# A $p$-adic local monodromy theorem 

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#### Abstract

We produce a canonical filtration for locally free sheaves on an open $p$-adic annulus equipped with a Frobenius structure. Using this filtration, we deduce a conjecture of Crew on $p$-adic differential equations, analogous to Grothendieck's local monodromy theorem (also a consequence of results of André and of Mebkhout). Namely, given a finite locally free sheaf on an open $p$-adic annulus with a connection and a compatible Frobenius structure, the module admits a basis over a finite cover of the annulus on which the connection acts via a nilpotent matrix.


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## 1. Introduction

1.1. Crew's conjecture on p-adic local monodromy. The role of $p$-adic differential equations in algebraic geometry was first pursued systematically by Dwork; the modern manifestation of this role comes via the theory of isocrystals and $F$-isocrystals, which over a field of characteristic $p>0$ attempt to play the part of local systems for the classical topology on complex varieties and lisse sheaves for the $l$-adic topology when $l \neq p$. In order to get a usable theory, however, an additional "overconvergence" condition must be imposed, which has no analogue in either the complex or l-adic cases. For example, the cohomology of the affine line is infinite dimensional if computed using convergent isocrystals, but has the expected dimension if computed using overconvergent isocrystals. This phenomenon was generalized by Monsky and Washnitzer [MW] into a cohomology theory for smooth affine varieties, and then generalized further by Berthelot to the theory of rigid cohomology, which has good behavior for arbitrary varieties (see for example [Be1]).

Unfortunately, the use of overconvergent isocrystals to date has been hampered by a gap in the local theory of these objects; for example, it obstructed the proof of finite dimensionality of Berthelot's rigid cohomology with arbitrary coefficients (the case of constant coefficients was treated by Berthelot in [Be2]). This gap can be described as a $p$-adic analogue of Grothendieck's local monodromy theorem for $l$-adic cohomology.

The best conceivable analogue of Grothendieck's theorem would be that an $F$-isocrystal becomes a successive extension of trivial isocrystals after a finite étale base extension. Unfortunately, this assertion is not correct; for
example, it fails for the pushforward of the constant isocrystal on a family of ordinary elliptic curves degenerating to a supersingular elliptic curve (and for the Bessel isocrystal described in Section 1.5 over the affine line).

The correct analogue of the local monodromy theorem was formulated conjecturally by Crew [Cr2, §10.1], and reformulated in a purely local form by Tsuzuki [T2, Th. 5.2.1]; we now introduce some terminology and notation needed to describe it. (These definitions are reiterated more precisely in Chapter 2.) Let $k$ be a field of characteristic $p>0$, and $\mathcal{O}$ a finite totally ramified extension of a Cohen ring $C(k)$. The Robba ring $\Gamma_{\mathrm{an}, \mathrm{con}}$ is defined as the set of formal Laurent series over $\mathcal{O}\left[\frac{1}{p}\right]$ which converge on some open annulus with outer radius 1 ; its subring $\Gamma_{\text {con }}$ consists of series which take integral values on some open annulus with outer radius 1 , and is a discrete valuation ring. (See Chapter 3 to find out where the notation comes from.) We say a ring endomorphism $\sigma: \Gamma_{\mathrm{an}, \mathrm{con}} \rightarrow \Gamma_{\mathrm{an}, \mathrm{con}}$ is a Frobenius for $\Gamma_{\mathrm{an}, \text { con }}$ if it is a composition power of a map preserving $\Gamma_{\text {con }}$ and reducing modulo a uniformizer of $\Gamma_{\text {con }}$ to the $p$-th power map. For example, one can choose $t \in \Gamma_{\text {con }}$ whose reduction is a uniformizer in the ring of Laurent series over $k$, then set $t^{\sigma}=t^{q}$. Note that one cannot hope to define a Frobenius on the ring of analytic functions on any fixed open annulus with outer radius 1 , because for $\eta$ close to 1 , functions on the annulus of inner radius $\eta$ pull back under $\sigma$ to functions on the annulus of inner radius $\eta^{1 / p}$. Instead, one must work over an "infinitely thin" annulus of radius 1 .

Given a ring $R$ in which $p \neq 0$ and an endomorphism $\sigma: R \rightarrow R$, we define a $\sigma$-module as a finite locally free module $M$ equipped with an $R$-linear map $F: M \otimes_{R, \sigma} R \rightarrow M$ that becomes an isomorphism over $R\left[\frac{1}{p}\right]$; the tensor product notation indicates that $R$ is viewed as an $R$-module via $\sigma$. For the rings considered in this paper, a finite locally free module is automatically free; see Proposition 2.5. Then $F$ can be viewed as an additive, $\sigma$-linear map $F: M \rightarrow M$ that acts on any basis of $M$ by a matrix invertible over $R\left[\frac{1}{p}\right]$.

We define a $(\sigma, \nabla)$-module as a $\sigma$-module plus a connection $\nabla: M \rightarrow$ $M \otimes \Omega_{R / \mathcal{O}}^{1}$ (that is, an additive map satisfying the Leibniz rule $\nabla(c \mathbf{v})=c \nabla(\mathbf{v})+$ $\mathbf{v} \otimes d c$ ) that makes the following diagram commute:


We say a $(\sigma, \nabla)$-module over $\Gamma_{\text {an,con }}$ is quasi-unipotent if, after tensoring $\Gamma_{\text {an,con }}$ over $\Gamma_{\text {con }}$ with a finite extension of $\Gamma_{\text {con }}$, the module admits a filtration by $(\sigma, \nabla)$-submodules such that each successive quotient admits a basis of elements in the kernel of $\nabla$. (If $k$ is perfect, one may also insist that the extension
of $\Gamma_{\text {con }}$ be residually separable.) With this notation, Crew's conjecture is resolved by the following theorem, which we will prove in a more precise form as Theorem 6.12.

Theorem 1.1 (Local monodromy theorem). Let $\sigma$ be any Frobenius for the Robba ring $\Gamma_{\mathrm{an}, \text { con }}$. Then every $(\sigma, \nabla)$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}$ is quasi-unipotent.

Briefly put, a $p$-adic differential equation on an annulus with a Frobenius structure has quasi-unipotent monodromy. It is worth noting (though not needed in this paper) that for a given $\nabla$, whether there exists a compatible $F$ does not depend on the choice of the Frobenius map $\sigma$. This follows from the existence of change of Frobenius functors [T2, Th. 3.4.10].

The purpose of this paper is to establish some structural results on modules over the Robba ring yielding a proof of Theorem 1.1. Note that Theorem 1.1 itself has been established independently by André [A2] and by Mebkhout $[\mathrm{M}]$. However, as we describe in the next section, the methods in this paper are essentially orthogonal to the methods of those authors. In fact, the different approaches provide different auxiliary information, various pieces of which may be of relevance in other contexts.
1.2. Frobenius filtrations and Crew's conjecture. Before outlining our approach to Crew's conjecture, we describe by way of contrast the common features of the work of André and Mebkhout. Both authors build upon the results of a series of papers by Christol and Mebkhout [CM1], [CM2], [CM3], [CM4] concerning properties of modules with connection over the Robba ring. Most notably, in [CM4] they produced a canonical filtration (the "filtration de pentes"), defined whether or not the connection admits a Frobenius structure. André and Mebkhout show (in two different ways) that when a Frobenius structure is present, the graded pieces of this filtration can be shown to be quasi-unipotent.

The strategy in this paper is in a sense completely orthogonal to the aforementioned approach. (For a more detailed comparison between the various approaches to Crew's conjecture, see the November 2001 Seminaire Bourbaki talk of Colmez [Co].) Instead of isolating the connection data, we isolate the Frobenius structure and prove a structure theorem for $\sigma$-modules over the Robba ring. This can be accomplished by a "big rings" argument, where one first proves that $\sigma$-modules can be trivialized over a large auxiliary ring, and then "descends" the construction back to the Robba ring. (Isolating Frobenius in this manner is not unprecedented; for example, this is the approach of Katz in [Ka].)

The model for our strategy of trivializing $\sigma$-modules over an auxiliary ring is the Dieudonné-Manin classification of $\sigma$-modules over a complete discrete valuation ring $R$ of mixed characteristic ( $0, p$ ) with algebraically closed residue
field. (This classification is a semilinear analogue of the diagonalization of matrices over an algebraically closed field, except that here there is no failure of semisimplicity.) We give a quick statement here, deferring the precise formulation to Section 5.2. For $\lambda \in \mathcal{O}\left[\frac{1}{p}\right]$ and $d$ a positive integer, let $M_{\lambda, d}$ denote the $\sigma$-module of rank $d$ over $R\left[\frac{1}{p}\right]$ on which $F$ acts by a basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{d}$ as follows:

$$
\begin{aligned}
& F \mathbf{v}_{1}=\mathbf{v}_{2} \\
& \vdots \\
& F \mathbf{v}_{d-1}=\mathbf{v}_{d} \\
& F \mathbf{v}_{d}=\lambda \mathbf{v}_{1} .
\end{aligned}
$$

Define the slope of $M_{\lambda, d}$ to be $v_{p}(\lambda) / d$. Then the Dieudonné-Manin classification states (in part) that over $R\left[\frac{1}{p}\right]$, every $\sigma$-module is isomorphic to a direct sum $\oplus_{j} M_{\lambda_{j}, d_{j}}$, and the slopes that occur do not depend on the decomposition.

If $R$ is a discrete valuation ring of mixed characteristic $(0, p)$, we may define the slopes of a $\sigma$-module over $R\left[\frac{1}{p}\right]$ as the slopes in a Dieudonné-Manin decomposition over the maximal unramified extension of the completion of $R$. However, this definition cannot be used immediately over $\Gamma_{\text {an,con }}$, because that ring is not a discrete valuation ring. Instead, we must first reduce to considering modules over $\Gamma_{\text {con }}$. Our main theorem makes it possible to do so. Again, we give a quick formulation here and prove a more precise result later as Theorem 6.10. (Note: the filtration in this theorem is similar to what Tsuzuki [T2] calls a "slope filtration for Frobenius structures".)

Theorem 1.2. Let $M$ be a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}$. Then there is a canonical filtration $0=M_{0} \subset M_{1} \subset \cdots \subset M_{l}=M$ of $M$ by saturated $\sigma$-submodules such that:
(a) each quotient $M_{i} / M_{i-1}$ is isomorphic over $\Gamma_{\mathrm{an}, \mathrm{con}}$ to a nontrivial $\sigma$ module $N_{i}$ defined over $\Gamma_{\text {con }}\left[\frac{1}{p}\right]$;
(b) the slopes of $N_{i}$ are all equal to some rational number $s_{i}$;
(c) $s_{1}<\cdots<s_{l}$.

The relevance of this theorem to Crew's conjecture is that $(\sigma, \nabla)$-modules over $\Gamma_{\text {con }}\left[\frac{1}{p}\right]$ with a single slope can be shown to be quasi-unipotent using a result of Tsuzuki [T1]. The essential case is that of a unit-root $(\sigma, \nabla)$-module over $\Gamma_{\text {con }}$, in which all slopes are 0 . Tsuzuki showed that such a module becomes isomorphic to a direct sum of trivial $(\sigma, \nabla)$-modules after a finite base extension, and even gave precise information about what extension is needed. This makes it possible to deduce the local monodromy theorem from Theorem 1.2.
1.3. Applications. We now describe some consequences of the results of this paper, starting with some applications via Theorem 1.1. One set of consequences occurs in the study of Berthelot's rigid cohomology (a sort of "grand unified theory" of $p$-adic Weil cohomologies). For example, Theorem 1.1 can be used to establish finite dimensionality of rigid cohomology with coefficients in an overconvergent $F$-isocrystal; see [Cr2] for the case of a curve and [Ke7] for the general case. It can also be used to generalize the results of Deligne's "Weil II" to overconvergent $F$-isocrystals; this is carried out in [Ke8], building on work of Crew [Cr1], [Cr2]. In addition, it can be used to treat certain types of "descent", such as Tsuzuki's full faithfulness conjecture [T3], which asserts that convergent morphisms between overconvergent $F$-isocrystals are themselves overconvergent; see [Ke6].

Another application of Theorem 1.1 has been found by Berger $[\mathrm{Bg}]$, who has exposed a close relationship between $F$-isocrystals and $p$-adic Galois representations. In particular, he showed that Fontaine's "conjecture de monodromie $p$-adique" for $p$-adic Galois representations (that every de Rham representation is potentially semistable) follows from Theorem 1.1.

Further applications of Theorem 1.2 exist that do not directly pass through Theorem 1.1. For example, André and di Vizio [AdV] have formulated a $q$-analogue of Crew's conjecture, in which the single differential equation is replaced by a formal deformation. They have established this analogue using Theorem 6.10 plus a $q$-analogue of Tsuzuki's unit-root theorem (Proposition 6.11), and have deduced a finiteness theorem for rigid cohomology of $q-F$-isocrystals. (It should also be possible to obtain these results using a $q$-analogue of the Christol-Mebkhout theorem, and indeed André and di Vizio have made progress in this direction; however, at the time of this writing, some technical details had not yet been worked out.)

We also plan to establish, in a subsequent paper, a conjecture of Shiho [Sh, Conj. 3.1.8], on extending overconvergent $F$-isocrystals to log- $F$-isocrystals after a generically étale base change. This result appears to require a more sophisticated analogue of Theorem 6.10, in which the "one-dimensional" Robba ring is replaced by a "higher-dimensional" analogue. (One might suspect that this conjecture should follow from Theorem 1.1 and some clever geometric arguments, but the situation appears to be more subtle.) Berthelot (private communication) has suggested that a suitable result in this direction may help in constructing Grothendieck's six operations in the category of arithmetic $\mathcal{D}$-modules, which would provide a $p$-adic analogue of the constructible sheaves in étale cohomology.
1.4. Structure of the paper. We now outline the strategy of the proof of Theorem 1.2, and in the process describe the structure of the paper. We note in passing that some of the material appears in the author's doctoral dissertation [Ke1], written under Johan de Jong, and/or in a sequence of unpublished
preprints $[\mathrm{Ke} 2]$, $[\mathrm{Ke} 3]$, [Ke4], [Ke5]. However, the present document avoids any logical dependence on unpublished results.

In Chapter 2, we recall some of the basic rings of the theory of $p$-adic differential equations; they include the Robba ring, its integral subring and the completion of the latter (denoted the "Amice ring" in some sources). In Chapter 3, we construct some less familiar rings by augmenting the classical constructions. These augmentations are inspired by (and in some cases identical to) the auxiliary rings used by de Jong [dJ] in his extension to equal characteristic of Tate's theorem [Ta] on $p$-divisible groups over mixed characteristic discrete valuation rings. (They also resemble the "big rings" in Fontaine's theory of $p$-adic Galois representations, and coincide with rings occurring in Berger's work.) In particular, a key augmentation, denoted $\Gamma_{\text {an,con }}^{\text {alg }}$, is a sort of "maximal unramified extension" of the Robba ring, and a great effort is devoted to showing that it shares the Bézout property with the Robba ring; that is, every finitely generated ideal in $\Gamma_{\text {an, con }}^{\text {alg }}$ is principal. (This chapter is somewhat technical; we suggest that the reader skip it on first reading, and refer back to it as needed.)

With these augmented rings in hand, in Chapter 4 we show that every $\sigma$-module over the Robba ring can be equipped with a canonical filtration over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$; this amounts to an "overconvergent" analogue of the Dieudonné-Manin classification. From this filtration we read off a sequence of slopes, which in case we started with a quasi-unipotent $(\sigma, \nabla)$-module agree with the slopes of Frobenius on a nilpotent basis; the Newton polygon with these slopes is called the special Newton polygon.

By contrast, in Chapter 5, we associate to a $(\sigma, \nabla)$-module over $\Gamma_{\text {con }}$ the Frobenius slopes produced by the Dieudonné-Manin classification. The Newton polygon with these slopes is called the generic Newton polygon. Following [dJ], we construct some canonical filtrations associated with the generic Newton polygon. This chapter is logically independent of Chapter 4 except at its conclusion, when the two notions of Newton polygon are compared. In particular, we show that the special Newton polygon lies above the generic Newton polygon with the same endpoint, and obtain additional structural consequences in case the Newton polygons coincide.

Finally, in Chapter 6, we take the "generic" and "special" filtrations, both defined over large auxiliary rings, and descend them back to the Robba ring itself. The basic strategy here is to separate positive and negative powers of the series parameter, using the auxiliary filtrations to guide the process. Starting with a $\sigma$-module over the Robba ring, the process yields a $\sigma$-module over $\Gamma_{\text {con }}$ whose generic and special Newton polygons coincide. The structural consequences mentioned above yield Theorem 1.2; by applying Tsuzuki's theorem on unit-root $(\sigma, \nabla)$-modules (Proposition 6.11), we deduce a precise form of Theorem 1.1.
1.5. An example: the Bessel isocrystal. To clarify the remarks of the previous section, we include a classical example to illustrate the different structures we have described, especially the generic and special Newton polygons. Our example is the Bessel isocrystal, first studied by Dwork [Dw]; our description is a summary of the discussion of Tsuzuki [T2, Ex. 6.2.6] (but see also André [A1]).

Let $p$ be an odd prime, and put $\mathcal{O}=\mathbb{Z}_{p}[\pi]$, where $\pi$ is a $(p-1)$-st root of $-p$. Choose $\eta<1$, and let $R$ be the ring of Laurent series in the variable $t$ over $\mathcal{O}$ convergent for $|t|>\eta$. Let $\sigma$ be the Frobenius lift on $\mathcal{O}$ such that $t^{\sigma}=t^{p}$. Then for suitable $\eta$, there exists a $(\sigma, \nabla)$-module $M$ of rank two over $R$ admitting a basis $\mathbf{v}_{1}, \mathbf{v}_{2}$ such that

$$
\begin{aligned}
& F \mathbf{v}_{1}=A_{11} \mathbf{v}_{1}+A_{12} \mathbf{v}_{2} \\
& F \mathbf{v}_{2}=A_{21} \mathbf{v}_{1}+A_{22} \mathbf{v}_{2} \\
& \nabla \mathbf{v}_{1}=t^{-2} \pi^{2} \mathbf{v}_{2} \otimes d t \\
& \nabla \mathbf{v}_{2}=t^{-1} \mathbf{v}_{1} \otimes d t .
\end{aligned}
$$

Moreover, the matrix $A$ satisfies

$$
\operatorname{det}(A)=p \quad \text { and } \quad A \equiv\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \quad(\bmod p)
$$

It follows that the two generic Newton slopes are nonnegative (because the entries of $A$ are integral), their sum is 1 (by the determinant equation), and the smaller of the two is zero (by the congruence). Thus the generic Newton slopes are 0 and 1.

On the other hand, if $y=(t / 4)^{1 / 2}$, define

$$
f_{ \pm}=1+\sum_{n=1}^{\infty}( \pm 1)^{n} \frac{(1 \times 3 \times \cdots \times(2 n-1))^{2}}{(8 \pi)^{n} n!} y^{n}
$$

and set

$$
\mathbf{w}_{ \pm}=f_{ \pm} \mathbf{e}_{1}+\left(y \frac{d f_{ \pm}}{d y}+\left(\frac{1}{2} \mp \pi y^{-1}\right) f_{ \pm}\right) \mathbf{e}_{2} .
$$

Then

$$
\nabla \mathbf{w}_{ \pm}=\left(\frac{-1}{2} \pm \pi y^{-1}\right) \mathbf{w}_{ \pm} \otimes \frac{d y}{y}
$$

Using the compatibility between the Frobenius and connection structures, we deduce that

$$
F \mathbf{w}_{ \pm}=\alpha_{ \pm} y^{-(p-1) / 2} \exp \left( \pm \pi\left(y^{-1}-y^{-\sigma}\right)\right) \mathbf{w}_{ \pm}
$$

for some $\alpha_{+}, \alpha_{-} \in \mathcal{O}\left[\frac{1}{p}\right]$ with $\alpha_{+} \alpha_{-}=2^{1-p} p$. By the invariance of Frobenius under the automorphism $y \rightarrow-y$ of $\Gamma_{\mathrm{an}, \mathrm{con}}[y]$, we deduce that $\alpha_{+}$and $\alpha_{-}$have the same valuation.

It follows (see [Dw, §8]) that $M$ is unipotent over

$$
\Gamma_{\mathrm{an}, \mathrm{con}}\left[y^{1 / 2}, z\right] /\left(z^{p}-z-y\right)
$$

and the two slopes of the special Newton polygon are equal, necessarily to $1 / 2$ since their sum is 1 . In particular, the special Newton polygon lies above the generic Newton polygon and has the same endpoint, but the two polygons are not equal in this case.

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## 2. A few rings

In this chapter, we set some notation and conventions, and define some of the basic rings used in the local study of $p$-adic differential equations. We also review the basic properties of rings in which every finitely generated ideal is principal (Bézout rings), and introduce $\sigma$-modules and $(\sigma, \nabla)$-modules.
2.1. Notation and conventions. Recall that for every field $K$ of characteristic $p>0$, there exists a complete discrete valuation ring with fraction field of characteristic 0 , maximal ideal generated by $p$, and residue field isomorphic to $K$, and that this ring is unique up to noncanonical isomorphism. Such a ring is called a Cohen ring for $K$; see [Bo] for the basic properties of such rings. If $K$ is perfect, the Cohen ring is unique up to canonical isomorphism, and coincides with the ring $W(K)$ of Witt vectors over $K$. (Note in passing: for $K$ perfect, we use brackets to denote Teichmüller lifts into $W(K)$.)

Let $k$ be a field of characteristic $p>0$, and $C(k)$ a Cohen ring for $k$. Let $\mathcal{O}$ be a finite totally ramified extension of $C(k)$, let $\pi$ be a uniformizer of $\mathcal{O}$, and fix once and for all a ring endomorphism $\sigma_{0}$ on $\mathcal{O}$ lifting the absolute Frobenius $x \mapsto x^{p}$ on $k$. Let $q=p^{f}$ be a power of $p$ and put $\sigma=\sigma_{0}^{f}$. (In principle, one could dispense with $\sigma_{0}$ and simply take $\sigma$ to be any ring endomorphism lifting the $q$-power Frobenius. The reader may easily verify that the results of this paper carry over, aside from some cosmetic changes in Section 2.2; for instance, the statement of Proposition 2.1 must be adjusted slightly.) Let $v_{p}$ denote the valuation on $\mathcal{O}\left[\frac{1}{p}\right]$ normalized so that $v_{p}(p)=1$, and let $|\cdot|$ denote the norm on $\mathcal{O}\left[\frac{1}{p}\right]$ given by $|x|=p^{-v_{p}(x)}$.

Let $\mathcal{O}_{0}$ denote the fixed ring of $\mathcal{O}$ under $\sigma$. If $k$ is algebraically closed, then the equation $u^{\sigma}=\left(\pi^{\sigma} / \pi\right) u$ in $u$ has a nonzero solution modulo $\pi$, and so by a variant of Hensel's lemma (see Proposition 3.17) has a nonzero solution in $\mathcal{O}$. Then $(\pi / u)$ is a uniformizer of $\mathcal{O}$ contained in $\mathcal{O}_{0}$, and hence $\mathcal{O}_{0}$ has the
same value group as $\mathcal{O}$. That being the case, we can and will take $\pi \in \mathcal{O}_{0}$ in case $k$ is algebraically closed.

We wish to alert the reader to several notational conventions in force throughout the paper. The first of these is "exponent consolidation". The expression $\left(x^{-1}\right)^{\sigma}$, for $x$ a ring element or matrix and $\sigma$ a ring endomorphism, will often be abbreviated $x^{-\sigma}$. This is not to be confused with $x^{\sigma^{-1}}$; the former is the image under $\sigma$ of the multiplicative inverse of $x$, the latter is the preimage of $x$ under $\sigma$ (if it exists). Similarly, if $A$ is a matrix, then $A^{T}$ will denote the transpose of $A$, and the expression $\left(A^{-1}\right)^{T}$ will be abbreviated $A^{-T}$.

We will use the summation notation $\sum_{i=m}^{n} f(i)$ in some cases where $m>n$, in which case we mean 0 for $n=m-1$ and $-\sum_{i=n+1}^{m-1} f(i)$ otherwise. The point of this convention is that $\sum_{i=m}^{n} f(i)=f(n)+\sum_{i=m}^{n-1} f(i)$ for all $n \in \mathbb{Z}$.

We will perform a number of calculations involving matrices; these will always be $n \times n$ matrices unless otherwise specified. Also, $I$ will denote the $n \times n$ identity matrix over any ring, and any norm or valuation applied to a matrix will be interpreted as the maximum or minimum, respectively, over entries of the matrix.
2.2. Valued fields. Let $k((t))$ denote the field of Laurent series over $k$. Define a valued field to be an algebraic extension $K$ of $k((t))$ for which there exist subextensions $k((t)) \subseteq L \subseteq M \subseteq N \subseteq K$ such that:
(a) $L=k^{1 / p^{m}}((t))$ for some $m \in\{0,1, \ldots, \infty\}$;
(b) $M=k_{M}((t))$ for some separable algebraic extension $k_{M} / k^{1 / p^{m}}$;
(c) $N=M^{1 / p^{n}}$ for some $n \in\{0,1, \ldots, \infty\}$;
(d) $K$ is a separable totally ramified algebraic extension of $N$.
(Here $F^{1 / p^{\infty}}$ means the perfection of the field $F$, and $K / N$ totally ramified means that $K$ and $N$ have the same residue field.) Note that $n$ is uniquely determined by $K$ : it is the largest integer $n$ such that $t$ has a $p^{n}$-th root in $K$. If $n<\infty$ (e.g., if $K / k((t))$ is finite), then $L, M, N$ are also determined by $K$ : $k_{M}^{1 / p^{n}}$ must be the integral closure of $k$ in $K$, which determines $k_{M}$, and $k^{1 / p^{m}}$ must be the maximal purely inseparable subextension of $k_{M} / k$.

The following proposition shows that the definition of a valued field is only restrictive if $k$ is imperfect. It also guides the construction of the rings $\Gamma^{K}$ in Section 3.1.

Proposition 2.1. If $k$ is perfect, then any algebraic extension $K / k((t))$ is a valued field.

Proof. Normalize the valuation $v$ on $k((t))$ so that $v(t)=1$. Let $k_{M}$ be the integral closure of $k$ in $K$, and define $L=k((t))$ and $M=k_{M}((t))$. Then (a) holds for $m=0$ and (b) holds because $k$ is perfect.

Let $n$ be the largest nonnegative integer such that $t$ has a $p^{n}$-th root in $K$, or $\infty$ if there is no largest integer. Put

$$
N=\bigcup_{i=0}^{\infty}\left(K \cap M^{1 / p^{i}}\right)
$$

Since $t^{1 / p^{i}} \in K$ for all $i \leq n$ and $k_{M}$ is perfect, we have $M^{1 / p^{n}} \subseteq N$. On the other hand, suppose $x^{1 / p^{i}} \in\left(K \cap M^{1 / p^{i}}\right) \backslash\left(K \cap M^{1 / p^{i-1}}\right)$; that is, $x \in M$ has a $p^{i}$-th root in $K$ but has no $p$-th root in $M$. Then $v(x)$ is relatively prime to $p$, so that we can find integers $a$ and $b$ such that $y=x^{a} / t^{b p^{i}} \in M$ has a $p^{i}$-th root in $K$ and $v(y)=1$. We can write every element of $M$ uniquely as a power series in $y$, so every element of $M$ has a $p^{i}$-th root in $K$. In particular, $t$ has a $p^{i}$-th root in $K$, and so $i \leq n$. We conclude that $N=M^{1 / p^{n}}$, verifying (c).

If $y \in K^{p} \cap N$, then $y=z^{p}$ for some $z \in K$ and $y^{p^{i}} \in M$ for some $i$. Then $z^{p^{i+1}} \in M$, so $z \in N$. Since $K^{p} \cap N=N^{p}$, we see that $K / N$ is separable. To verify that $K / N$ is totally ramified, let $K_{0}$ be any finite subextension of $K / k((t))$ and let $U$ be the maximal unramified subextension of $K_{0} /\left(K_{0} \cap N\right)$. We now recall two basic facts from [Se] about finite extensions of fields complete with respect to discrete valuations.

1. $K_{0} / U$ is totally ramified, because $K_{0} /\left(K_{0} \cap N\right)$ and its residue field extension are both separable.
2. There is a unique unramified extension of $K_{0} \cap N$ yielding any specified separable residue field extension.

Since $K_{0} \cap N$ is a power series field, we can make unramified extensions of $K_{0} \cap N$ with any specified residue field extension by extending the constant field $K_{0} \cap k_{M}$. By the second assertion above, $U /\left(K_{0} \cap N\right)$ must then be a constant field extension. However, $k_{M}$ is integrally closed in $K$, and so $U=K_{0} \cap N$ and $K_{0} /\left(K_{0} \cap N\right)$ is totally ramified by the first assertion above. Since $K$ is the union of its finite subextensions over $k((t))$, we conclude that $K / N$ is totally ramified, verifying (d).

The proposition fails for $k$ imperfect, as there are separable extensions of $k((t))$ with inseparable residue field extensions. For example, if $c$ has no $p$-th root in $k$, then $K=k((t))[x] /\left(x^{p}-x-c t^{-p}\right)$ is separable over $k((t))$ but induces an inseparable residue field extension. Thus $K$ cannot be a valued field, as valued fields contain their residue field extensions.

We denote the perfect and algebraic closures of $k((t))$ by $k((t))^{\text {perf }}$ and $k((t))^{\text {alg }}$; these are both valued fields. We denote the separable closure of $k((t))$ by $k((t))^{\text {sep }}$; this is a valued field only if $k$ is perfect, as we saw above.

We say a valued field $K$ is nearly finite separable if it is a finite separable extension of $k^{1 / p^{i}}((t))$ for some integer $i$. (That is, any inseparability is concentrated in the constant field.) This definition allows us to approximate certain separability assertions for $k$ perfect in the case of general $k$, where some separable extensions of $k((t))$ fail to be valued fields. For example,

$$
\begin{aligned}
& k^{1 / p}((t))[x] /\left(x^{p}-x-c t^{-p}\right) \\
& \quad=k^{1 / p}((t))[x] /\left(\left(x-c^{1 / p} t^{-1}\right)^{p}-\left(x-c^{1 / p} t^{-1}\right)-c^{1 / p} t^{-1}\right)
\end{aligned}
$$

is a nearly separable valued field. In general, given any separable extension of $k((t))$, taking its compositum with $k^{1 / p^{i}}((t))$ for sufficiently large $i$ yields a nearly separable valued field.
2.3. The "classical" case $K=k((t))$. The definitions and results of Chapter 3 generalize previously known definitions and results in the key case $K=k((t))$. We treat this case first, both to allow readers familiar with the prior constructions to get used to the notation of this paper, and to provide a base on which to build additional rings in Chapter 3.

For $K=k((t))$, let $\Gamma^{K}$ be the ring of bidirectional power series $\sum_{i \in \mathbb{Z}} x_{i} u^{i}$, with $x_{i} \in \mathcal{O}$, such that $\left|x_{i}\right| \rightarrow 0$ as $i \rightarrow-\infty$. Note that $\Gamma^{K}$ is a discrete valuation ring complete under the $p$-adic topology, whose residue field is isomorphic to $K$ via the map $\sum x_{i} u^{i} \mapsto \sum \overline{x_{i}} t^{i}$ (where the bar denotes reduction modulo $\pi$ ). In particular, if $\pi=p$, then $\Gamma^{K}$ is a Cohen ring for $K$.

For $n$ in the value group of $\mathcal{O}$, we define the "naïve partial valuations"

$$
v_{n}^{\text {naive }}\left(\sum x_{i} u^{i}\right)=\min _{v_{p}\left(x_{i}\right) \leq n}\{i\},
$$

with the maximum to be $+\infty$ if no such $i$ exist. These partial valuations obey some basic rules:

$$
\begin{aligned}
v_{n}(x+y) & \geq \min \left\{v_{n}(x), v_{n}(y)\right\}, \\
v_{n}(x y) & \geq \min _{m}\left\{v_{m}(x)+v_{n-m}(y)\right\} .
\end{aligned}
$$

In both cases, equality always holds if the minimum is achieved exactly once.
Define the levelwise topology on $\Gamma^{K}$ by declaring the collection of sets

$$
\left\{x \in \Gamma^{K}: v_{n}^{\text {naive }}(x)>c\right\}
$$

for each $c \in \mathbb{Q}$ and each $n$ in the value group of $\mathcal{O}$, to be a neighborhood basis of 0 . The levelwise topology is coarser than the $\pi$-adic topology, and the Laurent polynomial ring $\mathcal{O}\left[u, u^{-1}\right]$ is dense in $\Gamma^{K}$ under the levelwise topology; thus any levelwise continuous endomorphism of $\Gamma^{K}$ is determined by the image of $u$.

The ring $\Gamma_{\text {con }}^{K}$ is the subring of $\Gamma^{K}$ consisting of those series $\sum_{i \in \mathbb{Z}} x_{i} u^{i}$ satisfying the more stringent convergence condition

$$
\liminf _{i \rightarrow-\infty} \frac{v_{p}\left(x_{i}\right)}{-i}>0 .
$$

It is also a discrete valuation ring with residue field $K$, but is not $\pi$-adically complete.

Using the naïve partial valuations, we can define actual valuations on certain subrings of $\Gamma_{\text {con }}^{K}$. For $r>0$, let $\Gamma_{r, \text { naive }}^{K}$ be the set of $x=\sum x_{i} u^{i}$ in $\Gamma_{\text {con }}^{K}$ such that $\lim _{n \rightarrow \infty} r v_{n}^{\text {naive }}(x)+n=\infty$; the union of the subrings over all $r$ is precisely $\Gamma_{\text {con }}^{K}$. (Warning: the rings $\Gamma_{r, \text { naive }}^{K}$ for individual $r$ are not discrete valuation rings, even though their union is.) On this subring, we have the function

$$
w_{r}^{\text {naive }}(x)=\min _{n}\left\{r v_{n}^{\text {naive }}(x)+n\right\}=\min _{i}\left\{r i+v_{p}\left(x_{i}\right)\right\}
$$

which can be seen to be a nonarchimedean valuation as follows. It is clear that $w_{r}^{\text {naive }}(x+y) \geq \min \left\{w_{r}^{\text {naive }}(x), w_{r}^{\text {naive }}(y)\right\}$ from the inequality $v_{n}(x+y) \geq$ $\min \left\{v_{n}(x), v_{n}(y)\right\}$. As for multiplication, it is equally clear that $w_{r}^{\text {naive }}(x y) \geq$ $w_{r}^{\text {naive }}(x)+w_{r}^{\text {naive }}(y)$; the subtle part is showing equality. Choose $m$ and $n$ minimal so that $w_{r}^{\text {naive }}(x)=r v_{m}^{\text {naive }}(x)+m$ and $w_{r}^{\text {naive }}(y)=r v_{n}^{\text {naive }}(y)+n$; then

$$
r v_{m+n}^{\text {naive }}(x y)+m+n \geq \min _{i}\left\{r v_{i}^{\text {naive }}(x)+i+r v_{m+n-i}^{\text {naive }}(y)+m+n-i\right\} .
$$

The minimum occurs only once, for $i=m$, and so equality holds, yielding $w_{r}^{\text {naive }}(x y)=w_{r}^{\text {naive }}(x)+w_{r}^{\text {naive }}(y)$.

Since $w_{r}^{\text {naive }}$ is a valuation, we have a corresponding norm $|\cdot|_{r}^{\text {naive }}$ given by $|x|_{r}^{\text {naive }}=p^{-w_{r}^{\text {naive }}(x)}$. This norm admits a geometric interpretation: the ring $\Gamma_{r, \text { naive }}^{K}\left[\frac{1}{p}\right]$ consists of power series which converge and are bounded for $p^{-r} \leq|u|<1$, where $u$ runs over all finite extensions of $\mathcal{O}\left[\frac{1}{p}\right]$. Then $|\cdot|_{r}^{\text {naive }}$ coincides with the supremum norm on the circle $|u|=p^{-r}$.

Recall that $\sigma_{0}: \mathcal{O} \rightarrow \mathcal{O}$ is a lift of the $p$-power Frobenius on $k$. We choose an extension of $\sigma_{0}$ to a levelwise continuous endomorphism of $\Gamma^{K}$ that maps $\Gamma_{\text {con }}^{K}$ into itself, and which lifts the $p$-power Frobenius on $k((t))$. In other words, choose $y \in \Gamma_{\text {con }}^{K}$ congruent to $u^{p}$ modulo $\pi$, and define $\sigma_{0}$ by

$$
\sum_{i} a_{i} u^{i} \mapsto \sum_{i} a_{i}^{\sigma_{0}} y^{i}
$$

Define $\sigma=\sigma_{0}^{f}$, where $f$ is again given by $q=p^{f}$.
Let $\Gamma_{\text {an,con }}^{K}$ be the ring of bidirectional power series $\sum_{i} x_{i} u^{i}$, now with $x_{i} \in \mathcal{O}\left[\frac{1}{p}\right]$, satisfying

$$
\liminf _{i \rightarrow-\infty} \frac{v_{p}\left(x_{i}\right)}{-i}>0, \quad \liminf _{i \rightarrow+\infty} \frac{v_{p}\left(x_{i}\right)}{i} \geq 0
$$

In other words, for any series $\sum_{i} x_{i} u^{i}$ in $\Gamma_{\mathrm{an}, \mathrm{con}}^{K}$, there exists $\eta>0$ such that the series converges for $\eta \leq|u|<1$. This ring is commonly known as the Robba ring. It contains $\Gamma_{\text {con }}^{K}\left[\frac{1}{p}\right]$, as the subring of functions which are analytic and bounded on some annulus $\eta \leq|u|<1$, but neither contains nor is contained in $\Gamma^{K}$.

We can view $\Gamma^{K}$ as the $\pi$-adic completion of $\Gamma_{\text {con }}^{K}$; our next goal is to identify $\Gamma_{\text {an,con }}^{K}$ with a certain completion of $\Gamma_{\text {con }}^{K}\left[\frac{1}{p}\right]$. Let $\Gamma_{\text {an }, r \text {, naive }}^{K}$ be the ring of series $x \in \Gamma_{\text {an,con }}^{K}$ such that $r v_{n}^{\text {naive }}(x)+n \rightarrow \infty$ as $n \rightarrow \pm \infty$. Then $\Gamma_{\text {an,con }}^{K}$ is visibly the union of the rings $\Gamma_{\mathrm{an}, r, \text { naive }}^{K}$ over all $r>0$. We equip $\Gamma_{\mathrm{an}, r, \text { naive }}^{K}$ with the Fréchet topology for the norms $|\cdot|_{s}^{\text {naive }}$ for $0<s \leq r$. These topologies are compatible with the embeddings $\Gamma_{\text {an }, r \text {, naive }}^{K} \hookrightarrow \Gamma_{\text {an }, s, \text { naive }}^{K}$ for $0<s<r$ (that is, the topology on $\Gamma_{\mathrm{an}, r, \text { naive }}^{K}$ coincides with the subspace topology for the embedding), and so by taking the direct limit we obtain a topology on $\Gamma_{\text {an,con }}^{K}$, which by abuse of language we will also call the Fréchet topology. (A better name might be the "limit-of-Fréchet topology".) Note that $\Gamma_{r, \text { naive }}^{K}\left[\frac{1}{p}\right]$ is dense in $\Gamma_{\mathrm{an}, r, \text { naive }}^{K}$ for each $r$, so that $\Gamma_{\mathrm{con}}^{K}\left[\frac{1}{p}\right]$ is dense in $\Gamma_{\mathrm{an}, \text { con }}^{K}$.

Proposition 2.2. The ring $\Gamma_{\mathrm{an}, r, \text { naive }}^{K}$ is complete (for the Fréchet topology).

Proof. Let $\left\{x_{i}\right\}$ be a Cauchy sequence for the Fréchet topology, consisting of elements of $\Gamma_{r \text {,naive }}^{K}\left[\frac{1}{p}\right]$. This means that for $0<s \leq r$ and any $c>0$, there exists $N$ such that $w_{s}^{\text {naive }}\left(x_{i}-x_{j}\right) \geq c$ for $i, j \geq N$. Write $x_{i}=\sum_{l} x_{i, l} u^{l}$; then for each $l,\left\{x_{i, l}\right\}$ forms a Cauchy sequence. More precisely, for $i, j \geq N$,

$$
s l+v_{p}\left(x_{i, l}-x_{j, l}\right) \geq c .
$$

Since $\mathcal{O}$ is complete, we can take the limit $y_{l}$ of $\left\{x_{i, l}\right\}$, and it will satisfy $s l+v_{p}\left(x_{i, l}-y_{l}\right) \geq c$ for $i \geq N$. Thus if we can show $y=\sum_{l} y_{l} u^{l} \in \Gamma_{\text {an }, r, \text { naive }}^{K}$, then $\left\{x_{i}\right\}$ will converge to $y$ under $|\cdot|_{s}^{\text {naive }}$ for each $s$.

Choose $s \leq r$ and $c>0$; we must show that $s l+v_{p}\left(y_{l}\right) \geq c$ for all but finitely many $l$. There exists $N$ such that $s l+v_{p}\left(x_{i, l}-y_{l}\right) \geq c$ for $i \geq N$. Choose a single such $i$; then

$$
\begin{aligned}
s l+v_{p}\left(y_{l}\right) & \geq \min \left\{s l+v_{p}\left(x_{i, l}-y_{l}\right), s l+v_{p}\left(x_{i, l}\right)\right\} \\
& \geq \min \left\{c, s l+v_{p}\left(x_{i, l}\right)\right\} .
\end{aligned}
$$

Since $x_{i} \in \Gamma_{r, \text { naive }}^{K}\left[\frac{1}{p}\right], s l+v_{p}\left(x_{i, l}\right) \geq c$ for all but finitely many $l$. For such $l$, we have $s l+v_{p}\left(y_{l}\right) \geq c$, as desired. Thus $y \in \Gamma_{\text {an }, r \text {, naive }}^{K}$; as noted earlier, $y$ is the limit of $\left\{x_{i}\right\}$ under each $|\cdot|_{s}^{\text {naive }}$, and so is the Fréchet limit.

We conclude that each Cauchy sequence with elements in $\Gamma_{r, \text { naive }}^{K}\left[\frac{1}{p}\right]$ has a limit in $\Gamma_{\text {an }, r, \text { naive }}^{K}$. Since $\Gamma_{r, \text { naive }}^{K}\left[\frac{1}{p}\right]$ is dense in $\Gamma_{\text {an }, r, \text { naive }}^{K}$ (one sequence converging to $\sum_{i} x_{i} u^{i}$ is simply $\left.\left\{\sum_{i \leq j} x_{i} u^{i}\right\}_{j=0}^{\infty}\right), \Gamma_{\mathrm{an}, r, \text { naive }}^{K}$ is complete for the Fréchet topology, as desired.

Unlike $\Gamma^{K}$ and $\Gamma_{\text {con }}^{K}, \Gamma_{\text {an,con }}^{K}$ is not a discrete valuation ring. For one thing, $\pi$ is invertible in $\Gamma_{\mathrm{an}, \mathrm{con}}^{K}$. For another, there are plenty of noninvertible elements of $\Gamma_{\mathrm{an}, \text { con }}^{K}$, such as

$$
\prod_{i=1}^{\infty}\left(1-\frac{u^{p^{i}}}{p^{i}}\right)
$$

For a third, $\Gamma_{\text {an,con }}^{K}$ is not Noetherian; the ideal $\left(x_{1}, x_{2}, \ldots\right)$, where

$$
x_{j}=\prod_{i=j}^{\infty}\left(1-\frac{u^{p^{i}}}{p^{i}}\right)
$$

is not finitely generated. However, as long as we restrict to "finite" objects, $\Gamma_{\text {an,con }}^{K}$ behaves well: a theorem of Lazard [L] (see also [Cr2, Prop. 4.6] and our own Section 3.6) states that $\Gamma_{\mathrm{an}, \text { con }}^{K}$ is a Bézout ring, which is to say every finitely generated ideal is principal.

For $L$ a finite extension of $k((t))$, we have $L \cong k^{\prime}\left(\left(t^{\prime}\right)\right)$ for some finite extension $k^{\prime}$ of $k$ and some uniformizer $t^{\prime}$, and so one could define $\Gamma^{L}, \Gamma_{\text {con }}^{L}$, $\Gamma_{\text {an,con }}^{L}$ abstractly as above. However, a better strategy will be to construct these in a "relative" fashion; the results will be the same as the abstract rings, but the relative construction will give us more functoriality, and will allow us to define $\Gamma^{L}, \Gamma_{\text {con }}^{L}, \Gamma_{\text {an, con }}^{L}$ even when $L$ is an infinite algebraic extension of $k((t))$. We return to this approach in Chapter 3.

The rings defined above occur in numerous other contexts, and so it is perhaps not surprising that there are several sets of notation for them in the literature. One common set is

$$
\mathcal{E}=\Gamma^{k((t))}\left[\frac{1}{p}\right], \quad \mathcal{E}^{\dagger}=\Gamma_{\text {con }}^{k((t)))}\left[\frac{1}{p}\right], \quad \mathcal{R}=\Gamma_{\text {an, } \mathrm{con}}^{k((t))} .
$$

The peculiar-looking notation we have set up will make it easier to deal systematically with a number of additional rings to be defined in Chapter 3.
2.4. More on Bézout rings. Since $\Gamma_{\text {an,con }}^{K}$ is a Bézout ring, as are trivially all discrete valuation rings, it will be useful to record some consequences of the Bézout property.

Lemma 2.3. Let $R$ be a Bézout ring. If $x_{1}, \ldots, x_{n} \in R$ generate the unit ideal, then there exists a matrix $A$ over $R$ with determinant 1 such that $A_{1 i}=x_{i}$ for $i=1, \ldots, n$.

Proof. We prove this by induction on $n$, the case $n=1$ being evident. Let $d$ be a generator of $\left(x_{1}, \ldots, x_{n-1}\right)$. By the induction hypothesis, we can find an $(n-1) \times(n-1)$ matrix $B$ of determinant 1 such that $B_{1 i}=x_{i} / d$ for $i=1, \ldots, n-1$; extend $B$ to an $n \times n$ matrix by setting $B_{n n}=1$ and
$B_{\text {in }}=B_{n i}=0$ for $i=1, \ldots, n-1$. Since $\left(d, x_{n}\right)=\left(x_{1}, \ldots, x_{n}\right)$ is the unit ideal, we can find $e, f \in R$ such that $d e-f x_{n}=1$. Define the matrix

$$
C=\left(\begin{array}{ccccc}
d & 0 & \cdots & 0 & x_{n} \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & & \ddots & & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
f & 0 & \cdots & 0 & e
\end{array}\right) ; \quad \text { that is, } \quad C_{i j}= \begin{cases}d & i=j=1 \\
1 & 2 \leq i=j \leq n-1 \\
e & i=j=n \\
x_{n} & i=1, j=n \\
f & i=n, j=1 \\
0 & \text { otherwise } .\end{cases}
$$

Then we may take $A=C B$.
Given a finite free module $M$ over a domain $R$, we may regard $M$ as a subset of $M \otimes_{R} \operatorname{Frac}(R)$; given a subset $S$ of $M$, we define the saturated span $\operatorname{SatSpan}(S)$ of $S$ as the intersection of $M$ with the $\operatorname{Frac}(R)$-span of $S$ within $M \otimes_{R} \operatorname{Frac}(R)$. Note that the following lemma does not require any finiteness condition on $S$.

Lemma 2.4. Let $M$ be a finite free module over a Bézout domain $R$. Then for any subset $S$ of $M, \operatorname{SatSpan}(S)$ is free and admits a basis that extends to a basis of $M$; in particular, $\operatorname{SatSpan}(S)$ is a direct summand of $M$.

Proof. We proceed by induction on the rank of $M$, the case of rank 0 being trivial. Choose a basis $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ of $M$. If $S \subseteq\{0\}$, there is nothing to prove; otherwise, choose $\mathbf{v} \in S \backslash\{0\}$ and write $\mathbf{v}=\sum_{i} c_{i} \mathbf{e}_{i}$. Since $R$ is a Bézout ring, we can choose a generator $r$ of the ideal $\left(c_{1}, \ldots, c_{n}\right)$. Put $\mathbf{w}=\sum_{i}\left(c_{i} / r\right) \mathbf{e}_{i}$; then $\mathbf{w} \in \operatorname{SatSpan}(S)$ since $r \mathbf{w}=\mathbf{v}$. By Lemma 2.3, there exists an invertible matrix $A$ over $R$ with $A_{1 i}=c_{i} / r$. Put $\mathbf{y}_{j}=\sum_{i} A_{j i} \mathbf{e}_{i}$ for $j=2, \ldots, n$; then $\mathbf{w}$ and the $\mathbf{y}_{j}$ form a basis of $M$ (because $A$ is invertible), so that $M / \operatorname{SatSpan}(\mathbf{w})$ is free. Thus the induction hypothesis applies to $M / \operatorname{SatSpan}(\mathbf{w})$, where the saturated span of the image of $S$ admits a basis $\mathbf{x}_{1}, \ldots, \mathbf{x}_{r}$. Together with $\mathbf{w}$, any lifts of $\mathbf{x}_{1}, \ldots, \mathbf{x}_{r}$ to $M$ form a basis of $\operatorname{SatSpan}(S)$ that extends to a basis of $M$, as desired.

Note that the previous lemma immediately implies that every finite torsionfree module over $R$ is free. (If $M$ is torsion-free and $\phi: F \rightarrow M$ is a surjection from a free module $F$, then $\operatorname{ker}(\phi)$ is saturated, so that $M \cong F / \operatorname{ker}(\phi)$ is free.) A similar argument yields the following vitally important fact.

Proposition 2.5. Let $R$ be a Bézout domain. Then every finite locally free module over $R$ is free.

Proof. Let $M$ be a finite locally free module over $R$. Since $\operatorname{Spec} R$ is connected, the localizations of $M$ all have the same rank $r$. Choose a surjection
$\phi: F \rightarrow M$, where $F$ is a finite free $R$-module, and let $N=\operatorname{SatSpan}(\operatorname{ker}(\phi))$. Then we have a surjection $M \cong F / \operatorname{ker}(\phi) \rightarrow F / N$, and $F / N$ is free. Tensoring $\phi$ with $\operatorname{Frac}(R)$, we obtain a surjection $F \otimes_{R} \operatorname{Frac}(R) \rightarrow M \otimes_{R} \operatorname{Frac}(R)$ of vector spaces of dimensions $n$ and $r$. Thus the kernel of this map has rank $n-r$, which implies that $N$ has rank $n-r$ and $F / N$ is free of rank $r$.

Now localizing at each prime $\mathfrak{p}$ of $R$, we obtain a surjection $M_{\mathfrak{p}} \rightarrow(F / N)_{\mathfrak{p}}$ of free modules of the same rank. By a standard result, this map is in fact a bijection. Thus $M \rightarrow F / N$ is locally bijective, hence is bijective, and $M$ is free as desired.

The following lemma is a weak form of Galois descent for Bézout rings; its key value is that it does not require that the ring extension be finite.

Lemma 2.6. Let $R_{1} / R_{2}$ be an extension of Bézout domains and $G$ a group of automorphisms of $R_{1}$ over $R_{2}$, with fixed ring $R_{2}$. Assume that every $G$-stable, finitely generated ideal of $R_{1}$ contains a nonzero element of $R_{2}$. Let $M_{2}$ be a finite free module over $R_{2}$ and $N_{1}$ a saturated $G$-stable submodule of $M_{1}=M_{2} \otimes_{R_{2}} R_{1}$ stable under $G$. Then $N_{1}$ is equal to $N_{2} \otimes_{R_{2}} R_{1}$ for a saturated submodule (necessarily unique) $N_{2}$ of $M_{2}$.

Proof. We induct on $n=\operatorname{rank} M_{2}$, the case $n=0$ being trivial. Let $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ be a basis of $M_{2}$, and let $P_{1}$ be the intersection of $N_{1}$ with the span of $\mathbf{e}_{2}, \ldots, \mathbf{e}_{n}$; since $N_{1}$ is saturated, $P_{1}$ is a direct summand of $\operatorname{SatSpan}\left(\mathbf{e}_{2}, \ldots, \mathbf{e}_{n}\right)$ by Lemma 2.4 and hence also of $M_{1}$. By the induction hypothesis, $P_{1}=$ $P_{2} \otimes_{R_{2}} R_{1}$ for a saturated submodule $P_{2}$ of $M_{2}$ (necessarily a direct summand by Lemma 2.4). If $N_{1}=P_{1}$, we are done. Otherwise, $N_{1} / P_{1}$ is a $G$-stable, finitely generated ideal of $R_{1}$ (since $N_{1}$ can be identified with finitely generated by Lemma 2.4), and so contains a nonzero element $c$ of $R_{2}$. Pick $\mathbf{v} \in N_{1}$ reducing to $c$; that is, $\mathbf{v}-c \mathbf{e}_{1} \in \operatorname{SatSpan}\left(\mathbf{e}_{2}, \ldots, \mathbf{e}_{n}\right)$.

Pick generators $\mathbf{w}_{1}, \ldots, \mathbf{w}_{m}$ of $P_{2}$; since $P_{2}$ is a direct summand of $\operatorname{SatSpan}\left(\mathbf{e}_{2}, \ldots, \mathbf{e}_{n}\right)$, we can choose $\mathbf{x}_{1}, \ldots, \mathbf{x}_{n-m-1}$ in $M_{2}$ so that $\mathbf{e}_{1}, \mathbf{w}_{1}, \ldots$, $\mathbf{w}_{m}, \mathbf{x}_{1}, \ldots, \mathbf{x}_{n-m-1}$ is a basis of $M_{2}$. In this basis, we may write $\mathbf{v}=c \mathbf{e}_{1}+$ $\sum_{i} d_{i} \mathbf{w}_{i}+\sum_{i} f_{i} \mathbf{x}_{i}$, where $c$ is the element of $R_{2}$ chosen above. Put $\mathbf{y}=$ $\mathbf{v}-\sum_{i} d_{i} \mathbf{w}_{i}$. For any $\tau \in G$, we have $\mathbf{y}^{\tau}=c \mathbf{e}_{1}+\sum_{i} f_{i}^{\tau} \mathbf{x}_{i}$, and so on one hand, $\mathbf{y}^{\tau}-\mathbf{y}$ is a linear combination of $\mathbf{x}_{1}, \ldots, \mathbf{x}_{n-m-1}$. On the other hand, $\mathbf{y}^{\tau}-\mathbf{y}$ belongs to $N_{1}$ and so is a linear combination of $\mathbf{w}_{1}, \ldots, \mathbf{w}_{m}$. This forces $\mathbf{y}^{\tau}-\mathbf{y}=0$ for all $\tau \in G$; since $G$ has fixed ring $R_{2}$, we conclude $\mathbf{y}$ is defined over $R_{2}$. Thus we may take $N_{2}=\operatorname{Sat} \operatorname{Span}\left(\mathbf{y}, \mathbf{w}_{1}, \ldots, \mathbf{w}_{m}\right)$.

Note that the hypothesis that every $G$-stable finitely generated ideal of $R_{1}$ contains a nonzero element of $R_{2}$ is always satisfied if $G$ is finite: for any nonzero $r$ in the ideal, $\prod_{\tau \in G} r^{\tau}$ is nonzero and $G$-stable, and so belongs to $R_{2}$.
2.5. $\sigma$-modules and $(\sigma, \nabla)$-modules. The basic object in the local study of $p$-adic differential equations is a module with connection and Frobenius structure. In our approach, we separate these two structures and study the Frobenius structure closely before linking it with the connection. To this end, in this section we introduce $\sigma$-modules and $(\sigma, \nabla)$-modules, and outline some basic facts of what might be dubbed "semilinear algebra". These foundations, in part, date back to Katz [Ka] and were expanded by de Jong [dJ].

For $R$ an integral domain in which $p \neq 0$, and $\sigma$ a ring endomorphism of $R$, we define a $\sigma$-module over $R$ to be a finite locally free $R$-module $M$ equipped with an $R$-linear map $F: M \otimes_{R, \sigma} R \rightarrow M$ that becomes an isomorphism over $R\left[\frac{1}{p}\right]$; the tensor product notation indicates that $R$ is viewed as an $R$-module via $\sigma$. Note that we will only use this definition when $R$ is a Bézout ring, in which case every finite locally free $R$-module is actually free by Proposition 2.5. Then to specify $F$, it is equivalent to specify an additive, $\sigma$-linear map from $M$ to $M$ that acts on any basis of $M$ by a matrix invertible over $R\left[\frac{1}{p}\right]$. We abuse notation and refer to this map as $F$ as well; since we will only use the $\sigma$-linear map in what follows (with one exception: in proving Proposition 6.11), there should not be any confusion induced by this.

Now suppose $R$ is one of $\Gamma^{K}, \Gamma^{K}\left[\frac{1}{p}\right], \Gamma_{\text {con }}^{K}, \Gamma_{\text {con }}^{K}\left[\frac{1}{p}\right]$ or $\Gamma_{\text {an, con }}^{K}$ for $K=k((t))$. Let $\Omega_{R}^{1}$ be the free module over $R$ generated by a single symbol $d u$, and let $d: R \rightarrow \Omega_{R}^{1}$ be the $\mathcal{O}$-linear derivation given by the formula

$$
d\left(\sum_{i} x_{i} u^{i}\right)=\sum_{i} i x_{i} u^{i-1} d u
$$

We define a $(\sigma, \nabla)$-module over $R$ to be a $\sigma$-module $M$ plus a connection $\nabla: M \rightarrow M \otimes_{R} \Omega_{R}^{1}$ (i.e., an additive map satisfying the Leibniz rule $\nabla(c \mathbf{v})=$ $c \nabla(\mathbf{v})+\mathbf{v} \otimes d c$ for $c \in R$ and $\mathbf{v} \in M)$ that makes the following diagram commute:


Warning: this definition is not the correct one in general. For larger rings $R$, one must include the condition that $\nabla$ is integrable. That is, writing $\nabla_{1}$ for the induced map $M \otimes_{R} \Omega_{R}^{1} \rightarrow M \otimes_{R} \wedge^{2} \Omega_{R}^{1}$, we must have $\nabla_{1} \circ \nabla=0$. This condition is superfluous in our context because $\Omega_{R}^{1}$ has rank one, so $\nabla_{1}$ is automatically zero.

A morphism of $\sigma$-modules or $(\sigma, \nabla)$-modules is a homomorphism of the underlying $R$-modules compatible with the additional structure in the obvious fashion. An isomorphism of $\sigma$-modules or $(\sigma, \nabla)$-modules is a morphism
admitting an inverse; an isogeny is a morphism that becomes an isomorphism over $R\left[\frac{1}{p}\right]$.

Direct sums, tensor products, exterior powers, and subobjects are defined in the obvious fashion, as are duals if $p^{-1} \in R$; quotients also make sense provided that the quotient $R$-module is locally free. In particular, if $M_{1} \subseteq M_{2}$ is an inclusion of $\sigma$-modules, the saturation of $M_{1}$ in $M_{2}$ is also a $\sigma$-submodule of $M_{1}$; if $M_{1}$ itself is saturated, the quotient $M_{2} / M_{1}$ is locally free and hence is a $\sigma$-module.

Given $\lambda$ fixed by $\sigma$, we define the twist of a $\sigma$-module $M$ by $\lambda$ as the $\sigma$-module with the same underlying module but whose Frobenius has been multiplied by $\lambda$.

We say a $\sigma$-module $M$ is standard if it is isogenous to a $\sigma$-module with a basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ such that $F \mathbf{v}_{i}=\mathbf{v}_{i+1}$ for $i=1, \ldots, n-1$ and $F \mathbf{v}_{n}=\lambda \mathbf{v}_{1}$ for some $\lambda \in R$ fixed by $\sigma$. (The restriction that $\lambda$ is fixed by $\sigma$ is included for convenience only.) If $M$ is actually a $(\sigma, \nabla)$-module, we say $M$ is standard as a $(\sigma, \nabla)$-module if the same condition holds with the additional restriction that $\nabla \mathbf{v}_{i}=0$ for $i=1, \ldots, n$ (i.e., the $\mathbf{v}_{i}$ are "horizontal sections" for the connection). If $\mathbf{v}$ is a nonzero element of $M$ such that $F \mathbf{v}=\lambda \mathbf{v}$ for some $\lambda$, we say $\mathbf{v}$ is an eigenvector of $M$ of eigenvalue $\lambda$ and slope $v_{p}(\lambda)$.

Warning: elsewhere in the literature, the slope may be normalized differently, namely as $v_{p}(\lambda) / v_{p}(q)$. (Recall that $q=p^{f}$.) Since we will hold $q$ fixed, this normalization will not affect our results.

From Lemma 2.6, we have the following descent lemma for $\sigma$-modules. (The condition on $G$-stable ideals is satisfied because $R_{1} / R_{2}$ is an unramified extension of discrete valuation rings.)

Corollary 2.7. Let $R_{1} / R_{2}$ be an unramified extension of discrete valuation rings, and let $\sigma$ be a ring endomorphism of $R_{1}$ carrying $R_{2}$ into itself. Let $\operatorname{Gal}^{\sigma}\left(R_{1} / R_{2}\right)$ be the group of automorphisms of $R_{1}$ over $R_{2}$ commuting with $\sigma$; assume that this group has fixed ring $R_{2}$. Let $M_{2}$ be a $\sigma$-module over $R_{2}$ and $N_{1}$ a saturated $\sigma$-submodule of $M_{1}=M_{2} \otimes_{R_{2}} R_{1}$ stable under $\operatorname{Gal}^{\sigma}\left(R_{1} / R_{2}\right)$. Then $N_{1}=N_{2} \otimes_{R_{2}} R_{1}$ for some $\sigma$-submodule $N_{2}$ of $M_{2}$.

## 3. A few more rings

In this chapter, we define a number of additional auxiliary rings used in our study of $\sigma$-modules. Again, we advise the reader to skim this chapter on first reading and return to it as needed.
3.1. Cohen rings. We proceed to generalizing the constructions of Section 2.3 to valued fields. This cannot be accomplished using Witt vectors because $k((t))$ and its finite extensions are not perfect. To get around this, we
fix once and for all a levelwise continuous Frobenius lift $\sigma_{0}$ on $\Gamma^{k((t))}$ carrying $\Gamma_{\text {con }}^{k((t))}$ into itself; all of our constructions will be made relative to the choice of $\sigma_{0}$.

Recall that a valued field $K$ is defined to be an algebraic extension of $k((t))$ admitting subextensions $k((t)) \subseteq L \subseteq M \subseteq N \subseteq K$ such that:
(a) $L=k^{1 / p^{m}}((t))$ for some $m \in\{0,1, \ldots, \infty\}$;
(b) $M=k_{M}((t))$ for some separable algebraic extension $k_{M} / k^{1 / p^{m}}$;
(c) $N=M^{1 / p^{n}}$ for some $n \in\{0,1, \ldots, \infty\}$;
(d) $K$ is a separable totally ramified algebraic extension of $N$.

We will associate to each valued field $K$ a complete discrete valuation ring $\Gamma^{K}$ unramified over $\mathcal{O}$, equipped with a Frobenius lift $\sigma_{0}$ extending the definition of $\sigma_{0}$ on $\Gamma^{k((t))}$. This assignment will be functorial in $K$.

Let $\mathcal{C}$ be the category of complete discrete valuation rings unramified over $\mathcal{O}$, in which morphisms are unramified morphisms of rings (i.e., morphisms which induce isomorphisms of the value groups). If $R_{0}, R_{1} \in \mathcal{C}$ have residue fields $k_{0}, k_{1}$ and a homomorphism $\phi: k_{0} \rightarrow k_{1}$ is given, we say the morphism $f: R_{0} \rightarrow R_{1}$ is compatible (with $\phi$ ) if the diagram

commutes.
Lemma 3.1. Let $k_{1} / k_{0}$ be a finite separable extension of fields, and take $R_{0} \in \mathcal{C}$ with residue field $k_{0}$. Then there exists $R_{1} \in \mathcal{C}$ with residue field $k_{1}$ and a compatible morphism $R_{0} \rightarrow R_{1}$.

Proof. By the primitive element theorem, there exists a monic separable polynomial $\bar{P}(x)$ over $k_{0}$ and an isomorphism $k_{1} \cong k_{0}[x] /(\bar{P}(x))$. Choose a monic polynomial $P(x)$ over $R_{0}$ lifting $\bar{P}(x)$ and set $R_{1}=R_{0}[x] /(P(x))$. Then the inclusion $R_{0} \rightarrow R_{0}[x]$ induces the desired morphism $R_{0} \rightarrow R_{1}$.

Lemma 3.2. Let $k_{0} \rightarrow k_{1} \rightarrow k_{2}$ be homomorphisms of fields, with $k_{1} / k_{0}$ finite separable. For $i=0,1,2$, take $R_{i} \in \mathcal{C}$ with residue field $k_{i}$. Let $f: R_{0} \rightarrow$ $R_{1}$ and $g: R_{0} \rightarrow R_{2}$ be compatible morphisms. Then there exists a unique compatible morphism $h: R_{1} \rightarrow R_{2}$ such that $g=h \circ f$.

Proof. As in the previous proof, choose a monic separable polynomial $\bar{P}(x)$ over $k_{0}$ and an isomorphism $k_{1} \cong k_{0}[x] /(\bar{P}(x))$. Let $y$ be the image of $x+(\bar{P}(x))$ in $k_{1}$, and let $z$ be the image of $y$ in $k_{2}$.

Choose a monic polynomial $P(x)$ over $R_{0}$ lifting $\bar{P}(x)$, and view $R_{0}$ as a subring of $R_{1}$ and $R_{2}$ via $f$ and $g$, respectively. By Hensel's lemma, there exist unique roots $\alpha$ and $\beta$ of $P(x)$ in $R_{1}$ and $R_{2}$ reducing to $y$ and $z$, respectively, so that $h$ must satisfy $h(\alpha)=\beta$ if it exists. Then $R_{0}[x] /(P(x)) \cong R_{1}$ by the map sending $x+(P(x))$ to $\alpha$ and $R_{0}[x] /(P(x)) \hookrightarrow R_{2}$ by the map sending $x+(P(x))$ to $\beta$; so there exists a unique $h: R_{1} \rightarrow R_{2}$ such that $h(\alpha)=\beta$, and this gives the desired morphism.

Corollary 3.3. If $k_{1} / k_{0}$ is finite Galois, and $R_{i} \in \mathcal{C}$ has residue field $k_{i}$ for $i=0,1$, then for any compatible morphism $f: R_{0} \rightarrow R_{1}$, the group of $f$-equivariant automorphisms of $R_{1}$ is isomorphic to $\operatorname{Gal}\left(k_{1} / k_{0}\right)$.

Proof. Apply Lemma 3.2 with $k_{0} \rightarrow k_{1}$ the given embedding and $k_{1} \rightarrow k_{1}$ an element of $\operatorname{Gal}\left(k_{1} / k_{0}\right)$; the resulting $h$ is the corresponding automorphism.

Corollary 3.4. If $k_{1} / k_{0}$ is finite separable, $\phi$ is an endomorphism of $k_{1}$ mapping $k_{0}$ into itself, $R_{i} \in \mathcal{C}$ has residue field $k_{i}$ for $i=0,1$, and $f: R_{0} \rightarrow R_{1}$ is a compatible morphism, then any compatible endomorphism of $R_{0}$ (for $\phi$ ) admits a unique $f$-equivariant extension to $R_{1}$.

Proof. If $e: R_{0} \rightarrow R_{0}$ is the given endomorphism, apply Lemma 3.2 with $g=f \circ e$.

For $m$ a nonnegative integer, let $\mathcal{O}_{m}$ be a copy of $\mathcal{O}$. Then the assignment $k^{1 / p^{m}} \leadsto \mathcal{O}_{m}$ is functorial via the morphism $\sigma_{0}^{i}$ compatible with $k^{1 / p^{m}} \rightarrow$ $k^{1 / p^{m+i}}$; thus we can define $\mathcal{O}_{\infty}$ as the completed direct limit of the $\mathcal{O}_{m}$. For any finite separable extension $k_{M}$ of $k^{1 / p^{m}}$, choose $\mathcal{O}_{M}$ in $\mathcal{C}$ according to Lemma 3.1, to obtain a compatible morphism $\mathcal{O}_{m} \rightarrow \mathcal{O}_{M}$; note that $\mathcal{O}_{M}$ is unique up to canonical isomorphism by Lemma 3.2. Moreover, this assignment is functorial in $k_{M}$ (again by Lemma 3.2 ); so again we may pass to infinite extensions by taking the completed direct limit.

Now suppose $K$ is a nearly finite valued field, and that $L, m, M, k_{M}, N, n$ are as in the definition of valued fields; note that these are all uniquely determined by $K$. Define $\mathcal{O}_{M}$ associated to $k_{M}$ as above, define $\Gamma^{M}$ as the ring of power series $\sum_{i \in \mathbb{Z}} a_{i} u^{i}$, with $a_{i} \in \mathcal{O}_{M}$, such that $\left|a_{i}\right| \rightarrow 0$ as $i \rightarrow-\infty$, and identify $\Gamma^{M} / \pi \Gamma^{M}$ with $M=k_{M}((t))$ via the map $\sum_{i} a_{i} u^{i} \mapsto \sum_{i} \overline{a_{i}} t^{i}$. Define $\Gamma^{N}$ as a copy of $\Gamma^{M}$, but with $\Gamma^{M}$ embedded via $\sigma_{0}^{n}$ (which makes sense since $n<\infty$ ), and identify the residue field of $\Gamma^{N}$ with $N$ compatibly. Define $\Gamma^{K}$ as a copy of $\Gamma^{N}$ with its residue field identified with $K$ via some continuous $k_{M}^{1 / p^{n}}$. algebra isomorphism $K \cong N$ (which exists because both fields are power series fields over $k_{M}^{1 / p^{n}}$ by the Cohen structure theorem). Once this choice is made, there exists a levelwise continuous $\mathcal{O}$-algebra morphism $\Gamma^{N} \rightarrow \Gamma^{K}$ compatible
with the embedding $N \hookrightarrow K$. The assignments of $\Gamma^{M}, \Gamma^{N}, \Gamma^{K}$ are functorial, again by Lemma 3.2, so again we may extend the definition to infinite $K$ by completion.

Note that if $K / k((t))$ is nearly finite, then $\Gamma^{K}$ is equipped with a levelwise topology, and the embeddings provided by functoriality are levelwise continuous. Moreover, $\sigma_{0}$ extends uniquely to each $\Gamma^{K}$ by Corollary 3.4, and the functorial morphisms are $\sigma_{0}$-equivariant.

If $k$ and $K$ are perfect and $\mathcal{O}=C(k)=W(k)$, then $\Gamma^{K}$ is canonically isomorphic to the Witt ring $W(K)$. Under that isomorphism, $\sigma_{0}$ corresponds to the Witt vector Frobenius, which sends each Teichmüller lift to its $p$-th power. For general $\mathcal{O}$, we have $\Gamma^{K} \cong W(K) \otimes_{W(k)} \mathcal{O}$.

We will often fix a field $K$ (typically $k((t))$ itself) and write $\Gamma$ instead of $\Gamma^{K}$. In this case, we will frequently refer to $\Gamma^{L}$ for various canonical extensions $L$ of $K$, such as the separable closure $K^{\text {sep }}$, the perfect closure $K^{\text {perf }}$, and the algebraic closure $K^{\text {alg }}$. In all of these cases, we will drop the $K$ from the notation where it is understood, writing $\Gamma^{\text {perf }}$ for $\Gamma^{K^{\text {perf }}}$ and so forth.
3.2. Overconvergent rings. Let $K$ be a valued field. Let $v_{K}$ denote the valuation on $K$ extending the valuation on $k((t))$, normalized so that $v_{K}(t)=1$. Again, let $q=p^{f}$, and put $\sigma=\sigma_{0}^{f}$ on $\Gamma^{K}$. We define a subring $\Gamma_{\text {con }}^{K}$ of $\Gamma^{K}$ of "overconvergent" elements; the construction will not look quite like the construction of $\Gamma_{\text {con }}^{k((t))}$ from Section 2.3, so we must check that the two are consistent.

First assume $K$ is perfect. For $x \in \Gamma^{K}\left[\frac{1}{p}\right]$, write $x=\sum_{i=m}^{\infty} \pi^{i}\left[\overline{x_{i}}\right]$, where $m v_{p}(\pi)=v_{p}(x)$, each $\overline{x_{i}}$ belongs to $K$ and the brackets denote Teichmüller lifts. For $n$ in the value group of $\mathcal{O}$, we define the "partial valuations"

$$
v_{n}(x)=\min _{v_{p}\left(\pi^{i}\right) \leq n}\left\{v_{K}\left(\overline{x_{i}}\right)\right\} .
$$

These partial valuations obey two rules analogous to those for their naïve counterparts, plus a third that has no analogue:

$$
\begin{aligned}
v_{n}(x+y) & \geq \min \left\{v_{n}(x), v_{n}(y)\right\}, \\
v_{n}(x y) & \geq \min _{m}\left\{v_{m}(x)+v_{n-m}(y)\right\}, \\
v_{n}\left(x^{\sigma}\right) & =q v_{n}(x) .
\end{aligned}
$$

Again, equality holds in the first two lines if the minimum is achieved exactly once.

For each $r>0$, let $\Gamma_{r}^{K}$ denote the subring of $x \in \Gamma^{K}$ such that $\lim _{n \rightarrow \infty}\left(r v_{n}(x)+n\right)=\infty$. On $\Gamma_{r}^{K}\left[\frac{1}{p}\right] \backslash\{0\}$, we define the function

$$
w_{r}(x)=\min _{n}\left\{r v_{n}(x)+n\right\} ;
$$

then $w_{r}$ is a nonarchimedean valuation by the same argument as for $w_{r}^{\text {naive }}$ given in Section 2.3. Define $\Gamma_{\text {con }}^{K}=\cup_{r>0} \Gamma_{r}^{K}$.

The rings $\Gamma_{r}^{K}$ will be quite useful, but one must handle them with some caution, for the following reasons:
(a) The map $\sigma: \Gamma^{K} \rightarrow \Gamma^{K}$ sends $\Gamma_{\text {con }}^{K}$ into itself, but does not send $\Gamma_{r}^{K}$ into itself; rather, it sends $\Gamma_{r}^{K}$ into $\Gamma_{r / q}^{K}$.
(b) The ring $\Gamma_{\text {con }}^{K}$ is a discrete valuation ring, but the rings $\Gamma_{r}^{K}$ are not.
(c) The ring $\Gamma_{r}^{K}$ is complete for $w_{r}$, but not for the $p$-adic valuation.

For $K$ arbitrary, we want to define $\Gamma_{\text {con }}^{K}$ as $\Gamma_{\text {con }}^{\text {alg }} \cap \Gamma^{K}$. This intersection is indeed a discrete valuation ring (so again its fraction field is obtained by adjoining $\frac{1}{p}$ ), but it is not clear that its residue field is all of $K$. Indeed, it is a priori possible that the intersection is no larger than $\mathcal{O}$ itself! In fact, this pathology does not occur, as we will see below.

To make that definition, we must also check that $\Gamma_{\mathrm{con}}^{\mathrm{alg}} \cap \Gamma^{k((t))}$ coincides with the ring $\Gamma_{\text {con }}^{k((t))}$ defined earlier. This is obvious in a special case: if $\sigma_{0}(u)=u^{p}$, then $u$ is a Teichmüller lift in $\Gamma_{\mathrm{con}}^{\mathrm{alg}}$, and in this case one can check that the partial valuations and naïve partial valuations coincide. In general they do not coincide, but in a sense they are not too far apart. The relationship might be likened to that between the naïve and canonical heights on an abelian variety over a number field.

Put $z=u^{\sigma} / u^{q}-1$. By the original definition of $\sigma$ on $\Gamma^{k((t))}, v_{p}(z)>0$ and $z \in \Gamma_{\text {con }}^{k((t))}$. That means we can find $r>0$ such that $q^{-1} r v_{n}^{\text {naive }}(z)+n>0$ for all $n$; for all $s \leq q^{-1} r$, we then have $w_{s}^{\text {naive }}\left(u^{\sigma} / u^{q}\right)=0$.

Lemma 3.5. Choose $r>0$ such that $q^{-1} r v_{n}^{\text {naive }}(z)+n>0$ for all $n$. For $x=\sum_{i} x_{i} u^{i}$ in $\Gamma_{r, \text { naive }}^{k(t))}$, if $0<s \leq q r$ and $w_{s}^{\text {naive }}(x) \geq c$, then $w_{s / q}^{\text {naive }}\left(x^{\sigma}\right) \geq c$.

Proof. We have

$$
\begin{aligned}
w_{s / q}^{\text {naive }}\left(x_{i}^{\sigma}\left(u^{i}\right)^{\sigma}\right) & =w_{s / q}^{\text {naive }}\left(x_{i} u^{q i}\left(u^{\sigma} / u^{q}\right)^{i}\right) \\
& =w_{s / q}^{\text {naive }}\left(x_{i} u^{q i}\right)+w_{s / q}^{\text {naive }}\left(\left(u^{\sigma} / u^{q}\right)^{i}\right) \\
& =w_{s}^{\text {naive }}\left(x_{i} u^{i}\right)
\end{aligned}
$$

since $w_{s / q}^{\text {naive }}\left(u^{\sigma} / u^{q}\right)=0$ whenever $s / q \leq r / q$.
Given that $w_{s}^{\text {naive }}(x) \geq c$, it follows that $w_{s}^{\text {naive }}\left(x_{i} u^{i}\right) \geq c$ for each $i$, and by the above argument, that $w_{s / q}^{\text {naive }}\left(x_{i}^{\sigma}\left(u^{i}\right)^{\sigma}\right) \geq c$. We conclude that $w_{s / q}^{\text {naive }}\left(x^{\sigma}\right) \geq c$, as desired.

Lemma 3.6. Choose $r>0$ such that $q^{-1} r v_{n}^{\text {naive }}(z)+n>0$ for all $n$. For $x=\sum_{i} x_{i} t^{i} \in \Gamma_{r, \text { naive }}^{k((t))}$ and $0<s \leq r$, if $s v_{j}^{\text {naive }}(x)+j \geq c$ for all $j \leq n$, then $s v_{j}(x)+j \geq c$ for all $j \leq n$.

Proof. Note that $v_{0}=v_{0}^{\text {naive }}$, so that the desired result holds for $n=0$; we prove the general result by induction on $n$. Suppose, as the induction hypothesis, that if $s v_{j}^{\text {naive }}(x)+j \geq c$ for all $j<n$, then $s v_{j}(x)+j \geq c$ for all $j<n$. Before deducing the desired result, we first study the special case $x=u$ in detail (but using the induction hypothesis in full generality).

Choose $i$ large enough that

$$
v_{p}\left([t]-\left(u^{\sigma^{-i}}\right)^{q^{i}}\right)>n .
$$

Then

$$
\begin{aligned}
v_{n}(u) & \geq \min \left\{v_{n}([t]), v_{n}(u-[t])\right\} \\
& =\min \left\{1, v_{n}\left(u-\left(u^{\sigma^{-i}}\right)^{q^{i}}\right)\right\} .
\end{aligned}
$$

Applying $\sigma^{i}$ yields

$$
q^{i} v_{n}(u) \geq \min \left\{q^{i}, v_{n}\left(u^{\sigma^{i}}-u^{q^{i}}\right)\right\}
$$

Since $u \in \Gamma_{r, \text { naive }}^{k(t))}$ and $w_{r}^{\text {naive }}(u)=r$ trivially, we may apply Lemma 3.5 to $u, u^{\sigma}, \ldots, u^{\sigma^{i-1}}$ in succession to obtain

$$
w_{r / q^{i}}^{\text {naive }}\left(u^{\sigma^{i}}\right) \geq r .
$$

Since $w_{r / q^{i}}^{\text {naive }}\left(u^{q^{i}}\right)=r$, we conclude that $w_{r / q^{i}}^{\text {naive }}\left(u^{\sigma^{i}}-u^{q^{i}}\right) \geq r$.
Let $y=\left(u^{\sigma^{i}}-u^{q^{i}}\right) / \pi$. Then for $j \leq n-v_{p}(\pi)$,

$$
\begin{aligned}
\left(r / q^{i}\right) v_{j}^{\text {naive }}(y)+j & =\left(r / q^{i}\right) v_{j+v_{p}(\pi)}^{\text {naive }}(y \pi)+j+v_{p}(\pi)-v_{p}(\pi) \\
& \geq w_{r / q^{i}}^{\text {naive }}(y \pi)-v_{p}(\pi) \\
& \geq r-v_{p}(\pi)
\end{aligned}
$$

By the induction hypothesis, we conclude that $\left(r / q^{i}\right) v_{n-v_{p}(\pi)}(y)+n-v_{p}(\pi) \geq$ $r-v_{p}(\pi)$, and so $\left(r / q^{i}\right) v_{n}(y \pi)+n \geq r$. From above, we have

$$
\begin{aligned}
q^{i} v_{n}(u) & \geq \min \left\{q^{i}, v_{n}\left(u^{\sigma^{i}}-u^{q^{i}}\right)\right\} \\
& \geq \min \left\{q^{i}, q^{i}-q^{i} n / r\right\} \\
& =q^{i}-q^{i} n / r .
\end{aligned}
$$

Thus $r v_{n}(u)+n \geq r$. Since $v_{n}(u) \leq 1$, we also have $s v_{n}(u)+n \geq s$ for $s \leq r$; that is, the desired conclusion holds for the special case $x=u$. By the multiplication rule for partial valuations (and the same argument with $u$ replaced by $u^{-1}$ ), we also have $s v_{n}\left(u^{i}\right)+n \geq s i$ for all $i$.

With the case $x=u$ in hand, we now prove the desired conclusion for general $x$. We are given $s v_{j}^{\text {naive }}(x)+j \geq c$ for $j \leq n$; by the induction hypothesis, all that we must prove is that $s v_{n}(x)+n \geq c$.

The assumption $s v_{j}^{\text {naive }}(x)+j \geq c$ implies that $s v_{j}^{\text {naive }}\left(x_{i} u^{i}\right)+j \geq c$ for all $j \leq n$, which is to say, if $v_{p}\left(x_{i}\right) \leq n$ then $s i+v_{p}\left(x_{i}\right) \geq c$. For $j=v_{p}\left(x_{i}\right)$, we have

$$
\begin{aligned}
s v_{n}\left(x_{i} u^{i}\right)+n & =s v_{n-j}\left(u^{i}\right)+n-j+j \\
& \geq s i+j \\
& \geq c .
\end{aligned}
$$

We conclude that $s v_{n}(x)+n \geq c$, completing the induction.
We next refine the previous result as follows.
Lemma 3.7. Choose $r>0$ such that $r v_{n}^{\text {naive }}(z)+n>0$ for all $n$. If $x \in \Gamma^{k((t))}$, then for any $s \leq r, \min _{j \leq n}\left\{s v_{j}^{\text {naive }}(x)+j\right\}=\min _{j \leq n}\left\{s v_{j}(x)+j\right\}$ for all $n$. In particular, $w_{s}^{\text {naive }}(x)=w_{s}(x)$ if either one is defined.

That is, the naïve valuations $w_{s}^{\text {naive }}$ are not so simple-minded after all; as long as $s$ is not too large, they agree with the more canonically defined $w_{s}$.

Proof. Lemma 3.6 asserts that $\min _{j \leq n}\left\{s v_{j}(x)+j\right\} \geq \min _{j \leq n}\left\{s v_{j}^{\text {naive }}(x)+\right.$ $j\}$, so it remains to prove the reverse inequality, which we do by induction on $n$. If $\min _{j \leq n}\left\{s v_{j}^{\text {naive }}(x)+j\right\}$ is achieved by some $j<n$, then by the induction hypothesis,

$$
\begin{aligned}
\min _{j \leq n}\left\{s v_{j}^{\text {naive }}(x)+j\right\} & =\min _{j \leq n-v_{p}(\pi)}\left\{s v_{j}^{\text {naive }}(x)+j\right\} \\
& \geq \min _{j \leq n-v_{p}(\pi)}\left\{s v_{j}(x)+j\right\} \\
& \geq \min _{j \leq n}\left\{s v_{j}(x)+j\right\}
\end{aligned}
$$

Suppose then that $\min _{j \leq n}\left\{s v_{j}^{\text {naive }}(x)+j\right\}$ is achieved only for $j=n$. Put $x=\sum x_{i} u^{i}$; by definition, $v_{n}^{\text {naive }}(x)$ is the smallest integer $i$ with $v_{p}\left(x_{i}\right) \leq n$. In fact, we must have $v_{p}\left(x_{i}\right)=n$, or else we have $s v_{j}^{\text {naive }}(x)+j<s v_{n}^{\text {naive }}(x)+n$ for $j=v_{p}\left(x_{i}\right)$. Therefore $v_{n}\left(x_{i} u^{i}\right)=v_{n}^{\text {naive }}\left(x_{i} u^{i}\right)=i$.

For $j<n, s v_{j}^{\text {naive }}\left(x-x_{i} u^{i}\right)+j=s v_{j}^{\text {naive }}(x)+j>s i+n$. On the other hand, $v_{n}^{\text {naive }}(x)=v_{n}^{\text {naive }}\left(x_{i} u^{i}\right)=i$ and $v_{n}^{\text {naive }}\left(x-x_{i} u^{i}\right)>i$. Thus for all $j \leq n$,

$$
s v_{j}^{\text {naive }}\left(x-x_{i} u^{i}\right)+j>s i+n
$$

by Lemma 3.6, $s v_{n}\left(x-x_{i} u^{i}\right)+n>s i+n$ and so $v_{n}\left(x-x_{i} u^{i}\right)>i=v_{n}\left(x_{i} u^{i}\right)$. Therefore $v_{n}(x)=v_{n}\left(x_{i} u^{i}\right)=i$, so that

$$
\min _{j \leq n}\left\{s v_{j}(x)+j\right\} \leq s v_{n}(x)+n=s i+n=\min _{j \leq n}\left\{s v_{j}^{\text {naive }}(x)+j\right\}
$$

yielding the desired inequality.

Corollary 3.8. $\Gamma_{\operatorname{con}}^{\text {alg }} \cap \Gamma^{k((t))}=\Gamma_{\text {con }}^{k((t))}$.
We now define $\Gamma_{\text {con }}^{K}=\Gamma_{\text {con }}^{\text {alg }} \cap \Gamma^{K}$, and Corollary 3.8 assures us that this definition is consistent with our prior definition for $K=k((t))$. To show that $\Gamma_{\text {con }}^{\text {alg }} \cap \Gamma^{K}$ is "large" for any $K$, we need one more lemma, which will end up generalizing a standard fact about $\Gamma_{\text {con }}^{k((t))}$.

Lemma 3.9. For any valued field $K, \Gamma_{\text {con }}^{K}$ is Henselian.
Proof. By a lemma of Nagata [ $\mathrm{N}, 43.2$ ], it suffices to show that if $P(x)=$ $x^{d}+a_{1} x^{d-1}+\cdots+a_{d}$ is a polynomial over $\Gamma_{\text {con }}^{K}$ such that $a_{1} \not \equiv 0(\bmod \pi)$ and $a_{i} \equiv 0(\bmod \pi)$ for $i>1$, then $P(x)$ has a root $y$ in $\Gamma_{\text {con }}^{K}$ such that $y \equiv-a_{1}(\bmod \pi)$. By replacing $P(x)$ by $P\left(-x / a_{1}\right)$, we may reduce to the case $a_{1}=-1$; by Hensel's lemma, $P$ has a root $y$ in $\Gamma^{K}$ congruent to 1 modulo $\pi$, and $P^{\prime}(y) \equiv d y^{d-1}-(d-1) y^{d-2} \equiv 1(\bmod \pi)$.

Choose a constant $c>0$ such that $v_{n}\left(a_{i}\right) \geq-c n$ for all $n$, and define the sequence $\left\{y_{j}\right\}_{j=0}^{\infty}$ by the Newton iteration, putting $y_{0}=1$ and $y_{j+1}=$ $y_{j}-P\left(y_{j}\right) / P^{\prime}\left(y_{j}\right)$. Then $\left\{y_{j}\right\}$ converges $\pi$-adically to $y$; we now show by induction on $j$ that $v_{n}\left(y_{j}\right) \geq-c n$ for all $n$ and all $j$. Namely, this is obvious for $y_{0}$, and given $v_{n}\left(y_{j}\right) \geq-c n$ for all $n$, it follows that $v_{n}\left(P\left(y_{j}\right)\right) \geq-c n$, $v_{n}\left(P^{\prime}\left(y_{j}\right)\right) \geq-c n$, and $v_{n}\left(1 / P^{\prime}\left(y_{j}\right)\right) \geq-c n$ (the last because $v_{0}\left(P^{\prime}\left(y_{j}\right)\right)=0$ ). These together imply $v_{n}\left(y_{j+1}\right) \geq-c n$ for all $n$, completing the induction. We conclude that $y \in \Gamma_{\text {con }}^{K}$ and $\Gamma_{\text {con }}^{K}$ is Henselian, as desired.

We can now prove the following.
Proposition 3.10. For any valued field $K, \Gamma_{\text {con }}^{K}$ has residue field $K$.
Proof. We have already shown this for $K=k((t))$ by Corollary 3.8. If $K / k((t))$ is nearly finite, then $K$ uniquely determines $L, m, M, k_{M}, N, n$ as in the definition of valued fields. Now $M=k_{M}((t))$ for some finite extension $k_{M}$ of $k^{1 / p^{m}}$, so that Corollary 3.8 also implies that $\Gamma_{\text {con }}^{M}$ has residue field $M$. Also, $N=M^{1 / p^{n}}$ for some integer $n$, so that for any $\bar{x} \in M$, we can find $y \in \Gamma_{\text {con }}^{M}$ which lifts $\bar{x}^{p^{n}}$, and then $y^{\sigma^{-n}} \in \Gamma_{\text {con }}^{N}$ lifts $\bar{x}$.

Choose a monic polynomial $P$ over $\Gamma_{\mathrm{con}}^{N}$ lifting a monic separable polynomial $\bar{P}$ for which $K \cong N[x] /(\bar{P}(x))$ (again, possible by the primitive element theorem). By Hensel's lemma $P$ has a root $y$ in $\Gamma^{K}$, and $\Gamma^{K} \cong \Gamma^{N}[y] /(P(y))$. But since $\Gamma_{\mathrm{con}}^{\mathrm{alg}}$ is Henselian and $P$ has coefficients in $\Gamma_{\mathrm{con}}^{\mathrm{alg}}, y \in \Gamma_{\mathrm{con}}^{\mathrm{alg}}$. Thus the residue field of $\Gamma_{\text {con }}^{K}$ contains $N$ and $y$, and hence is all of $K$.

This concludes the proof for $K$ nearly finite over $k((t))$. A general valued field $K$ is the union of its nearly finite valued subfields $K_{1}$, and $\Gamma_{\text {con }}^{K}$ contains (but does not equal) the direct limit of the $\Gamma_{\text {con }}^{K_{1}}$. Thus its residue field contains the union of the $K_{1}$, and hence is equal to $K$.

If $L / K$ is a finite extension of valued fields, then $\Gamma^{L}$ is a finite unramified extension of $\Gamma^{K}$. The minimal polynomial over $\Gamma^{K}$ of any element of $\Gamma_{\text {con }}^{L}$ has coefficients in $\Gamma^{K} \cap \Gamma_{\text {con }}^{L}=\Gamma_{\text {con }}^{K}$; hence $\Gamma_{\text {con }}^{L}$ is integral over $\Gamma_{\text {con }}^{K}$. In fact it is a finite unramified extension of henselian discrete valuation rings.
3.3. Analytic rings: generalizing the Robba ring. In this section, we generalize the construction of the Robba ring. Besides the classical case where $K$ is a finite extension of $k((t))$, we will be especially interested in the case $K=k((t))^{\text {alg }}$, which will give a sort of "maximal unramified extension" of the standard Robba ring. (That ring also appears in [Bg], as the ring $\tilde{\mathbf{B}}_{\text {rig }}^{\dagger}$.)

Proposition 3.11. Suppose the valued field $K$ is either
(a) nearly finite over $k((t))$ or (b) perfect.

Then there exists $r_{0}>0$ such that for $0<r<r_{0}, \Gamma_{r}^{K}=\Gamma_{r}^{\text {alg }} \cap \Gamma^{K}$ has units congruent to every nonzero element of $K$.

Proof. For (a), there is no harm in assuming $K / k((t))$ is finite separable. Let $u$ be a lift to $\Gamma_{\text {con }}^{K}$ of a uniformizer $\bar{u}$ of $K$, and choose $r_{0}>0$ so that $u$ is a unit in $\Gamma_{r}^{K}$. Let $\mathcal{O}^{\prime}$ be the integral closure of $\mathcal{O}$ in $\Gamma^{K}$; its residue field is the integral closure $k^{\prime}$ of $k$ in $K$.

For any $c_{i} \in \mathcal{O}^{\prime}$, the series $1+\sum_{i=1}^{\infty} c_{i} u^{i}$ converges with respect to $w_{r}$ (hence levelwise) to a unit of $\Gamma_{r}^{K}$, because we can formally invert the series and the result also converges with respect to $w_{r}$. Any nonzero element of $K$ can be written as a nonzero element of $k^{\prime}$ times a power of $\bar{u}$ times a series in $\bar{u}$ with leading term 1 , thus can be lifted as an invertible element of $\mathcal{O}^{\prime}$ times a power of $u$ times a series of the form $1+\sum_{i=1}^{\infty} c_{i} u^{i}$. The result is invertible in $\Gamma_{r}^{K}$, as desired.

For (b), we can choose any $r_{0}>0$, since every Teichmüller lift belongs to $\Gamma_{r}^{K}$.

Note that the conclusion of the proposition need not hold for other valued fields. For example, it fails for $K=k((t))^{\text {sep }}$ if $\sigma_{0}(u)=u^{p}$ for some $u \in \Gamma_{\text {con }}^{k((t))}$ lifting $t$ : define a sequence $\left\{y_{i}\right\}_{i=1}^{\infty}$ of elements of $K$ by setting $y_{i}$ to be a root of $y_{i}^{p}-y_{i}=u^{-i}$. Then it can be shown that $y_{i}$ has a lift in $\Gamma_{r}^{K}$ only if $r<\frac{1}{i}(p /(p-1))^{2}$, and so there is no way to choose $r$ uniformly.

For the rest of this section, we assume that the hypotheses of Proposition 3.11 are satisfied. Recall that for $0<s \leq r$, we have defined the valuation $w_{s}$ on $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ by

$$
w_{s}(x)=\min _{n}\left\{n+s v_{n}(x)\right\},
$$

the minimum taken as $n$ runs over the value group of $\mathcal{O}$. We define a corresponding norm $|\cdot|_{s}$ by $|x|_{s}=p^{-w_{s}(x)}$.

While $\Gamma_{r}^{K}$ is complete under $|\cdot|_{r}, \Gamma_{r}^{K}\left[\frac{1}{p}\right]$ is not, and so we can attempt to complete it. In fact, we can define a Fréchet topology on $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ using the $w_{s}$ for $0<s \leq r$, and define $\Gamma_{\mathrm{an}, r}^{K}$ as the Fréchet completion of $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$. That is, $\Gamma_{\mathrm{an}, r}^{K}$ consists of equivalence classes of sequences of elements of $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ which are simultaneously Cauchy for all of the norms $|\cdot|_{s}$.

Set $\Gamma_{\mathrm{an}, \mathrm{con}}^{K}=\cup_{r>0} \Gamma_{\mathrm{an}, r}^{K}$. Echoing a warning from the previous section, we note that $\Gamma_{\mathrm{a}}^{K}$,con admits an action of $\sigma$, but each $\Gamma_{\mathrm{an}, r}^{K}$ is mapped not into itself, but into $\Gamma_{\mathrm{an}, r / q}^{K}$. More precisely, we have $w_{r / q}\left(x^{\sigma}\right)=w_{r}(x)$ for all $x \in \Gamma_{\mathrm{an}, r}^{K}$.

In case $K=k((t))$, we defined another ring called $\Gamma_{\text {an,con }}^{K}$ in Section 2.3. Fortunately, these rings coincide: for $r$ sufficiently small, by Corollary 3.8 we have $\Gamma_{r}^{K}=\Gamma_{r, \text { naive }}^{K}$ and so $\Gamma_{\mathrm{an}, r}^{K}=\Gamma_{\mathrm{a}, r, r \text { naive }}^{K}$ by Proposition 2.2.

Since $\Gamma_{\text {an,con }}^{K}$ is defined from $\Gamma_{\text {con }}^{K}$ by a canonical completion process, it inherits as much functoriality as is possible given the restricted applicability of Proposition 3.11. For example, if $L / K$ is a finite totally ramified extension, then $\Gamma_{\mathrm{an}, \mathrm{con}}^{L}$ is an integral extension of $\Gamma_{\mathrm{an}, \text { con }}^{K}$; in fact, one has a canonical identification of $\Gamma_{\text {an,con }}^{L}$ with $\Gamma_{\text {con }}^{L} \otimes_{\Gamma_{\text {con }}^{K}} \Gamma_{\text {an,con }}^{K}$, which in case $L / K$ is Galois gives an action of $\operatorname{Gal}(L / K)$ on $\Gamma_{\text {an,con }}^{L}$ with fixed ring $\Gamma_{\text {an,con }}^{K}$. Likewise, if $K$ is perfect, then the union $\cup_{L} \Gamma_{\text {an }, r}^{L}$ running over all nearly finite subextensions $L$ of $K$ is dense in $\Gamma_{\mathrm{an}, r}^{K}$ for each $r>0$, so $\cup_{L} \Gamma_{\mathrm{an}, \text { con }}^{L}$ is dense in $\Gamma_{\mathrm{an}, \text { con }}^{K}$.

We can extend the functions $v_{n}$ to $\Gamma_{\mathrm{an}, r}^{K}$ by continuity: if $x_{i} \rightarrow x$ in the Fréchet topology, then $v_{n}\left(x_{i}\right)$ either stabilizes at some finite value or tends to $+\infty$ as $i \rightarrow \infty$, and we may put $v_{n}(x)=\lim _{i \rightarrow \infty} v_{n}\left(x_{i}\right)$. Likewise, we can extend the functions $w_{s}$ to $\Gamma_{\mathrm{an}, r}^{K}$ by continuity, and again one has the formula

$$
w_{s}(x)=\min _{n}\left\{n+s v_{n}(x)\right\},
$$

as $n$ runs over the value group of $\mathcal{O}$. One also has

$$
\lim _{n \rightarrow \pm \infty}\left(n+s v_{n}(x)\right)=\infty
$$

for any $0<s<r$. For $n \rightarrow-\infty$, this follows from the corresponding limiting statement for $s=r$. For $n \rightarrow \infty$, note that if the limit did not tend to infinity, $x$ could not be written as a limit under $|\cdot|_{s}$ of elements of $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$.

It is not so easy to prove anything about the ring $\Gamma_{\mathrm{an}, \mathrm{con}}^{K}$ just from the above definition, since it is inconvenient even to write down elements of this ring. To this end, we isolate a special class of elements, which we call semiunits, and use them as building blocks to represent more general ring elements.

We define a semi-unit of $\Gamma_{r}^{K}$ (resp. of $\Gamma_{\mathrm{an}, r}^{K}$ ) as an element $u$ of $\Gamma_{r}^{K}$ (resp. of $\Gamma_{\mathrm{an}, r}^{K}$ ) which is either zero, or which satisfies the following conditions:
(a) $v_{n}(u)=\infty$ for $n<0$;
(b) $v_{0}(u)<\infty$;
(c) $r v_{n}(u)+n>r v_{0}(u)$ for $n>0$.

In particular, if $u \in \Gamma_{r}^{K}$, then $u$ is a semi-unit if either $u=0$ or $u$ is a unit in $\Gamma_{r}^{K}$, hence the terminology; more generally, in $\Gamma_{\mathrm{an}, r}^{K}$, the nonzero semiunits form a multiplicative group. In particular, under the condition of Proposition 3.11, every element of $K$ lifts to a semi-unit in $\Gamma_{r}^{K}$. Note that if $u$ is a semi-unit in $\Gamma_{\mathrm{an}, r}^{K}$, it is also a semi-unit in $\Gamma_{\mathrm{an}, s}^{K}$ for any $0<s<r$. Also be aware that if $K / k((t))$ is infinite, a semi-unit $u$ in $\Gamma_{\mathrm{an}, r}^{K}$ need not belong to $\Gamma_{r}^{K}$ even though $v_{p}(u) \geq 0$. (If $R$ is the subring of $x \in \Gamma_{\mathrm{an}, r}^{K}$ with $v_{p}(x) \geq 0$, then $R / \pi R$ is isomorphic to the completion of $K$ with respect to $v_{K}$.)

If $K$ is perfect, we define a strong semi-unit of $\Gamma_{r}^{K}$ (resp. of $\Gamma_{\mathrm{an}, r}^{K}$ ) as an element $u$ of $\Gamma_{r}^{K}$ (resp. of $\Gamma_{\mathrm{an}, r}^{K}$ ) which is either zero, or satisfies the following conditions:
(a) $v_{n}(u)=\infty$ for $n<0$;
(b) $v_{0}(u)<\infty$;
(c) $v_{n}(u)=v_{0}(u)$ for $n>0$.

Every Teichmüller lift is a strong semi-unit, so every element of $K$ lifts to a strong semi-unit in $\Gamma_{r}^{K}$.

Let $\left\{u_{i}\right\}_{i=-\infty}^{\infty}$ be a doubly infinite sequence of semi-units in $\Gamma_{r}^{K}$ (resp. in $\left.\Gamma_{\mathrm{an}, r}^{K}\right)$. Then we say $\left\{u_{i}\right\}$ is a semi-unit decomposition of $x$ in $\Gamma_{r}^{K}$ (resp. in $\Gamma_{\mathrm{an}, r}^{K}$ ) if $w_{r}\left(u_{i} \pi^{i}\right) \leq w_{r}\left(u_{j} \pi^{j}\right)$ whenever $i>j$ and $u_{i}, u_{j} \neq 0$, and if $\sum_{i=-M}^{N} u_{i} \pi^{i}$ converges to $x$ in the Fréchet topology as $M, N \rightarrow \infty$. We express this more succinctly by saying that $\sum u_{i} \pi^{i}$ is a semi-unit decomposition of $x$. Analogously, if $K$ is perfect and the $u_{i}$ are strong semi-units, we say $\sum u_{i} \pi^{i}$ is a strong semi-unit decomposition of $x$ if $v_{0}\left(u_{i}\right)<v_{0}\left(u_{j}\right)$ whenever $i>j$ and $u_{i}, u_{j} \neq 0$, and if $\sum_{i=-M}^{N} u_{i} \pi^{i}$ converges to $x$ in the Fréchet topology as $M, N \rightarrow \infty$.

If $\sum u_{i} \pi^{i}$ is a semi-unit decomposition of $x \in \Gamma_{\mathrm{an}, r}^{K}$, then for each $i$ such that $u_{i} \neq 0$, we may set $n=i v_{p}(\pi)$ and obtain $r v_{n}(x)+n=r v_{n}\left(u_{i} \pi^{i}\right)+n$; that is, $v_{n}(x)=v_{n}\left(u_{i} \pi^{i}\right)$. Since $r v_{n}(x)+n \rightarrow \infty$ as $n \rightarrow \infty$ for any $x \in \Gamma_{\mathrm{an}, r}^{K}$, we must then have $u_{i}=0$ for $i$ sufficiently large. There is no analogous phenomenon for strong semi-unit decompositions, however: for each $i$ such that $u_{i} \neq 0$, we set $n=i v_{p}(\pi)$ and obtain $v_{n}(x)=v_{n}\left(u_{i} \pi^{i}\right)$, but $v_{n}(x)$ may continue to decrease forever as $n \rightarrow \infty$, and so the $u_{i}$ need not eventually vanish.

Lemma 3.12. Each element $x$ of $\Gamma_{r}^{K}$ admits a semi-unit decomposition. If $K$ is perfect, each element $x$ of $\Gamma_{r}^{K}$ admits a strong semi-unit decomposition.

Proof. Without loss of generality (dividing by a suitable power of $\pi$ ), we may reduce to the case where $x \not \equiv 0(\bmod \pi)$. We define a sequence of semiunits $\left\{y_{i}\right\}_{i=0}^{\infty}$ such that $x \equiv \sum_{i=0}^{j} y_{i} \pi^{i}\left(\bmod \pi^{j+1}\right)$, as follows. Let $y_{0}$ be a
semi-unit congruent to $x$ modulo $\pi$. Given $y_{0}, \ldots, y_{j}$, let $y_{j+1}$ be a semi-unit congruent to $\left(x-\sum_{i=0}^{j} y_{i} \pi^{i}\right) / \pi^{j+1}$ modulo $\pi$.

The sum $\sum_{i=0}^{\infty} y_{i} \pi^{i}$ now converges to $x$, but we do not have the necessary comparison between $w_{r}\left(y_{i} \pi^{i}\right)$ and $w_{r}\left(y_{j} \pi^{j}\right)$, so we must revise the decomposition. We say $i$ is a corner if $w_{r}\left(y_{i} \pi^{i}\right)=\min _{j \leq i}\left\{w_{r}\left(y_{j} \pi^{j}\right)\right\}$. We now set $u_{i}=0$ if $i$ is not a corner; if $i$ is a corner, let $l$ be the next largest corner (or $\infty$ if there is none), and put $u_{i}=\sum_{j=i}^{l-1} y_{j} \pi^{j-i}$. By the definition of a corner, $w_{r}\left(y_{j} \pi^{j-i}\right)>w_{r}\left(y_{i}\right)$ for $i<j<l$, so that $u_{i}$ is a semi-unit. Moreover, if $i$ and $j$ are corners and $i>j$, then $w_{r}\left(u_{i} \pi^{i}\right)=w_{r}\left(y_{i} \pi^{i}\right) \leq w_{r}\left(y_{j} \pi^{j}\right)=w_{r}\left(u_{j} \pi^{j}\right)$; and the sum $\sum_{i=0}^{\infty} u_{i} \pi^{i}$ is merely the sum $\sum_{i=0}^{\infty} y_{i} \pi^{i}$ with the terms regrouped, so it still converges to $x$. Thus $\sum_{i=0}^{\infty} u_{i} \pi^{i}$ is a semi-unit decomposition of $x$.

If $K$ is perfect, we perform the revision slightly differently. We say $i$ is a corner if $v_{0}\left(y_{i}\right)<v_{0}\left(y_{j}\right)$ for all $j<i$. Again, we set $u_{i}=0$ if $i$ is not a corner, and if $i$ is a corner and $l$ is the next largest corner, we set $u_{i}=\sum_{j=i}^{l-i} y_{j} \pi^{j-i}$. Clearly $u_{i}$ is a strong semi-unit for each $i$, and the sum $\sum_{i=0}^{\infty} u_{i} \pi^{i}$ converges to $x$. If $i>j$ are corners, then $v_{0}\left(u_{i}\right)=v_{0}\left(y_{i}\right)<v_{0}\left(y_{j}\right)=v_{0}\left(u_{j}\right)$. Thus $\sum_{i=0}^{\infty} u_{i} \pi^{i}$ is a strong semi-unit decomposition of $x$.

Proposition 3.13. Every element of $\Gamma_{\mathrm{an}, r}^{K}$ admits a semi-unit decomposition.

Proof. For $x \in \Gamma_{\mathrm{an}, r}^{K}$, let $\sum_{l=0}^{\infty} x_{l}$ be a series of elements of $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ that converges under $|\cdot|_{r}$ to $x$, such that $w_{r}\left(x_{l}\right)<w_{r}\left(x_{l+1}\right)$. (For example, choose $x_{0}$ such that $w_{r}\left(x-x_{0}\right)>w_{r}(x)$, then choose $x_{1}$ such that $w_{r}\left(x-x_{0}-x_{1}\right)>$ $w_{r}\left(x-x_{0}\right)$, and so forth.)

For $l=0,1, \ldots$ and $i \in \mathbb{Z}$, we define elements $y_{i l}$ of $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ recursively in $l$, such that for any $l$, only finitely many of the $y_{i l}$ are nonzero, as follows. Apply Lemma 3.12 (after multiplying by a suitable power of $\pi$ ) to produce a semi-unit decomposition of $x_{0}+\cdots+x_{l}-\sum_{j<l} \sum_{i} y_{i j} \pi^{i}$. For each of the finitely many terms $u_{i} \pi^{i}$ of this decomposition with $u_{i} \neq 0$ and $w_{r}\left(u_{i} \pi^{i}\right)<w_{r}\left(x_{l+1}\right)$, put $y_{i l}=u_{i}$; for all other $i$, put $y_{i l}=0$. Then

$$
w_{r}\left(x_{0}+\cdots+x_{l}-\sum_{j \leq l} \sum_{i} y_{i j} \pi^{i}\right) \geq w_{r}\left(x_{l+1}\right)
$$

In particular, the doubly infinite sum $\sum_{l} \sum_{i} y_{i l} \pi^{i}$ converges under $|\cdot|_{r}$ to $x$. If we set $z_{i}=\sum_{l} y_{i l}$, the series $\sum_{i} z_{i} \pi^{i}$ converges under $|\cdot|_{r}$ to $x$.

Note that $w_{r}\left(x_{l}\right) \leq w_{r}\left(y_{i l} \pi^{i}\right)<w_{r}\left(x_{l+1}\right)$ whenever $y_{i l} \neq 0$. Thus for any fixed $i$, the values of $w_{r}\left(y_{i l} \pi^{i}\right)$, taken over all $l$ such that $y_{i l} \neq 0$, form a strictly increasing sequence. If $j$ is the first such index, we then have $w_{r}\left(y_{i j} \pi^{i}\right)<$ $w_{r}\left(\sum_{l>j} y_{i l} \pi^{i}\right)$, and so $z_{i}$ is a semi-unit.

Define $u_{i}$ to be zero if $w_{r}\left(z_{i} \pi^{i}\right)>w_{r}\left(z_{j} \pi^{j}\right)$ for some $j<i$; otherwise, let $l$ be the smallest integer greater than $i$ such that $w_{r}\left(z_{l} \pi^{l}\right) \leq w_{r}\left(z_{i} \pi^{i}\right)$ (or $\infty$ if
none exists), and put $u_{i}=\sum_{j=i}^{l-1} z_{j} \pi^{j-i}$. Then the series $\sum_{i} u_{i} \pi^{i}$ also converges under $|\cdot|_{r}$ to $x$, and if $u_{i} \neq 0$, then $u_{i}$ is a semi-unit and $w_{r}\left(u_{i} \pi^{i}\right)=w_{r}\left(z_{i} \pi^{i}\right)$. It follows that $w_{r}\left(u_{i} \pi^{i}\right) \leq w_{r}\left(u_{j} \pi^{j}\right)$ whenever $i>j$ and $u_{i}, u_{j} \neq 0$. This in turn implies that if $u_{i} \neq 0$ and $n=v_{p}\left(\pi^{i}\right)$, then $v_{n}\left(u_{i} \pi^{i}\right)=v_{n}(x)$.

We finally check that $\sum_{i} u_{i} \pi^{i}$ converges under $|\cdot|_{s}$ for $0<s<r$. The fact that $s v_{n}(x)+n \rightarrow \infty$ as $n \rightarrow \pm \infty$ implies that $s v_{v_{p}\left(\pi^{i}\right)}\left(u_{i} \pi^{i}\right)+v_{p}\left(\pi^{i}\right) \rightarrow \infty$ as $i \rightarrow \pm \infty$. Since $u_{i}$ is a semi-unit, $w_{s}\left(u_{i} \pi^{i}\right)=s v_{v_{p}\left(\pi^{i}\right)}\left(u_{i} \pi^{i}\right)+v_{p}\left(\pi^{i}\right)$, so $w_{s}\left(u_{i} \pi^{i}\right) \rightarrow \infty$ as $i \rightarrow \pm \infty$. Thus the sum $\sum_{i} u_{i} \pi^{i}$ converges under $|\cdot|_{s}$ for $0<s<r$, and the limit must equal $x$ because the sum converges to $x$ under $|\cdot|_{r}$. Therefore $\sum_{i} u_{i} \pi^{i}$ is a semi-unit decomposition, as desired.

Proposition 3.14. If $K$ is perfect, every element of $\Gamma_{\mathrm{an}, r}^{K}$ admits a strong semi-unit decomposition.

Proof. As in the previous proof, for $x \in \Gamma_{\mathrm{an}, r}^{K}$, let $\sum_{l=0}^{\infty} x_{l}$ be a series of elements of $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ that converges under $|\cdot|_{r}$ to $x$, such that $w_{r}\left(x_{l}\right)<w_{r}\left(x_{l+1}\right)$.

For $l=0,1, \ldots$ and $i \in \mathbb{Z}$, we define elements $y_{i l}$ of $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ recursively in $l$, such that for any $l$, only finitely many of the $y_{i l}$ are nonzero, as follows. Apply Lemma 3.12 to produce a strong semi-unit decomposition of $x_{0}+\cdots+x_{l}-$ $\sum_{j<l} \sum_{i} y_{i j} \pi^{i}$. For each of the finitely many terms $u_{i} \pi^{i}$ of this decomposition with $u_{i} \neq 0$ and $w_{r}\left(u_{i} \pi^{i}\right)<w_{r}\left(x_{l+1}\right)$, put $y_{i l}=u_{i}$; for all other $i$, put $y_{i l}=0$. Then

$$
w_{r}\left(x_{0}+\cdots+x_{l}-\sum_{j \leq l} \sum_{i} y_{i j} \pi^{i}\right) \geq w_{r}\left(x_{l+1}\right) .
$$

In particular, the doubly infinite sum $\sum_{l} \sum_{i} y_{i l} \pi^{i}$ converges under $|\cdot|_{r}$ to $x$. If we set $z_{i}=\sum_{l} y_{i l}$, the series $\sum_{i} z_{i} \pi^{i}$ converges under $|\cdot|_{r}$ to $x$.

Note that $w_{r}\left(x_{l}\right) \leq w_{r}\left(y_{i l} \pi^{i}\right)<w_{r}\left(x_{l+1}\right)$ whenever $y_{i l} \neq 0$. Thus for any fixed $i$, the values of $v_{0}\left(y_{i l}\right)$, taken over all $l$ such that $y_{i l} \neq 0$, form a strictly increasing sequence. If $j$ is the first such index, we then have $v_{0}\left(y_{i j}\right)<$ $v_{0}\left(\sum_{l>j} y_{i l}\right)$, and so $z_{i}$ is a strong semi-unit.

Define $u_{i}$ to be zero if $v_{0}\left(z_{i}\right) \geq v_{0}\left(z_{j}\right)$ for some $j<i$; otherwise, let $l$ be the smallest integer such that $v_{0}\left(z_{l}\right)<v_{0}\left(z_{i}\right)$ (or $\infty$ if none exists), and put $u_{i}=\sum_{j=i}^{l-1} z_{j} \pi^{j-i}$. Then the series $\sum_{i} u_{i} \pi^{i}$ also converges under $|\cdot|_{r}$ to $x$, and if $u_{i} \neq 0$, then $u_{i}$ is a strong semi-unit and $v_{0}\left(u_{i}\right)=v_{0}\left(z_{i}\right)$. It follows that $v_{0}\left(u_{i}\right)<v_{0}\left(u_{j}\right)$ whenever $i>j$ and $u_{i}, u_{j} \neq 0$. This in turn implies that if $u_{i} \neq 0$ and $n=v_{p}\left(\pi^{i}\right)$, then $v_{n}\left(u_{i} \pi^{i}\right)=v_{n}(x)$.

We finally check that $\sum_{i} u_{i} \pi^{i}$ converges under $|\cdot|_{s}$ for $0<s<r$, by the same argument as in the previous proof. Namely, the fact that $s v_{n}(x)+n \rightarrow \infty$ as $n \rightarrow \pm \infty$ implies that $s v_{v_{p}\left(\pi^{i}\right)}\left(u_{i} \pi^{i}\right)+v_{p}\left(\pi^{i}\right) \rightarrow \infty$ as $i \rightarrow \pm \infty$. Since $u_{i}$ is a strong semi-unit, $w_{s}\left(u_{i} \pi^{i}\right)=s v_{v_{p}\left(\pi^{i}\right)}\left(u_{i} \pi^{i}\right)+v_{p}\left(\pi^{i}\right)$, so that $w_{s}\left(u_{i} \pi^{i}\right) \rightarrow \infty$
as $i \rightarrow \pm \infty$. Thus the sum $\sum_{i} u_{i} \pi^{i}$ converges under $|\cdot|_{s}$ for $0<s<r$, and the limit must equal $x$ because the sum converges to $x$ under $|\cdot|_{r}$. Therefore $\sum_{i} u_{i} \pi^{i}$ is a strong semi-unit decomposition, as desired.

Although (strong) semi-unit decompositions are not unique, in a certain sense the "leading terms" are unique. To make sense of this remark, we first need a "leading coefficient map" for $K$.

Lemma 3.15. For $K$ a valued field, there exists a homomorphism $\lambda$ : $K^{*} \rightarrow\left(k^{\mathrm{alg}}\right)^{*}$ such that $\lambda(c)=c$ for all $c \in k^{\text {alg }} \cap K$ and $\lambda(x)=1$ if $v_{K}(x-1)>0$.

For instance, if $K=k((t))$, we could take $\lambda(x)$ to be the leading coefficient of $x$.

Proof. There is no loss of generality in enlarging $K$, so we may assume $K=k((t))^{\text {alg }}$. Define $t_{0}=t$, and for $i>0$, let $t_{i}$ be an $i$-th root of $t_{i-1}$. With this choice, for any $d \in \mathbb{Q}$ we can define $t^{d}$ as $t_{i}^{i!d}$ for any $i \geq d$; the expression does not depend on $i$.

Now for each $x \in K^{*}$, there exists a unique $c \in\left(k^{\text {alg }}\right)^{*}$ such that

$$
v_{K}\left(\frac{x}{c t^{v_{K}(x)}}-1\right)>0
$$

set $\lambda(x)=c$.
Choosing a map $\lambda$ as in Lemma 3.15, we define the leading terms map $L_{r}: \Gamma_{\mathrm{an}, r}^{K} \rightarrow \cup_{n=1}^{\infty} k^{\mathrm{alg}}\left[t^{1 / n}, t^{-1 / n}\right]$ as follows. For $x \in \Gamma_{\mathrm{an}, r}^{K}$ nonzero, find a finite sum $y=\sum_{j} u_{j} \pi^{j}$ such that each $u_{j}$ is a semi-unit, $w_{r}\left(u_{j} \pi^{j}\right)=w_{r}(x)$ for all $j$ such that $u_{j} \neq 0$, and $w_{r}(x-y)>w_{r}(x)$. Then put $L_{r}(x)=\sum_{j} \lambda\left(\overline{u_{j}}\right) t^{v_{0}\left(u_{j}\right)}$; this definition does not depend on the choice of $y$. Moreover, the leading terms map is multiplicative; that is, $L_{r}(x y)=L_{r}(x) L_{r}(y)$.

We define the upper degree and lower degree of a nonzero element of $\cup_{n=1}^{\infty} k^{\mathrm{alg}}\left[t^{1 / n}, t^{-1 / n}\right]$ as the largest and smallest powers of $t$, respectively, occurring in the element; we define the length of an element as the upper degree minus the lower degree. We extend all of these definitions to $\Gamma_{\mathrm{an}, r}^{K}$ through the $\operatorname{map} L_{r}$.

Warning: if $K$ is not nearly finite over $k((t))$, then the subring of $x \in$ $\Gamma_{\mathrm{an}, \mathrm{con}}^{K}$ with $v_{n}(x)=\infty$ for $n<0$ is a complete discrete valuation ring containing $\Gamma_{\text {con }}^{K}$, but it is actually much bigger than $\Gamma_{\text {con }}^{K}$. In fact, its residue field is the completion of $K$ with respect to the valuation $v_{K}$.

As noted earlier, a theorem of Lazard asserts that $\Gamma_{\text {an,con }}^{K}$ is a Bézout ring (every finitely generated ideal is principal) for $K=k((t))$; the same is true for $K$ a nearly finite extension of $k((t))$, since $K \cong k^{\prime}\left(\left(t^{\prime}\right)\right)$ for some uniformizer $t^{\prime}$ and some field $k^{\prime}$. We will generalize the Bézout property to
$\Gamma_{\text {an,con }}^{K}$ for $K / k((t))$ infinite in Section 3.6; for now, we deduce from Lemma 2.6 the following descent lemma for $\sigma$-modules. (The condition on $G$-stable ideals is satisfied because $G=\operatorname{Gal}(L / K)$ here is finite.)

Corollary 3.16. Let $L / K$ be a finite Galois extension of valued fields nearly finite over $k((t))$. Let $M$ be a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}^{K}$ and $N$ a saturated $\sigma$-submodule of $M \otimes_{\Gamma_{\text {an }, \text { con }}^{K}} \Gamma_{\text {an }, \text { con }}^{L}$ stable under $\operatorname{Gal}(L / K)$. Then $N$ is equal to $P \otimes_{\Gamma_{\text {an, con }}^{K}} \Gamma_{\text {an,con }}^{L}$ for some saturated $\sigma$-submodule $P$ of $M$.
3.4. Some $\sigma$-equations. We record here the behavior of some simple equations involving $\sigma$. For starters, we have the following variant of Hensel's lemma.

Proposition 3.17. Let $R$ be a complete discrete valuation ring, unramified over $\mathcal{O}$, with separably closed residue field, and let $\sigma$ be a $q$-power Frobenius lift. For $c_{0}, \ldots, c_{n} \in R$ with $c_{0}$ not divisible by $\pi$ and $x \in R$, define $f(x)=c_{0} x+c_{1} x^{\sigma}+\cdots+c_{n} x^{\sigma^{n}}$. Then for any $x, y \in R$ for which $f(x) \equiv y$ $(\bmod \pi)$, there exists $z \in R$ congruent to $x$ modulo $\pi$ for which $f(z)=y$. Moreover, if $R$ has algebraically closed residue field, then the same holds if any of $c_{0}, \ldots, c_{n}$ is not divisible by $\pi$.

Proof. Define a sequence $\left\{z_{l}\right\}_{l=1}^{\infty}$ of elements of $R$ such that $z_{1}=x$, $z_{l+1} \equiv z_{l}\left(\bmod \pi^{l}\right)$ and $f\left(z_{l}\right) \equiv y\left(\bmod \pi^{l}\right)$; then the limit $z$ of the $z_{l}$ will have the desired property. Given $z_{l}$, put $a_{l}=\left(y-f\left(z_{l}\right)\right) / \pi^{l}$, and choose $b_{l} \in R$ such that

$$
c_{0} b_{l}+c_{1} b_{l}^{q}\left(\pi^{\sigma} / \pi\right)^{l}+\cdots+c_{n} b_{l}^{q^{n}}\left(\pi^{\sigma^{n}} / \pi\right)^{l} \equiv a_{l} \quad(\bmod \pi) ;
$$

this is possible because either $R$ has algebraically closed residue field, or $c_{0} \neq 0$ and the polynomial at left must be separable. Put $z_{l+1}=z_{l}+\pi^{l} b_{l}$; then $f\left(z_{l+1}\right) \equiv f\left(z_{l}\right)+f\left(\pi^{l} b_{l}\right) \equiv y\left(\bmod \pi^{l+1}\right)$, as desired.

We next consider similar equations over some other rings. The following result will be vastly generalized by Proposition 5.11 later.

Proposition 3.18. Suppose $x \in \Gamma_{\mathrm{con}}^{\mathrm{alg}}$ (resp. $x \in \Gamma_{\mathrm{an}, \text { con }}^{\mathrm{alg}}$ with $v_{n}(x)=\infty$ for $n<0$ ) is not congruent to 0 modulo $\pi$. Then there exists a nonzero $y \in \Gamma_{\mathrm{con}}^{\mathrm{alg}}$ (resp. $y \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ with $v_{n}(y)=\infty$ for $n<0$ ) such that $y^{\sigma}=x y$.

Proof. Put $R=\Gamma_{\text {con }}^{\mathrm{alg}}$ (resp. let $R$ be the subring of $x \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ with $v_{n}(x)=\infty$ for $\left.n<0\right)$ and let $S$ be the completion of $R$. By Proposition 3.17, we can find nonzero $y \in S$ such that $y^{\sigma}=x y$; we need to show that $y \in R$. Choose $r>0$ and $c \in \mathbb{R}$ such that $r v_{n}(x)+n \geq c$ for all $n$. We then show that $r(q-1) v_{n}(y)+n \geq c$ by induction on $n$. Now,

$$
q v_{n}(y)=v_{n}\left(y^{\sigma}\right) \geq \min _{m \leq n}\left\{v_{m}(x)+v_{n-m}(y)\right\} .
$$

If the minimum is achieved for $m=0$ (which includes the base case $n=0$ ), then $(q-1) v_{n}(y) \geq v_{0}(x)$, so $r(q-1) v_{n}(y)+n \geq r v_{0}(x)+n \geq c$. If the minimum is achieved for some $m>0$, then by the induction hypothesis

$$
\begin{aligned}
r(q-1) v_{n}(y) & \geq \frac{r(q-1)}{q} v_{m}(x)+\frac{r(q-1)}{q} v_{n-m}(y) \\
& \geq \frac{(q-1)(c-m)}{q}+\frac{(c-n+m)}{q} \\
& \geq \frac{(q-1)(c-n)}{q}+\frac{(c-n)}{q} \geq c-n
\end{aligned}
$$

so that $r(q-1) v_{n}(y)+n \geq c$. Thus the induction goes through, and demonstrates that $y \in R$, as desired.

Finally, we consider a class of equations involving the analytic rings. We suppress $K$ from all superscripts for convenience, writing $\Gamma_{\text {con }}$ for $\Gamma_{\text {con }}^{K}$ and so forth.

Proposition 3.19. Let $K$ be a valued field (satisfying the condition of Proposition 3.11 in case $\Gamma_{\mathrm{an}, \mathrm{con}}$ is referenced).
(a) Assume $K$ is separably closed (resp. algebraically closed). For $\lambda \in \mathcal{O}$ a unit and $x \in \Gamma_{\text {con }}$ (resp. $x \in \Gamma_{\text {an,con }}$ ), there exists $y \in \Gamma_{\text {con }}$ (resp. $\left.y \in \Gamma_{\mathrm{an}, \mathrm{con}}\right)$ such that $y^{\sigma}-\lambda y=x$. Moreover, if $x \in \Gamma_{\mathrm{con}}\left[\frac{1}{p}\right]$, then any such $y$ belongs to $\Gamma_{\text {con }}\left[\frac{1}{p}\right]$.
(b) Assume $K$ is perfect. For $\lambda \in \mathcal{O}$ not a unit and $x \in \Gamma_{\text {con }}$ (resp. $x \in$ $\left.\Gamma_{\mathrm{an}, \mathrm{con}}\right)$, there exists $y \in \Gamma_{\mathrm{con}}$ (resp. $y \in \Gamma_{\mathrm{an}, \mathrm{con}}$ ) such that $y^{\sigma}-\lambda y=x$. Moreover, we can take $y$ nonzero in $\Gamma_{\mathrm{an}, \mathrm{con}}$ even if $x=0$.
(c) For $\lambda \in \mathcal{O}$ not a unit and $x \in \Gamma_{\mathrm{an}, \mathrm{con}}$, there is at most one $y \in \Gamma_{\mathrm{an}, \mathrm{con}}$ such that $\lambda y^{\sigma}-y=x$, and if $x \in \Gamma_{\mathrm{con}}$, then $y \in \Gamma_{\mathrm{con}}$ as well.
(d) For $\lambda \in \mathcal{O}$ not a unit and $x \in \Gamma_{\text {an,con }}$ such that $v_{n}(x) \geq 0$ for all $n$, there exists $y \in \Gamma_{\text {an,con }}$ such that $\lambda y^{\sigma}-y=x$.

Proof. (a) If $x \in \Gamma_{\text {con }}$, then Proposition 3.17 implies that there exists $y \in \Gamma$ such that $y^{\sigma}-\lambda y=x$. To see that in fact $y \in \Gamma_{\text {con }}$, note that if $v_{n}(y) \leq 0$, the fact that

$$
q v_{n}(y)=v_{n}\left(y^{\sigma}\right)=v_{n}(\lambda y+x) \geq \min \left\{v_{n}(x), v_{n}(y)\right\}
$$

implies that $q v_{n}(y) \geq v_{n}(x)$; while if $v_{n}(y)>0$, the fact that

$$
v_{n}(y)=v_{n}(\lambda y)=v_{n}\left(y^{\sigma}-x\right) \geq \min \left\{q v_{n}(y), v_{n}(x)\right\}
$$

implies that $v_{n}(y) \geq v_{n}(x)$, which also implies $q v_{n}(y) \geq v_{n}(x)$. Hence $y \in \Gamma_{\text {con }}$ and $w_{q r}(y) \geq w_{r}(x)$.

For $x \in \Gamma_{\text {an,con }}$ (with $K$ algebraically closed), choose $r>0$ such that $x \in \Gamma_{\mathrm{an}, r}$, and let $x=\sum_{i=-\infty}^{\infty} u_{i} \pi^{i}$ be a strong semi-unit decomposition. As above, there exists $y_{i} \in \Gamma_{\mathrm{an}, q r}$ with $y_{i}^{\sigma}-\lambda\left(\pi / \pi^{\sigma}\right)^{i} y_{i}=u_{i}\left(\pi / \pi^{\sigma}\right)^{i}$ such that $v_{n}\left(y_{i}\right)=\infty$ for $n<0$ and $w_{q r}\left(y_{i}\right) \geq w_{r}\left(u_{i}\right)$. This implies that $\sum_{i=-\infty}^{\infty} y_{i} \pi^{i}$ converges with respect to $|\cdot|_{s}$ for $0<s \leq r$; let $y$ be its Fréchet limit. Then

$$
\begin{aligned}
y^{\sigma}-\lambda y & =\sum_{i} y_{i}^{\sigma}\left(\pi^{i}\right)^{\sigma}-\lambda y_{i} \pi^{i} \\
& =\sum_{i} \lambda y_{i} \pi^{i}+u_{i} \pi^{i}-\lambda y_{i} \pi^{i} \\
& =\sum_{i} u_{i} \pi^{i}=x,
\end{aligned}
$$

so that $y$ is the desired solution.
To verify the last assertion, we may assume $k$ is algebraically closed and $\pi^{\sigma}=\pi$. Suppose $x \in \Gamma_{\mathrm{con}}\left[\frac{1}{p}\right]$ and $y \in \Gamma_{\text {an,con }}$ satisfy $y^{\sigma}-\lambda y=x$. By what we have shown above, there also exists $z \in \Gamma_{\operatorname{con}}\left[\frac{1}{p}\right]$ such that $z^{\sigma}-\lambda z=x$, so that $(y-z)^{\sigma}=\lambda(y-z)$. This equation yields $q v_{n}(y-z)=v_{n}(y-z)$ for all $n$, and so $v_{n}(y-z)=0$ or $\infty$ for all $n$. We cannot have $v_{n}(y-z)=0$ for all $n$, and so there is a smallest such $n$; we may assume $n=0$ without loss of generality. Then every solution $w$ of $w^{\sigma}=\lambda w$ in $\Gamma_{\mathrm{an}, \mathrm{con}}$ with $v_{n}(w)=\infty$ for $n<0$ is congruent to some element of $\mathcal{O}$ modulo $\pi$. In particular, we can find $c_{0}, c_{1}, \cdots \in \mathcal{O}$ such that $\sum_{j=0}^{l} c_{j} \pi^{j} \equiv y-z\left(\bmod \pi^{l+1}\right)$, since once $c_{0}, \ldots, c_{l}$ have been computed, we can take $w=(y-z) \pi^{-l-1}-\sum_{j=0}^{l} c_{j} \pi^{j-l-1}$, and there must be some $c_{l+1} \in \mathcal{O}$ congruent to $w$ modulo $\pi$. Thus $y-z \in \mathcal{O} \subseteq \Gamma_{\text {con }}\left[\frac{1}{p}\right]$, and so $y \in \Gamma_{\text {con }}\left[\frac{1}{p}\right]$.
(b) If $x \in \Gamma_{\text {con }}$, then the series

$$
\sum_{i=0}^{\infty} \lambda^{\sigma^{-1}} \cdots \lambda^{\sigma^{-i}} x^{\sigma^{-i-1}}
$$

converges $\pi$-adically to an element $y \in \Gamma$ satisfying

$$
\begin{aligned}
y^{\sigma}-\lambda y & =\sum_{i=0}^{\infty} \lambda \lambda^{\sigma^{-1}} \cdots \lambda^{\sigma^{-i+1}} x^{\sigma^{-i}}-\sum_{i=0}^{\infty} \lambda \lambda^{\sigma^{-1}} \cdots \lambda^{\sigma^{-i}} x^{\sigma^{-i-1}} \\
& =\sum_{i=0}^{\infty} \lambda \lambda^{\sigma^{-1}} \cdots \lambda^{\sigma^{-i+1}} x^{\sigma^{-i}}-\sum_{i=1}^{\infty} \lambda \lambda^{\sigma^{-1}} \cdots \lambda^{\sigma^{-i+1}} x^{\sigma^{-i}} \\
& =x .
\end{aligned}
$$

To see that in fact $y \in \Gamma_{\text {con }}$, choose $r>0$ and $c \leq 0$ such that $w_{r}(x) \geq c$; that is, $r v_{n}(x)+n \geq c$ for all $n \geq 0$. If $v_{n}(x) \leq 0$, then $r v_{n}\left(x^{\sigma^{-i}}\right)+n=$ $\left(r / q^{i}\right) v_{n}(x)+n \geq r v_{n}(x)+n \geq c$; if $v_{n}(x) \geq 0$, then $r v_{n}\left(x^{\sigma^{-i}}\right)+n \geq 0 \geq c$. In
any case, we have $w_{r}\left(x^{\sigma^{-i}}\right) \geq c$ for all $i$. Since $w_{r}\left(\lambda^{\sigma^{-i}}\right)=w_{r}(\lambda)>0$ for all $i$, we conclude that the series defining $y$ converges under $|\cdot|_{r}$, and so its limit $y$ in $\Gamma$ must actually lie in $\Gamma_{\text {con }}$.

Suppose now that $x \in \Gamma_{\text {an,con }}$; by Proposition 3.14, there exists a strong semi-unit decomposition $x=\sum_{n} \pi^{n} u_{n}$ of $x$. Let $N$ be the largest value of $n$ for which $v_{0}\left(u_{n}\right) \geq 0$, and put

$$
x_{-}=\sum_{n=-\infty}^{N} \pi^{n} u_{n}, \quad x_{+}=\sum_{n=N+1}^{\infty} \pi^{n} u_{n} .
$$

As above, we can construct $y_{+} \in \Gamma_{\text {an,con }}$ with $v_{n}\left(y_{+}\right)=\infty$ for $n$ sufficiently small, such that $y_{+}^{\sigma}-\lambda y_{+}=x_{+}$. As for $x_{-}$, let $m$ be the greatest integer less than or equal to $N$ for which $u_{m} \neq 0$. For any fixed $r, w_{r}\left(x_{-}^{\sigma^{i}}\right)=w_{r}\left(\left(u_{m} \pi^{m}\right)^{\sigma^{i}}\right)$ for $i$ sufficiently large. The series

$$
-\sum_{i=0}^{\infty}\left(\lambda \lambda^{\sigma} \cdots \lambda^{\sigma^{i}}\right)^{-1} x_{-}^{\sigma^{i}}
$$

then converges under $|\cdot|_{r}$, since

$$
\begin{aligned}
w_{r}\left(\left(\lambda \lambda^{\sigma} \cdots \lambda^{\sigma^{i}}\right)^{-1} x_{-}^{\sigma^{i}}\right) & =-(i+1) w_{r}(\lambda)+w_{r}\left(x_{-}^{\sigma^{i}}\right) \\
& =-(i+1) w_{r}(\lambda)+r q^{i} v_{0}\left(u_{m}\right)+m v_{p}(\pi)
\end{aligned}
$$

tends to infinity with $i$. Since this holds for every $r$, the series converges in $\Gamma_{\text {an,con }}$ to a limit $y_{-}$, which satisfies

$$
\begin{aligned}
y_{-}^{\sigma}-\lambda y_{-} & =-\sum_{i=0}^{\infty}\left(\lambda^{\sigma} \cdots \lambda^{\sigma^{i+1}}\right)^{-1} x_{-}^{\sigma^{i+1}}+\sum_{i=0}^{\infty}\left(\lambda^{\sigma} \cdots \lambda^{\sigma^{i}}\right)^{-1} x_{-}^{\sigma^{i}} \\
& =-\sum_{i=1}^{\infty}\left(\lambda^{\sigma} \cdots \lambda^{\sigma^{i}}\right)^{-1} x_{-}^{\sigma^{i}}+\sum_{i=0}^{\infty}\left(\lambda^{\sigma} \cdots \lambda^{\sigma^{i}}\right)^{-1} x_{-}^{\sigma^{i}} \\
& =x_{-} .
\end{aligned}
$$

We conclude that $y=y_{+}+y_{-}$satisfies $y^{\sigma}-\lambda y=x$.
To prove the final assertion, let $u$ be any strong semi-unit with $v_{0}(u)>0$, and set

$$
y=\sum_{i=0}^{\infty} \lambda^{\sigma^{-1}} \cdots \lambda^{\sigma^{-i}} u^{\sigma^{-i-1}}+\sum_{i=0}^{\infty}\left(\lambda \lambda^{\sigma} \cdots \lambda^{\sigma^{i}}\right)^{-1} u^{\sigma^{i}}
$$

then the above arguments show that both series converge and $y^{\sigma}-\lambda y=u-u$ $=0$.
(c) We prove the second assertion first. Namely, assume $x \in \Gamma_{\text {con }}$ and $y \in \Gamma_{\text {an,con }}$ satisfy $\lambda y^{\sigma}-y=x$; we show that $y \in \Gamma_{\text {con }}$. First suppose $0<$ $v_{n}(y)<\infty$ for some $n<0$. Then

$$
v_{n}(y)=v_{n}(y+x)=v_{n}\left(\lambda y^{\sigma}\right) \geq v_{n}\left(y^{\sigma}\right)=q v_{n}(y),
$$

a contradiction. Thus $v_{n}(y)$ is either nonpositive or $\infty$ for all $n<0$. We cannot have $v_{n}(y) \leq 0$ for all $n$, since for some $r>0$ we have $r v_{n}(y)+n \rightarrow \infty$ as $n \rightarrow-\infty$. Thus $v_{n}(y)=\infty$ for some $y$. (Beware: this is not enough $a$ priori to imply that $y \in \Gamma_{\text {con }}\left[\frac{1}{p}\right]$ if $K$ is infinite over $k((t))$.) Choose $n$ minimal such that $v_{n}(y)<\infty$. If $n<0$, then $v_{n}(y)=v_{n}(y+x)=v_{n}\left(\lambda y^{\sigma}\right)=\infty$, a contradiction. Thus $n \geq 0$. We can now show that $y$ is congruent modulo $\pi^{i}$ to an element of $\Gamma_{\text {con }}$, by induction on $i$. The base case $i=0$ is vacuous; given $y \equiv y_{i}\left(\bmod \pi^{i}\right)$ for $y_{i} \in \Gamma_{\mathrm{con}}$, we have

$$
y=-x+\lambda y^{\sigma} \equiv-x+\lambda y_{i}^{\sigma} \quad\left(\bmod \pi^{i+1}\right)
$$

Thus the induction follows. Since $y$ is the $\pi$-adic limit of elements of $\Gamma_{\text {con }}$, we conclude $y \in \Gamma_{\text {con }}$.

For the first assertion, suppose $x \in \Gamma_{\mathrm{an}, \text { con }}$ and $y_{1}, y_{2} \in \Gamma_{\mathrm{an}, \text { con }}$ satisfy $\lambda y_{i}^{\sigma}-y_{i}=x$ for $i=1,2$. Then $\lambda\left(y_{1}-y_{2}\right)^{\sigma}-\left(y_{1}-y_{2}\right)=0$; by the previous paragraph, this implies $v_{n}\left(y_{1}-y_{2}\right)=\infty$ for $n<0$. But then $v_{p}\left(y_{1}-y_{2}\right)=$ $v_{p}(\lambda)+v_{p}\left(\left(y_{1}-y_{2}\right)^{\sigma}\right)$, a contradiction unless $y_{1}-y_{2}=0$.
(d) Since $v_{n}(x) \geq 0$ for all $n$, we have $v_{n}\left(x^{\sigma^{i}}\right)=q^{i} v_{n}(x) \geq v_{n}(x)$ for all nonnegative integers $i$. Thus $w_{s}\left(x^{\sigma^{i}}\right) \geq w_{s}(x)$ for all $s$, so the series

$$
y=-\sum_{i=0}^{\infty} \lambda \lambda^{\sigma} \cdots \lambda^{\sigma^{i-1}} x^{\sigma^{i}}
$$

converges with respect to each of the norms $|\cdot|_{s}$, and

$$
\begin{aligned}
\lambda y^{\sigma}-y & =-\sum_{i=0}^{\infty} \lambda \lambda^{\sigma} \cdots \lambda^{\sigma^{i}} x^{\sigma^{i+1}}+\sum_{i=0}^{\infty} \lambda \lambda^{\sigma} \cdots \lambda^{\sigma^{i-1}} x^{\sigma^{i}} \\
& =-\sum_{i=1}^{\infty} \lambda \lambda^{\sigma} \cdots \lambda^{\sigma^{i-1}} x^{\sigma^{i}}+\sum_{i=0}^{\infty} \lambda \lambda^{\sigma} \cdots \lambda^{\sigma^{i-1}} x^{\sigma^{i}} \\
& =x,
\end{aligned}
$$

and so $y$ is the desired solution.
3.5. Factorizations over analytic rings. We assume that the valued field $K$ satisfies the conditions of Proposition 3.11, so that the ring $\Gamma_{\mathrm{an}, \mathrm{con}}=\Gamma_{\mathrm{an}, \mathrm{con}}^{K}$ is defined. As noted earlier, $\Gamma_{\text {an,con }}$ is not Noetherian even for $K=k((t))$, but in this case Lazard [L] proved that $\Gamma_{\mathrm{an}, \text { con }}$ is a Bézout ring, that is, a ring in which every finitely generated ideal is principal. In this section and the next, we generalize Lazard's result as follows.

Theorem 3.20. Suppose the conclusion of Proposition 3.11 is satisfied for the valued field $K$ and the positive number $r$. Then every finitely generated ideal in $\Gamma_{\mathrm{an}, r}=\Gamma_{\mathrm{an}, r}^{K}$ is principal. In particular, every finitely generated ideal in $\Gamma_{\mathrm{an}, \mathrm{con}}$ is principal.

Our approach resembles that of Lazard, with "pure elements" standing in for the divisors in his theory. The approach requires a number of auxiliary results on factorizations of elements of $\Gamma_{\mathrm{an}, \mathrm{con}}$; for the most part (specifically, excepting Section 6.1), only Theorem 3.20 will be used in the sequel, not the auxiliary results.

For $x \in \Gamma_{\mathrm{an}, r}$ nonzero, define the Newton polygon of $x$ as the lower convex hull of the set of points $\left(v_{n}(x), n\right)$, minus any segments of slopes less than $-r$ on the left end and/or any segments of nonnegative slope on the right end of the polygon; see Figure 1 for an example. Define the slopes of $x$ as the negatives of the slopes of the Newton polygon of $x$. (The negation is to ensure that the slopes of $x$ are positive.) Also define the multiplicity of a slope $s \in(0, r]$ of $x$ as the positive difference in $y$-coordinates between the endpoints of the segment of the Newton polygon of slope $-s$, or 0 if there is no such segment. If $x$ has only one slope $s$, we say $x$ is pure (of slope $s$ ). (Beware: this notion of slope differs from the slope of an eigenvector of a $\sigma$-module introduced in Section 2.5, and the Newton polygon here does not correspond to either the generic or special Newton polygons we define later.)


Figure 1: An example of a Newton polygon

Lemma 3.21. The multiplicity of $s$ as a slope of $x$ is equal to $s$ times the length (upper degree minus lower degree) of $L_{s}(x)$, where $L_{s}$ is the leading terms map in $\Gamma_{\mathrm{an}, s}$.

Proof. Let $\sum_{i} u_{i} \pi^{i}$ be a semi-unit decomposition of $x$. Let $S$ be the set of $l$ which achieve $\min _{l}\left\{w_{s}\left(u_{l} \pi^{l}\right)\right\}$, and let $i$ and $j$ be the smallest and largest elements of $S$; then $L_{s}(x)=\sum_{l \in S} \lambda\left(\overline{u_{l}}\right) t^{v_{0}\left(u_{l}\right)}$ and the length of $L_{s}(x)$ is equal to $v_{0}\left(u_{i}\right)-v_{0}\left(u_{j}\right)$.

We now show that the endpoints of the segment of the Newton polygon of $x$ of slope $-s$ are $\left(v_{0}\left(u_{i}\right), v_{p}\left(\pi^{i}\right)\right)$ and $\left(v_{0}\left(u_{j}\right), v_{p}\left(\pi^{j}\right)\right)$. First of all, for $n=v_{p}\left(\pi^{i}\right)$, we have $s v_{n}(x)+n=s v_{0}\left(u_{i}\right)+i v_{p}(\pi)=w_{s}\left(u_{i} \pi^{i}\right)$; likewise for $n=v_{p}\left(\pi^{j}\right)$. Next,
we note that $w_{s}(x) \geq \min _{l}\left\{w_{s}\left(u_{l} \pi^{l}\right)\right\}=w_{s}\left(u_{i} \pi^{i}\right)$. Thus for any $n, s v_{n}(x)+n \geq$ $w_{s}\left(u_{i} \pi^{i}\right)$; this means that the line through $\left(v_{0}\left(u_{i}\right), v_{p}\left(\pi^{i}\right)\right)$ and $\left(v_{0}\left(u_{j}\right), v_{p}\left(\pi^{j}\right)\right)$ is a lower supporting line for the set of points $\left(v_{n}(x), n\right)$. Finally, note that for $n<v_{p}\left(\pi^{i}\right)$,

$$
\begin{aligned}
s v_{n}(x)+n & \geq \min _{l<i}\left\{s v_{n}\left(\pi^{l} u_{l}\right)+n\right\} \\
& \geq \min _{l<i}\left\{w_{s}\left(\pi^{l} u_{l}\right)\right\} \\
& >w_{s}(x) ;
\end{aligned}
$$

while for $n>v_{p}\left(\pi^{j}\right)$,

$$
\begin{aligned}
s v_{n}(x)+n & \geq \min \left\{\min _{l \in[i, j]}\left\{s v_{n}\left(\pi^{l} u_{l}\right)+n\right\}, \min _{l \notin[i, j]}\left\{s v_{n}\left(\pi^{l} u_{l}\right)+n\right\}\right\} \\
& \geq \min \left\{\min _{l \in[i, j]}\left\{s v_{n}\left(\pi^{l} u_{l}\right)+n\right\}, \min _{l \notin[i, j]}\left\{w_{s}\left(\pi^{l} u_{l}\right)\right\}\right\} .
\end{aligned}
$$

For $l \in[i, j], n>v_{p}\left(\pi^{l}\right)$ and $u_{l}$ is a semi-unit, so that $s v_{n}\left(\pi^{l} u_{l}\right)+n>$ $w_{s}\left(\pi^{l} u_{l}\right) \geq w_{s}(x)$; for $l \notin[i, j], w_{s}\left(\pi^{l} u_{l}\right)>w_{s}(x)$ by the choice of $i$ and $j$. Putting the inequalities together, we again conclude $s v_{n}(x)+n>w_{s}(x)$.

Therefore the endpoints of the segment of the Newton polygon of $x$ of slope $-s$ are $\left(v_{0}\left(u_{i}\right), v_{p}\left(\pi^{i}\right)\right)$ and $\left(v_{0}\left(u_{j}\right), v_{p}\left(\pi^{j}\right)\right)$. Thus the multiplicity of $s$ as a slope of $x$ is $v_{p}\left(\pi^{j}\right)-v_{p}\left(\pi^{i}\right)=s\left(v_{0}\left(u_{i}\right)-v_{0}\left(u_{j}\right)\right)$, which is indeed $s$ times the length of $L_{s}(x)$, as claimed.

Corollary 3.22. Let $x$ and $y$ be nonzero elements of $\Gamma_{\mathrm{an}, r}$. Then the multiplicity of a slope $s$ of $x y$ is the sum of its multiplicities as a slope of $x$ and of $y$.

Proof. This follows immediately from the previous lemma and the multiplicity of the leading terms map $L_{s}$.

Corollary 3.23. The units of $\Gamma_{\mathrm{an}, \mathrm{con}}$ are precisely those $x \neq 0$ with $v_{n}(x)$
$=\infty$ for some $n$.
Proof. A unit of $\Gamma_{\text {an,con }}$ must also be a unit in $\Gamma_{\mathrm{an}, r}$ for some $r$, and a unit of $\Gamma_{\mathrm{an}, r}$ must have all slopes of multiplicity zero. (Remember, in $\Gamma_{\mathrm{an}, r}$, any slopes greater than $r$ are disregarded.) If $v_{n}(x)<\infty$ for all $n$, then $x$ has infinitely many different slopes, so it still has slopes of nonzero multiplicity in $\Gamma_{\mathrm{an}, r}$ for any $r$, and so can never become a unit.

We again caution that the condition $v_{n}(x)=\infty$ does not imply that $x \in \Gamma_{\text {con }}\left[\frac{1}{p}\right]$, if $K$ is not finite over $k((t))$.

It will be convenient to put elements of $x$ into a standard (multiplicative) form, and so we make a statement to this effect as a lemma.

Lemma 3.24. For any $x \in \Gamma_{\mathrm{an}, r}$ nonzero, there exists a unit $u \in \Gamma_{\mathrm{an}, r}$ such that ux admits a semi-unit decomposition $\sum_{i} u_{i} \pi^{i}$ with $u_{0}=1$ and $u_{i}=0$ for $i>0$. Moreover, for such $u$,
(a) $v_{0}(u x)=0$;
(b) $w_{r}(u x)=0$;
(c) $r v_{n}(u x)+n>0$ for $n>0$;
(d) the Newton polygon of ux begins at $(0,0)$.

Proof. By Proposition 3.13, we can find a semi-unit decomposition $\sum_{i} u_{i}^{\prime} \pi^{i}$ of $x$; then $u_{i}^{\prime}=0$ for $i$ sufficiently large. Choose the largest $j$ such that $u_{j}^{\prime} \neq 0$, and put $u=\pi^{-j}\left(u_{j}^{\prime}\right)^{-1}$. Then $u x$ admits the semi-unit decomposition $\sum_{i} u_{i} \pi^{i}$ with $u_{i}=u_{i+j}^{\prime} / u_{j}^{\prime}$, so that $u_{0}=1$ and $u_{i}=0$ for $i>0$.

To verify (a), note that $r v_{0}(u x) \geq \min _{i}\left\{r v_{0}\left(u_{i} \pi^{i}\right)\right\} \geq 0$, and the minimum is only achieved for $i=0$ : for $i<0, r v_{0}\left(u_{i} \pi^{i}\right)>w_{r}\left(u_{i} \pi^{i}\right) \geq 0$ since $\sum_{i} u_{i} \pi^{i}$ is a semi-unit decomposition of $u x$. Thus $r v_{0}(u x)=0$, whence (a).

To verify (b), note that $w_{r}(u x) \geq \min _{i}\left\{w_{r}\left(u_{i} \pi^{i}\right)\right\}=0$, whereas $w_{r}(u x) \leq$ $r v_{0}(u x)=0$ from (a).

To verify (c), note that for $n>0$ and $m=v_{p}\left(\pi^{i}\right), r v_{n}\left(u_{i} \pi^{i}\right)+n>$ $r v_{m}\left(u_{i} \pi^{i}\right)+m \geq w_{r}\left(u_{i} \pi^{i}\right) \geq 0$, so that $r v_{n}(u x)+n \geq \min _{i \leq 0}\left\{r v_{n}\left(u_{i} \pi^{i}\right)+n\right\}$ $>0$.

To verify (d), first note that the line through $(0,0)$ of slope $-r$ is a lower supporting line of the set of points $\left(v_{n}(u x), n\right)$, since $r v_{n}(u x)+n \geq w_{r}(u x) \geq 0$ for $n \leq 0$. Thus $(0,0)$ lies on the Newton polygon, and the slope of the segment of the Newton polygon just to the right of $(0,0)$ is at least $-r$. We also have $r v_{n}(u x)+n>0$ for $n>0$, so the slope of the segment of the Newton polygon just to the left of $(0,0)$, if there is one, must be less than $-r$. Thus the first segment of slope at least $-r$ does indeed begin at $(0,0)$, as desired.

The next lemma may be viewed as a version of the Weierstrass preparation theorem.

Lemma 3.25. Let $x$ be a nonzero element of $\Gamma_{\mathrm{an}, r}$ whose largest slope is $s_{1}$ with multiplicity $m>0$. Then there exists $y \in \Gamma_{\mathrm{an}, r}$, pure of slope $s_{1}$ with multiplicity $m$, which divides $x$.

Proof. If $x$ is pure of slope $s_{1}$, there is nothing to prove. So assume that $x$ is not pure, and let $s_{2}$ be the second largest slope of $x$.

By Lemma 3.24, there exists a unit $u \in \Gamma_{\mathrm{an}, r}$ such that $u x$ admits a semiunit decomposition $\sum_{i} u_{i} \pi^{i}$ with $u_{0}=1$ and $u_{i}=0$ for $i>0$, the slopes of $x$ and $u x$ occur with the same multiplicities, and the first segment of the Newton polygon of $u x$ has left endpoint $(0,0)$. Since that segment has slope $-s_{1}$ and
multiplicity $m$, its right endpoint is $\left(m / s_{1},-m\right)$. Put $M=-m / v_{p}(\pi)$; then $w_{s_{1}}\left(u_{M} \pi^{M}\right)=0$ and $w_{s_{1}}\left(u_{i} \pi^{i}\right)>0$ for $i<M$.

We first construct a sort of "Mittag-Leffler" decomposition of $u x$. Put $X=u x \pi^{-M} u_{M}^{-1}$, and set $y_{0}=z_{0}=1$. Given $y_{l}$ and $z_{l}$ for some $l$, let $\sum_{i} w_{i} \pi^{i}$ be a semi-unit decomposition of $X-y_{l} z_{l}$. Put

$$
\begin{aligned}
& y_{l+1}=y_{l}+\sum_{v_{0}\left(w_{i}\right)<0} w_{i} \pi^{i}, \\
& z_{l+1}=z_{l}+\sum_{v_{0}\left(w_{i}\right) \geq 0} w_{i} \pi^{i} .
\end{aligned}
$$

Given $s$ with $s_{2}<s<s_{1}$, put $c_{s}=w_{s}(X-1)$, so that $c_{s}>0$. We show that for each $l$, $w_{s}\left(y_{l}-1\right) \geq c_{s}, w_{s}\left(z_{l}-1\right) \geq c_{s}$, and $w_{s}\left(X-y_{l} z_{l}\right) \geq(l+1) c_{s}$. These inequalities are clear for $l=0$. If they hold for $l$, then

$$
\begin{aligned}
w_{s}\left(y_{l+1}-1\right) & \geq \min \left\{w_{s}\left(y_{l}-1\right), w_{s}\left(y_{l+1}-y_{l}\right)\right\} \\
& \geq \min \left\{c_{s},(l+1) c_{s}\right\}=c_{s}
\end{aligned}
$$

and similarly $w_{s}\left(z_{l+1}-1\right) \geq c_{s}$. As for the third inequality, note that

$$
\begin{aligned}
X-y_{l+1} z_{l+1} & =X-y_{l} z_{l}+y_{l}\left(z_{l}-z_{l+1}\right)+z_{l+1}\left(y_{l}-y_{l+1}\right) \\
& =\left(y_{l}-1\right)\left(z_{l}-z_{l+1}\right)+\left(z_{l+1}-1\right)\left(y_{l}-y_{l+1}\right)
\end{aligned}
$$

since $X-y_{l} z_{l}=\left(y_{l+1}-y_{l}\right)+\left(z_{l+1}-z_{l}\right)$. Since $w_{s}\left(y_{l}-y_{l+1}\right) \geq(l+1) c_{s}$ and $w_{s}\left(z_{l}-z_{l+1}\right) \geq(l+1) c_{s}$, we conclude that

$$
\begin{aligned}
w_{s}\left(X-y_{l+1} z_{l+1}\right) & \geq \min \left\{w_{s}\left(\left(y_{l}-1\right)\left(z_{l}-z_{l+1}\right)\right), w_{s}\left(\left(z_{l+1}-1\right)\left(y_{l}-y_{l+1}\right)\right)\right\} \\
& \geq \min \left\{c_{s}+(l+1) c_{s}, c_{s}+(l+1) c_{s}\right\} \\
& =(l+2) c_{s}
\end{aligned}
$$

as desired. This completes the induction.
We do not yet know that either $\left\{y_{l}\right\}$ or $\left\{z_{l}\right\}$ converges in $\Gamma_{\mathrm{an}, r}$; to get to that point, we need to play the two sequences off of each other. Suppose $s_{3}$ satisfies $s_{2}<s_{3}<s_{1}$. Note that to get from $y_{l}$ to $y_{l+1}$, we add terms of the form $w_{i} \pi^{i}$, with $w_{i}$ a semi-unit, for which $v_{0}\left(w_{i}\right)<0$ but $s v_{0}\left(w_{i}\right)+v_{p}\left(\pi^{i}\right) \geq(l+1) c_{s}$ for $s_{2}<s<s_{1}$. This implies that

$$
s v_{0}\left(w_{i}\right)+v_{p}\left(\pi^{i}\right) \geq(l+1) c_{s_{3}}
$$

for all $s \leq s_{3}$. In particular, $w_{s}\left(y_{l+1}-y_{l}\right) \rightarrow \infty$ as $l \rightarrow \infty$, so that $\left\{y_{l}\right\}$ converges to a limit $y$ in $\Gamma_{\mathrm{an}, s}$ for any $s \leq s_{3}$. Moreover, for $s \leq s_{3}$,

$$
\begin{aligned}
w_{s}\left(y_{l+1}-1\right) & \geq \min \left\{w_{s}\left(y_{l}-1\right), w_{s}\left(y_{l+1}-y_{l}\right)\right\} \\
& \geq \min \left\{w_{s}\left(y_{l}-1\right),(l+1) c_{s_{3}}\right\}
\end{aligned}
$$

and so by induction on $l, w_{s}\left(y_{l}-1\right) \geq c_{s_{3}}$. Hence $y$ and each of the $y_{l}$ are units in $\Gamma_{\text {an, }, s_{3}}$, for any $s_{3}<s_{1}$.

On the flip side, to get from $z_{l}$ to $z_{l+1}$, we add terms of the form $w_{i} \pi^{i}$, with $i$ a semi-unit, for which $v_{0}\left(w_{i}\right)>0$ but $s v_{0}\left(w_{i}\right)+v_{p}\left(\pi^{i}\right) \geq(l+1) c_{s}$ for $s_{2}<s<s_{1}$. This implies that $s v_{0}\left(w_{i}\right)+v_{p}\left(\pi^{i}\right) \geq(l+1) c_{s_{3}}$ for all $s \geq s_{3}$. As in the previous paragraph, we deduce $w_{s}\left(z_{l+1}-z_{l}\right) \rightarrow \infty$ and $w_{s}\left(z_{l}-1\right)>0$ for $s_{2}<s \leq r$.

Put $z=X y^{-1}$ in $\Gamma_{\mathrm{an}, s_{3}}$. Since $w_{s_{3}}\left(y_{l}\right)=0$ for all $l$,

$$
\begin{aligned}
w_{s_{3}}\left(z_{l}-z\right) & =w_{s_{3}}\left(y y_{l} z_{l}-y y_{l} z\right) \\
& =w_{s_{3}}\left(y\left(y_{l} z_{l}-X\right)+\left(y-y_{l}\right) X\right) \\
& \geq \min \left\{w_{s_{3}}\left(y\left(y_{l} z_{l}-X\right)\right), w_{s_{3}}\left(\left(y-y_{l}\right) X\right)\right\}
\end{aligned}
$$

and both terms in braces tend to infinity with $l$. Thus $z_{l} \rightarrow z$ under $|\cdot|_{s_{3}}$.
For $s_{3} \leq s \leq r$, since $s v_{n}\left(z_{l}-z\right)+n \geq(l+1) c_{s_{3}}$ and $s v_{n}\left(z_{l}\right)+n \rightarrow \infty$ as $n \rightarrow \pm \infty$, for any given $l$ we have $s v_{n}(z)+n \geq(l+1) c_{s_{3}}$ for all but finitely many $n$. Since this holds for any $l$, we have $s v_{n}(z)+n \rightarrow \infty$ as $n \rightarrow \pm \infty$. As we already have $z \in \Gamma_{\mathrm{an}, s_{3}}$, this is enough to imply $z \in \Gamma_{\mathrm{an}, r}$. Meanwhile, put

$$
a_{l}=X\left(1+(1-z)+\cdots+(1-z)^{l}\right)=y\left(1-(1-z)^{l+1}\right),
$$

so that $w_{s}\left(a_{l}-y\right)=(l+1) w_{s}(1-z)$ for $s_{2}<s<s_{1}$. In particular, for each $n$, $v_{n}\left(a_{m}-y\right) \rightarrow \infty$ as $m \rightarrow \infty$, and so the inequalities

$$
v_{n}\left(a_{l}-y\right) \geq \min \left\{v_{n}\left(a_{l}-a_{l+1}\right), \ldots, v_{n}\left(a_{m-1}-a_{m}\right), v_{n}\left(a_{m}-y\right)\right\}
$$

for each $m$ yield, in the limit as $m \rightarrow \infty$, the inequality

$$
v_{n}\left(a_{l}-y\right) \geq \min \left\{v_{n}\left(a_{l}-a_{l+1}\right), v_{n}\left(a_{l+1}-a_{l+2}\right), \ldots\right\} .
$$

Now $w_{s}\left(a_{l+1}-a_{l}\right)=w_{s}\left(X(1-z)^{l+1}\right)=w_{s}(X)+(l+1) w_{s}(1-z)$ for $s_{2}<s \leq r$, and so $s v_{n}\left(a_{l+1}-a_{l}\right)+n \geq w_{s}(X)+(l+1) w_{s}(1-z)$. We conclude that

$$
s v_{n}\left(a_{l}-y\right)+n \geq w_{s}(X)+(l+1) w_{s}(1-z),
$$

so that $s v_{n}(y)+n \geq w_{s}(X)+(l+1) w_{s}(1-z)$ for all but finitely many $n$. Therefore $s v_{n}(y)+n \rightarrow \infty$ as $n \rightarrow \pm \infty$ for $s_{2}<s \leq r$. Again, since we already have $y \in \Gamma_{\mathrm{an}, s_{3}}$, we deduce that $y \in \Gamma_{\mathrm{an}, r}$.

Since $y$ is a unit in $\Gamma_{\text {an,s }}$ for any $s<s_{1}$, it has no slopes less than $s_{1}$. Since $w_{s}(1-z)>0$ for $s_{2}<s \leq r, z$ has no slopes greater than $s_{2}$. Since the slopes of $y$ and $z$ together must comprise the slopes of $x, y$ must have $s_{1}$ as a slope with multiplicity $m$ and no other slopes, as desired.

A slope factorization of a nonzero element $x$ of $\Gamma_{\mathrm{an}, r}$ is a Fréchet-convergent product $x=\prod_{j=1}^{N} x_{j}$ for $N$ a positive integer or $\infty$, where each $x_{j}$ is pure and the slopes $s_{j}$ of $x_{j}$ satisfy $s_{1}>s_{2}>\cdots$.

Lemma 3.26. Every nonzero element of $\Gamma_{\mathrm{an}, r}$ has a slope factorization.

Proof. Let $x$ be a nonzero element of $\Gamma_{\text {an }, r}$ with slopes $s_{1}, s_{2}, \ldots$ By Lemma 3.25 , we can find $y_{1}$ pure of slope $s_{1}$ dividing $x$ such that $x / y_{1}$ has largest slope $s_{2}$. Likewise, we can find $y_{2}$ pure of slope $s_{2}$ such that $y_{2}$ divides $x / y_{1}, y_{3}$ pure of slope $s_{3}$ such that $y_{3}$ divides $x /\left(y_{1} y_{2}\right)$, and so on.

If there are $N<\infty$ slopes, then $x$ and $y_{1} \cdots y_{N}$ have the same slopes, so that $x /\left(y_{1} \cdots y_{N}\right)$ must be a unit $u$, and $x=\left(u y_{1}\right) y_{2} \cdots y_{N}$ is a slope factorization. Suppose instead there are infinitely many slopes; then $s_{i} \rightarrow 0$ as $i \rightarrow \infty$. By Lemma 3.24, for each $i$ we can find a unit $a_{i}$ such that $a_{i} y_{i}$ admits a semiunit decomposition $\sum_{j} u_{i j} \pi^{j}$ with $u_{i 0}=1$ and $u_{i j}=0$ for $j>0$. For $j<0$, $s v_{n}\left(u_{i j} \pi^{j}\right)+n$ is minimized for $n=v_{p}\left(\pi^{j}\right)<0$ because $u_{i j}$ is a semi-unit; for $i$ sufficiently large, $s \geq s_{i}$, so that

$$
\begin{aligned}
s v_{v_{p}\left(\pi^{j}\right)}\left(u_{i j} \pi^{j}\right)+v_{p}\left(\pi^{j}\right) & =\frac{s}{s_{i}}\left(s_{i} v_{v_{p}\left(\pi^{j}\right)}\left(u_{i j} \pi^{j}\right)+v_{p}\left(\pi^{j}\right)\right)+\left(\frac{s}{s_{i}}-1\right)\left(-v_{p}\left(\pi^{j}\right)\right) \\
& \geq\left(\frac{s}{s_{i}}-1\right) v_{p}(\pi)
\end{aligned}
$$

which tends to infinity as $i \rightarrow \infty$. Hence $w_{s}\left(a_{i} y_{i}-1\right) \rightarrow \infty$ as $i \rightarrow \infty$; if we put $z_{j}=\prod_{i=1}^{j} a_{i} y_{i}$, then $\left\{z_{j}\right\}$ converges to a limit $z$, and $\left\{x / z_{j}\right\}$ converges to a limit $u$, such that $u z=x$. The slopes of $z$ coincide with the slopes of $x$; so $u$ must be a unit, and $\left(u a_{1} y_{1}\right) \prod_{i>1}\left(a_{i} y_{i}\right)$ is a slope factorization of $x$.

Lemma 3.27. Let $x$ be an element of $\Gamma_{\mathrm{an}, r}$ which is pure of slope $s$ and multiplicity $m$. Then for every $y \in \Gamma_{\mathrm{an}, r}$, there exists $z \in \Gamma_{\mathrm{an}, r}$ such that:
(a) $y-z$ is divisible by $x$;
(b) $w_{s}(z) \geq w_{s}(y)$;
(c) $v_{n}(z)=\infty$ for $n<0$.

Proof. Put $M=m / v_{p}(\pi)$. By Lemma 3.24, there exists a unit $u \in \Gamma_{\mathrm{an}, r}$ such that $x u$ admits a semi-unit decomposition $\sum_{i=-M}^{0} x_{i} \pi^{i}$ with $x_{0}=1$ and $s v_{0}\left(x_{-M}\right)=m$. Note that

$$
w_{r}\left(x_{-M} \pi^{-M}\right)=r v_{-m}\left(x_{-M} \pi^{-M}\right)-m=m\left(\frac{r}{s}-1\right)
$$

Let $\sum_{i} y_{i} \pi^{i}$ be a semi-unit decomposition of $y$.
We define the sequence $\left\{c_{l}\right\}_{l=0}^{\infty}$ of elements of $\Gamma_{\mathrm{an}, r}$ such that $v_{n}\left(c_{l}\right)=\infty$ for $n<0, w_{r}\left(c_{l}\right) \geq-l\left(v_{p}(\pi)+m(r / s-1)\right), w_{s}\left(c_{l}\right) \geq-l v_{p}(\pi)$, and

$$
c_{l} \equiv \pi^{-l} \quad(\bmod x)
$$

Put $c_{0}=1$ to start. Given $c_{l}$, let $\sum_{i} u_{i} \pi^{i}$ be a semi-unit decomposition of $c_{l}$; since $v_{n}\left(c_{l}\right)=\infty$ for $n<0$, we have $u_{i}=0$ for $i<0$. Now set

$$
c_{l+1}=\pi^{-1}\left(c_{l}-u x x_{-M}^{-1} \pi^{M} u_{0}\right)
$$

The congruence $c_{l+1} \equiv \pi^{-1} c_{l} \equiv \pi^{-l-1}(\bmod x)$ is clear from the definition. Since $u x x_{-M}^{-1} \pi^{M} \equiv 1(\bmod \pi)$, the term in parentheses has positive valuation, and so $v_{n}\left(c_{l+1}\right)=\infty$ for $n<0$. Since $w_{s}(u x)=w_{s}\left(x_{-M} \pi^{-M}\right)=0$ and $w_{s}\left(u_{0}\right) \geq w_{s}\left(c_{l}\right)$, we have $w_{s}\left(c_{l+1}\right) \geq w_{s}\left(\pi^{-1} c_{l}\right) \geq-(l+1) v_{p}(\pi)$. Finally, $w_{r}\left(u_{0}\right) \geq w_{r}\left(c_{l}\right), w_{r}(u x)=0$ and $w_{r}\left(x_{-M} \pi^{-M}\right)=m(r / s-1)$, so that

$$
\begin{aligned}
w_{r}\left(c_{l+1}\right) & \geq w_{r}\left(\pi^{-1}\right)+\min \left\{w_{r}\left(c_{l}\right), w_{r}\left(u x x_{-M}^{-1} \pi^{M} u_{0}\right)\right\} \\
& \geq-v_{p}(\pi)+w_{r}\left(c_{l}\right)-m(r / s-1) \\
& \geq-(l+1)\left(m(r / s-1)+v_{p}(\pi)\right) .
\end{aligned}
$$

We wish to show that $\sum_{i=-\infty}^{-1} y_{i} c_{-i}$ converges, so that its limit is congruent to $\sum_{i=-\infty}^{-1} y_{i} \pi^{i}$ modulo $x$. To this end, choose $t>0$ large enough that

$$
\operatorname{tr}_{p}(\pi)>m(r / s-1)+v_{p}(\pi) .
$$

Then $(1 / t) v_{n}(y)+n \rightarrow \infty$ as $n \rightarrow-\infty$, and so in particular there exists $c>0$ such that $(1 / t) v_{n}(y) \geq-n-c$ for $n<0$. For $n=v_{p}\left(\pi^{i}\right)$ where $i<0$ and $y_{i} \neq 0$, we have $v_{n}(y)=v_{0}\left(y_{i}\right)$, and thus $v_{0}\left(y_{i}\right) \geq-t i v_{p}(\pi)-t c$. Then

$$
\begin{aligned}
w_{r}\left(y_{i} c_{-i}\right) & =w_{r}\left(y_{i}\right)+w_{r}\left(c_{-i}\right) \\
& =\operatorname{rv}\left(y_{0}\right)+w_{r}\left(c_{-i}\right) \\
& \geq-\operatorname{tri} v_{p}(\pi)-\operatorname{tr} c+i\left(m(r / s-1)+v_{p}(\pi)\right)
\end{aligned}
$$

which tends to infinity as $i \rightarrow-\infty$. Thus $\sum_{i=-\infty}^{-1} y_{i} c_{-i}$ converges under $|\cdot|_{r}$; since $v_{n}\left(y_{i} c_{-i}\right)=\infty$ for $n<0$, the sum also converges under $|\cdot|_{s}$ for $0<s<r$. Now, the sum has a limit $z^{\prime} \in \Gamma_{\mathrm{an}, r} ;$ put $z=z^{\prime}+\sum_{i=0}^{\infty} y_{i} \pi^{i}$. Then $y-z=$ $\sum_{i=-\infty}^{-1} y_{i}\left(\pi^{i}-c_{-i}\right)$; since each term in the sum is divisible by $x$, so is the sum. This verifies (a). To verify (b), note that $w_{s}\left(y_{i} c_{-i}\right) \geq w_{s}\left(y_{i} \pi^{i}\right)$ for $i<0$; so $w_{s}\left(z^{\prime}\right) \geq w_{s}(y)$, and clearly $w_{s}\left(z-z^{\prime}\right) \geq w_{s}(y)$, so that $w_{s}(z) \geq w_{s}(y)$. To verify (c), simply note that each term in the sum defining $z$ satisfies the same condition.
3.6. The Bézout property for analytic rings. Again, we assume that the valued field $K$ satisfies the conditions of Proposition 3.11, so that $\Gamma_{\mathrm{an}, \text { con }}=$ $\Gamma_{\mathrm{an}, \text { con }}^{K}$ is defined. With the factorization results of the previous section in hand, we now focus on establishing the Bézout property for $\Gamma_{\mathrm{an}, \mathrm{con}}$ (Theorem 3.20). We proceed by establishing principality of successively more general classes of finitely generated ideals, culminating in the desired result.

Lemma 3.28. Let $x$ and $y$ be elements of $\Gamma_{\mathrm{an}, r}$, each with finitely many slopes, and having no slopes in common. Then the ideal $(x, y)$ is the unit ideal.

Proof. We induct on the sum of the multiplicities of the slopes of $x$ and $y$; the case where either $x$ or $y$ has total multiplicity zero is vacuous, as then
$x$ or $y$ is a unit and so $(x, y)$ is the unit ideal. So we assume that both $x$ and $y$ have positive total multiplicity.

If $x$ is not pure, then by Lemma 3.26 it factors as $x_{1} x_{2}$, where $x_{1}$ is pure and $x_{2}$ is not a unit. By the induction hypothesis, the ideals $\left(x_{1}, y\right)$ and $\left(x_{2}, y\right)$ are the unit ideal; in other words, $x_{1}$ and $x_{2}$ have multiplicative inverses modulo $y$. In that case, so does $x=x_{1} x_{2}$, and so $(x, y)$ is the unit ideal. The same argument applies in case $y$ is not pure.

It thus remains to treat the case where $x$ and $y$ are both pure. Let $s$ and $t$ be the slopes of $x$ and $y$, and let $m$ and $n$ be the corresponding multiplicities. Put $M=m / v_{p}(\pi)$ and $N=n / v_{p}(\pi)$. Without loss of generality, we may assume $s<t$. By Lemma 3.24, we can find units $u$ and $v$ such that $u x$ and $v y$ admit semi-unit decompositions $u x=\sum_{i=-M}^{0} x_{i} \pi^{i}$ and $v y=\sum_{i=-N}^{0} y_{i} \pi^{i}$.

Put

$$
X=u x \pi^{M} x_{-M}^{-1}, \quad Y=v y \pi^{N} y_{-N}^{-1}, \quad z=X-Y .
$$

We can read off information about the Newton polygon of $z$ by comparing $w_{r}(X)$ with $w_{r}(Y)$; see Figure 2 for an illustration. (In both diagrams, the dashed lines have slope $-r$.) If $w_{r}(X)<w_{r}(Y)$ (left side of Figure 2), then the highest vertex of the lower convex hull of the set of points $\left(v_{l}(z), l\right)$ occurs at $\left(v_{m}(X), m\right)$ and the lowest vertex has positive $y$-coordinate. Moreover, the slope of the first segment of the lower convex hull is at least $-s$. Thus the sum of all multiplicities of $z$ is strictly less than $m$, and $y$ and $z$ have no common slopes, so the induction hypothesis implies that $(x, y)=(y, z)$ is the unit ideal.


Figure 2: The Newton polygons of $X=u x \pi^{M} x_{-M}^{-1}$ and $Y=v y \pi^{N} y_{-N}^{-1}$
If $w_{r}(X) \geq w_{r}(Y)$ (right side of Figure 2), then the highest vertex of the lower convex hull of the set of points $\left(v_{l}(z), l\right)$ occurs at $\left(v_{n}(Y), n\right)$ and the lowest vertex has positive $y$-coordinate. Moreover, $\left(v_{m}(X), m\right)$ is also a vertex of the lower convex hull, and the line joining it to $\left(v_{n}(Y), n\right)$ is a support line of the lower convex hull. Thus the segment joining the two points is a segment of
the lower convex hull, of slope less than $-t$; the remainder of the lower convex hull consists of segments of slope at least $-s$, of total multiplicity less than $m$. By Lemma $3.26, z$ factors as $z_{1} z_{2}$, where $z_{1}$ is pure of some slope greater than $t$, and $z_{2}$ has all slopes less than or equal to $s$ and total multiplicity less than $m$. By the induction hypothesis, $\left(x, z_{1}\right)$ and $\left(y, z_{2}\right)$ both equal the unit ideal. But $\left(y, z_{1}\right)=\left(x, z_{1}\right)$ since $z_{1}$ divides $z=u x \pi^{M} x_{-M}^{-1}-v y \pi^{N} y_{-N}^{-1}$; thus $\left(y, z_{1} z_{2}\right)=(y, z)=(x, y)$ is also equal to the unit ideal.

We conclude that the induction goes through for all $x$ and $y$. This completes the proof.

Lemma 3.29. Let $x$ and $y$ be elements of $\Gamma_{\mathrm{an}, r}$ with $x, y$ pure of the same slope $s$. Then $(x, y)$ is either the unit ideal or is generated by a pure element of slope s.

Proof. (Thanks to Olivier Brinon for reporting an error in a previous version of this proof.) Assume without loss of generality that $v_{p}(x)=v_{p}(y)=$ 0 . We induct on the multiplicity $m$ of $s$ as a slope of $x$; put $M=m / v_{p}(\pi)$.

Pick $r^{\prime}$ with $r<r^{\prime}<r_{0}$, so that the conclusion of Proposition 3.11 applies to $r^{\prime}$ as well as to $r$. Since $\Gamma_{r^{\prime}}\left[\frac{1}{p}\right]$ is dense in $\Gamma_{\text {an, }, r}$, we can find an element $z \in \Gamma_{r^{\prime}}$ with $w_{r}(z-x)>w_{r}(x)+(1-s / r) m$. Choose a semiunit decomposition $\sum_{i \geq 0} x_{i}^{\prime} \pi^{i}$ of $z$ in $\Gamma_{r^{\prime}}$, and put $x^{\prime}=\sum_{i=0}^{M} x_{i}^{\prime} \pi^{i}$; then $x^{\prime}$ is pure of slope $s$ and multiplicity $m$, and $\sum_{i=0}^{M} x_{i}^{\prime} \pi^{i}$ is a semiunit decomposition of $x^{\prime}$ in $\Gamma_{r^{\prime}}$. Put $c=w_{r}\left(x^{\prime}-x\right)-w_{r}(x)>0$.

Put $y_{0}=y$. Given $y_{l}$ such that $y_{l} \equiv y(\bmod x)$, if $y_{l}=0$, set $y_{l+1}=y_{l}$; otherwise, choose $y_{l}^{\prime} \in \Gamma_{r^{\prime}}$ with $w_{r}\left(y_{l}^{\prime}-y_{l}\right) \geq w_{r}\left(y_{l}\right)+c$. Put $y_{l, 0}^{\prime}=y_{l}^{\prime}$. Given $y_{l, n}^{\prime} \in \Gamma_{r^{\prime}}$ with $y_{l, n}^{\prime} \equiv y_{l}^{\prime}\left(\bmod x^{\prime}\right)$, if $y_{l, n}^{\prime}=0$ or $y_{l, n}^{\prime}$ has total multiplicity less than $m$, set $y_{l, n+1}^{\prime}=y_{l, n}^{\prime}$. Otherwise, choose a semiunit decomposition $\sum_{j} u_{j}^{\prime} \pi^{j}$ of $y_{l, n}^{\prime}$ in $\Gamma_{r^{\prime}}$, and put

$$
\begin{aligned}
y_{l, n+1}^{\prime} & =y_{l, n}^{\prime}-\sum_{j \geq M} u_{j}^{\prime} \pi^{j-M}\left(x_{M}^{\prime}\right)^{-1} x^{\prime} \\
& =\sum_{j<M} u_{j}^{\prime} \pi^{j}+\sum_{j \geq M} u_{j}^{\prime} \pi^{j}\left(1-\left(x_{M}^{\prime}\right)^{-1} \pi^{-M} x^{\prime}\right)
\end{aligned}
$$

so that $w_{r^{\prime}}\left(y_{l, n+1}^{\prime}-\sum_{j<M} u_{j}^{\prime} \pi^{j}\right) \geq\left(1-s / r^{\prime}\right) v_{p}(\pi)+w_{r^{\prime}}\left(y_{l, n}^{\prime}\right)$. In particular, if $w_{r^{\prime}}\left(u_{j}^{\prime} \pi^{j}\right)<w_{r^{\prime}}\left(y_{l, n}^{\prime}\right)+\left(1-s / r^{\prime}\right) v_{p}(\pi)$ for some $j<M$, then the Newton polygon of $y_{l, n+1}^{\prime}$ has a vertex at height $j v_{p}(\pi)$ that blocks the presence of a vertex at any height $\geq m$. In particular, by Lemma $3.21, y_{l, n+1}^{\prime}$ has total multiplicity less than $m$.

Consequently, either the sequence $\left\{y_{l, n}^{\prime}\right\}_{n=0}^{\infty}$ stabilizes, or $w_{r^{\prime}}\left(y_{l, n}^{\prime}\right) \rightarrow \infty$ as $n \rightarrow \infty$. Since $w_{r}(z) \geq\left(r / r^{\prime}\right) w_{r^{\prime}}(z)$ for any $z \in \Gamma_{r^{\prime}}$, we can choose $n$ such that one of the following is true:
(a) $w_{r}\left(y_{l, n}^{\prime}\right)<w_{r}\left(y_{l}\right)+c$ and $y_{l, n}^{\prime}=y_{l, n+1}^{\prime}$;
(b) $w_{r}\left(y_{l, n}^{\prime}\right) \geq w_{r}\left(y_{l}\right)+c$.

In either case, put $y_{l+1}=y_{l}+x\left(y_{l, n}^{\prime}-y_{l}^{\prime}\right) / x^{\prime}$. In case (a), the total multiplicity of $y_{l+1}$ is less than $m$ (by Lemma 3.21 as above). We may then apply Lemma 3.26 to factor $y_{l+1}=z_{1} z_{2}$, where $z_{1}$ has no slopes equal to $s$ and $z_{2}$ is either a unit or pure of slope $s$. By Lemma 3.28, $x$ is coprime to $z_{1}$, so $(x, y)=\left(x, y_{l+1}\right)=$ $\left(x, z_{2}\right)$. Since $z_{2}$ has multiplicity less than $m$, we may apply the induction hypothesis to prove the lemma in this case.

In case (b), repeat the construction; if we never land in case (a), then $w_{r}\left(y_{l}\right) \rightarrow \infty$ as $l \rightarrow \infty$. Since $v_{p}\left(y_{l}\right) \geq 0$ for each $l$, the sequence $\left\{y_{l}\right\}$ converges to zero in $\Gamma_{\mathrm{an}, r}$. We may thus take $z=\sum_{l=0}^{\infty}\left(y_{l}-y_{l+1}\right) / x$ to produce an element $z \in \Gamma_{\mathrm{an}, r}$ with $x z=y$. Thus the ideal $(x, y)$ is generated by $x$, proving the desired result.

Corollary 3.30. For $x, y \in \Gamma_{\mathrm{an}, r}$ with $x$ pure of slope $s$, the ideal $(x, y)$ is principal.

Proof. By Lemma 3.27, there exists $z \in \Gamma_{\mathrm{an}, r}$ such that $y-z$ is divisible by $x$ and $v_{n}(z)=\infty$ for $n<0$. Thus $z$ has only finitely many slopes. By Lemma 3.26, we can factor $z$ as $z_{1} z_{2}$, where $z_{1}$ is pure of slope $s$ and $z_{2}$ has no slopes equal to $s$. Then $\left(x, z_{2}\right)$ is the unit ideal, so that $(x, y)=(x, z)=\left(x, z_{1}\right)$, which is principal by Lemma 3.29.

Lemma 3.31 (Principal parts theorem). Let $s_{n}$ be a decreasing sequence of positive rationals with limit 0 , and suppose $x_{n} \in \Gamma_{\mathrm{an}, r}$ is pure of slope $s_{n}$ for all $n$. Then for any sequence $y_{n}$ of elements of $\Gamma_{\mathrm{an}, r}$, there exists $y \in \Gamma_{\mathrm{an}, r}$ such that $y \equiv y_{n}\left(\bmod x_{n}\right)$ for all $n$.

Proof. As in the proof of Lemma 3.26, we can replace each $x_{n}$ with itself times a unit, in such a way that $\prod_{n} x_{n}$ converges. Put $x=\prod_{n} x_{n}$ and $u_{n}=x / x_{n}$. By Lemma 3.29, $x_{n}$ is coprime to each of $x_{1}, \ldots, x_{n-1}$. By Corollary 3.30, the ideal $\left(x_{n}, \prod_{i>n} x_{i}\right)$ is principal, but if it were not the unit ideal, any generator would both be pure of slope $s_{n}$ and have all slopes less than $s_{n}$. Thus $x_{n}$ is coprime to $\prod_{i>n} x_{i}$, hence also to $u_{n}$.

We construct a sequence $\left\{z_{n}\right\}_{n=1}^{\infty}$ such that $u_{n} z_{n} \equiv y_{n}\left(\bmod x_{n}\right)$ and $\sum u_{n} z_{n}$ converges for the Fréchet topology; then we may set $y=\sum u_{n} z_{n}$ and be done. For the moment, fix $n$ and choose $v_{n}$ with $u_{n} v_{n} \equiv y_{n}\left(\bmod x_{n}\right)$.

For $s>s_{n}$, we have $\left|1-x_{n}\right|_{s}<1$, and so the sequence $c_{m}=-1-\left(1-x_{n}\right)-$ $\cdots-\left(1-x_{n}\right)^{m}$ is Cauchy for the norm $|\cdot|_{s}$, and $\left|1+c_{m} x_{n}\right|_{s}=\left|1-x_{n}\right|_{s}^{m+1} \rightarrow 0$ under $|\cdot|_{s}$. In particular, for any $\varepsilon>0$, there exists $m$ such that $\left|1+c_{m} x_{n}\right|_{s}<\varepsilon$ for $s_{n-1} \leq s \leq r$.

Now choose $\varepsilon_{n}>0$ such that $\varepsilon_{n}\left|u_{n} v_{n}\right|_{s}<1 / n$ for $s_{n-1} \leq s \leq r$ (with $n$ still fixed), choose $m$ as above for $\varepsilon=\varepsilon_{n}$, and put $z_{n}=v_{n}\left(1+c_{m} x_{n}\right)$. Then $u_{n} z_{n} \equiv u_{n} v_{n} \equiv y_{n}\left(\bmod x_{n}\right)$. Moreover, for any $s>0$, we have $s \geq s_{n-1}$ for sufficiently large $n$ since the $s_{n}$ tend to zero. Thus for $n$ sufficiently large,

$$
\begin{aligned}
\left|u_{n} z_{n}\right|_{s} & =\left|u_{n} v_{n}\left(1+c_{m} x_{n}\right)\right|_{s} \\
& <\varepsilon_{n}\left|u_{n} v_{n}\right|_{s}<1 / n
\end{aligned}
$$

Hence $\sum_{n} u_{n} z_{n}$ converges with respect to $|\cdot|_{s}$ for $0<s \leq r$, and its limit $y$ has the desired property.

At long last, we are ready to prove the generalization of Lazard's result, that $\Gamma_{\mathrm{an}, r}$ is a Bézout ring.

Proof of Theorem 3.20. By induction on the number of generators of the ideal, it suffices to prove that if $x, y \in \Gamma_{\mathrm{an}, r}$ are nonzero, then the ideal $(x, y)$ is principal.

Pick a slope factorization $\prod_{j} y_{j}$ of $y$. By Corollary 3.30, we can choose a generator $d_{j}$ of $\left(x, y_{j}\right)$ for each $j$, such that $d_{j}$ is either 1 or is pure of the same slope as $y_{j}$. As in the proof of Lemma 3.26 , we can choose the $d_{j}$ so that $\prod_{j} d_{j}$ converges. Since the $d_{j}$ are pairwise coprime by Lemma $3.28, x$ is divisible by the product of any finite subset of the $d_{j}$, and hence by $\prod_{j} d_{j}$.

Choose $a_{j}$ and $b_{j}$ such that $a_{j} x+b_{j} y_{j}=d_{j}$, and apply Lemma 3.31 to find $z$ such that $z \equiv a_{j} \prod_{k \neq j} d_{k}\left(\bmod y_{j}\right)$ for each $j$. Then $z x-\prod_{j} d_{j}$ is divisible by each $y_{j}$, so it is divisible by $y$, and so $\prod_{j} d_{j}$ generates the ideal $(x, y)$. Thus $(x, y)$ is principal and the proof is complete.

Corollary 3.32. For $K$ a finite extension of $k((t))$, the $\operatorname{ring} \Gamma_{r}^{K}\left[\frac{1}{p}\right]$ is a Bézout ring.

Proof. For $x, y \in \Gamma_{r}^{K}\left[\frac{1}{p}\right]$, Theorem 3.20 implies that the ideal $(x, y)$ becomes principal in $\Gamma_{\mathrm{an}, r}^{K}$. Let $d$ be a generator; then $d$ must have finite total multiplicity, and so belongs to $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$.

Put $x^{\prime}=x / d$ and $y^{\prime}=y / d$, so that $\left(x^{\prime}, y^{\prime}\right)$ becomes the unit ideal in $\Gamma_{\mathrm{an}, r}^{K}$. By Lemma 3.26, $x^{\prime}$ factors in $\Gamma_{\mathrm{an}, r}^{K}$ as $a_{1} \ldots a_{l}$, where each $a_{i}$ is pure. Since each of those factors has finite total multiplicity, each lies in $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$.

Since $\left(x^{\prime}, y^{\prime}\right)$ is the unit ideal in $\Gamma_{\mathrm{an}, r}^{K}$, so is $\left(a_{i}, y^{\prime}\right)$ for each $i$. That is, there exist $b_{i}$ and $c_{i}$ in $\Gamma_{\mathrm{an}, r}^{K}$ such that $a_{i} b_{i}+c_{i} y^{\prime}=1$. Since $a_{i}$ is pure, Lemma 3.27 implies that $c_{i} \equiv d_{i}\left(\bmod a_{i}\right)$ for some $d_{i}$ with finite total multiplicity, which thus belongs to $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$. Now $d_{i} y^{\prime} \equiv 1\left(\bmod a_{i}\right)$, and $e_{i}=\left(d_{i} y^{\prime}-1\right) / a_{i}$ has finite total multiplicity, so itself lies in $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$. We now have the relation $a_{i} e_{i}+d_{i} y^{\prime}=1$ within $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$, so that $\left(a_{i}, y^{\prime}\right)$ is the unit ideal in $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$. Since this is true for each $i,\left(x^{\prime}, y^{\prime}\right)$ is also the unit ideal and so $(x, y)=(d)$.

We conclude that any ideal generated by two elements is principal. By induction, this implies that $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ has the Bézout property.

One presumably has the same result if $K$ is perfect, but it does not follow formally from Theorem 3.20 , since $\Gamma_{r}^{K}$ is not Fréchet complete in $\Gamma_{\mathrm{an}, r}^{K}$. That is, an element of $\Gamma_{\mathrm{an}, r}^{K}$ of finite total multiplicity need not lie in $\Gamma_{r}^{K}$. So one must repeat the arguments used to prove Theorem 3.20 working within $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$; as we have no use for the result, we leave this to the reader.

## 4. The special Newton polygon

In this chapter, we construct a Newton polygon for $\sigma$-modules over $\Gamma_{\text {an,con }}$, the "special Newton polygon". (A quite similar construction has been given by Hartl and Pink [HP].) More precisely, we give a slope filtration over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\text {alg }}$ that, in case the $\sigma$-module is quasi-unipotent, is precisely the filtration that makes it quasi-unipotent. The special Newton polygon is a numerical invariant of this filtration.

Throughout this chapter, we assume $K$ is a valued field satisfying the condition of Proposition 3.11. The choice of $K$ will only be relevant once or twice, as most of the time we will be working with $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}=\Gamma_{\mathrm{an}, \mathrm{con}}^{k((t))^{\text {alg }}}$. When this is the case, we will also assume $k$ is algebraically closed and that $\pi^{\sigma}=\pi$.

We will use without further comment the facts that every element of $\Gamma_{\text {an, con }}^{\text {alg }}$ has a strong semi-unit decomposition (Proposition 3.14) and that $\Gamma_{\text {an,con }}$ and $\Gamma_{\mathrm{an}, \text { con }}^{\text {alg }}$ are Bézout rings (Theorem 3.20). In particular, any $\sigma$-module over $\Gamma_{\mathrm{an}, \text { con }}$ or $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ is free by Proposition 2.5 , and so admits a basis.
4.1. Properties of eigenvectors. Recall that we call a nonzero element $\mathbf{v}$ of a $\sigma$-module $M$ an eigenvector if there exists $\lambda \in \mathcal{O}\left[\frac{1}{p}\right]$ such that $F \mathbf{v}=\lambda \mathbf{v}$. Also recall that if $\mathbf{v}$ an eigenvector, the slope of $\mathbf{v}$ is defined to be $v_{p}(\lambda)$. (Beware: this differs from the notion of slope used in Section 3.5.) Our method of constructing the special Newton polygon of a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ is to exhibit a basis of eigenvectors after enlarging $\mathcal{O}$ suitably. Before proceeding, it behooves us to catalog some basic properties of eigenvectors of $\sigma$-modules over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$. Some of these assertions will also hold more generally over $\Gamma_{\mathrm{an}, \text { con }}$ (i.e., for arbitrary $K$ ), so we distinguish between $\Gamma_{\mathrm{an}, \text { con }}$ and $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ in the statements below.

For $M$ a $\sigma$-module over $\Gamma_{\text {an,con }}$, we say $\mathbf{v} \in M$ is primitive if $\mathbf{v}$ extends to a basis of $M$. By Lemma 2.3, if $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ is a basis of $M$ and $\mathbf{v}=\sum c_{i} \mathbf{e}_{i}$, then $\mathbf{v}$ is primitive if and only if the $c_{i}$ generate the unit ideal.

LEMMA 4.1. Let $M$ be a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$. Then every eigenvector of $M$ is a multiple of a primitive eigenvector.

Proof. Suppose $F \mathbf{v}=\lambda \mathbf{v}$. Choose a basis $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$, put $\mathbf{v}=\sum_{i} c_{i} \mathbf{e}_{i}$, and let $I$ be the ideal generated by the $c_{i}$. Then $I$ is invariant under $\sigma$ and $\sigma^{-1}$. By Theorem 3.20, $I$ is principal; if $r$ is a generator of $I$, then $r^{\sigma}$ is also a generator. Put $r^{\sigma}=c r$, with $c$ a unit, and write $c=\mu d$, with $\mu \in \mathcal{O}\left[\frac{1}{p}\right]$, $v_{0}(d)<\infty$ and $v_{n}(d)=\infty$ for $n<0$. By Proposition 3.18, there exists a unit $s \in \Gamma_{\text {an, con }}^{\text {alg }}$ such that $s^{\sigma}=d s$; then $(r / s)^{\sigma}=\mu(r / s)$. Therefore $\sum_{i} s\left(c_{i} / r\right) \mathbf{e}_{i}$ is a primitive eigenvector of $M$ of which $\mathbf{v}$ is a multiple, as desired.

A sort of converse to the previous statement is the following.
Proposition 4.2. For $M$ a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$, if $M$ contains an eigenvector of eigenvalue $\lambda \in \mathcal{O}\left[\frac{1}{p}\right]$, then it contains an eigenvector of eigenvalue $\lambda \mu$ for any $\mu \in \mathcal{O}$.

Proof. Let $\mathbf{v} \in M$ be an eigenvector with $F \mathbf{v}=\lambda \mathbf{v}$. If $\mu$ is a unit, there exists a unit $c \in \mathcal{O}$ such that $c^{\sigma}=\mu c$. If $\mu$ is not a unit, then by Proposition 3.19(b) there exists a nonzero $c \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ such that $c^{\sigma}=\mu c$. In either case, we have $F(c \mathbf{v})=c^{\sigma} \lambda \mathbf{v}=\lambda \mu(c \mathbf{v})$.

Proposition 4.3. Let $0 \rightarrow M_{1} \rightarrow M \rightarrow M_{2} \rightarrow 0$ be an exact sequence of $\sigma$-modules over $\Gamma_{\mathrm{an}, \text { con }}^{\mathrm{alg}}$. Assume $M_{1}$ and $M_{2}$ have bases $\mathbf{v}_{1}, \ldots, \mathbf{v}_{m}$ and $\mathbf{w}_{1}, \ldots, \mathbf{w}_{n}$ of eigenvectors such that the slope of $\mathbf{v}_{i}$ is less than or equal to the slope of $\mathbf{w}_{j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then the exact sequence splits over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$.

Proof. Choose a basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{m}, \mathbf{x}_{1}, \ldots, \mathbf{x}_{n}$ of $M$ such that $\mathbf{x}_{j}$ projects onto $\mathbf{w}_{j}$ in $M_{2}$ for $j=1, \ldots, n$. Suppose $F \mathbf{v}_{i}=\lambda_{i} \mathbf{v}_{i}$ for some $\lambda_{i} \in \mathcal{O}\left[\frac{1}{p}\right]$. Then $F \mathbf{x}_{j}=\mu_{j} \mathbf{x}_{j}+\sum_{i=1}^{m} A_{i j} \mathbf{v}_{i}$ for some $\mu_{j} \in \mathcal{O}\left[\frac{1}{p}\right]$ and $A_{i j} \in \Gamma_{\mathrm{an}, \text { con }}^{\mathrm{alg}}$. If $\mathbf{y}_{j}=\mathbf{x}_{j}+\sum_{i=1}^{m} c_{i j} \mathbf{v}_{i}$, then

$$
F \mathbf{y}_{j}=\mu_{j} \mathbf{y}_{j}+\sum_{i=1}^{m}\left(\lambda_{i} c_{i j}^{\sigma}-\mu_{j} c_{i j}+A_{i j}\right) \mathbf{v}_{i} .
$$

By Proposition 3.19(a) and (b), we can choose $c_{i j} \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ for each $i, j$ so that $\lambda_{i} c_{i j}^{\sigma}-\mu_{j} c_{i j}+A_{i j}=0$. For this choice, $\mathbf{v}_{1}, \ldots, \mathbf{v}_{m}, \mathbf{y}_{1}, \ldots, \mathbf{y}_{n}$ form a basis of eigenvectors, so that the exact sequence splits as desired.

Proposition 4.4. Let $M$ be a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}$ with a basis $\mathbf{w}_{1}, \ldots, \mathbf{w}_{n}$ such that $F \mathbf{w}_{i}=\mu_{i} \mathbf{w}_{i}+\sum_{j<i} A_{i j} \mathbf{w}_{j}$ for some $\mu_{i} \in \mathcal{O}\left[\frac{1}{p}\right]$ and $A_{i j} \in \Gamma_{\mathrm{an}, \mathrm{con}}$. Then any eigenvector of $M$ has slope at least $\min _{i}\left\{v_{p}\left(\mu_{i}\right)\right\}$.

Proof. Let $\mathbf{v}$ be any eigenvector of $M$, with $F \mathbf{v}=\lambda \mathbf{v}$. Write $\mathbf{v}=\sum_{i} b_{i} \mathbf{w}_{i}$ for some $b_{i} \in \Gamma_{\text {an,con }}$. Suppose that $v_{p}(\lambda)<v_{p}\left(\mu_{i}\right)$ for all $i$. Then

$$
\sum_{i} \lambda b_{i} \mathbf{w}_{i}=F \mathbf{v}=\sum_{i} b_{i}^{\sigma} \mu_{i} \mathbf{w}_{i}+\sum_{i} b_{i}^{\sigma} \sum_{j<i} A_{i j} \mathbf{w}_{j} .
$$

Comparing the coefficients of $\mathbf{w}_{n}$ yields $\lambda b_{n}=\mu_{n} b_{n}^{\sigma}$, which implies $b_{n}=0$ by Proposition 3.19(c). Then comparison of the coefficients of $\mathbf{w}_{n-1}$ yields $\lambda b_{n-1}=\mu_{n-1} b_{n-1}^{\sigma}$, so that $b_{n-1}=0$. Continuing in this fashion, we deduce $b_{1}=\cdots=b_{n}=0$, a contradiction. Thus $v_{p}(\lambda) \geq v_{p}\left(\mu_{i}\right)$ for some $i$, as desired.

Recall that a sequence $\left(a_{1}, \ldots, a_{n}\right)$ of real numbers is said to majorize another sequence $\left(b_{1}, \ldots, b_{n}\right)$ if $a_{1}+\cdots+a_{n}=b_{1}+\cdots+b_{n}$ and for $i=$ $1, \ldots, n-1$, the sum of the $i$ smallest of $a_{1}, \ldots, a_{n}$ is less than or equal to the sum of the $i$ smallest of $b_{1}, \ldots, b_{n}$. Note that two sequences majorize each other if and only if they are equal up to permutation.

Proposition 4.5. Let $M$ be a $\sigma$-module over $\Gamma_{\text {an con }}$ with a basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ of eigenvectors, with $F \mathbf{v}_{i}=\lambda_{i} \mathbf{v}_{i}$ for $\lambda_{i} \in \mathcal{O}\left[\frac{1}{p}\right]$. Let $\mathbf{w}_{1}, \ldots, \mathbf{w}_{n}$ be a basis of $M$ such that $F \mathbf{w}_{i}=\mu_{i} \mathbf{w}_{i}+\sum_{j<i} A_{i j} \mathbf{w}_{j}$ for some $\mu_{i} \in \mathcal{O}\left[\frac{1}{p}\right]$ and $A_{i j} \in \Gamma_{\mathrm{an}, \mathrm{con}}$. Then the sequence $v_{p}\left(\mu_{1}\right), \ldots, v_{p}\left(\mu_{n}\right)$ majorizes the sequence $v_{p}\left(\lambda_{1}\right), \ldots, v_{p}\left(\lambda_{n}\right)$.

Proof. Assume without loss of generality that $v_{p}\left(\lambda_{1}\right) \geq \cdots \geq v_{p}\left(\lambda_{n}\right)$. Note that $v_{p}\left(\mu_{1}\right)+\cdots+v_{p}\left(\mu_{n}\right)=v_{p}\left(\lambda_{1}\right)+\cdots+v_{p}\left(\lambda_{n}\right)$ since both are equal to the slopes of primitive eigenvectors of $\wedge^{n} M$. Note also that $\wedge^{i} M$ satisfies the conditions of Proposition 4.4 for all $i$, using the exterior products of the $\mathbf{w}_{j}$ as the basis and the corresponding products of the $\mu_{j}$ as the diagonal matrix entries. (More precisely, view the exterior products as being partially ordered by the sum of indices; any total ordering of the products refining this partial order satisfies the conditions of the proposition.) Since $\mathbf{v}_{n-i+1} \wedge \cdots \wedge \mathbf{v}_{n}$ is an eigenvector of $\wedge^{i} M$ of slope $v_{p}\left(\lambda_{n-i+1}\right)+\cdots+v_{p}\left(\lambda_{n}\right)$, by Proposition 4.4 this slope is greater than or equal to the smallest valuation of an $i$-term product of the $\mu_{j}$, i.e., the sum of the $i$ smallest of $v_{p}\left(\mu_{1}\right), \ldots, v_{p}\left(\mu_{n}\right)$. This is precisely the desired majorization.

Corollary 4.6. Let $M$ be a $\sigma$-module over $\Gamma_{\text {an,con }}$. If $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ and $\mathbf{w}_{1}, \ldots, \mathbf{w}_{n}$ are bases of $M$ such that $F \mathbf{v}_{i}=\lambda_{i} \mathbf{v}_{i}$ and $F \mathbf{w}_{i}=\mu_{i} \mathbf{w}_{i}$ for some $\lambda_{i}, \mu_{i} \in \mathcal{O}\left[\frac{1}{p}\right]$, then the sequences $v_{p}\left(\lambda_{1}\right), \ldots, v_{p}\left(\lambda_{n}\right)$ and $v_{p}\left(\mu_{1}\right), \ldots, v_{p}\left(\mu_{n}\right)$ are permutations of each other.

Finally, we observe that the existence of an eigenvector of a specified slope does not depend on what ring of scalars $\mathcal{O}$ is used, so long as the value group of $\mathcal{O}$ contains the desired slope.

Proposition 4.7. Let $M$ be a $\sigma$-module over $\Gamma_{\text {an, con }}^{\text {alg }}$. Suppose $\lambda \in \mathcal{O}\left[\frac{1}{p}\right]$ occurs as the eigenvalue of an eigenvector of $M \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$ for some finite extension $\mathcal{O}^{\prime}$ of $\mathcal{O}$. Then $\lambda$ occurs as the eigenvalue of an eigenvector of $M$.

Proof. Since $k$ here is algebraically closed, we can choose a basis $\mu_{1}, \ldots, \mu_{m}$ of $\mathcal{O}^{\prime}$ over $\mathcal{O}$ consisting of elements fixed by $\sigma$. (Namely, let $\pi^{\prime}$ be a uniformizer of $\mathcal{O}^{\prime}$ fixed by $\sigma$, and take $\mu_{i}=\left(\pi^{\prime}\right)^{i-1}$.) Given an eigenvector $\mathbf{v}$ over $\mathcal{O}^{\prime}\left[\frac{1}{p}\right]$ with $F \mathbf{v}=\lambda \mathbf{v}$, we can write $\mathbf{v}=\mu_{1} \mathbf{w}_{1}+\cdots+\mu_{m} \mathbf{w}_{m}$ for a unique choice of $\mathbf{w}_{1}, \ldots, \mathbf{w}_{m} \in M$. Now

$$
0=F \mathbf{v}-\lambda \mathbf{v}=\mu_{1}\left(F \mathbf{w}_{1}-\lambda \mathbf{w}_{1}\right)+\cdots+\mu_{m}\left(F \mathbf{w}_{m}-\mathbf{w}_{m}\right)
$$

Since the representation $0=\mu_{1}(0)+\cdots+\mu_{m}(0)$ is unique, we must have $F \mathbf{w}_{i}=\lambda \mathbf{w}_{i}$ for $i=1, \ldots, m$. Since $\mathbf{v}$ is nonzero, at least one of the $\mathbf{w}_{i}$ must be nonzero, and it provides the desired eigenvector within $M$.
4.2. Existence of eigenvectors. In this section, we prove that every $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ has an eigenvector.

Proposition 4.8. For every $\sigma$-module $M$ over $\Gamma_{\mathrm{an}, \text { con }}^{\mathrm{alg}}$, there exist $\lambda \in \mathcal{O}_{0}$ and $\mathbf{v} \in M$, both nonzero, such that $F \mathbf{v}=\lambda \mathbf{v}$.

Note that once this assertion is established for a single $\lambda$, it holds for all $\lambda \in \mathcal{O}$ of sufficiently high valuation by Proposition 3.19(b).

Proof. Let $v$ denote the valuation on $k((t))^{\text {alg }}$ normalized so that $v(t)=1$. Let $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ be a basis for $M$, and suppose $F \mathbf{e}_{i}=\sum_{j} A_{i j} \mathbf{e}_{j}$. Choose $r>0$ so that the entries of $A_{i j}$ all lie in $\Gamma_{\mathrm{an}, r}^{\mathrm{alg}}$, and let $c$ be an integer less than $\min \left\{w_{r}(A), w_{r}\left(\left(A^{-1}\right)^{\sigma^{-1}}\right)\right\}$. For $0<s \leq r$, we define the valuations $w_{s}$ on $M$ in terms of the basis $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$. That is, $w_{s}\left(\sum_{i} c_{i} \mathbf{e}_{i}\right)=\min _{i}\left\{w_{s}\left(c_{i}\right)\right\}$.

Notice that for $\lambda \in \mathcal{O}_{0}$ and $u$ a strong semi-unit,

$$
\begin{aligned}
v_{0}(u) \geq \frac{-c+v_{p}(\lambda)}{(q-1) r} & \Longleftrightarrow v_{0}(u) r \leq-v_{p}(\lambda)+v_{0}(u) q r+c \\
& \Longrightarrow w_{r}\left(u \mathbf{e}_{i}\right)<w_{r}\left(\lambda^{-1} F\left(u \mathbf{e}_{i}\right)\right), \\
v_{0}(u) \leq \frac{q c+q v_{p}(\lambda)}{(q-1) r} & \Longleftrightarrow v_{0}(u) r \leq v_{p}(\lambda)+v_{0}(u) r / q+c \\
& \Longrightarrow w_{r}\left(u \mathbf{e}_{i}\right)<w_{r}\left(\lambda F^{-1}\left(u \mathbf{e}_{i}\right)\right) .
\end{aligned}
$$

Choose $\lambda \in \mathcal{O}_{0}$ of large enough valuation so that $-c+v_{p}(\lambda)<q c+q v_{p}(\lambda)$, and let $d$ be a rational number such that $d(q-1) r \in\left(-c+v_{p}(\lambda), q c+q v_{p}(\lambda)\right)$.

Define functions $a, b, f: M \rightarrow M$ as follows. Given $\mathbf{w} \in M$, write $\mathbf{w}=$ $\sum_{i=1}^{n} z_{i} \mathbf{e}_{i}$, let $z_{i}=\sum_{m} \pi^{m} u_{i, m}$ be a strong semi-unit decomposition for each $i$,
let $x_{i}$ be the sum of $\pi^{m} u_{i, m}$ over all $m$ such that $v_{0}\left(u_{i, m}\right)<d$, and put $y_{i}=$ $z_{i}-x_{i}$. Put $a(\mathbf{w})=\sum_{i=1}^{n} x_{i} \mathbf{e}_{i}, b(\mathbf{w})=\sum_{i=1}^{n} y_{i} \mathbf{e}_{i}$, and

$$
f(\mathbf{w})=\lambda^{-1} b(\mathbf{w})-F^{-1} a(\mathbf{w}) .
$$

(Note: the definitions of $a, b, f$ depend on the choices of semi-unit decompositions above, but this does not cause any trouble.) From the inequalities tabulated above, we have

$$
w_{r}\left(\lambda F^{-1} a(\mathbf{w})\right) \geq w_{r}(a(\mathbf{w}))+\varepsilon, \quad w_{r}\left(\lambda^{-1} F b(\mathbf{w})\right) \geq w_{r}(b(\mathbf{w}))+\varepsilon
$$

for $\varepsilon=\min \left\{w_{r}(A), w_{r}\left(\left(A^{-1}\right)^{\sigma^{-1}}\right\}-c>0\right.$. Therefore

$$
\begin{aligned}
w_{r}(f(\mathbf{w})) & =w_{r}\left(\lambda^{-1} b(\mathbf{w})-F^{-1} a(\mathbf{w})\right) \\
& \geq w_{r}\left(\lambda^{-1} \mathbf{w}\right) \\
w_{r}(F(f(\mathbf{w}))-\lambda f(\mathbf{w})+\mathbf{w}) & =w_{r}\left(F \lambda^{-1} b(\mathbf{w})-a(\mathbf{w})-b(\mathbf{w})+\lambda F^{-1} a(\mathbf{w})+\mathbf{w}\right) \\
& =w_{r}\left(\lambda^{-1} F b(\mathbf{w})+\lambda F^{-1} a(\mathbf{w})\right) \\
& \geq w_{r}(\mathbf{w})+\varepsilon
\end{aligned}
$$

for all nonzero $\mathbf{w} \in M$.
Now define sequences $\left\{\mathbf{v}_{l}\right\}_{l=0}^{\infty}$ and $\left\{\mathbf{w}_{l}\right\}_{l=0}^{\infty}$ as follows. First choose $T \in$ $k((t))^{\text {alg }}$ of valuation $d$, and set

$$
\mathbf{v}_{0}=\lambda^{-1}[T] \mathbf{e}_{1}+\left[T^{1 / q}\right] F^{-1} \mathbf{e}_{1},
$$

where the brackets again denote Teichmüller lifts. Then define $\mathbf{v}_{l}$ and $\mathbf{w}_{l}$ recursively by the formulas

$$
\mathbf{w}_{l}=F \mathbf{v}_{l}-\lambda \mathbf{v}_{l}, \quad \mathbf{v}_{l+1}=\mathbf{v}_{l}+f\left(\mathbf{w}_{l}\right) .
$$

For each $l, \mathbf{v}_{l}$ is defined over $\Gamma_{\mathrm{an}, r q}^{\mathrm{alg}}$ and $\mathbf{w}_{l}$ is defined over $\Gamma_{\mathrm{an}, r}^{\mathrm{alg}}$. By the final remark of the previous paragraph, we have

$$
w_{r}\left(\mathbf{v}_{l+1}-\mathbf{v}_{l}\right)=w_{r}\left(f\left(\mathbf{w}_{l}\right)\right) \geq w_{r}\left(\lambda^{-1} \mathbf{w}_{l}\right)
$$

and

$$
\begin{aligned}
w_{r}\left(\mathbf{w}_{l}\right) & =w_{r}\left(F \mathbf{v}_{l}-\lambda \mathbf{v}_{l}\right) \\
& =w_{r}\left(F \mathbf{v}_{l-1}+F f\left(\mathbf{w}_{l-1}\right)-\lambda \mathbf{v}_{l-1}-\lambda f\left(\mathbf{w}_{l-1}\right)\right) \\
& =w_{r}\left(F f\left(\mathbf{w}_{l-1}\right)-\lambda f\left(\mathbf{w}_{l-1}\right)+\mathbf{w}_{l-1}\right) \\
& \geq w_{r}\left(\mathbf{w}_{l-1}\right)+\varepsilon .
\end{aligned}
$$

Thus $w_{r}\left(\mathbf{w}_{l}\right)$ is a strictly increasing function of $l$ that tends to $\infty$, and $w_{r}\left(\mathbf{v}_{l+1}-\right.$ $\left.\mathbf{v}_{l}\right)$ also tends to $\infty$ with $l$.

We claim that in the Fréchet topology, $\mathbf{w}_{l}$ converges to 0 and so $\mathbf{v}_{l}$ converges to a limit $\mathbf{v}$, from which it follows that $F \mathbf{v}-\lambda \mathbf{v}=\lim _{l \rightarrow \infty} \mathbf{w}_{l}=0$. We first show that $w_{s}\left(\lambda F^{-1} a\left(\mathbf{w}_{l}\right)\right) \rightarrow \infty$ as $l \rightarrow \infty$ for $0<s \leq q r$.

Let $a\left(\mathbf{w}_{l}\right)=\sum_{i, m} \pi^{m} a_{l, i, m} \mathbf{e}_{i}$ be a strong semi-unit decomposition, in which we must have $v_{0}\left(a_{l, i, m}\right)<d$ whenever $a_{l, i, m} \neq 0$. Then

$$
\begin{aligned}
w_{s}\left(\lambda F^{-1} a\left(\mathbf{w}_{l}\right)\right) \geq & w_{s}\left(\lambda\left(A^{-1}\right)^{\sigma^{-1}}\right)+w_{s}\left(a\left(\mathbf{w}_{l}\right)^{\sigma^{-1}}\right) \\
= & w_{s}\left(\lambda\left(A^{-1}\right)^{\sigma^{-1}}\right)+w_{s / q}\left(a\left(\mathbf{w}_{l}\right)\right) \\
= & w_{s}\left(\lambda\left(A^{-1}\right)^{\sigma^{-1}}\right)+\min _{i, m}\left\{m v_{p}(\pi)+(s / q) v_{0}\left(a_{l, i, m}\right)\right\} \\
\geq & w_{s}\left(\lambda\left(A^{-1}\right)^{\sigma^{-1}}\right)+\min _{i, m}\left\{m v_{p}(\pi)+r v_{0}\left(a_{l, i, m}\right)\right\} \\
& +\min _{i, m}\left\{(-r+s / q) v_{0}\left(a_{l, i, m}\right)\right\} \\
> & w_{s}\left(\lambda\left(A^{-1}\right)^{\sigma^{-1}}\right)+w_{r}\left(a\left(\mathbf{w}_{l}\right)\right)-(r-s / q) d .
\end{aligned}
$$

In particular, $w_{s}\left(\lambda F^{-1} a\left(\mathbf{w}_{l}\right)\right) \rightarrow \infty$ as $l \rightarrow \infty$.
We next show that $w_{s}\left(\lambda^{-1} F b\left(\mathbf{w}_{l}\right)\right) \rightarrow \infty$ as $l \rightarrow \infty$ for $0<s \leq r$. Let $b\left(\mathbf{w}_{l}\right)=\sum_{i, m} \pi^{m} b_{l, i, m} \mathbf{e}_{i}$ be a strong semi-unit decomposition, necessarily with $v_{0}\left(b_{l, i, m}\right) \geq d$ whenever $b_{l, i, m} \neq 0$. Then

$$
\begin{aligned}
w_{s}\left(\lambda^{-1} F b\left(\mathbf{w}_{l}\right)\right) & \geq w_{s}\left(\lambda^{-1} A\right)+w_{s}\left(b\left(\mathbf{w}_{l}\right)^{\sigma}\right) \\
& =w_{s}\left(\lambda^{-1} A\right)+w_{s q}\left(b\left(\mathbf{w}_{l}\right)\right) \\
& =w_{s}\left(\lambda^{-1} A\right)+\min _{i, m}\left\{m v_{p}(\pi)+s q v_{0}\left(b_{l, i, m}\right)\right\} .
\end{aligned}
$$

Choose $e>0$ large enough so that $s(q-1) e+w_{s}\left(\lambda^{-1} A\right)>0$. If $v_{0}\left(b_{l, i, m}\right)<e$, then

$$
\begin{aligned}
m v_{p}(\pi)+s q v_{0}\left(b_{l, i, m}\right) & =m v_{p}(\pi)+r v_{0}\left(b_{l, i, m}\right)+(s q-r) v_{0}\left(b_{l, i, m}\right) \\
& \geq w_{r}\left(b\left(\mathbf{w}_{l}\right)\right)+h,
\end{aligned}
$$

where $h=(s q-r) d$ if $s q-r \geq 0$ and $h=(s q-r) e$ if $s q-r<0$. If $v_{0}\left(b_{l, i, m}\right) \geq e$, then

$$
\begin{aligned}
m v_{p}(\pi)+s q v_{0}\left(b_{l, i, m}\right) & =m v_{p}(\pi)+s v_{0}\left(b_{l, i, m}\right)+s(q-1) v_{0}\left(b_{l, i, m}\right) \\
& \geq w_{s}\left(b\left(\mathbf{w}_{l}\right)\right)+s(q-1) e .
\end{aligned}
$$

Suppose $\lim \inf _{l \rightarrow \infty} w_{s}\left(b\left(\mathbf{w}_{l}\right)\right)<L$ for some $L<\infty$. For $l$ sufficiently large, we have $w_{s}\left(\lambda F^{-1} a\left(\mathbf{w}_{l}\right)\right) \geq L$ and $w_{r}\left(b\left(\mathbf{w}_{l}\right)\right) \geq L-h-w_{s}\left(\lambda^{-1} A\right)$; by the previous paragraph, this implies

$$
\begin{aligned}
w_{s}\left(b\left(\mathbf{w}_{l+1}\right)\right) & \geq w_{s}\left(\mathbf{w}_{l+1}\right) \\
& =w_{s}\left(F \mathbf{v}_{l+1}-\lambda \mathbf{v}_{l+1}\right) \\
& =w_{s}\left(F \mathbf{v}_{l}+F f\left(\mathbf{w}_{l}\right)-\lambda \mathbf{v}_{l}-\lambda f\left(\mathbf{w}_{l}\right)\right) \\
& =w_{s}\left(\mathbf{w}_{l}+F f\left(\mathbf{w}_{l}\right)-\lambda f\left(\mathbf{w}_{l}\right)\right) \\
a\left(\mathbf{w}_{l}\right) & =w_{s}\left(\lambda^{-1} F b\left(\mathbf{w}_{l}\right)+\lambda F^{-1} a\left(\mathbf{w}_{l}\right)\right) \\
& \geq \min \left\{w_{s}\left(\lambda^{-1} A\right)+w_{s}\left(b\left(\mathbf{w}_{l}\right)\right)+s(q-1) e, L\right\} .
\end{aligned}
$$

We first deduce from this inequality that $w_{s}\left(b\left(\mathbf{w}_{l}\right)\right)$ is bounded below: pick any $l$, choose $C<L$ such that $w_{s}\left(b\left(\mathbf{w}_{l}\right)\right)>C$, then note that $w_{s}\left(b\left(\mathbf{w}_{l+1}\right)\right) \geq$ $\min \left\{L, C+w_{s}\left(\lambda^{-1} A\right)+s(q-1) e\right\}>C$. If we put $M=\lim \inf w_{s}\left(b\left(\mathbf{w}_{l}\right)\right)$, we thus have $-\infty<M<L$. However, in the inequality above, the limit inferior of the left side is $M$, while the limit inferior of the smaller right side is $\min \left\{L, M+w_{s}\left(\lambda^{-1} A\right)+s(q-1) e\right\}>M$. This contradiction shows that no $L$ can exist as above, and so $w_{s}\left(b\left(\mathbf{w}_{l}\right)\right) \rightarrow \infty$ and $w_{s}\left(\lambda^{-1} F b\left(\mathbf{w}_{l}\right)\right) \rightarrow \infty$.

From $w_{s}\left(\lambda F^{-1} a\left(\mathbf{w}_{l}\right)\right) \rightarrow \infty$ for $0<s \leq q r$, and $w_{s}\left(\lambda^{-1} F b\left(\mathbf{w}_{l}\right)\right) \rightarrow \infty$ for $0<s \leq r$, we conclude that $w_{s}\left(a\left(\mathbf{w}_{l}\right)\right) \rightarrow \infty$ and $w_{s}\left(b\left(\mathbf{w}_{l}\right)\right) \rightarrow \infty$ for $0<s \leq r$. Thus $\mathbf{w}_{l}$ converges to 0 in the Fréchet topology, and $\mathbf{v}_{l}$ converges to a limit $\mathbf{v}$ satisfying $F \mathbf{v}=\lambda \mathbf{v}$.

Finally, we check that $\mathbf{v} \neq 0$. First note that $w_{r}\left(\lambda^{-1}[T] \mathbf{e}_{1}\right)=d r-v_{p}(\lambda)$, while

$$
\begin{aligned}
w_{r}\left(\mathbf{v}_{0}-\lambda^{-1}[T] \mathbf{e}_{1}\right) & =w_{r}\left(\left[T^{1 / q}\right] F^{-1} \mathbf{e}_{1}\right) \\
& \geq d r / q+c \\
& >d r-v_{p}(\lambda)
\end{aligned}
$$

by our choice of $d$. Therefore $w_{r}\left(\mathbf{v}_{0}\right)=d r-v_{p}(\lambda)$. On the other hand,

$$
\begin{aligned}
w_{r}\left(\lambda^{-1} \mathbf{w}_{0}\right) & =w_{r}\left(\lambda^{-1} F \mathbf{v}_{0}-\mathbf{v}_{0}\right) \\
& =w_{r}\left(\lambda^{-2} F[T] \mathbf{e}_{1}+\lambda^{-1}[T] \mathbf{e}_{1}-\lambda^{-1}[T] \mathbf{e}_{1}-\left[T^{1 / q}\right] F^{-1} \mathbf{e}_{1}\right) \\
& =w_{r}\left(\lambda^{-2} F[T] \mathbf{e}_{1}-\left[T^{1 / q}\right] F^{-1} \mathbf{e}_{1}\right) \\
& \geq \min \left\{r d q+c-2 v_{p}(\lambda), r d / q+c\right\}
\end{aligned}
$$

We have just checked that the second term in braces is greater than $d r-v_{p}(\lambda)=$ $w_{r}\left(\mathbf{v}_{0}\right)$. As for the first term,

$$
r d q+c-2 v_{p}(\lambda)-\left(d r-v_{p}(\lambda)\right)=d r(q-1)+c-v_{p}(\lambda)
$$

is positive, again by the choice of $d$. Therefore $w_{r}\left(\lambda^{-1} \mathbf{w}_{0}\right)>w_{r}\left(\mathbf{v}_{0}\right)$.
Since we showed earlier that $w_{r}\left(\mathbf{w}_{l}\right)$ is a strictly increasing function of $l$, we have $w_{r}\left(\lambda^{-1} \mathbf{w}_{l}\right) \geq w_{r}\left(\lambda^{-1} \mathbf{w}_{0}\right)$ for $l \geq 0$. We also showed earlier that $w_{r}\left(\mathbf{v}_{l+1}-\mathbf{v}_{l}\right) \geq w_{r}\left(\lambda^{-1} \mathbf{w}_{l}\right)$ for $l \geq 0$. Thus $w_{r}\left(\mathbf{v}_{l+1}-\mathbf{v}_{l}\right) \geq w_{r}\left(\lambda^{-1} \mathbf{w}_{0}\right)$
for each $l$, and so $w_{r}\left(\mathbf{v}_{l}-\mathbf{v}_{0}\right) \geq w_{r}\left(\lambda^{-1} \mathbf{w}_{0}\right)$. It follows that $w_{r}\left(\mathbf{v}-\mathbf{v}_{0}\right) \geq$ $w_{r}\left(\lambda^{-1} \mathbf{w}_{0}\right)>w_{r}\left(\mathbf{v}_{0}\right)$; in particular, $\mathbf{v} \neq 0$, so that $\lambda$ and $\mathbf{v}$ satisfy the desired conditions.

Corollary 4.9. Every $\sigma$-module $M$ over $\Gamma_{\mathrm{an}, \text { con }}^{\mathrm{alg}}$ admits a basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ such that $\mathbf{v}_{i}$ is an eigenvector in $M / \operatorname{SatSpan}\left(\mathbf{v}_{1}, \ldots, \mathbf{v}_{i-1}\right)$ for $i=1, \ldots, n$.

Proof. By the proposition and Lemma 4.1, every $\sigma$-module over $\Gamma_{\mathrm{an} \text {,con }}^{\mathrm{alg}}$ contains a primitive eigenvector. The corollary now follows by induction on the rank of $M$.

Corollary 4.10. The set of slopes of eigenvectors of $M$, over all finite extensions of $\mathcal{O}$, is bounded below.

Proof. Combine the previous corollary with Proposition 4.4.
4.3. Raising the Newton polygon. In the previous section, we produced within any $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ a basis on which $F$ acts by a triangular matrix. By Proposition 4.5, if there is a basis of eigenvectors, the valuations of the diagonal entries of this matrix majorize the slopes of the eigenvectors. Thus to produce a basis of eigenvectors, we need to "raise the Newton polygon", i.e., find eigenvectors whose eigenvalues have smaller slopes than the ones we started with. In this section, we carry this process out by direct computation in an important special case; the general process, using this case in some basic steps, will follow in the next section.

By a Puiseux polynomial over a field $K$, we shall mean a formal expression of the form

$$
P(z)=\sum_{i \in I} c_{i} z^{i}
$$

where $I$ is a finite set of nonnegative rationals and $c_{i} \in K$ for each $i \in I$. If $K$ has a valuation $v_{K}$, we define the Newton polygon of a Puiseux polynomial, by analogy with the definition for an ordinary polynomial, as the lower convex hull of the set of points $\left(-i, v_{K}\left(c_{i}\right)\right)$. In fact, for some integer $n, P\left(z^{n}\right)$ is an ordinary polynomial; by comparing the Newton polygons of $P(z)$ and $P\left(z^{n}\right)$, and using the usual theory of Newton polygons of polynomials over fields complete with respect to a valuation, we obtain the following result.

Lemma 4.11. Let $P(z)$ be a Puiseux polynomial over the $t$-adic completion of $k((t))^{\text {alg }}$. Then $P$ has a root of valuation $l$ if and only if the Newton polygon of $P$ has a segment of slope $l$.

For $x \in \Gamma_{\mathrm{an}, \text { con }}^{\text {alg }}$ a strong semi-unit, we refer to $v_{0}(x)$ as the valuation of $x$.

LEMMA 4.12. Let $n$ be a positive integer, and let $x=\sum_{i=0}^{n} u_{i} \pi^{i}$ for some strong semi-units $u_{i} \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ of negative (or infinite) valuation, not all zero. Then the system of equations

$$
\begin{equation*}
a^{\sigma}=\pi a, \quad \pi b^{\sigma^{n}}=b-a x \tag{1}
\end{equation*}
$$

has a solution with $a, b \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ not both zero.
Proof. For $i \in\{0, \ldots, n\}$ for which $u_{i} \neq 0, l \in \mathbb{Z}$ and $m \in \mathbb{R}^{+}$, put

$$
f(i, l, m)=\left(v_{0}\left(u_{i}\right)+m q^{-l}\right) q^{-n(i+l)}
$$

Note that for fixed $i$ and $m, f(i, l, m)$ approaches 0 from below as $l \rightarrow+\infty$, and tends to $+\infty$ as $l \rightarrow-\infty$. Thus the minimum $h(m)=\min _{i, l}\{f(i, l, m)\}$ is welldefined. Observe that the map $h: \mathbb{R}^{+} \rightarrow \mathbb{R}$ is continuous and piecewise linear with everywhere positive slope, and $h(q m)=q^{-n} h(m)$ because $f(i, l+1, q m)=$ $q^{-n} f(i, l, m)$. Since $f(i, l, m)$ takes negative values for fixed $i, l$ and small $m$, $h(m)<0$ for some $m$, implying $h\left(q^{j} m\right)<0$ for all $j \in \mathbb{Z}$, so that $h$ takes only negative values. We conclude that $h$ is a continuous increasing bijection of $\mathbb{R}^{+}$ onto $\mathbb{R}^{-}$.

Pick $t \in \mathbb{R}^{+}$at which $h$ changes slope, let $S$ be the finite set of ordered pairs $(i, l)$ for which $f(i, l, t)<q^{-n} h(t)$, and let $T$ be the set of ordered pairs $(i, l)$ for which $f(i, l, t)<0$; then $T$ is infinite (and contains $S$ ), but the values of $l$ for pairs $(i, l) \in T$ are bounded below. For each pair $(i, l) \in T$, put $s(i, l)=\left\lfloor\log _{q^{n}}(h(t) / f(i, l, t))\right\rfloor$. This function has the following properties:
(a) $s(i, l) \geq 0$ for all $(i, l) \in T$;
(b) $f(i, l, t) q^{n s(i, l)} \in\left[h(t), q^{-n} h(t)\right)$ for all $(i, l) \in T$;
(c) $(i, l) \in S$ if and only if $(i, l) \in T$ and $s(i, l)=0$;
(d) for any $e>0$, there are only finitely many pairs $(i, l) \in T$ such that $s(i, l) \leq e$.

For $c \in \mathbb{R}$, let $U_{c}$ be the set of $z \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ such that $v_{m}(z)=\infty$ for $m<0$ and $v_{m}(z) \geq c$ for $m \geq 0$. Then the function

$$
r(z)=\sum_{(i, l) \in T} \pi^{s(i, l)} u_{i}^{\sigma^{-n i-n l+n s(i, l)}} z^{\sigma^{-n i-(n+1) l+n s(i, l)}}
$$

is well-defined (by (d) above, the series is $\pi$-adically convergent) and carries $U_{t}$ into $U_{h(t)}$ because for $z \in U_{t}$ and $m \geq 0$,

$$
\begin{aligned}
v_{m}\left(u_{i}^{\sigma^{-n i-n l+n s(i, l)}} z^{\left.\sigma^{-n i-(n+1) l+n s(i, l)}\right) \geq}\right. & q^{-n i-n l+n s(i, l)} v_{0}\left(u_{i}\right) \\
& +\min _{j}\left\{q^{-n i-(n+1) l+n(s, i, l)} v_{j}(z)\right\} \\
\geq & q^{n s(i, l)} q^{-n(i+l)}\left(v_{0}\left(u_{i}\right)+t q^{-l}\right) \\
= & q^{n s(i, l)} f(i, l, t) \\
\geq & h(t) .
\end{aligned}
$$

The reduction of $r(z)$ modulo $\pi$ is congruent to a finite sum over pairs $(i, l) \in S$, so it is a Puiseux polynomial in the reduction of $z$. Since $s(i, l)=0$ for all $(i, l) \in S$ and the values $-n i-(n+1) l$ are all distinct (because $i$ only runs over $\{0, \ldots, n\}$ ), we get a distinct monomial modulo $\pi$ for each pair $(i, l) \in S$.

We now consider the Newton polygon of the Puiseux polynomial given by the reduction of $r(z)-w$, for $w \in U_{h(t)}$. It is the convex hull of the points $\left(-q^{-n i-(n+1) l}, v_{0}\left(u_{i}\right) q^{-n i-n l}\right)$ for each $(i, l) \in S$, together with $\left(0, v_{0}(w)\right)$. The line $y=t x+h(t)$ either passes through or lies below the point corresponding to $(i, l)$, depending on whether $f(i, l, t)$ is equal to or strictly greater than $h(t)$. Moreover, $\left(0, v_{0}(w)\right)$ lies on or above the line because $v_{0}(w) \geq h(t)$. Since $h$ changes slope at $t$, there must be at least two points on the line; therefore the Newton polygon has a segment of slope $t$. By Lemma 4.11, the Puiseux polynomial has a root of valuation $t$. In other words, there exists $z \in U_{t}$ with $v_{0}(z)=t$ such that $r(z) \equiv w(\bmod \pi)$.

As a consequence of the above reasoning, we see that the image of $U_{t}$ is dense in $U_{f(t)}$ with respect to the $\pi$-adic topology. Since $U_{t}$ is complete, $U_{t}$ must surject onto $U_{f(t)}$. Moreover, we can take $w=0$ and obtain $z_{0} \in U_{t}$ with $v_{0}\left(z_{0}\right)=t$ such that $r\left(z_{0}\right) \equiv 0(\bmod \pi)$; in particular, $z_{0}$ is nonzero modulo $\pi$. We may then obtain $z_{1} \in U_{t}$ such that $r\left(z_{1}\right)=r\left(z_{0}\right) / \pi$. Put $z=z_{0}-\pi z_{1}$; then $z \not \equiv 0(\bmod \pi)$ and so is nonzero, but $r(z)=0$.

Now set $a=\sum_{l=-\infty}^{\infty} \pi^{l} z^{\sigma^{-l}}$; the sum converges in $\Gamma_{\mathrm{an}, \text { con }}^{\mathrm{alg}}$ because for $s>0, w_{s}\left(\pi^{l} z^{\sigma^{-l}}\right) \geq l v_{p}(\pi)+r q^{-l} t$ and the latter tends to $\infty$ as $l \rightarrow \pm \infty$ (because $t>0$ ). Then

$$
\begin{aligned}
a x & =\sum_{i=0}^{n} \sum_{l=-\infty}^{\infty} \pi^{i+l} u_{i} z^{\sigma^{-l}} \\
& =\sum_{(i, l) \in T} \pi^{i+l} u_{i} z^{\sigma^{-l}}+\sum_{(i, l) \notin T} \pi^{i+l} u_{i} z^{\sigma^{-l}} .
\end{aligned}
$$

Let $A$ and $B$ denote the two sums in the last line; then $v_{m}(B) \geq 0$ for all $m$, so by Proposition 3.19(d) (with $\sigma$ replaced by $\sigma^{n}$ ), $B$ can be written as $\pi b_{1}^{\sigma^{n}}-b_{1}$ for some $b_{1} \in \Gamma_{\text {an,con }}^{\text {alg }}$. On the other hand, we claim that $A$ can be rewritten as
$r(z)+\pi b_{2}^{\sigma^{n}}-b_{2}$ for

$$
\begin{aligned}
b_{2} & =\sum_{(i, l) \in T} \sum_{j=1}^{i+l-s(i, l)} \pi^{i+l-j} u_{i}^{\sigma^{-n j}} z^{\sigma^{-l-n j}} \\
& =\sum_{(i, l) \in T} \sum_{k=0}^{i+l-s(i, l)-1} \pi^{k+s(i, l)}\left(u_{i}^{\sigma^{-n i-n l+n s(i, l)}} z^{\sigma^{-n i-(n+1) l+n s(i, l)}}\right)^{\sigma^{n k}}
\end{aligned}
$$

(via the substitution $k=i+l-s(i, l)-j$ ); we must check that this series converges $\pi$-adically and that its limit is overconvergent. Note that as $l \rightarrow+\infty$ for $i$ fixed, $f(i, l, m)$ is asymptotic to $v_{0}\left(u_{i}\right) q^{-n(i+l)}$. Therefore $i+l-s(i, l)$ is bounded, so that the possible values of $k$ are uniformly bounded over all pairs $(i, l) \in T$. This implies on one hand that the series converges $\pi$-adically (since $l$ is bounded below over pairs $(i, l) \in T$ and $s(i, l) \rightarrow+\infty$ as $l \rightarrow+\infty)$, and on the other hand that $v_{m}\left(b_{2}\right)$ is bounded below uniformly in $m$ (since the quantity in parentheses in the second sum belongs to $\left.U_{h(t)}\right)$, so that $b_{2} \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$.

Having shown that the series defining $b_{2}$ converges, we can now verify that $b_{2}-\pi b_{2}^{\sigma^{n}}=r(z)-A$ : the quantity on the left is the sum over pairs $(i, l) \in T$ of a sum over $k$ which telescopes, leaving the term $k=0$ minus the term $k=i+l-s(i, l)$, or

$$
\pi^{s(i, l)} u_{i}^{\sigma^{-n i-n l+n s(i, l)}} z^{\sigma^{-n i-(n+1) l+n s(i, l)}}-\pi^{i+l} u_{i} z^{\sigma^{-l}}
$$

which when summed over pairs $(i, l) \in T$ yields $r(z)-A$.
Since $r(z)=0$ by construction, we have $\pi b^{\sigma^{n}}=b-a x$ for $b=-\left(b_{1}+b_{2}\right)$. Thus $(a, b)$ constitutes a solution of (1), as desired.

We apply the previous construction to study the system of equations

$$
\begin{equation*}
a^{\sigma}=\pi a, \quad \pi b^{\sigma^{n}}=b-a c \tag{2}
\end{equation*}
$$

where $c \in \Gamma_{\text {an,con }}^{\text {alg }}$ is given. Notice that replacing $c$ by $c+\pi^{n+1} y^{\sigma^{n}}-y$ does not alter whether (2) has a solution: for any $a$ such that $a^{\sigma}=\pi a$, if $\pi b^{\sigma^{n}}=b-a c$, then

$$
\pi(b-a y)^{\sigma^{n}}=(b-a y)-a\left(c+\pi^{n+1} y^{\sigma^{n}}-y\right)
$$

We begin by analyzing (2) in a restricted case.
Lemma 4.13. For any positive integer $n$ and any $c \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ such that $v_{m}(c) \geq-1$ for all $m$ and $v_{m}(c)=\infty$ for some $m$, there exist $a, b \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ not both zero, satisfying (2).

Proof. By multiplying $c$ by a power of $\pi$, we may reduce to the case where $v_{m}(c)=\infty$ for $m<0$. Define the sequences $c_{0}, c_{1}, \ldots$ and $d_{0}, d_{1}, \ldots$ as far as is possible by the following iteration. First put $c_{0}=c$ and $d_{0}=0$. Given $c_{i}$, if
$v_{0}\left(c_{i}\right)<-1 / q^{n}$, stop. Otherwise, let $d_{i}$ be a strong semi-unit congruent to $c_{i}$ modulo $\pi$ and put $c_{i+1}=\left(c_{i}+\pi^{n+1} d_{i}^{\sigma^{n}}-d_{i}\right) / \pi$. Note that $v_{m}\left(c_{i}\right) \geq-1$ and $v_{m}\left(d_{i}\right) \geq-1 / q^{n}$ for all $m \geq 0$ and all $i$.

If the iteration never terminates, then we have $c+\pi^{n+1} d^{\sigma^{n}}-d=0$ for $d=\sum_{i=0}^{\infty} d_{i} \pi^{i}$. In this case, apply Proposition $3.19(\mathrm{~b})$ to produce $a$ nonzero such that $a^{\sigma}=\pi a$ and set $b=a d$ to obtain a solution to (2).

If the iteration terminates at $c_{l}$, set $d=\sum_{i=0}^{l-1} d_{i} \pi^{i}$, so that $\pi^{l} c_{l}=c+$ $\pi^{n+1} d^{\sigma^{n}}-d$. Let $\sum_{j=0}^{\infty} u_{j} \pi^{j}$ be a strong semi-unit decomposition of $c_{l}$, necessarily having $v_{0}\left(u_{0}\right)<-1 / q^{n}$. Put $e=\sum_{j=n+1}^{\infty} u_{j}^{\sigma^{-n}} \pi^{j-n-1}$ and set $x=$ $c_{l}-\pi^{n+1} e^{\sigma^{n}}+e$. Then

$$
x=\sum_{j=1}^{n} \pi^{j} u_{j}+\left(u_{0}+\sum_{j=n+1}^{\infty} u_{j}^{\sigma^{-n}} \pi^{j-n-1}\right)
$$

and the quantity in parentheses is a strong semi-unit of the same valuation as $u_{0}$, since $v_{0}\left(u_{0}\right)<-1 / q^{n} \leq v_{0}\left(u_{j}^{\sigma^{-n}}\right)$ for all $j$. Thus $x$ satisfies the condition of Lemma 4.12, and so there exist $a^{\prime}, b^{\prime} \in \Gamma_{\mathrm{an} \text {, con }}^{\text {alg }}$ not both zero so that

$$
\left(a^{\prime}\right)^{\sigma}=\pi a^{\prime}, \quad \pi\left(b^{\prime}\right)^{\sigma^{n}}=b^{\prime}-a^{\prime} x
$$

We obtain a solution of (2) by setting $a=a^{\prime}, b=a^{\prime} d-\pi^{l} a^{\prime} e+\pi^{l} b^{\prime}$.
We now analyze (2) in general by reducing to the special case treated above.

LEMMA 4.14. For any positive integer $n$ and any $c \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$, there exist $a, b \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ not both zero such that (2) holds.

Proof. Let $\sum_{i} u_{i} \pi^{i}$ be a strong semi-unit decomposition of $c$, and let $N$ be the smallest integer such that $v_{0}\left(u_{N}\right)<0$, or $\infty$ if there is no such integer. By Proposition $3.19(\mathrm{~d})$, there exists $y \in \Gamma_{\text {an,con }}^{\text {alg }}$ such that $\pi^{n+1} y^{\sigma^{n}}-$ $y+\sum_{i=-\infty}^{N-1} u_{i} \pi^{i}=0$.

If $N=\infty$, then in fact $\pi^{n+1} y^{\sigma^{n}}-y+c=0$, so we obtain a solution of (2) by choosing $a$ nonzero with $a^{\sigma}=\pi a$ via Proposition $3.19(\mathrm{~b})$, and setting $b=a y$. Suppose hereafter that $N<\infty$.

For each $i \geq N$ for which $u_{i} \neq 0$, set $t_{i}=\left\lceil\log _{q^{n}}\left(-v_{0}\left(u_{i}\right)\right)\right\rceil$, so that $-1 \leq v_{0}\left(u_{i}^{\sigma^{-n t_{i}}}\right)<-1 / q^{n}$ for all such $i$. Then the sum

$$
z=\sum_{i=N}^{\infty} \sum_{j=1}^{t_{i}} u_{i}^{\sigma^{-n j}} \pi^{i-(n+1) j}
$$

is $\pi$-adically convergent: $-v_{0}\left(u_{i}\right)$ grows at most linearly in $i$, so that $t_{i}$ grows at most logarithmically and $i-(n+1) t_{i} \rightarrow \infty$ as $i \rightarrow \infty$. Moreover, since $t_{i}$ is bounded below, $v_{0}\left(u_{i}^{\sigma^{-n j}} \pi^{i-(n+1) j}\right)$ is as well; thus the $\operatorname{sum} z$ is in $\Gamma_{\mathrm{an}, \text { con }}^{\mathrm{alg}}$.

Put $c^{\prime}=c+\pi^{n+1}(y-z)^{\sigma^{n}}-(y-z)$; then

$$
\begin{aligned}
c^{\prime} & =\sum_{i=N}^{\infty}\left(u_{i} \pi^{i}+\sum_{j=1}^{t_{i}} u_{i}^{\sigma^{-n j}} \pi^{i-(n+1) j}-\sum_{j=1}^{t_{i}} u_{i}^{\sigma^{-n(j-1)}} \pi^{i-(n+1)(j-1)}\right) \\
& =\sum_{i=N}^{\infty} u_{i}^{\sigma^{-n t_{i}}} \pi^{i-(n+1) t_{i}}
\end{aligned}
$$

so that $v_{m}\left(c^{\prime}\right) \geq-1$ for all $m$. By Lemma 4.13, there exist $a^{\prime}, b^{\prime} \in \Gamma_{\mathrm{an}, \text { con }}^{\text {alg }} \operatorname{not}$ both zero such that

$$
\left(a^{\prime}\right)^{\sigma}=\pi a^{\prime}, \quad \pi\left(b^{\prime}\right)^{\sigma^{n}}=b^{\prime}-a^{\prime} c^{\prime}
$$

we obtain a solution of (2) by setting $a=a^{\prime}, b=b^{\prime}+a^{\prime}(y-z)$.

We now prove our basic result on raising the Newton polygon, i.e., reducing the slope of an eigenvector.

Proposition 4.15. Let $m$ and $n$ be positive integers, and let $M$ be a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ admitting a basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}, \mathbf{w}$ such that for some $c_{i} \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$,

$$
\begin{aligned}
F \mathbf{v}_{i} & =\mathbf{v}_{i+1} \quad(i=1, \ldots, n-1) \\
F \mathbf{v}_{n} & =\pi \mathbf{v}_{1} \\
F \mathbf{w} & =\pi^{-m} \mathbf{w}+c_{1} \mathbf{v}_{1}+\cdots+c_{n} \mathbf{v}_{n}
\end{aligned}
$$

Then there exists $\mathbf{y} \in M$ such that $F \mathbf{y}=\mathbf{y}$.
This will ultimately be a special case of our main results; what makes this case directly tractable is that if $\operatorname{SatSpan}\left(\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}\right)$ does not admit an $F$-stable complement in $M$ (i.e., is not a direct summand of $M$ in the category of $\sigma$-modules), then the map $\mathbf{y} \mapsto F \mathbf{y}-\mathbf{y}$ is actually surjective, as predicted by the expected behavior of the special Newton polygon.

Proof. Suppose $\mathbf{y}=d \mathbf{w}+b_{1} \mathbf{v}_{1}+\cdots+b_{n} \mathbf{v}_{n}$ satisfies $F \mathbf{y}=\mathbf{y}$, or in other words

$$
d \mathbf{w}+\sum_{i=1}^{n} b_{i} \mathbf{v}_{i}=\pi^{-m} d^{\sigma} \mathbf{w}+\sum_{i=1}^{n} d^{\sigma} c_{i} \mathbf{v}_{i}+\sum_{i=1}^{n-1} b_{i}^{\sigma} \mathbf{v}_{i+1}+\pi b_{n}^{\sigma} \mathbf{v}_{1}
$$

Comparing coefficients in this equation, we have $b_{i}^{\sigma}=b_{i+1}-d^{\sigma} c_{i+1}$ for $i=$ $1, \ldots, n-1$, as well as $\pi b_{n}^{\sigma}=b_{1}-d^{\sigma} c_{1}$ and $d^{\sigma}=\pi^{m} d$. If we use the first
$n$ relations to eliminate $b_{2}, \ldots, b_{n}$, we get

$$
\begin{aligned}
b_{1}^{\sigma^{n}} & =b_{2}^{\sigma^{n-1}}-d^{\sigma^{n}} c_{2}^{\sigma^{n-1}} \\
= & b_{3}^{\sigma^{n-2}}-d^{\sigma^{n-1}} c_{3}^{\sigma^{n-2}}-d^{\sigma^{n}} c_{2}^{\sigma^{n-1}} \\
& \quad \vdots \\
& =b_{n}^{\sigma}-d^{\sigma^{2}} c_{n}^{\sigma}-\cdots-d^{\sigma^{n}} c_{2}^{\sigma^{n-1}} \\
= & \pi^{-1} b_{1}-d\left(\pi^{m-1} c_{1}+\pi^{2 m} c_{n}^{\sigma}+\pi^{3 m} c_{n-1}^{\sigma^{2}}+\cdots+\pi^{n m} c_{2}^{\sigma^{n-1}}\right) .
\end{aligned}
$$

Let $c^{\prime}$ be the quantity in parentheses in the last line. We have shown that if $F \mathbf{y}=\mathbf{y}$ has a nonzero solution, then the system of equations

$$
\begin{equation*}
d^{\sigma}=\pi^{m} d, \quad \pi b_{1}^{\sigma^{n}}=b_{1}-\pi c^{\prime} d \tag{3}
\end{equation*}
$$

has a solution with $b_{1}, d$ not both zero. Conversely, from any nonzero solution of (3) we may construct a nonzero $\mathbf{y} \in M$ such that $F \mathbf{y}=\mathbf{y}$, by using the relations $b_{i}^{\sigma}=b_{i+1}-d^{\sigma} c_{i+1}$ to successively define $b_{2}, \ldots, b_{n}$.

By Proposition 3.19(b), we can find $e \in \Gamma_{\mathrm{an}, \text { con }}^{\mathrm{alg}}$ nonzero such that $e^{\sigma}=$ $\pi^{m-1} e$; we will construct a solution of (3) with $d=a e$ for some $a$ such that $a^{\sigma}=\pi a$. Namely, put $c=\pi c^{\prime} e$, and apply Lemma 4.14 to find $a, b \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$, not both zero, such that

$$
a^{\sigma}=\pi a, \quad \pi b^{\sigma^{n}}=b-a c .
$$

Then $b_{1}=b$ and $d=a e$ constitute a nonzero solution of (2); as noted above, this implies that there exists $\mathbf{y} \in M$ nonzero with $F \mathbf{y}=\mathbf{y}$, as desired.
4.4. Construction of the special Newton polygon. We now assemble the results of the previous sections into the following theorem, the main result of this chapter.

Theorem 4.16. Let $M$ be a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$. Then $M$ can be expressed as a direct sum of standard $\sigma$-submodules.

As the proof of this theorem is somewhat intricate, we break off parts of the argument into separate lemmas. In these lemmas, a "suitable extension" of $\mathcal{O}\left[\frac{1}{p}\right]$ means one whose value group contains whatever slope is desired to be the slope of an eigenvector. By Proposition 4.7, proving the existence of an eigenvector of prescribed slope over a single suitable extension implies the same over any suitable extension.

Lemma 4.17. Let $M$ be a $\sigma$-module over $\Gamma_{\text {an }}^{\text {alg }, \text { con }}$ of rank 1 , and suppose $F$ acts on some generator $\mathbf{v}$ via $F \mathbf{v}=c \mathbf{v}$. Then $M$ contains an eigenvector, and any primitive eigenvector has slope $v_{p}(c)$.

Note that $v_{n}(c)=\infty$ for some $n$ by Corollary 3.23 , so that $v_{p}(c)$ makes sense.

Proof. The existence of an eigenvector of slope $v_{p}(c)$ follows from Proposition 3.18 . The uniqueness of the slope follows from Corollary 4.6.

For $M$ of rank 1 , we call this unique slope the slope of $M$. Note that if $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of $\sigma$-modules and $L, M, N$ have ranks $l, m, n$, respectively, then the slope of $\wedge^{m} M$ is the sum of the slopes of $\wedge^{l} L$ and $\wedge^{n} N$. (This assertion will be vastly generalized by Proposition 5.13 later.)

Lemma 4.18. Let $M$ be a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ of rank 2 , and let $d$ be the slope of $\wedge^{2} M$. Then $M$ contains an eigenvector of slope $d / 2$ over a suitable extension of $\mathcal{O}\left[\frac{1}{p}\right]$.

Proof. We may assume without loss of generality that $d / 2$ belongs to the value group of $\mathcal{O}\left[\frac{1}{p}\right]$. Let $e$ be the smallest integer such that $M$ contains an eigenvector of slope $e v_{p}(\pi)$. (There is such an integer by Proposition 4.8, and there is a smallest one by Corollary 4.10.) By twisting, we may reduce to the case where $e=1$.

Put $m=1-\left(d / v_{p}(\pi)\right)$ and suppose by way of contradiction that $m>0$. Choose an eigenvector $\mathbf{v}$ with $F \mathbf{v}=\pi \mathbf{v}$, which is necessarily primitive by Lemma 4.1; then by Lemma 4.17 applied to $M / \operatorname{SatSpan}(\mathbf{v})$, we can find w such that $\mathbf{v}, \mathbf{w}$ form a basis of $M$ and $F \mathbf{w}=\pi^{-m} \mathbf{w}+c \mathbf{v}$ for some $c \in \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$. Now by Proposition $4.15, M$ contains an eigenvector $\mathbf{v}_{1}$ with $F \mathbf{v}_{1}=\mathbf{v}_{1}$, contradicting the definition of $e$.

Hence $m \leq 0$, which implies $d \geq v_{p}(\pi)$. Since $d / 2$ is also a multiple of $v_{p}(\pi)$, we must have $d / 2 \geq v_{p}(\pi)$; by Proposition $4.2, M$ contains an eigenvector of slope $d / 2$.

LEMMA 4.19. Let $M$ be a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ of rank $n$, and let $d$ be the slope of $\wedge^{n} M$. Then $M$ contains eigenvectors of all slopes greater than $d / n$ over suitable extensions of $\mathcal{O}\left[\frac{1}{p}\right]$.

Proof. We proceed by induction on $n$. The case $n=1$ follows from Lemma 4.17, and the case $n=2$ follows from Lemma 4.18. Suppose $n>2$ and that the lemma has been proved for all smaller values of $n$. Let $s$ be the greatest lower bound of the set of rational numbers that occur as slopes of eigenvectors of $M$ (over suitable extensions of $\left.\mathcal{O}\left[\frac{1}{p}\right]\right)$. Again, the set is nonempty by Proposition 4.8 and is bounded below by Corollary 4.10.

For each $\varepsilon>0$ such that $s+\varepsilon \in \mathbb{Q}$, over a suitable extension of $\mathcal{O}\left[\frac{1}{p}\right]$ there exist an eigenvector $\mathbf{v}$ of $M$ of slope $s+\varepsilon$ and (by the induction hypothesis) an eigenvector $\mathbf{w}$ of $M / \operatorname{SatSpan}(\mathbf{v})$ of slope at most $s^{\prime}=(d-s-\varepsilon) /(n-1)+\varepsilon$. The preimage of $\operatorname{SatSpan}(\mathbf{w})$ in $M$ has rank 2, and so is covered by the induction
hypothesis; it thus contains, for any $\delta>0$, an eigenvector of slope at most

$$
\frac{s+\varepsilon}{2}+\frac{d-s+(n-2) \varepsilon}{2(n-1)}+\delta
$$

over a suitable extension of $\mathcal{O}\left[\frac{1}{p}\right]$. Such an eigenvector is also an eigenvector of $M$, so its slope is at least $s$. Letting $\varepsilon$ and $\delta$ go to 0 in the resulting inequality yields

$$
\frac{s}{2}+\frac{d-s}{2(n-1)} \geq s
$$

which simplifies to $s \leq d / n$, as desired.
Lemma 4.20. Let $M$ be a $\sigma$-module over $\Gamma_{\text {an,con }}^{\text {alg }}$ of rank $n$, and let $d$ be the slope of $\wedge^{n} M$. Then $M$ contains an eigenvector of slope $d / n$ over a suitable extension of $\mathcal{O}\left[\frac{1}{p}\right]$.

Proof. We proceed by induction on $n$; again, the case $n=1$ follows from Lemma 4.17 and the case $n=2$ follows from Lemma 4.18. Without loss of generality, we may assume the value group of $\mathcal{O}$ contains $d / n$, and then that $d=0$.

By Lemma 4.19, there exists an eigenvector $\mathbf{v}$ of $M$ of slope $v_{p}(\pi) /(n-1)$ over $\mathcal{O}\left[\pi^{1 /(n-1)}\right]$; we may as well assume $F \mathbf{v}=\pi^{1 /(n-1)} \mathbf{v}$. Let $N$ be the saturated span of $\mathbf{v}$ and its conjugates over $\mathcal{O}\left[\frac{1}{p}\right]$; let $m$ be the rank of $N$ and $s$ the slope of $\wedge^{m} N$. Then $m \leq n-1$ and $s \leq m v_{p}(\pi) /(n-1)$. If $m<n-1$, then $0<m v_{p}(\pi) /(n-1)<v_{p}(\pi)$, so that $s \leq 0$ and the induction hypothesis implies that $N$ contains an eigenvector of slope 0 . The same argument applies if $m=n-1$ and $s<v_{p}(\pi)$.

Suppose instead that $m=n-1$ and $s=v_{p}(\pi)$. Write $\mathbf{v}=\mathbf{v}_{1}+$ $\pi^{-1 /(n-1)} \mathbf{v}_{2}+\cdots+\pi^{-(n-2) /(n-1)} \mathbf{v}_{n-1}$ with each $\mathbf{v}_{i}$ defined over $\Gamma_{\text {an,con }}^{\text {alg }}$ (with no extension of $\left.\mathcal{O}\left[\frac{1}{p}\right]\right)$; then $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n-1}$ are linearly independent in $N$, and we have $F \mathbf{v}_{i}=\mathbf{v}_{i+1}$ for $i=1, \ldots, n-1$ and $F \mathbf{v}_{n-1}=\pi \mathbf{v}_{1}$. In particular, the $\mathbf{v}_{i}$ must be a basis of $N$ or else $\wedge^{n-1} N$ would have slope less than $s$. The slope of $M / N$ is $-v_{p}(\pi)$, and so by Lemma 4.17, we can choose $\mathbf{w} \in M$ such that $F \mathbf{w} \equiv \pi^{-1} \mathbf{w}(\bmod N)$. Proposition 4.15 then implies that $M$ contains an eigenvector of slope 0 , as desired.

Proof of Theorem 4.16. We proceed by induction on the rank of $M$. If $\operatorname{rank} M=1$, then $M$ is standard by Lemma 4.17. Suppose $\operatorname{rank} M=n>1$, and that the proposition has been established for all $\sigma$-modules of rank less than $n$. For any rational number $c$, define the $\mathcal{O}$-index of $c$ as the smallest integer $m$ such that $m c$ lies in the value group of $\mathcal{O}\left[\frac{1}{p}\right]$. The set of rational numbers of $\mathcal{O}$-index less than or equal to $n$ which occur as slopes of eigenvectors of $M$ is discrete (obvious), nonempty (by Proposition 4.8), and bounded below (by Corollary 4.10) and thus has a smallest element $r$.

Let $d$ be the slope of $\wedge^{n} M$. By Lemma 4.20, we have $r \leq d / n$. Let $s$ be the $\mathcal{O}$-index of $r$, and let $\lambda$ be an element of a degree $s$ extension $\mathcal{O}^{\prime}\left[\frac{1}{p}\right]$ of $\mathcal{O}\left[\frac{1}{p}\right]$ such that $v_{p}(\lambda)=r$ and $\lambda^{s} \in \mathcal{O}\left[\frac{1}{p}\right]$. Choose an eigenvector $\mathbf{v}$ over $\mathcal{O}^{\prime}\left[\frac{1}{p}\right]$ with $F \mathbf{v}=\lambda \mathbf{v}$, and write $\mathbf{v}=\sum_{i=0}^{s-1} \lambda^{-i} \mathbf{w}_{i}$ for $\mathbf{w}_{i} \in M$, so that $F \mathbf{w}_{i}=\mathbf{w}_{i+1}$ for $i=0, \ldots, s-2$ and $F \mathbf{w}_{s-1}=\lambda^{s} \mathbf{w}_{0}$. Put $N=\operatorname{SatSpan}\left(\mathbf{w}_{0}, \ldots, \mathbf{w}_{s-1}\right)$ and $m=\operatorname{rank} N$; then $s \geq m$, and the slope of $\wedge^{m} N$ is at most $m r$, since $N$ is the saturated span of eigenvectors of slope $r$.

If $m=n$, then also $s=n$ and $\mathbf{w}_{0} \wedge \cdots \wedge \mathbf{w}_{n-1}$ is an eigenvector of $\wedge^{n} M$ of slope $r n$. Thus $r n \geq d$; since $r \leq d / n$ as shown earlier, we conclude $r=d / n$, $\mathbf{w}_{0}, \ldots, \mathbf{w}_{n-1}$ form a basis of $M$, and $M$ is standard, completing the proof in this case. Thus we assume $m<n$ hereafter.

Given that $m<n$, we may apply the induction hypothesis to $N$, deducing in particular that its smallest slope is at most $r$ and has $\mathcal{O}$-index not greater than $m$. This yields a contradiction unless that slope is $r$, which is only possible if the slope of $\wedge^{m} N$ is $m r$. In turn, $m r$ belongs to the value group of $\mathcal{O}\left[\frac{1}{p}\right]$ only if $m=s$. Thus $m=s$, and since $\mathbf{w}_{0} \wedge \cdots \wedge \mathbf{w}_{s-1}$ is an eigenvector of $N$ of slope $r s$, the $\mathbf{w}_{0}, \ldots, \mathbf{w}_{s-1}$ form a basis of $N$, and $N$ is standard.

Apply the induction hypothesis to $M / N$ to express it as a sum $P_{1} \oplus \cdots \oplus P_{l}$ of standard $\sigma$-submodules; note that the $\mathcal{O}$-index of the slope of $P_{i}$ divides the rank of $P_{i}$, and so is at most $n$. If $l=1$, then the slope of $P_{1}$ cannot be less than $r$ (else the slope of $\wedge^{n} M$ would be less than $d$ ); thus, by Proposition 4.3, $M$ can be split as a direct sum of $N$ with a standard $\sigma$-module. If $l>1$, let $M_{i}$ be the preimage of $P_{i}$ under the projection $M \rightarrow M / N$; again the slope of each $P_{i}$ cannot be less than $r$, else the induction hypothesis would imply that $M_{i}$ contains an eigenvector of slope less than $r$ and $\mathcal{O}$-index not exceeding $n$, a contradiction. Thus by Proposition 4.3 again, each $M_{i}$ can be split as a direct sum $N \oplus N_{i}$ of $\sigma$-submodules, and we may decompose $M$ as $N \oplus N_{1} \oplus \cdots \oplus N_{l}$. This completes the induction in all cases.

By Corollary 4.6, the multiset union of the slopes of the standard summands of a $\sigma$-module $M$ over $\Gamma_{a n, \text { con }}^{\text {alg }}$ (each summand contributing its slope as many times as its rank) does not depend on the decomposition. Thus we define the special Newton polygon of $M$ as the polygon with vertices $\left(i, y_{i}\right)$ $(i=0, \ldots, n)$, where $y_{0}=0$ and $y_{i}-y_{i-1}$ is the $i$-th smallest slope of $M$ (counting multiplicity). We extend this definition to $\sigma$-modules over $\Gamma_{\text {an,con }}$ by base extending to $\Gamma_{\mathrm{an}, \text { con }}^{\mathrm{alg}}$.

## 5. The generic Newton polygon

In this chapter, we recall the construction of the generic Newton polygon associated to a $\sigma$-module over $\Gamma$. The construction uses a classification result, the Dieudonné-Manin classification, for $\sigma$-modules over a complete dis-
crete valuation ring with algebraically closed residue field. This classification does not descend very well, and so we describe some weaker versions of the classification that can be accomplished under less restrictive conditions. These weaker versions either appear in or are inspired directly by [dJ].
5.1. Properties of eigenvectors. Throughout this section, let $R$ be a discrete valuation ring with residue field $k$ which is unramified over $\mathcal{O}$. Again, we call an element $\mathbf{v}$ of a $\sigma$-module $M$ over $R$ or $R\left[\frac{1}{p}\right]$ an eigenvector if there exists $\lambda \in \mathcal{O}$ or $\mathcal{O}\left[\frac{1}{p}\right]$, respectively, such that $F \mathbf{v}=\lambda \mathbf{v}$, and refer to $v_{p}(\lambda)$ as the slope of $\mathbf{v}$. We call an eigenvector primitive if it forms part of a basis of $M$, but this definition is not very useful: every eigenvector is an $\mathcal{O}$-multiple of a primitive eigenvector of the same slope. In fact, in contrast to the situation over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$, the slopes of eigenvectors over $R$ are "rigid".

Proposition 5.1. Let $M$ be a $\sigma$-module over $R\left[\frac{1}{p}\right]$, with $k$ algebraically closed. Suppose $M$ admits a basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ of eigenvectors. Then any eigenvector $\mathbf{w}$ is an $\mathcal{O}\left[\frac{1}{p}\right]$-linear combination of those $\mathbf{v}_{i}$ of the same slope. In particular, any eigenvector has the same slope as one of the $\mathbf{v}_{i}$.

Proof. Suppose $F \mathbf{v}_{i}=\lambda_{i} \mathbf{v}_{i}$ for some $\lambda_{i} \in \mathcal{O}\left[\frac{1}{p}\right]$, and write $\mathbf{w}=\sum_{i} c_{i} \mathbf{v}_{i}$ with $c_{i} \in R\left[\frac{1}{p}\right]$. If $F \mathbf{w}=\mu \mathbf{w}$ for $\mu \in \mathcal{O}\left[\frac{1}{p}\right]$, then equating the coefficients of $\mathbf{v}_{i}$ yields $\lambda_{i} c_{i}^{\sigma}=\mu c_{i}$. If $v_{p}\left(\lambda_{i}\right) \neq v_{p}(\mu)$, this forces $c_{i}=0$; if $v_{p}\left(\lambda_{i}\right)=v_{p}(\mu)$, it forces $c_{i} \in \mathcal{O}\left[\frac{1}{p}\right]$. This proves the claim.

By imitating the proof of Proposition 4.5 using Proposition 5.1 in lieu of Proposition 4.4, we obtain the following analogue of Corollary 4.6.

Proposition 5.2. Let $M$ be a $\sigma$-module over $R\left[\frac{1}{p}\right]$. Suppose $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ and $\mathbf{w}_{1}, \ldots, \mathbf{w}_{n}$ are bases of eigenvectors with $F \mathbf{v}_{i}=\lambda_{i} \mathbf{v}_{i}$ and $F \mathbf{w}_{i}=\mu_{i} \mathbf{w}_{i}$, for some $\lambda_{i}, \mu_{i} \in \mathcal{O}\left[\frac{1}{p}\right]$. Then the sequences $v_{p}\left(\lambda_{1}\right), \ldots, v_{p}\left(\lambda_{n}\right)$ and $v_{p}\left(\mu_{1}\right), \ldots$, $v_{p}\left(\mu_{n}\right)$ are permutations of each other.

In case $M$ has a full set of eigenvectors of one slope, we have the following decomposition result.

Proposition 5.3. Suppose $k$ is algebraically closed, and let $M$ be a $\sigma$-module over $R$ spanned by eigenvectors of a single slope over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$, for some finite extension $\mathcal{O}^{\prime}$ of $\mathcal{O}$. Then $M$ is isogenous to the direct sum of standard $\sigma$-modules of that slope.

Proof. Let $s$ be the common slope, and let $m$ be the smallest positive integer such that $m s$ is a multiple of $v_{p}(\pi)$. Since $k$ is algebraically closed, there exists $\lambda \in \mathcal{O}^{\prime}$ such that $\lambda^{m} \in \mathcal{O}$. Let $\mathcal{O}^{\prime \prime}$ be the integral closure of $\mathcal{O}$ in $\mathcal{O}\left[\frac{1}{p}\right](\lambda)$.

Note that $M$ is spanned over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$ by eigenvectors $\mathbf{v}$ with $F \mathbf{v}=\lambda \mathbf{v}$ because $k$ is algebraically closed: if $F \mathbf{w}=\mu \mathbf{w}$ for some $\mu$ with $v_{p}(\mu)=v_{p}(\lambda)$, we can find $c \in \mathcal{O}^{\prime}$ nonzero such that $c^{\sigma}=(\lambda / \mu) c$ and obtain a new eigenvector $\mathbf{v}=c \mathbf{w}$ with $F \mathbf{v}=\lambda \mathbf{v}$. We next verify that $M$ is also spanned over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime \prime}$ by eigenvectors $\mathbf{v}$ with $F \mathbf{v}=\lambda \mathbf{v}$. Let $\mu_{1}, \ldots, \mu_{n}$ be a basis of $\mathcal{O}^{\prime}$ over $\mathcal{O}^{\prime \prime}$ consisting of elements fixed by $\sigma$ (possible because $k$ is algebraically closed). If $\mathbf{v}$ is an eigenvector over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$ with $F \mathbf{v}=\lambda \mathbf{v}$, we can write $\mathbf{v}=\sum_{i} \mu_{i} \mathbf{w}_{i}$ for some $\mathbf{w}_{i}$ over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime \prime}$, and we must have $F \mathbf{w}_{i}=\lambda \mathbf{w}_{i}$ for each $i$. Thus $\mathbf{v}$ is in the span of the $\mathbf{w}_{i}$, so the span of eigenvectors of eigenvalue $\lambda$ over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime \prime}$ has full rank over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$, and thus has full rank over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime \prime}$.

Finally, we establish that $M$ is isogenous to a direct sum of standard $\sigma$-modules. Let $\mathbf{v}$ be an eigenvector of eigenvalue $\lambda$ over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime \prime}$; we can write $\mathbf{v}=\sum_{i=0}^{m-1} \mathbf{w}_{i} \lambda^{-i}$ for some $\mathbf{w}_{i} \in M$. Then $F \mathbf{w}_{i}=\mathbf{w}_{i+1}$ for $i=0, \ldots$, $m-2$ and $F \mathbf{w}_{m-1}=\lambda^{m} \mathbf{w}_{0}$, so the span of $\mathbf{w}_{0}, \ldots, \mathbf{w}_{m-1}$ is standard. (Notice that $\mathbf{w}_{0}, \ldots, \mathbf{w}_{m-1}$ must be linearly independent: if on the contrary their span had rank $d<m$, then by Lemma 5.4 below, $s v_{p}(\lambda)$ would belong to the value group of $\mathcal{O}$ for some $s \leq d<m$, contradiction.) Let $M_{1}$ be the standard submodule just produced. Next, choose an eigenvector of eigenvalue $\lambda$ linearly independent of $M_{1}$, and produce another standard submodule $M_{2}$. Then choose an eigenvector linearly independent of $M_{1} \oplus M_{2}$, and so on until $M$ is exhausted.
5.2. The Dieudonné-Manin classification. Again, let $R$ be a discrete valuation ring unramified over $\mathcal{O}$.

Lemma 5.4. Suppose that $R$ is complete with algebraically closed residue field. Given elements $a_{0}, \ldots, a_{n-1}$ of $R$ with $a_{0}$ nonzero, let $M$ be the $\sigma$-module with basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ such that

$$
\begin{aligned}
F \mathbf{v}_{i} & =\mathbf{v}_{i+1} \quad(i=1, \ldots, n-1) \\
F \mathbf{v}_{n} & =a_{0} \mathbf{v}_{1}+\cdots+a_{n-1} \mathbf{v}_{n}
\end{aligned}
$$

Suppose s belongs to the value group of $R$. Then the maximum number of linearly independent eigenvectors of slope $s$ in $M$ is less than or equal to the multiplicity $m$ of $s$ as a slope of the Newton polygon of the polynomial $x^{n}+$ $a_{n-1} x^{n-1}+\cdots+a_{0}$ over $R$. Moreover, if $m>0$, then $M$ admits an eigenvector of slope $s$.

Proof. Let $l=\min _{j}\left\{-j s+v_{p}\left(a_{n-j}\right)\right\}$ (with $a_{n}=1$ for consistency); then there exists an index $i$ such that $l=-j s+v_{p}\left(a_{n-j}\right)$ for $j=i, j=i+m$, and possibly for some values of $j \in\{i+1, \ldots, i+m-1\}$, but not for any other values.

Let $\lambda$ be an element of valuation $s$ fixed by $\sigma$. Suppose $\mathbf{w}=\sum_{j} c_{j} \mathbf{v}_{j}$ satisfies $F \mathbf{w}=\lambda \mathbf{w}$. Then $\lambda c_{1}=a_{0} c_{n}^{\sigma}$ and $\lambda c_{j}=a_{j-1} c_{n}^{\sigma}+c_{j-1}^{\sigma}$ for $j=2, \ldots, n$.

Solving for $c_{n}$ yields

$$
\begin{aligned}
c_{n}= & \lambda^{-1} a_{n-1} c_{n}^{\sigma}+\lambda^{-1} c_{n-1}^{\sigma} \\
= & \lambda^{-1} a_{n-1} c_{n}^{\sigma}+\lambda^{-2} a_{n-2}^{\sigma} \sigma_{n}^{\sigma^{2}}+\lambda^{-2} c_{n-2}^{\sigma^{2}} \\
& \vdots \\
= & \lambda^{-1} a_{n-1} c_{n}^{\sigma}+\lambda^{-2} a_{n-2}^{\sigma} c_{n}^{\sigma^{2}}+\cdots+\lambda^{-n+1} a_{1}^{\sigma^{n-2}} c_{n}^{\sigma^{n-1}}+\lambda^{-n+1} c_{1}^{\sigma^{n-1}} \\
= & \lambda^{-1} a_{n-1} c_{n}^{\sigma}+\lambda^{-2} a_{n-2}^{\sigma} c_{n}^{\sigma^{2}}+\cdots+\lambda^{-n+1} a_{1}^{\sigma^{n-2}} c_{n}^{\sigma^{n-1}}+\lambda^{-n} a_{0}^{\sigma^{n-1}} c_{n}^{\sigma^{n}} .
\end{aligned}
$$

In other words, $f\left(c_{n}\right)=0$, where

$$
f(x)=-x+\frac{a_{n-1}}{\lambda} x^{\sigma}+\frac{a_{n-2}^{\sigma}}{\lambda^{2}} x^{\sigma^{2}}+\cdots+\frac{a_{0}^{\sigma^{n-1}}}{\lambda^{n}} x^{\sigma^{n}} .
$$

The coefficients of $f$ of minimal valuation are on $x^{\sigma^{i}}, x^{\sigma^{i+m}}$, and possibly some in between.

Now suppose $\mathbf{w}_{1}, \ldots, \mathbf{w}_{m+1}$ are linearly independent eigenvectors of $M$ with $F \mathbf{w}_{h}=\lambda \mathbf{w}_{h}$ for $h=1, \ldots, m+1$. Write $\mathbf{w}_{h}=\sum_{j} c_{h j} \mathbf{v}_{j}$. Then $c_{1 n}, \ldots, c_{(m+1) n}$ are linearly independent over $\mathcal{O}_{0}$ : if there were a relation $\sum_{h} d_{h} c_{h n}=0$ with $d_{h} \in \mathcal{O}_{0}$ not all zero, we would have

$$
\lambda \sum_{h} d_{h} c_{h j}=\left(\sum_{h} d_{h} c_{h(j-1)}\right)^{\sigma}+a_{j-1}\left(\sum_{h} d_{h} c_{h n}\right)^{\sigma} \quad(j=2, \ldots, n)
$$

and successively deduce $\sum_{h} d_{h} c_{h j}=0$ for $j=n-1, \ldots, 1$. That would mean $\sum_{h} d_{h} \mathbf{w}_{h}=0$, but the $\mathbf{w}_{h}$ are linearly independent.

By replacing the $\mathbf{w}_{h}$ with suitable $\mathcal{O}_{0}$-linear combinations, we can ensure that the $c_{h n}$ are in $R$ and their reductions modulo $\pi$ are linearly independent over $\mathbb{F}_{q}$. Now on one hand, the reduction of $\left(\lambda^{i} / a_{n-i}^{\sigma^{i-1}}\right) f(x)$ modulo $\pi$ is a polynomial in $x$ of the form $b_{i+m} x^{q^{i+m}}+\cdots+b_{i} x^{q^{i}}$, which has only $q^{m}$ distinct roots in $R / \pi R$. On the other hand, the $\mathbb{F}_{q}$-linear combinations of the reductions of the $c_{h n}$ yields $q^{m+1}$ distinct roots in $R / \pi R$, a contradiction.

We conclude that the multiplicity of $s$ as a slope of $M$ is at most $m$; this establishes the first assertion. To establish the second, note that if $m>0$, then there exists $c_{n} \neq 0$ such that $f\left(c_{n}\right)=0$ by Proposition 3.17; letting $c_{n}$ be this root, one can then solve for $c_{n-1}, \ldots, c_{1}$ and produce an eigenvector $\mathbf{v}$ with $F \mathbf{v}=\lambda \mathbf{v}$.

Using this lemma, we can establish the Dieudonné-Manin classification theorem (for which see also Katz [Ka]). We first state it not quite in the standard form. Note: a "basis up to isogeny" means a maximal linearly independent set.

Proposition 5.5. Suppose $R$ is complete with algebraically closed residue field. Then every $\sigma$-module $M$ over $R$ has a basis up to isogeny of eigenvectors of nonnegative slopes over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$ for some finite extension $\mathcal{O}^{\prime}$ of $\mathcal{O}$ (depending on $M$ ).

Proof. We proceed by induction on $n=\operatorname{rank} M$. Let $\mathbf{v}$ be any nonzero element of $M$, and let $m$ be the smallest integer such that $\mathbf{v}, F \mathbf{v}, \ldots, F^{m} \mathbf{v}$ are linearly dependent. Then $N=\operatorname{Sat} \operatorname{Span}\left(\mathbf{v}, F \mathbf{v}, \ldots, F^{m-1} \mathbf{v}\right)$ is a $\sigma$-submodule of $M$, and Lemma 5.4 implies that it has a primitive eigenvector $\mathbf{v}_{1}$ of nonnegative slope over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$ for some $\mathcal{O}^{\prime}$ (since the corresponding polynomial has a root of nonnegative valuation there). By the induction hypothesis, we can choose $\mathbf{w}_{2}, \ldots, \mathbf{w}_{n}$ over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime \prime}$ for some $\mathcal{O}^{\prime \prime}$, whose images in $M / \operatorname{SatSpan}\left(\mathbf{v}_{1}\right)$ form a basis up to isogeny of eigenvectors of nonnegative slopes. We then have $F \mathbf{v}_{1}=\lambda_{1} \mathbf{v}_{1}$, where we may take $\lambda_{1}$ fixed by $\sigma$, and $F \mathbf{w}_{i}=\lambda_{i} \mathbf{w}_{i}+c_{i} \mathbf{v}_{1}$ for some $\lambda_{i} \in \mathcal{O}$ and $c_{i} \in R$. Apply Proposition 3.17 to find $a_{i} \in R$ such that $\lambda_{1} c_{i}+\lambda_{1} a_{i}^{\sigma}-\lambda_{i} a_{i}=0$, and set $\mathbf{v}_{i}=\lambda_{1} \mathbf{w}_{i}+a_{i} \mathbf{v}_{1}$ for $i=2, \ldots, n$; then $F \mathbf{v}_{i}=\lambda_{i} \mathbf{v}_{i}$, and so $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ form a basis up to isogeny of eigenvectors of nonnegative slope over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime \prime}$, as desired.

From this statement we deduce the Dieudonné-Manin classification theorem in its more standard form.

Theorem 5.6 (Dieudonné-Manin). Suppose $R$ is complete with algebraically closed residue field. Then every $\sigma$-module over $R$ is canonically isogenous to the direct sum of $\sigma$-modules, each with a single slope, with all of these slopes distinct. Moreover, every $\sigma$-module of a single slope is isogenous to $a$ direct sum of standard $\sigma$-modules of that slope.

Proof. Let $M$ be a $\sigma$-module over $R$. For each slope $s$ that occurs in a basis up to isogeny of eigenvectors produced by Proposition 5.5 over $R \otimes \mathcal{O} \mathcal{O}^{\prime}$, let $M_{s}$ be the span of all eigenvectors of $M$ over $R \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$ of slope $s$. Then $M_{s}$ is invariant under $\operatorname{Gal}\left(\mathcal{O}^{\prime} / \mathcal{O}\right)$, so that by Galois descent, $M_{s}$ descends to a $\sigma$-submodule of $M$. Moreover, $M_{s}$ is isogenous to a direct sum of standard $\sigma$-modules of slope $s$ by Proposition 5.3. This proves the desired result.

Given a $\sigma$-module $M$ over a discrete valuation ring $R$ unramified over $\mathcal{O}$, we can embed $R$ into a complete discrete valuation ring over which $M$ has a basis up to isogeny of eigenvectors by Proposition 5.5. (First complete the direct limit of $R \xrightarrow{\sigma} R \xrightarrow{\sigma} \cdots$, then take its maximal unramified extension, then complete again, then tensor with a suitable $\mathcal{O}^{\prime}$ over $\mathcal{O}$.) By Proposition 5.2, the slopes and multiplicities do not depend on the choice of the basis. Define the generic slopes of $M$ as the slopes of the eigenvectors in the basis, and the generic Newton polygon of $M$ as the polygon with vertices $\left(i, y_{i}\right)$ for $i=$ $0, \ldots, \operatorname{rank} M$, where $y_{0}=0$ and $y_{i}-y_{i-1}$ is the $i$-th smallest generic slope
of $M$ (counting multiplicity). If $M$ has all slopes equal to 0 , we say $M$ is unit-root.

With the definition of the generic Newton polygon in hand, we can refine the conclusion of Lemma 5.4 as follows.

Proposition 5.7. Given elements $a_{0}, \ldots, a_{n-1}$ of $R$ with $a_{0}$ nonzero, let $M$ be the $\sigma$-module with basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ such that

$$
\begin{aligned}
F \mathbf{v}_{i} & =\mathbf{v}_{i+1} \quad(i=1, \ldots, n-1) \\
F \mathbf{v}_{n} & =a_{0} \mathbf{v}_{1}+\cdots+a_{n-1} \mathbf{v}_{n}
\end{aligned}
$$

Then the generic Newton polygon of $M$ coincides with the the Newton polygon of the polynomial $x^{n}+a_{n-1} x^{n-1}+\cdots+a_{0}$ over $R$.

Proof. The two Newton polygons have the same length $n$, and every number occurs at least as often as a slope of the polynomial as it occurs as a slope of $M$ by Lemma 5.4. Thus all multiplicities must coincide.

For our purposes, the principal consequence of this fact is the following.
Proposition 5.8. Let $M$ be a $\sigma$-module over $R\left[\frac{1}{p}\right]$ with all slopes nonnegative. Then $M$ is isomorphic to a $\sigma$-module defined over $R$.

Proof. We proceed by induction on $n=\operatorname{rank} M$. Let $\mathbf{v} \in M$ be nonzero, and let $m$ be the smallest integer such that $\mathbf{v}, F \mathbf{v}, \ldots, F^{m} \mathbf{v}$ are linearly dependent. Then $F^{m} \mathbf{v}=a_{0} \mathbf{v}+\cdots+a_{m-1} F^{m-1} \mathbf{v}$ for some $a_{0}, \ldots, a_{m-1} \in R\left[\frac{1}{p}\right]$; by Proposition 5.7, the $a_{i}$ belong to $R$. Let $N=\operatorname{SatSpan}\left(\mathbf{v}, F \mathbf{v}, \ldots, F^{m-1} \mathbf{v}\right)$; by the induction hypothesis, $M / N$ is isomorphic to a $\sigma$-module defined over $R$. So we can choose $\mathbf{w}_{1}, \ldots, \mathbf{w}_{n-m}$ that form a basis of $M$ together with $\mathbf{v}, F \mathbf{v}, \ldots, F^{m-1} \mathbf{v}$, such that for $i=1, \ldots, n-m, F \mathbf{w}_{i}$ equals an $R\left[\frac{1}{p}\right]$-linear combination of the $F^{j} \mathbf{v}$ plus an $R$-linear combination of the $\mathbf{w}_{j}$. For $\lambda$ sufficiently divisible by $\pi$, the basis $\lambda \mathbf{v}, \lambda F \mathbf{v}, \ldots, \lambda F^{m-1} \mathbf{v}, \mathbf{w}_{1}, \ldots, \mathbf{w}_{n-m}$ has the property that the image of each basis vector under Frobenius is an $R$-linear combination of basis vectors. This gives the desired isomorphism.

We close the section with another method for reading off the generic Newton polygon of a $\sigma$-module, inspired by an observation of Buzzard and Calegari [BC, Lemma 5]. (We suspect it may date back earlier, possibly to Manin.)

Proposition 5.9. Let $M$ be a $\sigma$-module over a discrete valuation ring $R$. Suppose $M$ has a basis on which $F$ acts by the matrix $A$, where $A D^{-1}$ is congruent to the identity matrix modulo $\pi$ for some diagonal matrix $D$ over $\mathcal{O}$. Then the slopes of the generic Newton polygon of $M$ equal the valuations of the diagonal entries of $D$.

Proof. Without loss of generality we may assume $R$ is complete with algebraically closed residue field. We produce a sequence of matrices $\left\{U_{l}\right\}_{l=1}^{\infty}$ such that $U_{1}=I, U_{l+1} \equiv U_{l}\left(\bmod \pi^{l}\right)$ and $U_{l}^{-1} A U_{l}^{\sigma} D^{-1} \equiv I\left(\bmod \pi^{l}\right)$; the $\pi$-adic limit $U$ of the $U_{l}$ will satisfy $A U^{\sigma}=U D$, proving the proposition. The conditions for $l=1$ are satisfied by the assumption that $A D^{-1} \equiv I(\bmod \pi)$.

Suppose $U_{l}$ has been defined. Put $V=U_{l}^{-1} A U_{l}^{\sigma} D^{-1}-I$. Define a matrix $W$ whose entry $W_{i j}$, for each $i$ and $j$, is a solution of the equation $W_{i j}-D_{i i} W_{i j}^{\sigma} D_{j j}^{-1}=V_{i j}$ with $\min \left\{v_{p}\left(W_{i j}\right), v_{p}\left(D_{i i} W_{i j}^{\sigma} D_{j j}^{-1}\right)\right\}=v_{p}\left(V_{i j}\right)$ (such a solution exists by Proposition 3.17). Then $W$ and $D W^{\sigma} D^{-1}$ are both congruent to 0 modulo $\pi^{l}$. Put $U_{l+1}=U_{l}(I+W)$; then

$$
\begin{aligned}
U_{l+1}^{-1} A U_{l+1}^{\sigma} D^{-1} & =(I+W)^{-1} U_{l}^{-1} A U_{l}^{\sigma}(I+W)^{\sigma} D^{-1} \\
& =(I+W)^{-1} U_{l}^{-1} A U_{l}^{\sigma} D^{-1}\left(I+D W^{\sigma} D^{-1}\right) \\
& =(I+W)^{-1}(I+V)\left(I+D W^{\sigma} D^{-1}\right) \\
& \equiv I-W+V+D W^{\sigma} D^{-1}=I \quad\left(\bmod \pi^{l+1}\right) .
\end{aligned}
$$

Thus the conditions for $U_{l+1}$ are satisfied, and the proposition follows.
5.3. Slope filtrations. The Dieudonné-Manin classification holds over $\Gamma^{K}$ only if $K$ is algebraically closed, and even then does not descend to $\Gamma_{\text {con }}^{K}$ in general. In this section, we exhibit two partial versions of the classification that hold with weaker conditions on the coefficient ring. One (the descending filtration) is due to de Jong [dJ, Prop. 5.8]; for symmetry, we present independent proofs of both results.

The following filtration result applies for any $K$ but does not descend to $\Gamma_{\text {con }}$.

Proposition 5.10 (Ascending generic filtration). Let $K$ be a valued field. Then any $\sigma$-module $M$ over $\Gamma=\Gamma^{K}$ admits a unique filtration $M_{0}=0 \subset M_{1} \subset$ $\cdots \subset M_{m}=M$ by $\sigma$-submodules such that

1. for $i=1, \ldots, m, M_{i-1}$ is saturated in $M_{i}$ and $M_{i} / M_{i-1}$ has all generic slopes equal to $s_{i}$, and
2. $s_{1}<\cdots<s_{m}$.

Moreover, if $K$ is separably closed and $k$ is algebraically closed, each $M_{i} / M_{i-1}$ is isogenous to a direct sum of standard $\sigma$-modules.

Warning: this proof uses the object $\Gamma^{\text {sep }}$ even though this has only so far been defined for $k$ perfect. Thus we must give an ad hoc definition here. For any finite separable extension $L$ over $K$, Lemma 3.1 produces a finite extension of $\Gamma^{K}$ with residue field $L$, and Lemma 3.2 allows us to identify that extension with a subring of $\Gamma^{\text {alg }}$. We define $\Gamma^{\text {sep }}$ as the completed union of these subrings; note that $\Gamma^{\text {perf }} \cap \Gamma^{\text {sep }}=\Gamma$.

Proof. By the Dieudonné-Manin classification (Theorem 5.6), $M$ is canonically isogenous to a direct sum of $\sigma$-submodules, each of a different single slope. By Corollary 2.7, these submodules descend to $\Gamma^{\text {perf }}$; let $M_{1}$ be the submodule of minimum slope. It suffices to show that $M_{1}$ is defined over $\Gamma$, as an induction on rank will then yield the general result. Moreover, it is enough to establish this when $M_{1}$ has rank 1: if $M_{1}$ has rank $d$, then the lowest slope submodule of $\wedge^{d} M$ is the rank 1 submodule $\wedge^{d} M_{1}$, and if $\wedge^{d} M_{1}$ is defined over $\Gamma$, then so is $M_{1}$.

Now suppose that $M_{1}$ has rank 1 ; this implies by Proposition 5.7 that the lowest slope of $M$ belongs to the value group of $\mathcal{O}$. By applying an isogeny, twisting, and applying Proposition 5.8 we may reduce to the case where the lowest slope is 0 . Let $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ be a basis of $M$ and let $A$ be the matrix such that $F \mathbf{e}_{l}=\sum_{j l} A_{j l} \mathbf{e}_{j}$.

Let $\mathbf{v}$ be an eigenvector of $M$ over $\Gamma^{\text {alg }}$ with $F \mathbf{v}=\mathbf{v}$. We will show that $\mathbf{v}$ is congruent to an element of $M \otimes_{\Gamma} \Gamma^{\text {sep }}$ modulo $\pi^{m}$ for each $m$, by induction on $m$. The case $m=0$ is vacuous, so assume the result is known for some $m$; that is, $\mathbf{v}=\mathbf{w}+\pi^{m} \mathbf{x}$ with $\mathbf{w} \in M \otimes_{\Gamma} \Gamma^{\text {sep }}$ and $\mathbf{x} \in M \otimes_{\Gamma} \Gamma^{\text {alg }}$. Then $0=F \mathbf{v}-\mathbf{v}=(F \mathbf{w}-\mathbf{w})+\pi^{m}(F \mathbf{x}-\mathbf{x})$; that is, $F \mathbf{x}-\mathbf{x}$ belongs to $M \otimes_{\Gamma} \Gamma^{\text {sep }}$. Write $\mathbf{x}=\sum_{j} c_{j} \mathbf{e}_{j}$ and $F \mathbf{x}-\mathbf{x}=\sum d_{j} \mathbf{e}_{j}$, and let $s$ be the smallest nonnegative integer such that the reduction of $c_{j}$ modulo $\pi$ lies in $\left(K^{\text {sep }}\right)^{1 / q^{s}}$ for all $j$. Then $d_{j}=-c_{j}+\sum_{l} A_{j l} c_{l}^{\sigma}$; if $s>0$, then writing $c_{j}=-d_{j}+\sum_{l} A_{j l} c_{l}^{\sigma}$ shows that the reduction of $c_{j}$ lies in ( $\left.K^{\text {sep }}\right)^{1 / q^{s-1}}$ for all $j$, a contradiction. Thus $s=0$, and $\mathbf{x}$ is congruent modulo $\pi$ to an element of $M \otimes_{\Gamma} \Gamma^{\text {sep }}$, completing the induction.

We conclude that $\mathbf{v} \in M \otimes_{\Gamma} \Gamma^{\text {sep }}$. Thus $M_{1}$ is defined both over $\Gamma^{\text {perf }}$ and over $\Gamma^{\text {sep }}$, so it is in fact defined over $\Gamma^{\text {perf }} \cap \Gamma^{\text {sep }}=\Gamma$, as desired. This proves the needed result, except for the final assertion. In case $K$ is separably closed, one can repeat the above argument over a suitable finite extension of $\mathcal{O}$ to show that each $M_{i} / M_{i-1}$ is spanned by eigenvectors, then apply Proposition 5.3.

The following filtration result applies over $\Gamma_{\text {con }}$, not just over $\Gamma$, but requires that $K$ be perfect.

Proposition 5.11 (Descending generic filtration). Let $K$ be a perfect valued field over $k$. Then any $\sigma$-module $M$ over $\Gamma_{\mathrm{con}}=\Gamma_{\mathrm{con}}^{K}$ admits a unique filtration $M_{0}=0 \subset M_{1} \subset \cdots \subset M_{m}=M$ by $\sigma$-submodules such that

1. for $i=1, \ldots, m, M_{i-1}$ is saturated in $M_{i}$ and $M_{i} / M_{i-1}$ has all generic slopes equal to $s_{i}$, and
2. $s_{1}>\cdots>s_{m}$.

Moreover, if $K$ is algebraically closed, each $M_{i} / M_{i-1}$ is isogenous to a direct sum of standard $\sigma$-modules.

Proof. By the Dieudonné-Manin classification (Theorem 5.6), $M$ is canonically isogenous to a direct sum of $\sigma$-submodules, each of a different single slope. By Corollary 2.7, these submodules descend to $\Gamma$; let $M_{1}$ be the submodule of maximum slope. It suffices to show that $M_{1}$ is defined over $\Gamma_{\text {con }}$, as an induction on rank will then yield the general result. Moreover, it is enough to establish this when $M_{1}$ has rank 1: if $M_{1}$ has rank $d$, then the lowest slope submodule of $\wedge^{d} M$ is the rank 1 submodule $\wedge^{d} M_{1}$, and if $\wedge^{d} M_{1}$ is defined over $\Gamma_{\text {con }}$, then so is $M_{1}$.

Now suppose that $M_{1}$ has rank 1 ; this implies that the highest slope of $M$ belongs to the value group of $\mathcal{O}$. Choose $\lambda \in \mathcal{O}$ whose valuation equals that slope. Let $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ be a basis of $M \otimes_{\Gamma_{\text {con }}} \Gamma^{\text {alg }}$, in which $F \mathbf{v}_{1}=\lambda \mathbf{v}_{1}$ and the remaining $\mathbf{v}_{i}$ span the submodules of $M$ of lower slopes. Choose $\mathbf{w}_{i} \in M \otimes_{\Gamma_{\text {con }}} \Gamma_{\text {con }}^{\text {alg }}$ sufficiently close $\pi$-adically to $\mathbf{v}_{i}$ for $i=1, \ldots, n$ so that the matrix $B$ with $\lambda \mathbf{w}_{i}=\sum_{j} B_{i j} F \mathbf{w}_{j}$ has entries in $\Gamma^{\text {alg }}$ and

$$
B_{i j} \equiv\left\{\begin{array}{ll}
1 & i=j=1 \\
0 & \text { otherwise }
\end{array} \quad(\bmod \pi)\right.
$$

this is possible because the congruence holds for $\mathbf{w}_{i}=\mathbf{v}_{i}$. Then the $\mathbf{w}_{i}$ form a basis of $M \otimes_{\Gamma_{\text {con }}} \Gamma_{\text {con }}^{\text {alg }}$.

Write $\mathbf{v}_{1}=\sum_{i} c_{i} \mathbf{w}_{i}$, so that $c_{i}^{\sigma}=\sum_{j} B_{j i} c_{j}$. Since $v_{0}(B) \geq 0$, we can find $r$ such that $w_{r}(B) \geq 0$. We now show that $r v_{h}\left(c_{i}\right)+h \geq 0$ for all $i$ and $h$, by induction on $h$. The case $h=0$ holds because $c_{i} \equiv 0(\bmod \pi)$. Suppose this holds with $h$ replaced by any smaller value. Then the equality $c_{i}^{\sigma}=\sum_{j} B_{j i} c_{j}$ implies

$$
q v_{h}\left(c_{i}\right) \geq \min _{l, j}\left\{v_{l}\left(B_{j i}\right)+v_{h-l}\left(c_{j}\right)\right\} .
$$

Choose $j, l$ for which the minimum is achieved. If $l=0$, then we must have $i=j=1$, in which case $v_{0}\left(B_{11}\right)=0$ and $q v_{h}\left(c_{1}\right) \geq v_{h}\left(c_{1}\right)$, whence $v_{h}\left(c_{1}\right) \geq 0$ and $r v_{h}\left(c_{1}\right)+h \geq 0$ as well. If the minimum occurs for some $l>0$, then

$$
\begin{aligned}
r v_{h}\left(c_{i}\right)+h & \geq r q^{-1}\left(v_{l}\left(B_{j i}\right)+v_{h-l}\left(c_{j}\right)\right)+h \\
& \geq r q^{-1}\left(v_{l}\left(B_{j i}\right)+v_{h-l}\left(c_{j}\right)\right)+q^{-1} h \\
& \geq q^{-1}\left(r v_{l}\left(B_{j i}\right)+l+r v_{h-l}\left(c_{j}\right)+(h-l)\right) \\
& \geq q^{-1}(0+0)=0
\end{aligned}
$$

by the induction hypothesis. Therefore $r v_{h}\left(c_{i}\right)+h \geq 0$ for all $h$, so that $c_{i} \in \Gamma_{\text {con }}^{\text {alg }}$ for each $i$.

We conclude that $\mathbf{v}_{1} \in M \otimes_{\Gamma} \Gamma_{\text {con }}^{\text {alg }}$. Thus $M_{1}$ is defined both over $\Gamma$ and over $\Gamma_{\text {con }}^{\text {alg }}$, and so it is in fact defined over $\Gamma \cap \Gamma_{\text {con }}^{\text {alg }}=\Gamma_{\text {con }}$, as desired. This proves the desired result, except for the final assertion. In case $K$ is algebraically closed, one can repeat the above argument over a suitable finite extension of $\mathcal{O}$ to show that each $M_{i} / M_{i-1}$ is spanned by eigenvectors, then apply Proposition 5.3.

Although we will not use the following result explicitly, it is worth pointing out.

Corollary 5.12. Let $K$ be a valued field, for $k$ algebraically closed. Then any $\sigma$-module $M$ over $\Gamma_{\text {con }}^{K}$, all of whose generic slopes are equal, is isogenous over $\Gamma_{\mathrm{con}}^{\mathrm{sep}}$ to a direct sum of standard $\sigma$-modules.

Proof. In this case, the ascending and descending filtrations coincide, and so both are defined over $\Gamma^{K} \cap \Gamma_{\text {con }}^{\text {perf }}=\Gamma_{\text {con }}^{K}$ and the eigenvectors are defined over $\Gamma_{\text {con }}^{\text {sep }} \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$ for some finite extension $\mathcal{O}^{\prime}$ of $\mathcal{O}$. Thus the claim follows from Proposition 5.3.
5.4. Comparison of the Newton polygons. A $\sigma$-module over $\Gamma_{\text {con }}$ can be base-extended both to $\Gamma$ and to $\Gamma_{\text {an,con }}$; as a result, it admits both a generic and a special Newton polygon. In this section, we compare these two polygons. The main results are that the special polygon lies above the generic polygon, and that when the two coincide, the $\sigma$-module admits a partial decomposition over $\Gamma_{\text {con }}$ (reminiscent of the Newton-Hodge decomposition of [Ka]).

Throughout this section, $K$ is an arbitrary valued field, which we suppress from the notation.

Proposition 5.13. Let $M$ and $N$ be $\sigma$-modules over $\Gamma_{\text {con }}$. Let $r_{1}, \ldots, r_{m}$ and $s_{1}, \ldots, s_{n}$ be the generic (resp. special) slopes of $M$ and $N$.

1. The generic (resp. special) slopes of $M \oplus N$ are $r_{1}, \ldots, r_{m}, s_{1}, \ldots, s_{n}$.
2. The generic (resp. special) slopes of $M \otimes N$ are $r_{i}+s_{j}$ for $i=1, \ldots, m$ and $j=1, \ldots, n$.
3. The generic (resp. special) slopes of $\wedge^{l} M$ are $r_{i_{1}}+\cdots+r_{i_{l}}$ for $1 \leq i_{1}<$ $\cdots<i_{l} \leq m$.
4. The generic (resp. special) slopes of $M^{*}$ are $-r_{1}, \ldots,-r_{m}$.

Proof. These results follow immediately from the definition of the generic (resp. special) Newton slopes as the valuations of the eigenvalues of a basis of eigenvectors of $M$ over $\Gamma^{\text {alg }}$ (resp. $\Gamma_{\mathrm{an}, \text { con }}^{\mathrm{alg}}$ ).

Proposition 5.14. Let $M$ be a $\sigma$-module over $\Gamma_{\text {con }}$. Then the special Newton polygon lies above the generic Newton polygon, and both have the same endpoint.

Proof. The Newton polygons coincide for $M$ of rank 1 because $M$ has an eigenvector over $\Gamma_{\mathrm{con}}^{\text {alg }}$ by Proposition 3.18. Thus the Newton polygons of $\wedge^{n} M$ coincide for $n=\operatorname{rank} M$; that is, the Newton polygons of $M$ have the same
endpoint. By the descending slope filtration (Proposition 5.11), $M$ admits a basis $\mathbf{w}_{1}, \ldots, \mathbf{w}_{n}$ over $\Gamma_{\text {con }}^{\text {alg }}$ such that modulo $\operatorname{SatSpan}\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{i-1}\right), \mathbf{w}_{i}$ is an eigenvector whose slope is the $i$-th largest generic slope of $M$. Let $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ be a basis of eigenvectors of $M$ over $\Gamma_{\mathrm{an}, \text { con }}^{\mathrm{alg}}$; then by Proposition 4.5, the sequence of valuations of the eigenvalues of the $\mathbf{w}_{i}$ majorizes that of the $\mathbf{v}_{i}$. In other words, the sequence of generic slopes majorizes the sequence of special slopes, whence the comparison of Newton polygons.

Proposition 5.15. Let $0 \rightarrow M_{1} \rightarrow M \rightarrow M_{2} \rightarrow 0$ be an exact sequence of $\sigma$-modules over $\Gamma_{\text {con }}$. Suppose the least generic slope of $M_{2}$ is greater than the greatest generic slope of $M_{1}$. Then the special Newton polygon of $M$ is equal to the union of the special Newton polygons of $M_{1}$ and $M_{2}$.

Proof. The least generic slope of $M_{2}$ is less than or equal to its least special slope, and the greatest generic slope of $M_{1}$ is greater than or equal to its greatest special slope, both by Proposition 5.14. Thus we may apply Proposition 4.3 over $\Gamma_{\text {an, con }}^{\text {alg }}$ (after extending $\mathcal{O}$ suitably) to deduce the desired result.

It is perhaps not surprising that when the generic and special Newton polygons coincide, one gets a slope filtration that descends farther than usual.

Proposition 5.16. Let $M$ be a $\sigma$-module over $\Gamma_{\text {con }}$ whose generic and special Newton polygons coincide. Then $M$ admits an ascending slope filtration over $\Gamma_{\text {con }}$.

Proof. We need to show that the ascending slope filtration of Proposition 5.10 is defined over $\Gamma_{\text {con }}$; it is enough to verify this after enlarging $\mathcal{O}$. This lets us assume that $k$ is algebraically closed, and that the value group of $\mathcal{O}$ contains all of the slopes of $M$. By Theorem 4.16, we can find a basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ of eigenvectors of $M$ over $\Gamma_{\mathrm{an}, \text { con }}^{\text {alg }}$, with $F \mathbf{v}_{i}=\lambda_{i} \mathbf{v}_{i}$ for $\lambda_{i} \in \mathcal{O}_{0}\left[\frac{1}{p}\right]$ such that $v_{p}\left(\lambda_{1}\right) \geq \cdots \geq v_{p}\left(\lambda_{n}\right)$. By Proposition 5.11 (the descending slope filtration), we can find a basis up to isogeny $\mathbf{w}_{1}, \ldots, \mathbf{w}_{n}$ of $M$ over $\Gamma_{\text {con }}^{\text {alg }}$ such that $F \mathbf{w}_{i}=\lambda_{i} \mathbf{w}_{i}+\sum_{j<i} A_{i j} \mathbf{w}_{j}$ for some $A_{i j} \in \Gamma_{\text {con }}^{\text {alg }}$.

Write $\mathbf{v}_{n}=\sum_{i} b_{i} \mathbf{w}_{i}$ with $b_{i} \in \Gamma_{\text {an }, \text { con }}^{\text {alg }}$; applying $F$ to both sides, we have $\lambda_{n} b_{i}=\lambda_{i} b_{i}^{\sigma}+\sum_{j>i} b_{j}^{\sigma} A_{j i}$ for $i=1, \ldots, n$. By Proposition 3.19(a) and (c), we obtain $b_{i} \in \Gamma_{\text {con }}^{\text {alg }}\left[\frac{1}{p}\right]$ for $i=n, n-1, \ldots, 1$, and so $\mathbf{v}_{n}$ is defined over $\Gamma_{\text {con }}^{\text {alg }}\left[\frac{1}{p}\right]$.

By repeating the above reasoning, we see that the image of $\mathbf{v}_{i}$ in $M / \operatorname{SatSpan}\left(\mathbf{v}_{i+1}, \ldots, \mathbf{v}_{n}\right)$ is defined over $\Gamma_{\operatorname{con}}^{\text {alg }}\left[\frac{1}{p}\right]$ for $i=n, \ldots, 1$. Thus the ascending slope filtration is defined over $\Gamma_{\mathrm{con}}^{\text {alg }}\left[\frac{1}{p}\right]$. Since it is also defined over $\Gamma$ by Proposition 5.10, it is in fact defined over $\Gamma \cap \Gamma_{\mathrm{con}}^{\mathrm{alg}}\left[\frac{1}{p}\right]=\Gamma_{\mathrm{con}}$, as desired.

## 6. From a slope filtration to quasi-unipotence

In this chapter we construct a canonical filtration of a $\sigma$-module over $\Gamma_{\text {an,con }}^{k((t))}$. We do this by partially descending the special slope filtration obtained over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ in Chapter 4. More specifically, we show that by changing the basis over a nearly finite extension of $\Gamma_{\text {an,con }}$, we can make Frobenius act by a matrix with entries in a nearly finite extension of $\Gamma_{\text {con }}$, whose generic Newton polygon coincides with the special Newton polygon, allowing the use of Proposition 5.16. This will yield the desired filtration (Theorem 6.10), from which we deduce the $p$-adic local monodromy theorem (Theorem 1.1) using the quasi-unipotence of unit-root $(\sigma, \nabla)$-modules over $\Gamma_{\text {con }}$; the latter is a theorem of Tsuzuki [T1] (for which see also Christol [Ch]).
6.1. Approximation of matrices. We collect some results that allow us to approximate matrices from a large ring with matrices from smaller rings. Note: we will need the notions of slopes and Newton polygons from Section 3.5.

Lemma 6.1. Let $K$ be a nearly finite extension of $k((t))$ and suppose $\Gamma_{r}^{K}$ contains a unit lifting a uniformizer of $K$. Then for any $x, y \in \Gamma_{r}^{K}\left[\frac{1}{p}\right], x$ is coprime to $y+\pi^{j}$ for all sufficiently large integers $j$.

Proof. Suppose on the contrary that $x$ and $y+\pi^{j}$ fail to be coprime for $j=j_{1}, j_{2}, \ldots$ By Corollary 3.32, the ideal $\left(x, y+\pi^{j_{l}}\right)$ in $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ is principal; let $d_{l}$ be a generator. Note that $\left(y+\pi^{j_{i}}, y+\pi^{j_{l}}\right)$ contains the unit $\pi^{j_{i}}-\pi^{j_{l}}$ for $i \neq l$, and so it is the unit ideal; this means the $d_{l}$ are pairwise coprime, and $x$ is divisible by $d_{1} \cdots d_{l}$ for any $l$. But $x$ has only finite total multiplicity while each $d_{l}$ has nonzero total multiplicity, a contradiction. Hence $x$ is coprime to $y+\pi^{j}$ for $j$ sufficiently large, as desired.

By an elementary operation on a matrix over a ring, we mean one of the following operations:
(a) adding a multiple of one row to another;
(b) multiplying one row by a unit of the ring;
(c) interchanging two rows.

An elementary matrix is one obtained from the identity matrix by a single elementary operation; multiplying a matrix on the right by an elementary matrix has the same effect as performing the corresponding elementary operation.

Lemma 6.2. Pick s such that $0<s<r$, and let $U$ be a matrix over $\Gamma_{\mathrm{an}, r}^{\mathrm{alg}}$ such that $w_{l}(\operatorname{det}(U)-1)>0$ for $s \leq l \leq r$. Then there exists an invertible matrix $V$ over $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$, for some nearly finite extension $K$ of $k((t))$, such that
$w_{l}(U V-I)>0$ for $s \leq l \leq r$. Moreover, if $U$ is defined over $\Gamma_{\text {an,r }}^{k((t))}$ and $t$ lifts to a semi-unit in $\Gamma_{r}^{k((t))}$, then we may take $K=k((t))$.

Although we only will apply this when $U$ is invertible, we need to formulate the more general statement in order to carry out the induction.

Proof. We induct on $n$, the case $n=1$ being vacuous. Let $M_{i}$ denote the cofactor of $U_{n i}$ in $U$, so that $\operatorname{det}(U)=\sum_{i} M_{i} U_{n i}$; note that $M_{i}=$ $\left(U^{-1}\right)_{i n} \operatorname{det}(U)$ in $\operatorname{Frac}\left(\Gamma_{\mathrm{an}, r}^{\mathrm{alg}}\right)$. Let $d$ be a generator of the ideal $\left(M_{1}, \ldots, M_{n}\right)$ in $\Gamma_{\mathrm{an}, r}^{\mathrm{alg}}$. Then $d$ divides $\operatorname{det}(U)$; by the hypothesis that $w_{l}(\operatorname{det}(U)-1)>0$ for $s \leq l \leq r$, the largest slope of $\operatorname{det}(U)$ is less than $s$, and so the largest slope of $d$ is also less than $s$. By Lemma 3.24, there exists a unit $u \in \Gamma_{\mathrm{an}, r}^{\mathrm{alg}}$ such that $w_{l}(u d-1)>0$ for $s \leq l \leq r$.

Let $\alpha_{1}, \ldots, \alpha_{n}$ be elements of $\Gamma_{\mathrm{an}, r}^{\mathrm{alg}}$ such that $\sum_{i} \alpha_{i} M_{i}=u d$. Choose $\beta_{1}, \ldots, \beta_{n-1}$ and $\beta_{n}^{\prime} \in \Gamma_{r}^{L}\left[\frac{1}{p}\right]$, for some nearly finite extension $L$ of $k((t))$, so that for $s \leq l \leq r$,
and

$$
w_{l}\left(\beta_{i}-\alpha_{i}\right)>-\max _{i}\left\{w_{l}\left(M_{i}\right)\right\} \quad(i=1, \ldots, n-1)
$$

$$
w_{l}\left(\beta_{n}^{\prime}-\alpha_{n}\right)>\max _{i}\left\{w_{l}\left(M_{i}\right)\right\}
$$

By Lemma 6.1, we can find $j$ for which $\beta_{n}=\beta_{n}^{\prime}+\pi^{j}$ has the properties that $w_{l}\left(\beta_{n}-\alpha_{n}\right)>\max _{i}\left\{w_{l}\left(M_{i}\right)\right\}$ for $s \leq l \leq r$ and $\left(\beta_{1}, \ldots, \beta_{n}\right)$ is the unit ideal in $\Gamma_{r}^{L}\left[\frac{1}{p}\right]$. (Both hold for $j$ sufficiently large.)

By Corollary 3.32, $\Gamma_{r}^{L}\left[\frac{1}{p}\right]$ is a Bézout ring. Thus Lemma 2.3 can be applied to produce a matrix $A$ over $\Gamma_{r}^{L}\left[\frac{1}{p}\right]$ of determinant 1 such that $A_{n i}=\beta_{i}$ for $i=1, \ldots, n$. Put $U^{\prime}=U A^{-1}$, and let $M_{n}^{\prime}$ be the cofactor of $U_{n n}^{\prime}$ in $U^{\prime}$. Then

$$
\begin{aligned}
M_{n}^{\prime} & =\left(\left(U^{\prime}\right)^{-1}\right)_{n n} \operatorname{det}\left(U^{\prime}\right) \\
& =\left(A U^{-1}\right)_{n n} \operatorname{det}(U) \operatorname{det}\left(A^{-1}\right) \\
& =\sum_{i} A_{n i}\left(U^{-1}\right)_{i n} \operatorname{det}(U) \\
& =\sum_{i} \beta_{i} M_{i}
\end{aligned}
$$

so that

$$
M_{n}^{\prime}-1=u d-1+\sum_{i}\left(\beta_{i}-\alpha_{i}\right) M_{i}
$$

and hence $w_{l}\left(M_{n}^{\prime}-1\right)>0$ for $s \leq l \leq r$.
Apply the induction hypothesis to the upper left $(n-1) \times(n-1)$ submatrix of $U^{\prime}$, let $V^{\prime}$ be the resulting matrix, and enlarge $L$ if needed so that $V^{\prime}$ has entries in $\Gamma_{\mathrm{an}, r}^{L}$. Extend $V^{\prime}$ to an $n \times n$ matrix by setting $V_{n n}^{\prime}=1$ and $V_{n i}^{\prime}=V_{i n}^{\prime}=0$ for $i=1, \ldots, n-1$. Then for $s \leq l \leq r, w_{l}\left(\left(U^{\prime} V^{\prime}-I\right)_{i j}\right)>0$ for $1 \leq i, j \leq n-1$. Moreover, $w_{l}\left(\operatorname{det}\left(V^{\prime}\right)-1\right)>0$, so $w_{l}\left(\operatorname{det}\left(U^{\prime} V^{\prime}\right)-1\right)>0$ as well.

We now exhibit a sequence of elementary operations which can be performed on $U^{\prime} V^{\prime}$ to obtain a new matrix $W$ over $\Gamma_{\text {an }, r}^{\text {alg }}$ with $w_{l}(W-I)>0$ for $s \leq l \leq r$; it may clarify matters to regard the procedure as an "approximate Gaussian elimination". First, define a sequence of matrices $\left\{X^{(h)}\right\}_{h=0}^{\infty}$ by $X^{(0)}=U^{\prime} V^{\prime}$ and

$$
X_{i j}^{(h+1)}= \begin{cases}X_{i j}^{(h)} & i<n \\ X_{n j}^{(h)}-\sum_{m=1}^{n-1} X_{n m}^{(h)} X_{m j}^{(h)} & i=n\end{cases}
$$

note that $X^{(h+1)}$ is obtained from $X^{(h)}$ by subtracting $X_{n m}^{(h)}$ times the $m$-th row from the $n$-th row for $m=1, \ldots, n-1$. At each step, $\min _{1 \leq j \leq n-1}\left\{w_{l}\left(X_{n j}^{(h)}\right)\right\}$ increases by at least $\min _{1 \leq i, j \leq n-1}\left\{w_{l}\left(\left(U^{\prime} V^{\prime}-I\right)_{i j}\right)\right\}$; thus for $h$ sufficiently large,

$$
w_{l}\left(X_{n j}^{(h)}\right)>\max \left\{0, \max _{1 \leq i \leq n-1}\left\{-w_{l}\left(X_{i n}^{(h)}\right)\right\}\right\} \quad(s \leq l \leq r ; j=1, \ldots, n-1)
$$

Pick such an $h$ and set $X=X_{h}$. Then $w_{l}\left((X-I)_{i j}\right)>0$ for $1 \leq i \leq n$ and $1 \leq j \leq n-1, w_{l}\left(X_{i n} X_{n j}\right)>0$ for $1 \leq i, j \leq n-1$, and $w_{l}(\operatorname{det}(X)-1)>0$. These together imply $w_{l}\left(X_{n n}-1\right)>0$.

Next, define a sequence of matrices $\left\{W^{(h)}\right\}_{h=0}^{\infty}$ by $W^{(0)}=X$ and

$$
W_{i j}^{(h+1)}= \begin{cases}W_{i j}^{(h)}-W_{i n}^{(h)} W_{n j}^{(h)} & i<n \\ W_{i j}^{(h)} & i=n\end{cases}
$$

note that $W^{(h+1)}$ is obtained from $W^{(h)}$ by subtracting $W_{i n}^{(h)}$ times the $n$-th row from the $i$-th row for $i=1, \ldots, n-1$. At each step, $w_{l}\left(X_{i n}^{(h)}\right)$ increases by at least $w_{l}\left(X_{n n}^{(h)}-1\right)$; thus for $h$ sufficiently large,

$$
w_{l}\left(W_{i n}^{(h)}\right)>0 \quad(s \leq l \leq r ; i=1, \ldots, n-1) .
$$

Pick such an $h$ and set $W=W_{h}$; then $w_{l}(W-I)>0$ for $s \leq l \leq r$.
To conclude, note that by construction, $\left(U^{\prime} V^{\prime}\right)^{-1} W$ is a product of elementary matrices over $\Gamma_{\mathrm{an}, r}^{\mathrm{alg}}$ of type (a). By suitably approximating each elementary matrix by one defined over $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ for a suitable extension $K$ of $L$, we get a matrix $X$ such that $w_{l}\left(U^{\prime} V^{\prime} X-I\right)>0$ for $s \leq l \leq r$. We may thus take $V=A^{-1} V^{\prime} X$.

We will need a refinement of the above result.
Lemma 6.3. Pick s such that $0<s<r$, and let $U$ be a matrix over $\Gamma_{\mathrm{an}, r}^{\mathrm{alg}}$ such that $w_{l}(\operatorname{det}(U)-1)>0$ for $s \leq l \leq r$. Then for any $c>0$, there exists a nearly finite extension $K$ of $k((t))$ and an invertible matrix $V$ over $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ such that $w_{l}(U V-I) \geq c$ for $s \leq l \leq r$. Moreover, if $U$ is defined over $\Gamma_{\mathrm{an}, r}^{k((t))}$ and $t$ lifts to a semi-unit in $\Gamma_{r}^{k((\bar{t}))}$, then we may take $K=k((t))$.

Proof. Put

$$
s^{\prime}=s\left(1+c / v_{p}(\pi)\right)^{-1}
$$

Apply Lemma 6.2 to obtain a nearly finite extension $L$ of $k((t))$ and an invertible matrix $V^{\prime}$ over $\Gamma_{r}^{L}\left[\frac{1}{p}\right]$ (with $L=k((t))$ in case $U$ is defined over $\Gamma_{\text {an, }}^{k((t))}$ ) such that $w_{l}\left(U V^{\prime}-I\right)>0$ for $s^{\prime} \leq l \leq r$.

Choose semi-unit decompositions $\sum_{h} W_{i j h} \pi^{h}$ of $\left(U V^{\prime}\right)_{i j}-I$ for $1 \leq i$, $j \leq n$. For $s \leq l \leq r$ and $m<0$ in the value group of $\mathcal{O}$, we deduce from $w_{s^{\prime}}\left(U V^{\prime}-I\right)>0$ that

$$
\begin{aligned}
l v_{m}\left(U V^{\prime}-I\right)+m & =\left(l / s^{\prime}\right)\left(s^{\prime} v_{m}\left(U V^{\prime}-I\right)+m\right)-m\left(l / s^{\prime}-1\right) \\
& >-m\left(l / s^{\prime}-1\right) \\
& >v_{p}(\pi)\left(s / s^{\prime}-1\right) \\
& =c
\end{aligned}
$$

Define a matrix $X$ by $X_{i j}=\sum_{h \geq 0} W_{i j h} \pi^{h}$; then $U V^{\prime}-I-X=\sum_{h<0} W_{i j h} \pi^{h}$, so that for $s \leq l \leq r$,

$$
w_{l}\left(U V^{\prime}-I-X\right)=\min _{m<0}\left\{l v_{m}\left(U V^{\prime}-I\right)+m\right\} \geq c
$$

By construction, $v_{m}(X)=\infty$ for $m<0$ and $v_{0}(X)>0$. Thus $I+X$ is invertible over $\Gamma_{\mathrm{an}, r}^{\mathrm{alg}}$. Choose a matrix $W$ over $\Gamma_{r}^{K}$, for some extension $K$ of $L$ (with $K=k((t))$ if $U$ is defined over $\left.\Gamma_{\text {an }, r}^{k((t))}\right)$, such that $w_{l}\left(W-(I+X)^{-1}\right) \geq c$ for $s \leq l \leq r$. Then $W$ is invertible over $\Gamma_{r}^{K}$, and for $s \leq l \leq r$,

$$
\begin{aligned}
& w_{l}\left(U V^{\prime} W-I\right) \\
& \quad=w_{l}\left(\left(U V^{\prime}-I-X\right) W+(I+X)\left(W-(I+X)^{-1}\right)\right) \\
& \quad \geq \min \left\{w_{l}\left(U V^{\prime}-I-X\right)+w_{l}(W), w_{l}(I+X)+w_{l}\left(W-(I+X)^{-1}\right)\right\} \\
& \quad \geq c
\end{aligned}
$$

We may thus take $V=V^{\prime} W$.
6.2. Some matrix factorizations. Throughout this section, we take $K=$ $k((t))$ and omit it from the notation; note also the use of the naïve partial valuations. Let $\Gamma_{u}$ and $\Gamma_{\mathrm{an}, u}$ denote the subrings of $\Gamma_{\mathrm{con}}$ and $\Gamma_{\mathrm{an}, \mathrm{con}}$, respectively, consisting of elements $x$ of the form $\sum_{i=0}^{\infty} x_{i} u^{i}$.

Lemma 6.4. For $r>0$ and $c>0$, let $A$ be a matrix over $\Gamma_{\mathrm{an}, r}$ such that $w_{r}^{\text {naive }}(A-I) \geq c$. Then there exists a unique pair of matrices $U=I+\sum_{i=1}^{\infty} U_{i} u^{i}$ over $\Gamma_{\mathrm{an}, r}$ and $V=\sum_{i=0}^{\infty} V_{i} u^{-i}$ over $\Gamma_{r}$ such that $w_{r}^{\text {naive }}(U-I)>0, w_{r}^{\text {naive }}(V-I)>0$, and $A=U V$. Moreover, these matrices satisfy $w_{r}^{\text {naive }}(U-I) \geq c$ and $w_{r}^{\text {naive }}(V-I) \geq c$.

Proof. Define a sequence of matrices $\left\{B^{(j)}\right\}_{j=0}^{\infty}$ as follows. Begin by setting $B^{(0)}=I$. Given $B^{(j)}$ for some $j$, put $A\left(B^{(j)}\right)^{-1}=\sum_{i=-\infty}^{\infty} X_{i}^{(j)} u^{i}, C^{(j)}=$ $\sum_{i \leq 0} X_{i}^{(j)} u^{i}, D^{(j)}=\sum_{i>0} X_{i}^{(j)} u^{i}$, and put $B^{(j+1)}=C^{(j)} B^{(j)}$.

Since $w_{r}^{\text {naive }}(A-I) \geq c$, we have $w_{r}^{\text {naive }}\left(C^{(0)}-I\right) \geq c$ and $w_{r}^{\text {naive }}\left(D^{(0)}\right) \geq c$ as well. Thus $w_{r}^{\text {naive }}\left(A\left(B^{(1)}\right)^{-1}-I\right) \geq c$, and by induction one has $w_{r}^{\text {naive }}\left(C^{(j)}-I\right) \geq c$ and $w_{r}^{\text {naive }}\left(D^{(j)}\right) \geq c$ for all $j$. But we can do better, by showing by induction that $w_{r}^{\text {naive }}\left(C^{(j)}-I\right) \geq(j+1) c$ and $w_{r}^{\text {naive }}\left(D^{(j+1)}-\right.$ $\left.D^{(j)}\right) \geq(j+2) c$ for $j \geq 0$. Given $w_{r}^{\text {naive }}\left(C^{(j)}-I\right) \geq(j+1) c$, we have

$$
\begin{aligned}
A\left(B^{(j+1)}\right)^{-1}-I & =A\left(B^{(j)}\right)^{-1}\left(C^{(j)}\right)^{-1}-I \\
& =\left(C^{(j)}+D^{(j)}\right)\left(C^{(j)}\right)^{-1}-I \\
& =D^{(j)}\left(C^{(j)}\right)^{-1} \\
& =D^{(j)}+D^{(j)}\left(\left(C^{(j)}\right)^{-1}-I\right) .
\end{aligned}
$$

Since $D^{(j)}$ has only positive powers of $u, C^{(j+1)}$ is equal to the sum of the terms of $D^{(j)}\left(\left(C^{(j)}\right)^{-1}-I\right)$ involving nonpositive powers of $u$. In particular,

$$
w_{r}^{\text {naive }}\left(C^{(j+1)}-I\right) \geq w_{r}^{\text {naive }}\left(D^{(j)}\left(\left(C^{(j)}\right)^{-1}-I\right)\right) \geq c+(j+1) c=(j+2) c ;
$$

likewise, $D^{(j+1)}-D^{(j)}$ consists of terms from $D^{(j)}\left(\left(C^{(j)}\right)^{-1}-I\right)$, so that $w_{r}^{\text {naive }}\left(D^{(j+1)}-D^{(j)}\right) \geq(j+2) c$. This completes the induction.

Since $C^{(j)}$ converges to $I$, we see that $B^{(j)}$ converges to a limit $V$ such that $w_{r}^{\text {naive }}(V-I) \geq c$. Under $w_{r}^{\text {naive }}, I+D^{(j)}$ also converges to a limit $U$ such that $w_{r}^{\text {naive }}(U-I) \geq c$, and $A\left(B^{(j)}\right)^{-1}-I-D^{(j)}$ converges to 0 . Therefore $A V^{-1}=U$ has entries in $\Gamma_{\text {an, } r \text {, naive }}$, and $U$ and $V$ satisfy the desired conditions.

This establishes the existence of the desired factorization. To establish uniqueness, suppose we have a second decomposition $A=U^{\prime} V^{\prime}$ with $U^{\prime}-I$ only involving positive powers of $u, V^{\prime}$ only involving negative powers of $u$, $w_{r}^{\text {naive }}\left(U^{\prime}-I\right)>0$, and $w_{r}^{\text {naive }}\left(V^{\prime}-I\right)>0$. Within the completion of $\Gamma_{r}\left[\frac{1}{p}\right]$ with respect to $|\cdot|_{r}$, the matrices $U, V, U^{\prime}, V^{\prime}$ are invertible and $\left(U^{\prime}\right)^{-1} U=$ $V^{\prime} V^{-1}$. On the other hand, $\left(U^{\prime}\right)^{-1} U-I$ involves only positive powers of $u$, while $V^{\prime} V^{-1}-I$ involves no positive powers of $u$. This is only possible if $\left(U^{\prime}\right)^{-1} U-I=V^{\prime} V^{-1}-I=0$, which yields $U=U^{\prime}$ and $V=V^{\prime}$.

The following proposition may be of interest outside of its use to prove the results of this paper. For example, Berger's proof [Bg, Cor. 0.3] that any crystalline representation is of finite height uses a lemma from [Ke1] equivalent to this.

Proposition 6.5. Let $A$ be an invertible matrix over $\Gamma_{\mathrm{an}, \mathrm{con}}$. Then there exist invertible matrices $U$ over $\Gamma_{\mathrm{an}, u}$ and $V$ over $\Gamma_{\text {con }}\left[\frac{1}{p}\right]$ such that $A=U V$. Moreover, if $w_{r}^{\text {naive }}(A-I)>0$ for some $r>0$, there are unique choices of $U$ and $V$ respectively such that $U-I$ involves only positive powers of $u, V$
involves no positive powers of $u$, $w_{r}^{\text {naive }}(U-I)>0$ and $w_{r}^{\text {naive }}(V-I)>0$; for these $U$ and $V, \min \left\{w_{r}^{\text {naive }}(U-I), w_{r}^{\text {naive }}(V-I)\right\} \geq w_{r}^{\text {naive }}(A-I)$.

Proof. By Lemma 6.2, there exists an invertible matrix $W$ over $\Gamma_{\text {con }}\left[\frac{1}{p}\right]$ such that $w_{r}^{\text {naive }}(A W-I)>0$. Apply Lemma 6.4 to write $A W=U_{1} V_{1}$ for matrices $U_{1}$ over $\Gamma_{\text {an }, u}$ and $V_{1}$ over $\Gamma_{\text {con }}$, and to write $(A W)^{-T}=U_{2} V_{2}$ for matrices $U_{2}$ over $\Gamma_{\text {an }, u}$ and $V_{2}$ over $\Gamma_{\text {con }}$. Now $I=(A W)^{T}(A W)^{-T}=V_{1}^{T} U_{1}^{T} U_{2} V_{2}$, and so $V_{1}^{-T} V_{2}^{-1}=U_{1}^{T} U_{2}$ has entries in $\Gamma_{\text {con }} \cap \Gamma_{\mathrm{an}, u}=\Gamma_{u}$. Moreover, $U_{1}^{T} U_{2}-I$ involves only positive powers of $u$, and so $U_{1}^{T} U_{2}$ is invertible over $\Gamma_{u}$ and $U_{1}$ is invertible over $\Gamma_{\mathrm{an}, u}$. Our desired factorization is now $A=U V$ with $U=U_{1}$ and $V=V_{1} W^{-1}$. If $w_{r}^{\text {naive }}(A-I)>0$, we may take $W=I$ above and deduce the uniqueness from Lemma 6.4.

So far we have exhibited factorizations that separate positive and negative powers of $u$. We use these to give a factorization that separates a matrix over $\Gamma_{\mathrm{an}, \mathrm{con}}$ into a matrix over $\Gamma_{\text {con }}$ times a matrix with only positive powers of $u$, in such a way that the closer the original matrix is to being defined over $\Gamma_{\text {con }}$, the smaller the positive matrix will be.

Proposition 6.6. Let $A$ be an invertible matrix over $\Gamma_{\mathrm{an}, r}$ such that $w_{r}^{\text {naive }}(A-I)>0$. Then there exists a canonical pair of invertible matrices $U$ over $\Gamma_{\mathrm{an}, u}$ and $V$ over $\Gamma_{\mathrm{con}}$ such that $A=U V, U-I$ has only positive powers of $u, V-I \equiv 0(\bmod \pi), w_{r}^{\text {naive }}(V-I) \geq w_{r}^{\text {naive }}(A-I)$ and

$$
w_{r}^{\text {naive }}(U-I) \geq \min _{m \leq 0}\left\{r v_{m}^{\text {naive }}(A-I)+m\right\}
$$

Here "canonical" does not mean "unique". It means that the construction of $U$ and $V$ depends only on $A$ and not on $r$.

Proof. Write $A-I=\sum_{i} A_{i} u^{i}$, and let $X$ be the sum of $A_{i}$ over all $i$ for which $v_{p}\left(A_{i}\right)>0$. Then

$$
\begin{aligned}
w_{r}^{\text {naive }}\left(A(I+X)^{-1}-I\right) & \geq w_{r}^{\text {naive }}(A-I-X)+w_{r}^{\text {naive }}\left((I+X)^{-1}\right) \\
& =\min _{v_{p}\left(A_{i}\right) \leq 0}\left\{v_{p}\left(A_{i}\right)+r i\right\} \\
& =\min _{m \leq 0}\left\{r v_{m}^{\text {naive }}(A-I)+m\right\}
\end{aligned}
$$

Apply Proposition 6.5 to factor $A(I+X)^{-1}$ as $B C$, where

$$
\min \left\{w_{r}^{\text {naive }}(B-I), w_{r}^{\text {naive }}(C-I)\right\} \geq \min _{m \leq 0}\left\{r v_{m}^{\text {naive }}(A-I)+m\right\}
$$

$B-I$ involves only positive powers of $u$, and $C$ involves no positive powers of $u$; the desired matrices are $U=B$ and $V=C(I+X)$.
6.3. Descending the special slope filtration. In this section, we refine the decomposition given by Theorem 4.16 in the case of a $\sigma$-module defined over $\Gamma_{\text {an,con }}^{k((t))}$, to obtain our main filtration theorem.

Lemma 6.7. For $K$ a valued field and $r>0$ satisfying the conclusion of Proposition 3.11, let $U$ be a matrix over $\Gamma_{\mathrm{an}, r}^{K}$ and $V$ a matrix over $\Gamma_{r}^{K}$ such that $w_{r}(V-I)>0$ and $v_{p}(V-I)>0$. Then

$$
\min _{m \leq 0}\left\{r v_{m}(U V-I)+m\right\}=\min _{m \leq 0}\left\{r v_{m}(U-I)+m\right\} .
$$

Proof. In one direction, we have

$$
\begin{aligned}
\min _{m \leq 0}\left\{r v_{m}(U V-I)+m\right\} & =\min _{m \leq 0}\left\{r v_{m}((U-I) V+(V-I))+m\right\} \\
& =\min _{m \leq 0}\left\{r v_{m}((U-I) V)+m\right\} \\
& \geq \min _{m \leq 0, l \geq 0}\left\{r v_{l}(V)+l+r v_{m-l}(U-I)+(m-l)\right\} \\
& \geq \min _{m \leq 0}\left\{r v_{m}(U-I)+m\right\},
\end{aligned}
$$

the last inequality holding because $w_{r}(V)=0$. The reverse direction is implied by the above inequality with $U$ and $V$ replaced by $U V$ and $V^{-1}$.

The key calculation is the following proposition. In fact, it should be possible to give a condition of this form that guarantees that a $\sigma$-module has a particular special Newton polygon. However, we have not found such a condition so far.

Proposition 6.8. Let $K$ be a nearly finite extension of $k((t))$ and $r>0$ a number for which there exists a semi-unit $u$ in $\Gamma_{q r}^{K}$ lifting a uniformizer of $K$. Let $A$ be an invertible matrix over $\Gamma_{\mathrm{an}, r}$, and suppose that there exists a diagonal matrix $D$ over $\mathcal{O}$ such that

$$
w_{r}\left(A D^{-1}-I\right)>\max _{i, j}\left\{v_{p}\left(D_{i i}\right)-v_{p}\left(D_{j j}\right)\right\} .
$$

Then there exists an invertible matrix $U$ over $\Gamma_{\mathrm{an}, q r}$ such that $w_{r}(U-I)>0$, $U-I$ involves only positive powers of $u, U^{-1} A U^{\sigma} D^{-1}$ is invertible over $\Gamma_{r}$ and $v_{p}\left(U^{-1} A U^{\sigma} D^{-1}-I\right)>0$.

Proof. There is no loss of generality in assuming $K=k((t))$. Then by Lemma 3.7, for $s \leq q r$ and $x \in \Gamma_{\mathrm{an}, q r}, w_{s}(x)=w_{s}^{\text {naive }}(x)$ and $\min _{m \leq 0}\left\{s v_{m}(x)+m\right\}=\min _{m \leq 0}\left\{s v_{m}^{\text {naive }}(x)+m\right\}$. This allows us to apply the results of the previous section.

Put $c=\max _{i, j}\left\{v_{p}\left(D_{i i}\right)-v_{p}\left(D_{j j}\right)\right\}$ and $d=w_{r}\left(A D^{-1}-I\right)$, and define sequences $\left\{A_{i}\right\},\left\{U_{i}\right\},\left\{V_{i}\right\}$ for $i=0,1, \ldots$ as follows. Begin with $A_{0}=A$. Given $A_{i}$, factor $A_{i} D^{-1}$ as $U_{i} V_{i}$ as per Proposition 6.6, and set $A_{i+1}=U_{i}^{-1} A_{i} U_{i}^{\sigma}$, so that $A_{i+1} D^{-1}=V_{i}\left(D U_{i}^{\sigma} D^{-1}\right)$.

Note that the application of Proposition 6.6 is only valid if $w_{r}\left(A_{i} D^{-1}-I\right)>0$. In fact, we will show that

$$
\min _{m \leq 0}\left\{r v_{m}\left(A_{i} D^{-1}-I\right)+m\right\} \geq d+i((q-1) d-c)
$$

and

$$
w_{r}\left(A_{i} D^{-1}-I\right) \geq d-c>0
$$

by induction on $i$. Both assertions hold for $i=0$. Given that they hold for $i$, we have $w_{r}\left(U_{i}-I\right) \geq \min _{m \leq 0}\left\{r v_{m}\left(A_{i} D^{-1}-I\right)+m\right\}$ by Proposition 6.6. On one hand, we have

$$
\begin{aligned}
w_{r}\left(D U_{i}^{\sigma} D^{-1}-I\right) & \geq w_{q r}\left(U_{i}-I\right)-c \\
& =\min _{m}\left\{q r v_{m}^{\text {naive }}\left(U_{i}-I\right)+m\right\}-c \\
& \geq \min _{m}\left\{r v_{m}^{\text {naive }}\left(U_{i}-I\right)+m\right\}-c \\
& =w_{r}\left(U_{i}-I\right)-c \\
& \geq \min _{m \leq 0}\left\{r v_{m}\left(A_{i} D^{-1}-I\right)+m\right\}-c \\
& \geq d-c ;
\end{aligned}
$$

since $w_{r}\left(V_{i}-I\right) \geq w_{r}\left(A_{i} D^{-1}-I\right) \geq d-c$, we conclude $w_{r}\left(A_{i+1} D^{-1}-I\right) \geq d-c$. On the other hand, by Lemma 6.7, we have

$$
\begin{aligned}
\min _{m \leq 0}\left\{r v_{m}\left(A_{i+1} D^{-1}-I\right)+m\right\} & =\min _{m \leq 0}\left\{r v_{m}\left(V_{i}\left(D U_{i}^{\sigma} D^{-1}\right)-I\right)+m\right\} \\
& =\min _{m \leq 0}\left\{r v_{m}\left(D U_{i}^{\sigma} D^{-1}-I\right)+m\right\} \\
& \geq \min _{m \leq 0}\left\{r q v_{m}\left(U_{i}-I\right)+m\right\}-c \\
& \geq q \min _{m \leq 0}\left\{r v_{m}\left(U_{i}-I\right)+m\right\}-c \\
& \geq q \min _{m \leq 0}\left\{r v_{m}\left(A_{i} D^{-1}-I\right)+m\right\}-c \\
& \geq q d+q i((q-1) d-c)-c \\
& \geq d+(i+1)((q-1) d-c) .
\end{aligned}
$$

This completes the induction and shows that the sequences are well-defined.
We have now shown $\min _{m \leq 0}\left\{r v_{m}\left(A_{i} D^{-1}-I\right)+m\right\} \rightarrow \infty$ as $i \rightarrow \infty$. By Proposition 6.6, this implies $w_{r}\left(U_{i}-I\right) \rightarrow \infty$ as $i \rightarrow \infty$, and so $w_{s}\left(U_{i}-I\right) \rightarrow \infty$ for $r \leq s \leq q r$ since $U_{i}-I$ involves only positive powers of $u$.

We next consider $s \leq r$, for which $w_{s}\left(V_{i}-I\right) \geq d-c>0$ for all $i$. By Lemma 6.7,

$$
\begin{aligned}
\min _{m \leq 0}\left\{s v_{m}\left(A_{i+1} D^{-1}-I\right)+m\right\} & =\min _{m \leq 0}\left\{s v_{m}\left(V_{i} D U_{i}^{\sigma} D^{-1}-I\right)+m\right\} \\
& =\min _{m \leq 0}\left\{s v_{m}\left(D U_{i}^{\sigma} D^{-1}-I\right)+m\right\} \\
& \geq w_{s q}\left(U_{i}-I\right)-c .
\end{aligned}
$$

For $r / q \leq s \leq r$, we already have $w_{s q}\left(U_{i}-I\right)-c \rightarrow \infty$ as $i \rightarrow \infty$, which yields $\min _{m \leq 0}\left\{s v_{m}\left(A_{i+1} D^{-1}-I\right)+m\right\} \rightarrow \infty$ as $i \rightarrow \infty$; by similar reasoning,
$w_{s}\left(A_{i+1} D^{-1}-I\right) \geq d-c$ for large $i$. By Proposition 6.6 (and the fact that the decomposition therein does not depend on $s$ ), we deduce $w_{s}\left(U_{i+1}-I\right) \rightarrow \infty$ as $i \rightarrow \infty$. But now we can repeat the same line of reasoning for $r / q^{2} \leq s \leq r / q$, and then for $r / q^{3} \leq s \leq r / q^{2}$, and so on. Hence $w_{s}\left(U_{i}-I\right) \rightarrow \infty$ for all $s>0$.

We define $U$ as the convergent product $U_{0} U_{1} \cdots$; note that $U$ is invertible because the product $\cdots U_{1}^{-1} U_{0}^{-1}$ also converges. Moreover,

$$
A_{i} D^{-1}=\left(U_{0} \cdots U_{i-1}\right)^{-1} A\left(U_{0} \cdots U_{i-1}\right)^{\sigma} D^{-1}
$$

converges to $U^{-1} A U^{\sigma} D^{-1}$ as $i \rightarrow \infty$. But for $m \leq 0$, we already have $r v_{m}\left(A_{i} D^{-1}-I\right)+m \rightarrow \infty$ as $i \rightarrow \infty$, so that $v_{m}\left(U^{-1} A U^{\sigma} D^{-1}-I\right)=\infty$. Hence $U^{-1} A U^{\sigma} D^{-1}$ and its inverse have entries in $\Gamma_{r}$ and $U^{-1} A U^{\sigma} D^{-1}$ is congruent to $I$ modulo $\pi$, as desired.

This lemma, together with the results of the previous chapters, allows us to deduce an approximation to our desired result, but only so far over an unspecified nearly finite extension of $k((t))$.

Proposition 6.9. Let $M$ be a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}=\Gamma_{\mathrm{an}, \mathrm{con}}^{k((t))}$ whose special Newton slopes lie in the value group of $\mathcal{O}$. Then there exists a nearly finite extension $K$ of $k((t))$ such that $M \otimes_{\Gamma_{\text {an, con }}} \Gamma_{\mathrm{an}, \mathrm{con}}^{K}$ is isomorphic to $M_{1} \otimes_{\Gamma_{\text {con }}^{K}}$ $\Gamma_{\mathrm{an}, \mathrm{con}}^{K}$ for some $\sigma$-module $M_{1}$ over $\Gamma_{\text {con }}^{K}\left[\frac{1}{p}\right]$ whose generic and special Newton polygons coincide.

If $k$ is perfect, we can take $K$ to be separable over $k((t))$, but this is not necessary for our purposes.

Proof. Pick a basis of $M$ and let $A$ be the matrix via which $F$ acts on this basis. By Theorem 4.16, there exists an invertible matrix $X$ over $\Gamma_{\text {an }, \text { con }}^{\text {alg }}$ such that $A=X D X^{-\sigma}$ for some diagonal matrix $D$ over $\mathcal{O}$. Choose $r>0$ such that $A$ is invertible over $\Gamma_{r}$ and $X$ is invertible over $\Gamma_{\mathrm{an}, r q}^{\mathrm{alg}}$.

Choose $c>\max _{i j}\left\{v_{p}\left(D_{i i}\right)-v_{p}\left(D_{j j}\right)\right\}$. By Lemma 6.3 applied to $X^{T}$, there exists a nearly finite extension $K$ of $k((t))$ and an invertible matrix $V$ over $\Gamma_{r}^{K}\left[\frac{1}{p}\right]$ such that $w_{l}(V X-I) \geq 2 c$ for $r \leq l \leq q r$. By replacing $K$ by a suitable inseparable extension, we can ensure that $\Gamma_{q r}^{K}$ contains a semi-unit lifting a uniformizer of $K$.

Observe that

$$
\left(V A V^{-\sigma}\right) D^{-1}=(V X) D(V X)^{-\sigma} D^{-1}
$$

Since $w_{r}(V X-I) \geq 2 c$ and

$$
w_{r}\left(D(V X)^{-\sigma} D^{-1}-I\right) \geq w_{q r}(V X-I)-c \geq c
$$

we have $w_{r}\left(V A V^{-\sigma} D^{-1}-I\right) \geq c$. By Proposition 6.8, there exists an invertible matrix $U$ over $\Gamma_{\mathrm{an}, q r}^{K}$ such that $U^{-1} V A V^{-\sigma} U^{\sigma} D^{-1}$ has entries in $\Gamma_{r}^{K}$ and is congruent to $I$ modulo $\pi$. Put $W=V^{-1} U$; then we can change basis in $M$ so
that $F$ acts on the new basis via the matrix $W^{-1} A W^{\sigma}$. Let $M_{1}$ be the $\Gamma_{\text {con }}^{K}\left[\frac{1}{p}\right]-$ span of the basis elements; by Proposition 5.9, the generic Newton slopes of $M_{1}$ are the valuations of the entries of $D$, so they coincide with the special Newton slopes. Thus $M_{1}$ is the desired $\sigma$-module.

By descending a little bit more, we now deduce the main result of the paper, a slope filtration theorem for $\sigma$-modules over the Robba ring.

Theorem 6.10. Let $M$ be a $\sigma$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}=\Gamma_{\mathrm{an}, \mathrm{con}}^{k((t))}$. Then there is a filtration $0=M_{0} \subset M_{1} \subset \cdots \subset M_{l}=M$ of $M$ by saturated $\sigma$-submodules such that:
(a) for $i=1, \ldots, l$, the quotient $M_{i} / M_{i-1}$ has a single special slope $s_{i}$;
(b) $s_{1}<\cdots<s_{l}$;
(c) each quotient $M_{i} / M_{i-1}$ contains an $F$-stable $\Gamma_{\text {con }}\left[\frac{1}{p}\right]$-submodule $N_{i}$ of the same rank, which spans $M_{i} / M_{i-1}$ over $\Gamma_{\mathrm{an}, \mathrm{con}}$, and which has all generic slopes equal to $s_{i}$.

Moreover, conditions (a) and (b) determine the filtration uniquely, and the $N_{i}$ in (c) are also unique.

Proof. Let $s_{1}$ be the lowest special slope of $M$ and $m$ its multiplicity. We prove that there exists a saturated $\sigma$-submodule $M_{1}$ of rank $m$ whose special slopes all equal $s_{1}$, that $M_{1}$ contains an $F$-stable $\Gamma_{\text {con }}\left[\frac{1}{p}\right]$-submodule $N_{1}$ of the same rank, which spans $M_{1}$ over $\Gamma_{\text {an,con }}$, and whose generic slopes equal $s_{1}$, and that these properties uniquely characterize $M_{1}$ and $N_{1}$. This implies the desired result by induction on the rank of $M$. (Once $M_{1}$ is constructed, apply the induction hypothesis to $M / M_{1}$.)

We first establish the existence of $M_{1}$. Let $\mathcal{O}^{\prime}$ be a Galois extension of $\mathcal{O}$ to which $\sigma$ extends whose value group contains all of the special slopes of $M$. By Proposition 6.9, for some valued field $K$ nearly finite and normal over $k((t)), M$ is isomorphic over $\Gamma_{\text {an,con }}^{K} \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$ to a $\sigma$-module $M^{\prime}$ defined over $\Gamma_{\text {con }}^{K}\left[\frac{1}{p}\right] \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$ whose generic and special Newton polygons are equal. By Proposition 5.16, $M^{\prime}$ admits an ascending slope filtration over $\Gamma_{\text {con }}^{K} \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$, and so $M$ admits one over $\Gamma_{\text {an,con }}^{K} \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$; let $Q_{1}$ and $P_{1}$ be the respective first steps of these filtrations. Then the slope of $P_{1}$ is $s_{1}$ with multiplicity $m$. Moreover, the top exterior power of $P_{1}$ is defined both over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ (because the lowest slope of $\wedge^{m} M$ is $s_{1} m$, which is in the value group of $\mathcal{O}$ ) and over $\Gamma_{\text {an,con }}^{K} \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$, and hence over their intersection $\Gamma_{\mathrm{an}, \mathrm{con}}^{K}$. Thus $P_{1}$ is defined over $\Gamma_{\mathrm{an}, \text { con }}^{K}$.

Let $K_{1}$ be the maximal purely inseparable subextension of $K / k((t))$ (necessarily a valued field), and let $M_{1}$ be the saturated span of the images of $P_{1}$ under $\operatorname{Gal}\left(K / K_{1}\right)$; by Corollary $3.16, M_{1}$ descends to $\Gamma_{\text {an,con }}^{K_{1}}$, and its rank is
at least $m$. Also, over $\Gamma_{\mathrm{an} \text {,con }}^{\mathrm{alg}} \otimes_{\mathcal{O}} \mathcal{O}^{\prime}, M_{1}$ is spanned by eigenvectors of slope $s_{1}$, so the special slopes of $M_{1}$ are all at most $s_{1}$ by Proposition 4.5. Thus $M_{1}$ has the single slope $s_{1}$ with multiplicity $m$.

We must still check that $M_{1}$ descends from $\Gamma_{\text {an,con }}^{K_{1}}$ to $\Gamma_{\text {an,con }}$. Let $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ be a basis of $M$ and let $\mathbf{v}_{1}, \ldots, \mathbf{v}_{m}$ be a basis of $M_{1}$. Then we can write $\mathbf{v}_{i}=\sum_{j} c_{i j} \mathbf{e}_{j}$ for some $c_{i j} \in \Gamma_{\mathrm{an}, \text { con }}^{K_{1}}$. Since $K_{1} / k((t))$ is purely inseparable, $K_{1}^{q^{d}} \subseteq k((t))$ for some integer $d$; for any such $d, F^{d} \mathbf{v}_{1}, \ldots, F^{d} \mathbf{v}_{m}$ is a basis of $M_{1}$ and $F^{d} \mathbf{v}_{i}=\sum_{j} c_{i j}^{\sigma^{d}} F^{d} \mathbf{e}_{j}$. Since each $c_{i j}^{\sigma^{d}}$ belongs to $\Gamma_{\text {an,con }}$, each $F^{d} \mathbf{v}_{i}$ belongs to $M$; thus $M_{1}$ descends to $\Gamma_{\text {an,con }}$.

We next establish existence of an $F$-stable $\Gamma_{\text {con }}\left[\frac{1}{p}\right]$-submodule $N_{1}$ of $M_{1}$, having the same rank and spanning $M_{1}$ over $\Gamma_{\mathrm{an}, \mathrm{con}}$, and having all generic slopes equal to $s_{1}$. Note that $Q_{1}$, defined above, is an $F$-stable $\left(\Gamma_{\text {con }}^{K}\left[\frac{1}{p}\right] \otimes_{\mathcal{O}} \mathcal{O}^{\prime}\right)$ submodule of $M_{1} \otimes_{\Gamma_{\text {an, con }}} \Gamma_{\text {an,con }}^{K} \otimes_{\mathcal{O}} \mathcal{O}^{\prime}=P_{1}$ with the properties desired of $N_{1}$. Moreover, $Q_{1} \otimes_{\Gamma_{\text {con }}^{K}} \Gamma_{\text {con }}^{\text {alg }}$ is equal to the $\left(\Gamma_{\text {con }}^{\text {alg }}\left[\frac{1}{p}\right] \otimes_{\mathcal{O}} \mathcal{O}^{\prime}\right)$-span of the eigenvectors of $M$ of slope $s_{1}$, which is invariant under $\operatorname{Gal}\left(k((t))^{\text {alg }} / k((t))^{\text {perf }}\right) \times \operatorname{Gal}\left(\mathcal{O}^{\prime} / \mathcal{O}\right)$. Thus $Q_{1}$ is invariant under $\operatorname{Gal}\left(K / K_{1}\right) \times \operatorname{Gal}\left(\mathcal{O}^{\prime} / \mathcal{O}\right)$; by Galois descent, it descends to $\Gamma_{\text {con }}^{K_{1}}\left[\frac{1}{p}\right]$, and thus to $\Gamma_{\text {con }}\left[\frac{1}{p}\right]$ (again, by applying Frobenius repeatedly). This yields the desired $N_{1}$.

With the existence of $M_{1}$ and $N_{1}$ in hand, we check uniqueness. For $M_{1}$, note that $M_{1} \otimes_{\Gamma_{\mathrm{an}, \text { con }}} \Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$ is equal to the $\left(\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}} \otimes_{\mathcal{O}} \mathcal{O}^{\prime}\right)$-span of the eigenvectors of $M$ of slope $s_{1}$, because otherwise some eigenvector of slope $s_{1}$ would survive quotienting by $M_{1}$, contradicting Proposition 4.4 because the quotient has all slopes greater than $s_{1}$. This description uniquely determines $M_{1}$. For $N_{1}$, note that $N_{1} \otimes_{\Gamma_{\text {con }}} \Gamma_{\text {con }}^{\text {alg }} \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$ is equal to the $\left(\Gamma_{\text {con }}^{\text {alg }}\left[\frac{1}{p}\right] \otimes_{\mathcal{O}} \mathcal{O}^{\prime}\right)$-span of the eigenvectors of $M$ of slope $s_{1}$, because it contains a basis of eigenvectors of slope $s_{1}$ by Proposition 5.11. This description uniquely determines $N_{1}$.

Thus $M_{1}$ and $N_{1}$ exist and are unique; as noted above, induction on the rank of $M$ now completes the proof.

One consequence of this proposition is that if $k$ is perfect, the lowest slope eigenvectors of a $\sigma$-module over $\Gamma_{\mathrm{an}, \text { con }}$ are defined not just over $\Gamma_{\mathrm{an}, \mathrm{con}}^{\mathrm{alg}}$, but over the subring $\Gamma_{\mathrm{an}, \mathrm{con}} \otimes_{\Gamma_{\text {con }}} \Gamma_{\text {con }}^{\mathrm{sep}}$. (If $k$ is not perfect, then $\Gamma_{\text {con }}^{\mathrm{sep}}$ does not really make sense, but we can replace it with $\Gamma_{\text {con }}^{\mathrm{alg}}$ to get a weaker but still nontrivial statement.)
6.4. The connection to the unit-root case. In this section, we deduce Theorem 1.1 from Theorem 6.10. To exploit the extra data of a connection provided by a $(\sigma, \nabla)$-module, we invoke Tsuzuki's finite monodromy theorem for unit root $F$-crystals [T1, Th. 5.1.1], as follows. (Another proof of the theorem appears in [Ch], and yet another in [Ke1]. However, none of these proves the theorem at quite the level of generality we seek, so we must fiddle a bit with the statement.)

Recall that a valued field $K / k((t))$ is said to be nearly finite separable if it is a finite separable extension of $k^{1 / p^{m}}((t))$ for some nonnegative integer $m$ (and that not all finite separable extensions of $k((t))$ are valued fields).

Proposition 6.11. Let $M$ be a unit-root ( $\sigma, \nabla$ )-module of rank $n$ over $\Gamma_{\text {con }}=\Gamma_{\text {con }}^{k((t))}$. For any nearly finite extension $K$ of $k((t))$, if there exists a basis of $M \otimes \Gamma_{\text {con }} \Gamma_{\text {con }}^{K}$ on which $F$ acts via a matrix $A$ with $v_{p}(A-I)>1 /(p-1)$, then the kernel of $\nabla$ on $M \otimes_{\Gamma_{\text {con }}} \Gamma_{\text {con }}^{K}$ has rank $n$ over $\mathcal{O}$ and is $F$-stable. Moreover, such a $K$ can always be chosen which is separable over $k((t))$ if $k$ is perfect, or nearly separable if $k$ is imperfect.

Proof. The theorem of Tsuzuki [T1, Th. 5.1.1] establishes the first assertion for $k$ algebraically closed; in fact, it produces a basis of eigenvectors in the kernel of $\nabla$. The first assertion in general follows from this case by a relatively formal argument, given below. Note that the kernel of $\nabla$ is always $F$-stable, so we do not have to establish this separately.

We now treat general $k$ by a "compactness" argument. For simplicity of notation, let us assume $K=k((t))$, and let $\mathcal{O}^{\prime}$ be the completion of the maximal unramified extension of the direct limit of $\mathcal{O} \xrightarrow{\sigma} \mathcal{O} \xrightarrow{\sigma} \cdots$. Then Tsuzuki's theorem provides a basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ of the kernel of $\nabla$ over $\Gamma_{\text {con }}^{k^{\text {alg }}((t))}$, and we must produce a basis of the kernel of $\nabla$ over $\Gamma_{\text {con }}$. Let $\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}$ be a basis of $M$ and put $\mathbf{v}_{i}=\sum_{j, l} c_{i, j, l} l^{l} \mathbf{e}_{j}$. Put $d_{j, l}=\min _{i}\left\{v_{p}\left(c_{i, j, l}\right) / v_{p}(\pi)\right\}$, and whenever $d_{j, l}<\infty$, write $c_{i, j, l}$ as $\pi^{d_{j, l}} f_{i, j, l}$.

The fact that $\nabla \mathbf{v}_{i}=0$ for $i=1, \ldots, n$ can be rewritten as a set of "quasilinear" equations in the $f_{i, j, l}$. That is, for $h=1,2, \ldots$, we have equations of the form

$$
\sum_{i, j, l} g_{h, i, j, l} f_{i, j, l}=0
$$

for certain $g_{h, i, j, l} \in \mathcal{O}$, such that for any $h$ and $m$, only finitely many of the $g_{h, i, j, l}$ are nonzero modulo $\pi^{m}$. We are given that these equations have $n$ linearly independent solutions over $\mathcal{O}^{\prime}$, and wish to prove they have $n$ linearly independent solutions over $\mathcal{O}$.

For each finite set $S$ of triples $(i, j, l)$, let $T_{S}(\mathcal{O})$ (resp. $\left.T_{S}\left(\mathcal{O}^{\prime}\right)\right)$ be the set of functions $f: S \rightarrow \mathcal{O}$ (resp. $f: S \rightarrow \mathcal{O}^{\prime}$ ), mapping a pair $(i, j, l) \in S$ to $f_{i, j, l}$, which can be extended to a simultaneous solution of any finite subset of the equations modulo any power of $\pi$. If we put the $T_{S}$ into an inverse system under inclusion on $S$, then the restriction maps are all surjective, and solutions to the complete set of equations are precisely elements of the inverse limit. However, each equation modulo each power of $\pi$ involves only finitely many variables, so $T_{S}$ is defined by linear conditions on the $f_{i, j, l}$. Thus $T_{S}\left(\mathcal{O}^{\prime}\right)=T_{S}(\mathcal{O}) \otimes \mathcal{O} \mathcal{O}^{\prime}$. Since the solutions of the system over $\mathcal{O}^{\prime}$ have rank $n$, we have $\operatorname{rank}_{\mathcal{O}^{\prime}} T_{S}\left(\mathcal{O}^{\prime}\right)$ $=n$ for $S$ sufficiently large. Thus the same holds over $\mathcal{O}$, which produces $n$
$\mathcal{O}$-linearly independent elements of the inverse limit, hence of the kernel of $\nabla$ over $\Gamma_{\text {con }}$. This establishes the first assertion of the proposition for general $k$.

Finally, we show that $K$ can be taken to be (nearly) separable over $k((t))$. By Proposition 5.10 (where the ad hoc definition of $\Gamma^{\text {sep }}$ was given), $M \otimes_{\Gamma_{\text {con }}} \Gamma^{\text {sep }}$ admits a basis up to isogeny of eigenvectors $\mathbf{w}_{1}, \ldots, \mathbf{w}_{n}$. By the DieudonnéManin classification in the form of Proposition 5.5 (and the fact that the unique slope is already in the value group), the kernel of $\nabla$ on $M \otimes_{\Gamma_{\text {con }}} \Gamma_{\text {con }}^{K}$ admits a basis up to isogeny of eigenvectors over some unramified extension $\mathcal{O}^{\prime}$ of $\mathcal{O}$; by the proof of Proposition 5.10, the residue field extension of $\mathcal{O}^{\prime}$ over $\mathcal{O}$ is separable. Thus $\mathcal{O}^{\prime} \subseteq \Gamma^{\text {sep }}$, and so each $\mathbf{v}_{i}$ in the kernel of $\nabla$ is a $\Gamma^{\text {sep }}\left[\frac{1}{p}\right]-$ linear combination of the $\mathbf{w}_{i}$. Hence the $\mathbf{v}_{i}$ are defined over $\Gamma^{\operatorname{sep}}\left[\frac{1}{p}\right] \cap \Gamma_{\text {con }}^{K}$. If $k$ is perfect, this intersection equals $\Gamma_{\text {con }}^{K_{1}}$ for $K_{1}$ the maximal separable subextension of $K$ over $k((t))$. If $k$ is imperfect, $K_{1}$ may fail to be a valued field. Instead, choose an integer $i$ for which the maximal purely inseparable subextension of the residue field extension of $K_{1}$ over $k((t))$ is contained in $k^{1 / p^{i}}$. Then the compositum $K_{2}$ of $K_{1}$ and $k^{1 / p^{i}}((t))$ is a nearly separable valued field, and the $\mathbf{v}_{i}$ are defined over $\Gamma_{\text {con }}^{K_{2}}$, as desired.

Theorem 1.1 follows immediately from the next theorem, which refines the results of Theorem 6.10 in the presence of a connection, by Tsuzuki's theorem.

TheOrem 6.12. Let $M$ be $a(\sigma, \nabla)$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}=\Gamma_{\mathrm{an}, \mathrm{con}}^{k(t)}$. Then the filtration of Theorem 6.10 satisfies the following additional properties:
(d) each $M_{i}$ is a $(\sigma, \nabla)$-submodule;
(e) each $N_{i}$ is $\nabla$-stable;
(f) there exists a nearly finite separable extension $K / k((t))$ (separable in case $k$ is perfect) such that each $N_{i}$ is spanned by the kernel of $\nabla$ over $\Gamma_{\mathrm{con}}^{K}\left[\frac{1}{p}\right]$;
(g) if $k$ is algebraically closed, $N_{i}$ is isomorphic over $\Gamma_{\text {con }}^{K}\left[\frac{1}{p}\right]$ to a direct sum of standard $(\sigma, \nabla)$-modules.

Proof. Again by induction on the rank of $M$, it suffices to prove (d), (e), (f), (g) for $i=1$. For (d) and (e), we may assume without loss of generality (by enlarging $\mathcal{O}$, then twisting) that the special slopes of $M$ belong to the value group of $\mathcal{O}$ and that $s_{1}=0$.

By Proposition 5.8, we can choose a basis for $N_{1}$ on which $F$ acts by an invertible matrix $X$ over $\Gamma_{\text {con }}$. Extend this basis to a basis of $M$; then $F$ acts on the resulting basis via some block matrix over $\Gamma_{\mathrm{an}, \mathrm{con}}$ of the form $\left(\begin{array}{cc}X & Y \\ 0 & Z\end{array}\right)$. View $\nabla$ as a map from $M$ to itself by identifying $x \in M$ with $x \otimes d u \in M \otimes_{\Gamma_{\mathrm{an}, \mathrm{con}}} \Omega^{1} ;$ then $\nabla$ acts on the chosen basis of $M$ by some block
matrix $\left(\begin{array}{cc}P & Q \\ R & S\end{array}\right)$ over $\Gamma_{\text {an,con }}$. The relation $\nabla \circ F=(F \otimes d \sigma) \circ \nabla$ translates into the matrix equation

$$
\left(\begin{array}{cc}
P & Q \\
R & S
\end{array}\right)\left(\begin{array}{cc}
X & Y \\
0 & Z
\end{array}\right)+\frac{d}{d u}\left(\begin{array}{cc}
X & Y \\
0 & Z
\end{array}\right)=\frac{d u^{\sigma}}{d u}\left(\begin{array}{cc}
X & Y \\
0 & Z
\end{array}\right)\left(\begin{array}{cc}
P & Q \\
R & S
\end{array}\right)^{\sigma} .
$$

The lower left corner of the matrix equation yields $R X=\frac{d u^{\sigma}}{d u} Z R^{\sigma}$. We can write $X=U^{-1} U^{\sigma}$ with $U$ over $\Gamma_{\mathrm{con}}^{\mathrm{alg}}$ by Proposition 5.11 (since $M_{1}$ has all slopes equal to 0 ) and $Z=V^{-1} D V^{\sigma}$ with $V$ over $\Gamma_{\mathrm{an} \text {,con }}^{\text {alg }}$ and $D$ a scalar matrix over $\mathcal{O}$ whose entries have positive valuation (because $M_{1}$ is the lowest slope piece of $M)$. We can write $\frac{d u^{\sigma}}{d u}=\mu x$ for some $\mu \in \mathcal{O}$ and $x$ an invertible element of $\Gamma_{\text {con }}$; since $u^{\sigma} \equiv u^{q}(\bmod \pi)$, we have $|\mu|<1$. By Proposition 3.18, there exists $y \in \Gamma_{\text {con }}^{\text {alg }}$ nonzero such that $y^{\sigma}=x y$. Now rewrite the equation $R X=\frac{d u^{\sigma}}{d u} Z R^{\sigma}$ as

$$
y V R U^{-1}=\mu D\left(y V R U^{-1}\right)^{\sigma}
$$

by Proposition 3.19(c) applied entrywise to this matrix equation, we deduce $y V R U^{-1}=0$ and so $R=0$. In other words, $M_{1}$ is stable under $\nabla$, and (d) is verified.

We next check that $N_{1}$ is $\nabla$-stable; this fact is due to Berger $[\mathrm{Bg}$, Lemme V.14], but our proof is a bit different. Put $X_{1}=\frac{d X}{d u}$; then the top left corner of the matrix equation yields $P X+X_{1}=\frac{d u^{\sigma}}{d u} X P^{\sigma}$, or

$$
y U P U^{-1}+y U X_{1} U^{-\sigma}=\mu\left(y U P U^{-1}\right)^{\sigma} .
$$

By Proposition 3.19(c), each entry of $y U P U^{-1}$ lies in $\Gamma_{\text {con }}^{\mathrm{alg}}$, so that the entries of $P$ lie in $\Gamma_{\text {con }}^{\text {alg }} \cap \Gamma_{\text {an,con }}=\Gamma_{\text {con }}$. Thus $N_{1}$ is stable under $\nabla$, and (e) is verified.

To check (f), we must relax the simplifying assumptions. If they do happen to hold, then $N_{1}$ is a unit-root $(\sigma, \nabla)$-module over $\Gamma_{\text {con }}$, so for some (nearly) finite separable extension $K$ of $k((t))$, the kernel of $\nabla$ on $N_{1} \otimes_{\Gamma_{\text {con }}} \Gamma_{\text {con }}^{K}$ has full rank. Without the simplifying assumptions, we only have that the kernel of $\nabla$ has full rank in $N_{1} \otimes_{\Gamma_{\text {con }}} \Gamma_{\text {con }}^{K} \otimes_{\mathcal{O}} \mathcal{O}^{\prime}$ for some finite extension $\mathcal{O}^{\prime}$ of $\mathcal{O}$. However, decomposing kernel elements with respect to a basis of $\mathcal{O}^{\prime}$ over $\mathcal{O}$ produces elements of the kernel of $\nabla$ in $N_{1} \otimes_{\Gamma_{\text {con }}} \Gamma_{\text {con }}^{K}$ which span $M$, so that the kernel has full rank over $N_{1} \otimes_{\Gamma_{\text {con }}} \Gamma_{\text {con }}^{K}$. Thus (f) is verified.

Finally, suppose $k$ is algebraically closed. As noted in the proof of Proposition 6.11, the kernel of $\nabla$ is always $F$-stable. By the Dieudonné-Manin classification (Theorem 5.6), it is isogenous as a $\sigma$-module to a direct sum of standard $\sigma$-modules. This gives a decomposition of $N_{1} \otimes_{\Gamma_{\text {con }}} \Gamma_{\text {con }}^{K}$ as a direct sum of standard $(\sigma, \nabla)$-modules. Thus $(\mathrm{g})$ is verified and the proof is complete.
6.5. Logarithmic form of Crew's conjecture. An alternate formulation of the local monodromy theorem can be given, that eschews the filtration and instead describes a basis of the original module given by elements of the kernel of $\nabla$. The tradeoff is that these elements are defined not over a Robba ring, but over a "logarithmic" extension thereof. As this is the most useful formulation in some applications, we give it explicitly.

For $r>0$, the series $\log (1+x)=x-x^{2} / 2+\cdots$ converges under $|\cdot|_{r}$ whenever $|x|_{r}<1$. Thus if $x \in \Gamma_{\text {con }}$ satisfies $|x-1|_{r}<1$, then $\log (1+x)$ is well-defined and $\log (1+x+y+x y)=\log (1+x)+\log (1+y)$.

For $K$ a valued field nearly finite separable over $k((t))$, choose $u \in \Gamma_{\text {con }}^{K}$ which lifts a uniformizer of $K$, put $\Gamma_{\log , a n, c o n}^{K}=\Gamma_{\mathrm{an}, \mathrm{con}}^{K}[\log u]$, and extend $\sigma$ and $\frac{d}{d u}$ to $\Gamma_{\text {log,an,con }}^{K}$ as follows:

$$
\begin{aligned}
(\log u)^{\sigma} & =q \log u+\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i}\left(\frac{u^{\sigma}}{u^{1}}-1\right)^{i} \\
\frac{d}{d u}(\log u) & =\frac{1}{u} .
\end{aligned}
$$

Theorem 6.13. Let $M$ be a $(\sigma, \nabla)$-module over $\Gamma_{\mathrm{an}, \mathrm{con}}=\Gamma_{\mathrm{an}, \mathrm{con}}^{k(t))}$. Then for some (nearly) finite separable extension $K$ of $k((t)), M$ admits a basis over $\Gamma_{\log , a n, c o n}^{K}$ of elements of the kernel of $\nabla$. Moreover, if $k$ is algebraically closed, $M$ can be decomposed over $\Gamma_{\text {log,an,con }}^{K}$ as the direct sum of standard $(\sigma, \nabla)$-submodules.

Proof. By Theorem 6.12, there exists a basis $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ of $M$ over $\Gamma_{\mathrm{an}, \text { con }}^{K}$, for some nearly finite separable extension $K$ of $k((t))$, such that

$$
\nabla \mathbf{v}_{i} \in \operatorname{SatSpan}\left(\mathbf{v}_{1}, \ldots, \mathbf{v}_{i-1}\right) \otimes \Omega^{1}
$$

Choose a lift $u \in \Gamma_{\text {con }}^{K}$ of a uniformizer of $K$, view $\nabla$ as a map from $M$ to itself by identifying $\mathbf{v} \in M$ with $\mathbf{v} \otimes d u$, and write $\nabla \mathbf{v}_{i}=\sum_{j<i} A_{i j} \mathbf{v}_{i}$ for some $A_{i j} \in \Gamma_{\text {an }, \text { con }}^{K}$.

Define a new basis $\mathbf{w}_{1}, \ldots, \mathbf{w}_{n}$ of $M$ over $\Gamma_{\text {log,an,con }}^{K}$ as follows. First put $\mathbf{w}_{1}=\mathbf{v}_{1}$. Given $\mathbf{w}_{1}, \ldots, \mathbf{w}_{i-1}$ with the same span as $\mathbf{v}_{1}, \ldots, \mathbf{v}_{i-1}$ such that $\nabla \mathbf{w}_{j}=0$ for $j=1, \ldots, i-1$, put $\nabla \mathbf{v}_{j}=c_{i, 1} \mathbf{w}_{1}+\cdots+c_{i, i-1} \mathbf{w}_{i-1}$ and write $c_{i, j}=$ $\sum_{l, m} d_{i, j, l, m} u^{l}(\log u)^{m}$. Now recall from calculus that every expression of the form $u^{l}(\log u)^{m}$, with $m$ a nonnegative integer, can be written as the derivative with respect to $u$ of a linear combination of such expressions. (If $l=-1$, the expression is the derivative of a power of $\log u$ times a scalar. Otherwise, integration by parts can be used to reduce the power of the logarithm.) Thus there exist $e_{i, j} \in \Gamma_{\text {log ,an,con }}^{K}$ such that $\frac{d}{d u} e_{i, j}=c_{i, j}$. Put $\mathbf{w}_{i}=\mathbf{v}_{i}-\sum_{j<i} e_{i, j} \mathbf{w}_{j}$; then $\nabla \mathbf{w}_{i}=0$. This process thus ends with a basis $\mathbf{w}_{1}, \ldots, \mathbf{w}_{n}$ of elements of the kernel of $\nabla$.

As in the proof of Proposition 6.11, the kernel of $\nabla$ is $F$-stable. Thus if $k$ is algebraically closed, we may apply the Dieudonné-Manin classification (Theorem 5.6) to decompose $M$ over $\Gamma_{\mathrm{log}, \text { an,con }}^{K}$ as the sum of standard $(\sigma, \nabla)$ modules, as desired.

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## References

[A1] Y. André, Représentations galoisiennes et opérateurs de Bessel p-adiques, Ann. Inst. Fourier (Grenoble) 52 (2002), 779-808.
[A2] , Filtrations de type Hasse-Arf et monodromie p-adique, Invent. Math. 148 (2002), 285-317.
$[\mathrm{Bg}]$ L. Berger, Représentations p-adiques et équations différentielles, Invent. Math. 148 (2002), 219-284.
[AdV] Y. André and L. di Vizio, $q$-difference equations and $p$-adic local monodromy, to appear in Astérisque; preprint available at http://picard.ups-tlse.fr/~divizio.
[Be1] P. Berthelot, Géométrie rigide et cohomologie des variétés algébriques de caractéristique $p$, in Introductions aux Cohomologies p-adiques (Luminy, 1984), Mém. Soc. Math. France 23 (1986), 7-32.
[Be2] P. Berthelot, Finitude et pureté cohomologique en cohomologie rigide (with an appendix in English by A.J. de Jong), Invent. Math. 128 (1997), 329-377.
[Bo] N. Bourbaki, Algebre Commutative, Chap. IX-X, Masson (Paris), 1983.
[BC] K. BuZZARD and F. CALEGARI, Slopes of overconvergent 2-adic modular forms, preprint, available at http://abel.math.harvard.edu/ $\sim \mathrm{fcale} /$, to appear in Compositio Math.
[Ch] G. Christol, About a Tsuzuki theorem, in p-adic Functional Analysis (Ioannina, 2000), Lecture Notes in Pure and Applied Math. 222, Dekker, New York, 2001, 6374.
[CM1] G. Christol and Z. Mebkhout, Sur le théorème de l'indice des équations différentielles p-adiques I, Ann. Inst. Fourier 43 (1993), 1545-1574.
[CM2] , Sur le théorème de l'indice des équations différentielles p-adiques II, Ann. of Math. 146 (1997), 345-410.
[CM3] , Sur le théorème de l'indice des équations différentielles p-adiques III, Ann. of Math. 151 (2000), 385-457.
[CM4] _ Sur le théorème de l'indice des équations différentielles p-adiques IV, Invent. Math. 143 (2001), 629-672.
[Co] P. Colmez, Les conjectures de monodromie p-adiques, Astérisque 290 (2003), 53-101.
[Cr1] R. Crew, F-isocrystals and their monodromy groups, Ann. Sci. École Norm. Sup. 25 (1992), 429-464.
[Cr2] , Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve, Ann. Sci. École Norm. Sup. 31 (1998), 717-763.
[dJ] A. J. De Jong, Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic, Invent. Math. 134 (1998), 301-333.
[Dw] B. Dwork, Bessel functions as p-adic functions of the argument, Duke Math. J. 41 (1974), 711-738.
[HP] U. Hartl and R. Pink, Vector bundles with a Frobenius structure on the punctured unit disc, Compositio Math. 140 (2004), 689-716.
[Ka] N. Katz, Slope filtration of F-crystals, Astérisque 63 (1979), 113-163. 113-163.
[Ke1] K. S. Kedlaya, Descent theorems for overconvergent $F$-crystals, Ph.D. thesis, Massachusetts Institute of Technology, 2000.
[Ke2] , Unipotency and semistability of overconvergent $F$-crystals, preprint, arXiv: math.AG/0102173.
[Ke3] —, Descent of morphisms of overconvergent $F$-crystals, preprint, arXiv: math.AG/0105244.
[Ke4] -, The Newton polygons of overconvergent $F$-crystals, preprint, arXiv: math.AG/0106192.
$[\mathrm{Ke} 5] \quad$, Quasi-unipotence of overconvergent $F$-crystals, preprint, arXiv: math.AG/0106193.
[Ke6] —, Full faithfulness for overconvergent $F$-isocrystals, in Geometric Aspects of Dwork Theory (Volume II), de Gruyter, Berlin, 2004, 819-835, arXiv: math. AG/0110125.
$[\mathrm{Ke} 7] \quad$, Finiteness of rigid cohomology with coefficients, preprint, arXiv: math.AG/0208027.
[Ke8] $\quad$, Fourier transforms and p-adic "Weil II", preprint, arXiv: math.NT/0210149.
[L] M. Lazard, Les zéros des fonctions analytiques d'une variable sur un corps valué complet, Publ. Math. IHES 14 (1962), 47-75.
[M] Z. Mebkhout, Analogue p-adique du théorème de Turrittin et le théorème de la monodromie $p$-adique, Invent. Math. 148 (2002), 319-351.
[MW] P. Monsky and G. Washnitzer, Formal cohomology. I, Ann. of Math. 88 (1968), 181-217.
[N] M. Nagata, Local Rings, Interscience Tracts in Pure and Applied Mathematics, No. 13, John Wiley \& Sons, New York, 1962.
[Se] J-P. Serre, Local Fields, Graduate Texts in Mathematics 67, Springer-Verlag, New York, 1979.
[Sh] A. Shiho, Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology, J. Math. Sci. Univ. Tokyo 9 (2002), 1-163.
[Ta] J. T. Tate, p-divisible groups, Proc. Conf. on Local Fields (Driebergen, 1966), Springer (Berlin), 1967, 158-183.
[T1] N. Tsuzuki, Finite local monodromy of overconvergent unit-root $F$-crystals on a curve, Amer. J. Math. 120 (1998), 1165-1190.
[T2] $\quad$, Slope filtration of quasi-unipotent overconvergent $F$-isocrystals, Ann. Inst. Fourier (Grenoble) 48 (1998), 379-412.
[T3] , Morphisms of $F$-isocrystals and the finite monodromy theorem for unit-root F-isocrystals, Duke Math. J. 111 (2002), 385-418.
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