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Holomorphic extensions of representations: (I) automorphic functions

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Abstract

Let G be a connected, real, semisimple Lie group contained in its complexification $G_{\mathbb{C}}$, and let K be a maximal compact subgroup of G. We construct a $K_{\mathbb{C}}$-G double coset domain in $G_{\mathbb{C}}$, and we show that the action of G on the K-finite vectors of any irreducible unitary representation of G has a holomorphic extension to this domain. For the resultant holomorphic extension of K-finite matrix coefficients we obtain estimates of the singularities at the boundary, as well as majorant/minorant estimates along the boundary. We obtain L^∞ bounds on holomorphically extended automorphic functions on G/K in terms of Sobolev norms, and we use these to estimate the Fourier coefficients of combinations of automorphic functions in a number of cases, e.g. of triple products of Maaß forms.

Introduction

Complex analysis played an important role in the classical development of the theory of Fourier series. However, even for $\mathrm{Sl}(2,\mathbb{R})$ contained in $\mathrm{Sl}(2,\mathbb{C})$, complex analysis on $\mathrm{Sl}(2,\mathbb{C})$ has had little impact on the harmonic analysis of $\mathrm{Sl}(2,\mathbb{R})$. As the K-finite matrix coefficients of an irreducible unitary representation of $\mathrm{Sl}(2,\mathbb{R})$ can be identified with classical special functions, such as hypergeometric functions, one knows they have holomorphic extensions to some domain. So for any infinite dimensional irreducible unitary representation of $\mathrm{Sl}(2,\mathbb{R})$, one can expect at most some proper subdomain of $\mathrm{Sl}(2,\mathbb{C})$ to occur. It is less clear that there is a universal domain in $\mathrm{Sl}(2,\mathbb{C})$ to which the action of G on K-finite vectors of every irreducible unitary representation has holomorphic extension. One goal of this paper is to construct such a domain for a real, connected, semisimple Lie group G contained in its complexification $G_{\mathbb{C}}$. It is important to have a maximal domain, and towards this goal we show that this one is maximal in some directions.

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Although defined in terms of subgroups of $G_{\mathbb{C}}$, the domain is natural also from the geometric viewpoint. This theme is developed more fully in [KrStII] where we show that the quotient of the domain by $K_{\mathbb{C}}$ is bi-holomorphic to a maximal Grauert tube of G/K with the adapted complex structure, and where we show that it also contains a domain bi-holomorphic but not isometric with a related bounded symmetric domain. Some implications of this for the harmonic analysis of G/K are also developed there.

However, the main goal of this paper is to use the holomorphic extension of K-finite vectors and their matrix coefficients to obtain estimates involving automorphic functions. To our knowledge, Sarnak was the first to use this idea in the paper [Sa94]. For example, with it he obtained estimates on the Fourier coefficients of polynomials of Maaß forms for $G = \mathrm{SO}(3,1)$. Sarnak also conjectured the size of the exponential decay rate for similar coefficients for $\mathrm{Sl}(2,\mathbb{R})$. Motivated by Sarnak's work, Bernstein-Reznikov, in [BeRe99], verified this conjecture, and in the process introduced a new technique involving G-invariant Sobolev norms. As an application of the holomorphic extension of representations and with a more representation-theoretic treatment of invariant Sobolev norms, we shall verify a uniform version of the conjecture for all real rank-one groups. As the representation-theoretic techniques are general, we are able also to obtain estimates for the decay rate of Fourier coefficients of Rankin-Selberg products of Maaß forms for $G = \mathrm{Sl}(n,\mathbb{R})$, and to give a conceptually simple proof of results of Good, [Go81a,b], on the growth rate of Fourier coefficients of Rankin-Selberg products for co-finite volume lattices in $\mathrm{Sl}(2,\mathbb{R})$.

It is a pleasure to acknowledge Nolan Wallach's influence on our work by his idea of viewing automorphic functions as generalized matrix coefficients, and to thank Steve Rallis for bringing the Bernstein-Reznikov work to our attention, as well as for encouraging us to pursue this project. To the referee goes our gratitude for a careful reading of our manuscript that resulted in the correction of some oversights, as well as a notable improvement of our estimates on automorphic functions for $\mathrm{Sl}(3,\mathbb{R})$.

1. The double coset domain

To begin we recall some standard structure theory in order to be able to define the domain that will be important for the rest of the paper. Any standard reference for structure theory, such as [Hel78], is adequate.

Let $\mathfrak{g}$ be a real, semisimple Lie algebra with a Cartan involution θ. Denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition. Take $\mathfrak{a} \subseteq \mathfrak{p}$ a maximal abelian subspace and let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \mathfrak{a}^*$ be the corresponding root system. Related to this root system is the root space decomposition according to the simultaneous eigenvalues of $\mathrm{ad}(H), H \in \mathfrak{a}$:

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha;$$

here $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and $\mathfrak{g}^\alpha = \{X \in \mathfrak{g}\colon (\forall H \in \mathfrak{a})\ [H, X] = \alpha(H)X\}$. For the choice of a positive system $\Sigma^+ \subseteq \Sigma$ one obtains the nilpotent Lie algebra $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$. Then one has the Iwasawa decomposition on the Lie algebra level

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Let $G_{\mathbb{C}}$ be a simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, where for a real Lie algebra $\mathfrak{l}$, by $\mathfrak{l}_{\mathbb{C}}$ we mean its complexification. We denote by $G, A, A_{\mathbb{C}}, K, K_{\mathbb{C}}, N$ and $N_{\mathbb{C}}$ the analytic subgroups of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}, \mathfrak{a}, \mathfrak{a}_{\mathbb{C}}, \mathfrak{k}, \mathfrak{k}_{\mathbb{C}}, \mathfrak{n}$ and $\mathfrak{n}_{\mathbb{C}}$. If $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ then it is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and the corresponding analytic subgroup $U = \exp(\mathfrak{u})$ is a maximal compact, and in this case, simply connected, subgroup of $G_{\mathbb{C}}$.

For these choices one has for G the Iwasawa decomposition, that is, the multiplication map

$$K \times A \times N \to G, \quad (k, a, n) \mapsto kan$$

is an analytic diffeomorphism. In particular, every element $g \in G$ can be written uniquely as $g = \kappa(g)a(g)n(g)$ with each of the maps $\kappa(g) \in K$, $a(g) \in A$, $n(g) \in N$ depending analytically on $g \in G$.

We shall be concerned with finding a suitable domain in $G_{\mathbb{C}}$ on which this decomposition extends holomorphically. Of course, various domains having this property have been obtained by several individuals. What distinguishes the one here is its $K_{\mathbb{C}}$-G double coset feature as well as a type of maximality. First we note the following:

LEMMA 1.1. *The multiplication mapping*

$$\Phi\colon K_{\mathbb{C}} \times A_{\mathbb{C}} \times N_{\mathbb{C}} \to G_{\mathbb{C}}, \quad (k, a, n) \mapsto kan$$

has everywhere surjective differential.

Proof. Obviously one has $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Then following Harish-Chandra, since Φ is left $K_{\mathbb{C}}$ and right $N_{\mathbb{C}}$-equivariant it suffices to check that $d\Phi(\mathbf{1}, a, \mathbf{1})$ is surjective for all $a \in A_{\mathbb{C}}$. Let $\rho_a(g) = ga$ be the right translation in $G_{\mathbb{C}}$ by the element a. Then for $X \in \mathfrak{k}_{\mathbb{C}}$, $Y \in \mathfrak{a}_{\mathbb{C}}$ and $Z \in \mathfrak{n}_{\mathbb{C}}$ one has

$$d\Phi(\mathbf{1}, a, \mathbf{1})(X, Y, Z) = d\rho_a(\mathbf{1})(X + Y + \mathrm{Ad}(a)Z),$$

from which the surjectivity follows. □

To describe the domain we extend $\mathfrak{a}$ to a θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ so that $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ with $\mathfrak{t} \subseteq \mathfrak{m}$. Let $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ be the corresponding root system of $\mathfrak{g}$. Then it is known that $\Delta|_{\mathfrak{a}} \backslash \{0\} = \Sigma$.

Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple restricted roots corresponding to the positive roots Σ^+. We define elements $\omega_1, \dots, \omega_n$ of $\mathfrak{a}^*$ as follows, using the restriction of the Cartan-Killing form to $\mathfrak{a}$:

$$(\forall 1 \leq i, j \leq n) \qquad \begin{cases} \langle \omega_j, \alpha_i \rangle = 0 & \text{if } i \neq j \\ \frac{2\langle \omega_i, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 1 & \text{if } \alpha_i \in \Delta \\ \frac{\langle \omega_i, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 1 & \text{if } \alpha_i \notin \Delta \text{ and } 2\alpha_i \notin \Sigma \\ \frac{\langle \omega_i, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 2 & \text{if } \alpha_i \notin \Delta \text{ and } 2\alpha_i \in \Sigma. \end{cases}$$

Using standard results in structure theory relating Δ and Σ one can show that $\omega_1, \dots, \omega_n$ are algebraically integral for $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. The last piece of structure theory we shall recall is the little Weyl group. We denote by $\mathcal{W}_{\mathfrak{a}} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ the *Weyl group* of $\Sigma(\mathfrak{a}, \mathfrak{g})$.

We are ready to define a first approximation to the double coset domain. We set

$$\mathfrak{a}_{\mathbb{C}}^1 = \{X \in \mathfrak{a}_{\mathbb{C}} \colon (\forall 1 \leq k \leq n)(\forall w \in \mathcal{W}_{\mathfrak{a}}) \ |\operatorname{Im} \omega_k(w.X)| < \frac{\pi}{4}\}$$

and

$$\mathfrak{a}_{\mathbb{C}}^0 = 2\mathfrak{a}_{\mathbb{C}}^1.$$

On the group side we let $A_{\mathbb{C}}^0 = \exp(\mathfrak{a}_{\mathbb{C}}^0)$ and $A_{\mathbb{C}}^1 = \exp(\mathfrak{a}_{\mathbb{C}}^1)$. Clearly $\mathcal{W}_{\mathfrak{a}}$ leaves each of $\mathfrak{a}_{\mathbb{C}}^0$, $\mathfrak{a}_{\mathbb{C}}^1$, $A_{\mathbb{C}}^0$ and $A_{\mathbb{C}}^1$ invariant.

If $\alpha \in \mathfrak{a}_{\mathbb{C}}^*$ is analytically integral for $A_{\mathbb{C}}$, then we set $a^\alpha = e^{\alpha(\log a)}$ for all $a \in A_{\mathbb{C}}$. Since $G_{\mathbb{C}}$ is simply connected, the elements ω_j are analytically integral for $A_{\mathbb{C}}$ and so we have a^{ω_k} well defined.

Next we introduce the domains

$$A_{\mathbb{C}}^{0,\leq} = \{a \in A_{\mathbb{C}} \colon (\forall 1 \leq k \leq n) \operatorname{Re}(a^{\omega_k}) > 0\},$$

and

$$A_{\mathbb{C}}^{1,\leq} = (A_{\mathbb{C}}^{0,\leq})^{\frac{1}{2}} = \{a \in A_{\mathbb{C}} \colon (\forall 1 \leq k \leq n) |\arg(a^{\omega_k})| < \frac{\pi}{4}\}.$$

Note that $A_{\mathbb{C}}^0 \subseteq A_{\mathbb{C}}^{0,\leq}$ and $A_{\mathbb{C}}^1 \subseteq A_{\mathbb{C}}^{1,\leq}$.

LEMMA 1.2. (i) *For $\Omega \subseteq A_{\mathbb{C}}$ open, $K_{\mathbb{C}}\Omega N_{\mathbb{C}}$ is open in $G_{\mathbb{C}}$. In particular, the sets $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$, $K_{\mathbb{C}}A_{\mathbb{C}}^1N_{\mathbb{C}}$, $K_{\mathbb{C}}A_{\mathbb{C}}^{1,\leq}N_{\mathbb{C}}$, $K_{\mathbb{C}}A_{\mathbb{C}}^0N_{\mathbb{C}}$ and $K_{\mathbb{C}}A_{\mathbb{C}}^{0,\leq}N_{\mathbb{C}}$ are open in $G_{\mathbb{C}}$.*

(ii) *$K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ is dense in $G_{\mathbb{C}}$.*

Proof. This is an immediate consequence of Lemma 1.1 as Φ is a morphism of affine algebraic varieties with everywhere submersive differential. □

PROPOSITION 1.3. *Let $G_{\mathbb{C}}$ be a simply connected, semisimple, complex Lie group. Then the multiplication mapping*

$$\Phi \colon K_{\mathbb{C}} \times A_{\mathbb{C}}^{0,\leq} \times N_{\mathbb{C}} \to G_{\mathbb{C}}, \quad (k, a, n) \mapsto kan$$

is an analytic diffeomorphism onto its open image $K_{\mathbb{C}}A_{\mathbb{C}}^{0,\leq}N_{\mathbb{C}}$.

Proof. In view of the preceding lemmas, it suffices to show that Φ is injective. Suppose that $kan = k'a'n'$ for some $k, k' \in K_{\mathbb{C}}$, $a, a' \in A_{\mathbb{C}}^{0,\leq}$ and $n, n' \in N_{\mathbb{C}}$. Denote by Θ the holomorphic extension of the Cartan involution of G to $G_{\mathbb{C}}$. Then we get that

$$\Theta(kan)^{-1}kan = \Theta(k'a'n')^{-1}k'a'n'$$

or equivalently

$$\Theta(n^{-1})a^2n = \Theta((n')^{-1})(a')^2n'.$$

Now the subgroup $\overline{N}_{\mathbb{C}} = \Theta(N_{\mathbb{C}})$ corresponds to the analytic subgroup with Lie algebra $\overline{\mathfrak{n}}_{\mathbb{C}} = \bigoplus_{\alpha\in-\Sigma^+}\mathfrak{g}_{\mathbb{C}}^{\alpha}$. As a consequence of the injectivity of the map

$$\overline{N}_{\mathbb{C}} \times A_{\mathbb{C}} \times N_{\mathbb{C}} \to \overline{N}_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}, \quad (\overline{n}, a, n) \mapsto \overline{n}an$$

we conclude that $n = n'$ and $a^2 = (a')^2$. We may assume that $a, a' \in \exp(i\mathfrak{a})$.

To complete the proof of the proposition it remains to show that $a^2 = (a')^2$ for $a, a' \in A_{\mathbb{C}}^{0,\leq}$ implies that $a = a'$. Let $X_1, \ldots, X_n$ in $\mathfrak{a}_{\mathbb{C}}$ be the dual basis to $\omega_1, \ldots, \omega_n$. We can write $a = \exp(\sum_{j=1}^n \varphi_j X_j)$ and $a' = \exp(\sum_{j=1}^n \varphi_j' X_j)$ for complex numbers φ_j, φ_j' satisfying $|\operatorname{Im}\varphi_j| < \frac{\pi}{2}$, $|\operatorname{Im}\varphi_j'| < \frac{\pi}{2}$. Then $a^2 = (a')^2$ implies that

$$e^{2\varphi_j} = a^{2\omega_j} = (a')^{2\omega_j} = e^{2\varphi_j'}$$

and hence $\varphi_j = \varphi_j'$ for all $1 \leq j \leq n$, concluding the proof of the proposition. □

Thus every element $z \in K_{\mathbb{C}}A_{\mathbb{C}}^{0,\leq}N_{\mathbb{C}}$ can be uniquely written as $z = \kappa(z)a(z)n(z)$ with $\kappa(z) \in K_{\mathbb{C}}$, $a(z) \in A_{\mathbb{C}}^{0,\leq}$ and $n(z) \in N_{\mathbb{C}}$ all depending holomorphically on z. Next we define domains using the restricted roots. We set

$$\mathfrak{b}^0 = \{X \in \mathfrak{a}\colon (\forall\alpha \in \Sigma)\ |\alpha(X)| < \pi\}.$$

and

$$\mathfrak{b}^1 = \frac{1}{2}\mathfrak{b}^0.$$

Clearly both $\mathfrak{b}^0$ and $\mathfrak{b}^1$ are $\mathcal{W}_{\mathfrak{a}}$-invariant. We set $\mathfrak{b}_{\mathbb{C}}^j = \mathfrak{a} + i\mathfrak{b}^j$ and $B_{\mathbb{C}}^j = \exp(\mathfrak{b}_{\mathbb{C}}^j)$ for $j = 0, 1$. Let $\mathfrak{a}^0 = i(\mathfrak{a}_{\mathbb{C}}^0 \cap i\mathfrak{a})$. Then, from the classification of restricted root systems and standard facts about the associated fundamental weights, one can verify that $\mathfrak{a}^0 \subseteq \mathfrak{b}^0$. For a comparison of these domains we provide below the illustrations for two rank 2 algebras.

LEMMA 1.4. *Let $\omega \subseteq i\mathfrak{b}^1$ be a nonempty, open, $\mathcal{W}_{\mathfrak{a}}$-invariant, convex set. Then the set*

$$K_{\mathbb{C}}\exp(\omega)G$$

is open in $G_{\mathbb{C}}$.

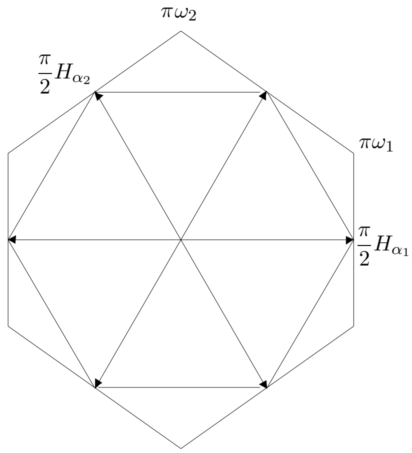

Figure 1

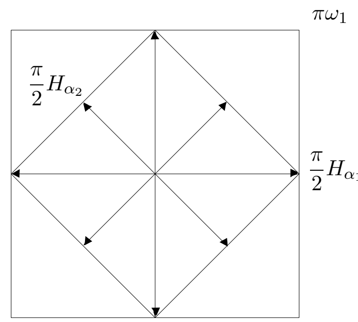

Figure 2

Figure 1 corresponds to $\mathfrak{sl}(3,\mathbb{R})$ and Figure 2 to $\mathfrak{sp}(2,\mathbb{R})$. The region enclosed by an outer polygon corresponds to $\mathfrak{b}^0$ while that enclosed by an inner polygon corresponds to $\mathfrak{a}^0$. The H_{α_i} denote the coroots of α_i and we identify the ω_i as elements of $\mathfrak{a}$ via the Cartan-Killing form.

Proof. Set $W = \mathrm{Ad}(K)\omega$. Since ω is open, convex, and $\mathcal{W}_{\mathfrak{a}}$-invariant, Kostant's nonlinear convexity theorem shows that W is an open, convex set in $i\mathfrak{p}$. Note that $K_{\mathbb{C}}\exp(\omega)G = K_{\mathbb{C}}\exp(W)G$. Now [AkGi90, p. 4-5] shows that the multiplication mapping

$$m\colon K_{\mathbb{C}} \times \exp(W) \times G \to G_{\mathbb{C}}, \quad (k,a,g) \mapsto kag$$

has everywhere surjective differential. From that the assertion follows. □

For each $1 \leq k \leq n$ we write (π_k, V_k) for the real, finite-dimensional, highest weight representation of G with highest weight ω_k. We choose a scalar product $\langle \cdot, \cdot \rangle$ on V_k which satisfies $\langle \pi_k(g)v, w\rangle = \langle v, \pi_k(\Theta(g)^{-1})w\rangle$ for all $v, w \in V_k$ and $g \in G_{\mathbb{C}}$. We denote by v_k a normalized highest weight vector of (π_k, V_k).

Lemma 1.5. *For all $1 \leq k \leq n$, $a \in A^1_{\mathbb{C}}$ and $m \in \overline{N}$,*

$$\mathrm{Re}\left(\langle \pi_k(\theta(m)^{-1}a^2m)v_k, v_k\rangle\right) > 0.$$

Proof. Fix $1 \leq k \leq n$, a and $m \in \overline{N}$, and note that $a^2 \in A^0_{\mathbb{C}}$. Now,

$$\langle \pi_k(\theta(m)^{-1}a^2m)v_k, v_k\rangle = \langle \pi_k(a^2)\pi_k(m)v_k, \pi_k(m)v_k\rangle. \tag{1.1}$$

Let $\mathcal{P}_k \subseteq \mathfrak{a}^*$ denote the set of $\mathfrak{a}$-weights of (π_k, V_k). Then (1.1) implies that there exist nonnegative numbers c_β, $\beta \in V_k$, such that

$$\langle \pi_k(\theta(m)^{-1}a^2m)v_k, v_k\rangle = \sum_{\beta\in\mathcal{P}_k} c_\beta a^{2\beta}.$$

Recall that

$$\mathcal{P}_k \subseteq \operatorname{conv}(\mathcal{W}_{\mathfrak{a}}\omega_k).$$

Since $\mathfrak{a}_{\mathbb{C}}^0$ is convex and Weyl group invariant, to finish the proof it suffices to show that $\operatorname{Re}(a^{2\omega_k}) > 0$ for all $a \in A_{\mathbb{C}}^1$. But this is immediate from the definition of $\mathfrak{a}_{\mathbb{C}}^1$. ☐

LEMMA 1.6. *Let $(b_j)_{j\in\mathbb{N}}$ be a convergent sequence in $A_{\mathbb{C}}$ and $(n_j)_{j\in\mathbb{N}}$ an unbounded sequence in $N_{\mathbb{C}}$. Then the sequence*

$$\Big(\Theta(n_j)^{-1} b_j n_j\Big)_{j\in\mathbb{N}}$$

is unbounded in $G_{\mathbb{C}}$.

Proof. Let $d(\cdot,\cdot)$ be a left invariant metric on $G_{\mathbb{C}}$. Then

$$d(\Theta(n_j)^{-1} b_j^2 n_j, \mathbf{1}) = d(b_j^2 n_j, \Theta(n_j)),$$

and we see that $\lim_{j\to\infty} d(\Theta(n_j)^{-1} b_j^2 n_j, \mathbf{1}) = \infty$ (this follows for example by embedding $\operatorname{Ad}(G_{\mathbb{C}})$ into $\operatorname{Sl}(m,\mathbb{C})$, where we can arrange matters so that $A_{\mathbb{C}}$ maps into the diagonal matrices and $N_{\mathbb{C}}$ in the upper triangular matrices). ☐

PROPOSITION 1.7. (i) *$K_{\mathbb{C}} A_{\mathbb{C}}^1 G$ is open in $G_{\mathbb{C}}$.*

(ii) *$K_{\mathbb{C}} A_{\mathbb{C}}^1 G \subseteq K_{\mathbb{C}} A_{\mathbb{C}}^{1,\leq} N_{\mathbb{C}}$.*

(iii) *For all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the mappings*

$$A_{\mathbb{C}}^1 \times G \to \mathbb{C}, \quad (a,g) \mapsto a(ag)^{\lambda},$$

$$A_{\mathbb{C}}^1 \times G \to K_{\mathbb{C}}, \quad (a,g) \mapsto \kappa(ag)$$

are analytic, and holomorphic in the first variable.

Proof. (i) appears in Lemma 1.2. For (ii) take an $a \in A_{\mathbb{C}}^1$. First we show that $a\overline{N} \subseteq K_{\mathbb{C}} A_{\mathbb{C}} N_{\mathbb{C}}$. Fix $m \in \overline{N}$ and let

$$\begin{aligned} \Omega &= \{a \in A_{\mathbb{C}}^1 : am \in K_{\mathbb{C}} A_{\mathbb{C}} N_{\mathbb{C}}\} \\ &= \{a \in A_{\mathbb{C}}^1 : \Theta(m)^{-1} a^2 m \in \overline{N}_{\mathbb{C}} A_{\mathbb{C}} N_{\mathbb{C}}\}. \end{aligned}$$

Then Ω is open and nonempty. We have to show that $\Omega = A_{\mathbb{C}}^1$. Suppose the contrary. Then there exists a sequence $(a_j)_{j\in\mathbb{N}}$ in Ω such that $a_0 = \lim_{j\to\infty} a_j \in A_{\mathbb{C}}^1 \backslash \Omega$.

Let $a \in \Omega$. Then by Proposition 1.3 we find unique $k \in K_{\mathbb{C}}$, $b \in A_{\mathbb{C}}$ and $n \in N_{\mathbb{C}}$ such that $am = kbn$ or, in other words,

$$\Theta(m)^{-1} a^2 m = \Theta(n)^{-1} b^2 n.$$

Taking matrix-coefficients with fundamental representations we thus get that

$$b^{2\omega_k} = \langle \pi_k(\Theta(n)^{-1} b^2 n) v_k, v_k \rangle = \langle \pi_k(\Theta(m)^{-1} a^2 m) v_k, v_k \rangle \tag{1.2}$$

for all $1 \leq k \leq n$. Applied to our sequence $(a_j)_{j\in\mathbb{N}}$ we get elements $k_j \in K_\mathbb{C}$, $b_j \in A_\mathbb{C}$ and $n_j \in N_\mathbb{C}$ with $a_j m = k_j b_j n_j$. Lemma 1.5 together with (1.2) imply that $(b_j)_{j\in\mathbb{N}}$ is bounded. If necessary, by taking a subsequence, we may assume that $b_0 = \lim_{j\to\infty} b_j$ exists in $A_\mathbb{C}$. Since $\Theta(m)^{-1}a_0^2 m \not\in \overline{N}_\mathbb{C} A_\mathbb{C} N_\mathbb{C}$, the sequence $(n_j)_{j\in\mathbb{N}}$ is unbounded in $N_\mathbb{C}$. Hence $\Big(\Theta(n_j)^{-1}b_j n_j\Big)_{j\in\mathbb{N}}$ is an unbounded sequence in $G_\mathbb{C}$ by Lemma 1.6. But this contradicts the fact that $\Big(\Theta(m)^{-1}a_j^2 m\Big)_{j\in\mathbb{N}}$ is bounded. Thus we have proved that $a\overline{N} \subseteq K_\mathbb{C} A_\mathbb{C} N_\mathbb{C}$ for all $a \in A_\mathbb{C}^1$. But now (1.2) together with Lemma 1.5 actually shows that $b \in A_\mathbb{C}^{1,\leq}$, hence $a\overline{N} \subseteq K_\mathbb{C} A_\mathbb{C}^{1,\leq} N_\mathbb{C}$ for all $a \in A_\mathbb{C}^1$. The Bruhat decomposition of G gives $G = \bigcup_{w\in\mathcal{W}_\mathfrak{a}} \overline{N} wMAN$ with $M = Z_K(A)$. Since $A_\mathbb{C}^1$ is $N_K(A)$-invariant, we get that $aG \subseteq K_\mathbb{C} A_\mathbb{C}^{1,\leq} N_\mathbb{C}$. Then (ii) is now clear while (iii) is a consequence of (ii) and Proposition 1.3. □

Next we are going to prove a significant extension of Proposition 1.7. We will conclude the proof in the following section.

THEOREM 1.8. *Let G be a classical semisimple Lie group. Then the following assertions hold*:

(i) $K_\mathbb{C} B_\mathbb{C}^1 G$ *is open in* $G_\mathbb{C}$;

(ii) $B_\mathbb{C}^1 G \subseteq K_\mathbb{C} A_\mathbb{C} N_\mathbb{C}$;

(iii) *there exists an analytic function*

$$B_\mathbb{C}^1 \times G \to \mathfrak{a}_\mathbb{C}, \quad (a,g) \mapsto H(ag),$$

holomorphic in the first variable, such that $ag \in K_\mathbb{C} \exp H(ag) N_\mathbb{C}$ *for all* $a \in B_\mathbb{C}^1$ *and* $g \in G$;

(iv) *there exists an analytic function*

$$\kappa \colon B_\mathbb{C}^1 \times G \to K_\mathbb{C}, \quad (a,g) \mapsto \kappa(ag),$$

holomorphic in the first variable, such that $ag \in \kappa(ag) A_\mathbb{C} N_\mathbb{C}$ *for all* $a \in B_\mathbb{C}^1$ *and* $g \in G$.

Proof. (i) follows from Lemma 1.2. (ii) follows from Proposition 2.5, Proposition 2.6 and Proposition 2.9 in the next section.

(iii) Set $L = K_\mathbb{C} \cap A_\mathbb{C}$ and note that L is a discrete subgroup of $G_\mathbb{C}$. Then the first part of the proof of Lemma 1.3 shows that we have a biholomorphic diffeomorphism

$$(K_\mathbb{C} \times_L A_\mathbb{C}) \times N_\mathbb{C} \to K_\mathbb{C} A_\mathbb{C} N_\mathbb{C}, \quad ([k,a],n) \mapsto kan.$$

In particular, we get a holomorphic middle projection

$$\widetilde{a}: K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}} \to A_{\mathbb{C}}/L, \quad kan \mapsto aL,$$

and so, by (ii), an analytic mapping

$$\widetilde{\Phi}: B^1_{\mathbb{C}} \times G \to A_{\mathbb{C}}/L, \quad (a,g) \mapsto \widetilde{a}(ag).$$

Now $\mathfrak{a}_{\mathbb{C}} \to A_{\mathbb{C}}/L$, via the map $X \mapsto \exp(X)L$, is the universal cover of $A_{\mathbb{C}}/L$. To complete the proof of (iii) it remains to show that $\widetilde{\Phi}$ lifts to a continuous map with values in $\mathfrak{a}_{\mathbb{C}}$. Since $\exp: \mathfrak{a}^1_{\mathbb{C}} \to A^1_{\mathbb{C}}$ is injective, Proposition 1.7 implies that $\widetilde{\Phi}|_{A^1_{\mathbb{C}} \times G}$ lifts to a continuous map Ψ with values in $\mathfrak{a}_{\mathbb{C}}$. Since the exponential function restricted to $\mathfrak{b}^1_{\mathbb{C}}$ is injective (cf. Remark 1.9.), $B^1_{\mathbb{C}}$ is simply connected and so for every simply connected set $U \subseteq G$ we get a continuous lift of $\widetilde{\Phi}|_{B^1_{\mathbb{C}} \times U}$ extending $\Psi|_{A^1_{\mathbb{C}} \times U}$. By the uniqueness of liftings we get a continuous lift of $\widetilde{\Phi}$ completing the proof of (iii).

(iv) In view of (ii), we get an analytic map

$$\widetilde{\kappa}: B^1_{\mathbb{C}} \times G \to K_{\mathbb{C}}/L, \quad (a,g) \mapsto \widetilde{\kappa}(ag)$$

even holomorphic in the first variable and such that $ag \in \widetilde{\kappa}(ag)A_{\mathbb{C}}N_{\mathbb{C}}$. Thus in order to prove the assertion in (iv), it suffices that $\widetilde{\kappa}$ lifts to a continuous map $\kappa: B^1_{\mathbb{C}} \times G \to K_{\mathbb{C}}$. But this is proved as in (iii). □

Remark 1.9. The simply connected hypothesis on $G_{\mathbb{C}}$ that has been made is not necessary. More generally, if G is classical, semisimple and contained in its complexification, then Theorem 1.8 is valid. Indeed, let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, $\mathfrak{g}_{\mathbb{C}}$ its complexification and let $G_{\mathbb{C}}$ be a simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. As before, let G be the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$.

Let now G_1 be another connected Lie group with Lie algebra $\mathfrak{g}$ and suppose that G_1 sits in its complexification $G_{1,\mathbb{C}}$. Write $G_1 = K_1A_1N_1$ for the Iwasawa decomposition of G_1 corresponding to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Set $B^1_{1,\mathbb{C}} = A_1 \exp_{G_{1,\mathbb{C}}}(i\mathfrak{b}^1)$. Since $G_{\mathbb{C}}$ is simply connected, we have a covering homomorphism

$$\pi: G_{\mathbb{C}} \to G_{1,\mathbb{C}}.$$

Hence Theorem 1.8 (ii) implies that

$$B^1_{1,\mathbb{C}}G_1 \subseteq K_{1,\mathbb{C}}A_{1,\mathbb{C}}N_{1,\mathbb{C}}.$$

To see that Theorem 1.8 (iii), (iv) remains true for G_1 contained in $G_{1,\mathbb{C}}$ one needs that $B^1_{1,\mathbb{C}}$ is simply connected. But this will follow from the fact that $\exp_{G_{1,\mathbb{C}}}: \mathfrak{b}^1_{\mathbb{C}} \to B^1_{1,\mathbb{C}}$ is injective. To see this, note that this map is injective if and only if the map

$$f: \mathfrak{b}^1 \to A_{1,\mathbb{C}}, \ X \mapsto \exp_{G_{1,\mathbb{C}}}(X)$$

is injective. If f were not injective, then there would exist an element $X \in \mathfrak{b}^0$, $X \neq 0$, such that $\exp_{G_{1,\mathbb{C}}}(X) = \mathbf{1}$. Hence $\alpha(X) \in i2\pi\mathbb{Z}$ for all $\alpha \in \Sigma$ (cf. [Hel78, Ch. VII,§4, Prop. 4.1]), a contradiction to $X \in \mathfrak{b}^0\backslash\{0\}$.

The next proposition will be used in a later section. It has independent interest as it can be considered as a principle of convex inclusions and as such is related to Kostant's nonlinear convexity theorem.

Suppose that E is a subset in a complex vector space V. We denote by $\operatorname{conv} E$ the convex hull of E and by $\operatorname{cone} E = \mathbb{R}^+E$ the cone generated by E.

Proposition 1.10. *Let $0 \in \omega \subseteq \mathfrak{b}^0$ be a connected subset. Set $\mathfrak{b}_{\mathbb{C}}^{\omega} = \mathfrak{a}+i\omega$ and $B_{\mathbb{C}}^{\omega} = \exp(\mathfrak{b}_{\mathbb{C}}^{\omega})$. Then,*

$$B_{\mathbb{C}}^{\omega}G \subseteq K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}} \Rightarrow B_{\mathbb{C}}^{\operatorname{conv}\omega}G \subseteq K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}.$$

Proof. Fix $g \in G$. It suffices to show the existence of a holomorphic function

$$f_g\colon B_{\mathbb{C}}^{\operatorname{conv}\omega} \to \mathfrak{a}_{\mathbb{C}}, \quad a \mapsto f_g(a)$$

such that $ag \in K_{\mathbb{C}}\exp(f_g(a))N_{\mathbb{C}}$ for $a \in B_{\mathbb{C}}^{\operatorname{conv}\omega}$ holds. We already know from Theorem 1.8(iii) that a holomorphic function $\widetilde{f}_g\colon B_{\mathbb{C}}^{\omega} \to \mathfrak{a}_{\mathbb{C}}$ with $ag \in K_{\mathbb{C}}\exp(\widetilde{f}_g(a))N_{\mathbb{C}}$ for $a \in B_{\mathbb{C}}^{\omega}$ exists. Now $B_{\mathbb{C}}^{\operatorname{conv}\omega}$ is the holomorphic hull of $B_{\mathbb{C}}^{\omega}$ and so $\widetilde{f}_g$ extends to a holomorphic mapping $f_g\colon B_{\mathbb{C}}^{\operatorname{conv}\omega} \to \mathfrak{a}_{\mathbb{C}}$.

It remains to show that $ag \in K_{\mathbb{C}}\exp(f_g(a))N_{\mathbb{C}}$ for $a \in B_{\mathbb{C}}^{\operatorname{conv}\omega}$. If not, then we find a convergent sequence $(a_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} a_n = a_0 \in B_{\mathbb{C}}^{\operatorname{conv}\omega}$, $a_ng \in K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ but $a_0g \notin K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$. Hence we find a sequence $m_n \in N_{\mathbb{C}}$ such that

$$\Theta(g)^{-1}a_n^2g = \Theta(m_n)^{-1}f_g(a_n)^2m_n$$

but $\Theta(g)^{-1}a_0^2g \notin \overline{N}_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$. As $\Big(f_g(a_n)\Big)_{n\in\mathbb{N}}$ is bounded, we conclude (cf. Lemma 1.6) that $(m_n)_{n\in\mathbb{N}}$ is unbounded, a contradiction. □

2. Matrix calculations

We shall prove (ii) of Theorem 1.8 by various results about matrices. First we shall treat the group $G = \mathrm{Sl}(m,\mathbb{R})$, $m \geq 2$. Then we shall give a class of subgroups of $\mathrm{Sl}(m,\mathbb{R})$ whose roots have a hereditary property similar to one held by Levi factors of parabolic subgroups. This will allow us to take care of most of the classical groups. The remaining cases are treated at the end of this section.

Here we obviously have $G \subseteq G_{\mathbb{C}} = \mathrm{Sl}(m,\mathbb{C})$ with $G_{\mathbb{C}}$ simply connected. We let $\mathfrak{k} = \mathfrak{so}(m,\mathbb{R})$ and choose

$$\mathfrak{a} = \{\mathrm{diag}(x_1, x_2, \dots, x_m) \in M(m, \mathbb{R}) \colon \sum_{i=1}^{m} x_i = 0\}$$

as a maximal abelian subalgebra in $\mathfrak{p} = \mathrm{Symm}(m, \mathbb{R}) \cap \mathfrak{sl}(m, \mathbb{R})$. Define elements $\varepsilon_j \in \mathfrak{a}^*$ by setting

$$\varepsilon_j(\mathrm{diag}(x_1, \dots, x_m)) = x_j.$$

Then $\Sigma = \{\varepsilon_i - \varepsilon_j \colon 1 \leq i \neq j \leq m\}$ and we take $\Sigma^+ = \{\varepsilon_i - \varepsilon_j \colon i < j\}$ as a positive system. The associated system of simple restricted roots is given by

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m\}.$$

As $\mathfrak{g}$ is *split* we have $\Sigma = \Delta$. In particular, the ω_j, $1 \leq j \leq m-1$ are the usual fundamental weights and are given by

$$\omega_j = \varepsilon_1 + \ldots + \varepsilon_j, \qquad (1 \leq j \leq m-1).$$

The Weyl group $\mathcal{W}_{\mathfrak{a}}$ of $\Sigma(\mathfrak{a}, \mathfrak{g})$ is the group of permutations on the m elements $\varepsilon_1, \dots, \varepsilon_m$.

In matrix notation the nilpotent groups N and $\overline{N}$ are given by:

$$N = \{\begin{pmatrix} 1 & x_{12} & \dots & x_{1m} \\ & 1 & x_{23} \dots & x_{2m} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} : x_{ij} \in \mathbb{R}\}$$

and

$$\overline{N} = \{\begin{pmatrix} 1 & & & \\ x_{21} & 1 & & \\ \vdots & \ddots & \ddots & \\ x_{m1} & \dots & x_{m,m-1} & 1 \end{pmatrix} : x_{ij} \in \mathbb{R}\}.$$

For each $1 \leq j \leq m$ we set $e_j = (\delta_{k-j,l-j})_{k,l} \in \mathrm{diag}(m, \mathbb{R})$. Further we associate to each ω_j the element $X_{\omega_j} = \sum_{k=1}^{j} e_j - \frac{j}{m} \sum_{j=1}^{m} e_j$.

Lemma 2.1. (i)

$$\mathfrak{b}^0 = \mathrm{int}\left(\mathrm{conv}\left(\{\pm \pi w.X_{\omega_j} \colon w \in \mathcal{W}_{\mathfrak{a}},\ 1 \leq j \leq m-1\}\right)\right);$$

(ii)

$$\mathfrak{b}^0 \subseteq \mathfrak{a} \cap \left(\frac{m-1}{m} \bigoplus_{j=1}^{m}]-\pi, \pi[e_j\right).$$

Proof. (i) Set

$$\mathfrak{b}' = \mathrm{int}\left(\mathrm{conv}\left(\{\pm \pi w X_{\omega_j} \colon w \in \mathcal{W}_{\mathfrak{a}},\ 1 \leq j \leq m-1\}\right)\right).$$

Both $\overline{\mathfrak{b}^0}$ and $\overline{\mathfrak{b}'}$ are closed, convex and $\mathcal{W}_{\mathfrak{a}}$-invariant. Thus by the convexity of $\mathfrak{b}^0$ and $\mathfrak{b}'$ we have to show only that $\overline{\mathfrak{b}^0} = \overline{\mathfrak{b}'}$. Now $\mathcal{W}_{\mathfrak{a}}$ rotates the extreme points of both $\overline{\mathfrak{b}^0}$ and $\overline{\mathfrak{b}'}$, and the extreme points of $\overline{\mathfrak{b}'}$ are given by $\pm w.\pi X_{\omega_j}$. We shall prove the result by double containment.

"$\supseteq$": By the Krein-Milman Theorem it suffices to show that $\pm\pi X_{\omega_j} \in \overline{\mathfrak{b}^0}$ for all $1 \leq j \leq m-1$. Every $\alpha \in \Sigma^+$ can be written as $\alpha = \sum_{j=1}^{m-1} \delta_j(\varepsilon_j - \varepsilon_{j+1})$ with coefficients $\delta_j \in \{0,1\}$. Thus $\alpha(X_{\omega_j}) \in \{0,1\}$ and the inclusion "$\supseteq$" follows from the definition of $\mathfrak{b}^0$.

"$\subseteq$": Notice that $\omega_1, \ldots, \omega_{m-1}$ constitute a basis of $\mathfrak{a}^*$. Hence every $X \in \overline{\mathfrak{b}^0}$ can be written as $X = \sum_{j=1}^{m-1} \lambda_j X_{\omega_j}$ with coefficients $\lambda_j \in \mathbb{R}$. From the definition of $\mathfrak{b}'$ we may assume that $\lambda_j \geq 0$ for all $1 \leq j \leq m-1$. In particular, we see that

$$(\varepsilon_1 - \varepsilon_m)(X) = \sum_{j=1}^{m-1} (\varepsilon_j - \varepsilon_{j+1})(X) = \sum_{j=1}^{m-1} \lambda_j \in [0, \pi[,$$

concluding the proof of "$\subseteq$".

(ii) For $X = \mathrm{diag}(x_1, \ldots, x_m) = \sum_{j=1}^m x_j e_j \in \mathfrak{b}^0$,

$$-\pi < 2x_1 + x_2 + \ldots + x_m = x_1 - x_m < \pi,$$

and

$$-\pi < x_1 - x_j < \pi \qquad \text{for all } 2 \leq j \leq m-1.$$

By summing these inequalities we obtain

$$-(m-1)\pi < mx_1 < (m-1)\pi,$$

or equivalently $|x_1| < \frac{m-1}{m}\pi$. Similarly, $|x_j| < \frac{m-1}{m}\pi$ for all $1 \leq j \leq m$. □

Remark 2.2. Notice that $\mathfrak{a}^0$ is strictly smaller than $\mathfrak{b}^0$, although they have common boundary points (cf. Figure 1). In particular, Lemma 2.1 shows that

$$\partial\mathfrak{a}^0 \cap \partial\mathfrak{b}^0 \supseteq \{\frac{\pi}{2}(e_i - e_j) \colon 1 \leq i \neq j \leq m-1\}.$$

For every $1 \leq k \leq m$ we denote by $\Delta_k(A)$ the k^{th} principal minor of a matrix $A \in M(m, \mathbb{C})$. For every $g = (g_{ij})_{1 \leq i,j \leq m} \in M(m, \mathbb{C})$ and $1 \leq k \leq m$ we define $g_{(k)} \in M(k, \mathbb{C})$ by $g_{(k)} = (g_{ij})_{1 \leq i,j \leq k}$.

PROPOSITION 2.3. *Let $G = \mathrm{Sl}(m, \mathbb{R})$ with $G_{\mathbb{C}} = \mathrm{Sl}(m, \mathbb{C})$. Then for all $1 \leq k \leq m$, $a \in B^0_{\mathbb{C}}$ and $g \in \mathrm{Sl}(m, \mathbb{R})$ with $g_{(k)} \in \mathrm{Gl}(k, \mathbb{R})$,*

(i) $\Delta_k(gag^t) \neq 0$;

(ii) $\mathrm{Spec}\Big((gag^t)_{(k)}\Big) \subseteq \mathrm{cone}\Big(\mathrm{conv}(\mathrm{Spec}(a))\Big)$.

Proof. (i) Fixing $1 \leq k \leq m$, $a \in B_{\mathbb{C}}^0$ and $g \in \mathrm{Sl}(m, \mathbb{C})$ with $g_{(k)} \in \mathrm{Gl}(k, \mathbb{C})$, we write $a = \mathrm{diag}(r_1 e^{i\varphi_1}, \ldots, r_m e^{i\varphi_m})$ with $r_i > 0$, $-\frac{m-1}{m}\pi < \varphi_i < \frac{m-1}{m}\pi$ (cf. Lemma 2.1(ii)). Set

$$g = \begin{pmatrix} g_{(k)} & B \\ * & * \end{pmatrix}$$

with $g_{(k)} \in \mathrm{Gl}(k, \mathbb{R})$ and $B \in M(k \times (m-k), \mathbb{R})$. Then,

$$\begin{aligned} \Delta_k(gag^t) &= \Delta_k\Bigg(\begin{pmatrix} g_{(k)} & B \\ * & * \end{pmatrix} \mathrm{diag}(r_1 e^{i\varphi_1}, \ldots r_m e^{i\varphi_m}) \begin{pmatrix} g_{(k)}^t & * \\ B^t & * \end{pmatrix} \Bigg) \\ &= \det_k \Big(g_{(k)} \,\mathrm{diag}(r_1 e^{i\varphi_1}, \ldots, r_k e^{i\varphi_k}) g_{(k)}^t \\ &\qquad + B \,\mathrm{diag}(r_{k+1} e^{i\varphi_{k+1}}, \ldots, r_m e^{i\varphi_m}) B^t \Big). \end{aligned}$$

In order to show that $\Delta_k(gag^t) \neq 0$ we have to show that the $k \times k$-matrix

$$X_{(k)} = g_{(k)} \,\mathrm{diag}(r_1 e^{i\varphi_1}, \ldots, r_k e^{i\varphi_k}) g_{(k)}^t + B \,\mathrm{diag}(r_{k+1} e^{i\varphi_{k+1}}, \ldots, r_m e^{i\varphi_m}) B^t$$

is invertible.

Assume first that $k \leq m - k$. Then we can write $B = (B_1, B_2)$ with $B_1 \in M(k, \mathbb{R})$ and $B_2 \in M(k \times (m-2k), \mathbb{R})$. Hence we obtain that

$$\begin{aligned} B \,\mathrm{diag}(r_{k+1} e^{i\varphi_{k+1}}, \ldots, r_m e^{i\varphi_m}) B^t &= B_1 \,\mathrm{diag}(r_{k+1} e^{i\varphi_{k+1}}, \ldots, r_{2k} e^{i\varphi_{2k}}) B_1^t \\ &\quad + B_2 \,\mathrm{diag}(r_{2k+1} e^{i\varphi_{2k+1}}, \ldots, r_m e^{i\varphi_m}) B_2^t. \end{aligned}$$

Let $\langle \cdot, \cdot \rangle$ be the usual hermitian inner product on $\mathbb{C}^k$. In particular, if $v \in \mathbb{C}^k$, $v \neq 0$, then we get

$$\begin{aligned} \langle X_{(k)} v, v \rangle &= \langle \mathrm{diag}(r_1 e^{i\varphi_1}, \ldots, r_k e^{i\varphi_k}) g_{(k)}^t v, g_{(k)}^t v \rangle \\ &\quad + \langle \mathrm{diag}(r_{k+1} e^{i\varphi_{k+1}}, \ldots, r_{2k} e^{i\varphi_{2k}}) B_1^t v, B_1^t v \rangle \\ &\quad + \langle \mathrm{diag}(r_{2k+1} e^{i\varphi_{2k+1}}, \ldots, r_m e^{i\varphi_m}) B_2^t v, B_2^t v \rangle. \end{aligned}$$

So there exist numbers $c_1, \ldots, c_m \geq 0$, not all zero, such that

$$\langle X_{(k)} v, v \rangle = \sum_{j=1}^{m} c_j e^{i\varphi_j}. \tag{2.1}$$

Similarly one shows that (2.1) holds for the case $k \geq m - k$. Now (i) follows from (2.1) and Lemma 2.4 below.

(ii) Since $X_{(k)} = (gag^t)_{(k)}$, (ii) follows from (2.1). □

We denote by $\mathbb{C}^+ = \{z \in \mathbb{C} \colon z \notin]-\infty, 0]\}$ the split plane in $\mathbb{C}$.

Lemma 2.4. *Let* $\varphi_1, \ldots, \varphi_m \in \mathbb{R}$ *be such that* $\operatorname{diag}(\varphi_1, \ldots, \varphi_m) \in \mathfrak{b}^0$. *Then for all sequences of nonnegative numbers* $c_1, \ldots, c_m$, *not all zero,*

$$\sum_{j=1}^{m} c_j e^{i\varphi_j} \in \mathbb{C}^+.$$

In particular $\sum_{j=1}^{m} c_j e^{i\varphi_j} \neq 0$.

Proof. As $\mathfrak{b}^0$ is $\mathcal{W}_\mathfrak{a}$-invariant there is no loss of generality to assume that $\varphi_1 \leq \ldots \leq \varphi_m$. Then $0 \leq \varphi_j - \varphi_1 < \pi$ for all $1 \leq j \leq m$. Since $\sum_{j=1}^{m} \varphi_j = 0$ we have $\varphi_m \geq 0$. Thus$\sum_{j=1}^{m} c_j e^{i\varphi_j}$ is a sum of vectors not all zero in the real convex cone

$$C = \{z \in \mathbb{C} \colon \varphi_m - \pi < \arg(z) \leq \varphi_m\}$$

in $\mathbb{C}$. In particular $\sum_{j=1}^{m} c_j e^{i\varphi_j}$ is nonzero since the convex cone C is pointed (i.e. contains no affine lines). Since $0 \leq \varphi_m < \frac{m-1}{m}\pi$ (cf. Lemma 2.1(ii)) we also have $C \backslash \{0\} \subseteq \mathbb{C}^+$, concluding the proof of the lemma. □

Proposition 2.5. *For* $G = \mathrm{Sl}(m, \mathbb{R})$,

$$K_\mathbb{C} B_\mathbb{C}^1 G \subseteq K_\mathbb{C} A_\mathbb{C} N_\mathbb{C}.$$

Proof. Take $a \in B_\mathbb{C}^1$ and recall that $a^2 \in B_\mathbb{C}^0$. First we show that $a\overline{N} \subseteq K_\mathbb{C} A_\mathbb{C} N_\mathbb{C}$. Let $\overline{n} \in \overline{N}$. Then Proposition 2.3(i) says that all principal minors of the complex symmetric matrix $\overline{n}^t a^2 \overline{n}$ are nonzero. Hence a theorem of Jacobi (cf. [Koe83, p. 124]) implies that there exist unique elements $b_0 \in A_\mathbb{C}$ and $m \in N_\mathbb{C}$ such that

$$\overline{n}^t a^2 \overline{n} = m^t b_0 m.$$

Let $a_0 \in A_\mathbb{C}$ be such that $a_0^2 = b_0$. Then we have

$$a\overline{n} = k a_0 m$$

with $k \in K_\mathbb{C}$ given by $k = a\overline{n} m^{-1} a_0^{-1}$.

Using, as before, the Bruhat decomposition $G = \bigcup_{w \in \mathcal{W}_\mathfrak{a}} \overline{N} w MAN$, together with the $N_K(A)$-invariance of $B_\mathbb{C}^1$, we get that $aG \subseteq K_\mathbb{C} A_\mathbb{C} N_\mathbb{C}$ for all $g \in G$, completing the proof. □

With $G = \mathrm{Sl}(m, \mathbb{R})$ out of the way we want to use an observation that will allow us to obtain a proof of Theorem 1.8(ii) for appropriate subgroups. The groups that will be covered in this way are: $\mathrm{Sp}(n, \mathbb{R})$, $\mathrm{Sp}(p, q)$, $\mathrm{Sp}(n, \mathbb{C})$, $\mathrm{SU}(p, q)$, $\mathrm{SO}^*(2n)$, $\mathrm{Sl}(n, \mathbb{C})$ and $\mathrm{Sl}(n, \mathbb{H})$.

Recall that a Levi subalgebra $\mathfrak{m}$ of a standard parabolic subalgebra must be of the form $\mathfrak{m} = \mathfrak{m}(\Theta)$ for $\Theta \subseteq \Pi$. If, moreover, $\mathfrak{m}$ is θ-stable, then the Iwasawa decomposition for $\mathfrak{m}$ is compatible with that of $\mathfrak{g}$. More generally, for $\mathfrak{g} = \mathfrak{sl}(m, \mathbb{R})$ we consider θ-stable subalgebras $\mathfrak{g}_1 \subseteq \mathfrak{g}$ with a property that will give them Iwasawa decompositions compatible with that of $\mathfrak{g}$. Set

$\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{g}_1$ and $\mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{g}_1$ so that $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ is a Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{a}_1 \subseteq \mathfrak{p}_1$ be a maximal abelian subspace. Since we can extend $\mathfrak{a}_1$ to a maximal abelian subspace of $\mathfrak{p}$ and since all maximal abelian subspaces of $\mathfrak{p}$ are conjugate under $\mathrm{Ad}(K)$, we may assume that $\mathfrak{a}_1 \subseteq \mathfrak{a}$. Choose a positive system Σ_1^+ of $\Sigma_1 = \Sigma(\mathfrak{g}_1, \mathfrak{a}_1)$. Then we can find a positive system Σ^+ of Σ such that $\Sigma_1^+ \subseteq \Sigma^+|_{\mathfrak{a}_1}$. Write $\mathfrak{n}_1 = \bigoplus_{\alpha \in \Sigma_1^+} \mathfrak{g}_1^\alpha$ and note that $\mathfrak{n}_1 \subseteq \mathfrak{n}$.

We now impose the following condition on the restricted roots:

$$\Sigma|_{\mathfrak{a}_1} \backslash \{0\} = \Sigma_1. \tag{I}$$

It can be checked that (I) holds for example for the standard imbeddings of the subalgebras $\mathfrak{g}_1 = \mathfrak{sp}(n, \mathbb{R})$ (with $2n = m$), $\mathfrak{su}(p, q)$ (with $2p + 2q = m$), $\mathfrak{sp}(p, q)$ (with $2p + 2q = m$) or $\mathfrak{so}^*(2n)$ (with $2n = m$)(in all cases the fact that makes things work is that the restricted root system of $\mathfrak{g}_1$ is either of type C_n or BC_n). Further examples are $\mathfrak{g}_1 = \mathfrak{sl}(n, \mathbb{C})$ (with $2n = m$), $\mathfrak{sp}(n, \mathbb{C})$ (with $2n = m$) or $\mathfrak{sl}(n, \mathbb{H})$ (with $4n = m$) (here the explanation is that the root system Σ_1 is of type A). Set

$$\mathfrak{b}_1^0 = \{X \in \mathfrak{a}_1 \colon (\forall \alpha \in \Sigma_1)\ |\alpha(X)| < \pi\}$$

and

$$\mathfrak{b}_1^1 = \frac{1}{2} \mathfrak{b}_1^0.$$

Then condition (I) guarantees that

$$\mathfrak{b}_1^1 \subseteq \mathfrak{b}^1. \tag{2.2}$$

We denote by G_1 the analytic subgroup of G which is associated to $\mathfrak{g}_1$. We assume that G_1 is closed. Further we denote by K_1, A_1, N_1 and $\overline{N}_1$ the analytic subgroups of G_1 corresponding to $\mathfrak{k}_1$, $\mathfrak{a}_1$, $\mathfrak{n}_1$ and $\overline{\mathfrak{n}}_1$. Finally we set $B_{1,\mathbb{C}}^1 = \exp(\mathfrak{a}_1 + i\mathfrak{b}_1^1)$. In order to prove Theorem 1.8(ii) for the group G_1 we have to show that

$$B_{1,\mathbb{C}}^1 G_1 \subseteq K_{1,\mathbb{C}} A_{1,\mathbb{C}} N_{1,\mathbb{C}}$$

or equivalently

$$(\forall b \in B_{1,\mathbb{C}}^1)(\forall g \in G_1)(\exists a \in A_{1,\mathbb{C}}, m \in N_{1,\mathbb{C}}), \qquad g^t b g = m^t a m. \tag{2.3}$$

In view of (2.2) and the validity of (2.3) for G we deduce that for all $b \in B_{1,\mathbb{C}}^1$, $g \in G_1$ there exist unique elements $m = m(b, g) \in N_{\mathbb{C}}$, $a = a(b, g) \in A_{\mathbb{C}}$ such that $g^t b^2 g = m^t a m$. Moreover $a = a(b, g)$ and $m = m(b, g)$ are analytic functions in the variables $b \in B_{1,\mathbb{C}}^1$, $g \in G_1$. Since we already know that $a(A_{1,\mathbb{C}}^1, G_1) \subseteq A_{1,\mathbb{C}}$ and $m(A_{1,\mathbb{C}}^1, G_1) \subseteq N_{1,\mathbb{C}}$ (cf. Proposition 1.7), the analyticity of both a and m implies that $a(B_{1,\mathbb{C}}^1, G_1) \subseteq A_{1,\mathbb{C}}$ and $m(B_{1,\mathbb{C}}^1, G_1) \subseteq N_{1,\mathbb{C}}$ proving (2.2).

We summarize the above discussion with

PROPOSITION 2.6. *Assume that G is one of the groups* $\mathrm{Sl}(n,\mathbb{R})$, $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{Sp}(p,q)$, $\mathrm{SU}(p,q)$, $\mathrm{SO}^*(2n)$, $\mathrm{Sl}(n,\mathbb{C})$, $\mathrm{Sl}(n,\mathbb{H})$ *or* $\mathrm{Sp}(n,\mathbb{C})$. *Then*

$$K_{\mathbb{C}}B^1_{\mathbb{C}}G \subseteq K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}.$$

There remain the restricted root systems for $\mathfrak{g} = \mathfrak{so}(p,q)$ and $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$. So first we recall some facts concerning these root systems of type B_n and D_n.

$\mathbf{B_n}$: The root system B_n is given by

$$\Sigma = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq n\} \cup \{\pm\varepsilon_i : 1 \leq i \leq n\}.$$

A basis of Σ is

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}.$$

If $\mathfrak{g}$ is a split real Lie algebra with restricted root system Σ, then the ω_i are the fundamental weights associated to Π and given by

$$\omega_1 = \varepsilon_1,\ \omega_2 = \varepsilon_1 + \varepsilon_2,\ \ldots,\ \omega_{n-1} = \varepsilon_1 + \ldots + \varepsilon_{n-1},\ \omega_n = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n).$$

$\mathbf{D_n}$: The root system D_n is given by

$$\Sigma = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq n\}$$

and a basis of Σ is given by

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}.$$

If $\mathfrak{g}$ is a split real Lie algebra with restricted root system Σ, then the ω_i are the fundamental weights:

$$\omega_1 = \varepsilon_1,\ \omega_2 = \varepsilon_1 + \varepsilon_2,\ \ldots,\ \omega_{n-2} = \varepsilon_1 + \ldots + \varepsilon_{n-2}$$

and

$$\omega_{n-1} = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_{n-1} - \varepsilon_n),\ \omega_n = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n).$$

To indicate the dependence of $\mathfrak{b}^0$ on the root system, we shall write $\mathfrak{b}^0(\Sigma)$ for $\mathfrak{b}^0$.

LEMMA 2.7. *There exists* $\mathfrak{b}^0(D_n) = \mathfrak{b}^0(B_n)$.

Proof. This is immediate from the equality $\mathrm{conv}(B_n) = \mathrm{conv}(D_n)$. ☐

The final goal of this section is to prove the inclusion

$$B^1_{\mathbb{C}}G \subseteq K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}} \tag{2.4}$$

for $G = \mathrm{SO}(p,q)$ or $G = \mathrm{SO}(n,\mathbb{C})$.

We shall repeat the strategy used for the target group of type A_n. So assume for the moment that (2.4) holds for $G = \mathrm{SO}(n,n)$. Assume that $p \geq q$ and embed $G_1 = \mathrm{SO}(p,q)$ into $\mathrm{SO}(p,p)$ in the natural way (upper left corner block). Then if we restrict Σ to $\mathfrak{a}_1$ we get a root system of type B_q or D_q. Hence by our restriction procedure from the preceding section and Lemma 2.7, we get (2.4) also for the subgroup G_1. Thus it suffices to prove (2.4) for $G = \mathrm{SO}(n,n)$ and $G = \mathrm{SO}(n,\mathbb{C})$, with both $\mathfrak{so}(n,n)$ and $\mathfrak{so}(n,\mathbb{C})$ split.

In what follows $\mathfrak{g}$ denotes either $\mathfrak{so}(n,n)$ or $\mathfrak{so}(n,\mathbb{C})$. We embed $\mathfrak{g}$ into $\mathfrak{sl}(2n,\mathbb{R})$ as in the previous section. Then if we restrict the weights of $\mathfrak{sl}(2n,\mathbb{R})$ to $\mathfrak{g}$, we obtain a root system of type C_n or BC_n. We set

$$\mathfrak{b}^0_{\mathrm{res}} = \mathfrak{b}^0(C_n) = \mathfrak{b}^0(BC_n)$$

and

$$\mathfrak{b}^1_{\mathrm{res}} = \frac{1}{2}\mathfrak{b}^0_{\mathrm{res}}.$$

On the group side we define $B^j_{\mathrm{res},\mathbb{C}} = \exp(\mathfrak{a} + i\mathfrak{b}^j_{\mathrm{res}})$ for $j = 0,1$. In particular we get that

$$B^1_{\mathrm{res},\mathbb{C}} G \subseteq K_{\mathbb{C}} A_{\mathbb{C}} N_{\mathbb{C}}. \tag{2.5}$$

We write (π_n, V_n) for the n^{th} fundamental representation of $\widetilde{G}$ with highest weight $\omega_n = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n)$. We write $\mathcal{P}_n$ for the set of $\mathfrak{a}$-weights of (π_n, V_n) and set

$$\mathfrak{b}(\pi_n)^0 = \{X \in \mathfrak{a}\colon (\forall \alpha \in \mathcal{P}_n)\ |\alpha(X)| < \frac{\pi}{2}\}.$$

As usual we put $\mathfrak{b}(\pi_n)^1 = \frac{1}{2}\mathfrak{b}(\pi_n)^0$.

Lemma 2.8. *The following holds*:

$$\mathrm{conv}(\mathfrak{b}^0_{\mathrm{res}} \cup \mathfrak{b}(\pi_n)^0) \supseteq \mathfrak{b}^0.$$

Proof. We claim that the extreme points of $\overline{\mathfrak{b}^0}$ are given by

$$\mathrm{Ext}(\overline{\mathfrak{b}^0}) = \begin{cases} \{\pm\pi e_i, \frac{\pi}{2}(\pm e_1 \pm \ldots \pm e_n)\} & \text{for } n \geq 3, \\ \{\pm \pi e_i\} & \text{for } n = 2. \end{cases}$$

In fact we have $\mathfrak{b}^0 = \mathfrak{b}^0(B_n)$ by Lemma 2.7 and so $\mathrm{Ext}(\overline{\mathfrak{b}^0})$ is invariant under the Weyl group $\mathcal{W}(B_n) = (\mathbb{Z}_2)^n \rtimes S_n$. From that the claim follows.

Now we have $\frac{\pi}{2}(\pm e_1 \pm \ldots \pm e_n) \in \overline{\mathfrak{b}^0_{\mathrm{res}}}$ and $\pm\pi e_i \in \overline{\mathfrak{b}(\pi_n)^0}$. Hence the assertion of the lemma follows from the Krein-Milman theorem. $\square$

Proposition 2.9. *Assume that* $G = \mathrm{SO}(p,q)$ *or* $G = \mathrm{SO}(n,\mathbb{C})$. *Then for all* $a \in B^1_{\mathbb{C}}$,

$$K_{\mathbb{C}} B^1_{\mathbb{C}} G \subseteq K_{\mathbb{C}} A_{\mathbb{C}} N_{\mathbb{C}}.$$

Proof. From what we have already done it is enough to prove the inclusion for $G = \mathrm{SO}(n,n)$ or $G = \mathrm{SO}(n,\mathbb{C})$. By passing to a covering group if necessary we can also replace G by $\widetilde{G}$. Set $B_{\mathbb{C}}(\pi_n) = \exp(\mathfrak{a} + i\mathfrak{b}(\pi_n)^1)$. In view of Proposition 1.9, Lemma 2.7, (2.5) and Lemma 2.8 it remains to check that

$$B_{\mathbb{C}}(\pi_n)G \subseteq K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}. \tag{2.6}$$

Suppose that (2.6) is false. Then we can find a $g \in G$ and a convergent sequence $(a_j)_{j\in\mathbb{N}}$ in $B_{\mathbb{C}}(\pi_n)$ with $\lim_{j\to\infty} a_j = a_0 \in B_{\mathbb{C}}(\pi_n)$, $\Theta(g)^{-1}a_j^2 g \in \overline{N}_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ for all $j \in \mathbb{N}$ but $\Theta(g)^{-1}a_0^2 g \notin \overline{N}_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$. In particular we find elements $m_j \in N_{\mathbb{C}}$ and $b_j \in A_{\mathbb{C}}$ with

$$\Theta(g)^{-1}a_j^2 g = \Theta(m_j)^{-1}b_j m_j.$$

To arrive at a contradiction we have to show that $(b_j)_{j\in\mathbb{N}}$ is bounded (cf. Lemma 2.4). Let $\langle\cdot,\cdot\rangle$ denote an hermitian inner product on V_n with $\langle\pi_n(g)v, w\rangle = \langle v, \pi_n(\Theta(\overline{g})^{-1})w\rangle$ for all $g \in G_{\mathbb{C}},\ v, w \in V$. Let $Q \subseteq V_n\backslash\{0\}$ be a compact subset. Then the definition of $\mathfrak{b}(\pi_n)^0$ shows that

$$\inf_{v\in Q} \mathrm{Re}\langle\pi_n(\Theta(g)^{-1}a_j^2 g)v, v\rangle > 0. \tag{2.7}$$

If $v = \frac{\pi_n(m_j)^{-1}v_\alpha}{\|\pi_n(m_j)^{-1}v_\alpha\|}$ for a normalized weight vector v_α with weight α, we get

$$\frac{b_j^\alpha\langle v_\alpha, \pi_n(\overline{m_j}m_j^{-1})v_\alpha\rangle}{\|\pi_n(m_j^{-1})v_\alpha\|^2} = \langle\pi_n(\Theta(m_j)^{-1}b_j m_j)v, v\rangle = \langle\pi_n(\Theta(g)^{-1}a_j^2 g)v, v\rangle$$

for all $j \in \mathbb{N}$. In particular, (2.7) implies that there are constants $C_1, C_2 > 0$ such that

$$(\forall\alpha\in\mathcal{P}_n) \qquad C_1 > \frac{|b_j^\alpha|\cdot|\langle v_\alpha, \pi_n(\overline{m_j}m_j^{-1})v_\alpha\rangle|}{\|\pi_n(m_j^{-1})v_\alpha\|^2} > C_2.$$

Recall that the weight spaces of the spin-representation (π_n, V_n) are one-dimensional. Hence it follows that $\langle\pi(n)v_\alpha, v_\alpha\rangle = \langle v_\alpha, v_\alpha\rangle$ for all $n \in N_{\mathbb{C}}$ and all weight vectors $v_\alpha \in V_n$. For the same reason we get $\|\pi_n(m_j)(v_\alpha)\|^2 \geq 1$ for all m_j. In particular we obtain that

$$(\forall\alpha\in\mathcal{P}_n) \qquad |b_j^\alpha| > C \tag{2.8}$$

for some constant $C > 0$. Now we have $\mathcal{P}_n = -\mathcal{P}_n$ and so (2.8) actually implies that $(b_j)_{j\in\mathbb{N}}$ is bounded. □

It seems reasonable to expect that a better technique would show Theorem 1.8 to be valid also for the exceptional groups. Thus we formulate

Conjecture A. Let G be a semisimple Lie group with $G \subseteq G_{\mathbb{C}}$ and $G_{\mathbb{C}}$ simply connected. Then

$$B_{\mathbb{C}}^1 G \subseteq K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}.$$ □

Remark 2.10. In [KrStII] we clarify the geometry of the domain, thereby giving more evidence for its naturality. We show that the domain $K_{\mathbb{C}}\backslash K_{\mathbb{C}}B^1_{\mathbb{C}}G$ is bi-holomorphic to a maximal Grauert tube of $K\backslash G$ having complex structure the adapted one. We also show the existence of a subdomain of $K_{\mathbb{C}}\backslash K_{\mathbb{C}}B^1_{\mathbb{C}}G$ bi-holomorphic to a Hermitian symmetric space but not isometric.

3. Holomorphic extension of irreducible representations

We now come to our first application of the preceding construction, the holomorphic extension of representations. Additional applications of this will be given in subsequent sections for specific situations, such as principal series of representations, specific groups, or eigenfunctions on (locally) symmetric spaces. The notation from representation theory needed for this section is standard and may be found explained in, say, [Kn86].

Notation. As per Conjecture A we shall write Ω for $B^1_{\mathbb{C}}$ if G is classical, and $A^1_{\mathbb{C}}$ otherwise.

THEOREM 3.1. *Let G be a linear, simple Lie group and let (π, E) be an irreducible Banach representation of G. Then for any K-finite vector $v \in E_K$, the orbit map*

$$G \to E, \; g \mapsto \pi(g)v$$

extends to a G-equivariant holomorphic map on $G\Omega K_{\mathbb{C}}$.

Proof. Set $V = E_K$, the collection of K-finite vectors of (π, E).

Casselman's subrepresentation theorem (cf. [Wal88, 3.8]) gives the existence of a $(\mathfrak{g}, K)$-embedding of V into a principal series representation

$$V \to (\mathrm{Ind}^G_{P_{\min}}(\sigma \otimes \lambda \otimes \mathbf{1}), \mathcal{H}_{\sigma,\lambda}) \tag{3.1}$$

where $P_{\min} = MAN$ is a minimal parabolic subgroup. In the next section we recall the standard terminology for principal series; in summary we set $\pi_{\sigma,\lambda} = \mathrm{Ind}^G_{P_{\min}}(\sigma \otimes \lambda \otimes \mathbf{1})$; we write $(W_\sigma, \langle\cdot,\cdot\rangle_\sigma)$ for the representation Hilbert space of σ; we realize $\mathcal{H}_{\sigma,\lambda}$ as a Hilbert subspace of $L^2(K/M, W_\sigma)$, and we use induction from the right.

Write $\mathcal{H}$ for the completion of V in $\mathcal{H}_{\sigma,\lambda}$. Let us first assume that $E = \mathcal{H}$. Fix $v \in V$ and write $f_v\colon G \to \mathcal{H} \subseteq \mathcal{H}_{\sigma,\lambda}, \; g \mapsto \pi_{\sigma,\lambda}(g)v$ for the corresponding orbit map. Then we have for all $g \in G$ that

$$(f_v(g))(kM) = a(g^{-1}k)^{\lambda-\rho}v(\kappa(g^{-1}k)) \qquad (k \in K). \tag{3.2}$$

Hence it follows from either Theorem 1.8 (for $\Omega = B^1_{\mathbb{C}}$) or Proposition 1.7 (for $\Omega = A^1_{\mathbb{C}}$) that analytic continuation of (3.2) gives rise to a map

$$\widetilde{f}_v\colon G\Omega K_{\mathbb{C}} \to C^\infty(K/M, W_\sigma), \quad \Big(g \mapsto (kM \mapsto a(g^{-1}k)^{\lambda-\rho}v(\kappa(g^{-1}k)))\Big). \tag{3.3}$$

Note that $\widetilde{f}_v|_G = f_v$.

We claim that $\operatorname{im} \widetilde{f}_v \subseteq \mathcal{H}$. Write $\mathcal{H}^\perp$ for the orthogonal complement of $\mathcal{H}$ in the Hilbert space $L^2(K/M, W_\sigma)$. Choose $w \in \mathcal{H}^\perp$. In order to show that $\langle w, \operatorname{im} \widetilde{f}_v\rangle = \{0\}$, we may assume that w is a K-finite, continuous function on K/M. Consider the function

$$F\colon G\Omega K_{\mathbb{C}} \to \mathbb{C}, \quad g \mapsto \langle \widetilde{f}_v(g), w\rangle = \int_K a(g^{-1}k)^{\lambda-\rho}\langle v(\kappa(g^{-1}k)), w(k)\rangle_\sigma \; dk,$$

with the equality on the right-hand side following from (3.3). Since w is a bounded function, it is easy to see that F is holomorphic. Since $F|_G = 0$ and F is holomorphic we have $F = 0$. This concludes the proof of the claim.

Next we show that $\widetilde{f}_v$ is holomorphic. Since $V \subseteq C^\infty(K/M, W_\sigma)$ is dense in $\mathcal{H}$ and because weak holomorphicity implies holomorphicity, it is enough to show that for all $w \in V$ the analytically continued matrix coefficients

$$\pi_{v,w}\colon G\Omega K_{\mathbb{C}} \to \mathbb{C}, \quad g \mapsto \langle \widetilde{f}_v(g), w\rangle$$

are holomorphic. Again (3.3) gives that

$$\pi_{v,w}(g) = \int_K a(g^{-1}k)^{\lambda-\rho}\langle v(\kappa(g^{-1}k)), w(k)\rangle_\sigma \; dk \qquad (g \in G\Omega K_{\mathbb{C}})$$

and the holomorphicity of $\widetilde{f}_v$ follows. Before we can deduce the general case from the case $E = \mathcal{H}$ we need a little more refined information on the orbit maps. Note that $\operatorname{im} \widetilde{f}_v \subseteq \mathcal{H}^\infty$, $\mathcal{H}^\infty$ the G-module of smooth vectors (indeed $\operatorname{im} \widetilde{f}_v \subseteq \mathcal{H}^\omega$, $\mathcal{H}^\omega$ the analytic vectors). Thus $\widetilde{f}_v$ also induces a map $\widehat{f}_v\colon G\Omega K_{\mathbb{C}} \to \mathcal{H}^\infty$. Recall that the (Fréchet) topology on $\mathcal{H}^\infty$ is induced from the seminorms

$$\mathcal{H}^\infty \ni v \mapsto \|d\pi(u)v\| \qquad (u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})).$$

From the explicit formula (3.2) of the induced action, one then deduces that $\widehat{f}_v$ is continuous. In particular $\widehat{f}_v$ is holomorphic, since it is continuous and since for all w in the dense subspace $V \subseteq (\mathcal{H}^\infty)'$ the function $\langle \widehat{f}_v, w\rangle = \pi_{v,w}$ is holomorphic.

Finally we have to show how the general case follows from the case where $E = \mathcal{H}$. We use the Casselman-Wallach globalization theorem (cf. [Wal92, 11.6.7(2)]) which implies that the embedding (3.1) extends to a G-equivariant topological embedding on the level of smooth vectors:

$$(\pi, E^\infty) \to (\pi_{\sigma,\lambda}, \mathcal{H}^\infty_{\sigma,\lambda}).$$

Hence the Fréchet representations (π, E^∞) and $(\pi_{\sigma,\lambda}, \mathcal{H}^\infty)$ are equivalent. As $\widehat{f}_v$ was shown to be holomorphic for every $v \in V$, the proof of Theorem 3.1 is now complete. □

The holomorphic extension of the orbit map, $g \mapsto \pi(g)v$, raises the question of the dependence of $\|\pi(g)v\|$ on g. This we address in a subsequent section. The holomorphic extension of a representation also gives rise to a holomorphic extension of its K-finite matrix coefficients. In the next section, we obtain estimates for the holomorphically extended matrix coefficients.

4. Principal series representations

Integral formulas. We shall look in more detail at the growth properties of the holomorphic extension of matrix coefficients of principal series representations induced off a minimal parabolic subgroup. For now, we shall focus on the case of spherical principal series for two reasons: we shall use these results to obtain estimates on automorphic functions for locally symmetric spaces; the extension to the general case requires considering Eisenstein integrals and, albeit with many technicalities, given the holomorphic properties of the decompositions in Theorem 1.8, this presents no fundamentally new difficulties.

Set $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha \in \mathfrak{a}^*$ with $m_\alpha = \dim \mathfrak{g}^\alpha$. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we define a vector space

$$\mathcal{D}_\lambda = \{f \in C^\infty(G)\colon (\forall man \in MAN)(\forall g \in G)\ f(gman) = a^{\lambda-\rho} f(g)\}.$$

The group G acts on $\mathcal{D}_\lambda$ by left translation in the arguments, i.e., we obtain a representation $(\pi_\lambda, \mathcal{D}_\lambda)$ of G given by $(\pi_\lambda(g)f)(x) = f(g^{-1}x)$ for $g, x \in G$, $f \in \mathcal{D}_\lambda$. Besides this realization we shall need the standard realizations of these representations that are called the compact (resp. noncompact) picture. The *compact realization* has for Hilbert space

$$\mathcal{K}_\lambda = \overline{\mathcal{D}_\lambda|_K}^{L^2(K)} \subseteq L^2(K),$$

while the *noncompact realization* has

$$\mathcal{N}_\lambda = \overline{\mathcal{D}_\lambda|_{\overline{N}}}^{L^2(\overline{N},\ a(\overline{n})^{-2\operatorname{Re}(\lambda)}\ d\overline{n})} \subseteq L^2(\overline{N},\ a(\overline{n})^{-2\operatorname{Re}(\lambda)}\ d\overline{n}).$$

The representations $(\pi_\lambda, \mathcal{K}_\lambda)$ and $(\pi_\lambda, \mathcal{N}_\lambda)$ are continuous representations of G. Moreover, the mapping

$$f|_K \to f|_{\overline{N}} \qquad (f \in \mathcal{D}_\lambda)$$

extends to a unitary equivalence $(\pi_\lambda, \mathcal{K}_\lambda) \to (\pi_\lambda, \mathcal{N}_\lambda)$, provided $L^2(K)$ is obtained from a normalized Haar measure on K and $L^2(\overline{N})$ is obtained from a Haar measure $d\overline{n}$ which satisfies $\int_{\overline{N}} a(\overline{n})^{-2\rho}\ d\overline{n} = 1$. For $\lambda \in i\mathfrak{a}^*$ the representations $(\pi_\lambda, \mathcal{K}_\lambda)$ and $(\pi_\lambda, \mathcal{N}_\lambda)$ are unitary. We will write $(\pi_\lambda, \mathcal{H}_\lambda)$ if we do not want to emphasize a particular realization.

We recall that for $(\pi, \mathcal{H})$ a continuous representation of a Lie group G on some Hilbert space $\mathcal{H}$, a vector $v \in \mathcal{H}$ is called *analytic* if the orbit map $f_v\colon G \to \mathcal{H},\ g \mapsto \pi(g)v$ is analytic. Suppose that G is contained in its universal complexification $G_{\mathbb{C}}$, and denote by $g \mapsto \overline{g}$ the complex conjugation in $G_{\mathbb{C}}$ with respect to the real form G. Then for every analytic vector $v \in \mathcal{H}$ there exists a left G-invariant open neighborhood U of $\mathbf{1} \in G_{\mathbb{C}}$ with $U = \overline{U}$ such that f_v extends to a holomorphic map $\widetilde{f}_v: U \to \mathcal{H}, \quad g \mapsto \pi(g)v$. With π^* denoting the contragradient representation one has

$$\langle \pi(g)v, v\rangle = \langle v, \pi(\overline{g})^* v\rangle \tag{4.1}$$

for all $g \in U$.

For G a Lie-group, $K < G$ a compact subgroup, and (π, V) a continuous representation of G on some topological vector space V, the representation (π, V) is called *K-spherical* if $V^K \neq \{0\}$, $V^K = \{v \in V: (\forall k \in K)\pi(k)v = v\}$. For all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the induced representation $(\pi_\lambda, \mathcal{D}_\lambda)$ is K-spherical, $\dim \mathcal{D}_\lambda^K = 1$ and the function

$$f_0: G \to \mathbb{C}, \quad x \mapsto a(x)^{\lambda-\rho}$$

is a generator of $\mathcal{D}_\lambda^K$. Moreover we have

$$(\forall g, x \in G) \qquad (\pi_\lambda(g) f_0)(x) = a(g^{-1}x)^{\lambda-\rho}.$$

In the other realizations one has $v_0 = f_0|_K = \mathbf{1}_K \in \mathcal{K}_\lambda^K$, and $w_0 = f_0|_{\overline{N}} \in \mathcal{N}_\lambda^K$ given by $w_0(\overline{n}) = a(\overline{n})^{\lambda-\rho}$.

PROPOSITION 4.1. *Let $(\pi_\lambda, \mathcal{K}_\lambda)$ be the compact realization of a spherical principal series representation with parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and let $v_0 = \mathbf{1}_K \in \mathcal{K}_\lambda^K$. Then the orbit map*

$$F: G \to \mathcal{K}_\lambda, \quad g \mapsto \pi_\lambda(g)v_0$$

extends to a holomorphic map

$$\widetilde{F}: G\Omega K_{\mathbb{C}} \to \mathcal{K}_\lambda$$

on the open domain $G\Omega K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$.

Remark. Note that a slight modification of Theorem 3.1 to representations of finite length implies the proposition. But we shall give here a more direct proof avoiding the heavy machinery of representation theory.

Proof. We consider the map

$$\Phi: G \times K \to \mathbb{C}, \quad (g, k) \mapsto a(g^{-1}k)^{\lambda-\rho}$$

and note that $\Phi_\lambda(g, \cdot) = F(g)$. By Proposition 1.7 and Theorem 1.8 the function Φ extends to an analytic map

$$\widetilde{\Phi}: G\Omega K_{\mathbb{C}} \times K \to \mathbb{C}, \quad (z, k) \mapsto a(z^{-1}k)^{\lambda-\rho}$$

which is holomorphic in the first argument. It is obvious that $\widetilde{\Phi}(z, \cdot) \in L^2(K)$ for all $z \in G\Omega K_{\mathbb{C}}$. Let $P: L^2(K) \to \mathcal{K}_\lambda$ denote the orthogonal projection and define

$$\widetilde{F}: G\Omega K_{\mathbb{C}} \to \mathcal{K}_\lambda, \quad z \mapsto P(\widetilde{\Phi}(z, \cdot)).$$

Then $\widetilde{F}|_G = F$ and it remains to show that $\widetilde{F}$ is holomorphic. For that however it suffices to show that

$$G\Omega K_{\mathbb{C}} \to \mathbb{C}, \quad z \mapsto \langle \widetilde{F}(z), f\rangle$$

is holomorphic for all $f \in \mathcal{D}_\lambda|_K \subseteq C^\infty(K)$. But this in turn follows from

$$\langle \widetilde{F}(z), f\rangle = \int_K a(z^{-1}k)^{\lambda-\rho}\overline{f(k)}\ dk$$

by the compactness of K, the continuity of $\widetilde{\Phi}$ and the holomorphy of $\widetilde{\Phi}(\cdot, k)$. $\square$

If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $(\pi_\lambda, \mathcal{K}_\lambda)$ is the induced representation realized in the compact picture, then the matrix coefficient of the K-fixed vector with itself is the familiar zonal spherical function,

$$\varphi_\lambda(g) = \langle \pi_\lambda(g^{-1})v_0, v_0\rangle. \tag{4.2}$$

The holomorphic extension to $G\Omega K_{\mathbb{C}}$ of $\pi_\lambda(g^{-1})v_0$ gives a holomorphic extension of the matrix coefficient $\varphi_\lambda(g)$. However, this is not the largest domain of analyticity for $\varphi_\lambda(g)$. Since we will estimate the norm of $\pi_\lambda(g^{-1})v_0$ by means of $\varphi_\lambda(g)$, in order to obtain optimal estimates on the norm it will be important to have an expression that represents $\varphi_\lambda(g)$ in its entire domain of holomorphy. In terms of the pairing in the compact realization, $\varphi_\lambda(g)$ is given by the well-known integral formula

$$\varphi_\lambda(g) = \int_K a(gk)^{\lambda-\rho}\ dk.$$

By the K-bi-invariance of φ_λ and in light of Proposition 1.7, this defining integral formula for the spherical function can be extended to $K_{\mathbb{C}}\Omega K_{\mathbb{C}}$. But in general the integral formula need not extend to any larger domain (cf. Example 4.3). There are a couple of reasons for this. First, the integrand $k \mapsto a(a^{-1}k)^{\lambda-\rho}$ becomes singular if a leaves $A_{\mathbb{C}}^1$, and secondly, it is no longer possible to take holomorphic square roots (the ρ-exponent frequently involves a square root). We shall present an alternative integral formula valid on a domain about twice as large and this will be crucial for the estimates on the norm of $\pi_\lambda(g^{-1})v_0$.

To state the result we recall the notation Ω, viz. if G is classical, then $\Omega = B_{\mathbb{C}}^1$ and otherwise $\Omega = A_{\mathbb{C}}^1$. Consistent with this and the notation $B_{\mathbb{C}}^0$ (resp. $A_{\mathbb{C}}^0$), we use $\Omega^2 = B_{\mathbb{C}}^0$ if G is classical and otherwise set $\Omega^2 = A_{\mathbb{C}}^0$.

THEOREM 4.2. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and φ_λ be the spherical function with parameter λ associated to G/K.*

(i) *The spherical function φ_λ extends to a $K_{\mathbb{C}}$-bi-invariant function on $K_{\mathbb{C}}\Omega^2 K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ which is holomorphic when restricted to Ω^2.*

(ii) $(\forall b \in A)(\forall a \in \exp(i\mathfrak{a}) \cap \Omega)$,

$$\varphi_\lambda(ba^2) = \int_K a(bak)^{\lambda-\rho} \cdot \overline{a(ak)^{\lambda-\rho}} \cdot \overline{a(ak)^{-2\operatorname{Re}\lambda}}\, dk.$$

In particular, for $\lambda \in i\mathfrak{a}^*$ *we get for all* $a \in \exp(i\mathfrak{a}) \cap \Omega$

$$\varphi_\lambda(a^2) = \int_K |a(ak)^{2(\lambda-\rho)}|\, dk.$$

(iii) $(\forall b \in A)(\forall a \in \exp(i\mathfrak{a}) \cap \Omega)$,

$$\varphi_\lambda(ba^2) = \int_{\overline{N}} a(ba\overline{n})^{\lambda-\rho} \cdot \overline{a(a\overline{n})^{\lambda-\rho}} \cdot \overline{a(a\overline{n})^{-2\operatorname{Re}\lambda}}\, d\overline{n}.$$

In particular, for $\lambda \in i\mathfrak{a}^*$, *for all* $a \in \exp(i\mathfrak{a}) \cap \Omega$,

$$\varphi_\lambda(a^2) = \int_{\overline{N}} |a(a\overline{n})^{2(\lambda-\rho)}|\, d\overline{n}.$$

Proof. (i) It suffices to show that $\varphi_\lambda|_A$ extends to a holomorphic function on Ω^2. We will work with the compact realization $(\pi_\lambda, \mathcal{K}_\lambda)$. Let $a \in A$. Then (2.1) implies that

$$(\forall a \in A) \qquad \varphi_\lambda(a^2) = \langle \pi_\lambda(a^{-1})v_0, \pi_\lambda(a^{-1})^* v_0\rangle. \tag{4.3}$$

We now analytically continue the right-hand side of (4.3). Recall from [Kn86, p. 170] that for $f \in \mathcal{D}_\lambda$ and $x, g \in G$ one has

$$(\pi_\lambda(g)^* f)(x) = a(gx)^{-2\operatorname{Re}\lambda} f(gx). \tag{4.4}$$

Hence $\pi_\lambda(g)^* f = a(g\cdot)^{-2\operatorname{Re}\lambda}\pi_\lambda(g^{-1})f$ for all $g \in G$. Similarly as in Proposition 4.1 one shows that $g \mapsto \pi(g)^* v_0$ extends to a holomorphic $\mathcal{K}_\lambda$-valued map on $G\Omega K_\mathbb{C}$. Thus Proposition 4.1 implies that the function

$$A \to \mathbb{C}, \quad a \mapsto \langle \pi_\lambda(a^{-1})v_0, \pi_\lambda(a^{-1})^* v_0\rangle$$

extends to a holomorphic function on Ω. Since we have a unique holomorphic square root on Ω^2, namely

$$\Omega^2 \to \Omega,\ a = \exp(X) \mapsto \sqrt{a} = \exp(\frac{1}{2}X),$$

the assertion of (i) now follows from (4.3).

(ii) In view of the proof of (i), (ii) is immediate from the analytic extensions of (4.3) and (4.4) to $\exp(i\mathfrak{a}) \cap \Omega$.

(iii) This is proved as (ii) is by use of the noncompact realization $(\pi_\lambda, \mathcal{N}_\lambda)$ instead of $(\pi_\lambda, \mathcal{K}_\lambda)$. ∎

Example 4.3. We explicate the theorem for the group $G = \mathrm{Sl}(2,\mathbb{R})$. Clearly $G \subseteq G_{\mathbb{C}} = \mathrm{Sl}(2,\mathbb{C})$ and $G_{\mathbb{C}}$ is simply connected. We let $\mathfrak{k} = \mathfrak{so}(2)$,

$$\mathfrak{a} = \{\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} : x \in \mathbb{R}\} \quad \text{and} \quad \mathfrak{n} = \{\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} : n \in \mathbb{R}\}.$$

For $z \in \mathbb{C}^*$, $x \in \mathbb{C}$ and $\theta \in \mathbb{C}$ we set

$$a_z = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \in A_{\mathbb{C}}, \quad n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N_{\mathbb{C}}$$

and

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_{\mathbb{C}}.$$

Then

$$A^0_{\mathbb{C}} = B^0_{\mathbb{C}} = \{a_z : \mathrm{Re}(z) > 0\} \quad \text{and} \quad A^1_{\mathbb{C}} = \{a_z : |\arg(z)| < \frac{\pi}{4}\}.$$

Since $\omega = \omega_1 = \rho$, we may identify $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}$ by means of the isomorphism

$$\mathbb{R} \to \mathfrak{a}^*, \quad \lambda \mapsto \lambda\omega.$$

Let us consider the spherical function with parameter $\lambda \in i\mathfrak{a}^*$. Then Proposition A.1(i) in the appendix shows that (4.2), the defining integral formula for φ_λ, extends to $A^1_{\mathbb{C}}$ and we have

$$(\forall a_z \in A^1_{\mathbb{C}}) \qquad \varphi_\lambda(a_z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(z^2 + \sin^2\theta(\frac{1}{z^2} - z^2))^{\frac{1}{2}-\lambda}}. \tag{4.5}$$

It is easy to see that it is no longer possible to take consistently an analytic square root of $(z,\theta) \mapsto z^2 + \sin^2\theta(\frac{1}{z^2} - z^2)$ if $|\arg(z)|$ becomes larger than $\frac{\pi}{4}$. Also as this function has zeros, the integrand of the integral expression above becomes singular (although in this case the singularity is integrable). On the other hand Theorem 4.2(ii), (iii), together with Proposition A.1 imply, for all $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, $r > 0$, that

$$\varphi_\lambda(a_{re^{i\varphi}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{(r^2e^{i\varphi} + \sin^2\theta(\frac{1}{r^2}e^{-i\varphi} - r^2e^{i\varphi}))^{\frac{1}{2}-\lambda}(e^{-i\varphi} + \sin^2\theta(e^{i\varphi} - e^{-i\varphi}))^{\frac{1}{2}+\lambda}}$$

and

$$\varphi_\lambda(a_{re^{i\varphi}}) = \int_{-\infty}^{\infty} \frac{dx}{(r^2e^{i\varphi} + \frac{1}{r^2}e^{-i\varphi}x^2)^{\frac{1}{2}-\lambda}(e^{-i\varphi} + e^{i\varphi}x^2)^{\frac{1}{2}+\lambda}}.$$

If one examines the integral over K, one sees that the second factor is identically 1 when evaluated on the real group A, so that it comes into play only on the complex domain.

An upper estimate. We can give a soft upper estimate along the convex hull of extreme points of the domain.

Proposition 4.4. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary principal series representation of G. Let ω be $\mathfrak{b}^0$ if G is classical and $\mathfrak{a}^0$ otherwise. If $a, b \in \omega$, then*

$$\sup_{0\le t\le 1} \varphi_\lambda\Big(\exp(i(ta+(1-t)b))\Big) \le \max\{\varphi_\lambda(\exp(ia)), \varphi_\lambda(\exp(ib))\}.$$

Proof. Set $S_{[0,1]} = \{z \in \mathbb{C}\colon \operatorname{Re} z \in [0,1]\}$ and $X = \{za+(1-z)b\colon z \in S_{[0,1]}\}$. Then

$$S_{[0,1]} \to X, \quad z \mapsto g(z) = za + (1-z)b$$

defines a bi-holomorphism of complex manifolds with boundary. We set

$$f\colon S_{[0,1]} \to \mathbb{C}, \quad z \mapsto \varphi_\lambda(\exp(ig(z))).$$

Then f is holomorphic on $\operatorname{int} S_{[0,1]}$ and we claim that f is bounded. In fact, we have

$$f(z) = \langle \pi_\lambda(\exp(\operatorname{Im} g(z)))\pi_\lambda(\exp(-i\frac{1}{2}\operatorname{Re} g(z)))v_0, \pi_\lambda(\exp(-i\frac{1}{2}\operatorname{Re} g(z)))v_0\rangle,$$

and so by the unitarity of π_λ

$$|f(z)| \le \langle \pi_\lambda(\exp(-i\frac{1}{2}\operatorname{Re} g(z)))v_0, \pi_\lambda(\exp(-i\frac{1}{2}\operatorname{Re} g(z)))v_0\rangle = f(\operatorname{Re} z).$$

This implies our claim and so the assertion of the proposition follows from the Phragmen-Lindelöf principle. □

A radial lower estimate. A precise estimate of the nature of the singularity along the entire boundary of Ω appears difficult. However, for an approach to the boundary along the direction of roots (or co-roots), we can obtain estimates. In this regard see Remark 5.5. We recall our standing hypothesis that G is a semisimple Lie group contained in its complexification $G_{\mathbb{C}}$.

Let $H \in \mathfrak{a}$ and assume that there is a θ-invariant $\mathfrak{sl}(2,\mathbb{R})$-triple $\{H, X, \theta(X)\} \subseteq \mathfrak{g}$, i.e., $[H, X] = 2X$, $[H, \theta(X)] = -2\theta(X)$, $[X, \theta(X)] = H$.

We shall want to give estimates for the radial behaviour

$$\varphi_\lambda(\exp(i\frac{\pi}{2}(1-\varepsilon)H))$$

for $\varepsilon \to 0$ provided that $\frac{\pi}{2}H \in \partial\mathfrak{b}^0$ and $(\pi_\lambda, \mathcal{H}_\lambda)$ is unitarizable. We do this by restriction of the representation to a subgroup isomorphic to $\mathrm{Sl}(2,\mathbb{R})$ or $\mathrm{PSl}(2,\mathbb{R})$. The triple gives

$$\mathfrak{g}_0 = \operatorname{span}_{\mathbb{R}}\{H, X, \theta(X)\},$$

a θ-stable subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Its Cartan decomposition is given by $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ with $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$ and $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$. Further, we set $\mathfrak{a}_0 = \mathbb{R}H$. We denote the analytic subgroups of G corresponding to $\mathfrak{g}_0$, $\mathfrak{a}_0$ and $\mathfrak{k}_0$ by G_0, A_0 and K_0.

We write $\widehat{G}_0$ for the set of equivalence classes of unitary irreducible representations of G_0 (the *unitary dual of* G_0). Since G_0 is semisimple and thus of type I, there is a natural T_1-topology on $\widehat{G}_0$, the *hull-kernel topology*, which we denote by $\tau_{\rm hk}$, and a Borel measure μ on $\widehat{G}_0$ such that

$$(\pi_\lambda|_{G_0}, \mathcal{H}_\lambda) = \left(\int_{\widehat{G}_0}^{\oplus} \pi_\sigma \otimes I \; d\mu(\sigma), \int_{\widehat{G}_0}^{\oplus} \mathcal{H}_\sigma \widehat{\otimes} V_\sigma \; d\mu(\sigma) \right).$$

Here $(\pi_\sigma, \mathcal{H}_\sigma)$ denotes a representative of $\sigma \in \widehat{G}_0$. In particular, $v_0 \in \mathcal{H}_\lambda^K$ disintegrates as

$$v_0 = \int_{\widehat{G}_0} v_0^\sigma \; d\mu(\sigma)$$

with $1 = \|v_0\|^2 = \int_{\widehat{G}_0} \|v_0^\sigma\|^2 \; d\mu(\sigma)$. As each v_0^σ is K_0-fixed, for all σ with $v_0^\sigma \neq 0$,

$$(\forall g \in G_0) \qquad \varphi_\sigma^0(g) = \frac{1}{\|v_0^\sigma\|^2} \langle \pi_\sigma(g^{-1}) v_0^\sigma, v_0^\sigma \rangle$$

defines a spherical function on G_0.

In particular we get for all $a \in A_0$

(4.6)
$$\varphi_\lambda(a) = \langle \pi_\lambda(a^{-1}) v_0, v_0 \rangle = \int_{\widehat{G}_0} \langle \pi_\sigma(a^{-1}) v_0^\sigma, v_0^\sigma \rangle \; d\mu(\sigma) = \int_{\widehat{G}_0} \varphi_\sigma^\alpha(a) \|v_0^\sigma\|^2 d\mu(\sigma).$$

PROPOSITION 4.5. *Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$ and assume that $G \subseteq G_\mathbb{C}$. Suppose that $\{H, X, \theta(X)\}$ with $H \in \mathfrak{a}$ forms an $\mathfrak{sl}(2,\mathbb{R})$-triple in $\mathfrak{g}$. Assume that $\frac{\pi}{2} H \in \partial \mathfrak{b}^0$ and that $(\pi_\lambda, \mathcal{H}_\lambda)$ is unitarizable. Then we have*

$$\varphi_\lambda(\exp(i\frac{\pi}{2}(1-\varepsilon)H)) \geq C|\log \varepsilon|$$

for $0 < \varepsilon \leq 1$ and a constant $C > 0$.

Proof. Write $Y_0 = \{\sigma \in \widehat{G}_0 \colon \pi_\sigma \text{ is } K\text{-spherical}\}$. From the well-known details of the unitary dual of G_0 we know that there is a natural parametrization $i\mathbb{R} \cup]0,1[\to Y_0$. Moreover if we equip $i\mathbb{R} \cup]0,1[$ with its Euclidean topology, then this parametrization becomes continuous (this essentially follows from the fact that the assignments $\sigma \mapsto \varphi_\sigma^0(a)$ are continuous with respect to the Euclidean topology; cf. the technique of [Wal92, 14.12.3]). In particular this parametrization induces the (possibly) stronger Euclidean topology τ_e on Y_0. Hence if $\bigcup_{n\in\mathbb{N}} Q_n$ is an exhaustion of (Y_0, τ_e) by compact sets, then $\bigcup_{n\in\mathbb{N}} Q_n$ defines an exhaustion of quasicompact Borel sets of $(Y_0, \tau_{\rm hk})$. So we can find a Q_n with $\int_{Q_n} v_0^\sigma \; d\mu(\sigma) \neq 0$. It follows from (4.6) that we have

$$\varphi_\lambda(\exp(i\frac{\pi}{2}(1-\varepsilon)H)) = \int_{Y_0} \varphi_\sigma^0(\exp(i\frac{\pi}{2}(1-\varepsilon)H)) \|v_0^\sigma\|^2 \; d\mu(\sigma)$$

for $\varepsilon > 0$. In particular we get that

$$\varphi_\lambda(\exp(i\frac{\pi}{2}(1-\varepsilon)H)) \geq \int_{Q_n} \varphi_\sigma^0(\exp(i\frac{\pi}{2}(1-\varepsilon)H))\|v_0^\sigma\|^2 \, d\mu(\sigma)$$

for $\varepsilon > 0$. Since Q_n is compact, Theorem 5.1 (to follow) implies that there is a constant $C \geq 0$ such that $\varphi_\sigma^0(\exp(i\frac{\pi}{2}(1-\varepsilon)H)) \geq C|\log\varepsilon|$ for all $\sigma \in Q_n$. Hence we get that

$$\varphi_\lambda(\exp(i\frac{\pi}{2}(1-\varepsilon)H)) \geq C|\log\varepsilon| \int_{Q_n} \|v_0^\sigma\|^2 \, d\mu(\sigma),$$

proving the theorem. □

Set $\Sigma_0 = \{\alpha \in \Sigma : 2\alpha \notin \Sigma\}$. Let $\alpha \in \Sigma_0$ and $H_\alpha \in \mathfrak{a}$ be the corresponding *co-root*, i.e., $H_\alpha \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \cap \mathfrak{a}$ such that $\alpha(H_\alpha) = 2$.

Corollary 4.6. *Suppose that G is one of the groups* $\mathrm{Sl}(n,\mathbb{R})$, $\mathrm{Sl}(n,\mathbb{C})$, $\mathrm{Sl}(n,\mathbb{H})$, $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{SO}^*(2n)$ *or* $\mathrm{SU}(p,q)$. *Let $\alpha \in \Sigma_0$ and H_α be its co-root. Assume that $(\pi_\lambda, \mathcal{H}_\lambda)$ is unitarizable. Then there exists a constant C, depending only on λ, such that*

$$\varphi_\lambda(\exp(i\frac{\pi}{2}(1-\varepsilon)H_\alpha)) \geq C|\log\varepsilon|$$

for $0 < \varepsilon \leq 1$.

Proof. Since all restricted root systems are either of type A_n, C_n or BC_n we have $\frac{\pi}{2}H_\alpha \in \partial\mathfrak{b}^0$. Then the assertion follows from Proposition 4.5. □

5. Real rank one

Singularity of spherical functions. We consider Lie algebras of real rank one. For these we are able to obtain sharp asymptotic behaviour of the holomorphically extended spherical functions.

As $\mathfrak{g}$ has real rank one, $\dim\mathfrak{a} = 1$. As is the custom, we set $p = \dim\mathfrak{g}^\alpha$, $q = \dim\mathfrak{g}^{2\alpha}$, and $c = \frac{1}{4(p+2q)}$. Here $\overline{\mathfrak{n}} = \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^{-2\alpha}$ and $[\mathfrak{g}^{-\alpha}, \mathfrak{g}^{-2\alpha}] = \{0\}$.

We have the familiar formula for all $\overline{n} = \exp(X+Y) = \exp(X)\exp(Y)$, $X \in \mathfrak{g}^{-\alpha}$, $Y \in \mathfrak{g}^{-2\alpha}$:

$$a(\overline{n})^\rho = [(1+c\|X\|^2)^2 + 4c\|Y\|^2]^{\frac{p+2q}{4}}. \tag{5.1}$$

Here $\|Z\|^2 = -\kappa(Z, \theta Z)$ for all $Z \in \mathfrak{g}$ with κ denoting the Cartan-Killing form of $\mathfrak{g}$. Computations involving $a(\overline{n})^\rho$ have appeared many times. We include the following computations only because they involve the holomorphic extension of $a(\overline{n})^\rho$ and for this we have no convenient reference.

Let $A_\alpha \in \mathfrak{a}$ be defined by $\alpha(A_\alpha) = 1$. For convenience we shall identify $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}$ by means of the isomorphisms

$$\mathbb{C} \to \mathfrak{a}_\mathbb{C}, \qquad z \mapsto zA_\alpha,$$
$$\mathbb{C} \to \mathfrak{a}_\mathbb{C}^*, \qquad \lambda \mapsto \lambda\alpha.$$

Thus, $\varphi_\lambda(e^z) := \varphi_\lambda(\exp zA_\alpha)$.

Let $\Omega_\mathfrak{g} \subseteq \mathfrak{a}$ denote $\mathfrak{b}^0$ if $\mathfrak{g}$ is classical and $\mathfrak{a}^0$ otherwise.

Here and henceforth we use the notation $f(\varepsilon) \asymp g(\varepsilon)$ for two positive valued functions $f(\varepsilon)$, $g(\varepsilon)$ if there exist constants $c_1, c_2 > 0$ such that $c_1 f(\varepsilon) \leq g(\varepsilon) \leq c_2 f(\varepsilon)$ for all ε.

THEOREM 5.1. *Let G be a connected Lie group of real rank one contained in its universal complexification $G_\mathbb{C}$.*

(i) *For all $\lambda \in i\mathfrak{a}^*$ the maximal tube domain of definition of $\varphi_\lambda \circ \exp_A$ is given by*

$$T_{\lambda,\max} = \mathfrak{b}_\mathbb{C}^0.$$

(ii) *For $X \in \partial\mathfrak{b}^0$, and a fixed $\lambda \in \mathfrak{a}_\mathbb{C}^*$, there exist a $C > 0$ such that for $\varepsilon \to 0$, $\varepsilon > 0$*

$$|\varphi_\lambda(\exp(\pm i(1-\varepsilon)X)| \leq C \begin{cases} |\log\varepsilon| & \text{for } p = 1,\ q = 0, \\ \varepsilon^{-p+1} & \text{for } p > 1,\ q = 0, \\ |\log\varepsilon| & \text{for } q = 1, \\ \varepsilon^{-q+1} & \text{for } q > 1. \end{cases}$$

If in addition $\lambda \in i\mathfrak{a}^*$, then

$$|\varphi_\lambda(\exp(\pm i(1-\varepsilon)X)| \asymp \begin{cases} |\log\varepsilon| & \text{for } p = 1,\ q = 0, \\ \varepsilon^{-p+1} & \text{for } p > 1,\ q = 0, \\ |\log\varepsilon| & \text{for } q = 1, \\ \varepsilon^{-q+1} & \text{for } q > 1. \end{cases}$$

Remark 5.2. For some $\lambda \in \mathfrak{a}_\mathbb{C}^* \backslash i\mathfrak{a}^*$ it can happen that $T_{\lambda,\max} = \mathfrak{a}_\mathbb{C}$; i.e., the spherical function $\varphi_\lambda|_A$ extends holomorphically to $A_\mathbb{C}$. Simply consider $G = \mathrm{Sl}(2,\mathbb{C})$. Then $\mathfrak{g} \cong \mathfrak{so}(3,1)$, which in our previous notation corresponds to the case $p = 2$ and $q = 0$. The explicit formula for spherical functions on complex groups specialized to $\mathrm{Sl}(2,\mathbb{C})$ then reads

$$\varphi_\lambda(e^z) = \frac{1}{\lambda}\frac{e^{(\lambda)z} - e^{-(\lambda)z}}{e^z - e^{-z}}.$$

Hence we see that $T_{\lambda,\max} = \mathfrak{b}_\mathbb{C}^0$ for $\lambda \notin \mathbb{Z}$ while for $\lambda \in \mathbb{Z}$ one has $T_{\lambda,\max} = \mathfrak{a}_\mathbb{C}$.

The proof of Theorem 5.1 is computational and is presented in lemmas for the various cases.

With our parametrization it follows from (5.1) and Theorem 4.2(iii) that for all $\lambda \in i\mathfrak{a}^*$, $\varphi \in \Omega_{\mathfrak{g}}$,

$$\varphi_\lambda(e^{-i\varphi}) = e^{-i\lambda\varphi} \times \int_{\mathbb{R}^p}\int_{\mathbb{R}^q} \frac{dX\ dY}{[(1+ce^{i\varphi}\|X\|^2)^2+4ce^{2i\varphi}\|Y\|^2]^{\frac{p+2q}{4}-\frac{\lambda}{2}}[(1+ce^{-i\varphi}\|X\|^2)^2+4ce^{-2i\varphi}\|Y\|^2]^{\frac{p+2q}{4}+\frac{\lambda}{2}}}.$$

Using polar coordinates we thus obtain

$$\varphi_\lambda(e^{-i\varphi}) = \widetilde{c}e^{-i\lambda\varphi} \times \int_0^\infty\int_0^\infty \frac{u^{p-1}v^{q-1}du\ dv}{[(1+e^{i\varphi}u^2)^2+e^{2i\varphi}v^2]^{\frac{p+2q}{4}-\frac{\lambda}{2}}[(1+e^{-i\varphi}u^2)^2+e^{-2i\varphi}v^2]^{\frac{p+2q}{4}+\frac{\lambda}{2}}}$$

for a constant $\widetilde{c}$ depending on only p and q. Finally with the substitution $r = u^2$, $s = v^2$ we arrive at

$$\text{(5.2)}\quad \varphi_\lambda(e^{-i\varphi}) = Ce^{-i\lambda\varphi} \times \int_0^\infty\int_0^\infty \frac{r^{\frac{p-2}{2}}s^{\frac{q-2}{2}}dr\ ds}{[(1+e^{i\varphi}r)^2+e^{2i\varphi}s]^{\frac{p+2q}{4}-\frac{\lambda}{2}}[(1+e^{-i\varphi}r)^2+e^{-2i\varphi}s]^{\frac{p+2q}{4}+\frac{\lambda}{2}}}$$

for all $\lambda \in i\mathfrak{a}^*$, $\varphi \in \Omega_{\mathfrak{g}}$ and a constant C which is independent of λ. We distinguish three cases.

Case 1: $p = 1$, $q = 0$. In this case we have $G = \mathrm{Sl}(2,\mathbb{R})$, the root system is split (i.e. $\Delta = \Sigma$), and so $\omega = \omega_1 = \frac{1}{2}\alpha$. Hence

$$\mathfrak{a}^0_{\mathbb{C}} = \mathfrak{b}^0_{\mathbb{C}} = \{z \in \mathbb{C}: |\operatorname{Im} z| < \pi\}$$

and (5.2) boils down to $(\forall\lambda \in i\mathbb{R})(\forall -\pi < \varphi < \pi)$,

$$\text{(5.3)}\qquad \varphi_\lambda(e^{-i\varphi}) = Ce^{-i\lambda\varphi}\int_0^\infty \frac{dr}{\sqrt{r}(1+e^{i\varphi}r)^{\frac{1}{2}-\lambda}(1+e^{-i\varphi}r)^{\frac{1}{2}+\lambda}}.$$

LEMMA 5.3. *For $p = 1$, $q = 0$ and $\lambda \in i\mathfrak{a}^*$, $T_{\lambda,\max} = \mathfrak{b}^0_{\mathbb{C}} = \mathfrak{a}^0_{\mathbb{C}}$. Moreover, for a fixed λ the asymptotics at the boundary are given by*

$$\varphi_\lambda(e^{-i(\pi-\varepsilon)}) \asymp |\log\varepsilon|$$

for $\varepsilon \to 0$, $\varepsilon > 0$.

Proof. This is immediate from (5.3). □

Case 2: $p > 1$, $q = 0$. Here $\mathfrak{g} = \mathfrak{so}(p+1,1)$ is a classical Lie algebra and so $\Omega_{\mathfrak{g}} = \mathfrak{b}^0$. We have $\omega = \omega_1 = \alpha$ and so $\mathfrak{a}^0_{\mathbb{C}} = \{z \in \mathbb{C}: |\operatorname{Im} z| < \frac{\pi}{2}\}$ and $\mathfrak{b}^0_{\mathbb{C}} = 2\mathfrak{a}^0_{\mathbb{C}}$. Formula (5.2) simplifies to

$$\text{(5.4)}\qquad \varphi_\lambda(e^{-i\varphi}) = e^{-i\lambda\varphi}\int_0^\infty \frac{r^{\frac{p-2}{2}}dr}{(1+e^{i\varphi}r)^{\frac{p}{2}-\lambda}(1+e^{-i\varphi}r)^{\frac{p}{2}+\lambda}}$$

for all $\lambda \in i\mathbb{R}$ and $-\pi < \varphi < \pi$.

LEMMA 5.4. *For $p > 1$, $q = 0$ and for all $\lambda \in i\mathfrak{a}^*$, $T_{\lambda,\max} = \mathfrak{b}^0_{\mathbb{C}} = 2\mathfrak{a}^0_{\mathbb{C}} = \{z \in \mathbb{C}\colon |\operatorname{Im} z| < \pi\}$. Moreover, for a fixed λ the asymptotics at the boundary are given by*

$$\varphi_\lambda(e^{-i(\pi-\varepsilon)}) \asymp \frac{1}{\varepsilon^{p-1}}$$

for $\varepsilon \to 0$, $\varepsilon > 0$.

Proof. We will estimate $\varphi_\lambda(e^{-i(\pi-\varepsilon)})$ for $\varepsilon \to 0$ $(\varepsilon > 0)$. Since the unbounded contribution to the integral is local (at $r = 1$) we may henceforth assume that $\lambda = 0$. Then (5.4) gives

$$\begin{aligned}
\varphi_0(e^{-i(\pi-\varepsilon)}) &= \int_0^\infty \frac{r^{\frac{p-2}{2}}\,dr}{|(1+e^{i(\pi-\varepsilon)}r)|^p} \\
&\asymp \int_0^2 \frac{r^{\frac{p-2}{2}}\,dr}{|1+(-1+i\varepsilon)r|^p} \\
&\asymp \int_0^2 \frac{r^{\frac{p-2}{2}}\,dr}{(|1-r|+r\varepsilon)^p} \\
&\asymp \int_{-1}^1 \frac{(r+1)^{\frac{p-2}{2}}\,dr}{(|r|+(r+1)\varepsilon)^p} \\
&\asymp \int_{\frac{-1}{2}}^{\frac{1}{2}} \frac{dr}{(|r|+\varepsilon)^p} \\
&\asymp \varepsilon^{-(p-1)}.
\end{aligned}$$

In the calculation above we used the first order approximation $e^{i(\pi-\varepsilon)} \approx -1+i\varepsilon$ for $\varepsilon > 0$, $\varepsilon \to 0$ which, as one easily convinces oneself, is justified. □

Case 3: $p > 1$, $q > 0$. In this case we have $\omega = \omega_1 = 2\alpha$ and so $\mathfrak{a}^0_{\mathbb{C}} = \{z \in \mathbb{C}\colon |\operatorname{Im} z| < \frac{\pi}{4}\}$. Formula (5.2) and Theorem 4.2(iii) then imply for all $\lambda \in i\mathbb{R}$, $-\frac{\pi}{4} < \varphi < \frac{\pi}{4}$ and $t > 0$ that

$$(5.5)\ \varphi_\lambda(t^{-1}e^{-i\varphi}) = Ct^{\frac{p+2q}{2}}e^{-i\lambda\varphi} \times \int_0^\infty\int_0^\infty \frac{r^{\frac{p-2}{2}}s^{\frac{q-2}{2}}\,dr\ ds}{[(1+t^2e^{i\varphi}r)^2+t^4e^{2i\varphi}s]^{\frac{p+2q}{4}-\frac{\lambda}{2}}[(1+e^{-i\varphi}r)^2+e^{-2i\varphi}s]^{\frac{p+2q}{4}+\frac{\lambda}{2}}}.$$

In particular (5.5) implies that $\varphi_\lambda \circ \log_A$ extends to a holomorphic function on $\mathfrak{b}^0_{\mathbb{C}} = 2\mathfrak{a}^0_{\mathbb{C}}$. Again, this turns out to be the maximal domain, as we will show below.

LEMMA 5.5. *For $p > 1$, $q > 0$ and for all $\lambda \in i\mathfrak{a}^*$, $T_{\lambda,\max} = \mathfrak{b}^0_{\mathbb{C}} = 2\mathfrak{a}^0_{\mathbb{C}} = \{z \in \mathbb{C}\colon |\operatorname{Im} z| < \frac{\pi}{2}\}$. Moreover, for a fixed λ the asymptotics at the boundary* *are given by*

$$\varphi_\lambda(e^{-i(\pi-\varepsilon)}) \asymp \begin{cases} \frac{1}{\varepsilon^{q-1}} & \text{if } q>1, \\ |\log\varepsilon| & \text{if } q=1, \end{cases}$$

for $\varepsilon\to 0$, $\varepsilon>0$.

Proof. We will estimate $\varphi_\lambda(e^{i(\frac{\pi}{2}-\varepsilon)})$ for $\varepsilon\to 0$, $\varepsilon>0$. Since the unbounded contribution of the integral is local (near $r=0$ and $s=1$), we may henceforth assume that $\lambda=0$. Then (5.5) gives that

$$\begin{aligned}
\varphi_0(e^{-i(\frac{\pi}{2}-\varepsilon)}) &\asymp \int_0^\infty\int_0^\infty \frac{r^{\frac{p-2}{2}}s^{\frac{q-2}{2}}\ ds\ dr}{|(1+e^{i(\frac{\pi}{2}-\varepsilon)}r)^2+e^{i(\pi-2\varepsilon)}s|^{\frac{p+2q}{2}}}\\
&\asymp \int_0^\infty\int_0^\infty \frac{r^{\frac{p-2}{2}}s^{\frac{q-2}{2}}\ ds\ dr}{|1+2re^{i(\frac{\pi}{2}-\varepsilon)}+e^{i(\pi-2\varepsilon)}r^2+e^{i(\pi-2\varepsilon)}s|^{\frac{p+2q}{2}}}\\
&\asymp \int_0^{\frac12}\int_0^2 \frac{r^{\frac{p-2}{2}}s^{\frac{q-2}{2}}\ ds\ dr}{|1+2r(i+\varepsilon)+(-1+i2\varepsilon)r^2+(-1+i2\varepsilon)s|^{\frac{p+2q}{2}}}\\
&\asymp \int_0^{\frac12}\int_0^2 \frac{r^{\frac{p-2}{2}}s^{\frac{q-2}{2}}\ ds\ dr}{|(1+2r\varepsilon-r^2-s)+i2(r+\varepsilon(r^2+s))|^{\frac{p+2q}{2}}}\\
&\asymp \int_0^{\frac12}\int_0^2 \frac{r^{\frac{p-2}{2}}s^{\frac{q-2}{2}}\ ds\ dr}{\Big||1+2\varepsilon r-r^2-s|+2|\varepsilon(r^2+s)+r|\Big|^{\frac{p+2q}{2}}}\\
&\asymp \int_0^{\frac12}\int_{-1}^1 \frac{r^{\frac{p-2}{2}}(s+1)^{\frac{q-2}{2}}\ ds\ dr}{\Big(|2\varepsilon r-r^2-s|+2\varepsilon(r^2+s+1)+2r\Big)^{\frac{p+2q}{2}}}\\
&\asymp \int_0^{\frac12}\int_{-1}^1 \frac{r^{\frac{p-2}{2}}\ ds\ dr}{\Big(|2\varepsilon r-r^2-s|+2\varepsilon(r^2+s+1)+2r\Big)^{\frac{p+2q}{2}}}.
\end{aligned}$$

Elimination of the absolute value in the integrand gives

$$\begin{aligned}
\varphi_0(e^{-i(\frac{\pi}{2}-\varepsilon)}) &\asymp \int_0^{\frac12}\int_{-1}^{2\varepsilon r-r^2} \frac{r^{\frac{p-2}{2}}\ ds\ dr}{\Big(2\varepsilon r-r^2-s+2\varepsilon(r^2+s+1)+2r\Big)^{\frac{p+2q}{2}}}\\
&\quad+\int_0^{\frac12}\int_{2\varepsilon r-r^2}^1 \frac{r^{\frac{p-2}{2}}\ ds\ dr}{\Big(s-2\varepsilon r+r^2+2\varepsilon(r^2+s+1)+2r\Big)^{\frac{p+2q}{2}}}\\
&\asymp \int_0^{\frac12}\int_{-1}^{2\varepsilon r-r^2} \frac{r^{\frac{p-2}{2}}\ ds\ dr}{\Big(2\varepsilon r-r^2+2\varepsilon(r^2+1)+2r+s(-1+2\varepsilon)\Big)^{\frac{p+2q}{2}}}\\
&\quad+\int_0^{\frac12}\int_{2\varepsilon r-r^2}^1 \frac{r^{\frac{p-2}{2}}\ ds\ dr}{\Big(-2\varepsilon r+r^2+2\varepsilon(r^2+1)+2r+s(1+2\varepsilon)\Big)^{\frac{p+2q}{2}}}
\end{aligned}$$

$$
\begin{aligned}
&\asymp \int_0^{\frac{1}{2}} \frac{r^{\frac{p-2}{2}}\, dr}{(2\varepsilon + 2r + 4\varepsilon r^2)^{\frac{p+2q}{2}-1}} \\
&\asymp \int_0^{\frac{1}{2}} \frac{r^{\frac{p-2}{2}}\, dr}{(r+\varepsilon)^{\frac{p+2q}{2}-1}} \\
&\asymp \varepsilon^{-\frac{p+2q}{2}+1} \int_0^{\frac{1}{2}} \frac{r^{\frac{p-2}{2}}\, dr}{(\frac{r}{\varepsilon}+1)^{\frac{p+2q}{2}-1}} \\
&\asymp \varepsilon^{-\frac{p+2q}{2}+1} \varepsilon^{\frac{p-2}{2}} \int_0^{\frac{1}{2}} \frac{\left(\frac{r}{\varepsilon}\right)^{\frac{p-2}{2}}\, dr}{(\frac{r}{\varepsilon}+1)^{\frac{p+2q}{2}-1}} \\
&\asymp \varepsilon^{-\frac{p+2q}{2}+1} \varepsilon^{\frac{p-2}{2}} \varepsilon \int_0^{\frac{1}{2\varepsilon}} \frac{r^{\frac{p-2}{2}}\, dr}{(r+1)^{\frac{p+2q}{2}-1}} \\
&\asymp \varepsilon^{-q+1} \int_0^{\frac{1}{2\varepsilon}} \frac{r^{\frac{p-2}{2}}\, dr}{(r+1)^{\frac{p+2q}{2}-1}} \\
&\asymp \varepsilon^{-q+1} \int_1^{\frac{1}{2\varepsilon}} r^{-q} dr \\
&\asymp \begin{cases} \frac{1}{\varepsilon^{q-1}} & \text{if } q > 1, \\ |\log \varepsilon| & \text{if } q = 1. \end{cases} \qquad \square
\end{aligned}
$$

We remark that in order to obtain upper estimates only, the assumption that $\lambda \in i\mathfrak{a}^*$ was not used in view of the degree of generality of the formula in Theorem 4.2(iii). Collecting the preceding results we have proved Theorem 5.1.

Everything that we will have proved about radial limits, namely Theorem 5.1 and Theorem 4.5, is consistent with the following conjecture.

Conjecture B. Let $\alpha \in \Sigma_0$ and $H_\alpha \in \mathfrak{a}$ be the corresponding *co-root*; i.e., $H_\alpha \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \cap \mathfrak{a}$ such that $\alpha(H_\alpha) = 2$. Let $c_\alpha \in \mathbb{R}$ such that $c_\alpha H_\alpha \in \partial \mathfrak{b}^0$. Further, set $m_\alpha = \dim \mathfrak{g}^\alpha$ for all $\alpha \in \Sigma$. Then for all $\alpha \in \Sigma_0$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we have

$$
|\varphi_\lambda(\exp(i(1-\varepsilon)c_\alpha H_\alpha))| \asymp \begin{cases} \frac{1}{\varepsilon^{m_\alpha - 1}} & \text{if } m_\alpha > 1, \\ |\log \varepsilon| & \text{if } m_\alpha = 1. \end{cases}
$$

Remark 5.6. Correspondence with G. Heckman and E. Opdam suggests that the nature of the singularity of the holomorphically extended spherical function in co-root directions might be obtained from properties of the monodromy associated to solutions of the system of invariant differential operators.

Lower estimates. In a later application to automorphic functions we will also need lower estimates for the norm of the K-fixed vector in the holomorphically continued region, for all $\varepsilon > 0$, not only at the singularity. The result is obtained in a way similar to the preceding.

Proposition 5.7. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary spherical principal series representation of a group G of real rank one. Let $X \in \partial\mathfrak{b}^1$ and set*

$$(v_0)_\varepsilon = \pi_\lambda(\exp(i(1-\varepsilon)X))v_0.$$

Then there exists a constant C independent of λ such that

$$\|(v_0)_\varepsilon\|^2 = |\varphi_\lambda(\exp(-2i(1-\varepsilon)X))| \geq C \begin{cases} e^{(\pi-7\varepsilon)|\lambda|} & \text{for } q=0, \\ \varepsilon e^{(\frac{\pi}{2}-21\varepsilon)|\lambda|} & \text{for } q>0, \end{cases}$$

for all $0 < \varepsilon \leq 1$.

Proof. As usual we restrict ourselves to the case of λ imaginary.

Case 1: $q = 0$. Here we have that

$$\|(v_0)_\varepsilon\|^2 = \varphi_\lambda(e^{i(\pi-\varepsilon)}) = e^{-i\lambda(\pi-\varepsilon)} \int_0^\infty \frac{r^{\frac{p-2}{2}}}{|(1+e^{i(\pi-\varepsilon)}r)^{\frac{p}{2}-\lambda}|^2}.$$

By the Weyl group invariance of φ_λ we have $\varphi_\lambda = \varphi_{-\lambda}$ and so we may assume that $\lambda \in i\mathbb{R}^+$, i.e., $\lambda = i|\lambda|$. Then we get

$$\|(v_0)_\varepsilon\|^2 \geq e^{(\pi-\varepsilon)|\lambda|} \int_0^{\frac{1}{2}} \frac{r^{\frac{p-2}{2}}\ dr}{|(1+e^{i(\pi-\varepsilon)}r)^{\frac{p}{2}-\lambda}|^2}.$$

If z is a complex number, then we write $-\pi \leq \arg(z) < \pi$ for the argument of z and $m(z)$ for the modulus of z. Then for $0 \leq r \leq \frac{1}{2}$ we have $0 \leq \arg(1+e^{i(\pi-\varepsilon)}r) < 3\varepsilon$ and $m(1+e^{i(\pi-\varepsilon)}r) \leq 2$. Hence

$$\frac{1}{|(1+e^{i(\pi-\varepsilon)}r)^{\frac{p}{2}-\lambda}|^2} \geq 2^{-\frac{p}{2}} e^{-6\varepsilon|\lambda|}$$

and the assertion of the proposition for $q = 0$ follows.

Case 2: $q > 0$. Here we have that

$$\begin{aligned} \|(v_0)_\varepsilon\|^2 &= \varphi_\lambda(e^{i(\frac{\pi}{2}-\varepsilon)}) \\ &= e^{-i\lambda(\frac{\pi}{2}-\varepsilon)} \int_0^\infty \int_0^\infty \frac{r^{\frac{p-2}{2}} s^{\frac{p-2}{2}}\ dr\ ds}{|\left((1+e^{i(\frac{\pi}{2}-\varepsilon)}r)^2 + se^{i(\pi-2\varepsilon)}\right)^{\frac{p+2q}{4}-\frac{\lambda}{2}}|^2}. \end{aligned}$$

By the Weyl group invariance of φ_λ we may assume that $\lambda \in i\mathbb{R}^+$ and hence get

$$\|(v_0)_\varepsilon\|^2 \geq e^{(\frac{\pi}{2}-\varepsilon)|\lambda|} \int_0^{\frac{1}{2}} \int_0^{\varepsilon} \frac{r^{\frac{p-2}{2}} s^{\frac{p-2}{2}}\ dr\ ds}{\left|((1+e^{i(\frac{\pi}{2}-\varepsilon)}r)^2 + se^{i(\pi-2\varepsilon)})^{\frac{p+2q}{4}-\frac{\lambda}{2}}\right|^2}.$$

Now for $0 \leq r \leq \varepsilon$ and $0 \leq s \leq \frac{1}{2}$ we have $0 \leq \arg((1+e^{i(\frac{\pi}{2}-\varepsilon)}r)^2 + se^{i(\pi-2\varepsilon)}) \leq 10\varepsilon$ and $m((1+e^{i(\frac{\pi}{2}-\varepsilon)}r)^2 + se^{i(\pi-2\varepsilon)}) \leq 2$. Hence the assertion follows as in Case 1. ∎

6. Invariant seminorms

Bernstein and Reznikov, in [BeRe99], introduced the notion of a maximal invariant seminorm associated to Sobolev norms of vectors in representations. For the K-fixed vector of spherical principal series representations for $G = \mathrm{Sl}(2,\mathbb{R})$ they coupled this with some estimates on the holomorphically extended spherical functions into a beautiful technique to get estimates on Rankin-Selberg integrals for Maaß forms.

We shall extend their technique in several directions. First, by using a more representation theoretic viewpoint we will be able to treat the case of real rank one groups. When specialized to $G = \mathrm{Sl}(2,\mathbb{R})$ this will allow us to get a small improvement over the corresponding results in [BeRe99]. Secondly, in Section 9 we are able to consider some higher rank groups for which we obtain estimates on triple products of Maaß forms. These higher rank results are likely new, but should be viewed as a sample of the technique rather than as sharp results.

Definition 6.1. (cf. [BeRe99, App. A]).

(a) Let V be a real or complex vector space and $(N_i)_{i\in I}$ a family of seminorms on it. Then

$$(\inf_{i\in I} N_i)(v) := \inf_{\sum_{i\in I} v_i = v} \sum_{i\in I} N_i(v_i)$$

also defines a seminorm on V and satisfies $\inf_{i\in I} N_i \leq N_j$ for every $j \in I$.

(b) Let G be a semigroup acting on V and $N\colon V \to [0,\infty[$ a single seminorm. Then for $g \in G$ define a seminorm N_g by $N_g(v) = N(g\cdot v)$. As in (a) one obtains a seminorm N^G by setting

$$N^G = \inf_{g\in G} N_g.$$

Definition 6.2. Let $(\pi,\mathcal{H})$ be a unitary representation of a Lie group G on some Hilbert space $\mathcal{H}$. Let $\{X_1,\ldots X_n\}$ be a basis of $\mathfrak{g}$. Then the k^{th} *Sobolev norm* on $\mathcal{H}^\infty$ is defined by

$$S_k(v) = \sum_{0\leq m_1+\ldots+m_n\leq k} \|d\pi(X_1^{m_1}\ldots X_n^{m_n})v\| \qquad (v\in\mathcal{H}^\infty).$$

It is easy to see that a different choice of basis leads to an equivalent seminorm. We remark that S_k, $k>1$, is usually *not* G-invariant. As in (b) above, we set

$$S_k^G(v) = \inf_{g\in G} S_k(\pi(g)v).$$

Then it is a natural problem to estimate $S_k^G(\cdot)$ for the various representations of G. Fix an irreducible unitary representation of a semisimple Lie group G having a nonzero K-fixed vector v_0. Let $v \in \mathcal{H}_{\lambda,K}$ be a K-finite vector.

Recall from Proposition 4.1 that the orbit map $G \to \mathcal{H}_\lambda,\ g \mapsto \pi_\lambda(g)v$ extends to a holomorphic map on $G\Omega K_{\mathbb{C}}$. Write $\Omega = A\Omega_i$ with $\Omega_i \subseteq \exp(i\mathfrak{a})$, and notice that Ω_i has compact closure. We shall show for real rank one groups that $S_k^G(\pi_\lambda(a)v)$ is comparable to $\|\pi_\lambda(a)v\|$ uniformly in $a \in \Omega$ for all K-finite vectors. Similar results will be obtained for holomorphic discrete series in Section 8. But first we explain how for spherical principal series the case of an arbitrary K-finite vector v can be reduced to the spherical vector v_0.

Reduction to a spherical vector.

LEMMA 6.3. *Let $G = KAN$ be any Iwasawa decomposition and set $L = AN$. Suppose that $(\pi_\lambda, \mathcal{H}_\lambda)$ is an irreducible unitary representation of a semisimple Lie group G having a nonzero K-fixed vector v_0.*

(i) *The K-spherical vector v_0 is L-cyclic, i.e. $\mathcal{H}_\lambda = \overline{\operatorname{span}_{\mathbb{C}}\{\pi_\lambda(L)v_0\}}$.*

(ii) *If $\mathcal{H}_{\lambda,K}$ denotes the K-finite vectors of $(\pi_\lambda, \mathcal{H}_\lambda)$, then*

$$\mathcal{H}_{\lambda,K} = d\pi_\lambda(\mathcal{U}(\mathfrak{l}_{\mathbb{C}}))v_0,$$

where $\mathfrak{l}$ denotes the Lie algebra of L.

Proof. (i) This follows from $\pi_\lambda(L)v_0 = \pi_\lambda(G)v_0$ and the irreducibility of $(\pi_\lambda, \mathcal{H}_\lambda)$.

(ii) This is immediate from (i). □

Let $(\pi, \mathcal{H})$ be a Hilbert representation of G. For a closed subgroup $L < G$ write $\mathcal{H}_L^\infty$ for the smooth vectors for $\pi\,|_L$. If π is irreducible, then from the Casselman-Wallach theory of smooth globalizations of Harish-Chandra modules (cf. [Wal92, Ch. 11]) one has that $\mathcal{H}^\infty = \mathcal{H}_K^\infty$.

If $H < G$ is a subgroup, denote by $S_{k,H}$ the k^{th} Sobolev norm for the representation $\pi\,|_H$. In particular, the Fréchet topology on $\mathcal{H}^\infty$ is also induced by the Sobolev norms $(S_{k,K})_{k\in\mathbb{N}}$.

LEMMA 6.4. *Let $G = KAN$ be any Iwasawa decomposition and set $L = AN$. Suppose that $(\pi_\lambda, \mathcal{H}_\lambda)$ is an irreducible unitary representation of a semisimple Lie group G having a nonzero K-fixed vector v_0.*

(i) *For every $k \in \mathbb{N}$ there exist an $l \in \mathbb{N}$ and a constant $C > 0$ such that*

$$(\forall a \in \Omega_i) \qquad S_k(\pi_\lambda(a)v_0) \le C S_{l,L}(\pi_\lambda(a)v_0).$$

(ii) *For every $v \in \mathcal{H}_{\lambda,K}$ and $k \in \mathbb{N}$ there exist an $l \ge k$ and a constant $C > 0$ such that*

$$(\forall a \in \Omega_i) \qquad S_k(\pi_\lambda(a)v) \le C S_{l,L}(\pi_\lambda(a)v_0).$$

Proof. (i) We identify $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ with $\mathcal{S}(\mathfrak{g}_\mathbb{C})$. Then the natural grading of $\mathcal{S}(\mathfrak{g}_\mathbb{C})$ yields a direct sum decomposition $\mathcal{U}(\mathfrak{g}_\mathbb{C}) = \bigoplus_{k\in\mathbb{N}} \mathcal{U}(\mathfrak{g}_\mathbb{C})^k$. Fix a norm $\|\cdot\|$ on $\mathfrak{g}_\mathbb{C}$ and take its natural extension to $\mathcal{S}(\mathfrak{g}_\mathbb{C})$. For any $g \in G_\mathbb{C}$, $\operatorname{Ad}(g)$ maps $\mathcal{U}(\mathfrak{g}_\mathbb{C})^k$ to itself boundedly, so has a norm, say, $\|\operatorname{Ad}(g)\|_k$. If $X \in \mathcal{U}(\mathfrak{g}_\mathbb{C})^k$ with $\|X\| = 1$, then

$$\|X\pi_\lambda(a)v_0\| = \|\pi_\lambda(a)(\operatorname{Ad}(a)^{-1}X)v_0\| \leq \|\operatorname{Ad}(a^{-1})\|_k \sup_{\substack{Y\in\mathcal{U}(\mathfrak{g}_\mathbb{C})^k \\ \|Y\|\leq 1}} \|\pi_\lambda(a)Yv_0\|.$$

Here $C = \sup_{a\in\Omega_i} \|\operatorname{Ad}(a^{\pm 1})\|_k$ is finite by the relative compactness of Ω_i. Hence from Lemma 6.3 there exist an $l \in \mathbb{N}$ and an $r > 0$ such that

$$\|X\pi_\lambda(a)v_0\| \leq C \sup_{\substack{Y\in\mathcal{U}(\mathfrak{g}_\mathbb{C})^k \\ \|Y\|\leq 1}} \|\pi_\lambda(a)Yv_0\| \leq C \sup_{\substack{Z\in\mathcal{U}(\mathfrak{l}_\mathbb{C})^l \\ \|Z\|\leq r}} \|\pi_\lambda(a)Zv_0\|.$$

Now as $\mathfrak{l}$ is normalized by $\mathfrak{a}$, we get that

$$\|X\pi_\lambda(a)v_0\| \leq C^2 \sup_{\substack{Z\in\mathcal{U}(\mathfrak{l}_\mathbb{C})^l \\ \|Z\|\leq r}} \|Z\pi_\lambda(a)v_0\| \leq C' S_{l,L}(\pi_\lambda(a)v_0)$$

for some constant C' independent of X.

(ii) By Lemma 6.3(ii) there exists an $X \in \mathcal{U}(\mathfrak{l}_\mathbb{C})$ such that $v = Xv_0$. Since $\mathfrak{a}$ normalizes $\mathfrak{l}$ the assertion follows now from (i). □

Throughout this section we shall follow the custom that a constant 'C' depends on any quantifiers preceding it in the statement. Thus in the previous result (ii), 'C' depends on π_λ, k, and v but not on a.

Compressing Sobolev norms. For any choice of positive roots Σ^+ we set $\mathfrak{a}^+ = \{X \in \mathfrak{a} \colon (\forall \alpha \in \Sigma^+)\ \alpha(X) > 0\}$ and $\mathfrak{a}^- = -\mathfrak{a}^+$, and, on the group side, let $A^\pm = \exp(\mathfrak{a}^\pm)$.

LEMMA 6.5. *Let $(\pi, \mathcal{H})$ be a unitary representation of G and $v \in \mathcal{H}^\infty$. Then for $k \in \mathbb{N}_0$,*

(i) $S^{A^+}_{k,N}(v) = \|v\|$;

(ii) $S^G_{k,AN}(v) = S^G_{k,A}(v)$.

Proof. (i) Let $\{X_1, \ldots, X_s\}$ be a basis of root vectors of $\mathfrak{n}$ corresponding to roots $\alpha_1, \ldots, \alpha_s \in \Sigma^+$. Then for any $v \in \mathcal{H}^\infty$

$$S_{l,N}(v) = \|v\| + \sum_{1\leq m_1+\ldots+m_s\leq l} \|d\pi(X_1^{m_1} \cdot\cdot X_s^{m_s})v\|.$$

For $a = \exp(X) \in A$, $X \in \mathfrak{a}$,

$$\begin{aligned} &S_{l,N}(\pi(a)v) \\ &\quad= \|\pi(a)v\| + \sum_{1\le m_1+\ldots+m_s\le l} \|d\pi(X_1^{m_1}\cdot\cdot X_s^{m_s})\pi(a)v\| \\ &\quad= \|v\| + \sum_{1\le m_1+\ldots+m_s\le l} \|\pi(a)d\pi((\mathrm{Ad}(a^{-1})X_1)^{m_1}\cdot\cdot(\mathrm{Ad}(a^{-1})X_s)^{m_s})v\| \\ &\quad= \|v\| + \sum_{1\le m_1+\ldots+m_s\le l} e^{-\sum_{j=1}^s m_j\alpha_j(X)}\|d\pi(X_1^{m_1}\cdot\cdot X_s^{m_s})v\|. \end{aligned}$$

If we choose $X \in \mathfrak{a}^+$,

$$\begin{aligned} S_{l,N}^{A^+}(v) &\le \inf_{a\in A^+} S_{l,N}(\pi(a)v) \le \inf_{t>0} S_{l,N}(\pi(\exp(tX))v) \\ &= \inf_{t>0}\left(\|v\| + \sum_{1\le m_1+\ldots+m_s\le l} e^{-t\sum_{j=1}^s m_j\alpha_j(X)}\|d\pi(X_1^{m_1}\cdot\cdot X_s^{m_s})v\|\right) \\ &= \|v\|. \end{aligned}$$

On the other hand, clearly $\|v\| \le S_k^{A^+}(v)$. Thus $\|v\| = S_{l,N}^{A^+}(v)$ completing the proof of (i).

(ii) One has the obvious inequality $S_{k,AN}(v) \ge S_{k,A}(v)$, so that $S_{k,AN}^G(v) \ge S_{k,A}^G(v)$. On the other hand,

$$\begin{aligned} S_{k,AN}^G(v) &\le \inf_{g\in G} S_{k,AN}(\pi(g)v) \\ &\le \inf_{h\in A^+} S_{k,AN}(\pi(h)v) \\ &= S_{k,A}(v), \end{aligned}$$

so that $S_{k,AN}^G(v) \le S_{k,A}^G(v)$. □

The case of $G = \mathrm{Sl}(2,\mathbb{R})$. Our goal is to estimate $S_k^G(\pi(a)v_0)$ for all $a \in \Omega_i$. In this section we shall present extensive details for $G = \mathrm{Sl}(2,\mathbb{R})$ as this will be the model for the proof later for rank one groups. Here we will consider an irreducible unitary spherical principal series representation $(\pi_\lambda, \mathcal{H}_\lambda)$. The complementary series and nonspherical principal series representations can be shown similarly. Discrete series however will be obtained rather differently in Section 8.

We identify N with $\mathbb{R}$ via the mapping $n_x \mapsto x$ (see Appendix A for notation). We are going to work in the noncompact realization of π_λ on $L^2(N) = L^2(\mathbb{R})$. With $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the action of $\pi_\lambda(g)$ is given by

$$(\pi_\lambda(g)f)(x) = |cx+d|^{\lambda-1} f\left(\frac{ax+b}{cx+d}\right) \tag{6.1}$$

for all $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. For this module one has

$$\mathcal{H}_\lambda^\infty = \{f \in C^\infty(\mathbb{R}) : |x|^{\lambda-1} f(\frac{1}{x}) \in C^\infty(\mathbb{R})\}.$$

We use a usual basis for the Lie algebra of $\mathfrak{g}$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\mathfrak{a} = \mathbb{R}H$, $\mathfrak{n} = \mathbb{R}E$ and $\overline{\mathfrak{n}} = \mathbb{R}F$. With $U = E - F$ we have $\mathfrak{k} = \mathbb{R}U$. Differentiating (6.1) one obtains the formulas

$$d\pi_\lambda(H) = (\lambda - 1) - 2x\frac{d}{dx}, \tag{6.2}$$

$$d\pi_\lambda(E) = -\frac{d}{dx}, \tag{6.3}$$

$$d\pi_\lambda(F) = (1-\lambda)x + x^2\frac{d}{dx}, \tag{6.4}$$

$$d\pi_\lambda(U) = (\lambda - 1) - (1+x^2)\frac{d}{dx}, \tag{6.5}$$

$$d\pi_\lambda(E+F) = (1-\lambda)x - (1-x^2)\frac{d}{dx}. \tag{6.6}$$

We also define the *radial operators* by

$$(R_j f)(x) = (x^j \frac{d^j}{dx^j} f)(x)$$

and define the *radial Sobolev norms* by

$$S_{k,\mathrm{rad}}(f) = \sum_{j=0}^{k} \|R_j f\|.$$

From the action of $d\pi_\lambda(H)$ and R^j it is clear that there exists a constant $C > 0$, depending on k and λ, such that for all $f \in \mathcal{S}(\mathbb{R})$

$$\frac{1}{C} S_{k,\mathrm{rad}}(f) \le S_{k,A}(f) \le C S_{k,\mathrm{rad}}(f). \tag{6.7}$$

As remarked by the referee, in (6.2) and (6.4) the coefficient of the derivative term has a zero; consequently $S_k(v)$ cannot be majorized by $S_{k,A\overline{N}}(v)$ or by $S_{k,A}(v)$ in general. However, we shall show in the next proposition that there is such a relationship for the $G-$invariant Sobolev norms.

PROPOSITION 6.6. *Let $G = \mathrm{Sl}(2,\mathbb{R})$ and $(\pi_\lambda, \mathcal{H}_\lambda)$, $\lambda \in i\mathfrak{a}^*$, be an irreducible unitary spherical principal series representation. Then for every $k \in \mathbb{N}_0$ there exists a $C > 0$ such that for $v \in \mathcal{H}_\lambda^\infty$,*

$$S_k^G(v) \le C S_{k,A}^G(v).$$

Proof. The A action on $K/M \cong S^1$ has two fixed points, corresponding to the two Bruhat cells. In the noncompact realization N they become the origin and the point at infinity. We shall estimate $S_k^G(f)$ by using first a cut-off function at infinity, $\overline{\mathfrak{n}}$, and an elementary estimate there. Near the origin a dilated cutoff localizes sufficiently high derivatives of f to get an estimate. Away from the fixed points, motivated by an argument in [BeRe99] and classical Littlewood-Paley theory, we use a family of suitably dilated cutoff functions which compress the $\mathfrak{n}$ derivatives in the definition of G-invariant norm to *radial* derivatives thereby obtaining the desired estimate.

For $j \in \mathbb{Z}$ we denote by I_j the set $\{x \in \mathbb{R}: 2^{-j-1} \leq |x| \leq 2^{-j+1}\}$. For a function ψ on $\mathbb{R}$ we write $\psi_j(x) = \psi(2^j x)$. Notice that if ψ is supported in I_0 then ψ_j is supported in I_j, and

$$\operatorname{supp}(\psi_j) \ \cap \ \operatorname{supp}(\psi_{j+1}) \subseteq \pm[\frac{1}{2^{j+1}}, \frac{1}{2^j}].$$

We take a smooth, nonnegative function φ supported in I_0 and such that for every $m \in \mathbb{N}_0$,

$$\sum_{j=0}^{m} \varphi_j(x) = \begin{cases} 0 & \text{if } |x| \leq 2^{-m-1}, \\ 1 & \text{if } 2^{-m} \leq |x| \leq 1, \\ 0 & \text{if } 2 \leq |x|. \end{cases}$$

Choose a nonnegative function $\tau \in C^\infty(\mathbb{R})$ with support in $\{x \in \mathbb{R}: 1 \leq |x|\}$ such that $(\tau + \varphi)(x) = 1$ for $|x| \geq 1$. Finally for each $m \in \mathbb{N}$ define the function $\tau_m \in C_c^\infty(\mathbb{R})$ by $\tau_m = \mathbf{1} - \tau - \sum_{j=0}^m \varphi_j$. Notice that $\operatorname{supp} \tau_m \subseteq \{x \in \mathbb{R}: |x| \leq 2^{-m}\}$ and $\tau_m(x) = 1$ for $|x| \leq 2^{-m-1}$. From the properties of the φ_j and τ it is easy to see that for any $l \geq 1$, $\tau_m^{(l)}(x) = -2^{lm}\varphi^{(l)}(2^m x)$.

Let $f \in \mathcal{H}_\lambda^\infty$. Since

$$\begin{aligned} \mathbf{1} &= \tau + \mathbf{1} - \tau \\ &= \tau + \tau_m + \sum_{j=0}^{m} \varphi_j \\ &= \tau + \varphi + \tau_m + \sum_{j=1}^{m} \varphi_j, \end{aligned}$$

then

$$f = (\tau + \varphi) f + \tau_m f + \sum_{j=1}^{m} \varphi_j f.$$

For any choices of $g, g_1, \ldots, g_m \in G$, using the definition of S_k^G, we get

$$S_k^G(f) \leq S_k((\tau + \varphi) f) + S_k(\pi_\lambda(g)(\tau_m f)) + \sum_{j=1}^{m} S_k(\pi_\lambda(g_j)(\varphi_j f)). \tag{6.8}$$

First we consider the term $S_k((\tau+\varphi)f)$. From an examination of formulas (6.2)–(6.4) one sees that $S_k((\tau + \varphi)f) \leq C S_{k,\overline{N}}((\tau + \varphi)f)$ for all $f \in \mathcal{H}_\lambda^\infty$. (Throughout this proof C will denote a constant depending only on k, τ, φ and λ.) Hence we have

$$S_k((\tau + \varphi)f) \leq C S_{k,\overline{N}}((\tau + \varphi)f) \leq C S_{k,\overline{N}}(f)$$

for all $f \in \mathcal{H}_\lambda^\infty$. Majorizing this term in (6.8) we get

$$S_k^G(f) \leq CS_{k,\overline{N}}(f) + S_k((\pi_\lambda(g)\tau_m f)) + \sum_{j=1}^m S_k(\pi_\lambda(g_j)(\varphi_j f)) \tag{6.9}$$

for all $f \in \mathcal{H}_\lambda^\infty$.

Next we specify a good choice of the elements $g, g_1, \dots, g_m \in G$. For every $t > 0$ denote by b_t the element

$$b_t = \begin{pmatrix} \frac{1}{\sqrt{t}} & 0 \\ 0 & \sqrt{t} \end{pmatrix} \in A.$$

From (6.1) it follows that

$$(\pi_\lambda(b_t)f)(x) = t^{\frac{1}{2}(1-\lambda)} f(tx)$$

for all $t > 0$ and $x \in \mathbb{R}$. Take $g_j = b_{2^{-j}}$ for all $1 \leq j \leq m$ and $g = b_{2^{-(m+1)}}$. Notice that for every m all the $\pi_\lambda(g_j)(\varphi_j f)$ are supported in $[-2, 2]$, as is $\pi_\lambda(g)(\tau_m f)$. For any smooth function h supported in $[-2, 2]$ we can conclude from the formulas (6.2)–(6.5) that $S_k(h) \leq CS_{k,N}(h)$. Using this in (6.9) we get

$$S_k^G(f) \leq CS_{k,\overline{N}}(f) + CS_{k,N}(\pi_\lambda(g)(\tau_m f)) + C\sum_{j=1}^m S_{k,N}(\pi_\lambda(g_j)(\varphi_j f)) \tag{6.10}$$

for all $f \in \mathcal{H}_\lambda^\infty$.

Estimating $S_{k,N}(\pi_\lambda(g)(\tau_m f))$, we use Leibniz on $\tau_m f$ and L^∞ estimates on $\tau_m^{(j)} = -2^{jm}\varphi^{(j)}(2^m x)$. From (6.3) one sees that $S_{k,N}(h) = \sum_{l=0}^k \|h^{(l)}\|$. Then

(6.11)

$$\begin{aligned}
S_{k,N}(\pi_\lambda(g)(\tau_m f)) &= \sum_{l=0}^k \|\frac{d^l}{dx^l} 2^{-\frac{(m+1)}{2}(1-\lambda)}(\tau_m f)(2^{-(m+1)}\cdot)\| \\
&= \sum_{l=0}^k |2^{-\frac{(m+1)}{2}(1-\lambda)}| \\
&\quad \times \left[\int \left| \sum_{n=0}^l 2^{-(m+1)l} \binom{l}{l-n} \tau_m^{(l-n)}(2^{-(m+1)}x) f^{(n)}(2^{-(m+1)}x)\right|^2 dx\right]^{\frac{1}{2}} \\
&\leq \sum_{l=0}^k |2^{-\frac{(m+1)}{2}(1-\lambda)}| \\
&\quad \times \sum_{n=0}^l \left[\int_{|x|\leq 2} \left|2^{-(m+1)l} \binom{l}{l-n} \tau_m^{(l-n)}(2^{-(m+1)}x) f^{(n)}(2^{-(m+1)}x)\right|^2 dx\right]^{\frac{1}{2}} \\
&= \sum_{l=0}^k \left|2^{\frac{(m+1)}{2}\lambda}\right| \sum_{n=0}^l \left[\int_{|y|\leq \frac{1}{2^m}} \left|2^{-(m+1)l} \binom{l}{l-n} \tau_m^{(l-n)}(y) f^n(y)\right|^2 dy\right]^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\le \sum_{l=0}^{k} |2^{\frac{(m+1)}{2}\lambda}| \sum_{n=0}^{l} \binom{l}{l-n} \frac{\|2^{(l-n)m}\varphi^{(l-n)}\|_\infty}{2^{(m+1)l}} \Big[\int_{|y|\le\frac{1}{2^m}} |f^{(n)}(y)|^2\, dy\Big]^{\frac12} \\
&= \sum_{n=0}^{k} |2^{\frac{(m+1)}{2}\lambda}| \frac{1}{2^{mn}} \sum_{l=n}^{k} \binom{l}{l-n} \frac{\|\varphi^{(l-n)}\|_\infty}{2^{l}} \Big[\int_{|y|\le\frac{1}{2^m}} |f^{(n)}(y)|^2\, dy\Big]^{\frac12} \\
&= \sum_{n=0}^{k} |2^{\frac{(m+1)}{2}\lambda}| \frac{1}{2^{(m+1)n}} \sum_{j=0}^{k-n} \binom{j+n}{n} \frac{\|\varphi^{j}\|_\infty}{2^{j}} \Big[\int_{|y|\le\frac{1}{2^m}} |f^{(n)}(y)|^2\, dy\Big]^{\frac12} \\
&\le \Big(\sum_{j=0}^{k} \frac{\|\varphi^{(j)}\|_\infty}{j!2^j}\Big) \sum_{n=0}^{k} \frac{k!}{n!2^{(m+1)n}} \Big[\int_{|y|\le\frac{1}{2^m}} |f^{(n)}(y)|^2\, dy\Big]^{\frac12}.
\end{aligned}$$

Now k is fixed and each of the at most k derivatives $f^{(n)}$ is in L^2, hence the integrals can be made uniformly small. So for each f we can choose an m so that the last line above is at most $\|f\|$. Then we have

$$S_k^G(f) \le CS_{k,\overline{N}}(f) + C\|f\| + C\sum_{j=1}^{m} S_{k,N}(\pi_\lambda(g_j)(\varphi_j f))$$

for any $f \in \mathcal{H}_\lambda^\infty$. Thus from (6.10) we obtain

$$S_k^G(f) \le CS_{k,\overline{N}}(f) + C\|f\| + C\sum_{l=0}^{k}\sum_{j=1}^{m} \|\frac{d^l}{dx^l}(2^{-\frac{j}{2}(1-\lambda)}\varphi f(2^{-j}\cdot))\|. \tag{6.12}$$

As in (6.11), using Leibniz on φf, L^∞ estimates on $\varphi^{(j)}$, and majorizing the binomial coefficients, we get

$$\begin{aligned}
\sum_{l=0}^{k}\sum_{j=1}^{m} \|\frac{d^l}{dx^l}(2^{-\frac{j}{2}}\varphi f(2^{-j}\cdot))\| &\le C\sum_{l=0}^{k}\sum_{j=1}^{m} \Big(\int_{I_0} 2^{-j-2l}|f^{(l)}(2^{-j}x)|^2\, dx\Big)^{\frac12} \\
&= C\sum_{l=0}^{k}\sum_{j=1}^{m} \Big(\int_{I_j} 2^{-2l}|f^{(l)}(x)|^2\, dx\Big)^{\frac12} \\
&\le 4C\sum_{l=0}^{k}\sum_{j=1}^{m} \Big(\int_{I_j} |x^l f^{(l)}(x)|^2\, dx\Big)^{\frac12} \\
&\le 4CS_{k,\mathrm{rad}}(f) \le 4CS_{k,A}(f),
\end{aligned} \tag{6.13}$$

where the last inequality follows from (6.7) and again C depends only on τ, φ, k and λ. Thus we get from (6.12) and (6.13) that

$$S_k^G(f) \le CS_{k,\overline{N}}(f) + C\|f\| + CS_{k,A}(f) \le C\|f\| + CS_{k,A\overline{N}}(f)$$

for all $f \in \mathcal{H}_\lambda^\infty$. Now,

$$S_k^G \le CS_{k,A\overline{N}}^G$$

and, by Lemma 6.5(ii), $S_k^G \le CS_{k,A}^G$ as was to be shown. ∎

In $G = \mathrm{Sl}(2, \mathbb{R})$ the element

$$k_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

is in K and is a square root of the Weyl group element. It will turn out that k_0 provides a uniform minimizer for Sobolev norms.

THEOREM 6.7. *Let $G = \mathrm{Sl}(2, \mathbb{R})$ and $(\pi_\lambda, \mathcal{H}_\lambda)$, $\lambda \in i\mathfrak{a}^*$, be an irreducible unitary spherical principal series representation. Then for every $k \in \mathbb{N}_0$ there exists a $C > 0$, depending on k and λ, such that for all $a \in \Omega_i$*

$$S_{k,A}(\pi_\lambda(k_0)\pi_\lambda(a)v_0) \leq C\|\pi_\lambda(a)v_0\|.$$

In particular, for all $a \in \Omega_i$

$$S_k^G(\pi_\lambda(a)v_0) \leq C\|\pi_\lambda(a)v_0\|.$$

Proof. In view of Proposition 6.6 the second assertion follows from the first one. To prove the first assertion notice that

$$(\pi_\lambda(k_0)f)(x) = |x+1|^{\lambda-1} f\left(\frac{x-1}{x+1}\right) \tag{6.14}$$

for all $f \in L^2(\mathbb{R})$.

We parametrize Ω_i with a_ε and a_ε^{-1}, where

$$a_\varepsilon = \begin{pmatrix} e^{i\frac{\pi}{4}(1-\varepsilon)} & 0 \\ 0 & e^{-i\frac{\pi}{4}(1-\varepsilon)} \end{pmatrix}$$

for $0 < \varepsilon \leq 1$. Then, in the noncompact realization, $\pi_\lambda(a_\varepsilon)v_0$ is of the form $c(\lambda, \varepsilon)f_\varepsilon$ where

$$f_\varepsilon(x) = \frac{1}{(1 + e^{i\pi(1-\varepsilon)}x^2)^{\frac{1}{2}-\lambda}}$$

and $c(\lambda, \varepsilon)$ is a constant depending on λ and ε, and is uniformly bounded in ε (as can be seen from §5). Notice that the poles of f_ε, as $\varepsilon \to 0$, are at $x = \pm 1$. Thus if we take a smooth cut-off function $\tau \in C_c^\infty(\mathbb{R})$ with, say, $\tau|_{[-2,2]} = 1$, then

$$S_{k,A}(\pi_\lambda(k_0)f_\varepsilon) \leq S_{k,A}(\pi_\lambda(k_0)\tau f_\varepsilon) + S_k(\pi_\lambda(k_0)(1-\tau)f_\varepsilon) \leq S_{k,A}(\pi_\lambda(k_0)\tau f_\varepsilon) + C. \tag{6.15}$$

Here C is a positive constant independent of ε because, on the support of $(1-\tau)$, one has $\|(1-\tau)f_\varepsilon\| \asymp \|(1-\tau)x^{-1}\|$, with similar results on the norms of derivatives.

With $g_\varepsilon = \tau f_\varepsilon$, in view of (6.7), (6.14) and (6.15), it suffices to show that

$$S_{k,\mathrm{rad}}(\pi_\lambda(k_0)g_\varepsilon) \leq C\|\pi_\lambda(a_\varepsilon)v_0\|$$

for all ε and some constant $C > 0$. By the radial Sobolev norms and the estimate in Lemma 5.3, $\|g_\varepsilon\| \asymp \|\pi_\lambda(a_\varepsilon)v_0\| \asymp \sqrt{|\log\varepsilon|}$, it is enough to show that

$$\|R_j\pi_\lambda(k_0)g_\varepsilon\| \le C_j\sqrt{|\log\varepsilon|}$$

for all $j \geq 1$.

For $f \in C_c^\infty(\mathbb{R})$ and from (6.14),

$$\begin{aligned}(R_1\pi_\lambda(k_0)f)(x) &= 2x|x+1|^{\lambda-1}\frac{1}{(x+1)^2}f'\left(\frac{x-1}{x+1}\right)\\ &\quad +\varepsilon(x)(\lambda-1)x|x+1|^{\lambda-2}f\left(\frac{x-1}{x+1}\right)\end{aligned}$$

with $\varepsilon(x) = 1$ for $x > -1$ and $\varepsilon(x) = -1$ for $x < -1$. Disregarding the sign function $\varepsilon(x)$, and using induction we have

$$(R_j\pi_\lambda(k_0)f)(x) = x^j\sum_{m=0}^{j} c_m f_m^j(x)$$

for some constants c_m independent of f, and where

$$f_m^j(x) = |x+1|^{\lambda-1-j-m}f^{(m)}\left(\frac{x-1}{x+1}\right).$$

Thus to estimate $S_{k,\mathrm{rad}}(\pi_\lambda(k_0)g_\varepsilon) = \sum_{j=0}^k \|R_j\pi_\lambda(k_0)g_\varepsilon\|$ we must show that

$$|\langle x^j g^j_{\varepsilon,m}, x^j g^j_{\varepsilon,n}\rangle| \le C|\log\varepsilon| \tag{6.16}$$

for all $m, n \le j$.

Now, consider an expression of the form $|\langle x^j f_m^j, x^j f_n^j\rangle|$ where

(6.17)

$$\begin{aligned}|\langle x^j f_m^j, x^j f_n^j\rangle| &\le \int_{\mathbb{R}} x^{2j}\ |x+1|^{-2-m-n-2j}\ |f^{(m)}\left(\frac{x-1}{x+1}\right)|\ |f^{(n)}\left(\frac{x-1}{x+1}\right)|\ dx\\ &= 2\int_{\mathbb{R}}\left|\frac{x+1}{1-x}\right|^{2j}\ \left|\frac{x+1}{1-x}+1\right|^{-m-n-2j}\ |f^{(m)}(x)f^{(n)}(x)|\ dx\\ &= 2^{-(m+n)-2j+1}\int_{\mathbb{R}}|x+1|^{2j}\ |1-x|^{m+n}\ |f^{(m)}(x)f^{(n)}(x)|\ dx.\end{aligned}$$

Next, as $\varepsilon \to 0$, the functions g_ε have poles at $x = 1$ and $x = -1$. Similarly, as $\varepsilon \to 0$, $g_\varepsilon^{(m)}(x)$ has poles only at $x = \pm 1$ and of order at most $m+\frac{1}{2}$. Examining (6.17) with $f = g_\varepsilon$ we see that the factor $|x+1|^{2j}|1-x|^{(m+n)}|$ cancels poles. In particular

$$|x+1|^{2j}|1-x|^{m+n}|g_\varepsilon^{(m)}(x)g_\varepsilon^{(n)}(x)|$$

has poles at $x = \pm 1$ for $\varepsilon \to 0$ of order no more than that of g_ε. This establishes (6.16) and concludes the proof of the theorem. $\square$

Remark 6.8. (a) The second estimate, $S_k^G(\pi_\lambda(a)v_0) \leq C\|\pi_\lambda(a)v_0\|$, in Theorem 6.7 is optimal in the sense that by G-invariance one has $\|v\| \leq S_k^G(v)$ for any smooth vector v in a unitary representation $(\pi, \mathcal{H})$ of G.

(b) One can modify the proof of Theorem 6.7 to give the more general result

$$(\forall a \in \Omega_i) \qquad S_{k,A}(\pi_\lambda(k_0)\pi_\lambda(a)v) \leq C\|\pi_\lambda(a)v\|$$

for an arbitrary K-finite vector v.

(c) The estimate $S_k^G(\pi_\lambda(a_\varepsilon)v_0) \leq C\sqrt{|\log\varepsilon|}$ in Theorem 6.7 is a little sharper than the estimate (0.5) in [BeRe99], viz. $S_k^G(\pi_\lambda(a_\varepsilon)v_0) \leq C|\log\varepsilon|$.

(d) Theorem 6.7 can be easily generalized to complementary series using the results on spherical functions in Theorem 4.2.

Part of the method for $G = \mathrm{Sl}(2,\mathbb{R})$ generalizes to all groups of real rank one. For example, the element $k_0 \in K$ can be found in these groups and gives a uniform minimizer for $S_{k,A}$.

LEMMA 6.9. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Suppose that the restricted root system Σ satisfies one of the following assumptions*:

(1) *Σ is of type A_1 or BC_1, i.e., $\mathfrak{g}$ is of real rank one*;

(2) *Σ is of type C_n or BC_n for $n \geq 2$.*

Then there exists a $k_0 \in K$ such that

$$\mathrm{Ad}(k_0)\mathfrak{a} \subseteq \mathfrak{k} \oplus \overline{\mathfrak{n}} = \mathfrak{k} \oplus \mathfrak{n}.$$

Proof. First recall that all maximal abelian subspaces in $\mathfrak{p}$ are conjugate under $\mathrm{Ad}(K)$.

Suppose that (1) is satisfied. Then $\mathfrak{a}$ is one-dimensional. Pick a nonzero root vector $X_\alpha \in \mathfrak{g}^\alpha$. Then $\mathfrak{e} = \mathbb{R}(X_\alpha - \theta(X_\alpha))$ is a maximal abelian subspace in $\mathfrak{p}$ which lies in $\mathfrak{k} \oplus \overline{\mathfrak{n}}$. Hence there exists a $k_0 \in K$ such that $\mathrm{Ad}(k_0)\mathfrak{a} = \mathfrak{e}$.

Suppose then that (2) is satisfied. Since Σ is of type C_n or BC_n we can find a maximal set $\gamma_1, \ldots, \gamma_n$ of long strongly orthogonal roots. But via $\mathfrak{sl}(2,\mathbb{R})$-reduction, the assertion follows from the already established rank one case above. □

We shall make the standing assumption, for the rest of this subsection, that G has real rank one. We need to make the element k_0 more explicit. Let β denote the long positive root. Then we have $\beta = \alpha$ if $q = 0$, otherwise $\beta = 2\alpha$. Choose an $\mathfrak{sl}(2,\mathbb{R})$-triple $\{E, F, H\}$ in $\mathfrak{g}$ such that E lies in the root space $\mathfrak{g}^\beta$, $F = -\theta E$ and such that

$$H = [E, F] \qquad [H, E] = 2E \qquad [H, F] = -2F.$$

With $U = E - F$ we choose

$$k_0 = \exp(\frac{\pi}{4}U).$$

Then

$$\mathrm{Ad}(k_0)^{-1}H = E + F.$$

Notice that

$$\Omega_i = \{\exp(i\varphi H)|\varphi \in]-\frac{\pi}{4}, \frac{\pi}{4}[\}$$

and introduce elements a_ε by

$$a_\varepsilon = \exp(i\frac{\pi}{4}(1-\varepsilon)H).$$

PROPOSITION 6.10. *Suppose that G is of real rank one and that $(\pi, \mathcal{H})$ is an irreducible unitary representation with K-spherical vector v_0. Then for all $k \in \mathbb{N}_0$ there exists a constant $C > 0$ such that*

$$(\forall a \in \Omega_i) \qquad S_{k,A}(\pi(k_0)\pi(a)v_0) \leq C \sum_{j=0}^{k} |a^\beta + a^{-\beta}|^j S_j(\pi(a)v_0).$$

Proof. $S_{k,A}$ is given by

$$S_{k,A}(v) = \sum_{j=0}^{k} \|H^j v\|.$$

We are going to prove the proposition by induction on k. As the case $k = 0$ is obvious, we start with the case $k = 1$, so that

$$S_{1,A}(v) = \|v\| + \|Hv\|.$$

For $a \in \Omega_i$ we obtain

$$\begin{aligned} H\pi(k_0)\pi(a)v_0 &= \pi(k_0)(E+F)\pi(a)v_0 = \pi(k_0)\pi(a)(a^{-\beta}E + a^\beta F)v_0 \\ &= \pi(k_0)\pi(a)\Big((a^\beta + a^{-\beta})F + a^{-\beta}\underbrace{(E-F)}_{\in\mathfrak{k}}\Big)v_0 \\ &= \pi(k_0)\pi(a)\Big((a^\beta + a^{-\beta})F\Big)v_0 \\ &= \pi(k_0)\Big(a^{-\beta}(a^\beta + a^{-\beta})F\Big)\pi(a)v_0. \end{aligned}$$

Using the unitarity of π we get

$$\begin{aligned} S_{1,A}(\pi(k_0)\pi(a)v_0) &= \|\pi(a)v_0\| + \|H\pi(k_0)\pi(a)v_0\| \\ &= \|\pi(a)v_0\| + |a^\beta + a^{-\beta}|\ \|F\pi(a)v_0\|. \end{aligned}$$

Since $\|F\pi(a)v_0\| \leq CS_1(\pi(a)v_0)$, the proof of the $k = 1$ case is complete.

Suppose that the statement holds for $k-1$. We must show that

$$\|H^k\pi(k_0)\pi(a)v_0\| \le C\sum_{j=0}^{k}|a^\beta + a^{-\beta}|^j S_j(\pi(a)v_0).$$

As before we have

$$H^k\pi(k_0)\pi(a)v_0 = \pi(k_0)(E+F)^k\pi(a)v_0 = \pi(k_0)\pi(a)(a^{-\beta}E + a^\beta F)^k v_0.$$

Now we must arrange the expressions $(a^{-\beta}E + a^\beta F)^k$ in an appropriate way. With $U = E - F$,

$$(a^{-\beta}E + a^\beta F)^k = \Big((a^\beta + a^{-\beta})F + a^{-\beta}U\Big)^k.$$

Now using repeatedly the fact

$$(\forall X \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^j) \qquad UXv_0 = ([U,X] + XU)v_0 = [U,X]v_0,$$

with $[U,X] \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^{j-1}$, we obtain elements $Z_{j,a} \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^j$, uniformly bounded depending on a, such that

$$(a^{-\beta}E + a^\beta F)^k v_0 = \sum_{j=0}^{k}(a^\beta + a^{-\beta})^j Z_{j,a}v_0$$

just as in the $k=1$ case. □

In (6.6) one can see that the derivative term of $E+F$ has coefficient vanishing precisely at $x = \pm 1$, the eventual poles of f_ε. This is the key to the proof of Theorem 6.7. For the other rank one groups the singularity of f_ε lies on a hypersurface, such as $\|x\| = \pm 1$, and this cannot be dominated by a single operator $E+F$, nevertheless an argument of this type will be developed.

$\overline{N}$ vector fields from principal series representations. Before we can complete our discussions for the rank one case, we first have to provide some simple facts related to the Bruhat decomposition. The facts collected below hold for an arbitrary semisimple Lie group.

For every $k \in K$ we write $\lambda_k\colon K/M \to K/M,\ xM \mapsto kxM$ for the left translation on K/M. There is the standard action of G on K/M by

$$G \times K/M \to K/M, \quad (g, kM) \mapsto \kappa(gk)M.$$

Then every $X \in \mathfrak{g}$ defines a vector field $\widetilde{X}$ on K/M via

$$\widetilde{X}_m = \frac{d}{dt}\Big|_{t=0}\kappa(\exp(tX)m)$$

for all $m \in M$. Write $p_{\mathfrak{k}}\colon\ \mathfrak{g} \to \mathfrak{k}$ for the projection along $\mathfrak{a}+\mathfrak{n}$ and $p_{\mathfrak{k}/\mathfrak{m}}\colon \mathfrak{g} \to \mathfrak{k}/\mathfrak{m}$ for the composition of $p_{\mathfrak{k}}$ and the quotient mapping $\mathfrak{k} \to \mathfrak{k}/\mathfrak{m}$.

Lemma 6.11. *For all $X \in \mathfrak{g}$ and $m = kM \in K/M$,*

$$\widetilde{X}_m = d\lambda_k(\mathbf{1})p_{\mathfrak{k}/\mathfrak{m}}(Ad(k^{-1})X).$$

Proof. By definition we have that

$$\widetilde{X}_m = \frac{d}{dt}\Big|_{t=0} \kappa(\exp(tX)k)M = \frac{d}{dt}\Big|_{t=0} \kappa(k\exp(t\,\mathrm{Ad}(k^{-1})X))M.$$

Set $Y = \mathrm{Ad}(k^{-1})X$. Then there exist smooth curves $Y_{\mathfrak{k}}(t) \in \mathfrak{k}$, $Y_{\mathfrak{a}}(t) \in \mathfrak{a}$, $Y_{\mathfrak{n}}(t) \in \mathfrak{n}$ such that

$$\exp(tY) = \exp(Y_{\mathfrak{k}}(t))\exp(Y_{\mathfrak{a}}(t))\exp(Y_{\mathfrak{n}}(t))$$

for all $t \in \mathbb{R}$. Differentiation at $t = 0$ yields

$$Y = Y_{\mathfrak{k}}'(0) + Y_{\mathfrak{a}}'(0) + Y_{\mathfrak{n}}'(0)$$

and so $Y_{\mathfrak{k}}'(0) = p_{\mathfrak{k}}(Y)$. Hence we get that

$$\begin{aligned}\widetilde{X}_m &= \frac{d}{dt}\Big|_{t=0} \kappa(k\exp(Y_{\mathfrak{k}}(t))\exp(Y_{\mathfrak{a}}(t))\exp(Y_{\mathfrak{n}}(t)))M \\ &= d\lambda_k(\mathbf{1})\frac{d}{dt}\Big|_{t=0} \kappa(\exp(Y_{\mathfrak{k}}(t)))M \\ &= d\lambda_k(\mathbf{1})p_{\mathfrak{k}/\mathfrak{m}}(\mathrm{Ad}(k^{-1})X).\end{aligned}$$

□

The Bruhat decomposition of G

$$G = \bigcup_{w\in\mathcal{W}} \overline{N}wMAN$$

gives the familiar decomposition of the flag manifold into Schubert cells

$$K/M = \bigcup_{w\in\mathcal{W}} \kappa(\overline{N}wM)/M.$$

Lemma 6.12. *For all $m \in \kappa(\overline{N})M/M \subseteq K/M$,*

$$T_mK/M = \{\widetilde{X}_m : X \in \overline{\mathfrak{n}}\}.$$

Proof. This result is known but we include it for completeness. In view of Lemma 6.11, it suffices to show that for $k = \overline{n}man$, $\overline{n} \in \overline{N}, m \in M, a \in A$, $n \in N$,

$$p_{\mathfrak{k}}(\mathrm{Ad}(k^{-1})\overline{\mathfrak{n}}) + \mathfrak{m} = \mathfrak{k}$$

or equivalently

$$\mathrm{Ad}(k^{-1})\overline{\mathfrak{n}} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n} = \mathfrak{g}.$$

By the special choice of k,

$$\begin{aligned}\mathrm{Ad}(k^{-1})\overline{\mathfrak{n}}+\mathfrak{m}+\mathfrak{a}+\mathfrak{n} &= \mathrm{Ad}(man)^{-1}\Big(\mathrm{Ad}(\overline{n}^{-1})\overline{\mathfrak{n}}+\mathrm{Ad}(man)(\mathfrak{m}+\mathfrak{a}+\mathfrak{n})\Big)\\ &= \mathrm{Ad}(man)^{-1}(\overline{\mathfrak{n}}+\mathfrak{m}+\mathfrak{a}+\mathfrak{n})=\mathfrak{g}. \qquad \square\end{aligned}$$

LEMMA 6.13. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a spherical principal series representation of G realized in $L^2(K/M)$. Then for all $m \in \kappa(\overline{N})M/M \subseteq K/M$,*

$$\mathrm{span}_{\mathbb{C}}\{\{d\pi_\lambda(X)_m : X \in \overline{\mathfrak{n}}\} \cup \{\mathbf{1}\}\} \supseteq T_m(K/M).$$

Proof. Recall that the G-action on $\mathcal{H}_\lambda$ is given by

$$(\pi_\lambda(g)f)(kM) = f(\kappa(g^{-1}k)M)a(g^{-1}k)^{\lambda-\rho}.$$

Hence we get for all $X \in \mathfrak{g}$ that

$$\begin{aligned}(d\pi_\lambda(X)f)(kM) &= \frac{d}{dt}\Big|_{t=0} f(\kappa(\exp(-tX)k)M)a(\exp(-tX)k)^{\lambda-\rho}\\ &= -(\widetilde{X}f)(kM) + f(k)(\lambda-\rho)\Big(p_{\mathfrak{a}}(\mathrm{Ad}(k^{-1})X)\Big)\end{aligned}$$

with $p_{\mathfrak{a}}\colon \ \mathfrak{g} \to \mathfrak{a}$ the projection along $\mathfrak{k}+\mathfrak{n}$. In view of Lemma 6.12, this concludes the proof of the lemma. $\square$

Truncation at infinity. Recall that G is a semisimple Lie group of real rank one and $(\pi_\lambda, \mathcal{H}_\lambda)$, $\lambda \in i\mathfrak{a}^*$, a unitary principal series representation of G which is realized on $L^2(\overline{N})$, with $\overline{N} \cong \mathbb{R}^p \oplus \mathbb{R}^q$.

For $0 < \varepsilon \le 1$ let $a_\varepsilon \in \Omega_i$ be as before. Define a function on $\mathbb{R}^p \oplus \mathbb{R}^q$ by

$$f_\varepsilon(X,Y) = \frac{1}{[(1+e^{i(\frac{\pi}{2}-\varepsilon)}\|X\|^2)^2 + e^{i(\pi-2\varepsilon)}\|Y\|^2]^{\frac{p+2q}{4}-\frac{\lambda}{2}}}.$$

Then, in the noncompact realization of π_λ on $L^2(\overline{N})$, the vector $\pi_\lambda(a_\varepsilon)v_0$ is of the form $c(\lambda,\varepsilon)f_\varepsilon$ where $c(\lambda,\varepsilon)$ is a constant depending only on λ and ε and uniformly bounded in ε (cf. §5).

In order to estimate $S_k(\pi_\lambda(a_\varepsilon)v_0)$ first we show that the behaviour at infinity does not contribute to the singularity as $\varepsilon \to 0$.

LEMMA 6.14. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$, $\lambda \in i\mathfrak{a}^*$, be a unitary spherical principal series representation of a semisimple Lie group G of real rank one. Let $k \in \mathbb{N}$. Then there exists a constant $C > 0$ depending on λ and k such that*

$$(\forall a \in \Omega_i) \qquad S_k(\pi_\lambda(a)v_0) \le C S_{k,\overline{N}}(\tau\pi_\lambda(a)v_0)$$

for a function $\tau \in C_c^\infty(\overline{N})$ with support in $B := \{Z \in \overline{N} : \|Z\| \le 2\}$.

Proof. To begin we use the compact realization of $\mathcal{H}_\lambda$ as $L^2(K/M)$. Then $\mathcal{H}_\lambda^\infty = C^\infty(K/M)$ and the topology on $\mathcal{H}_\lambda^\infty$ is induced from the usual Sobolev norms $(S_{k,K})_{k\ge 0}$ and, in addition, each S_k is equivalent to $S_{k,K}$.

The Bruhat decomposition gives

$$K/M = \kappa(\overline{N})M/M \cup \{w\}$$

where $\mathcal{W}_{\mathfrak{a}} = \{\mathbf{1}, w\}$. Recall that

$$(\pi_\lambda(a)v_0)(k) = a(a^{-1}k)^{\lambda-\rho}.$$

Let $\varphi \in C^\infty(K/M)$ with $\varphi \equiv 1$ in a small neighborhood of w. We claim that we can make $\operatorname{supp}\varphi$ small enough so that

$$S_{k,K}(\varphi\pi_\lambda(a)v_0) \leq C$$

for all $a \in \Omega_i$. In fact, we have $(\pi_\lambda(a)v_0)(w) = e^{(\lambda-\rho)(\log wa^{-1}w)}$ from which our claim easily follows.

In order to estimate $S_k(\pi_\lambda(a)v_0)$ we have just seen that we can truncate the function $\pi_\lambda(a)v_0$ away from infinity. In particular, we claim that there exists a constant $C > 0$ such that

$$S_k(\pi_\lambda(a)v_0) \leq CS_{k,\overline{N}}((1-\varphi)\pi_\lambda(a)v_0)$$

for all $a \in \Omega_i$. In fact it follows from Lemma 6.13 that, uniformly for all $a \in \Omega_i$,

$$\begin{aligned} S_{k,K}(\pi_\lambda(a)v_0) &\leq S_{k,K}((1-\varphi)\pi_\lambda(a)v_0) + S_{k,K}(\varphi\pi_\lambda(a)v_0) \\ &\leq CS_{k,\overline{N}}((1-\varphi)\pi_\lambda(a)v_0) + C. \end{aligned}$$

Now set $\tau = 1-\varphi$ so that τ is in $C_c^\infty(\overline{N})$ and we can arrange the support so that $\operatorname{supp}\tau \subseteq B$. □

Local estimates for invariant Sobolev norms. As before G denotes a semisimple Lie group of real rank one and $(\pi_\lambda, \mathcal{H}_\lambda)$, $\lambda \in i\mathfrak{a}^*$, a unitary principal series representation of G realized on $L^2(\overline{N})$.

Recall that $\mathfrak{a} = \mathbb{R}A_\alpha$ and define for $t > 0$ the elements $b_t = \exp(\log tA_\alpha) \in A$. Then for $f \in L^2(\overline{N})$,

$$(\pi_\lambda(b_t)f)(X, Y) = t^{\frac{p+2q}{2}-\lambda} f(tX, t^2Y)$$

for all $t > 0$ and $(X, Y) \in \mathbb{R}^p \oplus \mathbb{R}^q$. Also note the action of $\overline{N}$ on $L^2(\overline{N})$:

$$(\pi_\lambda(Y)f)(X) = f(X - Y)$$

for all $X, Y \in \overline{N}$. We also will use the notation $u = \|X\|$ and $v = \|Y\|$ for $(X, Y) \in \mathbb{R}^p \oplus \mathbb{R}^q$.

Our goal is now to obtain estimates for $S_k^G(f)$ for functions with support in the ball $B = \{Z \in \overline{N}\colon \|Z\| \leq 2\}$. Especially we are interested in estimating $S_k^G(\tau f_\varepsilon)$. Now for $\varepsilon \to 0$ the singularity of f_ε lies on the sphere $u = 1$ for $q = 0$ while for $q > 0$ the singularity lies on $u = 0$ and $v = 1$. This makes it necessary to distinguish the cases $q = 0$ and $q > 0$.

For a multi-index $\gamma = (\gamma_1, \ldots, \gamma_{p+q}) \in \mathbb{N}_0^{p+q}$ we set $|\gamma| = \gamma_1 + \ldots + \gamma_{p+q}$ and define the differential operator

$$\partial^\gamma = \frac{\partial^{\gamma_1}}{\partial X_1^{\gamma_1}} \cdot \cdot \frac{\partial^{\gamma_p}}{\partial X_p^{\gamma_p}} \cdot \frac{\partial^{\gamma_{p+1}}}{\partial Y_1^{\gamma_{p+1}}} \cdot \cdot \frac{\partial^{\gamma_{p+q}}}{\partial Y_q^{\gamma_{p+q}}}.$$

The case of $q = 0$. $\overline{N} = \mathbb{R}^p$ in this case and the singularities of f_ε for $\varepsilon \to 0$ lie on the sphere $u = 1$.

LEMMA 6.15. *Suppose that* $q = 0$ *and let* $B = \{X \in \mathbb{R}^p \colon \|X\| \le 2\}$. *Then for every* $k \in \mathbb{N}_0$ *there exists a constant* $C > 0$ *depending on* k *and* λ *such that*

$$S_k^G(f) \le C \sum_{|\gamma| \le k} \| \, |u-1|^{|\gamma|} \partial^\gamma f \|$$

for all smooth functions f *with support in* B.

Proof. The proof is very similar to the proof of Proposition 6.6, just more technical and with more notation. We present the details for the case $p = 2$, the general case simply having more spherical coordinate variables.

Let $\tau_1, \ldots, \tau_n$ be smooth nonnegative functions with $\sum_{i=1}^n \tau_i = \mathbf{1}$ on B. Then for any smooth function f with support in B the definition of the invariant Sobolev norms implies that

$$S_k^G(f) \le \sum_{i=1}^n S_k(\pi_\lambda(g_i)(\tau_i f)) \tag{6.18}$$

for any choice of $g_1, \ldots, g_n \in G$.

Recall that for all smooth functions h with support in $16B$ we have

$$S_k(h) \le C S_{k,\overline{N}}(h)$$

for a constant C depending only on k and λ. If we choose our elements g_i such that $\mathrm{supp}(\pi_\lambda(g_i)\tau_i) \subseteq 16B$, we get from (6.18) that

$$S_k^G(f) \le C \sum_{i=1}^n S_{k,\overline{N}}(\pi_\lambda(g_i)(\tau_i f)). \tag{6.19}$$

Next we make a good choice of functions τ_i and elements g_i. We start with the τ_i.

Let $\varphi(x)$ be the one variable function from the proof of Proposition 6.6. For any $j \in \mathbb{N}_0$ define a function on $\mathbb{R}^p$ by

$$\varphi_j(X) = \varphi(2^j(1-u))$$

and notice that

$$\mathrm{supp}\, \varphi_j \subseteq \{X \in \mathbb{R}^p \colon 2^{-j-1} \le |u-1| \le 2^{-j+1}\} \ . \tag{6.20}$$

From now on we use the fact that $p = 2$. Then elements $X \in \mathbb{R}^2$ are written in polar coordinates as $X = (u\cos 2\pi\theta, u \sin 2\pi\theta)$ with $\theta \in \mathbb{R}$. Now choose a nonnegative smooth function $\psi(\theta)$ with support in $[0, 2]$ such that

$$\sum_{m\in\mathbb{Z}} \psi(m + \cdot) = \mathbf{1}.$$

Fix $j \geq 2$. For $l \in \mathbb{Z}$ we define functions on $\mathbb{R}$ by

$$\psi_{j,l}(\theta) = \psi(2^j\theta - l).$$

Notice that

$$\operatorname{supp}\psi_{j,l} \subseteq \left[\frac{l}{2^j}, \frac{l+2}{2^j}\right]. \tag{6.21}$$

As $j \geq 2$ these intervals have length at most $\frac{1}{2}$ and so these functions descend to smooth functions on the circle $\mathbb{R}/\mathbb{Z}$. In particular, we obtain that

$$\sum_{l=0}^{2^j} \psi_{j,l}(\theta) = 1$$

for all $\theta \in \mathbb{R}/\mathbb{Z}$.

Now we can define our partition of unity. We fix $m \geq 2$ and define for all $2 \leq j \leq m - 1$ and $0 \leq l \leq 2^j$ the functions

$$\tau_{j,l}(X) = \varphi_j(X)\psi_{j,l}(\theta)$$

where $X = (u \cos 2\pi\theta, u \sin 2\pi\theta)$.

Recall the one-variable functions τ_m from Proposition 6.6. We define functions $\tau_{m,l}$, $0 \leq l \leq 2^m$ by

$$\tau_{m,l}(X) = \tau_{m-1}(1 - u)\psi_{m-1,l}(\theta)\ .$$

We claim that we have for all $\gamma \in \mathbb{N}_0^p$ with $|\gamma| \leq k$

$$\|\partial^\gamma \tau_{j,l}\|_\infty \leq C2^{|\gamma|j} \tag{6.22}$$

for a constant C depending only on k, ψ and φ. In fact it follows from (6.20) that all the $\tau_{j,l}$ are supported away from the origin where the cartesian partial derivatives can be dominated by spherical partial derivatives $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial \theta}$. The claim follows then from the construction of the $\tau_{m,j}$.

It also follows from the construction of the $\tau_{j,l}$ that we have for all choices of j, l that

$$\operatorname{diam}(\operatorname{supp}\tau_{j,l}) \leq 16 \cdot 2^{-j}. \tag{6.23}$$

For a fixed f as in the statement of the lemma we may assume that

$$\sum_{j=2}^{m}\sum_{l=0}^{2^j} \tau_{j,l} = \mathbf{1} \qquad \text{on} \qquad \operatorname{supp} f.$$

Then from (6.19) we get

$$S_k^G(f) \le C \sum_{j=2}^{m} \sum_{l=0}^{2^j} S_{k,\overline{N}}(\pi_\lambda(g_{j,l})(\tau_{j,l} f)) \tag{6.24}$$

for elements $g_{j,l} \in G$ to be specified.

For each j, l we now pick an element $Z_{j,l} \in \operatorname{supp} \tau_{j,l}$. Then define

$$g_{j,l} = (-Z_{j,l}) b_{2^{-j}} \in A\overline{N} \subseteq G$$

and notice that

$$(\pi_\lambda(g_{j,l})h)(X) = 2^{-\frac{jp}{2}+\lambda} h(2^{-j}X + Z_{j,l}) \tag{6.25}$$

for all functions h and $X \in \mathbb{R}^p$. Then (6.20), (6.21), (6.23) and (6.25) imply that

$$\operatorname{supp}[\pi_\lambda(g_{j,l})\tau_{j,l}] \subseteq 16B.$$

We choose m large enough such that

$$\sum_{j=0}^{2^m} S_{k,\overline{N}}(\pi_\lambda(g_{m,j})\tau_{m,j} f) \le \|f\|$$

holds. This is possible in view of (6.22) and (6.25). Also from (6.22) we can obtain

$$\begin{aligned} S_k^G(f) &\le \|f\| \\ &\quad + C \sum_{|\gamma|\le k} \sum_{j=0}^{m-1} \sum_{l=0}^{2^j} \left(\int_{2^j(\operatorname{supp}\tau_{j,l} - Z_{j,l})} 2^{-2j|\gamma|} 2^{-jp} |(\partial^\gamma f)(2^{-j}X + Z_{j,l})|^2 dX \right)^{\frac{1}{2}} \\ &\le \|f\| + C \sum_{|\gamma|\le k} \sum_{j=0}^{m-1} \sum_{l=0}^{2^j} \left(\int_{\operatorname{supp}\tau_{j,l}} 2^{-2j|\gamma|} |(\partial^\gamma f)(X)|^2 \, dX \right)^{\frac{1}{2}}. \end{aligned}$$

Now on $\operatorname{supp}\tau_{j,l}$ we have $2^{-j} \le 2|u-1|$ by (6.20) and so it follows that

$$\begin{aligned} S_k^G(f) &\le \|f\| + C \sum_{|\gamma|\le k} \sum_{j=0}^{m-1} \sum_{l=0}^{2^j} \left(\int_{\operatorname{supp}\tau_{j,l}} |u-1|^{2|\gamma|} |(\partial^\gamma f)(X)|^2 \, dX \right)^{\frac{1}{2}} \\ &\le C \sum_{|\gamma|\le k} \| \, |u-1|^{|\gamma|} \partial^\gamma f \|, \end{aligned} \tag{6.26}$$

as was to be shown. □

The case of $q > 0$. Here we have $\overline{N} = \mathbb{R}^p \oplus \mathbb{R}^q$ and the singularity of f_ε lies on the sphere $u = 0$ and $v = 1$. The analog of Lemma 6.15 for this geometry is

LEMMA 6.16. *Suppose that $q > 0$ and let $B = \{Z \in \mathbb{R}^p \oplus \mathbb{R}^q : \|Z\| \le 2\}$. Then for every $k \in \mathbb{N}_0$ there exists a constant $C > 0$ depending on k and λ such that*

$$S_k^G(f) \le C \sum_{|\gamma| \le k} \| \, |u|^{|\gamma|} \partial^\gamma f \| \tag{1}$$

and

$$S_k^G(f) \le C \sum_{|\gamma| \le k} \| \, |v-1|^{|\gamma|} \partial^\gamma f \| \tag{2}$$

for all smooth functions f with support in B.

Proof. The proof is essentially the same as the one of Lemma 6.15 and we describe only the necessary modifications.

The partition of unity is now by the truncators

$$\tau_{j,l}(X,Y) = \varphi(2^j u)\varphi(2^{2j}(1-v))\psi_{2j,l}(\theta)$$

with θ the spherical variable of $\mathbb{R}^q$. Next, as done earlier leading to (6.25), choose

$$g_{j,l} = (-Z_{j,l})b_{2^{-j}} \in A\overline{N}$$

with $Z_{j,l} \in \mathbb{R}^q$ and in the support of $\varphi(2^{2j}(1-v))\psi_{2j,l}(\theta)$. Then in the last step (6.26) of Lemma 6.15 we can interpret 2^{-j} either as u or $|1-v|$. Accordingly (1) and (2) follow. $\square$

Sharp estimates for invariant Sobolev norms for real rank one.

THEOREM 6.17. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$, $\lambda \in i\mathfrak{a}^*$, be a unitary spherical principal series representation of a semisimple Lie group G of real rank one. Let $k \in \mathbb{N}$. Then there exists a constant $C > 0$ depending on λ and k such that*

$$(\forall a \in \Omega_i) \qquad S_k^G(\pi_\lambda(a)v_0) \le C\|\pi_\lambda(a)v_0\|.$$

Proof. For $m, n \in \mathbb{N}_0$ define a function on $\overline{N} = \mathbb{R}^p \times \mathbb{R}^q$ by

$$f_\varepsilon^{m,n}(X,Y) = \frac{1}{[(1+e^{i(\frac{\pi}{2}-\varepsilon)}\|X\|^2)^2 + e^{i(\pi-2\varepsilon)}\|Y\|^2]^{\frac{m+2n}{4}-\frac{\lambda}{2}}}.$$

Notice that $f_\varepsilon = f_\varepsilon^{p,q}$.

In view of Lemma 6.14, it is sufficient to prove that there exists a constant $C > 0$ such that

$$S_{k,\overline{N}}^G(\tau f_\varepsilon^{p,q}) \le C\|\pi_\lambda(a_\varepsilon)v_0)\| \tag{6.27}$$

for all $0 < \varepsilon \le 1$. Here $\tau \in C^\infty(\overline{N})$ is a fixed function with support in the ball $\|Z\| \le 2$, $Z \in \overline{N}$.

In order to keep track of what differentiation does to the function $f_\varepsilon^{p,q}$ we define a shift operator $\mathcal{L}$ by

$$\mathcal{L}(f_\varepsilon^{m,n}) = \begin{cases} f_\varepsilon^{m,n+2} & \text{for } n > 0 \\ f_\varepsilon^{m+2,0} & \text{for } n = 0. \end{cases}$$

Now for $Z \in \overline{\mathfrak{n}}$ the elements $d\pi_\lambda(Z)$ are the usual differentiations on $\overline{N} = \mathbb{R}^p \times \mathbb{R}^q$. Consider the action of $d\pi_\lambda(Z)$ on $\tau f_\varepsilon^{m,n}$. When applied to τ it essentially does not increase the function for $\varepsilon \to 0$. Applied to $f_\varepsilon^{p,q}$ it increases the exponent $\frac{p+2q}{4} - \frac{\lambda}{2}$ by 1 and multiplies with a function $P_\varepsilon(X, Y)$, uniformly bounded in ε, X and Y, hence can be bounded by a constant. Thus, for every $Z \in \overline{\mathfrak{n}}$ and $k \in \mathbb{N}_0$ we obtain the inequality

$$|Z^k \tau f_\varepsilon^{p,q}| \leq C |\tau \mathcal{L}^k (f_\varepsilon^{p,q})| \tag{2}$$

for all $0 < \varepsilon \leq 1$ and a constant $C > 0$ independent of ε.

We now distinguish the cases $q = 0$ and $q > 0$. We start with the $q = 0$ case. In view of (6.27), (6.28) and Lemma 6.15 it is sufficient to show that

$$\|\ \tau |u-1|^k f_\varepsilon^{p+2k,0}\| \leq C \|\tau f_\varepsilon\|$$

as $\|\tau f_\varepsilon\| \asymp \|\pi_\lambda(a_\varepsilon)v_0\|$. Now this estimate is proved as in Lemma 5.5.

Finally the $q > 0$ case is proved similarly by employing (6.27), (6.28), Lemma 6.16 (1) and the proof of Lemma 5.6. □

Remark 6.18. Using the reduction results in Lemma 6.3 and Lemma 6.4 one can prove for an arbitrary K-finite vector that

$$(\forall a \in \Omega_i) \qquad S_k^G(\pi_\lambda(a)v) \leq C\|\pi_\lambda(a)v\|.$$

Since we do not need this result in this paper and since the proof, albeit more complicated, is fundamentally the same, we omit the details.

However, calculations in the higher rank case prompt us to state the following conjecture.

Conjecture C. Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G and v a K-finite vector. Then there exists a constant $C > 0$ such that

$$(\forall a \in \Omega_i) \qquad S_k^G(\pi(a)v) \leq C\|\pi(a)v\|.$$

7. Applications to automorphic forms

For completeness we recall a few notions from automorphic forms. We denote by $\|g\|$, $g \in G$, the operator norm of $\operatorname{Ad} g$, and we write $\mathcal{Z}(\mathfrak{g})$ for the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$.

Definition 7.1. (cf. [Bo97, Ch. V]). Let $\Gamma < G$ be a discrete subgroup of co-finite volume. A smooth function $f\colon G \to \mathbb{C}$ is called an *automorphic form* for Γ if the following conditions are satisfied:

(Aut1) f is left Γ-invariant, i.e., $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma$, $g \in G$.

(Aut2) f is right K-finite, i.e., $\operatorname{span}_{\mathbb{C}}\{f(\cdot k)\colon k \in K\}$ is a finite dimensional subspace in $C^\infty(G)$.

(Aut3) f is $\mathcal{Z}(\mathfrak{g})$-finite, i.e., $\mathcal{Z}(\mathfrak{g})f$ is a finite-dimensional subspace of $C^\infty(G)$.

(Aut4) f is of polynomial growth, i.e., there exists an $n \in \mathbb{N}$ and a $C > 0$ such that

$$|f(g)| \leq C\|g\|^n$$

for all g in a Siegel set $\mathcal{S} \subseteq G$ for the group Γ.

If $(\pi, \mathcal{H})$ is a unitary representation of G, then we write $\mathcal{H}^\infty$ for the Fréchet submodule of smooth vectors. The space of distribution vectors $\mathcal{H}^{-\infty}$ is by definition the strong antidual of $\mathcal{H}^\infty$. If $\Gamma < G$ is a subgroup, then we write $(\mathcal{H}^{-\infty})^\Gamma$ for the Γ-invariants of $\mathcal{H}^{-\infty}$.

The following proposition is well known but, because it is crucial to the approach used, we include its short proof.

Proposition 7.2. *Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G, $\eta \in (\mathcal{H}^{-\infty})^\Gamma$ and $v \in \mathcal{H}$ a K-finite vector. Then the function*

$$\theta_{v,\eta}\colon G \to \mathbb{C}, \quad g \mapsto \langle \pi(g)v, \eta\rangle = \overline{\eta(\pi(g)v)}$$

is an automorphic form.

Proof. Since K-finite vectors are analytic, the function $\theta_{v,\eta}$ is defined. As η is Γ-invariant, (Aut1) follows, while (Aut2) is a consequence of the K-finiteness of v. Since $(\pi, \mathcal{H})$ is irreducible, (Aut3) is a consequence of Schur's Lemma. Finally, the fact that $\mathcal{H}^\infty$ has moderate growth (cf. [Wal92, 11.5.1]) implies (Aut4). □

Remark 7.3. There is also a very useful converse to Proposition 7.2, i.e., every automorphic form is a generalized matrix coefficient (cf. [Wal92, 11.9.2]).

Proposition 7.4. *Let G be a semisimple Lie group with $G \subseteq G_{\mathbb{C}}$. Let $\Gamma < G$ be a co-compact subgroup. Let $(\pi_\lambda, \mathcal{H}_\lambda)$, $\lambda \in i\mathfrak{a}^*$, be a unitary spherical principal series representation. Let $\eta \in (\mathcal{H}_\lambda^{-\infty})^\Gamma$ define an embedding*

$$\mathcal{H}_\lambda^\infty \to C^\infty(\Gamma\backslash G), \quad v \mapsto \theta_{v,\eta};\ \theta_{v,\eta}(\Gamma g) = \langle \pi_\lambda(g)v, \eta\rangle.$$

Then for all $k > \frac{1}{2}\dim G$ there exists a constant $C > 0$ such that

$$(\forall v \in \mathcal{H}_\lambda^\infty) \qquad \|\theta_{v,\eta}\|_\infty \leq C S_k(v).$$

In particular, since $\|\cdot\|_\infty$ *is* G*-invariant*

$$(\forall v \in \mathcal{H}_\lambda^\infty) \qquad \|\theta_{v,\eta}\|_\infty \leq CS_k^G(v).$$

Proof. This is the content of [BeRe99, Lemma 3.3 and Prop. B.2]. □

Combining Theorem 6.17, Proposition 7.4 and Remark 6.18 we obtain the following L^∞ estimate on automorphic forms.

THEOREM 7.5. *Let* $(\pi_\lambda, \mathcal{H}_\lambda)$, $\lambda \in i\mathfrak{a}^*$, *be a unitary spherical principal series representation of a semisimple Lie group* G *of real rank one. Let* $\eta \in (\mathcal{H}_\lambda^{-\infty})^\Gamma$. *For any* K*-finite vector* $v \in \mathcal{H}_{\lambda,K}$ *there is a constant* C *such that for all* $a \in \Omega_i$

$$\|\theta_{\pi_\lambda(a)v,\eta}\|_\infty \leq C\|\pi_\lambda(a)v\|.$$

Triple products of Maaß forms for real rank one. Let Γ be a co-compact discrete subgroup of G and set $Y = \Gamma\backslash G$ and $X = \Gamma\backslash G/K$. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we have defined the K-spherical principal series representation $(\pi_\lambda, \mathcal{D}_\lambda)$. If π_λ is unitarizable, then $\mathcal{K}_\lambda$ denotes the Hilbert completion of $\mathcal{D}_\lambda$ in the compact realization. Denote by $\widehat{G}$ the unitary dual of G and by $\widehat{G}_s \subset \widehat{G}$ the subset corresponding to the K-spherical representations, i.e., corresponding to the unitarizable K-spherical principal series. It is then convenient to consider $\widehat{G}_s$ as a subset of $\mathfrak{a}_\mathbb{C}^*$ by identifying the equivalence class of π_λ with λ.

For $\Gamma < G$ co-compact the Plancherel theorem for the right regular action of G on $L^2(\Gamma\backslash G)$ says

$$L^2(\Gamma\backslash G) \cong \widehat{\bigoplus}_{\pi\in\widehat{G}} m_\pi \mathcal{K}_\pi. \tag{7.1}$$

Here $m_\pi = \dim(\mathcal{K}_\pi^{-\infty})^\Gamma < \infty$ is the multiplicity of $(\pi, \mathcal{K}_\pi)$ in $L^2(\Gamma\backslash G)$. If $0 \neq \eta \in (\mathcal{K}_\pi^{-\infty})^\Gamma$, then the G-equivariant map

$$\mathcal{K}_\pi^\infty \to C^\infty(\Gamma\backslash G), \quad v \mapsto (\Gamma g \mapsto \langle \pi(g)v, \eta\rangle)$$

extends (up to multiplication by a scalar) to an isometry $\mathcal{K}_\pi \to L^2(\Gamma\backslash G)$.

Write $\mathcal{K}_\pi^K$ for the subspace of K-fixed elements and recall that $\dim \mathcal{K}_\pi^K = 1$ for $\pi \in \widehat{G}_s$ and zero otherwise. Taking K-fixed vectors in (7.1) we obtain that

$$L^2(\Gamma\backslash G/K) \cong \widehat{\bigoplus}_{\pi\in\widehat{G}_s} m_\pi \mathcal{K}_\pi^K. \tag{7.2}$$

We will identify $L^2(\Gamma\backslash G/K)$ with a subspace of $L^2(\Gamma\backslash G)$.

If $v_0 \in \mathcal{K}_\pi^K$ and $\eta \in (\mathcal{K}_\pi^{-\infty})^\Gamma$, then

$$\psi_{v_0,\eta}(\Gamma gK) = \langle \pi(g)v_0, \eta\rangle$$

defines an element in $C^\infty(\Gamma\backslash G/K)$. The function $\psi_{v_0,\eta}$ is referred to as a *Maaß form.*

Let $(\psi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of Maaß forms of $L^2(\Gamma\backslash G/K)$. Then each $\psi_n(\Gamma gK)$ is of the form $\langle \pi_{\lambda_n}(g)v_0^n, \eta\rangle$ for a specific $\lambda_n \in \widehat{G}_s$, a unit vector $v_0^n \in \mathcal{K}_{\lambda_n}^K$ and a specific $\eta \in (\mathcal{H}_{\lambda_n}^{-\infty})^\Gamma$.

Fix $X_0 \in \partial\Omega_i$ and for $v \in \mathcal{H}_{\lambda,K}$ and $0 < \varepsilon \leq 1$ set

$$v_\varepsilon = \pi_\lambda(\exp(i(1-\varepsilon)X_0))v.$$

If $\psi = \theta_{v_0,\eta}$ is a Maaß form, then we write $\psi_\varepsilon = \theta_{(v_0)_\varepsilon,\eta} \in L^2(Y)$ for all $0 < \varepsilon \leq 1$. Now, since ψ is continuous, we have $\psi^2 \in L^2(X) \subseteq L^2(Y)$. Hence we get

$$\psi^2 = \sum_{i\in I} c_i\psi_i \qquad \text{with} \qquad c_i = \langle \psi^2, \psi_i\rangle. \tag{7.3}$$

If one considers (7.3) as an identity in $L^2(Y)$, then analytic continuation yields

$$\psi_\varepsilon^2 = \sum_{i\in I} c_i \psi_{i,\varepsilon},$$

for all $0 < \varepsilon \leq 1$. Taking norms, we get

$$\|\psi_\varepsilon^2\|^2 = \sum_{i\in I} |c_i|^2 \|\psi_{i,\varepsilon}\|^2 = \sum_{i\in I} |c_i|^2 \|(v_0^i)_\varepsilon\|^2. \tag{7.4}$$

THEOREM 7.6. *Let G be a simple Lie group of real rank one and $\Gamma < G$ a co-compact discrete subgroup. Then for every Maaß form ψ, the coefficients c_i of the Fourier series of $\psi^2 = \sum_{i\in I} c_i\psi_i$ satisfy the following estimates.*

(i) *If $q = 0$, then there exists a constant $C > 0$ such that for all $T > 1$,*

$$\sum_{|\lambda_i|\leq T} |c_i|^2 e^{\pi|\lambda_i|} \leq C \cdot \begin{cases} T^{2p-2} & \text{if } p > 1, \\ (\log T)^2 & \text{if } p = 1. \end{cases}$$

(ii) *If $q > 0$, then there exists a constant $C > 0$ such that for all $T > 1$,*

$$\sum_{|\lambda_i|\leq T} |c_i|^2 e^{\frac{\pi}{2}|\lambda_i|} \leq C \begin{cases} T^{2q-1} & \text{if } q > 1, \\ T(\log T)^2 & \text{if } q = 1. \end{cases}$$

Proof. We start the proof with the identity (7.4):

$$\|\psi_\varepsilon^2\|^2 = \sum_{i\in I} |c_i|^2 \|(v_0^i)_\varepsilon\|^2.$$

Now $\|\psi_\varepsilon^2\|^2 \leq \|\psi_\varepsilon\|_\infty^2 \|\psi_\varepsilon\|^2 = \|\psi_\varepsilon\|_\infty^2 \|(v_0)_\varepsilon\|^2$.

(i) If $q = 0$, then we have by Theorem 7.5 and Theorem 5.1(ii) for $\varepsilon \to 0^+$,

$$\|\psi_\varepsilon^2\|^2 \leq C \begin{cases} \varepsilon^{2-2p} & \text{if } p > 1 \\ |\log\varepsilon|^2 & \text{if } p = 1 \end{cases}$$

for some constant C. On the other hand from Proposition 5.7 we get a lower bound. Thus

$$\sum_{i\in I}|c_i|^2 e^{\pi|\lambda_i|}e^{-7\varepsilon|\lambda_i|}\leq C\begin{cases}\varepsilon^{-2p+2} & \text{if } p>1,\\ |\log\varepsilon|^2 & \text{if } p=1.\end{cases}$$

Setting $\varepsilon = \frac{1}{T}$ and collecting the λ_i with $|\lambda_i| \leq T$, the assertion in (i) follows.

(ii) By Theorem 7.5, Theorem 5.1(ii) and Proposition 5.7 the proof goes as in (i). □

Remark 7.7. (a) For $q = 0$ and $p = 1$ the estimate in Theorem 7.6 (i) is a slight improvement of that obtained by Bernstein and Reznikov (cf. [BeRe99]), viz. $(\log T)^3$ compared to our $(\log T)^2$.

(b) For $q = 0$ and $p = 2$ (this corresponds to $G/K \cong \mathbb{H}^3$), Sarnak proved in [Sa94] that

$$|c_i| \leq C(|\lambda_i|^2+1)^{\frac{3}{2}}e^{-\frac{\pi}{2}|\lambda_i|}$$

for all i. Our estimate in Theorem 7.5(i) yields the slight improvement to

$$|c_i| \leq C|\lambda_i|e^{-\frac{\pi}{2}|\lambda_i|}.$$

(c) With a more detailed analysis one can improve on the lower estimate in Proposition 5.7 in the case of $q = 0$. One can show that

$$\|(v_0)_\varepsilon\|^2 = |\varphi_\lambda(\exp(-2i(1-\varepsilon)X))| \geq Ce^{(\pi-\varepsilon)|\lambda|}\begin{cases}|\log\varepsilon| & \text{for } p=1,\\ \varepsilon^{-p+1} & \text{for } p>1,\end{cases}$$

for all $0 < \varepsilon \leq 1$. In particular, this gives a small improvement on the estimates of the triple products.

(d) We have presented these techniques for co-compact Γ; however, they apply equally well to finite volume but not co-compact lattices. We illustrate this in the next section where we obtain estimates on triple products of cusp forms.

8. $G = \mathrm{Sl}(2,\mathbb{R})$

Analytic continuation of the discrete series. Let $G = \mathrm{Sl}(2,\mathbb{R})$ and choose $K = \mathrm{SO}(2,\mathbb{R})$ as a maximal compact subgroup. For every $m \in \mathbb{Z}$ define a character χ_m of K by setting

$$\chi_m\left(\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right) = e^{\mathrm{im}\theta} \qquad (\theta\in\mathbb{R}).$$

We will identify $\widehat{K}$ with $\mathbb{Z}$ by means of the above isomorphism.

For every $k \in \mathbb{N}$ there exists a unitary highest weight representation of G with highest weight $-k$ and K-weight spectrum $-k, -k-2, -k-4, \ldots$. For $k \geq 2$ we obtain the discrete series which can be realized in the holomorphic functions on the upper half-plane $X = \{z \in \mathbb{C}: \operatorname{Im} z > 0\}$. More precisely, a unitary highest weight representation $(\pi_k, \mathcal{H}_k)$ with highest weight $-k$, $k \geq 2$, is given by the Hilbert space

$$\mathcal{H}_k = \{f \in \mathcal{O}(X): \int_H |f(z)|^2 \frac{dx\,dy}{y^{2-k}} < \infty\}$$

and the action

$$(\pi_k(g)f)(z) = (cz+d)^{-k} f\Big(\frac{az+b}{cz+d}\Big) \qquad (g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}).$$

We recall from Example 4.3 the notation A, $A_{\mathbb{C}}$, and $A_{\mathbb{C}}^1$. For $\varepsilon > 0$ small we define elements $a_\varepsilon \in A_{\mathbb{C}}^1$ by

$$a_\varepsilon = \begin{pmatrix} e^{i\frac{\pi}{4}(1-\varepsilon)} & 0 \\ 0 & e^{-i\frac{\pi}{4}(1-\varepsilon)} \end{pmatrix}.$$

If v is a K- weight vector of $(\pi_k, \mathcal{H}_k)$, then we are interested in estimating $\|\pi_k(a_\varepsilon)v\|^2$ for $\varepsilon \to 0$. The estimates given thus far have been related to principal series representations. We remark that the method we shall follow applies more generally to unitary highest weight modules of other groups.

We have found that estimates for $\|\pi_k(a_\varepsilon)v\|^2$ are obtained more easily if we switch to the realization on the positive real axis. For $k \in \mathbb{N}$ we define the Hilbert space:

$$\mathcal{W}_k = L^2(\mathbb{R}^+, x^k \frac{dx}{x}) = \{f: \mathbb{R}^+ \to \mathbb{C}: \int_0^\infty |f(x)|^2 x^k \frac{dx}{x} < \infty\}.$$

Then for $k \geq 2$ the mapping

$$\Phi_k: \mathcal{W}_k \to \mathcal{H}_k, \quad f \mapsto \Phi_k(f);\ \Phi_k(f)(z) = \int_0^\infty e^{ixz} f(x) x^k \frac{dx}{x}$$

is, by the Paley-Wiener theorem, up to multiplication by a scalar, an isomorphism of Hilbert spaces. Thus we can transport the G-action on $\mathcal{H}_k$ and obtain a unitary G-action on $\mathcal{W}_k$. Call this representation $(\rho_k, \mathcal{W}_k)$. Of course, one also has a unitary highest weight representation $(\rho_1, \mathcal{W}_1)$ with highest weight -1.

We have for all $k \geq 1$

$$(\forall a > 0)(\forall f \in \mathcal{W}_k) \qquad (\rho_k(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix})f)(x) = a^k f(a^2 x).$$

For every $m \in \mathbb{Z}$ let $\mathcal{W}_k^m = \{v \in \mathcal{W}_k: (\forall k \in K)\ \pi_k(k)v = \chi_m(k)v\}$. Then

$$\mathcal{W}_k = \widehat{\bigoplus}_{j=0}^{\infty} \mathcal{W}_k^{-k-2j}$$

is the orthogonal decomposition of $\mathcal{W}_k$ into K-isotypical components. Moreover, one can show that

$$\mathcal{W}_k^{-k-2j} = \mathbb{C}\{e^{-x}p_j(x)\}$$

with $p_j(x)$ a polynomial of degree j.

For the discrete series of $\mathrm{Sl}(2,\mathbb{R})$ we come now to their analog of Theorem 5.1.

THEOREM 8.1. *Let $(\rho_k, \mathcal{H}_k)$ be a unitary highest weight representation of $G = \mathrm{Sl}(2,\mathbb{R})$ with lowest weight $-k \in -\mathbb{N}$. Then for every K-finite vector $v \in \mathcal{W}_k$ with weight $m \in -k - 2\mathbb{N}_0$,*

$$\|\rho_k(a_\varepsilon)v\|^2 \asymp \varepsilon^{-|m|}.$$

Proof. Let $m = -k - 2j$. Then v is a multiple of $f(x) = e^{-x}p_j(x)$. Hence it suffices to show that for $g(x) = e^{-x}x^n$ one has

$$\|\rho_k(a_\varepsilon)g\|^2 \asymp \varepsilon^{-2n-k}$$

for every $n \in \mathbb{N}_0$.

Clearly for fixed g the map

$$F\colon A_{\mathbb{C}}^1 \to \mathcal{W}_k;\ F(a)(x) \ = a^k g(a^2x)$$

is analytic. Since $F|_A = \rho_k|_A(\cdot)g$, we have that

$$(\rho_k(a)g)(x) = a^k g(a^2x)$$

for all $a \in A_{\mathbb{C}}^1$. Therefore

$$\begin{aligned}
\|\rho(a_\varepsilon)g\|^2 &= \int_0^\infty |e^{-a_\varepsilon^2 x}(a_\varepsilon^2 x)^n|^2 x^k \ \frac{dx}{x} \\
&= |a_\varepsilon|^{4n}\int_0^\infty e^{-2\,\mathrm{Re}(a_\varepsilon^2)x}x^{2n+k} \ \frac{dx}{x} \\
&\asymp (\mathrm{Re}\, a_\varepsilon^2)^{-2n-k} = (\mathrm{Re}\, e^{i\frac{\pi}{2}(1-\varepsilon)})^{-2n-k} \asymp \varepsilon^{-2n-k}.
\end{aligned}$$

□

Automorphic forms associated to the discrete series. Let $\Gamma < G$ be a lattice in G but not co-compact. Let $P < G$ be a parabolic subgroup of G and $P = MA_PN_P$ with $M = \{\pm I\}$ its Langlands decomposition. Call P *cuspidal* for Γ if $\Gamma \cap N_P \neq \{\mathbf{1}\}$.

Definition 8.2. An automorphic form $f\colon G \to \mathbb{C}$ is called a *cusp form* if for all cuspidal parabolic subgroups $P < G$,

$$(\forall g \in G) \qquad \int_{(N_P\cap\Gamma)\backslash N_P} f(ng)\ dn = 0.$$

Recall from [Bo97, Cor. 8.7] that every cusp form $f\colon G \to \mathbb{C}$ belongs to $L^p(\Gamma\backslash G)$ for all $1 \le p \le \infty$.

If $(\pi, \mathcal{H})$ is an irreducible unitary representation of G, with space of K-finite vectors $\mathcal{H}_K$, set

$$(\mathcal{H}^{-\infty})_c^\Gamma = \{\eta \in (\mathcal{H}^{-\infty})^\Gamma \colon \theta_{v,\eta} \text{ is a cusp form for all } v \in \mathcal{H}_K\}.$$

Note that if $v \in \mathcal{H}_K$, $v \neq 0$, then the irreducibility of $(\pi, \mathcal{H})$ implies that

$$(\mathcal{H}^{-\infty})_c^\Gamma = \{\eta \in (\mathcal{H}^{-\infty})^\Gamma \colon \theta_{v,\eta} \text{ is a cusp form}\}.$$

Let now $\eta \in (\mathcal{H}^{-\infty})_c^\Gamma$. Then the map

$$\mathcal{H}_K \to L^2(\Gamma\backslash G), \quad v \mapsto \theta_{v,\eta}$$

gives rise (up to scalar multiple) to an isometric embedding

$$\mathcal{H} \to L^2(\Gamma\backslash G).$$

For v a K-weight vector, as usual, we set $v_\varepsilon = \pi(a_\varepsilon)v$ for all $\varepsilon > 0$ small.

Theorem 8.3. *Let $(\pi_k, \mathcal{H}_k)$, $k \ge 2$, be a holomorphic discrete series representation. Let $\eta \in (\mathcal{H}_k^{-\infty})_c^\Gamma$ and $v \in \mathcal{H}_k$ a K-weight vector of weight m. Then for ε small the following assertions hold*:

(i) $\|\theta_{v_\varepsilon,\eta}\|_{L^2(\Gamma\backslash G)} \asymp \varepsilon^{-\frac{|m|}{2}}$;

(ii) $\|\theta_{v_\varepsilon,\eta}\|_\infty \le C\varepsilon^{-\frac{|m|}{2}}$ *for a constant C depending only on v.*

Proof. (i) is just a restatement of Theorem 8.1, since the embedding $\mathcal{H}_k \to L^2(\Gamma\backslash G)$ is (up to scalar multiple) isometric, see Proposition 7.4.

(ii) First we need a little notation. Let X_1, X_2, X_3 be a basis of $\mathfrak{g}$. The n^{th} Sobolev norm of the representation $(\pi_k, \mathcal{H}_k)$ is given by the equivalent norm

$$(\forall v \in \mathcal{H}_k^\infty) \qquad S_n(v) = \|(\mathbf{1} - \sum_{i=1}^3 d\pi_k(X_i)^2)^{\frac{n}{2}} v\|.$$

It follows from [BeRe99, Prop. 4.1] that

$$\|\theta_{v,\eta}\|_\infty \le CS_3(v) \quad \text{for all } v \in \mathcal{H}_k^\infty.$$

Moreover since $\|\cdot\|_\infty$ is G-invariant we obtain

$$\|\theta_{v,\eta}\|_\infty \le CS_3^G(v) \quad \text{for all } v \in \mathcal{H}_k^\infty,$$

where $S_k^G(\cdot) = \inf_{g\in G} S_k(\pi_k(g)\cdot)$ is the infimum seminorm (cf. Definition 6.1).

We will work with the realization $\mathcal{W}_k$ on $\mathbb{R}^+$ of π_k. Then

$$d\pi_k(\mathcal{U}(\mathfrak{g}_\mathbb{C})) \subseteq \mathbb{C}[x, \frac{d}{dx}]$$

which can be seen either by direct computation or as a special case of a general result on unitary highest weight representations (cf. [KrNe00]). Set $D^N = (\mathbf{1} - \sum_{i=1}^3 d\pi_k(X_i)^2)^N$, $N \in \mathbb{N}_0$. It suffices to show that

$$\inf_{g\in G} \|D^N \pi_k(g) v_\varepsilon\| \leq C\|v_\varepsilon\|$$

for any K-weight vector v and a constant C depending only on v and not on ε. We do this only for a highest weight vector v; the computation for the other K-types is similar. Write $D^N = \sum_{j,l=0}^N a_{jl}\ x^j \frac{d^l}{dx^l}$ and note that v is given by the function $v(x) = e^{-x}$. For $t > 0, \in \mathbb{R}$ we define elements $b_t, n_s \in G$ by

$$b_t = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}, \quad \text{and} \quad n_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Then for all $f \in \mathcal{W}_k$

$$(\pi_k(n_s b_t)f)(x) = e^{-isx} t^{\frac{k}{2}} f(tx).$$

Thus,

$$\begin{aligned}\Big(D^N \pi_k(n_s b_t) v_\varepsilon\Big)(x) &= \sum_{j,l=0}^N a_{jl}\ x^j \frac{d^l}{dx^l}\Big(a_\varepsilon^k t^{\frac{k}{2}} e^{-a_\varepsilon^2 tx} e^{-isx}\Big) \\ &= a_\varepsilon^k t^{\frac{k}{2}} \sum_{j,l=0}^N (-is - a_\varepsilon^2 t)^l a_{jl}\ x^j e^{-a_\varepsilon^2 tx} e^{-isx},\end{aligned}$$

and

$$\begin{aligned}\|D^N \pi_k(n_s b_t) v_\varepsilon\|^2 &= t^k \sum_{j,j',l,l'=0}^N a_{jl}\overline{a_{j'l'}}(is + a_\varepsilon^2 t)^l \overline{(is + a_\varepsilon^2 t)}^{l'} \\ &\quad \times \int_0^\infty x^{j+j'} e^{-2\operatorname{Re}(a_\varepsilon^2)tx} x^k\ \frac{dx}{x} \\ &= \sum_{j,j',l,l'=0}^N a_{jl}\overline{a_{j'l'}}(is + a_\varepsilon^2 t)^l \overline{(is + a_\varepsilon^2 t)}^{l'} t^{-(j+j')} \\ &\quad \times (2\operatorname{Re}(a_\varepsilon^2))^{-(j+j'+k)} \int_0^\infty x^{j+j'} e^{-x} x^k\ \frac{dx}{x} \\ &\leq C \sum_{j,j',l,l'=0}^N |is + a_\varepsilon^2 t|^{l+l'} t^{-(j+j')} \varepsilon^{-(j+j'+k)}.\end{aligned}$$

Taking $t = \frac{1}{\varepsilon}$ we get

$$\|D^N \pi_k(n_s b_t) v_\varepsilon\|^2 \leq C\varepsilon^{-k} \sum_{l,l'=0}^N |is + \frac{a_\varepsilon^2}{\varepsilon}|^{l+l'}.$$

Finally for $s = -\frac{1}{\varepsilon}$ the expression $|is + \frac{a_\varepsilon^2}{\varepsilon}|$ is bounded for all $0 < \varepsilon \leq 1$ and so we see that $\inf_{g\in G} \|D^N \pi_k(g).v_\varepsilon\| \leq C\varepsilon^{-\frac{k}{2}}$, completing the proof of (ii). □

Triple products in the co-compact case. We recall how to relate automorphic forms of weight m to automorphic functions on G/K. An automorphic form $f\colon G \to \mathbb{C}$ is called of *weight* $m \in \mathbb{Z}$ if

$$(\forall k \in K) \qquad f(gk) = \chi_m(k) f(g).$$

We identify X with G/K by means of the isomorphism $G/K \to X$, $gK \mapsto g.i$. For every $z \in X$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ set $\mu(g,z) = cz + d$ and recall that μ satisfies the cocycle relation $\mu(g_1 g_2, z) = \mu(g_1, g_2 \cdot z)\mu(g_2, z)$. For $m \in \mathbb{Z}$ we set $\mu_m = \mu^{-m}$ and note that $\mu_m(k,i) = \chi_m(k)$ for all $k \in K$. If $f\colon G \to \mathbb{C}$ is of weight m, then the function

$$F(gK) = \overline{\mu_m(g,i)} f(g) \tag{8.1}$$

defines an analytic function on G/K which satisfies

$$(\forall \gamma \in \Gamma) \qquad F(gK) = \overline{\mu_m(\gamma, gK)}^{-1} F(\gamma g K)$$

for all $gK \in G/K$.

We say that $f\colon G \to \mathbb{C}$ is an *anti-holomorphic automorphic form* if $F\colon G/K \to \mathbb{C}$ is an anti-holomorphic function. In the usual notation we write $M_k^0(\Gamma)$ for the anti-holomorphic cusp forms on $X = G/K$ of weight $k \in \mathbb{N}$. If $(\pi_k, \mathcal{H}_k)$ is a unitary highest weight representation, then we write v_k for a normalized highest weight vector. If $\Theta_{v_k,\eta}$ is the function on G/K associated to $\theta_{v_k,\eta}$ via (8.1), then it can be deduced with the help of Proposition 7.2 and [Wal92, 11.9.2] that the mapping

$$(\mathcal{H}_k^{-\infty})_c^\Gamma \to M_k^0(\Gamma), \quad \eta \mapsto \Theta_{v_k,\eta}$$

is an isomorphism of (finite dimensional) vector spaces.

Let $f\colon G \to \mathbb{C}$ be an automorphic form of weight m. Then $|f|$ factors to a function on G/K which we also denote by $|f|$. Then (8.1) gives

$$(\forall z = x + iy \in X) \qquad |f|^2(z) = y^{-m} |F(z)|^2. \tag{8.2}$$

Set

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and define $\rho \in \mathfrak{a}^*$ by $\rho(X_0) = 1$. In a slightly different way, we will identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ by means of the isomorphism

$$\mathbb{C} \mapsto \mathfrak{a}_{\mathbb{C}}^*, \quad z \mapsto z 2\rho.$$

Let $(\pi_k, \mathcal{H}_k)$, $k \in \mathbb{N}_0$, be a unitary highest weight representation of G and $\theta_{v,\eta}(\Gamma g) = \langle \pi_k(g) v, \eta \rangle$ an automorphic form of weight m. Then $|\theta_{v,\eta}|^2 \in$

$L^2(\Gamma\backslash G/K)$ and as in (7.3) we have

$$|\theta_{v,\eta}|^2 = \sum_{n\in\mathbb{N}} c_n \psi_n$$

with

$$c_n = \langle |\theta_{v,\eta}|^2, \psi_n\rangle.$$

As before, we are interested in estimating the coefficients c_n, the so called *triple products*. For this we use the strategy introduced by Bernstein and Reznikov as used in the proof of Theorem 7.6. Of course, here we must employ the estimates just obtained related to discrete series.

THEOREM 8.4. *Let Γ be a discrete co-compact subgroup of $G = \mathrm{Sl}(2,\mathbb{R})$. Let $\theta_{v,\eta}$ be an automorphic form of weight $m \in Z$ associated to a unitary highest weight representation $(\pi_k, \mathcal{H}_k)$, $k \in \mathbb{N}_0$, of G. If $|\theta_{v,\eta}|^2 = \sum_{n\in\mathbb{N}} c_n\psi_n$ is the orthogonal expansion of $|\theta_{v,\eta}|^2$ in Maaß forms, then there exists a constant $C > 0$ such that for all $T > 0$,*

$$\sum_{|\lambda_n|\leq T} |c_n|^2 e^{\pi|\lambda_n|} \leq CT^{2|m|}.$$

Proof. We start with the identity (7.4),

$$\|\theta^2_{v_\varepsilon,\eta}\|^2 = \sum_{n\in\mathbb{N}} |c_n|^2 \|v^n_{0,\varepsilon}\|^2.$$

In view of Theorem 8.3, the left-hand side can be estimated as

$$\|\theta^2_{v_\varepsilon,\eta}\|^2 \leq \|\theta_{v_\varepsilon,\eta}\|^2_\infty \|\theta_{v_\varepsilon,\eta}\|^2 \leq C_1\varepsilon^{-|m|}\varepsilon^{-|m|} = C_1\varepsilon^{-2|m|}$$

for a constant $C_1 > 0$ independent of $\varepsilon > 0$. On the other hand, from Proposition 5.7,

$$\|v^n_{0,\varepsilon}\|^2 \geq C_2 e^{(\pi-7\varepsilon)|\lambda_n|}$$

for a constant $C_2 > 0$ independent of n and ε. Thus

$$\sum_{n\in\mathbb{N}} |c_n|^2 e^{(\pi-7\varepsilon)|\lambda_n|} \leq C\varepsilon^{-2|m|}.$$

Taking $\varepsilon = \frac{1}{T}$ and collecting all c_n with $|\lambda_n| \leq T$ prove the theorem. □

Triple products for the noncocompact case. Let $P_1, \dots, P_N$ be a set of representatives of the Γ-conjugacy classes of cuspidal parabolic subgroups. Every P_j admits a Levi decomposition $P_j = MA_jN_j$. We choose A_j such that $\mathrm{Lie}(A_j)$ is orthogonal to $\mathfrak{k}$ with respect to the Cartan-Killing form on $\mathfrak{g}$. Write $a_j\colon N_jA_jK \to A_j$ for the middle projection in the Iwasawa decomposition $G = N_jA_jK$.

For every $1 \le j \le N$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ one defines the *Eisenstein series* by

$$E_j(\Gamma g, s) := \sum_{\gamma \in (\Gamma \cap P_i)\backslash\Gamma} e^{(1+s)\rho(\log a_j(\gamma g))} \qquad (g \in G).$$

For convenience we summarize the properties of Eisenstein series needed here, as well as the familiar structure of the Plancherel theorem for $L^2(\Gamma\backslash G)$. [Bo97] is a convenient reference. The Eisenstein series admit a meromorphic continuation in the variable s to the entire complex plane (cf. [Bo97, 11.9]). The meromorphic continuation of E_j will be also denoted by E_j. We note that $E_j(\cdot, s)$, when defined, is an automorphic form in the sense of Definition 7.1 (cf. [Bo97, Th. 10.4]); $E_j(\cdot, s)$ has no poles on $i\mathbb{R}$ (cf. [Bo97, Th. 11.13]).

We normalize the inner product on $\mathcal{K}_\lambda$ such that the spherical vector $v_0^\lambda(g) = a(g)^{-\lambda-\rho}$, $g \in G$, has norm 1. Then the fact that $E_j(\cdot, s)$ is an automorphic form together with [Wal92, 11.9.2] implies the existence of an $\eta_{j,s} \in (\mathcal{K}_s^{-\infty})^\Gamma$ such that

$$\langle \sigma_s(g) v_0^s, \eta_{j,s} \rangle = E_j(\Gamma g, 2s) \quad (g \in G). \tag{8.3}$$

Let $(\rho, L^2(\Gamma\backslash G))$ denote the right regular representation of G on $L^2(\Gamma\backslash G)$. We write $L^2(\Gamma\backslash G)_s$ for the G-invariant subspace generated by $L^2(\Gamma\backslash G/K)$, i.e.,

$$L^2(\Gamma\backslash G)_s = \overline{\operatorname{span}\{\rho(G) L^2(\Gamma\backslash G/K)\}}.$$

One has

$$L^2(\Gamma\backslash G)_s = L^2(\Gamma\backslash G)_{s,d} \oplus L^2(\Gamma\backslash G)_{s,c} \tag{8.4}$$

where

$$L^2(\Gamma\backslash G)_{s,d} = \widehat{\bigoplus}_{\pi \in \widehat{G}_s} m_\pi \mathcal{K}_\pi \tag{8.5}$$

with $m_\pi < \infty$ (cf. [Bo97, Th. 16.2, Th. 16.6]), and

$$L^2(\Gamma\backslash G)_{s,c} = \sum_{j=1}^N \int_{i\mathbb{R}}^{\oplus} \mathcal{K}_{s,j}\; ds \tag{8.6}$$

(cf. [Bo97, Th. 17.7]). In (8.6) the module $\mathcal{K}_{s,j}$ is isometrically equivalent to $\mathcal{K}_s$ and $\mathcal{K}_{s,j}^\infty$ is realized as the image of the G-equivariant embedding

$$\mathcal{H}_s^\infty \to C^\infty(\Gamma\backslash G), \quad v \mapsto (\Gamma g \mapsto \langle \sigma_s(g) v, \eta_{j,s} \rangle). \tag{8.7}$$

Let $(\psi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\Gamma\backslash G)_{s,d} \cap L^2(\Gamma\backslash G/K)$ of Maaß cusp forms. Then $\psi_n(\Gamma g)$ equals $\langle \sigma_{\lambda_n}(g) v_0^n, \eta \rangle$ for v_0^n a normalized K-fixed vector in $\mathcal{K}_{\lambda_n}$ and some element $\eta \in (\mathcal{K}_\lambda^{-\infty})^\Gamma$.

If $f \in L^2(\Gamma\backslash G/K)$, then (8.3)–(8.7) imply that

$$f = \sum_{n=1}^\infty \langle f, \psi_n \rangle \psi_n + \sum_{j=1}^N \int_{\mathbb{R}} \langle f, E_j(\cdot, 2is) \rangle E_j(\cdot, 2is)\; ds \tag{8.8}$$

and

$$\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, \psi_n\rangle|^2 + \sum_{j=1}^{N} \int_{\mathbb{R}} |\langle f, E_j(\cdot, 2is)\rangle|^2 \, ds. \tag{8.9}$$

THEOREM 8.5. *Let $(\pi_k, \mathcal{H}_k)$ be a unitary highest weight representation of G and $f = \theta_{v,\eta}$ an associated cusp form of weight m. Then $|f|^2 \in L^2(\Gamma\backslash G/K)$ and there exists a constant $C > 0$ such that for all $T > 0$,*

$$\sum_{|\lambda_n|\leq T} |\langle |f|^2, \psi_n\rangle|^2 e^{\pi|\lambda_n|} + \sum_{j=1}^{N} \int_{-T}^{T} |\langle |f|^2, E_j(\cdot, 2is)\rangle|^2 e^{\pi|s|} ds \leq CT^{2|m|}.$$

Proof. It is clear that $|f|$ is right K-invariant. Moreover, since cusp forms are rapidly decreasing (cf. [Bo97, Th. 7.5]), we have $|f|^2 \in L^2(\Gamma\backslash G/K)$. Hence (8.8) gives

$$|f|^2 = \sum_{n=1}^{\infty} \langle |f|^2, \psi_n\rangle \psi_n + \sum_{j=1}^{N} \int_{\mathbb{R}} \langle |f|^2, E_j(\cdot, 2is)\rangle E_j(\cdot, 2is) \, ds. \tag{8.10}$$

We analytically continue (8.10) as in the proof of Theorem 8.4. First notice that $E_j(\cdot, 2s)$ corresponds via the Plancherel theorem to $v_0^s \in \mathcal{K}_s$, our unit K-spherical vector in $\mathcal{K}_s$ (cf. (8.3), (8.7)). So we define $E_{j,\varepsilon}(\cdot, 2s)$ as the element corresponding to $v_{0,\varepsilon}^s$. Now analytic continuation of (8.10) in $L^2(\Gamma\backslash G)_s$ gives

$$|f_\varepsilon|^2 = \sum_{n=1}^{\infty} \langle |f|^2, \psi_n\rangle \psi_{n,\varepsilon} + \sum_{j=1}^{N} \int_{\mathbb{R}} \langle |f|^2, E_j(\cdot, 2is)\rangle E_{j,\varepsilon}(\cdot, i2s) \, ds. \tag{8.11}$$

Taking norms in (8.11) we arrive at

$$\||f_\varepsilon|^2\|^2 = \sum_{n=1}^{\infty} |\langle |f|^2, \psi_n\rangle|^2 \|v_{0,\varepsilon}^n\|^2 + \sum_{j=1}^{N} \int_{\mathbb{R}} |\langle |f|^2, E_j(\cdot, 2is)\rangle|^2 \|v_{0,\varepsilon}^s\|^2 \, ds. \tag{8.12}$$

In the proof of Theorem 8.4 we showed that $\||f_\varepsilon|^2\|^2 \leq C_1\varepsilon^{-2|m|}$. It follows from Proposition 5.7 that there exists a constant $C_2 > 0$ such that $\|v_{0,\varepsilon}^s\|^2 \geq C_2 e^{(\pi-7\varepsilon)|s|}$ and $\|v_{0,\varepsilon}^n\|^2 \geq C_2 e^{(\pi-7\varepsilon)|\lambda_n|}$ for all $s \in i\mathbb{R}$ and $n \in \mathbb{N}$. Thus

$$\sum_{n=1}^{\infty} |\langle |f|^2, \psi_n\rangle|^2 e^{(\pi-7\varepsilon)|\lambda_n|} + \sum_{j=1}^{N} \int_{\mathbb{R}} |\langle |f|^2, E_j(\cdot, 2is)\rangle|^2 e^{(\pi-7\varepsilon)|s|} \, ds \leq C\varepsilon^{-2|m|}. \tag{8.13}$$

Setting $\varepsilon = \frac{1}{T}$ in (8.13) and collecting the appropriate terms prove the theorem. ∎

Remark 8.6. In [Go81a,b] Good proved a special case of Theorem 8.5 with methods from analytic number theory. To be more specific, for $k > 2$, k even, and $f = \theta_{v_k,\eta}$, i.e., $|f|^2(z) = y^k |F(z)|^2$, $z \in X$, for some (anti-)holomorphic automorphic form F on the upper half-plane X, Good proved (cf. [Go81a, Th. 1]) the estimate given in Theorem 8.5 for such an f. A comparison of the proofs shows the effectiveness of the representation theoretic approach. We should point out that the number theory normalization of the Eisenstein series $E_j(\cdot, u)$ used in [Go81a] differs by a change of parameters $u = 2s - 1$ from the representation theory notation used in this paper.

Estimating Fourier coefficients of holomorphic cusp forms. Let $F \in M_k^0(\Gamma)$ be an anti-holomorphic cusp form on the upper half-plane X. Let P be a cuspidal parabolic subgroup for Γ. Replacing Γ with a certain G-conjugate we may assume that

$$\Gamma \cap N = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}.$$

Then $\overline{F}$ is a holomorphic cusp form on X and as such admits a Fourier expansion at infinity

$$\overline{F}(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i n z}.$$

As before we have

$$\forall z = x + iy \in X \qquad |f|^2(z) = y^k |\overline{F}(z)|^2$$

for some cusp form $f = \theta_{v_k,\eta}$ associated to a unitary highest weight representation $(\pi_k, \mathcal{H}_k)$ of G.

If $h \in C^\infty(\Gamma\backslash G)$ is rapidly decreasing, then we define its constant term

$$h_P : N\backslash G \to \mathbb{C}, \quad Ng \mapsto \int_{(\Gamma\cap N)\backslash N} h(ng)\; dn.$$

Since $|f|^2$ is a cusp form and hence rapidly decreasing, the Rankin-Selberg convolution theorem (cf. [Bo97, Prop. 10.10]) yields for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ that

$$(8.14) \qquad \langle |f|^2, E(\cdot, s)\rangle_{L^2(\Gamma\backslash G)} = \langle (|f|^2)_P, e^{(1+s)\rho \log a(\cdot)}\rangle_{L^2(N\backslash G)}$$

with E the Eisenstein series associated to P. From (8.14) and a straightforward calculation one gets the familiar Rankin-Selberg zeta function (cf. [Bu97, p. 71–72])

$$(8.15) \qquad \frac{1}{2}(4\pi)^{\frac{1}{2}-k-\frac{s}{2}}\Gamma(k - \frac{1}{2} + \frac{s}{2}) \sum_{n=1}^{\infty} |c_n|^2 n^{\frac{1}{2}-k-\frac{s}{2}} = \langle |f|^2, E(\cdot, s)\rangle$$

for all s with $\operatorname{Re} s > 1$. From Theorem 8.4 we obtain the estimate

$$(8.16) \qquad \int_{-T}^{T} |\langle |f|^2, E(\cdot, 2is\rangle|^2 e^{\pi|s|}\; ds \le CT^{2k}.$$

It was shown by Good in [Go81a, p. 544-547] (see also [Pe95, p. 121–122]) that a combination of (8.15) and (8.16) yields the following result:

THEOREM 8.7. *Let $\overline{F}(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i n z}$ be a holomorphic cusp form of weight $k \in \mathbb{N}$ with respect to an arbitrary discrete subgroup $\Gamma < G$ of co-finite volume and the property $\Gamma \cap N = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$. Then*

$$|c_n| << n^{\frac{k}{2} - \frac{1}{6} + \varepsilon}$$

for every $\varepsilon > 0$.

Fourier coefficients of Maaß forms. We shall give yet another application of holomorphic extension, but with a different technique. Here we view Whittaker functions as eigenfunctions of the invariant differential operators on the locally symmetric space and use their holomorphic extension. It is classical that Whittaker functions have such extensions but this seems to be a new use of it. To avoid technicalities we suppose that $G = \mathrm{Sl}(2,\mathbb{R})$ and $\Gamma < G$ is a discrete subgroup with co-finite volume. We assume that Γ admits at least one cusp and that

$$\Gamma \cap N = \{\begin{pmatrix} 1 & nc \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$$

for some $c > 0$. Let $\theta_{v,\eta}$ be a Maaß form on $\Gamma\backslash G/K$. For an element $z = x + iy \in \mathbb{C}$ with $y > 0$ we define $g_z = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$ and we note that $g_z \cdot i = z$. Then $\theta_{v,\eta}$ admits a partial Fourier series

$$\theta_{v,\eta}(\Gamma g_z) = a_0(y) + \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} a_m \sqrt{y} K_s(\frac{2\pi}{c}|m|y) e^{2\pi i \frac{m}{c} x}$$

where $s \in \mathbb{C}$ and

$$K_s(y) = \frac{1}{2}\int_0^{\infty} e^{-y(t+\frac{1}{t})/2} t^s \, \frac{dt}{t} \qquad (y > 0)$$

is the K-Bessel function.

THEOREM 8.8. *Let $\theta_{v,\eta}$ be a Maaß cusp form for $G = \mathrm{Sl}(2,\mathbb{R})$. Then the Fourier coefficients of $\theta_{v,\eta}$ satisfy*

$$(\forall N \geq 2) \qquad \sum_{\substack{|m|\leq N \\ m\neq 0}} \frac{|a_m|^2}{m} \leq C \log N.$$

Proof. For every $m \in \mathbb{Z}$ define a unitary character χ_m of the circle group $\Gamma \cap N\backslash N$ by

$$\chi_m\Big(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\Big) = e^{2\pi i \frac{m}{c} x}.$$

Then we have for all $m \in \mathbb{Z}$, $m \neq 0$, the identity

$$a_m \sqrt{y} K_s(\frac{2\pi}{c}|m|y) e^{2\pi i \frac{m}{c} x} = \int_{\Gamma\cap N\backslash N} \theta_{v,\eta}(n g_z)\chi_{-m}(n)\; dn \tag{8.17}$$

for $z = x + iy$ in the upper half-plane. Now we holomorphically extend both sides of (8.17). For $0 < \varepsilon \leq 1$ recall that

$$a_\varepsilon = \begin{pmatrix} e^{i\frac{\pi}{4}(1-\varepsilon)} & 0 \\ 0 & e^{-i\frac{\pi}{4}(1-\varepsilon)} \end{pmatrix} \in A_{\mathbb{C}}^1$$

and $y_\varepsilon = e^{i\frac{\pi}{2}(1-\varepsilon)}$ and note that $g_{y_\varepsilon} = a_\varepsilon$. Then by analytic continuation,

$$a_m \sqrt{y_\varepsilon} K_s(\frac{2\pi}{c}|m|y_\varepsilon) = \int_{\Gamma\cap N\backslash N} \theta_{v,\eta}(n a_\varepsilon)\chi_{-m}(n)\; dn$$

for all $0 < \varepsilon \leq 1$. For every ε define

$$f_\varepsilon\colon \Gamma\cap N\backslash N \to \mathbb{C}, \quad n \mapsto \theta_{v,\eta}(n a_\varepsilon).$$

It follows from Theorem 7.5 that $\|\theta_{v_\varepsilon,\eta}\|_\infty \leq C|\log\varepsilon|^{\frac{1}{2}}$ and so in particular f_ε is bounded and belongs to $L^2(\Gamma\cap N\backslash N)$. Therefore we get that

$$f_\varepsilon(n) = \sum_{m\in\mathbb{Z}} b_m \chi_m(n)$$

and

$$\sum_{m\in\mathbb{Z}} |b_m|^2 = \|f_\varepsilon\|_2^2 \leq \|f_\varepsilon\|_\infty^2 \leq C|\log\varepsilon|.$$

Since $b_m = a_m\sqrt{y_\varepsilon}K_s(\frac{2\pi}{c}|m|y_\varepsilon)$ we thus obtain that

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} |a_m|^2|y_\varepsilon|\Big|K_s(\frac{2\pi}{c}|m|y_\varepsilon)\Big|^2 \leq C|\log\varepsilon|.$$

In particular for all $N \in \mathbb{N}$ we get that

$$\sum_{\substack{|m|\leq N\\ m\neq 0}} |a_m|^2\Big|K_s(\frac{2\pi}{c}|m|y_\varepsilon)\Big|^2 \leq C|\log\varepsilon|.$$

Now choose $\varepsilon = \frac{1}{N}$. Then for all $|m| \leq N$,

$$|m|y_\varepsilon = |m|ie^{-i\frac{2}{N}} \approx |m|i + \frac{2|m|}{N}$$

and so by the asymptotic expansions of the Bessel functions there exists $C' > 0$ such that

$$\Big|K_s(\frac{2\pi}{c}|m|y_\varepsilon)\Big| \geq \frac{C'}{\sqrt{|m|}}$$

for $|m|$ large. This proves the theorem. □

Remark 8.9. The Ramanujan conjecture for Maaß forms says that the coefficients $|a_n|$ grow more slowly than n^ε for all $\varepsilon > 0$. Comparison with the result in Theorem 8.8 shows that our result is consistent with this conjecture and that our result is essentially sharp. A little more care with obtaining an asymptotic estimate of C log N, instead of a bound of C log N, together with a Tauberian theorem for logarithmic means give the equivalence of such a result with the order of the pole of the Rankin-Selberg zeta function. We thank Wenzi Luo for an informative conversation on this topic.

9. $G = \mathrm{Sl}(3, \mathbb{R})$

Now we are going to apply our techniques to a group of higher rank, namely $G = \mathrm{Sl}(3, \mathbb{R})$. This group is low dimensional enough for explicit computations to be possible, yet it also illustrates that the technique works in higher rank as well. In particular, we will verify part of Conjecture B and give a complete answer to the boundary behaviour of the analytically continued spherical functions in the direction of the extremal rays. Finally, with these estimates available we can give an application to triple products.

Let us briefly summarize the notation for this special case:

$$\mathfrak{a} = \{\mathrm{diag}(x_1, x_2, x_3)\colon x_i \in \mathbb{R};\ \sum_{j=1}^{3} x_j = 0\}$$

and

$$A = \{\mathrm{diag}(a_1, a_2, a_3)\colon a_i > 0;\ \prod_{j=1}^{3} a_j = 1\}.$$

The positive system is $\Sigma^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\}$ and the associated simple roots are given by

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}.$$

Here $\omega_1 = \varepsilon_1$ and $\omega_2 = \varepsilon_1 + \varepsilon_2$. The Weyl group, $\mathcal{W}_\mathfrak{a} \cong S_3$, in this case acts as the permutation group of $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

Define 3×3 matrices E_{ij} by $E_{ij} = (\delta_{k-i,l-j})_{k,l}$. Note that $\mathfrak{g}^{\varepsilon_i - \varepsilon_j} = \mathbb{R}E_{ij}$. Now $\overline{\mathfrak{n}}_1 = \mathfrak{g}^{\varepsilon_2 - \varepsilon_1}$ and $\overline{\mathfrak{n}}_2 = \mathfrak{g}^{\varepsilon_3 - \varepsilon_2} \oplus \mathfrak{g}^{\varepsilon_3 - \varepsilon_1}$ are subalgebras of $\overline{\mathfrak{n}}$ satisfying $\overline{\mathfrak{n}} = \overline{\mathfrak{n}}_1 + \overline{\mathfrak{n}}_2$, with $\overline{\mathfrak{n}}_2$ abelian. The map

$$\Phi\colon \mathbb{R}^3 \to \overline{N}, \quad (x, y, z) \mapsto \exp(xE_{21}) \exp(yE_{32}) \exp(zE_{31}) = \begin{pmatrix} 1 & & \\ x & 1 & \\ z & y & 1 \end{pmatrix}$$

is a diffeomorphism. We take a choice of Haar measure $d\overline{n}$ on $\overline{N}$ so that its pullback under Φ is the product of Lebesgue measures $dx\ dy\ dz$.

Lemma 9.1. *For all*

$$a = \operatorname{diag}(a_1, a_2, a_3) \in A$$

and

$$\overline{n} = \exp(xE_{21}) \exp(yE_{32}) \exp(zE_{31}) \in \overline{N}$$

the following assertions hold:

(i) $a(a\overline{n})^{2\omega_1} = a_1^2 + a_2^2 x^2 + a_3^2 z^2$;

(ii) $a(a\overline{n})^{2\omega_2} = a_1^2 a_2^2 + a_1^2 a_3^2 y^2 + a_2^2 a_3^2 (z+xy)^2 = a_3^{-2} + a_2^{-2} y^2 + a_1^{-2}(z+xy)^2$.

Proof. This is elementary. □

We can improve Theorem 1.8 (ii) slightly.

Lemma 9.2. *For $G = \mathrm{Sl}(3, \mathbb{R})$,*

$$B_{\mathbb{C}}^1 G \subseteq K_{\mathbb{C}} A_{\mathbb{C}}^{0,\leq} N_{\mathbb{C}}.$$

Proof. By the usual argument with the Bruhat decomposition it is enough to show that $B_{\mathbb{C}}^1 \overline{N} \subseteq K_{\mathbb{C}} A_{\mathbb{C}}^{0,\leq} N_{\mathbb{C}}$. This follows readily from Lemma 9.1 and Lemma 2.1(i). □

In view of the discussion leading to Theorem 4.2(i), we can say that all spherical functions φ_λ, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, extend to $K_{\mathbb{C}} B_{\mathbb{C}}^0 K_{\mathbb{C}}$.

Since $\mathfrak{g}$ is split, $\rho = \omega_1 + \omega_2$. Then Theorem 4.2(iii) together with Lemma 9.1 gives the following expression for spherical functions on $G = \mathrm{Sl}(3, \mathbb{R})$, valid for all $a \in \exp i\mathfrak{b}^0$ and all $\lambda = \lambda_1 \omega_1 + \lambda_2 \omega_2 \in i\mathfrak{a}^*$ (the only reason we restrict ourselves to λ imaginary is to keep the following formula manageable).

(9.1)

$$\varphi_\lambda(a) = \int_{\mathbb{R}^3} \frac{dx\, dy\, dz}{\left|(a_1 + a_2 x^2 + a_3 z^2)^{1-\lambda_1} (a_1 a_2 + a_1 a_3 y^2 + a_2 a_3 (z+xy)^2)^{1-\lambda_2}\right|}$$

For each $\alpha \in \Sigma$ we write $H_\alpha \in \mathfrak{a}$ for the co-root of α. Notice that $\frac{\pi}{2} H_\alpha \in \partial \mathfrak{b}^0$ for all $\alpha \in \Sigma$.

We consider the radial limits of φ_λ in the directions of the co-roots and the fundamental weights. We begin with the co-root directions.

Lemma 9.3. *Let $\alpha \in \Sigma$ and H_α be the corresponding co-root. Then, for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ for which $(\pi_\lambda, \mathcal{H}_\lambda)$ is unitarizable,*

$$\varphi_\lambda(\exp(i\frac{\pi}{2}(1-\varepsilon) H_\alpha)) \geq C |\log \varepsilon|$$

for a constant $C = C(\lambda)$.

Proof. This is a special case of Corollary 4.6. □

Next, for the fundamental weights we have

$$X_{\omega_1} = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) \quad \text{and} \quad X_{\omega_2} = \mathrm{diag}(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$$

and $\pm\pi X_{\omega_j} \in \partial\mathfrak{b}^0$.

For these weights we use the following interesting splitting of the spherical functions in extremal directions.

Proposition 9.4. *Let $G = \mathrm{Sl}(3,\mathbb{R})$ and $X \in \mathbb{R}X_{\omega_1} \cap \mathfrak{b}^0$. Put $a = \exp(iX)$. Then, for all spherical functions φ_λ, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$,*

$$\begin{aligned}\varphi_\lambda(a) &= C\int_{\mathbb{R}^2} \frac{1}{\left|(a_1 + a_2x^2 + a_3z^2)^{1-\lambda_1}(a_1a_2 + a_2a_3z^2)^{1-\lambda_2}\right|} \\ &\quad \times \frac{1}{(\overline{a_1} + \overline{a_2}x^2 + \overline{a_3}z^2)^{\mathrm{Re}\,\lambda_1}(\overline{a_1a_2} + \overline{a_2a_3}z^2)^{\mathrm{Re}\,\lambda_2}}\, dx\, dz\end{aligned}$$

with

$$C = \int_{\overline{N}'} a(\overline{n}')^{-\varepsilon_2+\varepsilon_3}\, d\overline{n}'$$

and $\overline{N}' = \exp(\mathfrak{g}^{\varepsilon_3-\varepsilon_2})$.

Proof. Set $b = \exp(i\frac{1}{2}X)$. Let $\overline{\mathfrak{n}}' = \mathfrak{g}^{\varepsilon_3-\varepsilon_2}$ and $\overline{\mathfrak{n}}'' = \mathfrak{g}^{\varepsilon_2-\varepsilon_1} \oplus \mathfrak{g}^{\varepsilon_3-\varepsilon_1}$ and note that $\overline{\mathfrak{n}} = \overline{\mathfrak{n}}' \oplus \overline{\mathfrak{n}}''$ is a direct sum of subalgebras. Hence $\overline{N} = \overline{N}'\overline{N}'' \cong \overline{N}' \times \overline{N}''$ with $\overline{N}' = \exp(\overline{\mathfrak{n}}')$ and $\overline{N}'' = \exp(\overline{\mathfrak{n}}'')$. Hence we get from Theorem 4.2(iii) that

$$\begin{aligned}\varphi_\lambda(b^2) &= \int_{\overline{N}} |a(b\overline{n})^{2(\lambda-\rho)}| \cdot \overline{a(b\overline{n})^{-2\,\mathrm{Re}\,\lambda}}\, d\overline{n} \\ &= \int_{\overline{N}'}\int_{\overline{N}''} |a(b\overline{n}'\overline{n}'')^{2(\lambda-\rho)}| \cdot \overline{a(b\overline{n}'\overline{n}'')^{-2\,\mathrm{Re}\,\lambda}}\, d\overline{n}'\, d\overline{n}''.\end{aligned}$$

According to the Iwasawa decomposition, $\overline{n}' = k'a'n'$. Since k' commutes with b,

$$\begin{aligned}\varphi_\lambda(b^2) &= \int_{\overline{N}'}\int_{\overline{N}''} |a(bk'a'n'\overline{n}'')^{2(\lambda-\rho)}| \cdot \overline{a(bk'a'n'\overline{n}'')^{-2\,\mathrm{Re}\,\lambda}}\, d\overline{n}'\, d\overline{n}'' \\ &= \int_{\overline{N}'}\int_{\overline{N}''} |a(ba'n'\overline{n}''n'^{-1})^{2(\lambda-\rho)}| \cdot \overline{a(ba'n'\overline{n}''(n')^{-1})^{-2\,\mathrm{Re}\,\lambda}}\, d\overline{n}'\, d\overline{n}''.\end{aligned}$$

Since N' normalizes $\overline{N}''$ in a unipotent way, that

$$\begin{aligned}\varphi_\lambda(b^2) &= \int_{\overline{N}'}\int_{\overline{N}''} |a(ba'\overline{n}'')^{2(\lambda-\rho)}| \cdot \overline{a(ba'\overline{n}'')^{-2\,\mathrm{Re}\,\lambda}}\, d\overline{n}'\, d\overline{n}'' \\ &= \int_{\overline{N}'} |a(\overline{n}')^{2(\lambda-\rho)}||a(\overline{n}')^{-2\,\mathrm{Re}\,\lambda} \\ &\quad \times \int_{\overline{N}''} |a(ba'\overline{n}''(a')^{-1})^{2(\lambda-\rho)}| \cdot \overline{a(ba'\overline{n}''(a')^{-1})^{-2\,\mathrm{Re}\,\lambda}}\, d\overline{n}'\, d\overline{n}'' \\ &= \int_{\overline{N}'} a(\overline{n}')^{-2\rho} \int_{\overline{N}''} |a(ba'\overline{n}''(a')^{-1})^{2(\lambda-\rho)}| \cdot \overline{a(ba'\overline{n}''(a')^{-1})^{-2\,\mathrm{Re}\,\lambda}}\, d\overline{n}'\, d\overline{n}''.\end{aligned}$$

Now the Jacobian of the map $\overline{N}'' \to \overline{N}''$, $\overline{n}'' \mapsto a'\overline{n}''(a')^{-1}$ is given by $(a')^\gamma$ with $\gamma = -((\varepsilon_1 - \varepsilon_2) + (\varepsilon_1 - \varepsilon_3))$. Hence,

$$\varphi_\lambda(b^2) = \int_{\overline{N}'} a(\overline{n}')^{-(\varepsilon_2-\varepsilon_3)}\, d\overline{n}' \int_{\overline{N}''} |a(b\overline{n}'')^{2(\lambda-\rho)}| \cdot \overline{a(b\overline{n}'')^{-2\operatorname{Re}\lambda}}\, d\overline{n}''.$$

Finally Lemma 9.1 gives

$$\begin{aligned}\int_{\overline{N}''} &|a(b\overline{n}'')^{2(\lambda-\rho)}| \cdot \overline{a(b\overline{n}'')^{-2\operatorname{Re}\lambda}}\, d\overline{n}'' \\ &= \int_{\mathbb{R}^2} \frac{1}{\left|(a_1 + a_2x^2 + a_3z^2)^{1-\lambda_1}(a_1a_2 + a_2a_3z^2)^{1-\lambda_2}\right|} \\ &\quad \times \frac{1}{(\overline{a_1} + \overline{a_2}x^2 + \overline{a_3}z^2)^{\operatorname{Re}\lambda_1}(\overline{a_1a_2} + \overline{a_2a_3}z^2)^{\operatorname{Re}\lambda_2}}\, dx\, dz,\end{aligned}$$

concluding the proof of the proposition. □

Lemma 9.5. *Let $\lambda \in \mathfrak{a}_\mathbb{C}$. Then for $j = 1, 2$,*

$$|\varphi_\lambda(\exp(\pm i\pi(1-\varepsilon)X_{\omega_j}))| \le C\frac{1}{\varepsilon}$$

for a constant $C > 0$. Furthermore if $\lambda \in i\mathfrak{a}^*$, then

$$|\varphi_\lambda(\exp(\pm i\pi(1-\varepsilon)X_{\omega_j}))| \asymp \frac{1}{\varepsilon}.$$

Proof. Notice that $s_{\varepsilon_1-\varepsilon_3}(X_{\omega_1}) = -X_{\omega_2}$ (i.e. for $\mathrm{Sl}(3,\mathbb{R})$ π_1 is contragredient to π_2) so that we have to consider only the radial limits in the direction of $\pm X_{\omega_1}$. We restrict ourselves to the case of $-X_{\omega_1}$. Then Proposition 9.4 gives

$$\begin{aligned}&\varphi_\lambda(\exp(i\pi(1-\varepsilon)X_{\omega_1})) \\ &= C(\lambda)\int_{\mathbb{R}^2} \frac{1}{|(e^{i\frac{2}{3}\pi(1-\varepsilon)} + e^{-i\frac{1}{3}\pi(1-\varepsilon)}x^2 + e^{-i\frac{1}{3}\pi(1-\varepsilon)}z^2)^{1-\lambda_1}|} \\ &\times \frac{1}{|(e^{i\frac{1}{3}\pi(1-\varepsilon)} + e^{-i\frac{2}{3}\pi(1-\varepsilon)}z^2)^{1-\lambda_2}|(e^{-i\frac{2}{3}\pi(1-\varepsilon)} + e^{i\frac{1}{3}\pi(1-\varepsilon)}x^2 + e^{i\frac{1}{3}\pi(1-\varepsilon)}z^2)^{\operatorname{Re}\lambda_1}} \\ &\times \frac{1}{(e^{-i\frac{1}{3}\pi(1-\varepsilon)} + e^{i\frac{2}{3}\pi(1-\varepsilon)}z^2)^{\operatorname{Re}\lambda_2}}\, dx\, dz.\end{aligned}$$

Hence we get

$$\begin{aligned}\varphi_\lambda(\exp(i\pi(1-\varepsilon)X_{\omega_1})) &= C(\lambda)\int_{\mathbb{R}^2} \frac{|e^{i\frac{1}{3}\pi(1-\varepsilon)(\lambda_2-\lambda_1)}|e^{i\frac{2}{3}\pi(1-\varepsilon)(\operatorname{Re}\lambda_2-\operatorname{Re}\lambda_1)}}{|(e^{i\pi(1-\varepsilon)} + x^2 + z^2)^{1-\lambda_1}|} \\ \times &\frac{1}{|(1+e^{-i\pi(1-\varepsilon)}z^2)^{1-\lambda_2}|(e^{-i\pi(1-\varepsilon)} + x^2 + z^2)^{\operatorname{Re}\lambda_1}(1+e^{i\pi(1-\varepsilon)}z^2)^{\operatorname{Re}\lambda_2}}\, dx\, dz.\end{aligned}$$

We see that the singularity of the integral is located at $x^2 + z^2 = 1$ and $z^2 = 1$.

Let us assume now that $\lambda \in i\mathfrak{a}^*$. The upper estimate for general $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ will be proved in the same way.

So for fixed $\lambda \in i\mathfrak{a}*$ we get

$$\begin{aligned}
\varphi_\lambda(\exp(-i\pi(1-\varepsilon)X_{\omega_1})) &\asymp \int_0^2 \int_0^2 \frac{dx\ dz}{|(e^{i\pi(1-\varepsilon)} + x^2 + z^2)(e^{i\pi(1-\varepsilon)} + z^2)|} \\
&\asymp \int_0^2 \int_0^2 \frac{dx\ dz}{|(-1 + i\varepsilon + x^2 + z^2)(-1 + i\varepsilon + z^2)|} \\
&\asymp \int_0^2 \int_0^2 \frac{dx\ dz}{(\varepsilon + |1 - x^2 - z^2|)(\varepsilon + |1 - z^2|)} \\
&\asymp \int_0^2 \int_{-1}^1 \frac{dx\ dz}{(\varepsilon + |x^2 + z^2 + 2z|)(\varepsilon + |z|)} \\
&\asymp \int_0^2 \int_0^1 \frac{dx\ dz}{(\varepsilon + x^2 + z^2 + 2z)(\varepsilon + z)} \\
&\asymp \int_0^1 \frac{dz}{(\varepsilon + z^2 + 2z)(\varepsilon + z)} \\
&\asymp \int_0^1 \frac{dz}{(\varepsilon + z)^2} \\
&\asymp \frac{1}{\varepsilon}.
\end{aligned}$$

□

Putting these results together we have

THEOREM 9.6. *Let $G = \mathrm{Sl}(3,\mathbb{R})$ and $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitarizable principal series representation of G. Then for the associated spherical functions*:

(i) *If $X \in \partial\mathfrak{b}^0$, then there exists a constant $C \geq 0$ such that*

$$|\varphi_\lambda(\exp(\pm i(1-\varepsilon)X))| \leq C\frac{1}{\varepsilon};$$

(ii) *If $X = \pm\pi X_{\omega_j} \in \mathfrak{a}$, $j = 1, 2$, and $\lambda \in i\mathfrak{a}^*$, then*

$$|\varphi_\lambda(\exp(i(1-\varepsilon)X))| \asymp \frac{1}{\varepsilon};$$

(iii) *If $X \in \partial\mathfrak{b}^0$, then for a constant $C > 0$*

$$|\varphi_\lambda(\exp(i(1-\varepsilon)X))| \geq C|\log\varepsilon|;$$

(iv) *For all $\lambda \in i\mathfrak{a}^*$ the domain $\mathfrak{b}_{\mathbb{C}}^0 = \mathfrak{a} + i\mathfrak{b}^0$ is the maximal connected tube domain $\mathfrak{a} + i\omega \subseteq \mathfrak{a}_{\mathbb{C}}$, $\omega \subseteq \mathfrak{a}$, containing 0 such that $\varphi_\lambda|_A$ extends holomorphically to $\exp(\mathfrak{a} + i\omega)$.*

Proof. For the proof it is helpful to keep Figure 1 in mind.

(i) For X an extreme point, the assertion is Lemma 9.5. The general case follows from Proposition 4.4 (cf. Figure 1).

(ii) Lemma 9.5.

(iii) For a co-root direction this inequality is Lemma 9.3. For X_{ω_j} this follows from (ii). An arbitrary $X \in \partial\mathfrak{b}^0$ is a convex combination of co-roots and X_{ω_j} (cf. Figure 1) so that the result follows from Phragmen-Lindelöf as in the proof of Proposition 4.4.

(iv) In view of (iii), this follows from Proposition 4.4 (cf. Figure 1). □

Remark 9.7. Theorem 9.6 gives an almost complete description of the boundary behaviour for the spherical functions for $G = \mathrm{Sl}(3,\mathbb{R})$. Generally we expect the following for generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$:

$$|\varphi_\lambda(\exp(i(1-\varepsilon)X))| \asymp \begin{cases} \frac{1}{\varepsilon} & \text{for } X \in \partial\mathfrak{b}^0 \text{ extremal} \\ |\log\varepsilon| & \text{for } X \in \partial\mathfrak{b}^0 \text{ not extremal.} \end{cases}$$

Lower estimates. For $G = \mathrm{Sl}(3,\mathbb{R})$ the best lower estimates one can get for spherical functions are in the direction of the fundamental weights. We illustrate only this case.

PROPOSITION 9.8. *For $G = \mathrm{Sl}(3,\mathbb{R})$ and $X \in \overline{\mathfrak{b}^0}$ an extreme point, i.e., $X = \pm\pi X_{\omega_j}$, there exists a constant $C > 0$ such that*

$$|\varphi_\lambda(\exp(i(1-\varepsilon)X))| \geq Ce^{(1-\varepsilon)\sup_{w\in\mathcal{W}_{\mathfrak{a}}} w.\operatorname{Im}\lambda(-iX)}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Proof. We may assume that $X = \pi X_{\omega_1} = \pi\,\mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$. In what follows we will also see that the assumption $\lambda \in i\mathfrak{a}^*$ is justified. Fix $0 < \varepsilon \leq 1$ and set $a = \mathrm{diag}(a_1, a_2, a_3) = \exp(i(1-\varepsilon)X)$. Recall from (9.1) that

$$\varphi_\lambda(a) = \int_{\mathbb{R}^3} \frac{dx\; dy\; dz}{\left|(a_1 + a_2x^2 + a_3z^2)^{1-\lambda_1}(a_1a_2 + a_1a_3y^2 + a_2a_3(z+xy)^2)^{1-\lambda_2}\right|}.$$

Hence the fact that a_1, a_2, a_3 are $\mathbb{R}$-collinear in $\mathbb{C}$ as well as that a_1a_2, a_1a_3, a_2a_3 are $\mathbb{R}$-collinear in $\mathbb{C}$ implies that

$$\begin{aligned} \varphi_\lambda(a) &\geq \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \\ &\quad \times \int_0^{\frac{1}{2}} \frac{dx\; dy\; dz}{\left|(a_1 + a_2x^2 + a_3z^2)^{1-\lambda_1}(a_1a_2 + a_1a_3y^2 + a_2a_3(z+xy)^2)^{1-\lambda_2}\right|} \\ &\geq Ca_1^{\lambda_1}(a_1a_2)^{\lambda_2} = Ce^{\lambda(i(1-\varepsilon)X)}. \end{aligned}$$

Now the assertion of the proposition follows from the Weyl group invariance of φ_λ in the λ-variable. □

Estimates on automorphic forms. Here Γ denotes a co-compact discrete subgroup of $G = \mathrm{Sl}(3,\mathbb{R})$ and $(\pi_\lambda, \mathcal{H}_\lambda)$ a unitary spherical principal series representation of G. Recall from Proposition 7.4 that each $\eta \in (\mathcal{H}_\lambda^{-\infty})^\Gamma$ defines an embedding

$$\mathcal{H}_\lambda^\infty \to C^\infty(\Gamma\backslash G), \quad v \mapsto \theta_{v,\eta}; \ \theta_{v,\eta}(\Gamma g) = \langle \pi_\lambda(g)v, \eta\rangle$$

with

$$(\forall v \in \mathcal{H}_\lambda^\infty) \qquad \|\theta_{v,\eta}\|_\infty \leq C S_k^G(v)$$

for $k > 4 = \frac{1}{2}\dim G$. As before v_0 denotes the normalized K-spherical vector of $(\pi_\lambda, \mathcal{H}_\lambda)$. Fix an extremal element $X \in \overline{\mathfrak{b}^1}$, i.e., $X = \pm\frac{\pi}{2}X_{\omega_j}$, and set

$$(v_0)_\varepsilon = \pi_\lambda(\exp(i(1-\varepsilon)X))v_0$$

for all $0 < \varepsilon \leq 1$. As in Section 6 we are interested in the G-invariant Sobolev norms $S_k^G((v_0)_\varepsilon)$ for $\varepsilon \to 0$.

Proposition 9.9. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$, $\lambda \in i\mathfrak{a}^*$, be a unitary principal series representation of $G = \mathrm{Sl}(3,\mathbb{R})$ and $\eta \in (\mathcal{H}_\lambda^{-\infty})^\Gamma$ for a co-compact discrete subgroup $\Gamma < G$. Then there exists a constant $C > 0$ such that*

$$\|\theta_{(v_0)_\varepsilon,\eta}\|_\infty \leq C\varepsilon^{-\frac{11}{2}}$$

for all $0 < \varepsilon \leq 1$.

Proof. We may assume that the extreme point $X \in \overline{\mathfrak{b}^1}$ is $-\frac{\pi}{2}X_{\omega_1}$. Since $\dim G = 8$, it follows from Proposition 7.4 that it suffices to estimate $S_5^G((v_0)_\varepsilon)$. We shall show that

$$S_5((v_0)_\varepsilon) \leq C\varepsilon^{-\frac{11}{2}}$$

for all $0 < \varepsilon \leq 1$.

More generally, we will consider $S_k((v_0)_\varepsilon)$ for an arbitrary $k \in \mathbb{N}_0$. Let $X_1, \cdots, X_8$ be a basis of $\mathfrak{g}$. Then

$$S_k(v) = \sum_{m_1+\cdots+m_8\leq k} \|X_1^{m_1}\cdots X_8^{m_8}v\|$$

for all smooth vectors v.

We will work with the noncompact realization of π_λ, i.e., $\mathcal{H}_\lambda = L^2(\overline{N})$. Then the vector $(v_0)_\varepsilon$ is of the form $c(\lambda,\varepsilon)F_\varepsilon$, where

$$F_\varepsilon(x,y,z) = \frac{1}{(e^{i(\pi-\varepsilon)} + x^2 + z^2)^{\frac{1-\lambda_1}{2}}(1 + y^2 + e^{-i(\pi-\varepsilon)}(z+xy)^2)^{\frac{1-\lambda_2}{2}}}$$

and $c(\lambda,\varepsilon)$ is a constant uniformly bounded in ε.

We will use results from the proof of Proposition 9.4. We denote the change of variables in Proposition 9.4 that yielded the splitting by $(x,y,z)\mapsto\varphi(x,y,z)$. Recall that φ was given by a composition of two maps $(x,y,z)\mapsto(x+yz,y,z)$ followed by $(x,y,z)\mapsto(g(y)x,y,g(y)^{-1}z)$ where $g(y)=1+y^2$. Then

$$U\colon L^2(\overline{N})\to L^2(\overline{N}),\quad f\mapsto f\circ\varphi$$

defines a unitary operator. Set

$$f_\varepsilon(x,y,z)=\frac{1}{(e^{i(\pi-\varepsilon)}+x^2+z^2)^{\frac{1-\lambda_1}{2}}(1+e^{-i(\pi-\varepsilon)}z^2)^{\frac{1-\lambda_2}{2}}}\cdot g(y)^{-\frac{1}{2}+(\lambda_1-\lambda_2)}\ .$$

Then the change of variable formula in Proposition 9.4 implies that

$$U(F_\varepsilon)(x,y,z)=f_\varepsilon(x,y,z).$$

Define a differential operator $U_j=U\circ X_j\circ U^{-1}$ for $j=1,\cdots,8$. For every k we define a seminorm N_k for functions $f\in C^\infty(\mathbb{R}^3)$ by

$$N_k(f)=\sum_{m_1+m_2\le k}\|U_1^{m_1}\cdots U_8^{m_8}f\|,$$

whenever $N_k(f)<\infty$. In particular, we have

$$S_k(F_\varepsilon)=N_k(f_\varepsilon) \tag{9.2}$$

Let $B=[-2,2]^3$ and let $\tau\in C_c^\infty(\mathbb{R}^3)$ with $\tau|_B=1$. We claim that

$$N_k(f_\varepsilon)\le N_k(\tau f_\varepsilon)+C \tag{9.3}$$

for all $\varepsilon>0$ and a constant $C>0$ independent of ε. Write

$$f(x,y,z)=\frac{1}{(1+x^2+z^2)^{\frac{1-\lambda_1}{2}}(1+z^2)^{\frac{1-\lambda_2}{2}}}\cdot g(y)^{-\frac{1}{2}+(\lambda_1-\lambda_2)}$$

and note that f corresponds to the vector v_0 after the change of variables. Clearly we have

$$N_k(f)=S_k(v_0)<\infty.$$

Now, outside the ball B we can compare f_ε with f and our claim (9.3) follows.

Now define the two variable function $g_\varepsilon(x,z)$ by

$$g_\varepsilon(x,z)=\frac{1}{(e^{i(\pi-\varepsilon)}+x^2+z^2)^{\frac{1-\lambda_1}{2}}(1+e^{-i(\pi-\varepsilon)}z^2)^{\frac{1-\lambda_2}{2}}}\ .$$

Let $\tau_2\in C_c(\mathbb{R}^2)$ be such that $\tau_2|_{[-2,2]^2}=1$. Then it follows from (9.2) and (9.3) that there exists a constant $C>0$ such that

$$S_k((v_0)_\varepsilon)\le C+C\sum_{j+l\le k}\|\frac{\partial^j}{\partial x^j}\frac{\partial^l}{\partial z^l}\tau_2 g_\varepsilon\|_{L^2(\mathbb{R}^2)}.$$

The same computation as in Lemma 9.5 then leads to

$$S_k((v_0)_\varepsilon) \asymp \varepsilon^{-\frac{1+2k}{2}}.$$

Specializing to $k = 5$ proves the proposition. □

Remark 9.10. (a) The estimate in Proposition 9.9 is certainly not optimal. The conjectured optimal estimate would be

$$\|\theta_{(v_0)_\varepsilon,\eta}\|_\infty \leq C\|(v_0)_\varepsilon\| \asymp \frac{1}{\sqrt{\varepsilon}}.$$

The major technical difficulty arises from the singularities of the spherical vector $(v_0)_\varepsilon$ for $\varepsilon \to 0$ considered as a function on $\overline{N}$. In the rank one case the singular locus always is a *compact* variety, whereas in this case the singular locus lies on a complicated unbounded variety. This is reminiscent of the theory of intertwining operators in rank one versus higher rank. The technique there (essentially due to Gindikin-Karpelevic and later Schiffmann) is the origin of the change of variables used in Proposition 9.4 but here it did not lead to a simple product structure.

Triple products. For a co-compact discrete subgroup $\Gamma < G$ recall that $Y = \Gamma\backslash G$, $X = \Gamma\backslash G/K$ and there is the Plancherel decomposition

$$L^2(Y) \cong \widehat{\bigoplus}_{\pi\in\widehat{G}} m_\pi \mathcal{H}_\pi$$

and

$$L^2(X) \cong \widehat{\bigoplus}_{\pi\in\widehat{G}_s} m_\pi \mathcal{H}_\pi^K$$

where $\widehat{G}_s$ denotes the subset of $\widehat{G}$ which corresponds to the unitary spherical representations.

As before we let $(\psi_i)_{i\in I}$ be an orthonormal basis of Maaß forms of $L^2(X)$, also considered as an orthogonal system in $L^2(Y)$. Note that $\psi_i(\Gamma g) = \theta_{v_0^i,\eta} = \langle\pi_{\lambda_i}(g)v_0, \eta\rangle$ for some unitary principal series representation $(\pi_{\lambda_i}, \mathcal{H}_{\lambda_i})$ and $\eta \in (\mathcal{H}_{\lambda_i}^{-\infty})^\Gamma$.

Let $X \in \overline{\mathfrak{b}^1}$ be an extreme point; if $\psi = \theta_{v_0,\eta}$ is a Maaß form, then $\psi_\varepsilon = \theta_{(v_0)_\varepsilon,\eta} \in L^2(Y)$ for all $0 < \varepsilon \leq 1$. As before we have the identity

$$\psi_\varepsilon^2 = \sum_{i\in I} c_i \psi_{i,\varepsilon},$$

for all $0 < \varepsilon \leq 1$. Taking the norms, we have again

$$\|\psi_\varepsilon^2\|^2 = \sum_{i\in I} |c_i|^2 \|\psi_{i,\varepsilon}\|^2 = \sum_{i\in I} |c_i|^2 \|(v_0^i)_\varepsilon\|^2. \tag{9.4}$$

We now introduce new coordinates on $\mathfrak{a}_\mathbb{C}^*$ by means of the simple roots; i.e., we will write $\lambda = \lambda_1'\alpha_1 + \lambda_2'\alpha_2$. As norm on $\mathfrak{a}_\mathbb{C}^*$ we use the maximum norm $\|\lambda\| = \max\{|\lambda_1'|, |\lambda_2'|\}$.

THEOREM 9.11. *Let $G = \mathrm{Sl}(3,\mathbb{R})$ and $\Gamma < G$ be a co-compact discrete subgroup. Then for every Maaß form ψ corresponding to a unitary principal series there exists a constant $C > 0$ such that for all $T > 1$, for the coefficients c_i of the Fourier series of $\psi^2 = \sum_{i\in I} c_i\psi_i$, one has*

$$\sum_{\|\lambda_i\|\leq T} |c_i|^2 e^{\pi\|\lambda_i\|} \leq CT^{12}.$$

Proof. Given the previous estimates, the pattern of proof follows that of Theorem 7.6. and is left to the reader. □

Remark 9.12. Friedberg, in [Fr87], determines precise gamma factors for the Rankin-Selberg convolution for $\Gamma = \mathrm{Sl}(3,\mathbb{Z})$. By Stirling's approximation it is easily seen that the exponential growth of these gamma factors differs from the exponential term in Theorem 9.11. A likely explanation for this is that the classical Rankin-Selberg integral is computed using Eisenstein series off a maximal parabolic subgroup. Our results have used Eisenstein series or Maaß forms associated to the minimal parabolic. We think that any relationship of the exponential factors must involve the embedding parameters of the representation into the principal series off the minimal parabolic. We thank the referee for bringing this point to our attention.

Generalizations to $\mathrm{Sl}(n,\mathbb{R})$. Much of what has been said so far for $\mathrm{Sl}(3,\mathbb{R})$ can be generalized easily to $G = \mathrm{Sl}(n,\mathbb{R})$, $n \geq 3$. We will be interested in the radial limits of the spherical functions φ_λ, $\lambda \in \mathfrak{a}_\mathbb{C}^*$, in the imaginary direction of the first fundamental weight

$$X_{\omega_1} = \mathrm{diag}(\frac{n-1}{n}, -\frac{1}{n}, \ldots, -\frac{1}{n}).$$

First we need the splitting formula for the spherical functions analogous to Proposition 9.4. The splitting will be accomplished for the choice of subalgebras $\overline{\mathfrak{n}}' = \bigoplus_{\substack{\alpha\in\Sigma^+ \\ \alpha(X_{\omega_1})=0}} \mathfrak{g}^{-\alpha}$ and $\overline{\mathfrak{n}}'' = \bigoplus_{\substack{\alpha\in\Sigma^+ \\ \alpha(X_{\omega_1})>0}} \mathfrak{g}^{-\alpha}$. Then $\overline{\mathfrak{n}} = \overline{\mathfrak{n}}' \oplus \overline{\mathfrak{n}}''$ and we note that $\overline{\mathfrak{n}}'' \cong \mathbb{R}^{n-1}$ is abelian. The splitting formula in Proposition 9.4 then generalizes to

(9.5)

$$\varphi_\lambda(\exp(i\pi(1-\varepsilon)X_{\omega_1})) = C(\lambda,\varepsilon)\int_{\mathbb{R}^{n-1}} \prod_{j=1}^{n-1} \frac{1}{f_j(\varepsilon, x_j, \ldots,\ x_{n-1})}\ dx_1 \ldots\ dx_{n-1},$$

where

$$f_j(\varepsilon, x_j, \ldots,\ x_{n-1}) = |e^{i(\pi-\varepsilon)} + x_j^2 + \ldots + x_{n-1}^2|(e^{-i(\pi-\varepsilon)} + x_j^2 + \ldots + x_{n-1}^2)^{\mathrm{Re}\,\lambda_j}$$

for all $1 \leq j \leq n-1$ and $C(\lambda, \varepsilon)$ is bounded when $\varepsilon \to 0$ for a fixed λ. With (9.5) one easily concludes as in Lemma 9.5 that

$$\varphi_\lambda(\exp(i\pi(1-\varepsilon)X_{\omega_1})) \leq C\frac{1}{\varepsilon^{n-2}} \tag{9.6}$$

for all $0 < \varepsilon \leq 1$ and a constant C only depending on λ. Also, the proof of the lower estimate in Proposition 9.8 immediately generalizes to

$$\varphi_\lambda(\exp(i\pi(1-\varepsilon)X_{\omega_1})) \geq Ce^{\pi(1-\varepsilon)\sup_{w\in\mathcal{W}_{\mathfrak{a}}} w.\lambda(X_{\omega_1})} \tag{9.7}$$

now for a constant C independent of λ. Also, the proof of Proposition 9.9 generalizes easily to $G = \mathrm{Sl}(n, \mathbb{R})$. For a co-compact subgroup $\Gamma < G$ we get that

$$\|\theta_{(v_0)_\varepsilon,\eta}\|_\infty^2 \leq C\varepsilon^{-(n+\left[\frac{n^2-1}{2}\right])} \tag{9.8}$$

for all $0 < \varepsilon < 1$. Clearly (9.5)–(9.8) also hold for $\pm\pi w.X_{\omega_1}$, $w \in \mathcal{W}_{\mathfrak{a}}$, instead of πX_{ω_1}. We introduce now a norm on $\mathfrak{a}_{\mathbb{C}}^*$ by setting $\|\lambda\| = \sup_{w\in\mathcal{W}_{\mathfrak{a}}} |\lambda(w.X_{\omega_1})|$. Then the method in the proof of Theorem 9.11 together with (9.5)–(9.8) give the following estimates on triple products:

THEOREM 9.13. *Let $G = \mathrm{Sl}(n, \mathbb{R})$ and $\Gamma < G$ be a co-compact discrete subgroup. Then for every Maaß form ψ corresponding to a unitary principal series the coefficients c_i of the Fourier series of $\psi^2 = \sum_{i\in I} c_i\psi_i$ there exists a constant $C > 0$ such that for all $T > 1$, one has*

$$\sum_{\|\lambda_i\|\leq T} |c_i|^2 e^{\pi\|\lambda_i\|} \leq CT^{2n+\left[\frac{n^2-1}{2}\right]-2}.$$

Appendix A: The case $G = \mathrm{Sl}(2, \mathbb{R})$

In this appendix we deal with the group $G = \mathrm{Sl}(2, \mathbb{R})$ where we can perform the most explicit computations. We think this is still of interest since it is the guiding example on which one can make conjectures for the general case. In particular, some of the ideas of the proofs in Section 1 are present in the explicit computations below.

For $G = \mathrm{Sl}(2, \mathbb{R})$ note that $G \subseteq G_{\mathbb{C}} = \mathrm{Sl}(2, \mathbb{C})$ and $G_{\mathbb{C}}$ is simply connected. We let $\mathfrak{k} = \mathfrak{so}(2)$,

$$\mathfrak{a} = \{\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} : x \in \mathbb{R}\} \quad \text{and} \quad \mathfrak{n} = \{\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} : n \in \mathbb{R}\}.$$

For $z \in \mathbb{C}^*$, $x \in \mathbb{C}$ and $\theta \in \mathbb{C}$ we set

$$a_z = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \in A_{\mathbb{C}}, \quad n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N_{\mathbb{C}}$$

and

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_{\mathbb{C}}.$$

Note that

$$A^0_{\mathbb{C}} = \{a_z : \operatorname{Re}(z) > 0\} \quad \text{and} \quad A^1_{\mathbb{C}} = \{a_z : |\arg(z)| < \frac{\pi}{4}\}.$$

Proposition A.1. *Let $G = \mathrm{Sl}(2, \mathbb{R})$. Then the following assertions hold*:

(i) *For all $a_z \in A^1_{\mathbb{C}}$ and $\theta \in \mathbb{R}$,*

$$a_z k_\theta \in K_{\mathbb{C}} a_{z'} N_{\mathbb{C}}$$

with $a'_z \in A^1_{\mathbb{C}}$ and z' defined by

$$z' = \sqrt{z^2 + \sin^2\theta(\frac{1}{z^2} - z^2)}.$$

(ii) $A^1_{\mathbb{C}} K \subseteq K_{\mathbb{C}} A^1_{\mathbb{C}} N_{\mathbb{C}}$.

Proof. (i) Set $I = \{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{4}\}$ and fix $\theta \in \mathbb{R}$. Let $\Omega = \{z \in I : a_z k_\theta \in K_{\mathbb{C}} A^1_{\mathbb{C}} N_{\mathbb{C}}\}$. By Lemma 1.4 the set Ω is open and not empty. We have to show that $\Omega = I$. For $z \in \Omega$ define analytic functions $z'(z)$, $\varphi(z)$, $x(z)$ such that

$$a_z k_\theta = k_{\varphi(z)} a_{z'(z)} n_{x(z)}.$$

Writing this identity in matrix form yields

$$\begin{pmatrix} z\cos\theta & z\sin\theta \\ -\frac{\sin\theta}{z} & \frac{\cos\theta}{z} \end{pmatrix} = \begin{pmatrix} z'\cos\varphi & z'x\cos\varphi + \frac{\sin\varphi}{z'} \\ -z'\sin\varphi & -z'x\sin\varphi + \frac{\cos\varphi}{z'} \end{pmatrix}.$$

Thus we get

$$z^2\cos^2\theta + \frac{\sin^2\theta}{z^2} = (z')^2\cos^2\varphi + (z')^2\sin^2\varphi$$

or equivalently

$$(z')^2 = z^2 + \sin^2\theta(\frac{1}{z^2} - z^2). \tag{A.1}$$

Taking real parts in (A.1) yields

$$\operatorname{Re}(z')^2 = (1 - \sin^2\theta)\operatorname{Re}(z^2) + \sin^2\theta\operatorname{Re}(\frac{1}{z^2}) > 0. \tag{A.2}$$

We conclude that

$$z': I \to I, \quad z \mapsto z'(z) := \sqrt{z^2 + \sin^2\theta(\frac{1}{z^2} - z^2)}$$

is a well-defined holomorphic map.

Assume that $\Omega \neq I$. Then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in Ω such that $z = \lim z_n \in I\backslash\Omega$.

From (A.2) we now conclude that $z' = \lim_{n\to\infty} z'(z_n)$ exists in I. Now (A.1) implies that the limits $\varphi = \lim_{n\to\infty}\varphi(z_n)$ and $x = \lim_{n\to\infty} x(z_n)$ exist. Thus

$$a_z k_\theta = \lim_{n\to\infty} a_{z_n} k_\theta = \lim_{n\to\infty} k_{\varphi(z_n)} a_{z'(z_n)} n_{x(z_n)} = k_\varphi a_{z'} n_x \in K_{\mathbb{C}} A_{\mathbb{C}}^1 N_{\mathbb{C}},$$

contradicting $a_z k_\theta \notin K_{\mathbb{C}} A_{\mathbb{C}}^1 N_{\mathbb{C}}$. This proves (i).

(ii) This follows from (i). □

Remark A.2. (convexity theorems). For a linear semisimple Lie group G with Iwasawa decomposition $G = KAN$ Kostant proved two convexity theorems (cf. [Kos73]): the *linear convexity theorem* which asserts

$$(\text{A.3}) \qquad (\forall X \in \mathfrak{a}) \qquad p_{\mathfrak{a}}(\mathrm{Ad}(K).X) = \mathrm{conv}(\mathcal{W}_{\mathfrak{a}}.X)$$

with $p_{\mathfrak{a}}: \mathfrak{p} \to \mathfrak{a}$ the orthogonal projection with respect to the Cartan-Killing form, and the *nonlinear convexity theorem* which can be stated as

$$(\text{A.4}) \qquad (\forall a \in A) \qquad a(aK) = \exp\Big(\mathrm{conv}(\mathcal{W}_{\mathfrak{a}}.\log(a))\Big).$$

For $G = \mathrm{Sl}(2,\mathbb{R})$ a simple calculation shows that (A.3) extends to

$$(\forall X \in \mathfrak{a}_{\mathbb{C}}) \qquad p_{\mathfrak{a}_{\mathbb{C}}}(\mathrm{Ad}(K).X) = \mathrm{conv}(\mathcal{W}_{\mathfrak{a}}.X)$$

with $p_{\mathfrak{a}_{\mathbb{C}}}: \mathfrak{p}_{\mathbb{C}} \to \mathfrak{a}_{\mathbb{C}}$ the complex linear extension of $p_{\mathfrak{a}}$ and we conjecture that (A.3) holds for all semisimple Lie groups G. Also it is natural to ask whether the nonlinear convexity theorem (A.4) generalizes to elements $a \in A_{\mathbb{C}}^1$ (this makes sense in view of $A_{\mathbb{C}}^1 K \subseteq K_{\mathbb{C}} A_{\mathbb{C}}^{1,\leq} N_{\mathbb{C}}$ (cf. Proposition A.1(ii))). The answer is no and we can already see this for $G = \mathrm{Sl}(2,\mathbb{R})$. Here we have $\mathcal{W}_{\mathfrak{a}} \cong \mathbb{Z}_2 = \{\mathbf{1}, s\}$ with $s.a = a^{-1}$ for all $a \in A_{\mathbb{C}}$. For $z, w \in \mathbb{C}$ let

$$l_{z,w} = \{\lambda z + (1-\lambda)w : 0 \leq \lambda \leq 1\}$$

be the line segment in $\mathbb{C}$ connecting z and w. Then Proposition A.1 shows that for all $z \in \mathbb{C}^*$ with $|\arg(z)| < \frac{\pi}{4}$ that

$$a(a_z K) = \{a_w : w \in (l_{z^2, z^{-2}})^{\frac{1}{2}}\}.$$

Note that for $z = e^{i\varphi}$, $0 < |\varphi| < \frac{\pi}{4}$ we have $\mathbf{1} \notin \{a_w : w \in (l_{z^2,z^{-2}})^{\frac{1}{2}}\}$, but $\mathbf{1} \in \exp(\{a_w : w \in l_{-i\varphi, i\varphi}\})$. Thus we see that (A.4) usually does not hold for elements $a \in A_{\mathbb{C}}^1 \backslash A$.

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