Annals of Mathematics, **159** (2004), 783–817

Radon inversion on Grassmannians via Gårding-Gindikin fractional integrals

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Abstract

We study the Radon transform $\mathcal{R}f$ of functions on Stiefel and Grassmann manifolds. We establish a connection between $\mathcal{R}f$ and Gårding-Gindikin fractional integrals associated to the cone of positive definite matrices. By using this connection, we obtain Abel-type representations and explicit inversion formulae for $\mathcal{R}f$ and the corresponding dual Radon transform. We work with the space of continuous functions and also with L^p spaces.

1. Introduction

Let $G_{n,k}, G_{n,k'}$ be a pair of Grassmann manifolds of linear k-dimensional and k'-dimensional subspaces of $\mathbb{R}^n$, respectively. Suppose that $1 \le k < k' \le n-1$. A "point" $\eta \in G_{n,k}$ ($\xi \in G_{n,k'}$) is a nonoriented k-plane (k'-plane) in $\mathbb{R}^n$ passing through the origin. The Radon transform of a sufficiently good function $f(\eta)$ on $G_{n,k}$ is a function $(\mathcal{R}f)(\xi)$ on the Grassmannian $G_{n,k'}$. The value of $(\mathcal{R}f)(\xi)$ at the k'-plane ξ is the integral of the k-plane function $f(\eta)$ over all k-planes η which are subspaces of ξ:

$$(\mathcal{R}f)(\xi) = \int\limits_{\{\eta:\eta\subset\xi\}} f(\eta) d_\xi \eta, \qquad \xi \in G_{n,k'}, \tag{1.1}$$

$d_\xi \eta$ being the canonical normalized measure on the space of planes η in ξ. In the present paper we focus on inversion formulae for $\mathcal{R}f$, leaving aside such important topics as range characterization, affine Grassmannians, the complex case, geometrical applications, and further possible generalizations. Concerning these topics, the reader is addressed to fundamental papers by I.M. Gel'fand (and collaborators), F. Gonzalez, P. Goodey, E.L. Grinberg, S. Helgason, T. Kakehi, E.E. Petrov, R.S. Strichartz, and others.

*This work was supported in part by NSF grant DMS-9971828. The second author also was supported in part by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany).

The first question is: For which triples (k, k', n) is the operator $\mathcal{R}$ injective? (In such cases we will seek an explicit inversion formula, not just a uniqueness result.) It is natural to assume that the transformed function depends on at least as many variables as the original function, i.e.,

$$\dim G_{n,k'} \geq \dim G_{n,k}. \tag{1.2}$$

(If this condition fails then $\mathcal{R}$ has a nontrivial kernel.) By taking into account that

$$\dim G_{n,k} = k(n-k),$$

we conclude that (1.2) is equivalent to $k+k' \leq n$ (for $k < k'$). Thus the natural framework for the inversion problem is

$$1 \leq k < k' \leq n-1, \qquad k+k' \leq n. \tag{1.3}$$

For $k = 1$, f is a function on the projective space $\mathbb{RP}^{n-1} \equiv G_{n,1}$ and can be regarded as an even function on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. In this context $(\mathcal{R}f)(\xi)$ represents the totally geodesic Radon transform, which has been inverted in a number of ways; see, e.g., [H1], [H2], [Ru2], [Ru3]. For $k > 1$ several approaches have been proposed. In 1967 Petrov [P1] announced inversion formulae assuming $k' + k = n$. His method employs an analog of plane wave decomposition. Alas, all proofs in Petrov's article were omitted. His inversion formulae contain a divergent integral that requires regularization. Another approach, based on the use of differential forms, was suggested by Gel'fand, Graev and Šapiro [GGŠ] in 1970 (see also [GGR]). A third approach was developed by Grinberg [Gr1], Gonzalez [Go] and Kakehi [K]. It employs harmonic analysis on Grassmannians and agrees with the classical idea of Blaschke-Radon-Helgason to apply a certain differential operator to the composition of the Radon transform and its dual; see [Ru4] for historical notes. The second and third approaches are applicable only when $k' - k$ is even (although Gel'fand's approach has been extended to the odd case in terms of the Crofton symbol and the Kappa operator [GGR]). Note also that the methods above deal with C^∞-functions and resulting inversion formulae are rather involved. Here we aim to give simple formulae which are valid for both odd and even cases and which extend classical formulae for rank one spaces.

Main results. Our approach differs from the aforementioned methods. It goes back to the original ideas of Funk and Radon, employing fractional integrals, mean value operators and the appropriate group of motions. See [Ru4] for historical details. Our task was to adapt this classical approach to Grassmannians. This method covers the full range (1.3), agrees completely with the case $k = 1$, and gives transparent inversion formulae for any integrable function f. Along the way we derive a series of integral formulae which are known in the case $k = 1$ and appear to be new for $k > 1$. These formulae may be useful in other contexts.

As a prototype we consider the case $k = 1$, corresponding to the totally geodesic Radon transform $\varphi(\xi) = (\mathcal{R}f)(\xi)$, $\xi \in G_{n,k'}$. For this case, the well-known inversion formula of Helgason [H1], [H2, p. 99] in slightly different notation reads as follows:

$$f(x) = c\Big[\Big(\frac{d}{d(u^2)}\Big)^{k'-1} \int\limits_0^u (M_v^*\varphi)(x) v^{k'-1}(u^2 - v^2)^{(k'-3)/2} dv\Big]_{u=1}. \tag{1.4}$$

Here $f(x)$ is an even function on S^{n-1}, $c = 2^{k'-1}/(k'-2)!\sigma_{k'-1}$, $\sigma_{k'-1}$ is the area of the unit sphere $S^{k'-1}$, $(M_v^*\varphi)(x)$ is the average of $\varphi(\xi)$ over all $(k'-1)$-geodesics $S^{n-1} \cap \xi$ at distance $\cos^{-1}(v)$ from x.

We extend (1.4) to the higher rank case $k > 1$ as follows. The key ingredient in (1.4) is the fractional derivative in square brackets. We substitute the one-dimensional Riemann-Liouville integral, arising in Helgason's scheme and leading to (1.4), for its higher rank counterpart:

$$(I_+^\alpha w)(r) = \frac{1}{\Gamma_k(\alpha)} \int\limits_0^r w(s)\,(\det(r-s))^{\alpha-(k+1)/2} ds, \quad \operatorname{Re}\alpha > (k-1)/2, \tag{1.5}$$

associated to $\mathcal{P}_k$, the cone of symmetric positive definite $k \times k$ matrices. Let us explain the notation in (1.5). Here $r = (r_{i,j})$ and $s = (s_{i,j})$ are "points" in $\mathcal{P}_k$, $ds = \prod_{i \le j} ds_{i,j}$, the integration is performed over the "interval"

$$\{s :\, s \in \mathcal{P}_k,\, r - s \in \mathcal{P}_k\},$$

and $\Gamma_k(\alpha)$ is the Siegel gamma function (see (2.4), (2.5) below). Integrals (1.5) were introduced by Gårding [Gå], who was inspired by Riesz [R1], Siegel [S], and Bochner [B1], [B2]. Substantial generalizations of (1.5) are due to Gindikin [Gi] who developed a deep theory of such integrals.

Given a function $f(r)$, $r = (r_{i,j}) \in \mathcal{P}_k$, we denote

$$(D_+ f)(r) = \det\left(\eta_{i,j}\,\frac{\partial}{\partial r_{i,j}}\right) f(r), \qquad \eta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 1/2 & \text{if } i \neq j, \end{cases} \tag{1.6}$$

so that $D_+ I_+^\alpha = I_+^{\alpha-1}$ [Gå] (see Section 2.2). Useful information about Siegel gamma functions, integrals (1.5), and their applications can be found in [FK], [Herz], [M], [T].

Another important ingredient in (1.4) is $(M_v^*\varphi)(x)$. This is the average of $\varphi(\xi)$ over the set of all $\xi \in G_{n,k'}$ satisfying $\cos\theta = v$, θ being the angle between the unit vector x and the orthogonal projection $\text{Pr}_\xi x$ of x onto ξ. This property leads to the following generalization.

Let $V_{n,k}$ be the Stiefel manifold of all orthonormal k-frames in Euclidean n-space. Elements of the Stiefel manifold can be regarded as $n \times k$ matrices x satisfying $x'x = I_k$, where x' is the transpose of x, and I_k denotes the identity

$k \times k$ matrix. Each function f on the Grassmannian $G_{n,k}$ can be identified with the relevant function $f(x)$ on $V_{n,k}$ which is $\mathrm{O}(k)$ right-invariant, i.e., $f(x\gamma) = f(x)$ $\forall \gamma \in \mathrm{O}(k)$ (the group of orthogonal $k \times k$ matrices). The right $\mathrm{O}(k)$ invariance of a function on the Stiefel manifold simply means that the function is invariant under change of basis within the span of a given frame, and hence "drops" to a well-defined function on the Grassmannian. The aforementioned identification enables us to reach numerous important statements and to achieve better understanding of the matter by working with functions of a matrix argument.

Definition 1.1. Given $\eta \in G_{n,k}$ and $y \in V_{n,\ell}$, $\ell \le k$, we define

$$\mathrm{Cos}^2(\eta, y) = y'\mathrm{Pr}_\eta y, \qquad \mathrm{Sin}^2(\eta, y) = y'\mathrm{Pr}_{\eta^\perp} y, \tag{1.7}$$

where $\eta^\perp$ denotes the $(n-k)$-subspace orthogonal to η.

Both quantities represent positive semidefinite $\ell \times \ell$ matrices. This can be readily seen if we replace the linear operator Pr_η by its matrix xx' where $x = [x_1, \dots, x_k] \in V_{n,k}$ is an orthonormal basis of η. Clearly,

$$\mathrm{Cos}^2(\eta, y) + \mathrm{Sin}^2(\eta, y) = I_\ell.$$

We introduce the following mean value operators

$$(M_r f)(\xi) = \int\limits_{\mathrm{Cos}^2(\xi,x)=r} f(x)\,dm_\xi(x), \qquad (M_r^*\varphi)(x) = \int\limits_{\mathrm{Cos}^2(\xi,x)=r} \varphi(\xi)\,dm_x(\xi), \tag{1.8}$$

$x \in V_{n,k}$, $\xi \in G_{n,k'}$, $r \in \mathcal{P}_k$; $dm_\xi(x)$ and $dm_x(\xi)$ are the relevant induced measures. A precise definition of these integrals is given in Section 3. According to this definition, $(M_r^*\varphi)(x)$ is well defined as a function of $\eta \in G_{n,k}$, and (up to abuse of notation) one can write $(M_r^*\varphi)(x) \equiv (M_r^*\varphi)(\eta)$. Operators (1.8) are matrix generalizations of the relevant Helgason transforms for $k = 1$ (cf. formula (35) in [H2, p. 96]). The mean value $M_r^*\varphi$ with the matrix-valued averaging parameter $r \in \mathcal{P}_k$ serves as a substitute for $M_v^*\varphi$ in (1.4). For $r = I_k$, operators (1.8) coincide with the Radon transform (1.1) and its dual, respectively (see §4).

Theorem 1.2. *Let $f \in L^p(G_{n,k})$, $1 \le p < \infty$. Suppose that $\varphi(\xi) = (\mathcal{R}f)(\xi)$, $\xi \in G_{n,k'}$, $1 \le k < k' \le n-1$, $k + k' \le n$, and denote*

$$\alpha = (k'-k)/2, \qquad \hat{\varphi}_\eta(r) = (\det(r))^{\alpha-1/2}(M_r^*\varphi)(\eta), \qquad c = \frac{\Gamma_k(k/2)}{\Gamma_k(k'/2)}. \tag{1.9}$$

Then for any integer $m > (k'-1)/2$,

$$f(\eta) = c \overset{(L^p)}{\lim_{r \to I_k}} (D_+^m I_+^{m-\alpha} \hat{\varphi}_\eta)(r), \tag{1.10}$$

the differentiation being understood in the sense of distributions. In particular, for $k'-k=2\ell,\ \ell\in\mathbb{N}$,

$$f = c \overset{(L^p)}{\lim_{r\to I_k}} (D_+^{\ell}\hat{\varphi}_\eta)(r). \tag{1.11}$$

If f is a continuous function on $G_{n,k}$, then the limit in (1.10) *and* (1.11) *can be treated in the* sup-*norm.*

This theorem gives a family of inversion formulae parametrized by the integer m. They generalize (1.4) to the higher rank case and $f\in L^p$. The equality (1.10) coincides with (1.4), if $k=1$, $m=k'$, and has the same structure. Moreover, (1.10) covers the full range (1.3), including even and odd cases for $k'-k$. A simple structure of the formula (1.10) is based on the fact that the analytic family $\{I_+^\alpha\}$ includes the identity operator, namely, $I_+^0=I$. Here one should take into account that $I_+^\alpha w$ for $\operatorname{Re}\alpha\le(k-1)/2$ is defined by analytic continuation (for sufficiently good w) or in the sense of distributions; see Section 2.2 and [Gi].

As in the classical Funk-Radon theory, Theorem 1.2 is preceded by a similar one for zonal functions. The results for this important special case are as follows.

Definition 1.3 (ℓ-zonal functions). Let $\mathrm{O}(n)$ be the group of orthogonal $n\times n$ matrices. Fix ℓ so that $1\le\ell\le n-1$. Given $\rho\in\mathrm{O}(n-\ell)$, let

$$g_\rho = \begin{bmatrix} \rho & 0 \\ 0 & I_\ell \end{bmatrix} \in \mathrm{O}(n).$$

A function $f(\eta)$ on $G_{n,k}$ is called *ℓ-zonal* if $f(g_\rho\eta)=f(\eta)$ for all $\rho\in\mathrm{O}(n-\ell)$.

If $\ell=k=1$ then an ℓ-zonal function depends only on one variable, sometimes called *height*.

In the following theorems we employ the notion of *rank* of a symmetric space. This can be defined in various equivalent ways, e.g., using Lie algebras, maximal totally geodesic flat subspaces or invariant differential operators [H3]. The rank of $G_{n,k}$ can be computed: $\operatorname{rank} G_{n,k}=\min(k,n-k)$. Rank comes up in the harmonic analysis of functions on Grassmannians, and the injectivity dimension criterion (1.3) can be motivated by means of rank considerations [Gr3]. Here we do not use the intrinsic definition of rank explicitly, but it surfaces autonomously in the analysis.

THEOREM 1.4. *Choose ℓ so that $1\le\ell\le\min(k,n-k)$ $(=\operatorname{rank} G_{n,k})$, and let $f(\eta)$ be an integrable ℓ-zonal function on $G_{n,k}$.*

(i) *There is a function $f_0(s)$ on $\mathcal{P}_\ell$ so that*

$$f(\eta) \overset{\text{a.e.}}{=} f_0(s), \qquad s=\mathrm{Cos}^2(\eta,\sigma_\ell), \qquad \sigma_\ell=\begin{bmatrix} 0 \\ I_\ell \end{bmatrix}\in V_{n,\ell},$$

and

$$\int\limits_{G_{n,k}} f(\eta)d\eta = \frac{\Gamma_\ell(n/2)}{\Gamma_\ell(k/2)\,\Gamma_\ell((n-k)/2)} \int\limits_0^{I_\ell} f_0(s)d\mu(s), \tag{1.12}$$

$$d\mu(s) = (\det(s))^{(k-\ell-1)/2}\,(\det(I_\ell - s))^{(n-k-\ell-1)/2}\,ds. \tag{1.13}$$

(ii) *If* $\ell \le k' - k$, $1 \le k < k' \le n-1$, *then the Radon transform* $(\mathcal{R}f)(\xi)$, $\xi \in G_{n,k'}$, *is represented by the Gårding-Gindikin fractional integral as follows*:

$$(\mathcal{R}f)(\xi) = c\,(\det(S))^{-(k'-\ell-1)/2}(I_+^\alpha \tilde{f}_0)(S), \tag{1.14}$$

where $\tilde{f}_0(s) = (\det(s))^{(k-\ell-1)/2} f_0(s)$,

$$\alpha = (k'-k)/2, \qquad S = \mathrm{Cos}^2(\xi, \sigma_\ell) \in \mathcal{P}_\ell, \qquad c = \Gamma_\ell(k'/2)/\Gamma_\ell(k/2).$$

Let us comment on this theorem. The identity (1.12) gives precise information about the weighted L^1 space to which $f_0(s)$ belongs. This information is needed to keep convergence of numerous integrals which arise in the analysis below under control. The condition $1 \le \ell \le \mathrm{rank}\, G_{n,k}$ is natural. It reflects the geometric fact that $G_{n,k}$ is isomorphic to $G_{n,n-k}$ and is necessary to ensure absolute convergence of the integral in the right-hand side of (1.12). The additional condition $\ell \le k' - k$ in (ii) is necessary for absolute convergence of the fractional integral in (1.14), but it is not needed for $(\mathcal{R}f)(\xi)$ because the latter exists pointwise almost everywhere for any integrable f. This obvious gap can be reduced if we restrict ourselves to the case when $(\mathcal{R}f)(\xi)$, as well as f, is a function on the cone $\mathcal{P}_\ell$. To this end we impose the extra condition $1 \le \ell \le \mathrm{rank}\, G_{n,k'}$ and get

$$1 \le \ell \le \min(\mathrm{rank}\, G_{n,k},\, \mathrm{rank}\, G_{n,k'}) = \min\,(k, n-k'). \tag{1.15}$$

This condition does not imply $\ell \le k' - k$. Hence we need a substitute for (1.14) which holds for ℓ satisfying (1.15) and enables us to invert $\mathcal{R}f$.

Theorem 1.5. *Let* ℓ *satisfy* $1 \le \ell \le \min(k, n-k')$, *and suppose that* $\varphi(\xi) = (\mathcal{R}f)(\xi)$, $\xi \in G_{n,k'}$, *where* $f(\eta)$ *is an integrable* ℓ*-zonal function on* $G_{n,k}$.

(i) *There exist functions* $f_0(s)$ *and* $F_0(S)$ *so that*

$$f(\eta) \overset{\text{a.e.}}{=} f_0(s), \quad s = \mathrm{Cos}^2(\eta, \sigma_\ell), \qquad \varphi(\xi) \overset{\text{a.e.}}{=} F_0(S), \quad S = \mathrm{Cos}^2(\xi, \sigma_\ell).$$

If $\hat{f}_0(s) = (\det(s))^{(k-\ell-1)/2} f_0(s)$ *and* $\hat{F}_0(S) = (\det(S))^{(k'-\ell-1)/2} F_0(S)$ *then*

$$I_+^{(n-k')/2} \hat{F}_0 = c\, I_+^{(n-k)/2} \hat{f}_0, \qquad c = \Gamma_\ell(k'/2)/\Gamma_\ell(k/2). \tag{1.16}$$

(ii) *The function $f_0(s)$ can be recovered by the formula*

$$f_0(s) = c^{-1}(\det(s))^{-(k-\ell-1)/2}\,(D_+^m I_+^{m-\alpha}\hat{F}_0)(s), \qquad \alpha = (k'-k)/2, \quad m \in \mathbb{N}, \quad m > (k'-1)/2, \tag{1.17}$$

where D_+^m is understood in the sense of distributions.

Natural analogs of Theorems 1.4 and 1.5 hold for the dual Radon transform. For $k = 1$, these results were obtained in [Ru2]. Unlike the case $k = 1$ (where pointwise differentiation is possible), we cannot do the same for $k > 1$. The treatment of D_+^m in the sense of distributions is unavoidable in the framework of the method (even for smooth f), because of convergence restrictions. The latter are intimately connected with the complicated structure of the boundary of $\mathcal{P}_k$ (or $\mathcal{P}_\ell$). It is important to note that in the ℓ-zonal case inversion formulae for the Radon transform and its dual hold without the assumption $k + k' \leq n$.

A few words about technical tools are in order. We were inspired by the papers of Herz [Herz] and Petrov [P2] (unfortunately the latter was not translated into English). The key role in our argument belongs to Lemma 2.2 which extends the notion of bispherical coordinates [VK, pp. 12, 22] to Stiefel manifolds and generalizes Lemma 3.7 from [Herz, p. 495].

The paper is organized as follows. Section 2 contains preliminaries and derivation of basic integral formulae. In the rank-one case these formulae are known to every analyst working on the sphere. We need their extension to Stiefel and Grassmann manifolds. In Section 2 we also prove part (i) of Theorem 1.4 (see Corollary 2.9). In Section 3 we introduce mean value operators, which can be regarded as matrix analogs of geodesic spherical means on S^{n-1}, and which play a key role in our consideration. In Section 4 we complete the proof of the main theorems. Theorem 4.6 covers part (ii) of Theorem 1.4, and a similar statement holds for the dual Radon transform $\mathcal{R}^*$. Theorem 4.10 implies (1.16) and the corresponding equality for $\mathcal{R}^*$. Inversion formulae (1.10), (1.11), (1.17), and an analog of (1.17) for $\mathcal{R}^*$ are proved at the end of the section.

Acknowledgements. The work was started in Summer 2000 when B. Rubin was visiting Temple University in Philadelphia. He expresses gratitude to his co-author, Professor Eric Grinberg, for the hospitality. Both authors are grateful to the referee for his comments and valuable suggestions owing to which the original text of the paper was essentially improved.

2. Preliminaries

2.1. *Notation, matrix spaces and Siegel gamma functions.* The main references for the following are [M, Ch. 2 and Appendix], [T, Ch. 4], [Herz]. We recall some basic facts and definitions. Let $\mathfrak{M}_{n,k}$ be the space of real matrices having n rows and k columns. One can identify $\mathfrak{M}_{n,k}$ with the real Euclidean space $\mathbb{R}^{nk}$ so that for $x = (x_{i,j})$ the volume element is $dx = \prod_{i=1}^{n}\prod_{j=1}^{k} dx_{i,j}$. In the following x' denotes the transpose of x, 0 (sometimes with subscripts) denotes zero entries; I_k is the identity $k \times k$ matrix; $e_1, \dots, e_n$ are the canonical coordinate unit vectors in $\mathbb{R}^n$.

Let S_k be the space of $k \times k$ real symmetric matrices $r = (r_{i,j})$, $r_{i,j} = r_{j,i}$. A matrix $r \in S_k$ is called positive definite (positive semidefinite) if $a'ra > 0$ ($a'ra \geq 0$) for all vectors $a \neq 0$ in $\mathbb{R}^k$; this is commonly expressed as $r > 0$ ($r \geq 0$). Given $r_1, r_2 \in S_k$, the inequality $r_1 > r_2$ means $r_1 - r_2 \in \mathcal{P}_k$. The following facts are well known; see, e.g., [M], [T]:

(i) If $r > 0$ then $r^{-1} > 0$.

(ii) For any matrix $x \in \mathfrak{M}_{n,k}, x'x \geq 0$.

(iii) If $r \geq 0$ then r is nonsingular if and only if $r > 0$.

(iv) If $r > 0,\ s > 0,\ r - s > 0$ then $s^{-1} - r^{-1} > 0$ and $\det(r) > \det(s)$.

(v) A symmetric matrix is positive definite (positive semidefinite) if and only if all its eigenvalues are positive (nonnegative).

(vi) If $r \in S_k$ then there exists an orthogonal matrix $\gamma \in \mathrm{O}(k)$ such that $\gamma' r \gamma = \mathrm{diag}(\lambda_1, \dots, \lambda_k)$ where each λ_j is real and equal to the j^{th} eigenvalue of r.

(vii) If r is a positive semidefinite $k \times k$ matrix then there exists a positive semidefinite $k \times k$ matrix, written as $r^{1/2}$, such that $r = r^{1/2} r^{1/2}$.

We hope that, with these properties in mind, the reader will find more transparent the numerous calculations with functions of a matrix variables that occur throughout the paper.

The set S_k of symmetric $k \times k$ matrices is a vector space of dimension $k(k+1)/2$ and is a measure space isomorphic to $\mathbb{R}^{k(k+1)/2}$ with the volume element $dr = \prod_{i \leq j} dr_{i,j}$. For $r \geq 0$ we shall use the notation $|r| = \det(r)$. Given positive semidefinite matrices r and R in S_k, the symbol $\int_r^R f(s) ds$ denotes integration over the set

$$\{s : s \in \mathcal{P}_k,\ r < s < R\}.$$

For $\Omega \subset \mathcal{P}_k$, the function space $L^p(\Omega)$ is defined in the usual way with respect to the measure dr. The set $\mathcal{P}_k$ is a convex cone in S_k. It is a symmetric space of the group $\mathrm{GL}(k, \mathbb{R})$ of non-singular $k \times k$ real matrices. The action of $g \in \mathrm{GL}(k, \mathbb{R})$ on $r \in \mathcal{P}_k$ is given by $r \to g' r g$. This action is transitive (but not simply transitive). The relevant invariant measure on $\mathcal{P}_k$ has the form

$$d\mu(r) = |r|^{-d} \prod_{1 \leq i \leq j \leq k} dr_{i,j}, \qquad d = (k+1)/2, \tag{2.1}$$

[T, p. 18]. Let T_k be the group of upper triangular matrices t of the form

$$t = \begin{bmatrix} t_1 & & & \\ & \cdot & t_{i,j} & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & t_k \end{bmatrix}, \qquad t_i > 0, \quad t_{i,j} \in \mathbb{R}. \tag{2.2}$$

Each $r \in \mathcal{P}_k$ has a unique representation $r = t't,\ t \in T_k$, so that

$$\int_{\mathcal{P}_k} f(r)dr = \int_0^\infty t_1^k dt_1 \int_0^\infty t_2^{k-1} dt_2 \dots \int_0^\infty t_k \tilde{f}(t_1, \dots, t_k)\, dt_k, \tag{2.3}$$

$$\tilde{f}(t_1, \dots, t_k) = 2^k \int_{-\infty}^\infty \dots \int_{-\infty}^\infty f(t't) \prod_{i<j} dt_{i,j}$$

[T, p. 22], [M, p. 592]. In this last integration the diagonal entries of the matrix t are given by the arguments of $\tilde{f}$, and the strictly upper triangular entries of t are variables of integration.

To the cone $\mathcal{P}_k$ one can associate the Siegel gamma function

$$\Gamma_k(\alpha) = \int_{\mathcal{P}_k} e^{-\mathrm{tr}(r)} |r|^{\alpha - d} dr, \qquad \mathrm{tr}(r) = \text{trace of } r. \tag{2.4}$$

By (2.3), it is easy to check [M, p. 62] that this integral converges absolutely for $\mathrm{Re}\,\alpha > d-1$, and represents the product of the usual Γ-functions:

$$\Gamma_k(\alpha) = \pi^{k(k-1)/4} \Gamma(\alpha)\Gamma(\alpha - \frac{1}{2}) \dots \Gamma(\alpha - \frac{k-1}{2}). \tag{2.5}$$

For the corresponding Beta function we have [Herz, p. 480]

$$\int_0^R |r|^{\alpha-d} |R - r|^{\beta - d} dr = B_k(\alpha, \beta) |R|^{\alpha+\beta-d}, \tag{2.6}$$

$$B_k(\alpha, \beta) = \frac{\Gamma_k(\alpha)\, \Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}; \qquad \mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > d-1; \qquad R \in \mathcal{P}_k.$$

2.2. *Gårding-Gindikin fractional integrals.* Let

$$Q = \{r \in \mathcal{P}_k : 0 < r < I_k\}$$

be the "unit interval" in $\mathcal{P}_k$. Let f be a function in $L^1(Q)$. The Gårding-Gindikin fractional integrals of f of order α are defined by

$$(I_+^\alpha f)(r) = \frac{1}{\Gamma_k(\alpha)} \int_0^r f(s) |r - s|^{\alpha - d} ds, \tag{2.7}$$

$$(I_-^\alpha f)(r) = \frac{1}{\Gamma_k(\alpha)} \int_r^{I_k} f(s)|s-r|^{\alpha-d} ds, \tag{2.8}$$

where $r \in Q$, $d = (k+1)/2$, $\mathrm{Re}\,\alpha > d-1$. Both integrals are finite for almost all $r \in Q$. To see this it suffices to show that the integrals $\int_0^{I_k} (I_\pm^\alpha f)(r)dr$ are finite for any nonnegative $f \in L^1(Q)$. By changing the order of integration, and evaluating inner integrals according to (2.6), we get

$$\int_0^{I_k} (I_+^\alpha f)(r)dr = c \int_0^{I_k} f(s)|I_k - s|^\alpha ds, \qquad \int_0^{I_k} (I_-^\alpha f)(r)dr = c \int_0^{I_k} f(s)|s|^\alpha ds,$$

$c = \Gamma_k(d)/\Gamma_k(\alpha+d)$. Since the right-hand sides of these equalities are majorized by const $||f||_{L^1(Q)}$, the statement follows.

The equality (2.6) also implies the semigroup property

$$I_\pm^\alpha I_\pm^\beta f = I_\pm^{\alpha+\beta} f, \qquad f \in L^1(Q), \qquad \mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > d-1. \tag{2.9}$$

For $s = (s_{i,j}) \in \mathcal{P}_k$, we define the following differential operators in the s-variable:

$$D_+ = \det\left(\eta_{i,j} \frac{\partial}{\partial s_{i,j}}\right), \quad \eta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 1/2 & \text{if } i \neq j, \end{cases} \qquad D_- = (-1)^{k^2} D_+. \tag{2.10}$$

If f is sufficiently good, then

$$D_\pm^m I_\pm^\alpha f = I_\pm^{\alpha-m} f, \qquad m \in \mathbb{N}, \quad \mathrm{Re}\,\alpha > m+d-1, \tag{2.11}$$

(see, e.g., [Gå]). Let $\mathcal{D}(Q)$ be the space of infinitely differentiable functions supported in Q. For $w \in \mathcal{D}(Q)$, the integrals $I_\pm^\alpha w$ can be extended to all $\alpha \in \mathbb{C}$ as entire functions of α, so that $I_\pm^0 w = w$, $I_\pm^\alpha I_\pm^\beta w = I_\pm^{\alpha+\beta} w$ and $D_\pm^m I_\pm^\alpha w = I_\pm^\alpha D_\pm^m w = I_\pm^{\alpha-m} w$ for all $\alpha, \beta \in \mathbb{C}$ and all $m \in \mathbb{N}$ [Gi]. This enables us to define $I_\pm^\alpha f$ for $f \in L^1(Q)$ and $\mathrm{Re}\,\alpha \le d-1$ in the sense of distributions by setting

$$(I_\pm^\alpha f, w) = \int_Q (I_\pm^\alpha f)(r)\overline{w(r)}dr = (f, \overline{I_\mp^\alpha \overline{w}}), \quad w \in \mathcal{D}(Q).$$

Note that explicit construction of the analytic continuation of $I_\pm^\alpha w$ is rather complicated if w does not vanish identically on the boundary of Q (cf. [Gå], [R1], [R2]). In order to invert $\varphi = I_+^\alpha f$ for $f \in L^1(Q)$ and $\mathrm{Re}\,\alpha > d-1$ in the sense of distributions, let $m \in \mathbb{N}$, $m - \mathrm{Re}\,\alpha > d-1$. By (2.9), $I_+^m f = I_+^{m-\alpha}\varphi$, and therefore

$$(f, w) \equiv \int_Q f(r)\overline{w(r)}dr = (D_+^m I_+^{m-\alpha}\varphi, w) \equiv (\varphi, I_-^{m-\alpha} D_-^m w).$$

2.3. *Stiefel manifolds.* Let $V_{n,k} = \{x \in \mathfrak{M}_{n,k} : x'x = I_k\}$ be the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$, $n \ge k$. For $n = k$, $V_{n,n} = \mathrm{O}(n)$

represents the orthogonal group in $\mathbb{R}^n$. The Stiefel manifold is a homogeneous space with respect to the action $V_{n,k} \ni x \to \gamma x \in V_{n,k}$, $\gamma \in \mathrm{O}(n)$, so that $V_{n,k} = \mathrm{O}(n)/\mathrm{O}(n-k)$. The group $\mathrm{O}(n)$ acts on $V_{n,k}$ transitively. The same is true for the group $\mathrm{SO}(n) = \{\gamma \in \mathrm{O}(n) : \det(\gamma) = 1\}$ provided $n > k$. It is known that $\dim V_{n,k} = k(2n-k-1)/2$. We fix invariant measures dx on $V_{n,k}$ and $d\gamma$ on $\mathrm{SO}(n)$ normalized by

$$\sigma_{n,k} \equiv \int\limits_{V_{n,k}} dx = \frac{2^k \pi^{nk/2}}{\Gamma_k(n/2)} \tag{2.12}$$

and $\int_{\mathrm{SO}(n)} d\gamma = 1$ [M, p. 70], [J, p. 57].

LEMMA 2.1 (polar decomposition). *Almost all $x \in \mathfrak{M}_{n,k}$, $n \geq k$ (specifically, all matrices $x \in \mathfrak{M}_{n,k}$ of rank k), can be decomposed as*

$$x = vr^{1/2}, \quad v \in V_{n,k}, \quad r = x'x \in \mathcal{P}_k \quad \textit{so that} \quad dx = 2^{-k}|r|^{(n-k-1)/2} dr dv.$$

This statement can be found in [Herz, p. 482], [GK, p. 93], [M, pp. 66, 591].

LEMMA 2.2 (bi-Stiefel decomposition). *Let k and ℓ be arbitrary integers satisfying $1 \leq k \leq \ell \leq n-1$, $k+\ell \leq n$. Almost all $x \in V_{n,k}$ can be represented in the form*

$$x = \begin{bmatrix} ur^{1/2} \\ v(I_k - r)^{1/2} \end{bmatrix}, \qquad u \in V_{\ell,k}, \quad v \in V_{n-\ell,k}, \quad r \in \mathcal{P}_k, \tag{2.13}$$

so that

$$\int\limits_{V_{n,k}} f(x)dx = \int\limits_0^{I_k} d\nu(r) \int\limits_{V_{\ell,k}} du \int\limits_{V_{n-\ell,k}} f\left(\begin{bmatrix} ur^{1/2} \\ v(I_k - r)^{1/2} \end{bmatrix}\right) dv, \tag{2.14}$$

$$d\nu(r) = 2^{-k}|r|^{\gamma}|I_k - r|^{\delta} dr, \qquad \gamma = \frac{\ell - k - 1}{2}, \quad \delta = \frac{n - \ell - k - 1}{2}. \tag{2.15}$$

Proof. For $k = 1$, this statement is well known and represents bispherical decomposition on the unit sphere; cf. [VK, pp. 12, 22]. For the general case related to Stiefel manifolds the proof is essentially the same as that of the slightly less general Lemma 3.7 from [Herz, p. 495]. For convenience of the reader we sketch this proof.

Let us check (2.13). If $x = \begin{bmatrix} a \\ b \end{bmatrix} \in V_{n,k}$, $a \in \mathfrak{M}_{\ell,k}$, $b \in \mathfrak{M}_{n-\ell,k}$, then $I_k = x'x = a'a + b'b$. By Lemma 2.1 for almost all a we have $a = ur^{1/2}$. Hence $b'b = I_k - r$, and therefore $b = v(I_k - r)^{1/2}$. This gives (2.13). The explicit meaning of "almost all" in Lemma 2.2 becomes clear from Lemma 2.1 having been applied to the matrices a and b.

In order to prove (2.14) we write it in the form

$$\int\limits_{V_{n,k}} f(x)dx = \int\limits_{0<a'a<I_k} |I_k - a'a|^{\delta} da \int\limits_{V_{n-\ell,k}} f\left(\begin{bmatrix} a \\ v(I_k - a'a)^{1/2} \end{bmatrix}\right) dv \tag{2.16}$$

and show the coincidence of the two measures, dx and $\tilde{d}x = |I_k - a'a|^{\delta} da dv$. Following [Herz], we consider the Fourier transforms

$$F_1(s) = \int\limits_{V_{n,k}} \mathrm{etr}(is'x)dx \quad \text{and} \quad F_2(s) = \int\limits_{V_{n,k}} \mathrm{etr}(is'x)\tilde{d}x,$$

where $s \in \mathfrak{M}_{n,k}$, $\mathrm{etr}(\Lambda) = e^{\mathrm{tr}(\Lambda)}$, and show that $F_1 = F_2$. To this end we employ the Bessel functions $A_\lambda(r)$ of Herz for which

$$\int\limits_{V_{n,k}} \mathrm{etr}(is'x)dx = 2^k \pi^{nk/2} A_{(n-k-1)/2}\left(\frac{1}{4}s's\right). \tag{2.17}$$

Let

$$s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \quad x = \begin{bmatrix} a \\ v(I_k - a'a)^{1/2} \end{bmatrix}; \quad s_1 \in \mathfrak{M}_{\ell,k},\ s_2 \in \mathfrak{M}_{n-\ell,k};\ a \in \mathfrak{M}_{\ell,k}.$$

Then $s'x = s_1'a + s_2'v(I_k - a'a)^{1/2}$, and we have

$$F_2(s) = \int\limits_{a'a<I_k} \mathrm{etr}(is_1'a)|I_k - a'a|^{\delta} da \int\limits_{V_{n-\ell,k}} \mathrm{etr}(is_2'v(I_k - a'a)^{1/2})dv.$$

By (2.17) (use the equality $\mathrm{tr}(is_2'vR) = \mathrm{tr}(iR^{-1}Rs_2'vR) = \mathrm{tr}(iRs_2'v)$ with $R = (I_k - a'a)^{1/2}$) the inner integral is evaluated as

$$2^k \pi^{(n-\ell)k/2} A_\delta\left(\frac{1}{4}Rs_2's_2R\right) = 2^k \pi^{(n-\ell)k/2} A_\delta\left(\frac{1}{4}s_2's_2R^2\right)$$

(the last equality holds because of the invariance property $A_\delta(R^{-1}rR) = A_\delta(r)$). Thus

$$F_2(s) = \int\limits_{a'a<I_k} \mathrm{etr}(is_1'a)\varphi(a'a)da,$$

$$\varphi(r) = 2^k \pi^{(n-\ell)k/2} |I_k - a'a|^{\delta} A_\delta\left(\frac{1}{4}s_2's_2(I_k - r)\right).$$

The function $F_2(s)$ can be transformed by the generalized Bochner formula

$$\int\limits_{\mathfrak{M}_{\ell,k}} \mathrm{etr}(iy'a)\varphi(a'a)da = \pi^{\ell k/2} g\left(\frac{1}{4}y'y\right),$$

$$g(\Lambda) = \int\limits_{\mathcal{P}_k} A_\gamma(\Lambda r)|r|^{\gamma}\varphi(r)dr, \qquad \gamma = \frac{\ell - k - 1}{2}, \quad y \in \mathfrak{M}_{\ell,k}$$

[Herz, p. 493], that yields

$$F_2(s) = 2^k \pi^{nk/2} \int\limits_0^{I_k} A_\gamma\left(\frac{1}{4}s_1' s_1 r\right) |r|^\gamma A_\delta\left(\frac{1}{4}s_2' s_2 (I_k - r)\right) |I_k - r|^\delta dr,$$

γ, δ being defined by (2.15). This integral can be evaluated using the formula (2.6) from [Herz, p. 487]. The result is

$$F_2(s) = 2^k \pi^{nk/2} A_{(n-k-1)/2}\left(\frac{1}{4}(s_1' s_1 + s_2' s_2)\right) = 2^k \pi^{nk/2} A_{(n-k-1)/2}\left(\frac{1}{4}s' s\right).$$

By (2.17), the latter coincides with $F_1(s)$. □

Remark 2.3. The assumptions $k + \ell \le n$ and $k \le \ell$ in Lemma 2.2 are necessary for absolute convergence of the integral $\int_0^{I_k}$ in the right-hand side of (2.14). It would be interesting to prove this lemma directly, without using the Fourier transform. Such a proof would be helpful in transferring Lemma 2.2 and many other results of the paper to the hyperbolic space (cf. [VK, pp. 12, 23], [BR], [Ru5] for the rank-one case).

LEMMA 2.4. *Let $x \in V_{n,k}$, $y \in V_{n,\ell}$; $1 \le k, \ell \le n$. If f is a function of $\ell \times k$ matrices then*

$$\frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(y'x) dx = \frac{1}{\sigma_{n,\ell}} \int\limits_{V_{n,\ell}} f(y'x) dy. \tag{2.18}$$

Proof. We should observe that formally the left-hand side is a function of y, while the right-hand side is a function of x. In fact, both are constant. To prove (2.18) let $G = \mathrm{SO}(n)$, $g \in G$, $g_1 = g^{-1}$. The left-hand side is

$$\int\limits_G f(y' g x) dg = \int\limits_G f((g_1 y)' x) dg_1$$

which equals the right-hand side. □

We shall need a "lower-dimensional" representation of integrals of the form

$$I_f = \int\limits_{V_{n,k}} f(A'x) dx, \qquad A \in \mathfrak{M}_{n,\ell}; \quad 0 < k < n, \ 0 < \ell < n. \tag{2.19}$$

For $k = \ell = 1$ such a representation is well known. In the following lemma we do not specify assumptions for the function f. For our purposes it suffices to assume only that the integral (2.19) is absolutely convergent and therefore well defined for all or almost all A. This enables us to give a proof which consists, in fact, of a number of applications of the Fubini theorem. Furthermore, for our purposes it suffices to consider matrices A for which $A'A$ is positive definite.

It means that we exclude those matrices for which the point $R = A'A$ lies on the boundary of the cone $\mathcal{P}_\ell$.

LEMMA 2.5. *For* $A \in \mathfrak{M}_{n,\ell}$, *let* $R = A'A \in \mathcal{P}_\ell$, $k + \ell \le n$, $\gamma = (|k-\ell|-1)/2$, $\delta = (n-k-\ell-1)/2$. *If* $k \le \ell$, *then*

$$I_f = \frac{\sigma_{n-\ell,k}}{2^k} \int\limits_0^{I_k} |r|^\gamma |I_k - r|^\delta dr \int\limits_{V_{\ell,k}} f(R^{1/2} u r^{1/2}) du. \tag{2.20}$$

If $k \ge \ell$, $c = 2^{-\ell} \sigma_{n,k} \sigma_{n-k,\ell}/\sigma_{n,\ell}$, *then*

$$I_f = c \int\limits_0^{I_\ell} |r|^\gamma |I_\ell - r|^\delta dr \int\limits_{V_{k,\ell}} f(R^{1/2} r^{1/2} u') du \tag{2.21}$$

$$= c|R|^{-\delta-k/2} \int\limits_0^{R} |r|^\gamma |R - r|^\delta dr \int\limits_{V_{k,\ell}} f(r^{1/2} u') du. \tag{2.22}$$

Proof. By Lemma 2.1, $A = vR^{1/2}$, $v \in V_{n,\ell}$. Since the group SO(n) acts transitively on $V_{n,k}$ we can set

$$v = g\omega_\ell, \qquad g \in \mathrm{SO}(n), \qquad \omega_\ell = \begin{bmatrix} I_\ell \\ 0 \end{bmatrix},$$

and obtain

$$I_f = \int\limits_{V_{n,k}} f(R^{1/2} v' x) dx = \int\limits_{V_{n,k}} f(R^{1/2} \omega_\ell' g' x) dx = \int\limits_{V_{n,k}} f(R^{1/2} \omega_\ell' x) dx$$

(we have changed variables $g'x \to x$). Now (2.20) follows by Lemma 2.2. If $k \ge \ell$, then (2.18) yields

$$I_f = \int\limits_{V_{n,k}} f(R^{1/2} v' x) dx = \frac{\sigma_{n,k}}{\sigma_{n,\ell}} \int\limits_{V_{n,\ell}} f(R^{1/2} v' x) dv.$$

Now we replace x by $\gamma\omega_k$, $\gamma \in \mathrm{SO}(n)$, $\omega_k = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$, and change the variable $v \to \gamma v$. This gives

$$I_f = \frac{\sigma_{n,k}}{\sigma_{n,\ell}} \int\limits_{V_{n,\ell}} f(R^{1/2} v' \gamma \omega_k) dv$$

$$= \frac{\sigma_{n,k}}{\sigma_{n,\ell}} \int\limits_{V_{n,\ell}} f(R^{1/2} v' \omega_k) dv = \frac{\sigma_{n,k}}{\sigma_{n,\ell}} \int\limits_{V_{n,\ell}} f(R^{1/2} (\omega_k' v)') dv.$$

We apply Lemma 2.2 again, but with k and ℓ interchanged. This gives (2.21). The proof of (2.22) is as follows.

$$\begin{aligned}
I_f &= \frac{\sigma_{n,k}}{\sigma_{n,\ell}} \int\limits_{V_{n,\ell}} f(R^{1/2}(\omega'_k v)')dv \\
&\overset{(2.16)}{=} \frac{\sigma_{n,k}}{\sigma_{n,\ell}} \int\limits_{o<a'a<I_\ell} |I_\ell - a'a|^\delta da \int\limits_{V_{n-k,\ell}} f\left(R^{1/2}\left(\omega'_k \begin{bmatrix} a \\ v(I_\ell - a'a)^{1/2} \end{bmatrix}\right)'\right) dv \\
&= \frac{\sigma_{n,k}\, \sigma_{n-k,\ell}}{\sigma_{n,\ell}} \int\limits_{o<a'a<I_\ell} |I_\ell - a'a|^\delta f(R^{1/2}a')da \\
&\quad (\text{set } s = aR^{1/2} \in \mathfrak{M}_{k,\ell} \text{ so that } ds = |R|^{k/2} da \quad [\text{M, p. 58}]) \\
&= \frac{\sigma_{n,k}\, \sigma_{n-k,\ell}}{\sigma_{n,\ell}\, |R|^{\delta+k/2}} \int\limits_{o<s's<R} |R - s's|^\delta f(s')ds.
\end{aligned}$$

It remains to apply Lemma 2.1. □

2.4. *The Grassmann manifolds.* Analysis on the Stiefel manifold $V_{n,k}$ is intimately connected with that on the Grassmannian $G_{n,k} = V_{n,k}/\mathrm{O}(k)$. Given $x \in V_{n,k}$, we denote by $\{x\}$ the subspace spanned by the columns of x. Note that $\eta = \{x\} \in G_{n,k}$. A function $f(x)$ on $V_{n,k}$ is $\mathrm{O}(k)$ right-invariant, i.e., $f(x\gamma) = f(x)\ \forall \gamma \in \mathrm{O}(k)$, if and only if there is a function $F(\eta)$ on $G_{n,k}$ so that $f(x) = F(\{x\})$. We endow $G_{n,k}$ with the normalized $\mathrm{O}(n)$ left-invariant measure $d\eta$ so that

$$\frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(x)dx = \int\limits_{G_{n,k}} F(\eta)d\eta. \tag{2.23}$$

For the sake of convenience, we shall identify $\mathrm{O}(k)$ right-invariant functions $f(x)$ on $V_{n,k}$ with the corresponding functions $F(\eta)$ on $G_{n,k}$, and use for both the same letter f. In the case of possible confusion, additional explanation will be given.

2.5. *Invariant functions.*

Definition 2.6. Let $\rho \in \mathrm{O}(n-\ell)$, $g_\rho = \begin{bmatrix} \rho & 0 \\ 0 & I_\ell \end{bmatrix} \in \mathrm{O}(n)$. A function $f(x)$ on $V_{n,k}$ $(F(\eta)$ on $G_{n,k})$ is called *ℓ-zonal* if $f(g_\rho x) = f(x)$ $(F(g_\rho \eta) = F(\eta))$ for all $\rho \in \mathrm{O}(n-\ell)$.

Lemma 2.7. *For $k+\ell \le n$ the following statements hold.*

(a) *A function $f(x)$ on $V_{n,k}$ is ℓ-zonal if and only if there is a function f_1 on $\mathfrak{M}_{\ell,k}$ such that $f(x) \overset{\text{a.e.}}{=} f_1(\sigma'_\ell x)$, $\quad \sigma_\ell = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} \in V_{n,\ell}$.*

(b) *Let $k \ge \ell$. A function $f(x)$ on $V_{n,k}$ is ℓ-zonal and $\mathrm{O}(k)$ right-invariant (simultaneously) if and only if there is a function f_0 on $\mathcal{P}_\ell$ such that $f(x) \overset{\text{a.e.}}{=} f_0(s)$, $s = \sigma'_\ell x x' \sigma_\ell = \sigma'_\ell \mathrm{Pr}_{\{x\}} \sigma_\ell$. Thus, $f_0(s) = f_1(s^{1/2} u'_0)$, $u'_0 = [0_{\ell\times(k-\ell)}, I_\ell]$, where f_1 is the function from* (a).

(c) *Let $k \ge \ell$. A function $F(\eta)$ on $G_{n,k}$ is ℓ-zonal if and only if there is a function F_0 (or $F_0^\perp$) on $\mathcal{P}_\ell$ such that $F(\eta) \overset{\text{a.e.}}{=} F_0(s)$, $s = \sigma'_\ell \mathrm{Pr}_\eta \sigma_\ell = \mathrm{Cos}^2(\eta, \sigma_\ell)$ (or $F(\eta) \overset{\text{a.e.}}{=} F_0^\perp(r)$, $r = \sigma'_\ell \mathrm{Pr}_{\eta^\perp} \sigma_\ell = \mathrm{Sin}^2(\eta, \sigma_\ell)$).*

Proof. (a) Let f be ℓ-zonal. We write $x = \begin{bmatrix} a \\ b \end{bmatrix}$, $\quad a \in \mathfrak{M}_{n-\ell,k}$, $b \in \mathfrak{M}_{\ell,k}$. Since $n-\ell \ge k$, Lemma 2.1 gives $a = v s^{1/2}$, $v \in V_{n-\ell,k}$, $s = a'a = I_k - b'b$. Thus for $\rho \in \mathrm{O}(n-\ell)$, we have $\rho a = \rho v s^{1/2}$. Let

$$r_v \in \mathrm{SO}(n-\ell) \quad \text{so that} \quad r_v : v_0 = \begin{bmatrix} I_k \\ 0_{(n-k-\ell)\times k} \end{bmatrix} \to v.$$

We set $\rho = r_v^{-1}$. Then

$$f(x) = f\left(\begin{bmatrix} v_0 s^{1/2} \\ b \end{bmatrix}\right) = f\left(\begin{bmatrix} v_0 (I_k - b'b)^{1/2} \\ b \end{bmatrix}\right) = f_1(b) = f_1(\sigma'_\ell x).$$

The converse statement in (a) is obvious.

(b) By (a), $f(x) = f_1(\sigma'_\ell x) = f_1((x'\sigma_\ell)')$, and Lemma 2.1 yields $x'\sigma_\ell = u s^{1/2}$, $u \in V_{k,\ell}$, $s = \sigma'_\ell x x' \sigma_\ell$. Let $u_0 = \begin{bmatrix} 0_{(k-\ell)\times \ell} \\ I_\ell \end{bmatrix}$, $\quad r_u \in \mathrm{O}(k)$, so that $r_u u_0 = u$. Since f is $\mathrm{O}(k)$ right-invariant, then

$$\begin{aligned} f(x) &= f(x r_u) = f_1(\sigma'_\ell x r_u) = f_1((r'_u x' \sigma_\ell)') = f_1((r'_u u s^{1/2})') \\ &= f_1((u_0 s^{1/2})') = f_1(s^{1/2} u'_0) = f_0(s). \end{aligned}$$

The converse statement is obvious.

(c) For $\eta \in G_{n,k}$, let $x \in V_{n,k}$ and $y \in V_{n,n-k}$ be orthonormal bases of η and $\eta^\perp$, respectively, i.e. $\eta = \{x\} = \{y\}^\perp$. The functions $\psi(x) = F(\{x\})$ and $\psi^\perp(y) = F(\{y\}^\perp)$ are ℓ-zonal. Moreover, ψ is $\mathrm{O}(k)$ right-invariant, and $\psi^\perp$ is $\mathrm{O}(n-k)$ right-invariant. Hence the result follows from (b). $\square$

Lemmas 2.7, 2.5, and the equality (2.12) imply the following

LEMMA 2.8. *Let* $1 \le k \le n-1, \quad 1 \le \ell \le \min(k, n-k)$,

$$d\mu(s) = |s|^\gamma |I_\ell - s|^\delta ds, \quad \gamma = (k-\ell-1)/2, \quad \delta = (n-k-\ell-1)/2, \tag{2.24}$$

$$c = \Gamma_\ell(n/2)/\Gamma_\ell(k/2)\Gamma_\ell((n-k)/2).$$

If $f(x) \in L^1(V_{n,k})$ *is* ℓ*-zonal and* $\mathrm{O}(k)$ *right-invariant, then there is a function* $f_0(s)$ *on* $\mathcal{P}_\ell$ *so that for almost all* x, $f(x) = f_0(s)$, $s = \sigma'_\ell x x' \sigma_\ell$, *and*

$$\frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(x) dx = c \int\limits_0^{I_\ell} f_0(s) d\mu(s). \tag{2.25}$$

This lemma implies the following corollary for functions on the Grassmannian.

COROLLARY 2.9. *If* $1 \le k \le n-1, \quad 1 \le \ell \le \min(k, n-k)$, *and* $f(\eta) \in L^1(G_{n,k})$ *is* ℓ*-zonal, then there is a function* $f_0(s)$ *on* $\mathcal{P}_\ell$ *so that for almost all* η,

$$f(\eta) = f_0(s), \quad s = \sigma'_\ell \mathrm{Pr}_\eta \sigma_\ell = \mathrm{Cos}^2(\eta, \sigma_\ell),$$

and

$$\int\limits_{G_{n,k}} f(\eta) d\eta = c \int\limits_0^{I_\ell} f_0(s) d\mu(s), \tag{2.26}$$

with $d\mu(s)$ *and* c *the same as in Lemma* 2.8.

This corollary proves part (i) of Theorem 1.4.

3. Mean value operators

Suppose that $1 \le k \le k' \le n-1, \ k + k' \le n$. We recall the notation

$$x_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix}, \qquad \eta_0 = \{x_0\} = \mathbb{R}e_{n-k+1} + \ldots + \mathbb{R}e_n, \qquad \xi_0 = \mathbb{R}e_1 + \ldots + \mathbb{R}e_{k'},$$

and set $G = \mathrm{SO}(n)$,

$$K = \left\{ \rho \in G : \rho = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad \alpha \in \mathrm{SO}(n-k), \ \beta \in \mathrm{SO}(k) \right\},$$

$$K' = \left\{ \tau \in G : \tau = \begin{bmatrix} \gamma & 0 \\ 0 & \delta \end{bmatrix}, \quad \gamma \in \mathrm{SO}(k'), \ \delta \in \mathrm{SO}(n-k') \right\},$$

so that K and K' are isotropy subgroups of the coordinate planes η_0 and ξ_0, respectively. The corresponding normalized invariant measures will be denoted by $d\rho$ and $d\tau$. Let

$$[0, I_k] = \{r \in \mathcal{P}_k : 0 < r < I_k\} \cup \{0\} \cup \{I_k\}.$$

Given $r \in [0, I_k]$, we set

$$x_r = \begin{bmatrix} 0_{(k'-k)\times k} \\ r^{1/2} \\ 0_{(n-k'-k)\times k} \\ (I_k - r)^{1/2} \end{bmatrix} \in V_{n,k}, \qquad \eta_r = \{x_r\} \in G_{n,k},$$

where 0 (with subscripts) denotes zero entries,

$$g_r = \begin{bmatrix} I_{k'-k} & 0 & 0 & 0 \\ 0 & (I_k - r)^{1/2} & 0 & r^{1/2} \\ 0 & 0 & I_{n-k'-k} & 0 \\ 0 & -r^{1/2} & 0 & (I_k - r)^{1/2} \end{bmatrix}.$$

It is easy to check that g_r represents a linear transformation preserving coordinate unit vectors $e_{k'+1}, \dots, e_{n-k}$ so that $g_r x_0 = x_r$, $g_r \eta_0 = \eta_r$. Moreover, $g_r' g_r = I_n$, which means that $g_r \in \mathrm{O}(n)$. The proof of the equality $g_r' g_r = I_n$ represents a routine multiplication of matrices, and we skip it. For the reader's convenience we only note that, when doing calculations, one should take into account that matrices $r^{1/2}$ and $(I_k - r)^{1/2}$ commute because they are diagonalizable by the same orthogonal transformation; see the proof of Theorem A9.3 in [M, p. 588]. To motivate the fact that the matrices $r^{1/2}$ and $(I_k - r)^{1/2}$ commute, we can also say that, at least for $r < I_k$, both matrices are power series in the matrix variable r, i.e., limits of polynomials; hence they commute.

Given $x \in V_{n,k}$, $\eta \in G_{n,k}$, $\xi \in G_{n,k'}$, let g_x, g_η, and $g_\xi \in G$ be arbitrary rotations satisfying $g_x x_0 = x$, $g_\eta \eta_0 = \eta$, $g_\xi \xi_0 = \xi$. For $f : V_{n,k} \to \mathbb{C}$ and $\varphi : G_{n,k'} \to \mathbb{C}$, we set $f_\xi(x) = f(g_\xi x)$, $\varphi_x(\xi) = \varphi(g_x \xi)$, $\varphi_\eta(\xi) = \varphi(g_\eta \xi)$. If f is a function on $G_{n,k}$ we denote $f_\xi(\eta) = f(g_\xi \eta)$.

For functions f on $V_{n,k}$ and φ on $G_{n,k'}$, we introduce the following mean value operators with the averaging parameter $r \in [0, I_k]$:

$$(M_r f)(\xi) = \int_{K'} f_\xi(\tau x_r) d\tau, \qquad (M_r^* \varphi)(x) = \int_K \varphi_x(\rho g_r^{-1} \xi_0) d\rho. \tag{3.1}$$

If f is a function on $G_{n,k}$ we set

$$(M_r f)(\xi) = \int_{K'} f_\xi(\tau g_r \eta_0) d\tau.$$

The mean value $M_r^*\varphi$ can be regarded as a function of $\eta \in G_{n,k}$. Up to abuse of notation we shall write

$$(M_r^*\varphi)(\eta) = \int\limits_K \varphi_\eta(\rho g_r^{-1}\xi_0)d\rho.$$

These mean value operators have a simple geometric interpretation. Namely, let $x = g_\xi \tau x_r \in V_{n,k}$. Multiplying matrices and making use of Definition 1.1, we get $\mathrm{Cos}^2(\xi, x) = r$. Similarly, if $x \in V_{n,k}$ and $\xi = g_x \rho g_r^{-1}\xi_0 \in G_{n,k'}$, then again $\mathrm{Cos}^2(\xi, x) = r$. Thus (3.1) can be written implicitly as
(3.2)

$$(M_r f)(\xi) = \int\limits_{\mathrm{Cos}^2(\xi,x)=r} f(x)dm_\xi(x), \qquad (M_r^*\varphi)(x) = \int\limits_{\mathrm{Cos}^2(\xi,x)=r} \varphi(\xi)dm_x(\xi).$$

LEMMA 3.1. *For $1 \le k \le k' \le n-1,\ k+k' \le n$,*

$$\int\limits_{G_{n,k'}} \varphi(\xi)(M_r f)(\xi)d\xi = \int\limits_{G_{n,k}} f(\eta)(M_r^*\varphi)(\eta)d\eta \tag{3.3}$$

provided that either of these integrals converges for f and φ replaced by $|f|$ and $|\varphi|$, respectively.

Proof. The left-hand side is

$$\begin{aligned}\int\limits_G \varphi(g\xi_0)(M_r f)(g\xi_0)dg &= \int\limits_{K'} d\tau \int\limits_G \varphi(g\xi_0)f(g\tau g_r\eta_0)dg\\ &= \int\limits_K d\rho \int\limits_{K'} d\tau \int\limits_G \varphi(g\xi_0)f(g\tau g_r\rho^{-1}\eta_0)dg\\ &= \int\limits_G f(\lambda\eta_0)d\lambda \int\limits_K \varphi(\lambda\rho g_r^{-1}\xi_0)d\rho\end{aligned}$$

as desired. □

LEMMA 3.2. *Suppose that $1 \le k \le k' \le n-1,\ k+k' \le n$, and let $d\nu(r)$ be the measure* (2.15) *with ℓ replaced by k', namely,*

$$d\nu(r) = 2^{-k}|r|^{(k'-k-1)/2}|I_k - r|^{(n-k'-k-1)/2}.$$

Then

$$\int\limits_{V_{n,k}} f(x)dx = c \int\limits_0^{I_k} (M_r f)(\xi) d\nu(r), \quad \forall \xi \in G_{n,k'}; \tag{3.4}$$

$$\int\limits_{G_{n,k'}} \varphi(\xi)d\xi = c \int\limits_0^{I_k} (M^*_r \varphi)(\eta) d\nu(r), \quad \forall \eta \in G_{n,k}, \tag{3.5}$$

$$c = \frac{\sigma_{k',k}\, \sigma_{n-k',k}}{\sigma_{n,k}}.$$

Proof. Replace in (2.14) ℓ by k', f by f_ξ, and set

$$u = \gamma \begin{bmatrix} 0_{(k'-k)\times k} \\ I_k \end{bmatrix}, \qquad v = \beta \begin{bmatrix} 0_{(n-k'-k)\times k} \\ I_k \end{bmatrix},$$

$\gamma \in \mathrm{SO}(k')$, $\delta \in \mathrm{SO}(n-k')$. Integration against $d\gamma d\delta$ (instead of $dudv$) gives (3.4). Let us prove (3.5). Denote the left-hand side of (3.5) by I and write it as $I = \int_G \tilde\varphi(g)dg$ where $\tilde\varphi(g) = \int_K \varphi_\eta(\rho g^{-1}\xi_0)d\rho$. Since $\tilde\varphi$ is K right-invariant, one can identify it with a certain function ψ on $G_{n,k} = G/K$ so that $\tilde\varphi(g) = \psi(g\eta_0)$. By (3.4),

$$I = \int\limits_{G_{n,k}} \psi(\eta)d\eta = \frac{\sigma_{k',k}\, \sigma_{n-k',k}}{\sigma_{n,k}} \int\limits_0^{I_k} d\nu(r) \int\limits_{K'} \psi(\tau g_r \eta_0) d\tau$$

where the inner integral reads

$$\int\limits_{K'} \tilde\varphi(\tau g_r)d\tau = \int\limits_{K'} d\tau \int\limits_K \varphi_\eta(\rho g_r^{-1} \tau^{-1} \xi_0) d\rho = \int\limits_K \varphi_\eta(\rho g_r^{-1} \xi_0) d\rho.$$

Thus we are done. □

Let us introduce another mean value operator on $V_{n,k}$ which serves as an approximate identity on $V_{n,k}$ (or on $G_{n,k}$), and which can be regarded as an analog of the spherical mean on S^{n-1}. For $x, y \in V_{n,k}$, we denote $f_x(y) = f(g_x y)$, where $g_x \in G$ satisfies $g_x x_0 = x$, $\quad x_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$. Assuming $2k \le n$, given a $k \times k$ matrix a such that $a'a \in [0, I_k]$, we set

$$(\mathcal{M}_a f)(x) = \frac{1}{\sigma_{n-k,k}} \int\limits_{V_{n-k,k}} f_x\left(\begin{bmatrix} u(I_k - a'a)^{1/2} \\ a \end{bmatrix}\right) du. \tag{3.6}$$

This can be written as $\int_{x'y=a} f(y)d\sigma_a(y)$ where $d\sigma_a(y)$ denotes the corresponding normalized measure on the "section" $\{y \in V_{n,k} : x'y = a\}$.

LEMMA 3.3. (i) *For* $x, z \in V_{n,k}$,

$$\int\limits_{\mathrm{SO}(n-k)} f_x(\alpha z) d\alpha = (\mathcal{M}_{x_0' z} f)(x). \tag{3.7}$$

(ii) *Let* $f^\gamma(x) = f(x\gamma), \quad \gamma \in \mathrm{O}(k)$. *Then*

$$\mathcal{M}_{a\gamma} f = \mathcal{M}_a f^\gamma. \tag{3.8}$$

(iii) *If* f *is* $\mathrm{O}(k)$ *right-invariant, then*

$$\mathcal{M}_{x_0' z} f = \mathcal{M}_r f, \qquad r^2 = x_0' z z' x_0 \in [0, I_k]. \tag{3.9}$$

Proof. (i) As in the proof of (2.13), we write

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \qquad z_2 = x_0' z, \quad z_1 = u(I_k - z_2' z_2)^{1/2}, \quad u \in V_{n-k,k}.$$

Then

$$\int\limits_{\mathrm{SO}(n-k)} f_x(\alpha z) d\alpha = \int\limits_{\mathrm{SO}(n-k)} f_x \left(\alpha \begin{bmatrix} u(I_k - z_2' z_2)^{1/2} \\ z_2 \end{bmatrix} \right) d\alpha$$

which gives (3.7).

(ii) We have

$$\int\limits_{V_{n-k,k}} f_x \left(\begin{bmatrix} u(I_k - a'a)^{1/2} \\ a \end{bmatrix} \gamma \right) du = \int\limits_{V_{n-k,k}} f_x \left(\begin{bmatrix} v(\gamma'(I_k - a'a)^{1/2}\gamma) \\ a\gamma \end{bmatrix} \right) dv, \tag{3.10}$$

$v = u\gamma$. Since $\gamma'(I_k - a'a)^{1/2}\gamma = (I_k - \gamma' a' a \gamma)^{1/2}$, (3.10) implies $\mathcal{M}_a f^\gamma = \mathcal{M}_{a\gamma} f$.

(iii) By Lemma 2.1, $x_0' z = (z' x_0)' = (vr)' = rv'$, $v \in \mathrm{O}(k)$, and (3.9) follows from (3.8). □

LEMMA 3.4. *Let* $f \in L^p(V_{n,k})$, $||\cdot||_p = ||\cdot||_{L^p(V_{n,k})}$, $2k \le n$.

(a) *If* $1 \le p \le \infty$, *then* $\sup_{0<a'a<I_k} ||\mathcal{M}_a f||_p \le ||f||_p$.

(b) *If* $1 \le p < \infty$, *then* $\lim\limits_{a \to I_k} ||\mathcal{M}_a f - f||_p = 0$.

(c) *If* $f \in C(V_{n,k})$ *and* $a \to I_k$, *then* $\mathcal{M}_a f \to f$ *uniformly on* $V_{n,k}$.

Proof. For $G = \mathrm{SO}(n)$, we have $||f||_p^p = \sigma_{n,k}||f(gz)||^p_{L^p(G)}, \forall z \in V_{n,k}$. Hence, by the generalized Minkowski inequality,

$$||\mathcal{M}_a f||_p = \Big(\sigma_{n,k} \int\limits_G |(\mathcal{M}_a f)(gx_0)|^p dg\Big)^{1/p}$$
$$= \frac{\sigma_{n,k}^{1/p}}{\sigma_{n-k,k}} \Big(\int\limits_G \Big| \int\limits_{V_{n-k,k}} f\left(g\left[\begin{array}{c} u(I_k - a'a)^{1/2} \\ a \end{array}\right]\right) du\Big|^p dg\Big)^{1/p}$$
$$\le \frac{\sigma_{n,k}^{1/p}}{\sigma_{n-k,k}} \int\limits_{V_{n-k,k}} \Big(\int\limits_G |f(g(...))|^p dg\Big)^{1/p} du = ||f||_p.$$

Let us prove (b). Denote $z_u = \left[\begin{array}{c} u(I_k - a'a)^{1/2} \\ a \end{array}\right]$. As above,

$$||\mathcal{M}_a f - f||_p \le \frac{\sigma_{n,k}^{1/p}}{\sigma_{n-k,k}} \int\limits_{V_{n-k,k}} ||f(gz_u) - f(gx_0)||_{L^p(G)} du$$

$$= \sigma_{n,k}^{1/p} \int\limits_{\mathrm{SO}(n-k)} ||f(g\gamma z_\omega) - f(gx_0)||_{L^p(G)} d\gamma, \qquad \omega = \left[\begin{array}{c} I_k \\ 0_{(n-2k)\times k} \end{array}\right]. \tag{3.11}$$

Replace $g\gamma \to g$ under the sign of the norm and denote

$$A_a = \left[\begin{array}{ccc} a & 0 & (I_k - a'a)^{1/2} \\ 0 & I_{n-2k} & 0 \\ -(I_k - a'a)^{1/2} & 0 & a \end{array}\right], \qquad \tilde{f} = f(gx_0). \tag{3.12}$$

Then $z_\omega = A_a x_0$, and the integral in (3.11) can be written as $||\tilde{f}(gA_a) - \tilde{f}(g)||_{L^p(G)}$. The latter tends to 0 as $a \to I_k$ (see [HR, Ch. 5, §20.4]). The statement (c) follows directly from (3.6). □

Lemma 3.5. *Let $1 \le k \le n-1,\ 2k \le n,\ \lambda = (n-2k-1)/2$. For any $x \in V_{n,k}$,*

$$\int\limits_{V_{n,k}} f(y)dy = \frac{\sigma_{n-k,k}}{2^k} \int\limits_0^{I_k} |I_k - r|^\lambda |r|^{-1/2} dr \int\limits_{\mathrm{O}(k)} (\mathcal{M}_{vr^{1/2}} f)(x) dv. \tag{3.13}$$

If f is $\mathrm{O}(k)$ right-invariant then for any $x \in V_{n,k}$,

$$\int\limits_{V_{n,k}} f(y)dy = \frac{\sigma_{n-k,k}\,\sigma_{k,k}}{2^k} \int\limits_0^{I_k} |I_k - r|^\lambda |r|^{-1/2} (\mathcal{M}_{r^{1/2}} f)(x) dr. \tag{3.14}$$

Proof. By (2.14) with $\ell = n-k$ and r replaced by $I_k - r$, the integral $I = \int_{V_{n,k}} f(y)dy$ can be written as

$$I = 2^{-k} \int\limits_0^{I_k} |I_k - r|^\lambda |r|^{-1/2}\, dr \int\limits_{V_{k,k}} dv \int\limits_{V_{n-k,k}} f_x \left(\begin{bmatrix} u(I_k - r)^{1/2} \\ vr^{1/2} \end{bmatrix} \right) du.$$

This coincides with (3.13). In order to derive (3.14) from (3.13), we make use of (3.9) and write $\mathcal{M}_{vr^{1/2}} f$ as $\mathcal{M}_{(vrv')^{1/2}} f$. Then we interchange integrals and replace $vrv' \to r$. $\square$

Remark 3.6. The case $a = 0$ in (3.6) is worth mentioning separately. In this case $(\mathcal{M}_0 f)(x) = \sigma_{n-k,k}^{-1} \int_{V_{n-k,k}} f_x \left(\begin{bmatrix} u \\ 0 \end{bmatrix} \right) du$ averages f over the set of all k-frames in the $(n-k)$-plane $\{x\}^\perp$. Thus $(\mathcal{M}_0 f)(x)$ represents the Radon transform of the form $(\mathcal{R}f)(\{x\}^\perp)$.

4. Radon transforms

The original Radon transform $(\mathcal{R}f)(\xi)$ was defined by (1.1) for functions $f \equiv f(\eta)$ on the Grassmannian $G_{n,k}$. For technical reasons we shall also use another transform $(\mathfrak{R}f)(\xi)$ in which $f \equiv f(x)$ is a function on the Stiefel manifold. If $f(x)$ is $\mathrm{O}(k)$ right-invariant then the two transforms coincide.

Let us proceed to give precise definitions. We denote $G = \mathrm{SO}(n)$,

$$\check{x}_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in V_{n,k}, \quad \check{\eta}_0 = \{\check{x}_0\} = \mathbb{R}e_1 + \ldots + \mathbb{R}e_k, \quad \xi_0 = \mathbb{R}e_1 + \ldots + \mathbb{R}e_{k'},$$

$$K_0 = \left\{ \rho \in G : \rho = \begin{bmatrix} I_k & 0 \\ 0 & \beta \end{bmatrix}, \quad \beta \in \mathrm{SO}(n-k) \right\},$$
$$K_0' = \left\{ \tau \in G : \tau = \begin{bmatrix} \gamma & 0 \\ 0 & I_{n-k'} \end{bmatrix}, \quad \gamma \in \mathrm{SO}(k') \right\}.$$

Definition 4.1. Suppose that $1 \le k < k' \le n-1$, $\xi \in G_{n,k'}$, g_ξ is an arbitrary rotation with the property $g_\xi \xi_0 = \xi$. If $f \equiv f(x)$, $x \in V_{n,k}$, we define

$$(\mathfrak{R}f)(\xi) = \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} f\left(g_\xi \begin{bmatrix} u \\ 0 \end{bmatrix} \right) du = \int\limits_{K_0'} f(g_\xi \tau \check{x}_0) d\tau. \tag{4.1}$$

If $f \equiv f(\eta)$, $\eta \in G_{n,k}$, we define

$$(\mathcal{R}f)(\xi) = \int\limits_{K_0'} f(g_\xi \tau \check{\eta}_0) d\tau. \tag{4.2}$$

In the first formula $f(x)$ is integrated over all k-frames x in ξ, whereas in the second one we integrate $f(\eta)$ over all subspaces η of ξ. We draw attention to a consistency of $\check{x}_0$, $\check{\eta}_0$ and K_0 in this definition. The expressions (4.1) and (4.2) are independent of the choice of rotation $g_\xi : \xi_0 \to \xi$. Furthermore, up to abuse of notation, one can write

$$(\mathfrak{R}f)(\xi) = (\mathcal{R}f)(\xi) \tag{4.3}$$

provided that in the right-hand side f is a function on $G_{n,k}$, and in the left-hand side f denotes the corresponding $\mathrm{O}(k)$ right-invariant function on $V_{n,k}$ (see Section 2.4). If f is not $\mathrm{O}(k)$ right-invariant, then (4.3) is replaced by

$$(\mathfrak{R}f)(\xi) = (\mathcal{R}\tilde{f})(\xi), \qquad \tilde{f}(\eta) = \int\limits_{\mathrm{SO}(k)} f\left(r_\eta \begin{bmatrix} \alpha & 0 \\ 0 & I_{n-k} \end{bmatrix} \check{x}_0\right) d\alpha, \tag{4.4}$$

$r_\eta \check{\eta}_0 = \eta$, $r_\eta \in G$. The function $\tilde{f}(\eta)$ is the average of $f(x)$ over all k-frames in η.

Definition 4.2. For a function $\varphi(\xi)$, $\xi \in G_{n,k'}$, the dual Radon transforms associated to (4.1), (4.2) are defined by

$$(\mathfrak{R}^*\varphi)(x) = \int\limits_{K_0} \varphi(r_x \rho \xi_0) d\rho, \qquad (\mathcal{R}^*\varphi)(\eta) = \int\limits_{K_0} \varphi(r_\eta \rho \xi_0) d\rho, \tag{4.5}$$

r_x and r_η being arbitrary rotations satisfying $r_x \check{x}_0 = x$ and $r_\eta \check{\eta}_0 = \eta$, respectively.

These transforms average φ over the set of all k'-subspaces containing $x \in V_{n,k}$ (or $\eta \in G_{n,k}$). The definition does not depend on the choice of rotations r_x, r_η, and therefore

$$(\mathcal{R}^*\varphi)(\{x\}) = (\mathfrak{R}^*\varphi)(x) \tag{4.6}$$

(one can take $r_{\{x\}} = r_x$).

LEMMA 4.3 (duality relations). *For $1 \le k < k' \le n-1$,*

$$\int\limits_{G_{n,k'}} \varphi(\xi)(\mathfrak{R}f)(\xi) d\xi = \frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(x)(\mathfrak{R}^*\varphi)(x) dx, \tag{4.7}$$

$$\int\limits_{G_{n,k'}} \varphi(\xi)(\mathcal{R}f)(\xi) d\xi = \int\limits_{G_{n,k}} f(\eta)(\mathcal{R}^*\varphi)(\eta) d\eta, \tag{4.8}$$

provided that either integral in the corresponding formula is finite for f and φ is replaced by $|f|$ and $|\varphi|$, respectively.

Proof. The left-hand side of (4.7) reads

$$\int\limits_{G} \varphi(g\xi_0)dg \int\limits_{K_0'} f(g\tau\check{x}_0)d\tau = \int\limits_{G} \varphi(g\xi_0)dg \int\limits_{K_0} d\rho \int\limits_{K_0'} f(g\tau\rho^{-1}\check{x}_0)d\tau$$
$$= \int\limits_{G} f(\lambda\check{x}_0)d\lambda \int\limits_{K_0} \varphi(\lambda\rho\xi_0)d\rho = \frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(x)(\mathfrak{R}^*\varphi)(x)dx.$$

The proof of (4.8) follows the same lines with $\check{x}_0$ replaced by $\check{\eta}_0$. □

COROLLARY 4.4. *For $1 \le k < k' \le n-1$,*

$$\int\limits_{G_{n,k'}} (\mathfrak{R}f)(\xi)d\xi = \frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(x)dx, \qquad \int\limits_{V_{n,k}} (\mathfrak{R}^*\varphi)(x)dx = \sigma_{n,k} \int\limits_{G_{n,k'}} \varphi(\xi)d\xi, \tag{4.9}$$

$$\int\limits_{G_{n,k'}} (\mathcal{R}f)(\xi)d\xi = \int\limits_{G_{n,k}} f(\eta)d\eta, \qquad \int\limits_{G_{n,k}} (\mathcal{R}^*\varphi)(\eta)d\eta = \int\limits_{G_{n,k'}} \varphi(\xi)d\xi, \tag{4.10}$$

and therefore the Radon transforms $\mathcal{R}f$, $\mathcal{R}^*\varphi$, $\mathfrak{R}f$, $\mathfrak{R}^*\varphi$ are well defined almost everywhere (*in the appropriate manifolds*) *for any integrable f and φ.*

Consider an important special case when f and φ are ℓ-zonal (see Definition 2.6). We shall see that there is an essential difference between the cases (a) $\ell \le k$ and (b) $\ell > k$ for $\mathfrak{R}f$, and (a) $\ell \le n-k'$ and (b) $\ell > n-k'$ for $\mathfrak{R}^*\varphi$. Namely, in the case (a) the Radon transforms and their duals are represented by Gårding-Gindikin fractional integrals associated to the cone $\mathcal{P}_\ell$, whereas in the case (b) such representations fail.

Assuming $x \in V_{n,k}$, $\xi \in G_{n,k'}$, $1 \le k < k' \le n-1$, we denote

$$\sigma_\ell = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} \in V_{n,\ell}, \qquad \alpha = (k'-k)/2,$$
$$\gamma = \frac{|k-\ell|-1}{2}, \quad \tilde{\gamma} = \frac{|n-k'-\ell|-1}{2}, \quad \delta = \frac{k'-k-\ell-1}{2};$$
$$s = \sigma_\ell' x x' \sigma_\ell, \quad S = \sigma_\ell' \mathrm{Pr}_\xi \sigma_\ell; \qquad r = I_\ell - s; \quad R = \sigma_\ell' \mathrm{Pr}_{\xi^\perp} \sigma_\ell = I_\ell - S.$$

THEOREM 4.5. (i) *Let $f(x)$ be an integrable function on $V_{n,k}$. Suppose that $f(x)$ has the form $f(x) \equiv f_0(\sigma_\ell' x x' \sigma_\ell)$, and denote $\tilde{f}_0(s) = |s|^\gamma f_0(s)$. If $1 \le \ell \le k'-k$, then*

$$(\mathfrak{R}f)(\xi) = \begin{cases} c_1 |S|^{-\delta-k/2} (I_+^\alpha \tilde{f}_0)(S), & \text{if } \ell \le k, \\ c_2 \int\limits_0^{I_k} |I_k - t|^\delta |t|^\gamma dt \int\limits_{V_{\ell,k}} f_0(S^{1/2} utu' S^{1/2}) du & \text{if } \ell > k; \end{cases} \tag{4.11}$$

$$c_1 = \frac{\Gamma_\ell(k'/2)}{\Gamma_\ell(k/2)}, \qquad c_2 = 2^{-k}\pi^{-\ell k/2}\frac{\Gamma_k(k'/2)}{\Gamma_k((k'-\ell)/2)}.$$

(ii) *Let* $\varphi(\xi)$ *be an integrable function on* $G_{n,k'}$. *Suppose that* $\varphi(\xi)$ *has the form* $\varphi(\xi) \equiv \varphi_0(\sigma'_\ell \mathrm{Pr}_{\xi^\perp}\sigma_\ell)$, *and denote* $\tilde\varphi_0(R) = |R|^{\tilde\gamma}\varphi_0(R)$. *If* $1 \le \ell \le \min(k, k'-k)$ *then*

(4.12)

$$(\mathfrak{R}^*\varphi)(x)$$
$$= \begin{cases} \tilde c_1 |r|^{-\delta-(n-k')/2}(I_+^\alpha \tilde\varphi_0)(r), & \textit{if } \ell \le n-k', \\ \tilde c_2 \int\limits_0^{I_{n-k'}} |I_{n-k'} - R|^\delta |R|^{\tilde\gamma} dR \int\limits_{V_{\ell,n-k'}} \varphi_0(r^{1/2}uRu'r^{1/2})du & \textit{if } \ell > n-k'; \end{cases}$$

$$\tilde c_1 = \frac{\Gamma_\ell((n-k)/2)}{\Gamma_\ell((n-k')/2)}, \quad \tilde c_2 = 2^{k'-n}\pi^{\ell(k'-n)/2}\frac{\Gamma_{n-k'}((n-k)/2)}{\Gamma_{n-k'}((n-k-\ell)/2)}.$$

Proof. (i) By (4.1),

$$(\mathfrak{R}f)(\xi) = \frac{1}{\sigma_{k',k}}\int\limits_{V_{k',k}} f\left(g_\xi \begin{bmatrix} u \\ 0 \end{bmatrix}\right) du = \frac{1}{\sigma_{k',k}}\int\limits_{V_{k',k}} f_0(z_u z'_u)du, \quad z_u = \sigma'_\ell g_\xi \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

Denote

$$a = g_\xi^{-1}\sigma_\ell = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad a_1 \in \mathfrak{M}_{k',\ell}, \quad a_2 \in \mathfrak{M}_{n-k',\ell}.$$

Then $z_u = a'_1 u$, and one can write (use Lemma 2.1)

$$(4.13) \quad (\mathfrak{R}f)(\xi) = \frac{1}{\sigma_{k',k}}\int\limits_{V_{k',k}} f_0(a'_1 uu'a_1)du = \frac{1}{\sigma_{k',k}}\int\limits_{V_{k',k}} f_0(S^{1/2}v'uu'vS^{1/2})du$$

where

$$a'_1 a_1 = \sigma'_\ell g_\xi \begin{bmatrix} I_{k'} & 0 \\ 0 & 0 \end{bmatrix} g_\xi^{-1}\sigma_\ell = \sigma'_\ell \mathrm{Pr}_\xi \sigma_\ell = S.$$

If $\ell \le k'-k$ then Lemma 2.5 yields the following equalities. In the case $\ell > k$:

$$(\mathfrak{R}f)(\xi) = \frac{\sigma_{k'-\ell,k}}{2^k \sigma_{k',k}}\int\limits_0^{I_k} |I_k - t|^\delta |t|^\gamma dt \int\limits_{V_{\ell,k}} f_0(S^{1/2}utu'S^{1/2})du.$$

In the case $\ell \le k$:

$$(\mathfrak{R}f)(\xi) = \frac{\sigma_{k'-k,\ell}\,\sigma_{k,\ell}}{2^\ell\,\sigma_{k',\ell}\,|S|^{\delta+k/2}}\int\limits_0^S |S-s|^\delta |s|^\gamma f_0(s)ds.$$

By (2.12) and (2.7), these equalities imply (4.11).

(ii) By (4.5),

$$\begin{aligned}(\mathfrak{R}^*\varphi)(x) &= \int\limits_{K_0} \varphi(r_x\rho^{-1}\xi_0)d\rho = \int\limits_{K_0} \varphi_0(\sigma'_\ell r_x \rho^{-1}\mathrm{Pr}_{\xi_0^\perp}\rho r_x^{-1}\sigma_\ell)d\rho \\ &= \int\limits_{K_0} \varphi_0(y'\rho^{-1}\Lambda\rho y)d\rho, \qquad y = r_x^{-1}\sigma_\ell \in V_{n,\ell}, \\ \Lambda &= \left[\begin{array}{cc} 0 & 0 \\ 0 & I_{n-k'} \end{array}\right].\end{aligned}$$

As in (2.13) we write

$$y = \left[\begin{array}{c} y_1 \\ y_2 \end{array}\right] = \left[\begin{array}{c} v_1 s^{1/2} \\ v_2 r^{1/2} \end{array}\right], \qquad v_1 \in V_{k,\ell}, \qquad v_2 \in V_{n-k,\ell},$$

$$s = y'_1 y_1 = y'\check{x}_0\check{x}'_0 y \in \mathcal{P}_\ell, \qquad r + s = I_\ell.$$

Then for

$$\rho = \left[\begin{array}{cc} I_k & 0 \\ 0 & \beta \end{array}\right], \qquad \omega = \left[\begin{array}{c} 0_{(k'-k)\times(n-k')} \\ I_{n-k'} \end{array}\right] \in V_{n-k,n-k'}, \tag{4.14}$$

we have $y'\rho^{-1}\Lambda\rho y = y'_2\beta'\omega\omega'\beta y_2 = r^{1/2}v'_2\beta'\omega\omega'\beta v_2 r^{1/2}$. This gives

$$\begin{aligned}(\mathfrak{R}^*\varphi)(x) &= \int\limits_{\mathrm{SO}(n-k)} \varphi_0(r^{1/2}v'_2\beta'\omega\omega'\beta v_2 r^{1/2})d\beta \\ &= \frac{1}{\sigma_{n-k,\ell}} \int\limits_{V_{n-k,\ell}} \varphi_0(r^{1/2}v'_2\omega\omega' v_2 r^{1/2})dv_2 \qquad (4.15) \\ &\overset{(2.18)}{=} \frac{1}{\sigma_{n-k,n-k'}} \int\limits_{V_{n-k,n-k'}} \varphi_0(A'\omega\omega' A)d\omega, \quad A = v_2 r^{1/2} \in \mathfrak{M}_{n-k,\ell}.\end{aligned}$$

The last integral has the form (2.19), and Lemma 2.5 gives the following equalities. In the case $n-k' < \ell$:

$$(\mathfrak{R}^*\varphi)(x) = \frac{\sigma_{n-k-\ell,n-k'}}{2^{n-k'}\sigma_{n-k,n-k'}} \int\limits_0^{I_{n-k'}} |R|^{\tilde{\gamma}}|I_{n-k'} - R|^\delta dR \int\limits_{V_{\ell,n-k'}} \varphi_0(r^{1/2}uRu'r^{1/2})du.$$

In the case $n-k' \geq \ell$:

$$(\mathfrak{R}^*\varphi)(x) = \frac{\sigma_{n-k',\ell}\,\sigma_{k'-k,\ell}}{2^\ell\,\sigma_{n-k,\ell}\,|r|^{\delta+(n-k')/2}} \int\limits_0^r |R|^{\tilde{\gamma}}|r-R|^\delta \varphi_0(R)dR.$$

Owing to (2.12) and (2.7), these coincide with (4.12). □

It is worth noting that for $\ell > k$, the rank of the matrix $s = \sigma'_\ell x x' \sigma_\ell$ is $\le k$. It follows that s is a boundary point of the cone $\mathcal{P}_\ell$. So is $R = \sigma'_\ell \mathrm{Pr}_{\xi^\perp} \sigma_\ell$ if $\ell > n - k'$.

By making use of (4.3), (4.6), Lemma 2.7(c), and Definition 1.1, one can reformulate Theorem 4.5 for Radon transforms $\mathcal{R}$, $\mathcal{R}^*$. We keep the notation used in Theorem 4.5.

THEOREM 4.6. *Suppose that f and φ are ℓ-zonal integrable functions on $G_{n,k}$ and $G_{n,k'}$, respectively. Let $1 \le k < k' \le n-1$, $\quad \eta \in G_{n,k}$, $\quad \xi \in G_{n,k'}$,*

$$s = \sigma'_\ell \mathrm{Pr}_\eta \sigma_\ell = \mathrm{Cos}^2(\eta, \sigma_\ell), \quad r = \sigma'_\ell \mathrm{Pr}_{\eta^\perp} \sigma_\ell = \mathrm{Sin}^2(\eta, \sigma_\ell),$$

$$S = \sigma'_\ell \mathrm{Pr}_\xi \sigma_\ell = \mathrm{Cos}^2(\xi, \sigma_\ell).$$

(i) *If $1 \le \ell \le \min(k, k'-k)$, then there is a function f_0 on $(0, I_\ell)$ such that $f(\eta) \overset{\text{a.e.}}{=} f_0(s)$ and*

$$(\mathcal{R}f)(\xi) = c_1 |S|^{-\delta - k/2} (I_+^\alpha \tilde{f}_0)(S), \qquad \tilde{f}_0(s) = |s|^\gamma f_0(s). \tag{4.16}$$

(ii) *If $1 \le \ell \le \min(n-k', k, k'-k)$, then there is a function φ_0 on $(0, I_\ell)$ such that $\varphi(\xi) \overset{\text{a.e.}}{=} \varphi_0(S)$ and*

$$(\mathcal{R}^*\varphi)(\eta) = \tilde{c}_1 |r|^{-\delta - (n-k')/2} (I_+^\alpha \tilde{\varphi}_0)(r), \qquad \tilde{\varphi}_0(S) = |S|^{\tilde{\gamma}} \varphi_0(S). \tag{4.17}$$

Theorems 4.5 and 4.6 are consistent with similar results in [Ru2] for totally geodesic transforms (the case $k = 1$).

Now we switch to the general (not necessarily zonal) case. Let M_r^* and $\mathcal{M}_a$ be the mean value operators from (3.1) and (3.6), respectively. For fixed $x \in V_{n,k}$, we denote

$$\psi(s) = |s|^{-1/2} \int\limits_{\mathrm{SO}(k)} (\mathcal{M}_{\delta s^{1/2}} f)(x)\, d\delta, \qquad s \in \mathcal{P}_k, \tag{4.18}$$

and suppose in the following that f is an integrable $\mathrm{O}(k)$ right-invariant function on $V_{n,k}$.

LEMMA 4.7. *Let $1 \le k < k' \le n-1$, $k + k' \le n$, $\alpha = (k'-k)/2$. If $2k \le k'$, then*

$$|r|^{\alpha - 1/2} M_r^* \mathfrak{R} f = \frac{\Gamma_k(k'/2)}{\Gamma_k(k/2)} (I_+^\alpha \psi)(r), \qquad r \in (0, I_k), \tag{4.19}$$

where $I_+^\alpha \psi$ is the Gårding-Gindikin fractional integral associated to the cone $\mathcal{P}_k$.

Proof. By (3.1), (4.1) and (3.7),

$$(M_r^*\mathfrak{R}f)(x) = \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} du \int\limits_K f_x\left(\rho g_r^{-1}\left[\begin{array}{c} u \\ 0 \end{array}\right]\right) d\rho \tag{4.20}$$
$$= \frac{1}{\sigma_{k',k}} \int\limits_{\mathrm{SO}(k)} d\delta \int\limits_{V_{k',k}} (\mathcal{M}_{\delta a(u)}f)(x)du,$$

$$a(u) = x_0' g_r^{-1}\left[\begin{array}{c} u \\ 0 \end{array}\right] = x_r'\left[\begin{array}{c} u \\ 0 \end{array}\right] = a_1' u, \quad a_1 = \left[\begin{array}{c} 0_{(k'-k)\times k} \\ r^{1/2} \end{array}\right] \in \mathfrak{M}_{k',k}. \tag{4.21}$$

For $2k \le k'$, due to (3.8) and $\mathrm{O}(k)$ right-invariance of f, by (2.22) we obtain

$$M_r^*\mathfrak{R}f = \frac{\sigma_{k'-k,k}\,\sigma_{k,k}}{2^k \sigma_{k',k}} |r|^{-(k'-k-1)/2} \int\limits_0^r |r-s|^{(k'-2k-1)/2} |s|^{-1/2} ds \int\limits_{\mathrm{SO}(k)} \mathcal{M}_{\delta s^{1/2}} f\, d\delta.$$

By (4.18) and (2.12), this gives (4.19). □

Remark 4.8. For $k=1$, the equality (4.19) is due to Helgason [H1], [H2]. The proof presented above extends the argument from [Ru2, Lemma 2.8(i)] to the matrix case. The obvious assumption $k' \ge 2$ in [H1], [H2] transforms into $k' \ge 2k$. For $k' < 2k$, the fractional integral in (4.19) diverges.

The equality (4.19) points the way to inversion formulae. First of all we have to eliminate the artificial restriction $k' \ge 2k$.

Lemma 4.9. *Let f be an integrable* $\mathrm{O}(k)$ *right-invariant function on* $V_{n,k}$,

$$\varphi(\xi) = (\mathfrak{R}f)(\xi), \quad \xi \in G_{n,k'}, \quad 1 \le k < k' \le n-1; \quad k+k' \le n.$$

For fixed $x \in V_{n,k}$, we denote $\hat{\varphi}(s) = |s|^{(k'-k-1)/2}(M_s^*\varphi)(x)$, $s \in \mathcal{P}_k$. Then

$$I_+^{(n-k')/2}\hat{\varphi} = \frac{\Gamma_k(k'/2)}{\Gamma_k(k/2)} I_+^{(n-k)/2}\psi, \tag{4.22}$$

with $\psi(s)$ defined by (4.18).

Proof. For $k' \ge 2k$, (4.22) follows immediately from (4.19) due to the semigroup property of fractional integrals. Once the result is known, we shall prove it in the maximal range of parameters. The idea is to use (2.22) twice, from the right to the left and from the left to the right, with different parameters.

Owing to (4.20) and (4.21), by Lemma 2.1 we have

$$\begin{aligned} M_r^* \mathfrak{R} f &= \frac{1}{\sigma_{k',k}} \int\limits_{\mathrm{SO}(k)} d\delta \int\limits_{V_{k',k}} \mathcal{M}_{\delta a_1' u} f\, du = \frac{1}{\sigma_{k',k}} \int\limits_{\mathrm{SO}(k)} d\delta \int\limits_{V_{k',k}} \mathcal{M}_{\delta s^{1/2} v' u} f\, du \\ &\qquad (s = a_1' a, \quad v \in V_{k',k}) \\ &\overset{(2.18)}{=} \frac{1}{\sigma_{k',k}} \int\limits_{\mathrm{SO}(k)} d\delta \int\limits_{V_{k',k}} \mathcal{M}_{\delta s^{1/2} v' u_0} f\, dv, \qquad u_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in V_{k',k}. \end{aligned}$$

Thus the left-hand side of (4.22) reads

$$\begin{aligned} &\frac{1}{\sigma_{k',k}\, \Gamma_k((n-k')/2)} \int\limits_0^r |r-s|^{(n-k'-k-1)/2} |s|^{(k'-k-1)/2} ds \\ &\qquad \times \int\limits_{\mathrm{SO}(k)} d\delta \int\limits_{V_{k',k}} \mathcal{M}_{\delta s^{1/2} v' u_0} f\, dv. \end{aligned}$$

By (2.22) (with k replaced by k' and ℓ replaced by k), this can be written as

$$\frac{2^k\, \sigma_{n,k}\, |r|^{(n-k-1)/2}}{\sigma_{k',k}\, \sigma_{n,k'}\, \sigma_{n-k',k}\, \Gamma_k((n-k')/2)} \int\limits_{\mathrm{SO}(k)} d\delta \int\limits_{V_{n,k'}} \mathcal{M}_{\delta A' y u_0} f\, dy, \tag{4.23}$$

with $A \in \mathfrak{M}_{n,k}$ so that $A'A = r$. Then we set

$$y = \gamma y_0, \qquad \gamma \in \mathrm{SO}(n), \qquad y_0 = \begin{bmatrix} I_{k'} \\ 0 \end{bmatrix} \in V_{n,k'},$$

and integrate in γ. Since $y_0 u_0 \in V_{n,k}$ we have

$$\int\limits_{V_{n,k'}} \mathcal{M}_{\delta A' y u_0} f\, dy = \sigma_{n,k'} \int\limits_{\mathrm{SO}(n)} \mathcal{M}_{\delta A' \gamma y_0 u_0} f\, d\gamma = \frac{\sigma_{n,k'}}{\sigma_{n,k}} \int\limits_{V_{n,k}} \mathcal{M}_{\delta A' z} f\, dz. \tag{4.24}$$

Now we plug (4.24) in (4.23) and apply (2.22) again (with $\ell = k$). This gives (4.22). The conditions for k, k', and n in the lemma agree with those needed for (2.22). □

In a similar way one can extend the range of parameters in Theorems 4.5 and 4.6. An analog of Theorem 4.5 reads as follows.

THEOREM 4.10. *Let $f(x)$ and $\varphi(\xi)$ be ℓ-zonal integrable functions on $V_{n,k}$ and $G_{n,k'}$, respectively*; $1 \le k < k' \le n-1$, $1 \le \ell \le \min(k, n-k')$. *Suppose that f is* $\mathrm{O}(k)$ *right-invariant, and set*

$$s = \sigma_\ell' x x' \sigma_\ell, \quad S = \sigma_\ell' \mathrm{Pr}_\xi \sigma_\ell, \quad r = I_\ell - s, \quad R = I_\ell - S, \quad \sigma_\ell = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix}.$$

(i) *There exist functions* $f_0(s)$ *and* $F_0(S)$ *on* $(0, I_\ell)$ *such that* $f(x) \overset{\text{a.e.}}{=} f_0(s)$, $(\mathfrak{R}f)(\xi) \overset{\text{a.e.}}{=} F_0(S)$, *and*

$$I_+^{(n-k')/2} \hat{F}_0 = \frac{\Gamma_\ell(k'/2)}{\Gamma_\ell(k/2)} I_+^{(n-k)/2} \hat{f}_0, \tag{4.25}$$

$$\hat{F}_0(S) = |S|^{(k'-\ell-1)/2} F_0(S), \qquad \hat{f}_0(s) = |s|^{(k-\ell-1)/2} f_0(s).$$

(ii) *There exist functions* $\varphi_0(R)$ *and* $\Phi_0(r)$ *on* $(0, I_\ell)$ *such that* $\varphi(\xi) \overset{\text{a.e.}}{=} \varphi_0(R)$, $(\mathfrak{R}^*\varphi)(x) \overset{\text{a.e.}}{=} \Phi_0(r)$, *and*

$$I_+^{k/2} \hat{\Phi}_0 = \frac{\Gamma_\ell((n-k)/2)}{\Gamma_\ell((n-k')/2)} I_+^{k'/2} \hat{\varphi}_0, \tag{4.26}$$

$$\hat{\Phi}_0(r) = |r|^{(n-k-\ell-1)/2} \Phi_0(r), \qquad \hat{\varphi}_0(R) = |R|^{(n-k'-\ell-1)/2} \varphi_0(R).$$

Proof. (i) By Lemma 2.7(b), $f(x)$ can be written as $f_0(s)$. Owing to (4.13), for any $v \in V_{k',\ell}$, Lemma 2.1 yields

$$\begin{aligned}
(\mathfrak{R}f)(\xi) &= \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} f_0(S^{1/2} v' u u' v S^{1/2}) du \\
&\overset{(2.18)}{=} \frac{1}{\sigma_{k',\ell}} \int\limits_{V_{k',\ell}} f_0(S^{1/2} v' u_0 u_0' v S^{1/2}) dv = F_0(s), \quad u_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in V_{k',k}.
\end{aligned}$$

Hence by (2.22) (with k replaced by k'), the left-hand side of (4.25) can be written as

$$\begin{aligned}
&\frac{1}{\sigma_{k',\ell}\, \Gamma_\ell((n-k')/2)} \int\limits_0^t |t-S|^{(n-k'-\ell-1)/2} |S|^{(k'-\ell-1)/2} dS \\
&\qquad\qquad \times \int\limits_{V_{k',\ell}} f_0(S^{1/2} v' u_0 u_0' v S^{1/2}) dv \\
&= \frac{2^\ell\, \sigma_{n,\ell}\, |t|^{(n-\ell-1)/2}}{\sigma_{k',\ell}\, \sigma_{n,k'}\, \sigma_{n-k',\ell}\, \Gamma_\ell((n-k')/2)} \int\limits_{V_{n,k'}} f_0(A' y u_0 u_0' y' A) dy
\end{aligned} \tag{4.27}$$

provided $k' + \ell \le n$, $A \in \mathfrak{M}_{n,\ell}$, $A'A = t \in \mathcal{P}_\ell$. As in (4.24), the integral in (4.27) can be replaced by $(\sigma_{n,k'}/\sigma_{n,k}) \int_{V_{n,k}} f_0(A' z z' A) dz$ and transformed by (2.22). Proceeding as in the proof of Lemma 4.6, we get (4.25).

(ii) Existence of φ_0 satisfying $\varphi(\xi) = \varphi_0(R)$ follows from Lemma 2.7(c) provided $\ell \le \min(k', n-k')$. By (4.15), $\mathfrak{R}^*\varphi$ has the form $\Phi_0(r)$, and by (2.22) (with k replaced by $n-k$), the left-hand side of (4.26) is represented as

(4.28)

$$\frac{1}{\sigma_{n-k,\ell}\,\Gamma_\ell(k/2)}\int_0^t |t-r|^{(k-\ell-1)/2}|r|^{(n-k-\ell-1)/2}dr \int_{V_{n-k,\ell}} \varphi_0(r^{1/2}v_2'\omega\omega' v_2 r^{1/2})dv_2$$
$$= \frac{2^\ell\,\sigma_{n,\ell}\,|t|^{(n-\ell-1)/2}}{\sigma_{n-k,\ell}\,\sigma_{n,n-k}\,\sigma_{k,\ell}\,\Gamma_\ell(k/2)}\int_{V_{n,n-k}} \varphi_0(A'y\omega\omega' y'A)dy$$

where $A\in\mathfrak{M}_{n,\ell}$, $A'A=t$, and ω has the same meaning as in (4.14). The equality (4.28) holds provided $\ell\le\min(k,n-k)$. The integral $\int_{V_{n,n-k}}$ in (4.28) can be written as

$$\frac{\sigma_{n,n-k}}{\sigma_{n,n-k'}}\int_{V_{n,n-k'}}\varphi_0(A'zz'A)dz. \tag{4.29}$$

We plug (4.29) in (4.28) and apply (2.22) (with k replaced by $n-k'$ and $1\le\ell\le\min(k',n-k')$). This gives (4.26). $\square$

Theorem 4.10 can be easily reformulated for $f(x)$ and $(\mathfrak{R}^*\varphi)(x)$ replaced by the corresponding functions on $G_{n,k}$. This will give us an extension of Theorem 4.6.

Having (4.22), (4.25) and (4.26) at our disposal, we can change the order of fractional integrals on both sides by making use of their semigroup property. That was impossible with (4.11), (4.12) and (4.19), because these formulae were derived with inevitable additional restrictions.

Corollary 4.11. *Let $x\in V_{n,k}$, $\xi\in G_{n,k'}$, $1\le k<k'\le n-1$, $\alpha=(k'-k)/2$, $m\in\mathbb{N}$, and suppose that the functions $\psi,\hat\varphi$, $\hat f_0$, $\hat F_0$, $\hat\varphi_0$, $\hat\Phi_0$ have the same meaning as in* (4.22), (4.25), (4.26).

(i) *If $f(x)\in L^1(V_{n,k})$, $\varphi(\xi)=(\mathfrak{R}f)(\xi)$, $k+k'\le n$, then for $m>(k'-1)/2$,*

$$I_+^{m-\alpha}\hat\varphi=\frac{\Gamma_k(k'/2)}{\Gamma_k(k/2)}I_+^m\psi. \tag{4.30}$$

(ii) *If $f\in L^1(V_{n,k})$ and $\varphi\in L^1(G_{n,k'})$ are ℓ-zonal, $1\le\ell\le\min(k,n-k')$, and f is* O(k) *right-invariant* (*i.e.* $f=f_0(s)$, $\varphi=\varphi_0(R)$, $\mathfrak{R}f=F_0(S)$, $\mathfrak{R}^*\varphi=\Phi_0(r)$; *see Theorem* 4.7), *then for $m>\alpha+(\ell-1)/2$,*

$$I_+^{m-\alpha}\hat F_0=\frac{\Gamma_\ell(k'/2)}{\Gamma_\ell(k/2)}I_+^m\hat f_0, \tag{4.31}$$

$$I_+^{m-\alpha}\hat\Phi_0=\frac{\Gamma_\ell((n-k)/2)}{\Gamma_\ell((n-k')/2)}I_+^m\hat\varphi_0. \tag{4.32}$$

Proof. Equalities (4.30)–(4.32) follow from (4.22), (4.25) and (4.26) if we apply $I_+^{m-(n-k)/2}$ (on $\mathcal{P}_k$) to (4.22), $I_+^{m-(n-k)/2}$ (on $\mathcal{P}_\ell$) to (4.25), and $I_+^{m-k'/2}$ (on $\mathcal{P}_\ell$) to (4.26). If m is not sufficiently large, the action of these operators is treated in the sense of distributions (see Sec. 2.2). The resulting equalities (4.30)-(4.32) still hold pointwise almost everywhere (on $(0, I_k)$ for (4.30), and on $(0, I_\ell)$ for (4.31) and (4.32)), because fractional integrals in these equalities are well defined and represent integrable functions. □

Proof of Theorem 1.2. By (4.30) and (4.18),

$$\psi(r) \equiv |r|^{-1/2} \int\limits_{\mathrm{SO}(k)} \mathcal{M}_{\delta r^{1/2}} f \, d\delta = \frac{\Gamma_k(k/2)}{\Gamma_k(k'/2)} (D_+^m I_+^{m-\alpha} \hat{\varphi})(r). \tag{4.33}$$

Note that owing to Remark 3.6, $\psi(r)$ behaves like $|r|^{-1/2}$ as $|r| \to 0$, and therefore (unlike the case $k = 1$) we cannot differentiate (4.30) pointwise, even for f smooth. Thus we have to invoke distributions. To complete the proof it remains to note that $\psi(r) \to f$ as $r \to I_k$ in the required sense. Indeed, by the generalized Minkowski inequality

$$\|\psi(r) - f\|_p \le |r|^{-1/2} \int\limits_{\mathrm{SO}(k)} ||\mathcal{M}_{\delta r^{1/2}} f - f||_p \, d\delta + [|r|^{-1/2} - 1] \, ||f||_p.$$

Because of Lemma 3.4(a), the integrand in the first term does not exceed $2||f||_p$. Hence, by the Lebesgue theorem on dominated convergence and Lemma 3.4(b,c), we obtain $\lim\limits_{r \to I_k} \psi(r) = f$ in the L^p-norm (for $f \in L^p$) or in the sup-norm (if f is a continuous function). □

In a similar way we get the following inversion formulae for the Radon transform and its dual in the ℓ-zonal case:

$$f_0(s) = \frac{\Gamma_\ell(k/2)}{\Gamma_\ell(k'/2)} |s|^{-(k-\ell-1)/2} (D_+^m I_+^{m-\alpha} \hat{F}_0)(s), \tag{4.34}$$

$$\varphi_0(R) = \frac{\Gamma_\ell((n-k')/2)}{\Gamma_\ell((n-k)/2)} |R|^{-(n-k'-\ell-1)/2} (D_+^m I_+^{m-\alpha} \hat{\Phi}_0)(R). \tag{4.35}$$

$$\hat{F}_0(S) = |S|^{(k'-\ell-1)/2} F_0(S), \qquad \hat{\Phi}_0(r) = |r|^{(n-k-\ell-1)/2} \Phi_0(r).$$

These equalities hold under assumptions of Corollary 4.11(ii) and follow from (4.31), (4.32). The formula (4.34) was presented in Theorem 1.4(ii) "in Grassmannian language".

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(Received January 4, 2002)