails and trivial left tails, and have stochastic dimensions. We do not know whether it follows that $\mathcal{L} \mathcal{R}$ is left tail trivial. For that reason, we introduce the notion of the *restricted composition* $\mathcal{L} \diamond \mathcal{R}$, which is the relation consisting of all pairs $(x, z) \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that there is some $y \in \mathbb{Z}^d$ with $x \mathcal{L} y$ and $y \mathcal{R} z$ and

$$(3.3) \quad \langle xz \rangle \leq \min\{\langle xy \rangle, \langle yz \rangle\}.$$

THEOREM 3.3. *Let $\mathcal{R}, \mathcal{L} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ be independent random relations.*

- (i) *If \mathcal{L} has trivial left tail and \mathcal{R} has trivial remote tail, then the restricted composition $\mathcal{L} \diamond \mathcal{R}$ has trivial left tail.*
- (ii) *If $\dim_S(\mathcal{L})$ and $\dim_S(\mathcal{R})$ exist, then*

$$\dim_S(\mathcal{L} \diamond \mathcal{R}) = \min\{\dim_S(\mathcal{L}) + \dim_S(\mathcal{R}), d\}.$$

(iii) If $\dim_S(\mathcal{L})$ and $\dim_S(\mathcal{R})$ exist, \mathcal{L} has trivial left tail, \mathcal{R} has trivial right tail and $\dim_S(\mathcal{L}) + \dim_R(\mathcal{R}) \geq d$, then $\mathbf{P}[x\mathcal{L} \diamond \mathcal{R}z] = 1$ for all $x, z \in \mathbb{Z}^d$.

Proof. (i) This is a consequence of Lemma 3.2.

(ii) Denote $\gamma' := \max\{0, d - \dim_S(\mathcal{L}) - \dim_R(\mathcal{R})\}$. The inequality $\mathbf{P}[x\mathcal{L} \diamond \mathcal{R}z] \geq \langle xz \rangle^{-\gamma'}$ follows from the proof of Corollary 2.9; this concludes the proof if $\gamma' = 0$. If $\gamma' > 0$, then the required upper bound for $\mathbf{P}[x\mathcal{L} \diamond \mathcal{R}z, y\mathcal{L} \diamond \mathcal{R}w]$ follows from Theorem 2.4, since $\mathcal{L} \diamond \mathcal{R}$ is a subrelation of $\mathcal{L}\mathcal{R}$.

(iii) Let \mathfrak{F}_n (respectively, \mathfrak{G}_n) be the (completed) σ -field generated by the events $u\mathcal{L}x$ (respectively, $x\mathcal{R}z$) as x ranges over $H_n^\infty(u)$. The event

$$A_n := \{\exists x \in H_n^{2n}(u) : u\mathcal{L}x, x\mathcal{R}z\}$$

is clearly in $\mathfrak{F}_n \vee \mathfrak{G}_n$. By Lemma 2.8, there is a constant C such that $\mathbf{P}[A_n] \geq 1/C > 0$, provided that n is sufficiently large. Let $A = \bigcap_{k \geq 1} \bigcup_{n \geq k} A_n$ be the event that there are infinitely many $x \in \mathbb{Z}^d$ satisfying $u\mathcal{L}x$ and $x\mathcal{R}z$. Then $\mathbf{P}[A] \geq 1/C$. By Lemma 3.2, $\mathbf{P}[A] = 1$. \square

COROLLARY 3.4. *Let $m \geq 2$, and let $\{\mathcal{R}_i\}_{i=1}^m$ be independent random relations in \mathbb{Z}^d such that $\dim_S(\mathcal{R}_i)$ exists for each $i \leq m$. Suppose that*

$$\sum_{i=1}^m \dim_S(\mathcal{R}_i) \geq d,$$

and in addition, \mathcal{R}_1 is left tail trivial, each of $\mathcal{R}_2, \dots, \mathcal{R}_{m-1}$ has a trivial remote tail, and \mathcal{R}_m is right tail trivial. Then $\mathbf{P}[u\mathcal{R}_1\mathcal{R}_2 \cdots \mathcal{R}_m z] = 1$ for all $u, z \in \mathbb{Z}^d$.

Proof. Let $\mathcal{L}_1 = \mathcal{R}_1$ and define inductively $\mathcal{L}_k = \mathcal{L}_{k-1} \diamond \mathcal{R}_k$ for $k = 2, \dots, m$, so that \mathcal{L}_k is a subrelation of $\mathcal{R}_1\mathcal{R}_2 \cdots \mathcal{R}_k$. It follows by induction from Theorem 3.3 that \mathcal{L}_k has a trivial left tail for each $k < m$, and

$$\dim_S(\mathcal{L}_k) = \min\left\{\sum_{i=1}^k \dim_S(\mathcal{R}_i), d\right\}.$$

By Theorem 3.3(iii), the restricted composition $\mathcal{L}_m = \mathcal{L}_{m-1} \diamond \mathcal{R}_m$ satisfies

$$(3.4) \quad \mathbf{P}[u\mathcal{L}_m z] = 1 \text{ for all } u, z \in \mathbb{Z}^d. \quad \square$$

4. Relevant USF properties

Basic to the understanding of the USF is a procedure from BLPS [2] that generates the (wired) USF on any transient graph; it is called “Wilson’s method rooted at infinity”, since it is based on an algorithm from Wilson [15] for picking uniformly a spanning tree in a finite graph. Let $\{v_1, v_2, \dots\}$ be an

arbitrary ordering of the vertices of a transient graph G . Let X_1 be a simple random walk started from v_1 . Let F_1 denote the loop-erasure of X_1 , which is obtained by following X_1 and erasing the loops as they are created. Let X_2 be a simple random walk from v_2 which stops if it hits F_1 , and let F_2 be the union of F_1 with the loop-erasure of X_2 . Inductively, let X_n be a simple random walk from v_n , which is stopped if it hits F_{n-1} , and let F_n be the union of F_{n-1} with the loop-erasure of X_n . Then $F := \bigcup_{n=1}^\infty F_n$ has the distribution of the (wired) USF on G . The edges in F inherit the orientation from the loop-erased walks creating them, and hence F may be thought of as an oriented forest. Its distribution does not depend on the ordering chosen for the vertices. See BLPS [2] for details.

We say that a random set A *stochastically dominates* a random set Q if there is a coupling μ of A and Q such that $\mu[A \supset Q] = 1$.

THEOREM 4.1 (Domination). *Let $F, F_0, F_1, F_2, \dots, F_m$ be independent samples of the wired USF in the graph G . Let $\mathcal{C}(x, F)$ denote the vertex set of the component of x in F . Fix a distinguished vertex v_0 in G , and write $\mathcal{C}_0 = \mathcal{C}(v_0, F)$. For $j \geq 1$, define inductively \mathcal{C}_j to be the union of all vertex components of F that are contained in, or adjacent to, \mathcal{C}_{j-1} . Let $Q_0 = \mathcal{C}(v_0, F_0)$. For $j \geq 1$, define inductively Q_j to be the union of all vertex components of F_j that intersect Q_{j-1} . Then \mathcal{C}_m stochastically dominates Q_m .*

Proof. For each $R > 0$ let $B_R := \{v \in \mathbb{Z}^d : |v| < R\}$, where $|v|$ is the graph distance from v_0 to v . Fix $R > 0$. Let B_R^W be the graph obtained from G by collapsing the complement of B_R to a single vertex, denoted v_R^* . Let F', F'_0, \dots, F'_m be independent samples of the uniform spanning tree (UST) in B_R^W . Define F^* to be F' without the edges incident to v_R^* , and define F_i^* for $i = 0, \dots, m$ analogously.

Write $\mathcal{C}_0^* = \mathcal{C}(v_0, F^*)$. For $j \geq 1$, define inductively \mathcal{C}_j^* to be the union of all vertex components of F^* that are contained in, or adjacent to, \mathcal{C}_{j-1}^* .

Let $Q_0^* = \mathcal{C}(v_0, F_0^*)$. For $j \geq 1$, define inductively Q_j^* to be the union of all vertex components of F_j^* that intersect Q_{j-1}^* .

We show by induction on j that \mathcal{C}_j^* stochastically dominates Q_j^* . The induction base where $j = 0$ is obvious. For the inductive step, assume that $0 \leq j < m$ and there is a coupling μ_j of F' and $(F'_0, F'_1, \dots, F'_j)$ such that $\mathcal{C}_j^* \supset Q_j^*$ holds μ_j -a.s. Let H be a set of vertices in B_R such that $\mathcal{C}_j^* = H$ with positive probability. Let \widehat{H}^c denote the subgraph of B_R^W spanned by the vertices $B_R \cup \{v_R^*\} \setminus H$. Let S_H be $F' \cap \widehat{H}^c$ conditioned on $\mathcal{C}_j^* = H$. Since \mathcal{C}_j^* is a union of components of F^* , it is clear that S_H has the same distribution as a UST on \widehat{H}^c . Therefore, by the negative association property of uniform spanning trees (see the discussion following Remark 5.7 in BLPS [2]), S_H stochastically dominates $F'_{j+1} \cap \widehat{H}^c$ (conditional on $\mathcal{C}_j^* = H$). Hence,

we may extend μ_j to a coupling μ_{j+1} of F' and $(F'_0, F'_1, \dots, F'_{j+1})$ such that $\mathcal{C}_j^* \supset Q_j^*$ and $H = \mathcal{C}_j^*$ satisfies $F^* \cap \widehat{H}^c \supset F_{j+1}^* \cap \widehat{H}^c$ a.s. with respect to μ_{j+1} . In such a coupling, we also have $\mathcal{C}_{j+1}^* \supset Q_{j+1}^*$ a.s.

This completes the induction step, and proves that \mathcal{C}_m^* stochastically dominates Q_m^* . Observe that \mathcal{C}_m^* converges weakly to \mathcal{C}_m and Q_m^* converges weakly to Q_m , as $R \rightarrow \infty$. This follows from the fact that F stochastically dominates F^* (see BLPS [2, Cor. 4.3(b)]) and $F^* \rightarrow F$ weakly as $R \rightarrow \infty$. The theorem follows. \square

THEOREM 4.2 (Stochastic dimension of USF). *Let F be the USF in \mathbb{Z}^d , where $d \geq 5$. Let \mathcal{U} be the relation*

$$\mathcal{U} := \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : x \text{ and } y \text{ are in the same component of } F\}.$$

Then \mathcal{U} has stochastic dimension 4.

Proof. We use Wilson’s method rooted at infinity and a technique from LPS [9]. For $x, z \in \mathbb{Z}^d$, consider independent simple random walk paths $\{X(i)\}_{i \geq 0}$ and $\{Z(j)\}_{j \geq 0}$ starting at x and z respectively. By running Wilson’s method rooted at infinity starting with x and then starting another random walk from z , we see that $\mathbf{P}[z\mathcal{U}x]$ is equal to the probability that the walk Z intersects the loop-erasure of the walk X . Given $m \in \mathbb{N}$, denote by $\{L_m(i)\}_{i=0}^{q_m}$ the path obtained from loop-erasing $\{X(i)\}_{i=0}^m$. Define

$$\begin{aligned} \tau_L(m) &:= \inf \left\{ k \in \{0, 1, \dots, q_m\} : L_m(k) \in \{X(i)\}_{i \geq m} \right\}, \\ \tau_L(m, n) &:= \inf \left\{ k \in \{0, 1, \dots, q_m\} : L_m(k) \in \{Z(j)\}_{j \geq n} \right\}. \end{aligned}$$

Consider the indicator variables $I_{m,n} := \mathbf{1}_{\{X(m)=Z(n)\}}$ and

$$J_{m,n} := I_{m,n} \mathbf{1}_{\{\tau_L(m,n) \leq \tau_L(m)\}}.$$

Observe that on the event $J_{m,n} = 1$, the path $\{Z(j)\}_{j \geq 0}$ intersects the loop-erasure of $\{X(i)\}_{i=0}^\infty$ at $L_m(\tau_L(m, n))$. Hence $\mathbf{P}[x\mathcal{U}z] = \mathbf{P}[\sum_{m,n} J_{m,n} > 0]$. Given $X(m) = Z(n)$, the law of $\{X(m+j)\}_{j \geq 0}$ is the same as the law of $\{Z(n+j)\}_{j \geq 0}$. Therefore,

$$\mathbf{P}[\tau_L(m, n) \leq \tau_L(m) | X(m) = Z(n)] \geq 1/2,$$

so that $\mathbf{E}[I_{m,n}] \geq \mathbf{E}[J_{m,n}] \geq \mathbf{E}[I_{m,n}]/2$. Let

$$\begin{aligned} \Phi &= \Phi_{xz} := \sum_{m=0}^\infty \sum_{n=0}^\infty I_{m,n}; \\ \Psi &= \Psi_{xz} := \sum_{m=0}^\infty \sum_{n=0}^\infty J_{m,n}. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{E}[\Psi] &\asymp \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathbf{E}[I_{m,n}] \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{v \in \mathbb{Z}^d} \mathbf{P}[X(m) = v] \mathbf{P}[Z(n) = v] \\ &= \sum_{v \in \mathbb{Z}^d} G(x, v) G(z, v), \end{aligned}$$

where G is the Green function for a simple random walk in \mathbb{Z}^d . Since $G(x, v) \asymp \langle xv \rangle^{2-d}$ (see, e.g., Spitzer [12]), we infer that (see Lemma 2.10)

$$(4.1) \quad \mathbf{E}[\Psi] \asymp \sum_{v \in \mathbb{Z}^d} \langle xv \rangle^{2-d} \langle zv \rangle^{2-d} \asymp \langle xz \rangle^{4-d}.$$

The second moment calculation for Φ is classical. Again, the relation $G(x, v) \asymp \langle xv \rangle^{2-d}$ is used:

$$\begin{aligned} \mathbf{E}[\Psi^2] &\leq \mathbf{E}[\Phi^2] \\ &\leq \sum_{y, w \in \mathbb{Z}^d} G(x, y) G(y, w) [G(z, y) G(y, w) + G(z, w) G(w, y)] \\ &\asymp \sum_{y \in \mathbb{Z}^d} G(x, y) G(z, y) + \sum_{y \in \mathbb{Z}^d} G(x, y) \sum_{w \in \mathbb{Z}^d} G(y, w)^2 G(z, w) \\ &\asymp \sum_{y \in \mathbb{Z}^d} \langle xy \rangle^{2-d} \langle zy \rangle^{2-d} + \sum_{y \in \mathbb{Z}^d} \langle xy \rangle^{2-d} \sum_{w \in \mathbb{Z}^d} \langle yw \rangle^{4-2d} \langle zw \rangle^{2-d}. \end{aligned}$$

The proof of Lemma 2.10 gives

$$\sum_{w \in \mathbb{Z}^d} \langle yw \rangle^{4-2d} \langle zw \rangle^{2-d} \asymp \langle yz \rangle^{2-d}.$$

Hence

$$\mathbf{E}[\Psi^2] \asymp \langle xz \rangle^{4-d} + \sum_{y \in \mathbb{Z}^d} \langle xy \rangle^{2-d} \langle yz \rangle^{2-d} \asymp \langle xz \rangle^{4-d}.$$

Therefore,

$$\mathbf{P}[x\mathcal{U}z] \geq \mathbf{P}[\Psi_{xz} > 0] \geq \frac{\mathbf{E}[\Psi_{xz}]^2}{\mathbf{E}[\Psi_{xz}^2]} \asymp \langle xz \rangle^{4-d}.$$

This verifies (2.1) for \mathcal{U} with $d - \alpha = 4$.

To verify (2.2), we must bound $\mathbf{P}[x\mathcal{U}z, y\mathcal{U}w]$. Let X, Z, Y, W be simple random walks starting from x, z, y, w , respectively. Let \hat{X} denote the set of vertices visited by X , and similarly for Z, Y and W . Then we may generate the

USF by first using the random walks X, Z, Y, W . On the event $x\mathcal{U}z \wedge y\mathcal{U}w \wedge \neg x\mathcal{U}y$, we must have $\widehat{X} \cap \widehat{Z} \neq \emptyset$ and $\widehat{Y} \cap \widehat{W} \neq \emptyset$. Hence,

$$\begin{aligned}
 (4.2) \quad \mathbf{P}[x\mathcal{U}z \wedge y\mathcal{U}w \wedge \neg x\mathcal{U}y] &\leq \mathbf{P}[\widehat{X} \cap \widehat{Z} \neq \emptyset] \mathbf{P}[\widehat{Y} \cap \widehat{W} \neq \emptyset] \\
 &\leq \sum_{a \in \mathbb{Z}^d} \mathbf{G}(x, a) \mathbf{G}(z, a) \sum_{b \in \mathbb{Z}^d} \mathbf{G}(y, b) \mathbf{G}(w, b) \\
 &\preceq \langle xz \rangle^{4-d} \langle yw \rangle^{4-d}.
 \end{aligned}$$

On the other hand, the event $x\mathcal{U}z \wedge y\mathcal{U}w \wedge x\mathcal{U}y$ is the event $U(x, y, z, w)$ that x, y, z, w are all in the same USF tree.

By Theorem 4.3 below, $\mathbf{P}[U(x, y, z, w)] \preceq \langle xyzw \rangle^{4-d}$. This together with (4.2) gives

$$\begin{aligned}
 \mathbf{P}[x\mathcal{U}z \wedge y\mathcal{U}w] &\leq \mathbf{P}[x\mathcal{U}z \wedge y\mathcal{U}w \wedge \neg x\mathcal{U}y] + \mathbf{P}[U(x, y, z, w)] \\
 &\preceq \langle xz \rangle^{4-d} \langle yw \rangle^{4-d} + \langle xyzw \rangle^{4-d},
 \end{aligned}$$

which verifies (2.2) with $\alpha = d - 4$, and completes the proof. □

THEOREM 4.3. *For any finite set $W \subset \mathbb{Z}^d$, denote by $U(W)$ the event that all vertices in W are in the same USF component. Then*

$$(4.3) \quad \mathbf{P}[U(W)] \preceq \langle W \rangle^{4-d},$$

where the implied constant depends only on d and the cardinality of W .

Proof. When $|W| = 2$, say $W = \{x, z\}$, (4.3) follows from (4.1), since $\mathbf{P}[x\mathcal{U}z] = \mathbf{P}[\Psi > 0] \leq \mathbf{E}[\Psi]$. We proceed by induction on $|W|$. For the inductive step, suppose that $|W| \geq 3$. For $x, y \in W$ denote by $U(W; x, y)$ the intersection of $U(W)$ with the event that the path connecting x, y in the USF is edge-disjoint from the oriented USF paths connecting the vertices in $V := W \setminus \{x, y\}$ to infinity. By considering all the possibilities for the vertex $z \in \mathbb{Z}^d$ where the oriented USF paths from x and y to ∞ meet, and running Wilson’s method rooted at infinity starting with random walks from the vertices in $V \cup \{z\}$ and following by random walks from x and y we obtain

$$(4.4) \quad \mathbf{P}[U(W; x, y)] \preceq \sum_{z \in \mathbb{Z}^d} \mathbf{P}[U(Vz)] \langle zx \rangle^{2-d} \langle zy \rangle^{2-d},$$

where $Vz := V \cup \{z\}$. By the induction hypothesis and Lemma 2.6,

$$\mathbf{P}[U(Vz)] \preceq \langle Vz \rangle^{4-d} \preceq \langle V \rangle^{4-d} \sum_{v \in V} \langle vz \rangle^{4-d}.$$

Inserting this into (4.4) and applying Lemma 2.12 with $\alpha = \beta = d - 2$, yields

$$\begin{aligned}
 \mathbf{P}[U(W; x, y)] &\preceq \langle V \rangle^{4-d} \sum_{v \in V} \sum_{z \in \mathbb{Z}^d} \langle vz \rangle^{4-d} \langle zx \rangle^{2-d} \langle zy \rangle^{2-d} \\
 &\preceq \langle V \rangle^{4-d} \sum_{v \in V} \langle vxy \rangle^{4-d} \preceq \langle W \rangle^{4-d},
 \end{aligned}$$

where the last inequality follows from the fact that for every $v \in V$, the union of a tree on $\{v, x, y\}$ and a tree on V is a tree on $W = V \cup \{x, y\}$. Finally, observe that

$$U(W) = \bigcup_{(x,y)} U(W; x, y),$$

where (x, y) runs over all pairs of distinct vertices in W . Indeed, on $U(W)$, denote by $m(W)$ the vertex where all oriented USF paths based in W meet, and pick x, y as the pair of vertices in W such that their oriented USF paths meet farthest (in the intrinsic metric of the tree) from $m(W)$. Consequently,

$$\mathbf{P}[U(W)] \leq \sum_{(x,y)} \mathbf{P}[U(W; x, y)] \asymp \langle W \rangle^{4-d}. \quad \square$$

Remark 4.4. The estimate in Theorem 4.3 is tight, up to constants; i.e., for any finite set $W \subset \mathbb{Z}^d$,

$$(4.5) \quad \mathbf{P}[U(W)] \gtrsim \langle W \rangle^{4-d},$$

where the implicit constant may depend only on d and the cardinality of W . Since we will not need this lower bound, we omit the proof.

THEOREM 4.5 (Tail triviality of the USF relation). *The relation \mathcal{U} of Theorem 4.2 has trivial left, right and remote tails.*

In BLPS [2] it was proved that the tail of the (wired or free) USF on every infinite graph is trivial. However, this is not the same as the tail triviality of the relation \mathcal{U} . Indeed, if the underlying graph G is a regular tree of degree greater than 2, then the relation \mathcal{U} determined by the (wired) USF in G does not have a trivial left tail.

Proof. The theorem clearly holds when $d \leq 4$, for then $\mathcal{U} = \mathbb{Z}^d \times \mathbb{Z}^d$ a.s. Therefore, restrict to the case $d > 4$. Let F be the USF in \mathbb{Z}^d . We start with the remote tail. Fix $x \in \mathbb{Z}^d$, $r > 0$, and let $\mathcal{S} = \{K_1 \subset F, K_2 \cap F = \emptyset\}$ be a cylinder event where K_1, K_2 are sets of edges in a ball $B(x, r)$ of radius r and center x . For $R > 0$, let Υ_R be the union of all one-sided-infinite simple paths in F that start at some vertex $v \in \mathbb{Z}^d \setminus B(x, R)$. By BLPS [2, Th. 10.1], almost surely each component of F has one end. (This implies that Υ_R is a.s. the same as the union of all oriented USF paths starting outside of $B(x, R)$.) Therefore, there exists a function $\phi(R)$ with $\phi(R) \rightarrow \infty$ as $R \rightarrow \infty$ such that

$$(4.6) \quad \lim_{R \rightarrow \infty} \mathbf{P}[\Upsilon_R \cap B(x, \phi(R)) \neq \emptyset] = 0.$$

Let $\tilde{\Upsilon}_R := \Upsilon_R$ if $\Upsilon_R \cap B(x, \phi(R)) = \emptyset$, and $\tilde{\Upsilon}_R := \emptyset$ otherwise. Note that $\tilde{\Upsilon}_R$ is a.s. determined by $F \setminus B(x, \phi(R))$, or more precisely by the set of edges

in F with both endpoints outside $B(x, \phi(R))$. By tail triviality of F itself, it therefore follows that

$$(4.7) \quad \lim_{R \rightarrow \infty} \mathbf{E} \left[\left| \mathbf{P}[\mathcal{S} | \tilde{\Upsilon}_R] - \mathbf{P}[\mathcal{S}] \right| \right] = 0.$$

For $R > r$, on the event $\Upsilon_R \cap B(x, \phi(R)) = \emptyset$, we have $\mathbf{P}[\mathcal{S} | \tilde{\Upsilon}_R] = \mathbf{P}[\mathcal{S} | \Upsilon_R]$. Hence, by (4.6),

$$\lim_{R \rightarrow \infty} \mathbf{E} \left[\left| \mathbf{P}[\mathcal{S} | \Upsilon_R] - \mathbf{P}[\mathcal{S} | \tilde{\Upsilon}_R] \right| \right] = 0.$$

With (4.7) this gives

$$(4.8) \quad \lim_{R \rightarrow \infty} \mathbf{E} \left[\left| \mathbf{P}[\mathcal{S} | \Upsilon_R] - \mathbf{P}[\mathcal{S}] \right| \right] = 0.$$

Since the remote tail of \mathcal{U} is determined by Υ_R for every R , the remote tail is independent of \mathcal{S} , whence it is trivial.

Now consider the left tail of \mathcal{U} . Let A be an event in $\mathfrak{F}_L(x)$. Let $\mathcal{S} = \{K_1 \subset F, K_2 \cap F = \emptyset\}$ as above, where K_1, K_2 are sets of edges in $B(x, r)$. To establish the triviality of $\mathfrak{F}_L(x)$, it is enough to show that A and \mathcal{S} are independent.

Let Υ'_R be the union of all the one-sided-infinite simple paths in F that start at some vertex in the outer boundary of $B(x, R)$. By Wilson’s method rooted at infinity we may construct F by first choosing Υ'_R , then the path in F , γ_R say, from x to Υ'_R and after that the paths starting at points outside $\Upsilon'_R \cup \gamma_R$. Let ζ_R be the endpoint of γ_R on Υ'_R . Recall that γ_R is obtained by loop erasure of a simple random walk path starting from x and stopped when it hits Υ'_R . Given Υ'_R , the paths in F to Υ'_R from the vertices in the complement of $\Upsilon'_R \cup B(x, R)$ are conditionally independent of γ_R . Therefore the conditional distribution of ζ_R given Υ_R is just the harmonic measure on Υ'_R for a simple random walk started at x . Given Υ_R , we shall write $\mu_y(z; \Upsilon_R)$ for the probability that a simple random walk started at y first hits Υ_R at z . Similarly, for a set W of vertices we write $\mu_y(W; \Upsilon_R) := \sum_{z \in W} \mu_y(z; \Upsilon_R)$.

It is clear that the pair (Υ_R, ζ_R) determines for which $z \notin B(x, R)$ the relation $x\mathcal{U}z$ holds. Therefore, the indicator function of A is measurable with respect to the pair (Υ_R, ζ_R) . Given Υ_R , let A_R denote the set of vertices $z \in \Upsilon_R$ such that A holds if $\zeta_R = z$. Then

$$(4.9) \quad \mathbf{P}[A] = \mathbf{P}[\zeta_R \in A_R] = \mathbf{E}[\mu_x(A_R; \Upsilon_R)].$$

A very similar relation holds for $\mathbf{P}[\mathcal{S}, A]$. Instead of choosing γ_R immediately after Υ'_R , we can first determine whether \mathcal{S} occurs after choosing Υ'_R and then choose γ_R . Again when Υ'_R is given, the paths from points outside Υ'_R and outside $B(x, R)$ are not influenced by \mathcal{S} , so that $\mathbf{P}[\mathcal{S} | \Upsilon'_R] = \mathbf{P}[\mathcal{S} | \Upsilon_R]$. We further remind the reader of the following fact (see, e.g., BLPS [2]). Suppose that $G = (V, E)$ is a finite graph and $E_1, E_2 \subset E$. Let T' be the (set of edges of the) UST in G . Then T' , conditioned on $T' \cap (E_1 \cup E_2) = E_1$ (assuming this

event has positive probability), is the union of E_1 with the set of edges of a UST on the graph obtained from G by contracting the edges in E_1 and deleting the edges in E_2 . Let H be the graph obtained from \mathbb{Z}^d by contracting the edges in K_1 and deleting the edges in K_2 . By first choosing Υ'_R and continuing with Wilson's algorithm we see that the conditional law of $F \setminus \Upsilon_R$ given Υ_R , is the law of a UST on the finite graph obtained from \mathbb{Z}^d by gluing all vertices in Υ_R to a single vertex. If we have a further condition on the occurrence of \mathcal{S} , then we should also contract the edges in K_1 and delete the edges in K_2 . Therefore, conditionally on Υ_R and the occurrence of \mathcal{S} , the path γ_R has the distribution of the loop-erasure of a simple random walk on H , starting at x and stopping when it hits Υ'_R . If we write $\mu_y^H(z; \Upsilon_R)$ for the probability that a simple random walk on H started at y first hits Υ_R at z , then we obtain, analogously to (4.9), that

$$\mathbf{P}[\mathcal{S}, A] = \mathbf{P}[\mathcal{S}, \zeta_R \in A_R] = \mathbf{E} \left[\mathbf{P}[\mathcal{S} | \Upsilon_R] \mu_x^H(A_R; \Upsilon_R) \right].$$

In view of (4.8) this gives

$$(4.10) \quad \left| \mathbf{P}[\mathcal{S}, A] - \mathbf{P}[\mathcal{S}] \mathbf{E} \left[\mu_x^H(A_R; \Upsilon_R) \right] \right| \leq \mathbf{E} \left[\left| \mathbf{P}[\mathcal{S} | \Upsilon_R] - \mathbf{P}[\mathcal{S}] \right| \right] \xrightarrow{R \rightarrow \infty} 0.$$

Given $\varepsilon > 0$, let $R_1 > r$ be such that a simple random walk started from any vertex outside $B(x, R_1)$ has probability at most ε to visit $B(x, r)$. Since every bounded harmonic function in \mathbb{Z}^d is constant, there exists $R_2 > R_1$ such that any harmonic function u on $B(x, R_2)$ that takes values in $[0, 1]$, satisfies the Harnack inequality

$$(4.11) \quad \sup_{y, z \in B(x, R_1+1)} |u(y) - u(z)| < \varepsilon.$$

(See, e.g., Theorem 1.7.1(a) in Lawler [7].) In particular, when $\Upsilon_R \cap B(x, R_2) = \emptyset$, we can apply this to the harmonic function $y \mapsto \mu_y(A_R; \Upsilon_R)$ to obtain

$$(4.12) \quad \sup_{y, y' \in B(x, R_1+1)} |\mu_y(A_R; \Upsilon_R) - \mu_{y'}(A_R; \Upsilon_R)| < \varepsilon$$

provided that $\Upsilon_R \cap B(x, R_2) = \emptyset$.

We need to show that $\mu_x(A_R, \Upsilon_R)$ is close to $\mu_x^H(A_R, \Upsilon_R)$. To this end, note that by the definition of R_1 , when a simple random walk on H from x first exits $B(x, R_1)$, it has probability at most ε to revisit $B(x, r)$. On the event that it does not, it may be considered as a random walk in \mathbb{Z}^d which does not visit $B(x, r)$. By (4.12) it follows that

$$(4.13) \quad |\mu_x^H(A_R; \Upsilon_R) - \mu_x(A_R; \Upsilon_R)| \leq 2\varepsilon,$$

on the event $\Upsilon_R \cap B(x, R_2) = \emptyset$. Finally take $R_3 > R_2$ sufficiently large so that

$$\forall R \geq R_3, \quad \mathbf{P}[\Upsilon_R \cap B(x, R_2) \neq \emptyset] < \varepsilon.$$

From (4.13) it follows that for $R \geq R_3$,

$$\left| \mathbf{E}[\mu_x^H(A_R; \Upsilon_R)] - \mathbf{E}[\mu_x(A_R; \Upsilon_R)] \right| < 3\varepsilon.$$

Combined with (4.10) and (4.9) this shows that for sufficiently large R

$$|\mathbf{P}[\mathcal{S}, A] - \mathbf{P}[\mathcal{S}] \mathbf{P}[A]| < 4\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude that A and \mathcal{S} are independent, which proves that \mathcal{U} has trivial left tail. By symmetry, the right tail is trivial too. \square

Remark 4.6. The above proof of left-tail triviality for \mathcal{U} is valid for the wired USF in any transient graph G such that there are no nonconstant bounded harmonic functions and a.s. each component of the USF has one end. Only the one-end property is needed for triviality of the remote tail. (For recurrent graphs, the USF is a.s. a tree, whence obviously the USF relation has trivial left, right and remote tails.)

The following lemma will be needed in the proof of the last statement of Theorem 1.1.

LEMMA 4.7. *Let $D \subset \mathbb{Z}^d$ be a finite connected set with a connected complement, and denote by \widehat{D}^c the subgraph of \mathbb{Z}^d spanned by the vertices in $\mathbb{Z}^d \setminus D$. Let F be the USF on \mathbb{Z}^d , and denote by Γ_D the event that there are no oriented edges in F from D^c to D . Then the distribution of $F \cap \widehat{D}^c$ conditioned on Γ_D , is the same as the distribution of the wired USF in the graph \widehat{D}^c .*

Proof. We first consider a finite version of this statement. Let $G = (V, E)$ be a finite connected graph, $\rho \in V$ a distinguished vertex, and $D \subset V \setminus \{\rho\}$. Let G_- be the subgraph of G spanned by $V \setminus D$. Denote by $S(G)$ the set of spanning trees in G , and let Γ_D^* be the set of $t \in S(G)$ such that for every $w \in D^c$, the path in t from w to ρ is disjoint from D .

Claim. Let T be a uniform spanning tree (UST) in G . Assume that $\Gamma_D^* \neq \emptyset$. Then conditioned on $T \in \Gamma_D^*$, the edge set $T \cap G_-$ is a UST in G_- .

To prove the claim, fix two trees $t_1, t_2 \in S(G_-)$, and for every $t \in \Gamma_D^*$ that contains t_1 , define $t' := (t \setminus t_1) \cup t_2$. For each vertex $v \in V$ there is a path in t' from v to ρ , that uses edges of $(t \setminus t_1)$ until it reaches D^c , and then uses edges of t_2 . Note that t cannot contain any edge of $t_2 \setminus t_1$, because t_1 plus any edge of \widehat{D}^c outside t_1 contains a circuit. Thus t and t' have the same number of edges. It follows that $t' \in S(G)$ and moreover, $t' \in \Gamma_D^*$. The map $t \mapsto t'$, is a bijection, because $t' \mapsto (t' \setminus t_2) \cup t_1$ is its inverse. This shows that t_1 and t_2 have the same number of extensions to spanning trees of G that are in Γ_D^* . Moreover any $t \in \Gamma_D^*$ is an extension of some $t_1 \in S(G_-)$. In other words, we have established the claim.

The lemma follows by consideration of the uniform spanning forest on \mathbb{Z}^d as a weak limit of uniform spanning trees in finite subgraphs (with wired complements), where ρ is chosen as the wired vertex. \square

Remark 4.8. If a finite connected set $D \subset \mathbb{Z}^d$ has a connected complement, then the lemma above implies that a.s., every component of the wired USF in $\mathbb{Z}^d \setminus D$ has one end. The proof of Theorem 4.5 can then be adapted to show that the wired USF relation in $\mathbb{Z}^d \setminus D$ has trivial remote, right, and left, tail σ -fields. Note that this can also be inferred from Remark 4.6. To verify that every bounded harmonic function on $\mathbb{Z}^d \setminus D$ is constant, we recall that the existence of nonconstant bounded harmonic functions on a graph is equivalent to the existence of two disjoint sets of vertices A, B such that with positive probability the random walk eventually stays in A , and the same holds for B . (If such A and B exist, then define the harmonic function $h(v)$ as the probability that a simple random walk started from v eventually stays in A . If h is a bounded harmonic function with $\sup h = 1, \inf h = 0$, say, then we may take $A := h^{-1}([0, 1/4])$ and $B := h^{-1}([3/4, 1])$.) This criterion is clearly unaffected by the removal of D from \mathbb{Z}^d .

5. Proofs of main results

Proof of Theorem 1.1. Denote $m = \lfloor \frac{d-1}{4} \rfloor$. By applying Corollary 3.4 to $m + 1$ independent copies of the USF relation, and invoking Theorems 4.2 and 4.5, we infer that a.s., every two vertices in \mathbb{Z}^d are related by the composition of these $m + 1$ copies. Now Theorem 4.1 yields the upper bound $\max\{N(x, y) : x, y \in \mathbb{Z}^d\} \leq m$ a.s.

For the final assertion of the theorem, it suffices to show that for every $x, y \in \mathbb{Z}^d$ and every $r > 0$, the event $\Xi(x, y, r, m)$ that there is a path in $\mathbb{Z}^d \setminus B_r$ from $T(x)$ to $T(y)$ with at most m edges outside F , satisfies

$$(5.1) \quad \mathbf{P}[\Xi(x, y, r, m)] = 1.$$

Let $\Delta_d(r)$ denote the collection of finite, connected sets $D \subset \mathbb{Z}^d$ such that $B_r \subset D$ and D^c is connected. For $D \in \Delta_d(r)$, denote by Γ_D the event that there are no oriented edges in F from D^c to D . We claim that for every $r > 0$,

$$(5.2) \quad \mathbf{P}\left[\bigcup_{D \in \Delta_d(r)} \Gamma_D\right] = 1.$$

Indeed, let \vec{B}_r denote the set of vertices $v \in \mathbb{Z}^d$ such that the oriented path from v to infinity in F enters B_r . Since a.s. each of the USF components has one end, \vec{B}_r and its complement are connected, and \vec{B}_r is a.s. finite. Thus $\Gamma_{\vec{B}_r}$ occurs and (5.2) holds.

Therefore, to prove (5.1), it suffices to show that

$$(5.3) \quad \forall D \in \Delta_d(r), \quad \mathbf{P}[\Xi(x, y, r, m) | \Gamma_D] = 1.$$

Fix $D \in \Delta_d(r)$, let G_- be the subgraph of \mathbb{Z}^d spanned by the vertices in $\mathbb{Z}^d \setminus D$, and let F_- denote the wired USF on G_- . By Lemma 4.7, conditioned on Γ_D , the law of $F \cap G_-$ is the same as that of F_- . Consequently, it suffices to show that a.s. every pair of vertices in G_- is connected by a path in G_- with at most m edges outside of F_- . The USF relation on G_- has trivial left, right and remote tails, by Remark 4.8. It is also clear that this relation has stochastic dimension 4. Therefore, the above proof for \mathbb{Z}^d applies also to G_- , and shows that every pair of vertices in G_- is connected by a path in G_- with at most m edges outside F_- . Since r is arbitrary and D is an arbitrary set in $\Delta_d(r)$, this proves the last assertion of the theorem, and also completes the proof for $5 \leq d \leq 8$.

For the case $d > 8$, it remains only to prove the lower bound on $\max N(x, y)$. This is done in the following proposition. \square

PROPOSITION 5.1. *The inequality*

$$\mathbf{P}[N(x, y) \leq k] \preccurlyeq \langle xy \rangle^{4k+4-d}$$

holds for all $x, y \in \mathbb{Z}^d$ and all $k \in \mathbb{N}$.

Proof. The proposition clearly holds for $k \geq \lfloor (d-1)/4 \rfloor$. So assume that $k < \lfloor (d-1)/4 \rfloor$. Consider a sequence $\{x_j, y_j\}_{j=0}^k$ in \mathbb{Z}^d with $x_0 = x$, $y_k = y$ and $|y_j - x_{j+1}| = 1$ for $j = 0, 1, \dots, k-1$. Let $A = A(x_0, y_0, \dots, x_k, y_k)$ be the event that $y_j \in T(x_j)$ for $j = 0, 1, \dots, k$ and $y_j \notin T(x_i)$ if $j \neq i$. Observe that

$$\{N(x, y) = k\} \subset \bigcup A(x_0, y_0, \dots, x_k, y_k),$$

where the union is over all such sequences $x_0, y_0, \dots, x_k, y_k$. To estimate the probability of A , we run Wilson’s algorithm rooted at infinity, beginning with random walks started at $x_0, y_0, \dots, x_k, y_k$. For A to hold, for each $j = 0, \dots, k$, the random walk started at x_j must intersect the path of the random walk started at y_j . Hence, as in (4.1),

$$(5.4) \quad \mathbf{P}[A] \preccurlyeq \prod_{j=0}^k \langle x_j y_j \rangle^{4-d}.$$

Since x_{j+1} and y_j are adjacent, this implies

$$(5.5) \quad \mathbf{P}[N(x, y) = k] \preccurlyeq \sum \prod_{j=0}^k \langle x_j x_{j+1} \rangle^{4-d},$$

where the sum extends over all sequences $\{x_j\}_{j=0}^{k+1} \subset \mathbb{Z}^d$ such that $x_0 = x$ and $x_{k+1} = y$.

We use Lemma 2.10 repeatedly in (5.5), first to sum over x_k , then over x_{k-1} , and so on, up to x_1 , and obtain $\mathbf{P}[N(x, y) = k] \asymp \langle xy \rangle^{4k+4-d}$, which completes the proof. \square

Proof of Theorem 1.2. The case $d \leq 4$ is from Pemantle [11]. The case $d = 5, 6, 7, 8$ follows from Theorem 1.1. For $d \geq 9$ and fixed $x, y \in \mathbb{Z}^d$ and $r > 0$ we infer from (5.4) that

$$\begin{aligned} \mathbf{P}[\exists v \in T(x), \exists w \in T(y) : |v| \geq n, |w| \geq n, |v - w| \leq r, T(x) \neq T(y)] \\ \leq \sum_{v,w} \mathbf{P}[A(x, v, y, w)] \asymp \sum_{v,w} \langle xv \rangle^{4-d} \langle yw \rangle^{4-d}, \end{aligned}$$

where the sums are over v, w satisfying $|v|, |w| \geq n$ and $|v - w| \leq r$. The latter sum is bounded by $C(x, y, r) n^{8-d}$, by Lemma 2.10. \square

6. Further examples and remarks

We now present some examples of relations having a stochastic dimension, and several questions.

Example 6.1. Fix some $\alpha \in (0, d)$. Let $\mathcal{R} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ be the random relation such that $\{x\mathcal{R}y\}_{\{x,y\} \subset \mathbb{Z}^d}$ are independent events, and $\mathbf{P}[x\mathcal{R}y] = \langle xy \rangle^{-\alpha}$. Then \mathcal{R} has stochastic dimension $d - \alpha$. This relation corresponds to *long range percolation* in \mathbb{Z}^d , where the degree of every vertex is infinite and the infinite cluster contains all vertices of \mathbb{Z}^d . Theorem 2.4 implies that the intrinsic diameter of this graph is $\lceil \frac{d}{d-\alpha} \rceil$ almost surely.

Example 6.2. Let $d \geq 3$, and for every $x \in \mathbb{Z}^d$, let S^x be a simple random walk starting from x , with $\{S^x\}_{x \in \mathbb{Z}^d}$ independent. Let $x\mathcal{S}y$ be the relation $\exists n \in \mathbb{N}$ such that $S^x(n) = y$. It is easily seen that \mathcal{S} has stochastic dimension 2. From Theorems 4.2, 3.4 and 4.5, we infer that a simple random walk a.s. intersects each component of an independent USF if and only if $d \leq 6$.

Example 6.3. For every $x \in \mathbb{Z}^d$, let S^x be as above. Define a relation \mathcal{Q} where $x\mathcal{Q}y$ if and only if $\exists n \in \mathbb{N}$ such that $S^x(n) = S^y(n)$. Then \mathcal{Q} has stochastic dimension 2 (provided $d \geq 2$). The proof is left to the reader.

Example 6.4. Fix $d \geq 2$. For every $x \in \mathbb{Z}^d$, let $S_*^x(\cdot)$ be a continuous-time simple random walk starting from x , where particles walk independently until they meet; when two particles meet, they coalesce, and the resulting particle carries both labels. Let $x\mathcal{W}y$ be the relation $\exists t > 0$ such that $S_*^x(t) = y$. Then \mathcal{W} has stochastic dimension 2.

Example 6.5 (USF generated by other walks). Let $\{S_n\}_{n \geq 0}$ be a symmetric random walk on \mathbb{Z}^d ; i.e., the increments $S_n - S_{n-1}$ are symmetric

i.i.d. \mathbb{Z}^d -valued variables. Say that this random walk has *Greenian index* α if the Green function $G_S(x, y) := \sum_{n=0}^{\infty} \mathbf{P}[S_n = y | S_0 = x]$ satisfies $G_S(x, y) \asymp \langle xy \rangle^{\alpha-d}$ for all $x, y \in \mathbb{Z}^d$. It is well-known that random walks with bounded increments of mean zero have Greenian index 2, provided that $d \geq 3$; for $d = 3$, the boundedness assumption may be relaxed to finite variance (see Spitzer [12]). Williamson [14] shows that for $0 < \alpha < \min\{2, d\}$, if a random walk $\{S_n\}$ in \mathbb{Z}^d with mean zero increments satisfies $\mathbf{P}[S_n - S_{n-1} = x] \sim c|x|^{-d-\alpha}$ (i.e., the ratio of the two expressions here tends to 1 as $|x| \rightarrow \infty$), then this random walk has Greenian index α . We can use Wilson’s method rooted at infinity to generate the wired spanning forest (WSF) based on a symmetric random walk with Greenian index α . (Note: this will not be a subgraph of the usual graph structure on \mathbb{Z}^d .) The arguments of the preceding section extend to show that the relation determined by the components of this WSF will have stochastic dimension 2α .

Example 6.6 (Minimal Spanning Forest). Let the edges of \mathbb{Z}^d have independent and identically distributed weights w_e , which are uniform random variables in $[0, 1]$. The *minimal spanning forest* is the subgraph of \mathbb{Z}^d obtained by removing every edge that has the maximal weight in some cycle; see, e.g., Newman and Stein [10].

CONJECTURE 6.7. *Let \mathcal{M} if x and y are in the same minimal spanning forest component. Then \mathcal{M} has stochastic dimension 8 in \mathbb{Z}^d if $d \geq 8$. This is a variation on a conjecture of Newman and Stein [10].*

Remark 6.8. It is natural to ask how the notion of stochastic dimension is related to other notions of dimension for subsets of a lattice. In this direction, we state the following. Let $\mathcal{L} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ be a random relation. Denote $\Gamma_o = \Gamma_o(\mathcal{L}) := \{z : o\mathcal{L}z\}$; if \mathcal{L} is an equivalence relation, then $\Gamma_o(\mathcal{L})$ is the equivalence class of the origin o , but we do not assume this.

PROPOSITION 6.9. *Suppose that $\mathcal{L} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ has stochastic dimension $\alpha \in (0, d)$. Denote by η_n the number of vertices in $\Gamma_o(\mathcal{L}) \cap B(o, n)$. Then*

- (i) *The laws of η_n/n^α are tight, but do not converge weakly to a point mass at 0.*
- (ii) *With positive probability, $\limsup_n \frac{\log \eta_n}{\log n} = \alpha$.*
- (iii) *If, moreover, \mathcal{L} has a trivial left tail, then*

$$\limsup_n \frac{\log \eta_n}{\log n} = \alpha \text{ a.s.}$$

Proof. (i) The definition of stochastic dimension and a standard calculation yield that $\mathbf{E}\eta_n \asymp n^\alpha$ and $\mathbf{E}\eta_n^2 \asymp n^{2\alpha}$. These estimates imply the asserted

tightness, and also that

$$(6.1) \quad \inf_n \mathbf{P}(\eta_n \geq \mathbf{E}\eta_n/2) > 0$$

(see, e.g., Kahane [5, p. 8]).

(ii) The bound $\mathbf{E}\eta_n \asymp n^\alpha$ implies that $\sum_k \eta_{2^k}/(2^{k\alpha}k^2) < \infty$ a.s. Thus, monotonicity of η_n yields that $\limsup_n \frac{\log \eta_n}{\log n} \leq \alpha$ a.s. The lower bound follows from (6.1).

(iii) This follows from (ii) and the definition of the left tail. □

When \mathcal{L} is the USF relation and $d > 4$, we have $\alpha = 4$, and the limsup in (iii) can be replaced by a limit (we omit the proof). With more work, it can be shown that the “discrete Hausdorff dimension” (in the sense of Barlow and Taylor [1]) of the USF component Γ_o is also 4, almost surely.

A harder task is to prove existence of the *scaling limit* for the USF in dimension $d > 4$, and show that this limiting object has Hausdorff dimension 4 almost surely.

An easily stated problem in this direction, is to show that

$$(6.2) \quad \mathbf{P}[\Gamma_o \cap B(x, k) \neq \emptyset] \asymp (k/|x|)^{d-4}$$

for $k < |x|$. At present, we can only establish such an estimate up to a power of $\log x$. Note that (6.2), if true, depends on properties of the USF beyond its stochastic dimension; the analog of (6.2) fails for the relation considered in Example 6.1.

Example 6.10. As noted in the introduction, for $5 \leq d \leq 8$, the USF defines a translation invariant random partition of \mathbb{Z}^d with infinitely many connected components, where any two components come within distance 1 infinitely often. A partition with these properties exists for any $d \geq 3$: In the hyperplane $H := \{x \in \mathbb{Z}^d : x_d = 0\}$, let L_0 be the lattice consisting of all vertices with all coordinates even, and let L be a random translate of L_0 in H , chosen uniformly among the 2^{d-1} possibilities. Let F_0 be the union of all lines perpendicular to H which contain a vertex of L , and complete F_0 to a spanning forest F using Wilson’s algorithm.

Example 6.11. Consider the random relation $\mathcal{P} \subset \mathbb{Z}^2 \times \mathbb{Z}^2$, where $x\mathcal{P}y$ if and only if x and y are in the same connected component of Bernoulli bond percolation at the critical parameter $p = p_c = 1/2$. It is known that a.s. all the components will be finite. If $\mathcal{L}, \mathcal{R} \subset \mathbb{Z}^2 \times \mathbb{Z}^2$ and $|\{y : x\mathcal{L}y\}| < \infty$ and $|\{y : x\mathcal{R}y\}| < \infty$ for all $x \in \mathbb{Z}^2$, then $\{y : x\mathcal{L}\mathcal{R}y\}$ is finite too. After inductive application of Theorem 2.4 to independent copies of \mathcal{P} , it follows that \mathcal{P} cannot have a positive stochastic dimension. However, the Russo-Seymour-Welsh Theorem implies that $\mathbf{P}[x\mathcal{P}y] \geq \langle xy \rangle^{-\beta}$ for some $\beta < \infty$.

Remark 6.12. Our results on the USF in \mathbb{Z}^d extend readily to other Cayley graphs of polynomial growth. Let G be such a graph, and let o be a vertex of G . By Gromov [3], G satisfies $|B(o, r)| \asymp r^d$ for some integer d (see also Woess [16, Lemma 3.14 and Th. 3.16]). Consequently, we may find an $a > 2$ such that $|B(o, a^{n+1}) \setminus B(o, a^n)| \asymp a^{nd}$ for $n \in \mathbb{N}$. Thus, in our above proofs, we may work with shells based on powers of a , in place of the shells H_n^N . By Hebisch and Saloff-Coste [4], the Green function estimates which we used in \mathbb{Z}^d apply to G . Moreover, any bounded harmonic function on G is a constant (see, e.g., [6]). This implies, in particular, that the free and the wired USF on G are the same, by Theorem 7.3 of BLPS [2] (so it is clear what we mean by the USF on G). By Theorem 10.1 of BLPS [2] a.s. each component of the USF on G has one end, unless G is quasi-isometric to \mathbb{Z} (in which case G is recurrent, $d = 1$, the USF is a tree a.s. and our results clearly hold). With these basic facts established, the proofs go through just as for \mathbb{Z}^d , with the case $d \leq 4$ handled by Corollary 9.6 of BLPS [2].

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