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The space of embedded minimal surfaces of fixed genus in a 3-manifold II; Multi-valued graphs in disks

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0. Introduction

This paper is the second in a series where we give a description of the space of all embedded minimal surfaces of fixed genus in a fixed (but arbitrary) closed 3-manifold. The key for understanding such surfaces is to understand the local structure in a ball and in particular the structure of an embedded minimal disk in a ball in $\mathbf{R}^3$. We show here that if the curvature of such a disk becomes large at some point, then it contains an almost flat multi-valued graph nearby that continues almost all the way to the boundary. This will be proved by showing the existence of small multi-valued graphs near points of large curvature and then using the extension result for multi-valued graphs proved in the first paper in this series.

There are two local models for embedded minimal disks (by an embedded disk, we mean a smooth injective map from the closed unit ball in $\mathbf{R}^2$ into $\mathbf{R}^3$). One model is the plane (or, more generally, a minimal graph), the other is a piece of a helicoid. In the first four papers of this series, we will show that every embedded minimal disk is either a graph of a function or is a double spiral staircase like a helicoid. Recall that a double spiral staircase consists of two spiral staircases that spiral together around a common axis, one inside the other. This will be done by showing that if the curvature is large at some point (and hence the surface is not a graph), then it is a double spiral staircase. To prove that it is a double spiral staircase, we will first prove that it is built out of N-valued graphs where N is a fixed number. These N-valued graphs are like a single spiral staircase connecting N floors. The existence of the N-valued graphs was initiated in the first paper and will be completed here. The third and fourth papers of this series will deal with how the multi-valued graphs fit together and, in particular, prove regularity of the set of points of large curvature – the axis of the double spiral staircase.

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To explain our results more precisely, we start by recalling the exact definition of a multi-valued graph. Let D_r be the disk in the plane centered at the origin and of radius r and let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbf{C} \setminus \{0\}$ with global polar coordinates (ρ, θ) so that $\rho > 0$ and $\theta \in \mathbf{R}$. An N-valued graph Σ over the annulus $D_{r_2} \setminus D_{r_1}$ (see Figure 1) is a (single-valued) graph over

$$\{(\rho, \theta) \in \mathcal{P} \,|\, r_1 < \rho < r_2 \text{ and } |\theta| \leq \pi N\} . \tag{0.1}$$

The multi-valued graphs that we consider will never close up; in fact they will all be embedded. Note that embedded is equivalent to requiring that the separation never vanishes. Here the separation is the function given by

$$w(\rho, \theta) = u(\rho, \theta + 2\pi) - u(\rho, \theta) .$$

If Σ is the helicoid, then $\Sigma \setminus x_3 - \text{axis} = \Sigma_1 \cup \Sigma_2$, where Σ_1, Σ_2 are ∞-valued graphs. Also, Σ_1 is the graph of the function $u_1(\rho, \theta) = \theta$ and Σ_2 is the graph of the function $u_2(\rho, \theta) = \theta + \pi$. In either case the separation $w = 2\pi$.

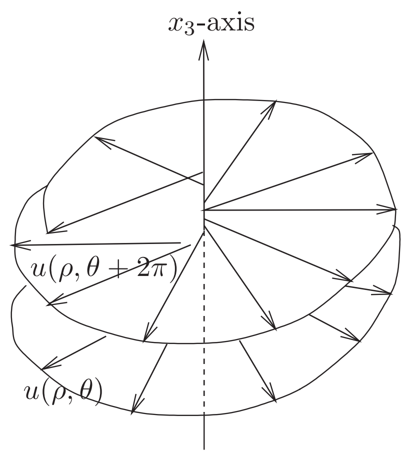

Figure 1: A multi-valued graph.

The main result of this paper is the following existence theorem for multi-valued graphs near points of large curvature:

THEOREM 0.2. *Given* $N \in \mathbf{Z}_+$, $\varepsilon > 0$, *there exist* C_1, $C_2 > 0$ *so that the following holds*:

Let $0 \in \Sigma^2 \subset B_R \subset \mathbf{R}^3$ *be an embedded minimal disk with* $\partial \Sigma \subset \partial B_R$. *If for some* $R > r_0 > 0$,

$$\max_{B_{r_0} \cap \Sigma} |A|^2 \geq 4\, C_1^2\, r_0^{-2} ,$$

then there exists (*after a rotation of* $\mathbf{R}^3$) *an* N-*valued graph* Σ_g *over* $D_{R/C_2} \setminus D_{2r_0}$ *with gradient* $\leq \varepsilon$ *and contained in* $\Sigma \cap \{x_3^2 \leq \varepsilon^2\, (x_1^2 + x_2^2)\}$.

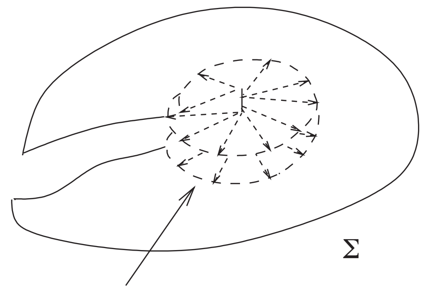

Figure 2: Theorem 0.4 — finding a small multi-valued graph in a disk near a point of large curvature.

This theorem is modeled by one-half of the helicoid and its rescalings. Recall that the helicoid is the minimal surface Σ^2 in $\mathbf{R}^3$ parametrized by

$$(s\,\cos t, s\,\sin t, t) \tag{0.3}$$

where $s, t \in \mathbf{R}$. By one-half of the helicoid we mean the multi-valued graph given by requiring that $s > 0$ in (0.3).

Theorem 0.2 will follow by combining a blow-up result with [CM3]. This blow-up result says that if an embedded minimal disk in a ball has large curvature at a point, then it contains a small, almost flat, multi-valued graph nearby; that is:

Theorem 0.4 (see Figure 2). *Given $N, \omega > 1$, and $\varepsilon > 0$, there exists $C = C(N, \omega, \varepsilon) > 0$ so that the following holds:*

Let $0 \in \Sigma^2 \subset B_R \subset \mathbf{R}^3$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_R$. If for some $0 < r_0 < R$,

$$\sup_{B_{r_0} \cap \Sigma} |A|^2 \leq 4\,|A|^2(0) = 4\,C^2\,r_0^{-2}\,,$$

then there exist $\bar{R} < r_0/\omega$ and (after a rotation of $\mathbf{R}^3$) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega \bar{R}} \setminus D_{\bar{R}}$ with gradient $\leq \varepsilon$, and $\mathrm{dist}_\Sigma(0, \Sigma_g) \leq 4\,\bar{R}$.

Recall that by the middle sheet Σ^M of an N-valued graph Σ we mean the portion over

$$\{(\rho, \theta) \in \mathcal{P} \,|\, r_1 < \rho < r_2 \text{ and } 0 \leq \theta \leq 2\,\pi\}\,. \tag{0.5}$$

The result that we need from [CM3] (combining theorem 0.3 and lemma II.3.8 there) is the following theorem:

Theorem 0.6 ([CM3]; see Figure 3). *Given* N_1 *and* $\tau > 0$, *there exist* $N, \Omega, \varepsilon > 0$ *so that the following holds*:

Let $\Sigma \subset B_{R_0} \subset \mathbf{R}^3$ *be an embedded minimal disk with* $\partial\Sigma \subset \partial B_{R_0}$. *If* $\Omega\, r_0 < 1 < R_0/\Omega$ *and* Σ *contains an* N*-valued minimal graph* Σ_g *over* $D_1 \setminus D_{r_0}$ *with gradient* $\leq \varepsilon$ *and*

$$\Sigma_g \subset \{x_3^2 \leq \varepsilon^2 (x_1^2 + x_2^2)\}\,,$$

then Σ *contains an* N_1*-valued graph* Σ_d *over* $D_{R_0/\Omega} \setminus D_{r_0}$ *with gradient* $\leq \tau$ *and* $(\Sigma_g)^M \subset \Sigma_d$.

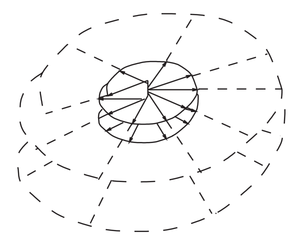

Figure 3: Theorem 0.6 — extending a small multi-valued graph in a disk at a point.

As a consequence of Theorem 0.2, we will show that if $|A|^2$ is blowing up at a point for a sequence of embedded minimal disks, then there is a smooth minimal graph through this point in the limit of a subsequence (Theorem 5.8 below).

Theorems 0.2, 0.4, 0.6, 5.8 are local and are for simplicity stated and proven only for $\mathbf{R}^3$ with the flat metric although they can (with only very minor changes) easily be seen to hold for a sufficiently small ball in any given fixed Riemannian 3-manifold.

The key step for proving Theorem 0.4 is to find many large pieces of Σ with a (scale-invariant) quadratic curvature bound (these pieces will be intrinsic sectors). To find such pieces, we use the upper bound on $|A|^2$, i.e.,

$$|A|^2 \leq 4\, C^2\, r_0^{-2}\,,$$

to prove that the area of intrinsic balls in Σ grows polynomially and, consequently, we get an average curvature bound. This average curvature bound and a curvature estimate for embedded disks (see Corollary 1.18 below) will give large pieces of Σ with the desired quadratic curvature bound. Using the lower bound on $|A|^2(0)$, i.e.,

$$|A|^2(0) = C^2\, r_0^{-2}\,,$$

we show that there are many such pieces so that two must be close together in $\mathbf{R}^3$; embeddedness implies that these are disjoint, hence almost stable, and therefore nearly flat. Piecing together these large flat pieces will then give the desired N-valued graph.

In Section 1, we will first estimate the area of a surface (not necessarily minimal) in terms of its total curvature. This should be seen as analogous to a Poincaré inequality (for functions), and will be used similarly. We next bound the curvature by the area for a minimal disk. This estimate is similar to a Caccioppoli inequality and, unlike the Poincaré type inequality, relies on minimality. We then apply these results to prove a curvature estimate for embedded minimal disks with bounded total curvature in an intrinsic ball.

In Section 2, we will collect some results on stability of minimal surfaces. The basic point is that two disjoint but nearby embedded minimal surfaces satisfying *a priori* curvature estimates must be nearly stable which leads to an improved curvature estimate, showing that the two surfaces are nearly "parallel graphs."

In Section 3, we obtain polynomial bounds for the area and total curvature of intrinsic balls in embedded minimal disks with bounded curvature. This polynomial growth leads to a doubling property, at least at many places, which will be used later to find large pieces with a scale-invariant curvature bound.

In Section 4, we combine the results of the previous sections to prove Theorem 0.4. The key is to find many large intrinsic sectors with a scale-invariant curvature bound as described above. This will give us a large almost stable sector and then an N-valued graph in it. We also prove a lower bound for the separation of a multi-valued graph on the scale at which it forms.

In Section 5, we use a simple rescaling ("blow-up") argument to deduce Theorem 0.2 from Theorem 0.4. Finally, we prove, as a consequence of Theorem 0.2, that if $|A|^2$ is blowing up at a point for a sequence of embedded minimal disks, then there is a smooth minimal graph through this point in the limit of a subsequence.

Let x_1, x_2, x_3 be the standard coordinates on $\mathbf{R}^3$ and $\Pi : \mathbf{R}^3 \to \mathbf{R}^2$ orthogonal projection to $\{x_3 = 0\}$. For $y \in S \subset \Sigma \subset \mathbf{R}^3$ and $s > 0$, the extrinsic and intrinsic balls and tubes are

(0.7)
$$B_s(y) = \{x \in \mathbf{R}^3 \,|\, |x - y| < s\}\,, \quad T_s(S) = \{x \in \mathbf{R}^3 \,|\, \mathrm{dist}_{\mathbf{R}^3}(x, S) < s\}\,,$$

(0.8)
$$\mathcal{B}_s(y) = \{x \in \Sigma \,|\, \mathrm{dist}_\Sigma(x, y) < s\}\,, \mathcal{T}_s(S) = \{x \in \Sigma \,|\, \mathrm{dist}_\Sigma(x, S) < s\}\,.$$

D_s denotes the disk $B_s(0) \cap \{x_3 = 0\}$ and K_Σ the sectional curvature of a smooth compact surface Σ. When Σ is immersed A_Σ will be its second fundamental form. When Σ is oriented, $\mathbf{n}_\Sigma$ is the unit normal.

1. Poincaré and Caccioppoli-type inequalities for area and curvature

In this section, we will first estimate the area of a surface (not necessarily minimal) in terms of its total curvature; see Corollary 1.5. This should be seen as analogous to a Poincaré inequality (for functions), and will be used similarly later in this paper. After that, we will bound the curvature by the area for a minimal disk; see Corollary 1.7. This inequality is similar to a Caccioppoli inequality and, unlike the Poincaré type inequality, relies on the fact that the surface is minimal. Finally, we will apply these inequalities to show a strengthened (intrinsic) version of a result of Schoen and Simon.

LEMMA 1.1. *If $\mathcal{B}_{r_0}(x) \subset \Sigma^2$ is disjoint from the cut locus of x, then*

$$\text{Length}(\partial \mathcal{B}_{r_0}) - 2\pi r_0 = -\int_0^{r_0} \int_{\mathcal{B}_\rho} \mathrm{K}_\Sigma \,, \tag{1.2}$$

$$\text{Area}(\mathcal{B}_{r_0}(x)) - \pi\, r_0^2 = -\int_0^{r_0} \int_0^{\tau} \int_{\mathcal{B}_\rho(x)} \mathrm{K}_\Sigma \,. \tag{1.3}$$

Proof. For $0 < t \leq r_0$, by the Gauss-Bonnet theorem,

$$\frac{d}{dt} \int_{\partial \mathcal{B}_t} 1 = \int_{\partial \mathcal{B}_t} k_g = 2\,\pi - \int_{\mathcal{B}_t} \mathrm{K}_\Sigma \,, \tag{1.4}$$

where k_g is the geodesic curvature of $\partial \mathcal{B}_t$. Integrating (1.4) gives the lemma. □

Throwing away the positive part of K_Σ in (1.3) gives the next corollary:

COROLLARY 1.5. *If $\mathcal{B}_{r_0}(x) \subset \Sigma^2$ is disjoint from the cut locus of x, then*

$$\text{Area}(\mathcal{B}_{r_0}(x)) \leq \pi\, r_0^2 - \frac{1}{2}\, r_0^2 \int_{\mathcal{B}_{r_0}(x)} \min\{\mathrm{K}_\Sigma, 0\} \,. \tag{1.6}$$

The next corollary specializes Lemma 1.1 to the case where Σ is minimal.

COROLLARY 1.7. *If $\Sigma^2 \subset \mathbf{R}^3$ is immersed and minimal, $\mathcal{B}_{r_0} \subset \Sigma^2$ is a disk, and $\mathcal{B}_{r_0} \cap \partial\Sigma = \emptyset$, then*

$$\begin{aligned} t^2 \int_{\mathcal{B}_{r_0 - 2\,t}} |A|^2 &\leq \int_{\mathcal{B}_{r_0}} |A|^2\, (r_0 - r)^2/2 = \int_0^{r_0} \int_0^{\tau} \int_{\mathcal{B}_\rho(x)} |A|^2 \\ &= 2\, (\text{Area}\, (\mathcal{B}_{r_0}) - \pi\, {r_0}^2) \leq r_0 \,\text{Length}(\partial \mathcal{B}_{r_0}) - 2\pi\, r_0^2 \,. \end{aligned} \tag{1.8}$$

Proof. Since Σ is minimal, the Gauss equation gives $|A|^2 = -2\,\mathrm{K}_\Sigma$ and hence, by Lemma 1.1,

$$\begin{aligned} t^2 \int_{\mathcal{B}_{r_0 - 2\,t}} |A|^2 &\leq t \int_0^{r_0 - t} \int_{\mathcal{B}_\rho} |A|^2 \leq \int_0^{r_0} \int_0^{\tau} \int_{\mathcal{B}_\rho} |A|^2 \\ &= 2\, (\text{Area}\, (\mathcal{B}_{r_0}) - \pi\, {r_0}^2) \,. \end{aligned} \tag{1.9}$$

The first equality in (1.8) follows from the co-area formula and integration by parts twice (i.e., $\int_0^{r_0} f(t)\, g''(t)\, dt = \int_0^{r_0} f''(t)\, g(t)\, dt$ with $f(t) = \int_0^t \int_{\mathcal{B}_s} |A|^2$ and $g(t) = (r_0 - t)^2/2$).

To get the last inequality in (1.8), note that $\frac{d^2}{dt^2}\text{Length}(\partial \mathcal{B}_t) \geq 0$ by (1.4); hence

$$t\,\frac{d}{dt}\text{Length}(\partial \mathcal{B}_t) \geq \text{Length}(\partial \mathcal{B}_t)$$

and consequently

$$\frac{d}{dt}\,(\text{Length}(\partial \mathcal{B}_t)/t) \geq 0\,.$$

From this, the last inequality in (1.8) follows easily. □

The next lemma gives a curvature estimate for intrinsic balls in an embedded minimal disk with small total curvature in an annulus.

LEMMA 1.10. *Given C, there exists $\varepsilon > 0$ so that if $\mathcal{B}_{9s} \subset \Sigma \subset \mathbf{R}^3$ is an embedded minimal disk with*

$$\int_{\mathcal{B}_{9s}} |A|^2 \leq C \text{ and } \int_{\mathcal{B}_{9s}\setminus\mathcal{B}_s} |A|^2 \leq \varepsilon\,, \tag{1.11}$$

then

$$\sup_{\mathcal{B}_s} |A|^2 \leq s^{-2}\,.$$

Proof. Observe first that for ε small, the curvature estimate of [CiSc] and (1.11) give

$$\sup_{\mathcal{B}_{8s}\setminus\mathcal{B}_{2s}} |A|^2 \leq C_1^2\,\varepsilon\, s^{-2}\,. \tag{1.12}$$

Combining (1.2) and the first inequality in (1.11) gives

$$\text{Length}(\partial \mathcal{B}_{2s}) \leq (4\pi + C)\,s\,. \tag{1.13}$$

We will next use (1.12) and (1.13) to show that, after rotating $\mathbf{R}^3$, $\mathcal{B}_{8s}\setminus\mathcal{B}_{2s}$ is (locally) a graph over $\{x_3 = 0\}$ and furthermore $|\Pi(\partial\mathcal{B}_{8\,s})| > 3\,s$. Combining these two facts with embeddedness, the lemma will then follow easily from Rado's theorem.

By (1.13), we have the diameter bound

$$\text{diam}(\mathcal{B}_{8s}\setminus\mathcal{B}_{2s}) \leq (12 + 2\pi + C/2)s\,.$$

Consequently, since

$$|\nabla \text{dist}_{\mathbf{S}^2}(\mathbf{n}(x), \mathbf{n})| \leq |A|\,,$$

integrating (1.12) gives

$$\sup_{x,x' \in \mathcal{B}_{8s}\setminus\mathcal{B}_{2s}} \text{dist}_{\mathbf{S}^2}(\mathbf{n}(x'), \mathbf{n}(x)) \leq C_1\,\varepsilon^{1/2}\,(12 + 2\pi + C/2)\,. \tag{1.14}$$

We can therefore rotate $\mathbf{R}^3$ so that

$$\sup_{\mathcal{B}_{8s}\setminus\mathcal{B}_{2s}} |\nabla x_3| \leq C_2\,\varepsilon^{1/2}\,(1 + C)\,. \tag{1.15}$$

Given $y \in \partial \mathcal{B}_{2s}$, let γ_y be the outward normal geodesic from y to $\partial \mathcal{B}_{8s}$ parametrized by arclength on $[0, 6s]$. Integrating (1.12) along γ_y gives

$$\int_{\gamma_y|_{[0,t]}} |k_g^{\mathbf{R}^3}| \leq \int_{\gamma_y|_{[0,t]}} |A| \leq C_1\, \varepsilon^{1/2}\, t/s\,, \tag{1.16}$$

where $k_g^{\mathbf{R}^3}$ is the geodesic curvature of γ_y as a curve in $\mathbf{R}^3$. Since

$$\left| \frac{d}{dt} \langle \gamma_y'(t), \gamma_y'(0) \rangle \right| \leq |k_g^{\mathbf{R}^3}|\,,$$

integrating (1.16) gives a bound for the oscillation of the unit normal γ_y'

$$\langle \gamma_y'(t), \gamma_y'(0) \rangle > 1 - C_3\, \varepsilon^{1/2}\,.$$

Integrating again gives the fact that the endpoints of γ_y project to distant points on the line in the $\gamma_y'(0)$ direction

$$\langle \gamma_y(6s) - \gamma_y(0), \gamma_y'(0) \rangle > (1 - C_3\, \varepsilon^{1/2})\, 6s\,. \tag{1.17}$$

Since $\gamma_y(0) \in \mathcal{B}_{2s} \subset B_{2s}$ and (1.15) implies that $\gamma_y'(0)$ is nearly horizontal, we see that, for ε small, (1.17) implies

$$|\Pi(\partial \mathcal{B}_{8\,s})| > 3\, s\,.$$

For the next step, see Figure 5. Combining $|\Pi(\partial \mathcal{B}_{8\,s})| > 3\, s$ and (1.15), it follows that, for ε small,

$$\Pi^{-1}(\partial D_{2s}) \cap \mathcal{B}_{8s}$$

is a collection of immersed multi-valued graphs over ∂D_{2s}. Since $\mathcal{B}_{8s}$ is embedded, $\Pi^{-1}(\partial D_{2s}) \cap \mathcal{B}_{8s}$ consists of disjoint embedded circles which are graphs over ∂D_{2s}; this is the only use of embeddedness. Since $x_1^2 + x_2^2$ is subharmonic on the disk $\mathcal{B}_{8s}$, these circles bound disks in $\mathcal{B}_{8s}$ which are then graphs by Rado's theorem (see, e.g., [CM1]). The lemma now follows easily from (1.15) and the mean value inequality. □

In [ScSi], Schoen and Simon proved a curvature estimate for embedded minimal disks with quadratic area growth in extrinsic balls. The next corollary generalizes this to intrinsic balls. Note that bounds on area and total curvature are essentially equivalent by (1.8).

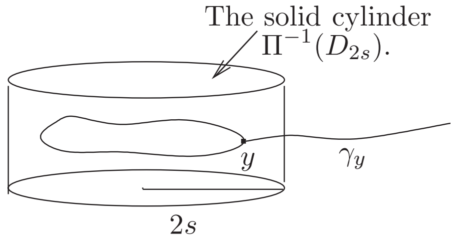

Figure 4: Proof of Lemma 1.10: By (1.15) and (1.16), each γ_y is almost a horizontal line segment of length $6s$. Therefore, $|\Pi(\partial \mathcal{B}_{8s})| > 3s$.

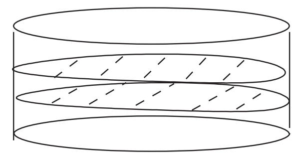

Figure 5: Proof of Lemma 1.10: $\Pi^{-1}(\partial D_{2s}) \cap \mathcal{B}_{8s}$ is a union of graphs over ∂D_{2s}. Each bounds a graph in Σ over D_{2s} by Rado's theorem.

Corollary 1.18. *Given a constant C_I, there exists C_P so that if $\mathcal{B}_{2s} \subset \Sigma \subset \mathbf{R}^3$ is an embedded minimal disk with*

$$\int_{\mathcal{B}_{2s}} |A|^2 \leq C_I \, , \tag{1.19}$$

then

$$\sup_{\mathcal{B}_s} |A|^2 \leq C_P \, s^{-2} \, .$$

Proof. Let $\varepsilon > 0$ be given by Lemma 1.10 with $C = C_I$ and then let N be the least integer greater than C_I/ε. Given $x \in \mathcal{B}_s$, there exists $1 \leq j \leq N$ with

$$\int_{\mathcal{B}_{9^{1-j}s}(x) \setminus \mathcal{B}_{9^{-j}s}(x)} |A|^2 \leq C_I/N \leq \varepsilon \, . \tag{1.20}$$

When we combine (1.19) and (1.20), Lemma 1.10 gives that

$$|A|^2(x) \leq (9^{-j} s)^{-2} \leq 9^{2N} \, s^{-2} \, . \qquad \square$$

We close this section with a generalization to surfaces of higher genus; see Theorem 1.22 below. This will not be used in this paper but will be useful in [CM6]. First, we need the following lemma:

Lemma 1.21. *Let Σ be a surface and $\sigma \subset \Sigma$ a simple closed curve with length $\leq C\, r_0$. If for all $y \in \sigma$ the ball $\mathcal{B}_{r_0}(y)$ is a disk disjoint from $\partial\Sigma$, then there is a broken geodesic $\sigma_1 \subset \Sigma$ homotopic to σ in $\mathcal{T}_{r_0}(\sigma)$ and with $\leq C+1$ breaks*; *see Figure* 6.

Furthermore, if Σ is an annulus with $\mathrm{K}_\Sigma \leq 0$ and σ separates $\partial\Sigma$, then σ_1 contains a simple curve σ_2 homotopic to σ with $\leq C+2$ breaks.

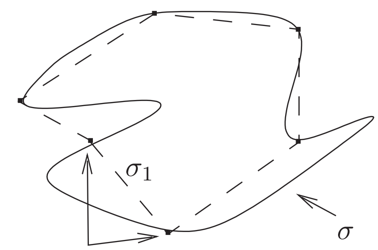

Figure 6: Lemma 1.21: A curve σ and a broken geodesic σ_1 homotopic to σ in $\mathcal{T}_{r_0}(\sigma)$.

Proof. Parametrize σ by arclength so that $\sigma(0) = \sigma(\text{Length}(\sigma))$. Let

$$0 = t_0 < \cdots < t_n = \text{Length}(\sigma)$$

be a subdivision with $t_{i+1} - t_i \leq r_0$ and $n \leq C + 1$. Since $\mathcal{B}_{r_0}(y)$ is a disk for all $y \in \sigma$, it follows that we can replace σ with a broken geodesic σ_1 with breaks at $\sigma(t_i) = \sigma(t_i)$ and with σ_1 homotopic to σ in $\mathcal{T}_{r_0}(\sigma)$.

Suppose also now that Σ is an annulus with $\mathrm{K}_\Sigma \leq 0$ and σ is topologically nontrivial. Let $[a, b]$ be a maximal interval so that $\sigma_1|_{[a,b]}$ is simple. We are done if $\sigma_1|_{[a,b]}$ is homotopic to σ. Otherwise, $\sigma_1|_{[a,b]}$ bounds a disk in Σ and the Gauss-Bonnet theorem implies that $\sigma_1|_{(a,b)}$ contains a break. Hence, replacing σ_1 by $\sigma_1 \setminus \sigma_1|_{(a,b)}$ gives a subcurve homotopic to σ but does not increase the number of breaks. Repeating this eventually gives σ_2. □

Given a surface Σ with boundary $\partial\Sigma$, we will define the *genus* of Σ ($\text{gen}(\Sigma)$) to be the genus of the closed surface $\hat{\Sigma}$ obtained by adding a disk to each boundary circle. For example, the disk and the annulus are both genus zero; on the other hand, a closed surface of genus g with k disks removed has genus g.

In contrast to Corollary 1.18 (and the results preceding it), the next result concerns surfaces intersected with extrinsic balls. Below, $\Sigma_{0,s}$ is the component of $B_s \cap \Sigma$ with $0 \in \Sigma_{0,s}$.

Theorem 1.22. *Given C_a, g, there exist C_c, C_r so that the following holds: If $0 \in \Sigma \subset B_{r_0}$ is an embedded minimal surface with $\partial\Sigma \subset \partial B_{r_0}$, $\text{gen}(\Sigma) \leq g$ and*

$$\text{Area}(\Sigma) \leq C_a\, r_0^2\,, \;\; \Sigma \setminus \Sigma_{0,s} \textit{ is topologically an annulus}$$
$$\textit{for each } C_r\, r_0 \leq s \leq r_0$$

then Σ is a disk and

$$\sup_{\Sigma_{0,C_r\, r_0}} |A|^2 \leq C_c\, r_0^{-2}\,.$$

Proof. By the co-area formula, we can find $r_0/2 \leq r_1 \leq 3r_0/4$ with

$$\text{Length}(\partial B_{r_1} \cap \Sigma) \leq 4\, C_a\, r_0 \,.$$

It is easy to see from the maximum principle that $\mathcal{B}_{r_0/4}(y)$ is a disk for each $y \in \partial B_{r_1} \cap \Sigma$ (with $C_r < 1/4$). Applying Lemma 1.21 to $\partial \Sigma_{0,r_1} \subset \Sigma \setminus \Sigma_{0,r_0/4}$, we get a simple broken geodesic

$$\sigma_2 \subset \mathcal{T}_{r_0/4}(\partial \Sigma_{0,r_1})$$

homotopic to $\partial \Sigma_{0,r_1}$ and with $\leq 16\, C_a + 2$ breaks. Consequently, the Gauss-Bonnet theorem gives

$$\int_{\Sigma_{0,r_0/4}} |A|^2 = -2 \int_{\Sigma_{0,r_0/4}} K_\Sigma \leq 8\,\pi\, g + 2 \int_{\sigma_2} |k_g| \leq 8\,\pi (g + 4\, C_a + 1)\,. \tag{1.23}$$

For $\varepsilon > 0$, arguing as in Corollary 1.18 gives r_2 with $\int_{\Sigma_{0,5r_2} \setminus \Sigma_{0,r_2}} |A|^2 \leq \varepsilon^2$ so, by [CiSc],

$$\sup_{\Sigma_{0,4r_2} \setminus \Sigma_{0,2r_2}} |A|^2 \leq C\, \varepsilon^2\, r_2^{-2}\,. \tag{1.24}$$

When we use the area bound, $\partial \Sigma_{0,3r_2}$ can be covered by $C C_a$ intrinsic balls $\mathcal{B}_{r_2/4}(x_i)$ with $x_i \in \partial \Sigma_{0,3r_2}$ (by the maximum principle, each $\mathcal{B}_{r_2/4}(x_i)$ is a disk). Hence, since $\partial \Sigma_{0,3r_2}$ is connected, any two points in $\partial \Sigma_{0,3r_2}$ can be joined by a curve in $\Sigma_{0,4r_2} \setminus \Sigma_{0,2r_2}$ of length $\leq C\, r_2$. Integrating (1.24) twice then gives a plane $P \subset \mathbf{R}^3$ with $\partial \Sigma_{0,3r_2} \subset T_{C\,\varepsilon\, r_2}(P)$. By the convex hull property, $0 \in \Sigma_{0,3r_2} \subset T_{C\,\varepsilon\, r_2}(P)$. Hence, since $\partial \Sigma_{0,3r_2}$ is connected and embedded, $\partial \Sigma_{0,3r_2}$ is a graph over the boundary of a convex domain for ε small. The standard existence theory and Rado's theorem give a minimal graph Σ_g with $\partial \Sigma_g = \partial \Sigma_{0,3r_2}$. By translating Σ_g above $\Sigma_{0,3r_2}$ and sliding it down to the first point of contact, and then repeating this from below, it follows easily from the strong maximum principle that $\Sigma_g = \Sigma_{0,3r_2}$, completing the proof. □

2. Finding large nearly stable pieces

We will collect here some results on stability of minimal surfaces which will be used later to conclude that certain sectors are nearly stable. The basic point is that two disjoint but nearby embedded minimal surfaces satisfying *a priori* curvature estimates must be nearly stable (made precise below). We start by recalling the definition of δ_s-stability. Let again $\Sigma \subset \mathbf{R}^3$ be an embedded oriented minimal surface.

Definition 2.1 (δ_s-stability). Given $\delta_s \geq 0$, set

$$L_{\delta_s} = \Delta + (1 - \delta_s)|A|^2\,, \tag{2.2}$$

so that L_0 is the usual Jacobi operator on Σ. A domain $\Omega \subset \Sigma$ is δ_s-*stable* if

$$\int \phi \, L_{\delta_s} \phi \leq 0$$

for any compactly supported Lipschitz function ϕ (i.e., $\phi \in C_0^{0,1}(\Omega)$).

It follows that Ω is δ_s-stable if and only if, for all $\phi \in C_0^{0,1}(\Omega)$, we have the δ_s-stability inequality:

$$(1 - \delta_s) \int |A|^2 \phi^2 \leq \int |\nabla \phi|^2 \, . \tag{2.3}$$

Since the Jacobi equation is the linearization of the minimal graph equation over Σ, standard calculations give:

LEMMA 2.4. *There exists $\delta_g > 0$ so that if Σ is minimal and u is a positive solution of the minimal graph equation over Σ (i.e., $\{x + u(x)\, \mathbf{n}_\Sigma(x) \,|\, x \in \Sigma\}$ is minimal) with*

$$|\nabla u| + |u| \, |A| \leq \delta_g \, ,$$

then $w = \log u$ satisfies on Σ

$$\Delta w = -|\nabla w|^2 + \operatorname{div}(a \nabla w) + \langle \nabla w, a \nabla w \rangle + \langle b, \nabla w \rangle + (c-1)|A|^2 \, , \tag{2.5}$$

for functions a_{ij}, b_j, c on Σ with $|a|, |c| \leq 3\,|A|\,|u| + |\nabla u|$ and $|b| \leq 2\,|A|\,|\nabla u|$.

The following slight modification of a standard argument (see, e.g., Proposition 1.26 of [CM1]) gives a useful sufficient condition for δ_s-stability of a domain:

LEMMA 2.6. *There exists $\delta > 0$ so that if Σ is minimal and $u > 0$ is a solution of the minimal graph equation over $\Omega \subset \Sigma$ with*

$$|\nabla u| + |u| \, |A| \leq \delta \, ,$$

then Ω is $1/2$-stable.

Proof. Set $w = \log u$ and choose a cutoff function $\phi \in C_0^{0,1}(\Omega)$. Applying Stokes' theorem to

$$\operatorname{div}(\phi^2 \, \nabla w - \phi^2 \, a \, \nabla w) \, ,$$

substituting (2.5), and using $|a|, |c| \leq 3\,\delta, |b| \leq 2\,\delta\,|\nabla w|$ we get

$$\begin{aligned}(1 - 3\,\delta) \int \phi^2 \, |A|^2 \leq & - \int \phi^2 \, |\nabla w|^2 + \int \phi^2 \, \langle \nabla w, b + a \, \nabla w \rangle \\ & + 2 \int \phi \langle \nabla \phi, \nabla w - a \, \nabla w \rangle \\ \leq & (5\delta - 1) \int \phi^2 \, |\nabla w|^2 + 2(1 + 3\delta) \int |\phi \, \nabla w| \, |\nabla \phi| \, .\end{aligned} \tag{2.7}$$

The lemma now follows easily from the absorbing inequality (i.e., $2xy \leq \varepsilon x^2 + y^2/\varepsilon$). □

We will use Lemma 2.6 to see that disjoint embedded minimal surfaces that are close are nearly stable (Corollary 2.13 below). Integrating

$$|\nabla \mathrm{dist}_{\mathbf{S}^2}(\mathbf{n}(x), \mathbf{n})| \leq |A|$$

on geodesics gives

$$\sup_{x' \in \mathcal{B}_s(x)} \mathrm{dist}_{\mathbf{S}^2}(\mathbf{n}(x'), \mathbf{n}(x)) \leq s \sup_{\mathcal{B}_s(x)} |A| \,. \tag{2.8}$$

By (2.8), we can choose $0 < \rho_2 < 1/4$ so that:

> If $\mathcal{B}_{2s}(x) \subset \Sigma$, $s \sup_{\mathcal{B}_{2s}(x)} |A| \leq 4\,\rho_2$, and $t \leq s$, then the component $\Sigma_{x,t}$ of $B_t(x) \cap \Sigma$ with $x \in \Sigma_{x,t}$ is a graph over $T_x\Sigma$ with gradient $\leq t/s$ and

$$\inf_{x' \in \mathcal{B}_{2s}(x)} |x' - x| / \mathrm{dist}_\Sigma(x, x') > 9/10 \,. \tag{2.9}$$

One consequence is that if $t \leq s$ and we translate $T_x\Sigma$ so that $x \in T_x\Sigma$, then

$$\sup_{x' \in \mathcal{B}_t(x)} |x' - T_x\Sigma| \leq t^2/s \,. \tag{2.10}$$

Lemma 2.11. *There exist $C_0, \rho_0 > 0$ so that the following holds*:

If $\rho_1 \leq \min\{\rho_0, \rho_2\}$ and $\Sigma_1, \Sigma_2 \subset \mathbf{R}^3$ are oriented minimal surfaces with $|A|^2 \leq 4$ on each Σ_i so that

$$x \in \Sigma_1 \setminus \mathcal{T}_{4\rho_2}(\partial\Sigma_1) \,,$$
$$y \in B_{\rho_1}(x) \cap \Sigma_2 \setminus \mathcal{T}_{4\rho_2}(\partial\Sigma_2) \,,$$
$$\mathcal{B}_{2\rho_1}(x) \cap \mathcal{B}_{2\rho_1}(y) = \emptyset \,,$$

then $\mathcal{B}_{\rho_2}(y)$ is the graph $\{z + u(z)\,\mathbf{n}(z)\}$ over a domain containing $\mathcal{B}_{\rho_2/2}(x)$ with $u \neq 0$ and $|\nabla u| + 4\,|u| \leq C_0\,\rho_1$.

Proof. Since $\rho_1 \leq \rho_2$, (2.9) implies that

$$\mathcal{B}_{2\rho_2}(x) \cap \mathcal{B}_{2\rho_2}(y) = \emptyset \,.$$

If $t \leq 9\rho_2/5$, then $|A|^2 \leq 4$ implies that the components $\Sigma_{x,t}, \Sigma_{y,t}$ of $B_t(x) \cap \Sigma_1, B_t(y) \cap \Sigma_2$, respectively, with $x \in \Sigma_{x,t}, y \in \Sigma_{y,t}$, are graphs with gradient $\leq t/(2\rho_2)$ over $T_x\Sigma_1, T_y\Sigma_2$ and have

$$\Sigma_{x,t} \subset \mathcal{B}_{2\rho_2}(x),\ \Sigma_{y,t} \subset \mathcal{B}_{2\rho_2}(y) \,.$$

The last conclusion implies that $\Sigma_{x,t} \cap \Sigma_{y,t} = \emptyset$. It now follows that $\Sigma_{x,t}, \Sigma_{y,t}$ are graphs over the same plane. Namely, if we set

$$\theta = \mathrm{dist}_{\mathbf{S}^2}(\mathbf{n}(x), \{\mathbf{n}(y), -\mathbf{n}(y)\}) \,,$$

then (2.10), $|x-y|<\rho_1$, and $\Sigma_{x,t}\cap\Sigma_{y,t}=\emptyset$ imply that

$$\rho_1-(t/2-\rho_1)\,\sin\theta+t^2/(2\rho_2)>-t^2/(2\rho_2)\,. \tag{2.12}$$

Hence,

$$\sin\theta<\rho_1/(t/2-\rho_1)+t^2/[(t/2-\rho_1)\rho_2]\,.$$

For ρ_0/ρ_2 small, $\mathcal{B}_{\rho_2}(y)$ is a graph with bounded gradient over $T_x\Sigma_1$. The lemma now follows easily from the Harnack inequality. $\square$

Combining Lemmas 2.6 and 2.11 gives the next corollary:

Corollary 2.13 (see Figure 7). *Given $C_0,\delta>0$, there exists $\varepsilon(C_0,\delta)>0$ so that if $p_i\in\Sigma_i\subset\mathbf{R}^3$ $(i=1,2)$ are embedded minimal surfaces with*

$$\Sigma_1\cap\Sigma_2=\emptyset\,,\ \mathcal{B}_{2R}(p_i)\cap\partial\Sigma_i=\emptyset\,,\ |p_1-p_2|<\varepsilon\,R\,,$$

and

$$\sup_{\mathcal{B}_{2R}(p_i)}|A|^2\leq C_0\,R^{-2}\,, \tag{2.14}$$

then $\mathcal{B}_R(\tilde{p}_i)\subset\tilde{\Sigma}_i$ is δ-stable where $\tilde{p}_i$ is the point over p_i in the universal cover $\tilde{\Sigma}_i$ of Σ_i.

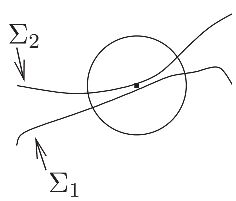

Figure 7: Corollary 2.13: Two sufficiently close disjoint minimal surfaces with bounded curvatures must be nearly stable.

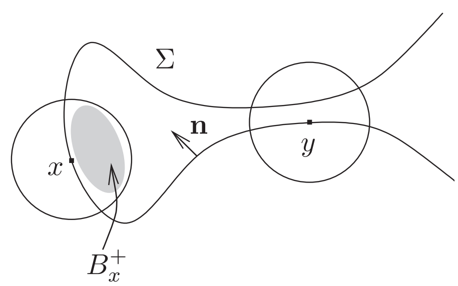

Figure 8: The set VB in (2.18). Here $x\in VB$ and $y\in\Sigma\setminus VB$.

The next result gives a decomposition of an embedded minimal surface with bounded curvature into a portion with bounded area and a union of disjoint 1/2-stable domains.

Lemma 2.15. *There exists C_1 so that the following holds:*

If $0\in\Sigma\subset B_{2R}\subset\mathbf{R}^3$ is an embedded minimal surface with $\partial\Sigma\subset\partial B_{2R}$, and $|A|^2\leq 4$, then there exist disjoint 1/2-stable subdomains $\Omega_j\subset\Sigma$ and a *function* $\chi \leq 1$ *which vanishes on* $B_R \cap \Sigma \setminus \cup_j \Omega_j$ *so that*

$$\text{Area}(\{x \in B_R \cap \Sigma \,|\, \chi(x) < 1\}) \leq C_1 \, R^3 \,, \tag{2.16}$$

$$\int_{\mathcal{B}_R} |\nabla \chi|^2 \leq C_1 \, R^3 \,. \tag{2.17}$$

Proof. We can assume that $R > \rho_2$ (otherwise $B_R \cap \Sigma$ is stable). Let $\delta > 0$ be from Lemma 2.6 and C_0, ρ_0 be from Lemma 2.11. Set

$$\rho_1 = \min\{\rho_0/C_0, \delta/C_0, \rho_2/4\} \,.$$

Given $x \in B_{2R-\rho_1} \cap \Sigma$, let Σ_x be the component of $B_{\rho_1}(x) \cap \Sigma$ with $x \in \Sigma_x$ and let B_x^+ be the component of $B_{\rho_1}(x) \setminus \Sigma_x$ which $\mathbf{n}(x)$ points into. See Figure 8. Set

$$VB = \{x \in B_R \cap \Sigma \,|\, B_x^+ \cap \Sigma \setminus \mathcal{B}_{4\,\rho_1}(x) = \emptyset\} \tag{2.18}$$

and let $\{\Omega_j\}$ be the components of $B_R \cap \Sigma \setminus \overline{VB}$. Choose a maximal disjoint collection $\{\mathcal{B}_{\rho_1}(y_i)\}_{1 \leq i \leq \nu}$ of balls centered in VB. Hence, the union of the balls $\{\mathcal{B}_{2\,\rho_1}(y_i)\}_{1 \leq i \leq \nu}$ covers VB. Further, the "half-balls" $B_{\rho_1/2}(y_i) \cap B_{y_i}^+$ are pairwise disjoint. To see this, suppose that $|y_i - y_j| < \rho_1$ but $y_j \notin \mathcal{B}_{2\rho_1}(y_i)$. Then, by (2.9), $y_j \notin \mathcal{B}_{8\rho_1}(y_i)$ so that $\mathcal{B}_{4\rho_1}(y_j) \notin B_{y_i}^+$ and $\mathcal{B}_{4\rho_1}(y_i) \notin B_{y_j}^+$; the triangle inequality then implies that

$$B_{\rho_1/2}(y_i) \cap B_{y_i}^+ \cap B_{\rho_1/2}(y_j) \cap B_{y_j}^+ = \emptyset \,,$$

as claimed. By (2.8)–(2.10), we see that each $B_{\rho_1/2}(y_i) \cap B_{y_i}^+$ has volume approximately ρ_1^3 and is contained in B_{2R} so that $\nu \leq C\,R^3$. Define the function χ on Σ by

$$\chi(x) = \begin{cases} 0 & \text{if } x \in VB \,, \\ \text{dist}_\Sigma(x, VB)/\rho_1 & \text{if } x \in \mathcal{T}_{\rho_1}(VB) \setminus VB \,, \\ 1 & \text{otherwise} \;. \end{cases} \tag{2.19}$$

Since $\mathcal{T}_{\rho_1}(VB) \subset \cup_{i=1}^{\nu} \mathcal{B}_{3\,\rho_1}(y_i)$, $|A|^2 \leq 4$, and $\nu \leq C\,R^3$, we get (2.16). Combining (2.16) and $|\nabla \chi| \leq \rho_1^{-1}$ gives (2.17) (taking C_1 larger).

It remains to show that each Ω_j is 1/2-stable. Fix j. By construction, if $x \in \Omega_j$, then there exists $y_x \in B_x^+ \cap \Sigma \setminus \mathcal{B}_{4\,\rho_1}(x)$ minimizing $|x - y_x|$ in $B_x^+ \cap \Sigma$. In particular, by Lemma 2.11, $\mathcal{B}_{\rho_2}(y_x)$ is the graph $\{z + u_x(z)\,\mathbf{n}(z)\}$ over a domain containing $\mathcal{B}_{\rho_2/2}(x)$ with $u_x > 0$ and $|\nabla u_x| + 4\,|u_x| \leq \min\{\delta, \rho_0\}$. Choose a maximal disjoint collection of balls $\mathcal{B}_{\rho_2/6}(x_i)$ with $x_i \in \Omega_j$ and let $u_{x_i} > 0$ be the corresponding functions defined on $\mathcal{B}_{\rho_2/2}(x_i)$. Since Σ is embedded (and compact) and $|u_{x_i}| < \rho_0$, Lemma 2.11 implies that

$$u_{x_i}(x) = \min_{t>0}\{x + t\,\mathbf{n}(x) \in \Sigma\}$$

for $x \in \mathcal{B}_{\rho_2/2}(x_i)$. Hence, $u_{x_{i_1}}(x) = u_{x_{i_2}}(x)$ for $x \in \mathcal{B}_{\rho_2/2}(x_{i_1}) \cap \mathcal{B}_{\rho_2/2}(x_{i_2})$. Note that $\mathcal{T}_{\rho_2/6}(\Omega_j) \subset \cup_i \mathcal{B}_{\rho_2/2}(x_i)$. We conclude that the u_{x_i}'s give a well-defined function $u_j > 0$ on $\mathcal{T}_{\rho_2/6}(\Omega_j)$ with $|\nabla u_j| + |u_j|\,|A| \leq \delta$. Finally, Lemma 2.6 implies that each Ω_j is 1/2-stable. $\square$

3. Total curvature and area of embedded minimal disks

Using the decomposition of Lemma 2.15, we next obtain polynomial bounds for the area and total curvature of intrinsic balls in embedded minimal disks with bounded curvature.

LEMMA 3.1. *There exists C_1 so that if $0 \in \Sigma \subset B_{2R}$ is an embedded minimal disk with $\partial \Sigma \subset \partial B_{2R}$ and $|A|^2 \leq 4$, then*

$$\int_0^R \int_0^t \int_{\mathcal{B}_s} |A|^2 \, ds \, dt = 2(\text{Area}(\mathcal{B}_R) - \pi R^2) \leq 6 \pi R^2 + 20 C_1 R^5 \, . \tag{3.2}$$

Proof. Let C_1, χ, and $\cup_j \Omega_j$ be given by Lemma 2.15. Define ψ on $\mathcal{B}_R$ by

$$\psi = \psi(\text{dist}_\Sigma(0, \cdot)) = 1 - \text{dist}_\Sigma(0, \cdot)/R \, ,$$

so that $\chi\psi$ vanishes off of $\cup_j \Omega_j$. Using $\chi\psi$ in the 1/2-stability inequality, the absorbing inequality and (2.17) gives

$$\begin{aligned} \int |A|^2 \chi^2 \psi^2 &\leq 2 \int |\nabla(\chi\psi)|^2 \\ &= 2 \int \left(\chi^2 |\nabla\psi|^2 + 2\chi\,\psi \langle \nabla\chi, \nabla\psi \rangle + \psi^2 |\nabla\chi|^2 \right) \\ &\leq 6\, C_1 R^3 + 3 \int \chi^2 |\nabla\psi|^2 \leq 6\, C_1 R^3 + 3\, R^{-2} \text{Area}(\mathcal{B}_R) \, . \end{aligned} \tag{3.3}$$

Using (2.16) and $|A|^2 \leq 4$, we get

$$\int |A|^2 \psi^2 \leq 4\, C_1 R^3 + \int |A|^2 \chi^2 \psi^2 \leq 10\, C_1 R^3 + 3\, R^{-2} \, \text{Area}\, (\mathcal{B}_R) \, . \tag{3.4}$$

The lemma follows from (3.4) and Corollary 1.7. $\square$

The polynomial growth allows us to find large intrinsic balls with a fixed doubling:

COROLLARY 3.5. *There exists C_2 so that given $\beta, R_0 > 1$, we get R_2 so that the following holds*:

If $0 \in \Sigma \subset B_{R_2}$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_{R_2}$ and

$$\sup_\Sigma |A|^2 \leq 4 |A|^2(0) = 4 \, ,$$

then there exists $R_0 < R < R_2/(2\,\beta)$ *with*

$$\int_{\mathcal{B}_{3\,R}} |A|^2 + \beta^{-10} \int_{\mathcal{B}_{2\,\beta\,R}} |A|^2 \leq C_2\, R^{-2}\, \text{Area}\,(\mathcal{B}_R)\,. \tag{3.6}$$

Proof. Set $\mathcal{A}(s) = \text{Area}(\mathcal{B}_s)$. Given m, Lemma 3.1 yields

$$\left(\min_{1 \leq n \leq m} \frac{\mathcal{A}((4\beta)^{2n}\, R_0)}{\mathcal{A}((4\beta)^{2n-2}\, R_0)} \right)^m \leq \frac{\mathcal{A}((4\beta)^{2m}\, R_0)}{\mathcal{A}(R_0)} \leq C_1'\, (4\beta)^{10m}\, R_0^3\,. \tag{3.7}$$

Fix m with $C_1'\, R_0^3 < 2^m$ and set $R_2 = 2\,(4\beta)^{2m}\, R_0$. By (3.7), there exists $R_1 = (4\beta)^{2n-2}\, R_0$ with $1 \leq n \leq m$ so that

$$\frac{\mathcal{A}((4\beta)^2\, R_1)}{\mathcal{A}(R_1)} \leq 2\,(4\beta)^{10}\,. \tag{3.8}$$

For simplicity, assume that $\beta = 4^q$ for $q \in \mathbf{Z}^+$. As in (3.7), (3.8), we get $0 \leq j \leq q$ with

$$\frac{\mathcal{A}(4^{j+1}\, R_1)}{\mathcal{A}(4^j\, R_1)} \leq \left[\frac{\mathcal{A}(4\beta\, R_1)}{\mathcal{A}(R_1)} \right]^{1/(q+1)} \leq 2^{1/(q+1)}\, 4^{10}\,. \tag{3.9}$$

Set $R = 4^j\, R_1$. Combining (3.8), (3.9), and Corollary 1.7 gives (3.6). □

4. The local structure near the axis

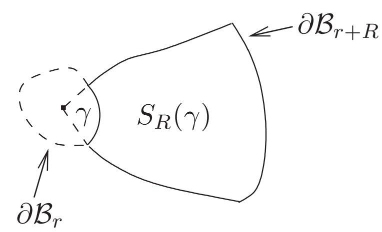

Figure 9: The intrinsic sector over a curve γ defined in (4.1).

Given $\gamma \subset \partial\mathcal{B}_r$, define the intrinsic sector (see Figure 9),

$$S_R(\gamma) = \{\exp_0(v)\,|\, r \leq |v| \leq r + R \text{ and } \exp_0(r\, v/|v|) \in \gamma\}\,. \tag{4.1}$$

The key for proving Theorem 0.4 is to find n large intrinsic sectors with a scale-invariant curvature bound. To do this:

- We first use Corollary 1.18 to bound $\text{Length}(\partial\mathcal{B}_R)/R$ from below for every $R \geq R_0$.
- Corollary 3.5 gives some $R_3 > R_0$ and n long disjoint curves $\tilde{\gamma}_i \subset \partial\mathcal{B}_{R_3}$ so that the sectors over $\tilde{\gamma}_i$ have bounded $\int |A|^2$.

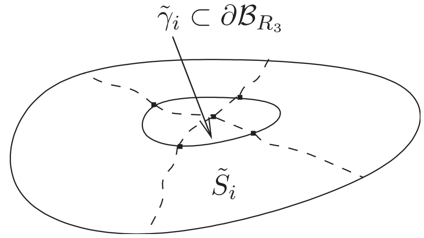

Figure 10: Equation (4.6) divides a punctured ball into sectors $\tilde{S}_i$.

- Corollary 1.18 gives the scale-invariant curvature bound.
- Once we have these sectors, for n large, two must be close and hence, by Lemmas 2.6 and 2.11, 1/2-stable.
- Finally, the N-valued graph is then given by corollary II.1.34 of [CM3] (see Corollary 4.2 below).

Corollary 4.2 ([CM3]). *Given $\omega > 8, 1 > \varepsilon > 0, C_0$, and N, there exist m_1, Ω_1 so that the following holds:*

If $0 \in \Sigma$ is an embedded minimal disk containing a curve $\gamma \subset \partial \mathcal{B}_{r_1}$ with

- $\int_\gamma k_g < C_0\, m_1$ *and* Length$(\gamma) = m_1\, r_1$,
- $\mathcal{T}_{r_1/8}(S_{\Omega_1^2\, \omega\, r_1}(\gamma))$ *is 1/2-stable,*

then (after a rotation of $\mathbf{R}^3$) $S_{\Omega_1^2\, \omega\, r_1}(\gamma)$ contains an N-valued graph Σ_N over $D_{\omega\, \Omega_1\, r_1} \setminus D_{\Omega_1\, r_1}$ with gradient $\leq \varepsilon$, $|A| \leq \varepsilon / r$, and $\operatorname{dist}_{S_{\Omega_1^2\, \omega\, r_1}(\gamma)}(\gamma, \Sigma_N) < 4\, \Omega_1\, r_1$.

Proof of Theorem 0.4. Rescale Σ by C/r_0 so that $|A|^2(0) = 1$ and $|A|^2 \leq 4$ on B_C.

Let C_2 be from Corollary 3.5 and then let $m_1, \Omega_1 > \pi$ be given by Corollary 4.2 with C_0 there $= 2\, C_2 + 2$. Fix a_0 large (to be chosen). By Corollaries 1.7, 1.18, there exists $R_0 = R_0(a_0)$ so that for any $R_3 \geq R_0$

$$a_0\, R_3 \leq R_3/4 \int_{\mathcal{B}_{R_3/2}} |A|^2 \leq \operatorname{Length}(\partial \mathcal{B}_{R_3})\,. \tag{4.3}$$

Set $\beta = 2\Omega_1^2\, \omega$. Corollaries 1.7, 3.5 give $R_2 = R_2(R_0, \beta)$ so that if $C \geq R_2$, then there is $R_0 < R_3 < R_2/(2\, \beta)$ with

$$\begin{aligned}\int_{\mathcal{B}_{3\, R_3}} |A|^2 + \beta^{-10} \int_{\mathcal{B}_{2\, \beta\, R_3}} |A|^2 &\leq C_2\, {R_3}^{-2} \operatorname{Area}(\mathcal{B}_{R_3}) \\ &\leq C_2 \operatorname{Length}(\partial \mathcal{B}_{R_3})/(2R_3)\,.\end{aligned} \tag{4.4}$$

Using (4.3), choose n so that

$$a_0\, R_3 \leq 4\, m_1\, n\, R_3 < \text{Length}(\partial \mathcal{B}_{R_3}) \leq 8\, m_1\, n\, R_3\,, \tag{4.5}$$

and fix $2n$ disjoint curves $\tilde{\gamma}_i \subset \partial \mathcal{B}_{R_3}$ with length $2\, m_1\, R_3$. Define the intrinsic sectors (see Figure 10)

$$\tilde{S}_i = \{\exp_0(v)\,|\, 0 < |v| \leq 2\,\beta\, R_3 \text{ and } \exp_0(R_3\, v/|v|) \in \tilde{\gamma}_i\}\,. \tag{4.6}$$

Since the $\tilde{S}_i$'s are disjoint, combining (4.4) and (4.5) gives

$$\sum_{i=1}^{2n} \left(\int_{\mathcal{B}_{3\, R_3} \cap \tilde{S}_i} |A|^2 + \beta^{-10} \int_{\tilde{S}_i} |A|^2 \right) \leq 4\, C_2\, m_1\, n\,. \tag{4.7}$$

Hence, after reordering the $\tilde{\gamma}_i$, we can assume that for $1 \leq i \leq n$

$$\int_{\mathcal{B}_{3\, R_3} \cap \tilde{S}_i} |A|^2 + \beta^{-10} \int_{\tilde{S}_i} |A|^2 \leq 4\, C_2\, m_1\,. \tag{4.8}$$

By the Riccati comparison theorem, there are curves $\gamma_i \subset \partial \mathcal{B}_{2R_3} \cap \tilde{S}_i$ with length $2\, m_1\, R_3$ so that if $y \in S_i = S_{\beta R_3}(\gamma_i) \subset \tilde{S}_i$, then $\mathcal{B}_{\text{dist}_\Sigma(0,y)/2}(y) \subset \tilde{S}_i$. Hence, by Corollary 1.18 and (4.8), for $y \in S_i$ and $i \leq n$,

$$\sup_{\mathcal{B}_{\text{dist}_\Sigma(0,y)/4}(y)} |A|^2 \leq C_3\, \text{dist}_\Sigma^{-2}(0,y)\,, \tag{4.9}$$

where $C_3 = C_3(\beta, m_1)$. For $i \leq n$, (4.8) and the Gauss-Bonnet theorem yield

$$\int_{\gamma_i} k_g \leq 2\,\pi + 2\, C_2\, m_1 < (2\, C_2 + 2)\, m_1\,. \tag{4.10}$$

By (4.9) and a Riccati comparison argument, there exists $C_4 = C_4(C_3)$ so that for $i \leq n$

$$1/(2R_3) \leq \min_{\gamma_i} k_g \leq \max_{\gamma_i} k_g \leq C_4/R_3\,. \tag{4.11}$$

Applying Lemma 2.11 repeatedly (and using (4.9)), it is easy to see that there exists $\alpha > 0$ so that if $i_1 < i_2 \leq n$ and

$$\text{dist}_{C^1([0,2m_1],\mathbf{R}^3)}(\gamma_{i_1}/R_3, \gamma_{i_2}/R_3) \leq \alpha\,, \tag{4.12}$$

then

$$\{z + u(z)\, \mathbf{n}(z)\,|\, z \in \mathcal{T}_{R_3/4}(S_{i_1})\} \subset \cup_{y \in S_{i_2}} \mathcal{B}_{\text{dist}_\Sigma(0,y)/4}(y)$$

for a function $u \neq 0$ with

$$|\nabla u| + |A|\, |u| \leq C_0'\, \text{dist}_{C^1([0,2m_1],\mathbf{R}^3)}(\gamma_{i_1}/R_3, \gamma_{i_2}/R_3)\ . \tag{4.13}$$

Here $\text{dist}_{C^1([0,2m_1],\mathbf{R}^3)}(\gamma_{i_1}/R_3, \gamma_{i_2}/R_3)$ is the scale-invariant C^1-distance between the curves.

Next, we use compactness to show that (4.12) must hold for n large. Namely, since each $\gamma_i/R_3 \subset B_2$ is parametrized by arclength on $[0, 2m_1]$ and has a uniform $C^{1,1}$ bound by (4.11), this set of maps is compact by the Arzela-Ascoli theorem. Hence, there exists n_0 so that if $n \geq n_0$, then (4.12) holds for some $i_1 < i_2 \leq n$. In particular, (4.13) and Lemma 2.6 imply that S_{i_1} is 1/2-stable for n large (now choose a_0, R_0, R_2). After rotating $\mathbf{R}^3$, Corollary 4.2 gives the N-valued graph $\Sigma_g \subset S_{i_1}$ over $D_{2\omega\,\Omega_1\,R_3} \setminus D_{2\Omega_1\,R_3}$ with gradient $\leq \varepsilon$, $|A| \leq \varepsilon/r$, and

$$\mathrm{dist}_{\Sigma}(0, \Sigma_g) \leq 8\,\Omega_1\,R_3\,.$$

Rescaling by r_0/C, we see that the theorem follows with $\bar{R} = 2\Omega_1\,R_3 r_0/C$. □

The next result simply combines the existence of a small multi-valued graph from Theorem 0.4 with the extension result from Theorem 0.6:

Corollary 4.14. *Given $N > 1$ and $\tau > 0$, there exist $\Omega > 1$ and $C > 0$ so that the following holds*:

Let $0 \in \Sigma^2 \subset B_R$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_R$. If for some r_0 with $R > r_0 > 0$,

$$\sup_{B_{r_0} \cap \Sigma} |A|^2 \leq 4\,|A|^2(0) = 4\,C^2\,r_0^{-2}\,,$$

then there exists (after a rotation) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{R/\Omega} \setminus D_{r_0}$ with gradient $\leq \tau$, $\mathrm{dist}_{\Sigma}(0, \Sigma_g) \leq 4\,r_0$, and $\Sigma_g \subset \{x_3^2 \leq \tau^2\,(x_1^2 + x_2^2)\}$.

Proof. This follows immediately by combining Theorems 0.4 and 0.6. □

The bound for the gradient of the multi-valued graph in Theorem 0.4 immediately implies an upper bound for the separation w. The next proposition obtains a lower bound for the initial separation, i.e., for the separation over $\partial D_{\bar{R}}$.

Proposition 4.15 (see Figure 11). *There exists $\beta > 0$ so that if $\Sigma_g \subset \Sigma$ is as in Theorem 0.4, then the separation between the sheets of Σ_g over $\partial D_{\bar{R}}$ is at least $\beta\,\bar{R}$.*

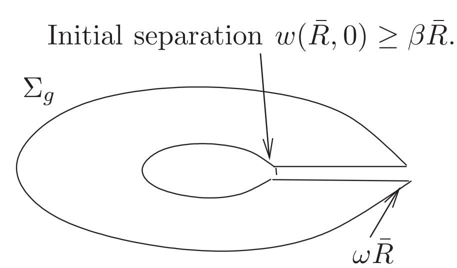

Figure 11: Proposition 4.15: The initial separation is inversely proportional to the maximum of $|A|$.

Proof. This follows easily from the curvature bound $\sup_{B_{r_0}\cap\Sigma} |A|^2 \leq 4\,C^2\,r_0^{-2}$, Lemma 2.11, the Harnack inequality, and estimates for 1/2-stable surfaces. □

5. The blow-up

Combining Corollary 4.14 and a simple rescaling ("blow-up") argument will give Theorem 0.2. This rescaling argument shows that if

$$\sup_{B_{\frac{r_0}{2}}\cap\Sigma} |A|^2 \geq 16\,C^2\,r_0^{-2}\,,$$

then there are a point y and radius $s > 0$ so that

$$\sup_{B_s(y)\cap\Sigma} |A|^2 \leq 4\,|A|^2(y) = 4\,C^2\,s^{-2}\,.$$

That is, the curvature at y is large (this just means that C should be thought of as a large constant) and is almost (up to the constant 4) the maximum on the ball $B_s(y)$. We will say that the pair (y, s) is a *blow-up pair*, where y is the *blow-up point* and s is its *scale*.

Lemma 5.1. *If $\Sigma \subset B_{r_0}$ is a surface with $\partial\Sigma \subset \partial B_{r_0}$ and*

$$\sup_{B_{\frac{r_0}{2}}\cap\Sigma} |A|^2 \geq 16\,C^2\,r_0^{-2}\,,$$

then there exist $y \in \Sigma$ and $s < r_0 - |y|$ such that

$$\sup_{B_s(y)\cap\Sigma} |A|^2 \leq 4\,|A|^2(y) = 4\,C^2\,s^{-2}\,. \tag{5.2}$$

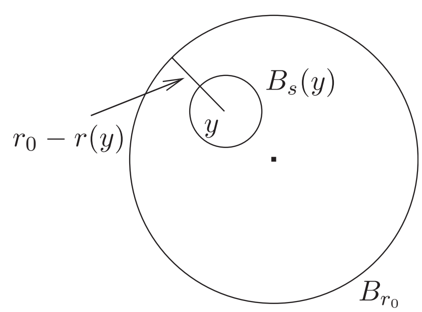

Figure 12: Existence of blow-up points, that is, pairs of points $y \in \Sigma$ and $s > 0$ satisfying (5.2).

Proof. Define a non-negative function F on $B_{r_0} \cap \Sigma$ by

$$F(x) = (r_0 - |x|)^2 \, |A|^2(x) \,,$$

so that F vanishes on $\partial B_{r_0} \cap \Sigma$. Let $y \in B_{r_0} \cap \Sigma$ be where the maximum of F is achieved and set

$$s = C/|A|(y) \,.$$

One easily checks that y and s have the required properties; see Figure 12. That is, clearly $|A|^2(y) = C^2 \, s^{-2}$ and since y is where the maximum of F is achieved, we have

$$(r_0 - |y|)^2 \, |A|^2(y) = \sup_{B_{r_0}} F \geq \left(\frac{r_0}{2}\right)^2 \sup_{B_{\frac{r_0}{2}} \cap \Sigma} |A|^2 \geq 4 \, C^2 \,.$$

Since $s^2 \, |A|^2(y) = C^2$, we see that $2s < (r_0 - |y|)$ so that

$$\sup_{z \in B_s(y)} \frac{r_0 - |y|}{r_0 - |z|} \leq 2 \,. \tag{5.3}$$

Using again the maximality of $F(y)$, we have for $z \in B_s(y) \cap \Sigma$ that

$$|A|^2(z) \leq \left(\frac{r_0 - |y|}{r_0 - |z|} \right)^2 |A|^2(y) \leq 4 \, |A|^2(y) \,, \tag{5.4}$$

where the last inequality used (5.3). □

Proof of Theorem 0.2. This follows immediately from Corollary 4.14 and Lemma 5.1. □

If $y_i \in \Sigma_i$ is a sequence of minimal disks with $y_i \to y$ and $|A|(y_i)$ blowing up, then we can take $r_0 \to 0$ in Theorem 0.2. Combining this with the sublinear growth of the separation between the sheets from [CM3], we will get in Theorem 5.8 a smooth limit through y.

Below $\Sigma_{r,s}^{0,2\pi} \subset \Sigma$ is the "middle sheet" over

$$\{(\rho, \theta) \, | \, 0 \leq \theta \leq 2\pi, \, r \leq \rho \leq s\} \,.$$

The sublinear growth is given by proposition II.2.12 of [CM3]:

Proposition 5.5 ([CM3], see Figure 13). *Given* $\alpha > 0$, *there exist* $\delta_p > 0$, $N_g > 5$ *so that the following holds*:

If Σ *is an* N_g*-valued minimal graph over* $D_{\mathrm{e}^{N_g} R} \setminus D_{\mathrm{e}^{-N_g} R}$ *with gradient* ≤ 1 *and* $0 < w < \delta_p \, R$ *is a solution of the minimal graph equation over* Σ *with* $|\nabla w| \leq 1$, *then for* $R \leq s \leq 2\,R$

$$\sup_{\Sigma_{R,2R}^{0,2\pi}} |A_\Sigma| + \sup_{\Sigma_{R,2R}^{0,2\pi}} |\nabla w|/w \leq \alpha/(4\,R) \,, \tag{5.6}$$

$$\sup_{\Sigma_{R,s}^{0,2\pi}} w \leq (s/R)^\alpha \sup_{\Sigma_{R,R}^{0,2\pi}} w \,. \tag{5.7}$$

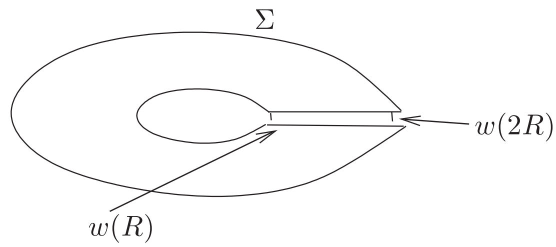

Figure 13: The sublinear growth of the separation w of the multi-valued graph Σ: $w(2R) \leq 2^\alpha\, w(R)$ with $\alpha < 1$.

We will next show that if $|A|^2$ is blowing up for a sequence of embedded minimal disks, then there is a smooth minimal graph through this point in the limit of a subsequence:

THEOREM 5.8 (see Figure 14). *There exists $\Omega > 1$ so that the following holds*:

Let $y_i \in \Sigma_i \subset B_R$ with $\partial \Sigma_i \subset \partial B_R$ be a sequence of embedded minimal disks where $y_i \to 0$. If $|A_{\Sigma_i}|(y_i) \to \infty$, then:

(1) *After a rotation and passing to a subsequence, there exist $\varepsilon_i \to 0$ and 2-valued minimal graphs $\Sigma_{d,i} \subset \{x_3^2 \leq x_1^2 + x_2^2\} \cap \Sigma_i$ over $D_{R/\Omega} \setminus D_{\varepsilon_i}$ with gradient ≤ 1.*

(2) *The $\Sigma_{d,i}$ converge (with multiplicity two) to a smooth minimal graph through 0.*

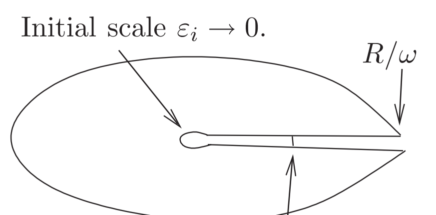

Figure 14: Theorem 5.8: As $|A_{\Sigma_i}|(y_i) \to \infty$ and $y_i \to y$, 2-valued graphs converge to a graph through y. (The upper sheets of the 2-valued graphs collapse to the lower sheets.)

Proof. The first claim follows immediately from taking $r_0 \to 0$ in Theorem 0.2 and the Arzela-Ascoli theorem.

The key for the second claim is to show that the separation goes to zero over $(\rho, 0)$ for any fixed $\rho > 0$. To see this, we use the sublinear growth of the separation (i.e., Proposition 5.5) to get

$$|w_i(\rho,0)| \leq \left(\frac{\rho}{\varepsilon_i}\right)^{\alpha} |w_i(\varepsilon_i,0)| \leq 2\,\pi\,\varepsilon_i^{1-\alpha}\,\rho^{\alpha}\,. \tag{5.9}$$

Note that the bound $|w_i(\varepsilon_i, 0)| \leq 2\,\pi\,\varepsilon_i$ came from integrating the bound gradient ≤ 1 around the circle of radius ε_i. It follows from (5.9) that the $\Sigma_{d,i}$ close up in the limit. In particular, the $\Sigma_{d,i}$ converge to a minimal graph Σ' over $D_{R/\Omega} \setminus \{0\}$ with gradient ≤ 1 and $\Sigma' \subset \{x_3^2 \leq x_1^2 + x_2^2\}$. By a standard removable singularity theorem, $\Sigma' \cup \{0\}$ is a smooth minimal graph over $D_{R/\Omega}$. $\square$

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