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Hilbert series, Howe duality and branching for classical groups

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Abstract

An extension of the Littlewood Restriction Rule is given that covers all pertinent parameters and simplifies to the original under Littlewood's hypotheses. Two formulas are derived for the Gelfand-Kirillov dimension of any unitary highest weight representation occurring in a dual pair setting, one in terms of the dual pair index and the other in terms of the highest weight. For a fixed dual pair setting, all the irreducible highest weight representations which occur have the same Gelfand-Kirillov dimension.

We define a class of unitary highest weight representations and show that each of these representations, L, has a Hilbert series $\mathrm{H}_L(q)$ of the form:

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^{\mathrm{GKdim}\, L}} R(q),$$

where $R(q)$ is an explictly given multiple of the Hilbert series of a finite dimensional representation B of a real Lie algebra associated to L. Under this correspondence $L \to B$, the two components of the Weil representation of the symplectic group correspond to the two spin representations of an orthogonal group. The article includes many other cases of this correspondence.

1. Introduction

(1.1) Let V be a complex vector space of dimension n with a nondegenerate symmetric or skew symmetric form. Let G be the group leaving the form invariant. Now, G is either the orthogonal group $\mathrm{O}(n)$ or the sympletic group $\mathrm{Sp}(\frac{n}{2})$ for n even. The representations F^λ of $\mathrm{Gl}(V)$ are parametrized by the partitions λ with at most n parts. In 1940, D. E. Littlewood gave a formula for the decomposition of F^λ as a representation of G by restriction.

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THEOREM 1 (Littlewood Restriction [Lit 1,2]). *Suppose that λ is a partition having at most $\frac{n}{2}$ (positive) parts.*

(i) *Suppose n is even and set $k = \frac{n}{2}$. Then the multiplicity of the finite dimensional $\mathrm{Sp}(k)$ representation V^μ with highest weight μ in F^λ equals*

$$\sum_{\xi} \dim \mathrm{Hom}_{\mathrm{GL}(n)}(F^\lambda, F^\mu \otimes F^\xi), \tag{1.1.1}$$

where the sum is over all nonnegative integer partitions ξ with columns of even length.

(ii) *Then the multiplicity of the finite dimensional $\mathrm{O}(n)$ representation E^ν in F^λ equals*

$$\sum_{\xi} \dim \mathrm{Hom}_{\mathrm{GL}(n)}(F^\lambda, F^\nu \otimes F^\xi), \tag{1.1.2}$$

where the sum is over all nonnegative integer partitions ξ with rows of even length.

Recently Gavarini [G] (see also [GP]) has given a new proof of this theorem based on Brauer algebra methods and has extended the result for the orthogonal group case. In that case the weaker hypothesis is: The sum of the number of parts of λ plus the number of parts of λ of length greater than one is bounded by n. In this article we describe some new results in character theory and an interpretation of these results through Howe duality. This will yield yet another proof of the Littlewood Restriction and more importantly a generalization valid for all parameters λ.

In 1977 Lepowski [L] gave resolutions of each finite dimensional representation of a semisimple Lie algebra in terms of generalized Verma modules associated to any parabolic subalgebra. This work extended the so-called BGG resolutions [BGG] from Borel subalgebras to general parabolic subalgebras. The first result of this article gives an analogue of the Lepowski result for unitarizable highest weight representations. To formulate this precisely we begin with some notation.

Let G be a simple connected real Lie group with maximal compactly embedded subgroup K with (G, K) a Hermitian symmetric pair and let $\mathfrak{g}$ and $\mathfrak{k}$ be their complexified Lie algebras. Fix a Cartan subalgebra $\mathfrak{h}$ of both $\mathfrak{k}$ and $\mathfrak{g}$ and let Δ (resp. $\Delta_{\mathfrak{k}}$) denote the roots of $(\mathfrak{g}, \mathfrak{h})$ (resp. $(\mathfrak{k}, \mathfrak{h})$). Let Δ_n be the complement so that $\Delta = \Delta_{\mathfrak{k}} \cup \Delta_n$. We call the elements in these two sets the compact and noncompact roots respectively. The Lie algebra $\mathfrak{k}$ contains a one dimensional center $\mathbb{C}z_0$. The adjoint action of z_0 on $\mathfrak{g}$ gives the decomposition: $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$, where $\mathfrak{k}$ equals the centralizer of z_0 and $\mathfrak{p}^\pm$ equals the ± 1 eigenspaces of ad z_0. Here $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$ is a maximal parabolic subalgebra. Let Δ^+ denote a fixed positive root system for which $\Delta^+ = \Delta_{\mathfrak{k}}^+ \cup \Delta_n^+$ and where Δ_n^+ is the set of roots corresponding to $\mathfrak{p}^+$. Let $\mathcal{W}$ (resp. $\mathcal{W}_{\mathfrak{k}}$) denote the Weyl group for $(\mathfrak{g}, \mathfrak{h})$ (resp. $(\mathfrak{k}, \mathfrak{h})$). We call the latter the Weyl group of $\mathfrak{k}$ and regard it as a subgroup of $\mathcal{W}$. Then $\mathcal{W} = \mathcal{W}_\mathfrak{k}\mathcal{W}^\mathfrak{k}$ where we define $\mathcal{W}^\mathfrak{k} = \{x \in \mathcal{W} | x\Delta^+ \supset \Delta_\mathfrak{k}^+\}$. Let ρ (resp. $\rho_\mathfrak{k}, \rho_n$) equal one half the sum over the set Δ^+ (resp. $\Delta_\mathfrak{k}^+, \Delta_n^+$). When the root system Δ contains only one root length we call the roots short. For any root α let $\alpha^\vee$ denote the coroot defined by $(\alpha^\vee, \xi) = \frac{2(\alpha,\xi)}{(\alpha,\alpha)}$.

Next we define the root systems and reductive Lie algebras associated to unitarizable highest weight representations of G. Suppose $L = L(\lambda + \rho)$ is a unitarizable highest weight representation of G with highest weight λ. Set $\Psi_\lambda = \{\alpha \in \Delta | (\alpha, \lambda + \rho) = 0\}$ and $\Psi_\lambda^+ = \Psi_\lambda \cap \Delta^+$. We call Ψ_λ the singularities of $\lambda + \rho$ and note that Ψ_λ^+ is a set of strongly orthogonal noncompact roots. Define $\mathcal{W}_\lambda$ to be the subgroup of the Weyl group $\mathcal{W}$ generated by the identity and all the reflections r_α which satisfy the following three conditions:

$$\text{(1.1.3)} \qquad \text{(i) } \alpha \in \Delta_n^+ \quad \text{and } (\lambda + \rho, \alpha^\vee) \in \mathbb{N}^* \quad \text{(ii) } \alpha \text{ is orthogonal to } \Psi_\lambda \ ,$$

$$\text{(iii) if some } \delta \in \Psi_\lambda \text{ is long then } \alpha \text{ is short} \ .$$

Let Δ_λ equal the subset of Δ of elements δ for which the reflection $r_\delta \in \mathcal{W}_\lambda$ and let $\Delta_{\lambda,\mathfrak{k}} = \Delta_\lambda \cap \Delta_\mathfrak{k}$, $\Delta_\lambda^+ = \Delta_\lambda \cap \Delta^+$ and $\Delta_{\lambda,\mathfrak{k}}^+ = \Delta_{\lambda,\mathfrak{k}} \cap \Delta^+$. Then in our setting Δ_λ and $\Delta_{\lambda,\mathfrak{k}}$ are abstract root systems and we let $\mathfrak{g}_\lambda$ (resp. $\mathfrak{k}_\lambda$) denote the reductive Lie algebra with root system Δ_λ (resp. $\Delta_{\lambda,\mathfrak{k}}$) and Cartan subalgebra $\mathfrak{h}$ equal to that of $\mathfrak{g}$. Then the pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$ is a Hermitian symmetric pair although not necessarily of the same type as $(\mathfrak{g}, \mathfrak{k})$. For example, if λ is the highest weight of either component of the Weil representation of $\mathrm{Sp}(n)$ then Δ_λ will be the root system of type D_n and the Hermitian symmetric pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$ will correspond to the real form $\mathrm{so}^*(2n)$. Let ρ_λ (resp. $\rho_{\mathfrak{k},\lambda}$) equal half the sum of the roots in Δ_λ^+ (resp. $\Delta_{\mathfrak{k},\lambda}^+$).

For any $\Delta_\mathfrak{k}^+$ (resp. $\Delta_{\lambda,\mathfrak{k}}^+$, Δ_λ^+)-dominant integral weight μ, let E_μ (resp. $E_{\mathfrak{k}_\lambda,\mu}, B_{\mathfrak{g}_\lambda,\mu}$) denote the finite dimensional $\mathfrak{k}$ (resp. $\mathfrak{k}_\lambda$, $\mathfrak{g}_\lambda$) module with highest weight μ. Set $\mathcal{W}_{\lambda,\mathfrak{k}} = \mathcal{W}_\lambda \cap \mathcal{W}_\mathfrak{k}$ and define:

(1.1.4)

$$\mathcal{W}_\lambda^\mathfrak{k} = \{x \in \mathcal{W}_\lambda | x\Delta_\lambda^+ \supset \Delta_{\lambda,\mathfrak{k}}^+\} \text{ and } \quad \mathcal{W}_\lambda^{\mathfrak{k},i} = \{x \in \mathcal{W}_\lambda^\mathfrak{k} | \mathrm{card}(x\Delta_\lambda^+ \cap -\Delta_\lambda^+) = i\} \ .$$

For any $\mathfrak{k}$-integral $\xi \in \mathfrak{h}^*$, let ξ^+ denote the unique element in the $\mathcal{W}_\mathfrak{k}$-orbit of ξ which is $\Delta_\mathfrak{k}^+$-dominant. For any $\mathfrak{k}$-dominant integral weight λ define the generalized Verma module with highest weight λ to be the induced module defined by: $N(\lambda + \rho) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{k} \oplus \mathfrak{p}^+)} E_\lambda$. Finally we define what will be an important hypothesis. We say that λ is *quasi-dominant* if $(\lambda + \rho, \alpha) > 0$ for all $\alpha \in \Delta^+$ with $\alpha \perp \Psi_\lambda$. Whenever λ is quasi-dominant then we find that there are close connections between the character theory and Hilbert series of $L(\lambda+\rho)$ and the finite dimensional $\mathfrak{g}_\lambda$-module $B_{\mathfrak{g}_\lambda,\lambda+\rho-\rho_\lambda}$. To simplify notation we set $B_\lambda = B_{\mathfrak{g}_\lambda, \lambda+\rho-\rho_\lambda}$. In the examples mentioned above where L is one of the two components of the Weil representation then the resulting B_λ are the two spin representations of $\mathrm{so}^*(2n)$.

THEOREM 2. *Suppose $L = L(\lambda+\rho)$ is a unitarizable highest weight module. Then L admits a resolution in terms of generalized Verma modules. Specifically, for $1 \le i \le r_\lambda = \mathrm{card}(\Delta_\lambda^+ \cap \Delta_n^+)$, set $\mathbf{C}_i^\lambda = \sum_{x \in \mathcal{W}_\lambda^{\mathfrak{k},i}} N((x(\lambda+\rho))^+)$. Then there is a resolution of L:*

$$0 \to \mathbf{C}_{r_\lambda}^\lambda \to \cdots \to \mathbf{C}_1^\lambda \to \mathbf{C}_0^\lambda \to L \to 0 \; . \tag{1.1.5}$$

The grading of $\mathcal{W}_\lambda^{\mathfrak{k}}$ plays an important role in this theorem. Note that the grading $\mathcal{W}_\lambda^{\mathfrak{k},i}$ is not the one inherited from $\mathcal{W}^{\mathfrak{k}}$. We have two applications of this theorem. The first will generalize the Littlewood Restriction Theorem while the second in the quasi-dominant setting will give an identity relating the Hilbert series of L and B_λ.

(1.2) Let L denote a unitarizable highest weight representation for $\mathfrak{g}$, one of the classical Lie algebras $\mathrm{su}(p,q), \mathrm{sp}(n,\mathbb{R})$ or $\mathrm{so}^*(2n)$. These Lie algebras occur as part of the reductive dual pairs:

$$\begin{array}{lll} \text{(i)} & \mathrm{Sp}(k) \times \mathrm{so}^*(2n) \text{ acting on } \mathcal{P}(M_{2k\times n}), \\ \text{(ii)} & \mathrm{O}(k) \times \mathrm{sp}(n) \text{ acting on } \mathcal{P}(M_{k\times n}) \text{ and} \\ \text{(iii)} & \mathrm{U}(k) \times \mathrm{u}(p,q) \text{ acting on } \mathcal{P}(M_{k\times n}), \end{array} \tag{1.2.1}$$

where $n = p+q$. Let $\mathcal{S} = \mathcal{P}(M_{2k\times n})$ or $\mathcal{P}(M_{k\times n})$ as in (1.2.1). We consider the action of two dual pairs on $\mathcal{S}$. The first is $\mathrm{GL}(m)\times\mathrm{GL}(n)$ with $m = 2k$ or k and the second is $G_1 \times G_2$, one of the two pairs (i) or (ii) in (1.2.1). In this setting G_1 is contained in $\mathrm{GL}(m)$ while $\mathrm{GL}(n)$ is the maximal compact subgroup of G_2. We can calculate the multiplicity of an irreducible $G_1 \times \mathrm{GL}(n)$ representation in $\mathcal{S}$ in two ways. The resulting identity is the branching formula.

For any integer partition $\lambda = (\lambda_1 \ge \cdots \ge \lambda_l)$ with at most l parts, let $F_{(l)}^\lambda$ be the irreducible representation of $\mathrm{GL}(l)$ indexed in the usual way by its highest weight. Similarly, for each nonnegative integer partition μ with at most l parts, let $V_{(l)}^\mu$ be the irreducible representation of $\mathrm{Sp}(k)$ with highest weight μ. Let $E_{(l)}^\nu$ denote the irreducible representation of $\mathrm{O}(l)$ associated to the nonnegative integer partition ν with at most l parts and having Ferrers diagram whose first two columns have lengths which sum to l or less. Our conventions for $\mathrm{O}(l)$ follow [GW, Ch. 10].

The theory of dual pairs gives three decompositions of $\mathcal{S}$: as a $\mathrm{GL}(m) \times \mathrm{GL}(n)$ representation,

$$\mathcal{S} = \sum_\lambda F_{(m)}^\lambda \otimes F_{(n)}^\lambda, \tag{1.2.2}$$

where the sum is over all nonnegative integer partitions having $\min\{m,n\}$ or fewer parts; as a $\mathrm{Sp}(k)\times \mathrm{so}^*(2n)$ representation,

$$\mathcal{S}=\sum_{\mu} V_{(k)}^{\mu}\otimes V_{\mu}^{(n)}, \tag{1.2.3}$$

where the sum is over all nonnegative integer partitions μ having $\min\{k,n\}$ or fewer parts; and as a $\mathrm{O}(k)\times \mathrm{sp}(n)$ representation,

$$\mathcal{S}=\sum_{\nu} E_{(k)}^{\nu}\otimes E_{\nu}^{(n)}, \tag{1.2.4}$$

where the sum is over all nonnegative integer partitions ν having $\min\{k,n\}$ or fewer parts and having a Ferrers diagram whose first two columns sum to k or less.

Several conventions regarding highest versus lowest weights and an affine shift coming from the dual pair action of $\mathfrak{k}$ introduce an involution on weights as follows. For an n-tuple $\tau=(\tau_1,\cdots,\tau_n)$, define:

$$\tau^{\sharp}=\begin{cases}(-\frac{k}{2}-\tau_n,\cdots,-\frac{k}{2}-\tau_1) & \text{for the } (\mathrm{O}(k),\mathrm{sp}(n)) \text{ case,}\\ (-k-\tau_n,\cdots,-k-\tau_1) & \text{for the } (\mathrm{Sp}(k),\mathrm{so}^*(2n)) \text{ case.}\end{cases} \tag{1.2.5}$$

Note that $(\tau^{\sharp})^{\sharp}=\tau$. Computing the multiplicity of $V_{(k)}^{\mu}\otimes F_{(n)}^{\lambda}$ in $\mathcal{S}$ and $E_{(k)}^{\nu}\otimes F_{(n)}^{\lambda}$ in $\mathcal{S}$ we obtain:

Theorem 3. (i) *The multiplicity of the* $\mathrm{Sp}(k)$ *representation* $V_{(k)}^{\mu}$ *in* $F_{(2k)}^{\lambda}$ *equals the multiplicity of* $F_{(n)}^{\lambda^{\sharp}}$ *in the unitarizable highest weight representation* $V_{\mu}^{(n)}$ *of* $\mathrm{so}^*(2n)$.

(ii) *The multiplicity of the* $\mathrm{O}(k)$ *representation* $E_{(k)}^{\nu}$ *in* $F_{(k)}^{\lambda}$ *equals the multiplicity of* $F_{(n)}^{\lambda^{\sharp}}$ *in the unitarizable highest weight representation* $E_{\nu}^{(n)}$ *of* $\mathrm{sp}(n)$.

In the cases where the unitarizable highest weight representation is the full generalized Verma module we call the parameter a generic point. A short calculation shows that the Littlewood hypothesis implies inclusion in the generic set. Then Theorem 3 implies Theorem 1.

For any partitions λ and μ with at most n parts, define constants:

$$C_{\mu}^{\lambda}=\sum_{\xi}\dim \mathrm{Hom}_{\mathrm{GL}(n)}(F_{(n)}^{\lambda},F_{(n)}^{\mu}\otimes F_{(n)}^{\xi}), \tag{1.2.6}$$

where the sum is over all nonnegative integer partitions ξ with rows of even length, and

$$D_{\mu}^{\lambda}=\sum_{\xi}\dim \mathrm{Hom}_{\mathrm{GL}(n)}(F_{(n)}^{\lambda},F_{(n)}^{\nu}\otimes F_{(n)}^{\xi}), \tag{1.2.7}$$

where the sum is over all nonnegative integer partitions ξ with columns of even length. We refer to these constants as the Littlewood coefficients and note that they can be computed by the Littlewood-Richardson rule.

For any $\mathfrak{k}$-integral $\xi \in \mathfrak{h}^*$ and $s \in \mathcal{W}$, define:

$$s \circledast \xi = \ (s(\xi+\rho))^{+} - \rho \ , \quad \text{and} \quad s \cdot \xi = (s \circledast \xi^{\sharp})^{\sharp} \ . \tag{1.2.8}$$

Theorems 2 and 3 combine to give:

Theorem 4. (i) *Given nonnegative integer partitions σ and μ with at most* $\min(k,n)$ *parts and with μ having a Ferrers diagram whose first two columns sum to k or less, then*

$$\dim \operatorname{Hom}_{\mathrm{O}(k)}(E^{\mu}_{(k)}, F^{\sigma}_{(k)}) = \sum_i \sum_{s \in \mathcal{W}^{\mathfrak{k},i}_{\mu^\sharp}} (-1)^i \ C^{\sigma}_{s\cdot\mu}. \tag{1.2.9}$$

(ii) *Given partitions σ and ν such that $\ell(\sigma) \leq \min(2k,n)$ and $\ell(\nu) \leq$* $\min(k,n)$,

$$\dim \operatorname{Hom}_{\mathrm{Sp}(k)}(V^{\nu}_{(k)}, F^{\sigma}_{(2k)}) = \sum_i \sum_{s \in \mathcal{W}^{\mathfrak{k},i}_{\nu^\sharp}} (-1)^i \ D^{\sigma}_{s\cdot\nu}. \tag{1.2.10}$$

An example is given at the end of Section 7 where the sum on the right reduces to a difference of two Littlewood coefficients.

(1.3) For any Hermitian symmetric pair $\mathfrak{g}, \mathfrak{k}$ and highest weight $\mathfrak{g}$-module M, let M_0 denote the $\mathfrak{k}$-submodule generated by any highest weight vector. Write $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$, where $\mathfrak{p}^+$ is spanned by the root spaces for positive noncompact roots, and set $M_j = \mathfrak{p}^- \cdot M_{j-1}$ for $j > 0$. Define the Hilbert series $H_M(q)$ of M by:

$$H(q) = H_M(q) = \sum_{j \geq 0} \dim \ M_j \ \ q^j \ . \tag{1.3.1}$$

Since the enveloping algebra of $\mathfrak{p}^-$ is Noetherian there are a unique integer d and a unique polynomial $R_M(q)$ such that:

$$H_M(q) = \frac{R_M(q)}{(1-q)^d} \quad \text{where} \quad R_M(q) = \sum_{0 \leq j \leq e} a_j \ q^j \ . \tag{1.3.2}$$

In this setting the integer d is the Gelfand-Kirillov dimension ([BK], [V]), $d = \mathrm{GKdim}(M)$ and $R_M(1)$ is called the Bernstein degree of M and denoted $\mathrm{Bdeg}(M)$. This polynomial $R_M(q)$ is a q-analogue of the Bernstein degree. For any $\mathfrak{g}_\lambda$-dominant integral μ we let $B^i_{\mathfrak{g}_\lambda,\mu}$ denote the grading of $B_{\mathfrak{g}_\lambda,\mu}$ as a $\mathfrak{g}_\lambda \cap \mathfrak{p}^-$-module as in (1.3.1) with $\mathfrak{p}^-$ replaced by $\mathfrak{g}_\lambda \cap \mathfrak{p}^-$. Define the Hilbert series of $B_{\mathfrak{g}_\lambda,\mu}$ by :

$$P(q) = P_\mu(q) = \sum \dim \ B^i_{\mathfrak{g}_\lambda,\mu} \ q^i \ . \tag{1.3.3}$$

THEOREM 5. *Suppose $L = L(\lambda+\rho)$ is unitarizable and $\lambda+\rho$ is quasi-dominant. Set d equal to the Gelfand-Kirillov dimension of L as given by Theorems* 6 *and* 7. *Then the Hilbert series of L is*:

$$\mathrm{H}_L(q) = \frac{\dim E_\lambda}{\dim E_{\mathfrak{k}_\lambda,\lambda}} \, \frac{P(q)}{(1-q)^d}\ . \tag{1.3.4}$$

Moreover the Bernstein degree of L is given by:

$$\mathrm{Bdeg}(L) = \frac{\dim E_\lambda}{\dim E_{\mathfrak{k}_\lambda,\lambda}} \dim B_\lambda. \tag{1.3.5}$$

THEOREM 6. *Suppose that L is a unitarizable highest weight representation occurring in one of the dual pairs settings* (1.2.1).

(i) *If $\mathfrak{g}$ is* $\mathrm{so}^*(2n)$, *then the Gelfand-Kirillov dimension of L equals $k(2n-2k-1)$ for $1 \le k \le [\frac{n-2}{2}]$ and equals $\binom{n}{2}$ otherwise.*

(ii) *If $\mathfrak{g}$ is* $\mathrm{sp}(n)$, *then the Gelfand-Kirillov dimension of L equals $\frac{k}{2}(2n-k+1)$ for $1 \le k \le n-1$ and equals $\binom{n+1}{2}$ otherwise.*

(iii) *If $\mathfrak{g}$ is* $\mathrm{u}(p,q)$, *then the Gelfand-Kirillov dimension of L equals $k(n-k)$ for $1 \le k \le \min\{p,q\}$ and equals pq otherwise.*

Note that in all cases the Gelfand-Kirillov dimension is dependent only on the dual pair setting given by k and n and is independent of λ otherwise. It is of course convenient to compute the Gelfand-Kirillov dimension of L directly from the highest weight. Let β denote the maximal root of $\mathfrak{g}$.

THEOREM 7. *Set $s = -\frac{2(\lambda,\beta)}{(\beta,\beta)}$. Then for* $\mathrm{so}^*(2n)$, *the Gelfand-Kirillov dimension of L is*

$$\begin{cases} \frac{s}{2}(2n-s-1) & \text{for } 2 \le s \le 2[\frac{n}{2}]-2 \\ \binom{n}{2} & \text{otherwise;} \end{cases}$$

for $\mathrm{sp}(n)$, *the Gelfand-Kirillov dimension of L is*

$$\begin{cases} s(2n-2s+1) & \text{for } 1 \le 2s \le n \\ \binom{n+1}{2} & \text{otherwise;} \end{cases}$$

and for $\mathrm{u}(p,q)$ *with $n = p+q$, the Gelfand-Kirillov dimension of L is*

$$\begin{cases} s(n-s) & \text{for } 1 \le s \le \min\{p,q\} \\ pq & \text{otherwise.} \end{cases}$$

(1.4) In Section 6 we apply Theorems 5 and 6 to determine the Gelfand-Kirillov dimension, Hilbert series and Bernstein degree of some well-known representations. We begin with the Wallach representations [W]. Let r equal

the split rank of $\mathfrak{g}$, let ζ be the fundamental weight of $\mathfrak{g}$ which is orthogonal to all the roots of $\mathfrak{k}$. Suppose $\mathfrak{g}$ is isomorphic to either $\mathrm{so}^*(2n)$, $\mathrm{sp}(n)$ or $\mathrm{su}(p,q)$ and set $c = 2,\ \frac{1}{2}$ or 1 depending on which of the three cases we are in. For $1 \leq j < r$ define the j^{th} Wallach representation W_j to be the unitarizable highest weight representation with highest weight $-jc\zeta$. For $\mathrm{so}^*(2n)$ the Hilbert series for the first Wallach representation is:
(1.4.1)

$$\mathrm{H}_L(q) = \frac{R(q)}{(1-q)^{2n-3}} = \frac{1}{(1-q)^{2n-3}}\ \frac{1}{n-2} \sum_{0\leq j\leq n-3} \binom{n-2}{n-3-j}\binom{n-2}{j}\, q^j.$$

For $\mathrm{sp}(n)$ the Hilbert series for the first Wallach representation is:

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^n} \sum_{0\leq t\leq [\frac{n}{2}]} \binom{n}{2t} q^t. \tag{1.4.2}$$

This is the Hilbert series for the half of the Weil representation generated by a one dimensional representation of $\mathfrak{k}$. The other part of the Weil representation has Hilbert series:

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^n} \sum_{0\leq t\leq [\frac{n}{2}]} \binom{n}{2t+1} q^t. \tag{1.4.3}$$

For $\mathrm{U}(p,q)$ the Hilbert series for the first Wallach representation is:

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^{n-1}} \sum_{0\leq t<\min\{p,q\}} \binom{p-1}{t}\binom{q-1}{t}\, q^t\ . \tag{1.4.4}$$

These examples are obtained from Theorem 5 by writing out respectively the Hilbert series of the $n-3^{\text{rd}}$ exterior power of the standard representation of $\mathrm{so}^*(2n-4)$, the two components of the spin representation of $\mathrm{so}^*(2n)$ and the $p-1^{\text{st}}$ fundamental representation of $\mathrm{U}(p-1,q-1)$. In these four examples the Bernstein degrees are: $\frac{1}{n-2}\binom{2n-4}{n-3}$, 2^{n-1}, 2^{n-1} and $\binom{n-2}{p-1}$. In Section 6 we give several other families of representations with interesting combinatorial expressions for the Hilbert series and Bernstein degrees including all highest weight representations with singular infinitesimal character and minimal Gelfand-Kirillov dimension.

Call a highest weight representation positive if all the nonzero coefficients of the polynomial $R_L(q)$ in (1.3.2) are positive. All Cohen-Macaulay $S(\mathfrak{p}^-)$-modules including the Wallach representations are positive but many unitary highest weight representations are not. From this perspective Theorem 5 introduces a large class of positive representations, those with quasi-dominant highest weight.

The representation theory of unitarizable highest weight modules was studied from several different points of view. Classifications were given in [EHW] and [J]. Studies of the cohomology and character theory can be found in [A], [C], [ES], [ES2] and [E]. Both authors thank Professor Nolan Wallach for his interest in this project as well as several critical suggestions. A form of Theorem 3 and its connection to the Littlewood Restriction Theorem are two of the results in the second author's thesis which was directed by Professor Wallach.

Upon completion of this article we have found several references related to the Littlewood branching rules. The earliest (1951) is by M. J. Newell [N] which describes his modification rules to extend the Littlewood branching rules to all parameters. A more recent article by S. Sundaram [S] generalizes the Littlewood branching to all parameters in the symplectic group case. In both articles the results take a very different form from what is presented here.

During the time this announcement has been refereed, there has been some related research which has appeared [NOTYK]. In this work the authors begin with a highest weight module L and then consider the associated variety $\mathcal{V}(L)$ as defined by Vogan. This variety is the union of $K_{\mathbb{C}}$-orbits and equals the closure of a single orbit. In [NOTYK] the Gelfand-Kirillov dimension and the Bernstein degree of L are recovered from the corresponding objects for the variety $\mathcal{V}(L)$. As an example of their technique they obtain the Gelfand-Kirillov dimension and the degree of the Wallach representations ([NOTYK, pp. 149–150]). Our results in this setting obtain these two invariants as well as the full Hilbert series since all the highest weights are quasi-dominant. The results of these two very different approaches have substantial overlap although neither subsumes the other.

Most of the results presented in this article were announced in [EW].

2. Unitarizable highest weight modules and standard notation

(2.1) Here we set down some notation used throughout the article and state some well-known theorems in the precise forms needed later. Let (G, K) be an irreducible Hermitian symmetric pair with real (resp. complexified) Lie algebras $\mathfrak{g}_o$ and $\mathfrak{k}_o$ (resp. $\mathfrak{g}$ and $\mathfrak{k}$) and Cartan involution θ. Let all the associated notation be as in (1.1). Let $\mathfrak{b}$ be the Borel subalgebra containing $\mathfrak{h}$ and the root spaces of Δ^+.

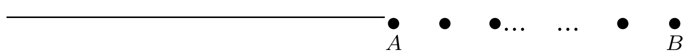
(2.2) For any $\Delta_{\mathfrak{k}}$ dominant integral weight λ let F_λ denote the irreducible finite dimensional representation of $\mathfrak{k}$ with highest weight λ. Define the generalized Verma modules by induction. Let $\mathfrak{p}^+$ act on F_λ by zero and then induce up from the enveloping algebra $\mathrm{U}(\mathfrak{q})$ to $\mathrm{U}(\mathfrak{g})$:

(2.2.1) $$N(\lambda+\rho) := N(F_\lambda) := \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{q})} F_\lambda\ .$$

We call $N(\lambda+\rho)$ the generalized Verma module with highest weight λ. Let $L(\lambda+\rho)$ denote the unique irreducible quotient of $N(\lambda+\rho)$. Since $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{p}^-$ and $\mathfrak{p}^-$ is abelian we can identify $N(\lambda+\rho)$ with $S(\mathfrak{p}^-) \otimes F_\lambda$, where the $S(\ )$

denotes the symmetric algebra. Therefore the natural grading of the symmetric algebra induces a grading $N(\lambda+\rho)^i$ of $N(\lambda+\rho)$. Different levels in the grade correspond to different eigenvalues of $\mathrm{ad}z_0$ and so any $\mathfrak{k}$-submodule of $N(\lambda+\rho)$ will inherit a grading by restriction. Suppose that $N(\lambda+\rho)$ is reducible with maximal submodule M. Then M inherits a grading and we define the level of reduction of $N(\lambda+\rho)$ to be the minimal j for which $M^j \neq 0$.

We say that $L(\lambda+\rho)$ is unitarizable if there exists a unitary representation of G whose $\mathrm{U}(\mathfrak{g})$ module of K-finite vectors is equivalent as a $\mathfrak{g}$-module to $L(\lambda+\rho)$. The unitarizable highest weight modules are central to all that we do here so we now describe much that is known about this set. The classification we follow is from [EHW]. Let λ be any $\mathfrak{k}$-dominant integral weight in $\mathfrak{h}^*$. Let β denote the unique maximal root. Choose $\zeta \in \mathfrak{h}^*$ orthogonal to the compact roots and with $(\zeta, \beta^\vee) = 1$. Consider the lines $\mathbb{L}(\lambda) = \{\lambda + z\zeta \mid z \in \mathbb{R}\}$, for $\mathfrak{k}$-dominant integral $\lambda \in \mathfrak{h}^*$. A normalization is chosen for each line so that $z = 0$ corresponds to the unique point with highest weight module a limit of discrete series module. When λ is such we write λ_0 in place of λ and the line is parametrized in the form $\{\lambda_0 + z\zeta | z \in \mathbb{R}\}$. Then $(\lambda_0 + \rho, \beta) = 0$ and the set of values z with $\lambda_0 + z\zeta$ unitarizable takes the form:
(2.2.2)



Let Λ denote the highest weights of all the unitarizable highest weight modules. Let Λ_r denote the subset of weights λ for which $N(\lambda+\rho)$ is reducible. We call these the unitary reduction points. These points correspond to the elements on the line (2.2.2) which are the equally spaced dots from A to B. The constants A and B are both positive.

The characteristics of the line and these equally spaced points are determined by two real root systems $Q(\lambda)$ and $R(\lambda)$ associated to each line $\mathbb{L}(\lambda)$. As defined in [EHW] $Q(\lambda) \subset R(\lambda)$ and equality holds in the equal root length cases. In all cases the number of reduction points on the line equals the split rank of $Q(\lambda)$ and the level of reduction is one at the rightmost dot and it increases by one each step until the level equals the split rank of $Q(\lambda)$ at the leftmost dot. For any reduction point λ let $l(\lambda)$ denote the level of reduction of that point and define the triple $a(\lambda) = (Q(\lambda), R(\lambda), l(\lambda))$. Let $\mathcal{A}$ denote the set of all such triples as λ ranges over the set of reduction points. For $a \in \mathcal{A}$, let Λ_a denote the set of all λ with $a(\lambda) = a$.

(2.3) Set $L = L(\lambda+\rho)$, $N = N(\lambda+\rho)$ and assume that L is unitarizable and N is reducible. Consider the short exact sequence $0 \to M \to N \to L \to 0$. From [DES] and [EJ] the subspace M has several canonical characterizations.

Let $\gamma_1 < \cdots < \gamma_l$ be Harish-Chandra's system of strongly orthogonal roots for Δ_n^+. That is, let γ_1 equal the unique simple noncompact root and let

$\Psi_1 = \{\gamma \in \Delta_n^+ - \{\gamma_1\} | \, \gamma \pm \gamma_1 \notin \Delta\}$. If $\Psi_1 = \emptyset$ then $l = 1$. Otherwise, let γ_2 be the smallest element of Ψ_1 and set $\Psi_2 = \{\gamma \in \Psi_1 - \{\gamma_2\} | \, \gamma \pm \gamma_2 \notin \Phi\}$. By induction if γ_j and Ψ_{j-1} have been defined set $\Psi_j = \{\gamma \in \Psi_{j-1} | \gamma \pm \gamma_i$ is not zero or a root for all $i, 1 \leq i \leq j\}$. Let γ_{j+1} be any minimal element in Ψ_j so long as this set is non empty. Define weights $\mu_i, 1 \leq i \leq l$, by $\mu_i = -(\gamma_1 + \gamma_2 + \cdots + \gamma_i)$. Set $\mathfrak{n}_{\mathfrak{k}} = \mathfrak{k} \cap [\mathfrak{b}, \mathfrak{b}]$. Let F_i denote the $\mathfrak{k}$ submodule of $S(\mathfrak{p}^-)$ with highest weight μ_i. Suppose that ξ and δ are $\mathfrak{k}$-dominant integral then $F_\xi \otimes F_\delta$ contains, with multiplicity one, the irreducible module with extreme weight $\xi - \delta$. We call this component of the tensor product the PRV component.

Proposition [EJ], [DES]. *Suppose L is unitarizable and not isomorphic to N and let d be the level of reduction of L. Then M is isomorphic to a quotient of the generalized Verma module $N(F_\nu)$ with F_ν equal to the* PRV *component of $F_d \otimes F_\lambda$.*

3. A BGG type resolution for unitarizable highest weight modules

(3.1) Each finite dimensional representation of a semisimple Lie algebra has a resolution in terms of sums of Verma modules [BGG]. Lepowski [L] gives a refinement resolving in terms of generalized Verma modules associated to a parabolic subalgebra. In this section we give a very similar resolution for unitarizable highest weight representations. Define subsets of the Weyl group by $\mathcal{W}^i = \{x \in \mathcal{W} | \mathrm{card}(x\Delta^+ \cap -\Delta^+) = i\}$ and set $\mathcal{W}^{\mathfrak{k},i} = \mathcal{W}^i \cap \mathcal{W}^{\mathfrak{k}}$.

Theorem [L]. *Suppose λ is $\mathfrak{g}$-dominant integral and E is the finite dimensional $\mathfrak{g}$-module $L(\lambda+\rho)$. For $0 \leq i \leq r = |\Delta_n^+|$, set $\mathbf{C}_i = \sum_{x \in \mathcal{W}^{\mathfrak{k},i}} N(x(\lambda+\rho))$. Then there exists a resolution of E:*

$$0 \to \mathbf{C}_r \to \cdots \to \mathbf{C}_1 \to \mathbf{C}_0 \to E \to 0\,. \tag{3.1.1}$$

(3.2) We next consider the case where E is replaced by the Weil representation. Suppose that $\mathfrak{g}$ is the symplectic Lie algebra $\mathrm{sp}(n)$. Then the Weil representation decomposes as the sum of two irreducible highest weight representations. Normalizing parameters as in [EHW] set ζ equal to the functional on $\mathfrak{h}$ orthogonal to all the compact roots and with $\frac{2(\beta,\zeta)}{(\beta,\beta)} = 1$. Here ζ is the fundamental weight corresponding to the long root in the Dynkin diagram and is usually denoted ω_n. Let ω_{n-1} be the adjacent fundamental weight. Then the two components of the Weil representation are $L' = L(-\frac{1}{2}\zeta + \rho)$ and $L'' = L(-\frac{3}{2}\zeta + \omega_{n-1} + \rho)$. Expressed in the usual Euclidean coordinates the highest weights are $(-\frac{1}{2}, -\frac{1}{2}, \cdots -\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2}, \cdots -\frac{1}{2}, -\frac{3}{2})$ respectively.

Theorem. *Let Δ_s^+ denote the short roots in Δ^+. Let $\mathcal{U}$ denote the index-two subgroup of $\mathcal{W}$ which corresponds to permutations and even numbers of sign changes and set $\mathcal{U}^{\mathfrak{k}} = \mathcal{U} \cap \mathcal{W}^{\mathfrak{k}}$ and $\mathcal{U}^{\mathfrak{k},i} = \{x \in \mathcal{U}^{\mathfrak{k}} | \mathrm{card}(x\Delta_s^+ \cap -\Delta_s^+) = i\}$. For $1 \leq i \leq r_\circ = \Delta_n^+ \cap \Delta_s^+$, define $\mathbf{C}_i' = \sum_{x \in \mathcal{U}^{\mathfrak{k},i}} N(x(-\frac{1}{2}\zeta + \rho))$ and $\mathbf{C}_i'' =$*

$\sum_{x\in\mathcal{U}^{\mathfrak{k},i}} N(x(-\frac{3}{2}\zeta+\omega_{n-1}+\rho))$. *Then there are resolutions of the components of the Weil representation,*

$$0 \to \mathbf{C}'_{r_\circ} \to \cdots \to \mathbf{C}'_1 \to \mathbf{C}'_0 \to L' \to 0 \ , \tag{3.2.1}$$

and

$$0 \to \mathbf{C}''_{r_\circ} \to \cdots \to \mathbf{C}''_1 \to \mathbf{C}''_0 \to L'' \to 0 \ . \tag{3.2.2}$$

Note that the grading $\mathcal{U}^{\mathfrak{k},i}$ is not the one inherited from $\mathcal{W}^{\mathfrak{k},i}$; in general $\mathcal{U}^{\mathfrak{k},i} \neq \mathcal{U} \cap \mathcal{W}^{\mathfrak{k},i}$.

Proof. The proof begins with a review of the proof of Theorem 3.1[L]. The canonical imbeddings of the Verma submodules into Verma modules are used to define what are called the standard maps between generalized Verma modules. Of course in some cases some of these induced maps can be zero. In any case these maps can be used to construct a complex with terms as in (3.1.1). Here $\mathcal{U}$ is the Weyl group of type D_n and the grading $\mathcal{U}^{\mathfrak{k},i}$ comes from that root system. Therefore Lepowski's argument applies by switching root systems from C_n to D_n. To prove that this complex is a resolution Lepowski relies on the known Kostant $\mathfrak{p}^-$-cohomology formulas for the finite module E. This same argument gives the proof in this setting when we replace the Kostant results with the cohomology formulas in the next theorem.

(3.3) Theorem [E, Th. 2.2]. *Suppose λ equals either λ' or λ'' as above and $L = L(\lambda+\rho)$. Then, for $i \in \mathbb{N}$, there exists the cohomology formula of $\mathfrak{k}$-modules*:

$$H^i(\mathfrak{p}^+, L) \cong \oplus_{x\in\mathcal{U}^{\mathfrak{k},i}} \quad F_{x(\lambda+\rho)-\rho} \ . \tag{3.3.1}$$

(3.4) We now turn to the corresponding results in the general case.

Proof of Theorem 2. We have two proofs of this result. The first proof begins with the standard maps, as in the proof of Theorem 3.2, and uses the constants associated with the root system Δ_λ to define a complex as in (3.1.1). Then the $\mathfrak{p}^+$-cohomology formulas [E, Th. 2.2] can be used in place of the Kostant formulas in the Lepowski [L] argument. This knowledge of the $\mathfrak{p}^+$-cohomology implies that the complex is in fact exact, which completes the first proof.

The second is a consequence of the proof of the $\mathfrak{p}^+$-cohomology formulas in [E]. In that article it is proved that every unitarizable highest weight module L was an element of a category of highest weight modules which was equivalent to another category of highest weight modules and this equivalence carried L to either the trivial representation or one of the two components of the Weil representation in the image category. Therefore the general result follows from Theorems 3.1 and 3.2 since this equivalence carries generalized Verma modules to generalized Verma modules.

4. Hilbert series for unitarizable highest weight modules

(4.1) For any highest weight module A define the character of A to be the formal sum: $\mathrm{char}(A) = \sum_\xi \dim(A_\xi)\mathbf{e}^\xi$, where the subscript denotes the weight subspace. For any weight λ and Weyl group element x, define:

$$x \circledast \lambda = (x(\lambda+\rho))^+ - \rho \ . \tag{4.1.1}$$

THEOREM. *Suppose $L = L(\lambda+\rho)$ is a unitarizable highest weight module. Then*

$$\mathrm{char}(L) = \frac{1}{\prod_{\alpha\in\Delta_n^+}(1-\mathbf{e}^{-\alpha})} \sum_{\substack{1\le i\le r_\lambda \\ x\in\mathcal{W}_\lambda^{\mathfrak{k},i}}} (-1)^i \ \mathrm{char}\ F_{x\circledast\lambda} \ . \tag{4.1.2}$$

Proof. This result is an immediate consequence of the resolution given in Theorem 2.

Since this sum can be rather complicated we now look at a courser invariant than the character. This is obtained by using the eigenspaces for the action of the central element z_o of $\mathfrak{k}$. In our setting the $1, 0$ and -1 eigenspaces under the adjoint action are $\mathfrak{p}^+, \mathfrak{k}$ and $\mathfrak{p}^-$ respectively. For each Weyl group element x let g_x denote the difference of eigenvalues defined: $g_x = \lambda(z_0) - (x \circledast \lambda)(z_0)$. Note that since z_0 is $\mathfrak{k}$ central, g_x also equals $(\lambda+\rho)(z_0) - (x(\lambda+\rho))(z_0)$. Let $S = \mathrm{U}(\mathfrak{p}^-)$. Then S is the symmetric algebra of $\mathfrak{p}^-$ and any irreducible highest weight module is finitely generated as an S module. So L has a Hilbert series.

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^{\dim\ \mathfrak{p}^+}} \sum_{\substack{1\le \iota\le r_\lambda \\ x\in\mathcal{W}_\lambda^{\mathfrak{k},i}}} (-1)^i \ \dim\ F_{x\circledast\lambda}\ q^{g_x} \ . \tag{4.1.3}$$

Define the degree of L, $\deg(L)$, to be the order of the pole at 1 in the rational expression (4.1.3). Then we have:

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^{\deg(L)}}\ R(q) \quad \text{with} \quad R(q) = \sum a_i\ q^i \tag{4.1.4}$$

which we refer to as the reduced form of the Hilbert series. In this setting the degree of L is also equal to the Gelfand-Kirillov dimension of L, $\mathrm{GKdim}(L)$. The set $\mathcal{W}^{\mathfrak{k},r_\lambda}$ contains one element, say $\{x_\circ\}$ and so by comparison of (4.1.3) and (4.1.4), the degree of the polynomial $R(q)$ equals $g_{x_\circ} - \dim\,\mathfrak{p}^+ + \mathrm{GKdim}(L)$.

As an illustration of an especially simple case where these formulas lead to something interesting, suppose that $\lambda = 0$. Then $\deg(L) = 0$ and $\mathrm{H}_L = 1$. This gives:

LEMMA. *For each of the Hermitian symmetric settings and for $0 \le i \le r = \dim(\mathfrak{p}^+)$,*

$$\sum_{x\in\mathcal{W}^{\mathfrak{k},i}} \dim\ F_{x(\rho)-\rho} \ = \binom{r}{i}. \tag{4.1.5}$$

Proof. Set $H(0) = 1$ in (4.1.3) and note that in this case $g_x = i$.

(4.2) For the remainder of this section we assume that $\mathfrak{g}$ is of type $\mathrm{so}^*(2n)$, $\mathrm{sp}(n, \mathbb{R})$ or $\mathrm{u}(p, q)$. These Lie algebras occur as part of the dual pair setting:

(4.2.1)
$$\begin{aligned} \mathrm{Sp}(k) \times \mathrm{so}^*(2n) &\text{ acting on } \mathcal{P}(M_{2k\times n}), \\ \mathrm{O}(k) \times \mathrm{sp}(n) &\text{ acting on } \mathcal{P}(M_{k\times n}) \text{ and} \\ \mathrm{U}(k) \times \mathrm{u}(p,q) &\text{ acting on } \mathcal{P}(M_{k\times n}), \end{aligned}$$

where $n = p + q$. In these cases the element z_0 equals $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ in the first two cases and $(\frac{q}{n}, \dots, \frac{q}{n}; \frac{-p}{n}, \dots, \frac{-p}{n})$ for $\mathrm{u}(p, q)$ where a p-tuple precedes the semi-colon and a q-tuple follows it.

The proof of Theorem 6 will rely on the following lemma regarding the decomposition of tensor products.

(4.3) We continue with the three cases in (4.2.1). Let E denote the irreducible finite dimensional $\mathfrak{g}$-module with highest weight ω_1, the first fundamental weight. Here $\omega_1 = (1, 0, \dots, 0)$ in the first two cases and $\omega_1 = (\frac{n-1}{n}, \frac{-1}{n}, \dots, \frac{-1}{n})$ for $\mathrm{u}(p, q)$. So the z_0-eigenvalues of E are $\pm\frac{1}{2}$ in the first two cases and $\frac{q}{n}$ and $\frac{-p}{n}$ in the $\mathrm{u}(p, q)$ case. Then E splits as a direct sum of two irreducible $\mathfrak{k}$-modules $E = E_+ \oplus E_-$ corresponding to the z_0-eigenvalues $\pm\frac{1}{2}$ in the first two cases and $\frac{q}{n}$ and $\frac{-p}{n}$ in the third. Set $b_+ = \frac{1}{2}$ or $\frac{q}{n}$ and $b_- = \frac{-1}{2}$ or $\frac{-p}{n}$ respectively in the first two and third cases.

LEMMA. *For any $\mathfrak{k}$-dominant integral weight ν, let F_ν denote the irreducible finite dimensional $\mathfrak{k}$-module with highest weight ν. Then as $\mathfrak{k}$-modules $E \otimes F_\nu = \sum_\gamma F_{\nu+\gamma}$, where the sum is over all weights γ of E for which $\nu + \gamma$ is $\mathfrak{k}$-dominant.*

Proof. The Weyl character formula gives:

$$\mathrm{char}(F_\nu) = (1/D) \sum_{x\in\mathcal{W}_\mathfrak{k}} \epsilon(x)\ \mathbf{e}^{x(\nu+\rho)}.$$

From this we have: $\mathrm{char}(E \otimes F_\nu) = (1/D) \sum_\gamma \sum_{x\in\mathcal{W}_\mathfrak{k}} \epsilon(x)\ \mathbf{e}^{x(\nu+\gamma+\rho)}$, where the sum is over the weights γ of E. A calculation shows that $\nu + \gamma + \rho$ is always dominant and so the only cancellation which can and will occur in this expression is for those γ for which $\nu + \gamma + \rho$ is singular. This is precisely the set for which $\nu + \gamma$ is not $\mathfrak{k}$-dominant. The Littlewood-Richardson rule gives an alternate proof.

(4.4) LEMMA. *Let $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_r$ be an enumeration of all the weights γ of E for which $\nu + \gamma$ is dominant. Then there is a filtration $E \otimes N(F_\nu) = B_1 \supset B_2 \supset \cdots \supset B_{r+1} = 0$ where $B_i/B_{i+1} \cong N(F_{\nu+\gamma_i+\rho})$, $1 \leq i \leq r$.*

Proof. Using the preceding lemma choose a $\mathfrak{b}$ stable filtration of $E \otimes F_\nu$ and then induce up from $\mathrm{U}(\mathfrak{b})$ to $\mathrm{U}(\mathfrak{g})$.

(4.5) LEMMA. *Suppose that $L = L(\lambda+\rho)$ is unitarizable. Let γ be a weight of E and assume that $\lambda+\gamma$ is $\mathfrak{k}$-dominant.*

(i) *Suppose the level of reduction of L is not one. Then $E \otimes L$ contains $L(\lambda+\gamma+\rho)$.*

(ii) *Suppose that L has level of reduction one and choose $\delta \in \Delta_n$ so that $\lambda-\delta$ is the highest weight of the* PRV *component in $\mathfrak{p}^- \otimes F_\lambda$. Assume that $\delta \neq -\gamma+\gamma'$ for any weight γ' of E. Then $E \otimes L$ contains $L(\lambda+\gamma+\rho)$.*

Proof. From Proposition 2.3, we have a right exact sequence $N(\nu+\rho) \to N(\lambda+\rho) \to L \to 0$. Tensoring with E we obtain the right exact sequence:

$$E \otimes N(\nu+\rho) \to E \otimes N(\lambda+\rho) \to E \otimes L \to 0 \ . \tag{4.5.1}$$

Therefore using (4.4), to prove that $L(\lambda+\gamma+\rho)$ does occur in $E \otimes L$ we merely check that it does not occur in $E \otimes N(\nu+\rho)$.

First suppose that L has a level of reduction l_0 not equal to one. If the level is zero then $L = N(\lambda+\rho)$ and (4.4) implies the result. So assume the level is greater than one. Let a denote the eigenvalue of z_0 on F_λ. Then z_0 acts by $a+b_+$ or $a+b_-$ on $F_{\lambda+\gamma}$. But the eigenvalues of z_0 acting on $E \otimes N(\nu+\rho)$ are less than or equal to $a-l_0+b_+$. In all cases $b_+ - b_- = 1$ and so these sets of eigenvalues do not intersect for $l_0 > 1$. So $L(\lambda+\gamma+\rho)$ cannot occur as a subquotient of $E \otimes N(\nu+\rho)$. This proves (i).

Now suppose the level of reduction is one and $E \otimes L$ does not contain $L(\lambda+\gamma+\rho)$. Then $\nu = \lambda-\delta$ and we know $L(\lambda+\gamma+\rho)$ must occur in $E \otimes N(\nu+\rho)$. By the preceding argument about eigenvalues of z_0, there exists γ' a weight of E_+ with $\lambda-\delta+\gamma' = \lambda+\gamma$. This gives $\delta = \gamma'-\gamma$ and completes the proof.

Proof of Theorem 6. It is most convenient to proceed case by case.

(4.6) *The* so$^*(2n)$ *case.* This is the easiest case both notationally and theoretically so we will begin here. Suppose that $L = L(\lambda+\rho)$ is a highest weight representation occurring in the dual pair setting (4.2.1) for $\mathrm{Sp}(k) \times \mathrm{so}^*(2n)$. Set $r = \min\{k, n\}$. Then from [KV], [EHW] or [DES], in Euclidean coordinates, λ has the form:

$$\lambda = (-k, -k, \dots, -k, -k-w_r, \dots, -k-w_1) \text{ with } w_1 \geq \cdots \geq w_r \geq 0 \ , \tag{4.6.1}$$

$$\lambda+\rho = (n-1-k, \dots, -k+r, -k+r-1-w_r, \dots, -k-w_1) \ . \tag{4.6.2}$$

Let $\underline{w} = (w_1, \dots, w_r)$ and let $\lambda(\underline{w})$ denote the expression in (4.6.1). Choose t maximal with $w_t \neq 0$ and set $x = n-t$. Then organizing into segments, we have:

$$\lambda = (\underbrace{-k, \dots, -k}_{x}, -k-w_t, \dots, -k-w_1) \ . \tag{4.6.3}$$

In the general case, for $k \geq n-1$, $N(\lambda+\rho)$ is irreducible and the lemma holds. So we may assume that $1 \leq k \leq n-2$ and thus the first two coordinates of λ are equal. The root system $Q(\lambda)$ associated to λ in [EHW] is either $\mathrm{su}(1,q), 1 \leq q \leq n-1$, or $\mathrm{so}^*(2p), 3 \leq p \leq n$. First suppose that $Q(\lambda) \cong \mathrm{su}(1,q)$. Then since the first two coordinates of λ are equal, $Q(\lambda)$ is a root system of rank either one or three with the set of simple roots either $\{-\beta\}$ or $\{-\beta, e_2-e_3, e_1-e_2\}$ where $\beta = e_1+e_2$ is the maximal root. If we are at a reduction point in this case then the level of reduction is one, $q=1$ or 3 and

$$\lambda+\rho = (1,0,\widehat{-1},\cdots) \text{ or } \lambda+\rho = (2,1,0,\widehat{-1},\cdots)\ . \tag{4.6.4}$$

Alternatively suppose that λ has level of reduction one and $Q(\lambda) \cong \mathrm{so}^*(2p), 3 \leq p \leq n$. From Section 9 of [EHW],

$$\lambda+\rho = (p-1,p-2.\cdots,1,0,\widehat{-1},\cdots)\ , \tag{4.6.5}$$

where the superscript $\widehat{\ }$ denotes omission of that term.

Recall from (2.2.2) the line $\mathbb{L}(\lambda)$ and the parametrization $\lambda = \lambda_0 + z\zeta$ for some real number z. Set $d(\lambda) = B - z$ with B as in (2.2.2). So $d(\lambda)$ is the distance from λ (identified with z) to the last reduction point B. From (4.6.5) and (4.6.4) and the fact that the distance is zero when the level of reduction is one, we conclude: $x = p$ and

$$\lambda+\rho+d(\lambda)\zeta = (p-1,p-2,\ldots,1,0,\widehat{-1},\cdots)\ . \tag{4.6.6}$$

So for all λ in the dual pair setting $\mathrm{Sp}(k)\times \mathrm{so}^*(2n)$ and for all $k, 1 \leq k \leq n-2$, we solve for $d(\lambda)$ to obtain:

$$d(\lambda) = 2k-2n+2x. \tag{4.6.7}$$

Lemma. *The Gelfand-Kirillov dimension of L equals $k(2n-2k-1)$, for $1 \leq k < [\frac{n}{2}]$ and equals $\binom{n}{2}$ otherwise. So in all cases it is independent of $\underline{w}$.*

Proof. We proceed by induction on $|\underline{w}| = \sum w_i$. First if this norm is zero then λ has all coordinates $-k$ and so this representation is the k^{th} point on the line containing the trivial module of $\mathfrak{g}$, a so-called Wallach representation [W]. For $1 \leq k < [\frac{n}{2}]$, this module is the coordinate ring for the variety of skew symmetric $n \times n$ matrices of rank less than or equal to $2k$ [DES]. Its dimension is $k(2n-2k-1)$. For $k \geq [\frac{n}{2}]$, L is not a reduction point and the Gelfand-Kirillov dimension of $L = \dim \mathfrak{p}^+ = \binom{n}{2}$. This proves the result when $\underline{w} = 0$.

Now suppose $\underline{w} \neq 0$. If $k \geq n-1$, then $N(\lambda+\rho)$ is irreducible and the lemma holds. So assume $1 \leq k \leq n-2$. Suppose L has level of reduction one. Then from the formulas (4.6.5) and (4.6.6) the leading $n-t$ coordinates form a consecutive string of descending integers which include 0 as the x^{th} coordinate and -1 does not occur. Moreover in these cases the root δ in (4.5) equals $e_{x-1}+e_x$. Set $\gamma = e_{x+1}$ and let $\underline{w}' = (w_1,\ldots,w_{t-1},w_t-1)$ and $\lambda' = \lambda(\underline{w}')$.

Let x' correspond to x when λ is replaced by λ'. If $w_t \geq 2$ then $x = x'$ and $d(\lambda') = d(\lambda) = 0$. Then the pairs λ, γ and $\lambda', -\gamma$ both satisfy the hypotheses of (4.5)(ii). Here the level of reduction is one and the δ are equal for both λ and λ'. If $w_t = 1$ then $x' = x + 1$ and so $d(\lambda') = d(\lambda) + 2 = 2$ and λ' does not have level of reduction one. We conclude that for all w_t, $L(\lambda')$ occurs in $E \otimes L(\lambda)$ and $L(\lambda)$ occurs in $E \otimes L(\lambda')$.

Next suppose λ has level of reduction $l \geq 2$. Then $d(\lambda) \geq 1$. Let $\underline{w}'$ and γ be as above. Then $x' = x$ or $x + 1$ and so $d(\lambda') \geq d(\lambda) \neq 0$. From this we conclude that $L(\lambda'+\rho)$ has level of reduction not equal to one. Thus by Lemma 4.5(i) we obtain the same inclusions as above: $L(\lambda')$ occurs in $E \otimes L(\lambda)$ and $L(\lambda)$ occurs in $E \otimes L(\lambda')$. By the induction hypothesis the lemma holds for λ'. Then the two inclusions in the tensor products imply that the Gelfand-Kirillov dimension of $L(\lambda)$ equals the Gelfand-Kirillov dimension of $L(\lambda')$. This implies they all have the same Gelfand-Kirillov dimension and completes the proof for the $\mathrm{so}^*(2n)$ case.

(4.7) *The* $\mathrm{sp}(n)$ *case.* Suppose that $L = L(\lambda + \rho)$ is a highest weight representation occurring in the dual pair setting (4.2.1) for $\mathrm{sp}(n)$. For some t-tuple $\underline{\mu} = (\mu_1, \ldots, \mu_t)$ with weakly decreasing coordinates, $t = \min\{k, n\}$, $s = |\{i|\mu_i > 0\}|$ and $j = |\{i|\mu_i > 1\}|$, we have: $s + j \leq k$ and

$$\lambda = \left(-\frac{k}{2}, -\frac{k}{2}, \ldots, -\frac{k}{2}, -\frac{k}{2} - \mu_t, \ldots, -\frac{k}{2} - \mu_1\right) . \tag{4.7.1}$$

Organizing into segments, we have:
(4.7.2)

$$\lambda = \Big(\underbrace{-\frac{k}{2}, \ldots, -\frac{k}{2}}_{x}, \underbrace{-\frac{k}{2} - 1, \ldots, -\frac{k}{2} - 1}_{y}, \underbrace{-\frac{k}{2} - 1 - b_j, \ldots, -\frac{k}{2} - 1 - b_1}_{j}\Big) ,$$

with $b_1 \geq b_2 \geq \cdots \geq b_j > 0$, $x + y + j = n$, $x \geq n - k + j$, $y \leq k - 2j$. We say that λ is of the first type if $y = 0$ and otherwise of the second type. To express the dependence on $\underline{\mu}$ we write $\lambda(\underline{\mu})$ in place of λ when necessary.

Now suppose that L has level of reduction one. From [EHW] we obtain the form: for some integers $1 \leq q \leq r \leq n$,

$$\lambda + \rho = \left(\frac{q+r}{2}, \frac{q+r}{2} - 1, \ldots, \widehat{\frac{-q+r}{2}}, \ldots, \frac{q-r}{2}, \widehat{\frac{q-r}{2}} - 1 \cdots\right), \tag{4.7.3}$$

with $x = q$ and $x + y = r$. Here the superscript $\widehat{\ }$ designates omission of that term in the segment. Recall from (2.2.2) the line $\mathbb{L}(\lambda)$ and the parametrization $\lambda = \lambda_0 + z\zeta$ for some real number z. Set $d(\lambda) = B - z$ with B as in (2.2.2). Then as in (4.6.7),

$$\lambda + \rho + d(\lambda)\zeta = \left(\frac{q+r}{2}, \cdots\right) . \tag{4.7.4}$$

Now, solving for $d(\lambda)$ we have, for all λ,

$$d(\lambda) = \frac{1}{2}(k - 2n + 2x + y). \tag{4.7.5}$$

From (4.7.3) and [EHW], we obtain:

LEMMA. *Suppose that L has level of reduction one and choose $\delta \in \Delta_n$ so that $\lambda - \delta$ is the highest weight of the* PRV *component in $\mathfrak{p}^- \otimes F_\lambda$. Then $\delta = e_q + e_r$. In both cases the nonzero coordinates of δ are disjoint from the coordinates of λ where the $b_i, 1 \le i \le j$, occur.*

(4.8) Suppose λ and μ are given as in (4.7.1). Assume for some integer p that $\mu_p > \mu_{p+1}$ if $1 \le p < t$ or $\mu_p > 0$ if $p = t$. Set $\underline{\mu}' = (\mu_1, \dots, \mu_p - 1, \dots, \mu_t)$. Let $\lambda = \lambda(\underline{\mu})$ and $\lambda' = \lambda(\underline{\mu}')$.

LEMMA. *Assume $t \neq n$. Let x, y, j be the indices given in* (4.7.2) *for λ and let x', y', j' be the indices given in* (4.7.2) *for λ'. Then $x' \ge x$, $x' + y' \ge x + y$ and $d(\lambda) \le d(\lambda')$. Moreover if both λ and λ' are reduction points then the level of reduction of $L(\lambda')$ minus the level of reduction of $L(\lambda)$ equals $2(x'-x)+y'-y$. In particular the levels of reduction at λ and λ' either stay the same or increase depending as the indices x and $x + y$ either stay the same or increase.*

Proof. The inequalities on x and $x+y$ are clear. These inequalities and the formula (4.7.5) imply the inequality for $d(\lambda)$. Let l denote the level of reduction for L. The last reduction point on the line (2.2.2) has level of reduction one and the level increases by one for each unitarizable representation until we reach the maximum (for that line) at the first reduction point. So when λ is a reduction point then $l = 2d(\lambda) + 1$ which implies the result.

(4.9) LEMMA. *The Gelfand-Kirillov dimension of L equals $\frac{k}{2}(2n - k + 1)$ for $1 \le k \le n$ and equals $\binom{n+1}{2}$ otherwise.*

Proof. We proceed by induction on $|\underline{\mu}|$ as in the proof of (4.6). First suppose that λ is of the first type. If $\underline{\mu}$ is zero then λ is on the line containing the trivial representation and for $1 \le k \le n$, L is realized on the coordinate ring of the variety of symmetric $n \times n$ matrices of rank less than or equal to k. The dimension of this variety is $\frac{k}{2}(2n - k + 1)$. For $k \ge n$, λ is not a reduction point and the Gelfand-Kirillov dimension of L equals $\binom{n+1}{2}$. This proves the formula in this case.

Now suppose λ is of the first type and $\underline{\mu} \neq 0$. Choose the maximal index p with $\mu_p \neq 0$ and let γ be the weight of E whose coordinate expression is all zeros except $+1$ as the $n + 1 - p^{\text{th}}$ coordinate. Let $\underline{\mu}' = (\mu_1, \dots, \mu_p - 1, \dots, \mu_r)$. Now suppose that L has level of reduction one. Then with notation as in Lemma 4.7, $\delta = 2e_x$ and $x < n + 1 - p$. From Lemmas 4.5(ii) and 4.7 we conclude that $L(\lambda')$ occurs in $E \otimes L(\lambda)$. Using Lemma 4.8 we find that if the indices x', y', j' for λ' are not equal to x, y, j, then the level of reduction for L' is greater than one. So if λ' has level of reduction one, then x, y, j equals x', y', j' and with the argument as above $L(\lambda)$ occurs in $E \otimes L(\lambda')$. On the other hand if these indices are not equal then the level of reduction of λ' is greater than one and so by (4.5)(i) we get the same inclusion: $L(\lambda)$ occurs in $E \otimes L(\lambda')$. In turn this implies they have the same degree. By the induction hypothesis the lemma holds for $\lambda' = \lambda(\underline{\mu}')$ and this completes the proof for λ of type one.

Next suppose that λ is of the second type. By essentially the same technique as above we prove that the degree is independent of the b_i chosen in (4.7.2). Suppose the b_i are not all 0 and choose the maximal index p with $b_p \neq 0$ and let γ be the weight of E whose coordinate expression is all zeros except $+1$ as the $n+1-p^{\text{th}}$ coordinate. Let $\underline{\mu}' = (\mu_1, \ldots, \mu_p - 1, \ldots, \mu_r)$. By the induction hypothesis the lemma holds for $\lambda' = \lambda(\underline{\mu}')$. As above Lemmas 4.5, 4.7 and 4.8 complete the argument proving $L(\lambda')$ occurs in $E \otimes L(\lambda)$ and $L(\lambda)$ occurs in $E \otimes L(\lambda')$. This proves the independence of the b_i.

To complete the proof we determine the degree formula when λ is of the second type and $b_i = 0$, $1 \leq i \leq j$. In this case λ has indices x, y, j with $j = 0$ and $y = n - x$.

$$\lambda = \Big(\underbrace{-\frac{k}{2}, \ldots, -\frac{k}{2}}_{x}, \underbrace{-\frac{k}{2} - 1, \ldots, -\frac{k}{2} - 1}_{n-x}\Big) . \tag{4.9.1}$$

Set $p = x + 1$ and let γ be the weight of E whose coordinate expression is all zeros except $+1$ as the p^{th} coordinate. Let $\underline{\mu}' = (\mu_1, \ldots, \mu_p - 1, \ldots, \mu_r)$. Suppose L has level of reduction one. Then from (4.7.3), we have:

$$\lambda + \rho = \Big(\underbrace{\frac{x+n}{2}, \frac{x+n}{2} - 1, \cdots,}_{x} \frac{\widehat{-x+n}}{2}, \ldots, \frac{x-n}{2}\Big) , \tag{4.9.2}$$

where as before superscript $\widehat{\ }$ designates omission of the term. From Lemma 4.7, $\delta = e_x + e_n$, $\gamma = e_p$ and so $\delta \neq -\gamma + \gamma'$ and thus by Lemma 4.5(ii), $L(\lambda')$ occurs in $E \otimes L(\lambda)$. By Lemma 4.8, $d(\lambda')$ is greater than zero and so $L(\lambda')$ does not have level of reduction one. So $L(\lambda)$ occurs in $E \otimes L(\lambda')$. By applying this shift λ to λ' successively $n - x$ times we obtain the parameter:

$$\lambda'' = \left(-\frac{k}{2}, -\frac{k}{2}, \ldots, -\frac{k}{2}\right) . \tag{4.9.3}$$

Each shift of the type λ to λ' increases the value of the function $d()$ by one. So we continue to get both inclusions in tensor products and thus the Gelfand-Kirillov dimension of L equals the Gelfand-Kirillov dimension of $L(\lambda'')$. The case of λ'' was handled above. The proof is complete for level of reduction one. If the level of reduction of L is not one then $d(\lambda') \geq d(\lambda) \geq \frac{1}{2}$ and the argument above applies with (4.5)(i) replacing (4.5)(ii). This proves (4.9).

(4.10) *The* u(p,q) *case.* Suppose that $L = L(\lambda+\rho)$ occurs in the dual pair $\mathrm{U}(k) \times \mathrm{u}(p,q)$ setting (4.2.1) acting on polynomials in nk variables with $n = p+q$. Choose r, s with $0 \le r \le p-1$, $0 \le s \le q-1$ and let $\underline{w} = (w_1, \dots, w_r)$ and $\underline{u} = (u_1, \dots, u_s)$ be weakly decreasing sequences of nonnegative integers with $w_r \neq 0$ and $u_s \neq 0$. Then in Euclidean coordinates, for some integers $l \ge 0$ and $m \ge 0$,

(4.10.1)

$$\lambda = (-k-l, \dots, -k-l, -k-l-w_r, \dots, -k-l-w_1; u_1+m, \dots, u_s+m, m, \dots, m)\,,$$

where k satisfies one of the four inequalities: $k \ge r+s$ if $l = m = 0$; $k \ge p+s$ if $l \neq 0$ and $m = 0$; $k \ge r+q$ if $l = 0$ and $m \neq 0$; and finally $k \ge p+q$ if $l \neq 0$ and $m \neq 0$. In (4.10.1) the semicolon designates the separation of the n-tuple into a p-tuple and q-tuple. We write $\lambda(\underline{w}, \underline{u})$ in place of λ if we need to emphasize the dependence on $\underline{w}$ and $\underline{u}$. Organizing into segments, we have:

$$\begin{aligned}(4.10.2)\qquad \lambda = (&\underbrace{-k-l, \dots, -k-l}_{x}, -k-l-w_r, \dots \\ &\dots, -k-l-w_1; u_1+m, \dots, u_s+m, \underbrace{m, \dots, m}_{y}).\end{aligned}$$

Now suppose in addition that L has level of reduction one. Then from [EHW] with $c = -\frac{n+1}{2} + y + m$,

(4.10.3)

$$\lambda + \rho = (\underbrace{x+c, \dots, 1+c}_{x}, \widehat{c}, \cdots ; \cdots, \widehat{1+c}, \underbrace{c, -1+c, \dots, -y+1+c}_{y})\ .$$

Comparing these last two formula, we see that if λ has level of reduction one then $k = n - l - m - x - y$.

Recall from (2.2.2) the line $\mathbb{L}(\lambda)$ and the parametrization $\lambda = \lambda_0 + z\zeta$ for some real number z. As was done above in the other two cases set $d(\lambda) = B - z$ with B as in (2.2.2). From (4.10.3) we obtain:

$$(4.10.4)\qquad \lambda + \rho + d(\lambda)\zeta = (x+c, \dots, -y+1+c)\ .$$

Then solving for $d(\lambda)$, for all λ,

$$(4.10.5)\qquad d(\lambda) = k + l + m - n + x + y.$$

Turning to the proof of Theorem 6 we begin by eliminating some of the easy cases. First note that the first reduction point on the line (2.2.2) occurs at $d(\lambda) = \min\{x,y\} - 1$. Suppose $m = 0$ and $l \ge 1$. Then $k \ge p+s = p+q-y$ and $d(\lambda) = k + l - n + x + y \ge l + x > x \ge \min\{x,y\}$. Suppose $m \ge 1$ and $l = 0$. Then $k \ge r+q = p+q-x$ and $d(\lambda) = k + m - n + x + y \ge m + y > y \ge \min\{x,y\}$. Finally suppose $m \ge 1$ and $l \ge 1$. Then $k \ge p+q = n$ and $d(\lambda) = k + l + m - n + x + y \ge l + m + x + y > \min\{x,y\}$. Therefore in all three cases $N(\lambda+\rho)$ is irreducible. So if λ is a reduction point then we may assume that both m and l are zero.

Lemma. *Suppose L is unitarizable with level of reduction one. Choose $\delta \in \Delta_n$ so that $\lambda - \delta$ is the highest weight of the* PRV *component in $\mathfrak{p}^- \otimes F_\lambda$. Then $\delta = e_x - e_{n+1-y}$. In particular the coordinates of δ are disjoint from the coordinates where the w_i and u_i occur.*

Proof. The properties of δ follow from (4.10.3).

(4.11) Choose integers $a, 1 \leq a \leq r$, and $b, 1 \leq b \leq s$, and assume either $w_a > w_{a+1}$ if $1 \leq a < r$; or $w_a > 0$ if $a = r$; and either $u_b > u_{b+1}$ if $1 \leq b < s$; or $u_b > 0$ if $b = s$. Set $\underline{w}' = (w_1, \ldots, w_a - 1, \ldots, w_r)$, $\underline{u}' = (u_1, \ldots, u_b - 1, \ldots, u_s)$, $\lambda' = \lambda(\underline{w}', \underline{u})$ and $\lambda'' = \lambda(\underline{w}, \underline{u}')$. Let x', y' and δ' (resp. x'', y'' and δ'') denote the indices x, y in (4.10.2) and the weight δ in (4.10) obtained when λ is replaced by λ' (resp. λ'').

Lemma. *With notation as above, $x' \geq x = x''$, $y'' \geq y = y'$. If λ does not have level of reduction one then both λ' and λ'' also do not have level of reduction one. If both λ and λ' (resp. λ'') have level of reduction one then $x = x'$ and $\delta = \delta'$ (resp. $y = y''$ and $\delta = \delta''$) .*

Proof. The inequalities on x and y are clear. Since λ has level of reduction one if and only if $d(\lambda) = 0$, the rest of the lemma follows from the formula for $d(*)$ in (4.10.5).

Let E_1 denote the first fundamental representation of $\mathrm{su}(p,q)$. Set $e = -\frac{1}{n}$ and $f = \frac{n-1}{n}$ and, for $1 \leq i \leq n$, set $\underline{f}_i$ equal to the n-tuple with the i^{th} coordinate f and all the others equal to e. The $\underline{f}_i$ are the weights of E_1. Let E denote the tensor product of E_1 with the central character $\frac{1}{n}(1, 1, \cdots, 1)$. The weights of E are the n-tuples e_i with 1 in the i^{th} coordinate and zeros elsewhere.

(4.12) Lemma. *The Gelfand-Kirillov dimension of $L(\lambda+\rho)$ equals $k(n-k)$ for all $1 \leq k \leq \min\{p,q\}$ and equals pq otherwise.*

Proof. First consider the three cases considered in (4.10) where l and m are not both zero. Then $k \geq \min\{p,q\}$, $N(\lambda + \rho)$ is irreducible and the Gelfand-Kirillov dimension is pq. So the lemma holds in these cases.

Now suppose $l = m = 0$. Suppose that $\underline{w}$ and $\underline{u}$ are not zero and let λ' and λ'' be defined as above. By (4.10) and (4.5), $L(\lambda' + \rho)$ occurs in $E \otimes L(\lambda + \rho)$ and $L(\lambda'' + \rho)$ occurs in $E^* \otimes L(\lambda + \rho)$. Now suppose $L(\lambda + \rho)$ has level of reduction one. Then by Lemma 4.11, if $L(\lambda' + \rho)$ also has level of reduction one, $\delta = \delta' = \delta'' = e_x - e_{n+1-y}$. If we set $\gamma = e_a$ or $-e_b$ and then apply (4.5)(ii), $L(\lambda+\rho)$ occurs in both $E^* \otimes L(\lambda'+\rho)$ and $E \otimes L(\lambda''+\rho)$. If $L(\lambda+\rho)$ does not have level of reduction one then we apply (4.5)(i) and obtain the same inclusions. Therefore the Gelfand-Kirillov dimension is independent of both $\underline{w}$ and $\underline{u}$.

Now suppose $\underline{w} = \underline{u} = 0$. Then the parameter λ lies on the line with the trivial representation and the Wallach representations. We know $N(\lambda+\rho)$ is irreducible unless $1 \leq k < \min(p,q)$ and these representations lie in the Wallach set. They are isomorphic to the coordinate ring of the variety of all $p \times q$ matrices having rank less than or equal to k. Now $\mathrm{Gl}(p) \times \mathrm{Gl}(q)$ acts on this space and a calculation of the stabilizer gives the dimension equal to $k(n-k)$. This completes the proof.

This completes the proof of Theorem 6 in all three cases.

(4.13) Corollary. *With notation now as in Theorem* 6*, suppose* $L(\lambda+\rho)$ *is an irreducible highest weight module occurring in one of the dual pair settings. Let* $\mathcal{J}$ *denote the annihilator of* L *in* $\mathrm{U}(\mathfrak{g})$*. Then in the three cases* $\mathrm{so}^*(2n), \mathrm{sp}(n)$ *and* $\mathrm{u}(p,q)$*; the Gelfand-Kirilov dimension of* $\mathrm{U}(\mathfrak{g})/\mathcal{J}$ *is,*

$$2k(2n-2k-1) \text{ (resp. } k(2n-k+1) \text{ , } 2k(n-k) \text{),}$$

where the restrictions on k *are as in* (4.2).

Proof. From Borho-Kraft [BK], the Gelfand-Kirillov dimension of $\mathrm{U}(\mathfrak{g})/\mathcal{J}$ is twice the Gelfand-Kirillov dimension of L.

5. Hilbert series of unitarizable and finite dimensional representations

The generalized BGG resolution is given as a sum indexed by the coset space $W^{\mathfrak{k}}$. These spaces have interesting combinatorial descriptions which will offer some detailed presentations of Hilbert series. They will also give the most direct route to the proof of a theorem which relates Hilbert series for unitary highest weight representations and those for finite dimensional representations. We describe the results separately in the three cases: $\mathrm{so}^*(2n)$, $\mathrm{sp}(n,\mathbb{R})$ and $\mathrm{u}(p,q)$.

(5.1) *The* $\mathrm{so}^*(2n)$ *case.* Let $L(\lambda+\rho)$ be any unitarizable highest weight representation and let $\lambda+\rho = (a_1, \ldots, a_n)$ be its expression in Euclidean coordinates as in (4.6). We partition these coordinates as follows. Define Θ^s to be the set of all positive coordinates a for which $-a$ is also a coordinate of $\lambda+\rho$. Let Θ be the set of all nonnegative coordinates c not in Θ^s. Let Ψ be a complementing set defined by:

(5.1.1) $$\lambda+\rho = (\Theta, \Theta^s, -\Theta^s, \Psi)^+ \ .$$

Choose indices $i_1 < \cdots < i_m$ so that

(5.1.2) $$\Theta = \{(\lambda+\rho)_{i_1}, \ldots, (\lambda+\rho)_{i_m}\} \ .$$

LEMMA. *Suppose that* λ *is a reduction point. Then* Θ *contains two or more coordinates. If* $|\Theta| = 2$, *then* $\mathfrak{g}_\lambda \cong \mathrm{sl}(2,\mathbb{R})$. *If* $m = |\Theta| \geq 3$ *then* $\Delta_\lambda^+ = \{e_{i_j} \pm e_{i_k} | 1 \leq j < k \leq m\}$ *and is of type* D_m.

Proof. From (4.6.5) and (4.6.6), $\lambda + \rho = (b, \ldots, c, \cdots)$ where the initial segment from b to c is a set of consecutive decreasing integers which includes 0. From this we see that Θ contains 0 and all coordinates in Ψ are negative with absolute value greater than all the coordinates in Θ. Since λ is a reduction point, $|\Theta| \geq 2$. Then $\mathcal{W}_\lambda$ is generated by the reflections s_α for roots $\alpha = e_{i_s} + e_{i_t}$, $1 \leq s < t \leq m$. From this, $\mathfrak{g}_\lambda$ is contained in $\mathrm{so}(2m)$. We now check the opposite inclusion. Set $\alpha_s = e_{i_s} + e_{i_{s+1}}, 1 \leq s < m$, and $\alpha_m = s_{\alpha_2}(e_{i_1} + e_{i_3}) = e_{i_1} - e_{i_2}$. Then the set of roots

$$\{\alpha_1, -\alpha_2, \alpha_3, \ldots, (-1)^m \alpha_{m-1}, -\alpha_m\}$$

is contained in Δ_λ and is a set of simple roots for the root system D_m. With the earlier inclusion, $\mathfrak{g}_\lambda$ is of type D_m.

The terms in the resolution of $L(\lambda + \rho)$ in Theorem 2 correspond to all subsets of Θ of even cardinality as follows. For any subset $\Phi \subset \Theta$ of even cardinality, define $\Phi^\vee$ by the identity $\Theta = \Phi \cup \Phi^\vee$ and set:

$$\lambda_\Phi + \rho = (\Phi^\vee, \Theta^s, -\Theta^s, -\Phi, \Psi)^+ \ . \tag{5.1.3}$$

Recalling (4.1.1), we have equality of sets:

$$\{\lambda_\Phi\} = \{s \circledast \lambda | s \in \mathcal{W}_\lambda^{\mathfrak{k}}\} \ , \tag{5.1.4}$$

where Φ is any subset of Θ of even cardinality. If $\lambda_\Phi = s \circledast \lambda$ then define ε_Φ to be 0 or 1 depending on whether the parity of the length of s in $\mathcal{W}_\lambda$ is even or odd. So ε_Φ has the same parity as the cardinality of $s\Delta_\lambda^+ \cap -\Delta_\lambda^+$.

(5.2) *The* $\mathrm{sp}(n)$ *case.* Let $L(\lambda + \rho)$ be any unitarizable highest weight representation and let $\lambda + \rho = (a_1, \ldots, a_n)$ be its expression in Euclidean coordinates. We partition these coordinates as follows. Define Θ^s to be the set of all positive coordinates a for which $-a$ is also a coordinate of $\lambda + \rho$. Let Θ be the set of all coordinates c not in Θ^s with $c > 0$ and all pairs of coordinates c, d both not in Θ^s, both nonzero and with $c + d > 0$. Let Ψ be a complementing set defined by:

$$\lambda + \rho = \begin{cases} (\Theta, \Theta^s, 0, -\Theta^s, \Psi)^+, & \text{if zero is a coordinate of } \lambda + \rho \\ (\Theta, \Theta^s, -\Theta^s, \Psi)^+, & \text{if zero is not a coordinate of } \lambda + \rho. \end{cases} \tag{5.2.1}$$

Choose indices $i_1 < \cdots < i_m$ so that

$$\Theta = \{(\lambda + \rho)_{i_1}, \ldots, (\lambda + \rho)_{i_m}\} \ . \tag{5.2.2}$$

Lemma. *Suppose λ is a reduction point. Then from* (4.7.3):

$$\lambda + \rho + d(\lambda)\zeta = \left(\frac{q+r}{2}, \frac{q+r}{2} - 1, \ldots, \widehat{\frac{-q+r}{2}}, \ldots, \frac{q-r}{2}, \widehat{\frac{q-r}{2} - 1} \cdots \right).$$

i. *Suppose $q = r$. If λ has level of reduction one, then $\Delta_\lambda^+ = \{2e_{i_1}\}$ and is of type A_1 for $m = 1$ while*

$$\Delta_\lambda^+ = \{e_{i_j} \pm e_{i_k} | 1 \leq j < k \leq m\} \cup \{2e_{i_j} | 1 \leq j \leq m\}$$

and is of type C_m for larger m. If λ has level of reduction greater than one, then $\Delta_\lambda^+ = \{e_{i_1} + e_{i_2}\}$ and is of type A_1 for $m = 2$ while $\Delta_\lambda^+ = \{e_{i_j} \pm e_{i_k} | 1 \leq j < k \leq m\}$ and is of type D_m for $m > 2$.

ii. *Suppose $r > q$ and let l denote the level of reduction. If $l = r - q + 1$ then $\Delta_\lambda^+ = \{2e_{i_1}\}$ and is of type A_1 for $m = 1$ while*

$$\Delta_\lambda^+ = \{e_{i_j} \pm e_{i_k} | 1 \leq j < k \leq m\} \cup \{2e_{i_j} | 1 \leq j \leq m\}$$

and is of type C_m for larger m. If $l \neq r - q + 1$ then $\Delta_\lambda^+ = \{e_{i_1} + e_{i_2}\}$ and is of type A_1 for $m = 2$ while $\Delta_\lambda^+ = \{e_{i_j} \pm e_{i_k} | 1 \leq j < k \leq m\}$ and is of type D_m for $m > 2$.

Proof. First suppose $q = r$. From (4.7.3) if λ has level of reduction one, $\lambda + \rho$ takes the form $(a, a-1, \cdots 1, \widehat{0}, \cdots)$. So Ψ is a set of negative integers with absolute value greater than a. It follows that $\Delta_\lambda^+ = \{2e_{i_1}\}$ and is of type A_1 for $m = 1$ while $\Delta_\lambda^+ = \{e_{i_j} \pm e_{i_k} | 1 \leq j < k \leq m\} \cup \{2e_{i_j} | 1 \leq j \leq m\}$ and is of type C_m for larger m. Again since λ is a reduction point, $m \geq 1$.

If the level of reduction is greater than one, then $\lambda + \rho$ will begin with either the integer a and include all the decreasing consecutive integers to zero or it will begin with the odd half integer a and include all the decreasing consecutive odd half integers to $\frac{1}{2}$. Again this implies that Ψ is a set of negative integers(or negative odd half integers) with absolute value greater than a. In either case $\mathcal{W}_\lambda$ will only contain permutations with an even number of sign changes. So Δ_λ will only contain short roots. With the argument exactly as in the so$^*(2n)$ case, $\Delta_\lambda^+ = \{e_{i_1} + e_{i_2}\}$ and is of type A_1 for $m = 2$ while $\Delta_\lambda^+ = \{e_{i_j} \pm e_{i_k} | 1 \leq j < k \leq m\}$ and is of type D_m for $m > 2$. Here, since λ is a reduction point, $m \geq 2$.

Now suppose $q < r$. From (4.7.3), $\lambda + \rho$ has the form:

$$(a, a-1, \ldots, \widehat{e}, \ldots, -b, \widehat{-b-1}, \cdots), \tag{5.2.3}$$

for some integers(or odd half integers) a, b and e with $a > b > 0$, $e \neq -b$ and $|e| \leq b$; by which we mean that the initial segment of $\lambda + \rho$ is a decreasing string of consecutive integers (or consecutive odd half integers) from a to $-b$ with one omission, the value e. From this form, all the elements of Ψ are negative and have absolute value greater than a. Let l denote the level of reduction of λ.

Then from (4.7.3), $e = \frac{1}{2}(r-q-l+1)$. Now consider cases depending on the value of e. If $e=0$, then Θ is a set of positive integers and $\Delta_\lambda^+ = \{2e_{i_1}\}$ and is of type A_1 for $m=1$ while $\Delta_\lambda^+ = \{e_{i_j} \pm e_{i_k} | 1 \le j < k \le m\} \cup \{2e_{i_j} | 1 \le j \le m\}$ and is of type C_m for larger m. If $e<0$, then either 0 is a coordinate of $\lambda+\rho$ or the coordinates are odd half integers. In either case $\mathcal{W}_\lambda$ is a set of permutations with even numbers of sign changes. So Δ_λ is contained in the short roots. Then as in the $\mathrm{so}^*(2n)$ case, $\Delta_\lambda^+ = \{e_{i_1} + e_{i_2}\}$ and is of type A_1 for $m=2$ while $\Delta_\lambda^+ = \{e_{i_j} \pm e_{i_k} | 1 \le j < k \le m\}$ and is of type D_m for $m>2$. Finally suppose $e>0$. Then either 0 is a coordinate of $\lambda+\rho$ or the coordinates are odd half integers. In either case $\mathcal{W}_\lambda$ is a set of permutations with even numbers of sign changes. From the form (5.2.3), the last coordinate in Θ, $(\lambda+\rho)_{i_m}$ equals $-e$ and we have:

$$(\lambda+\rho)_{i_1} > (\lambda+\rho)_{i_2} > \cdots > (\lambda+\rho)_{i_{m-1}} > |(\lambda+\rho)_{i_m}| = e\ . \tag{5.2.4}$$

So in this case as well, $\Delta_\lambda^+ = \{e_{i_1} + e_{i_2}\}$ and is of type A_1 for $m=2$ while $\Delta_\lambda^+ = \{e_{i_j} \pm e_{i_k} | 1 \le j < k \le m\}$ and is of type D_m for $m>2$.

In (5.2.1) the first formula gives an integral point because of the zero coordinate. In the second formula we have an integral subcase when the coordinates are integers and a half integral subcase when the coordinates are odd half integers. We refer to these as the first, second and third cases. Now, by the lemma the terms in the resolution of $L(\lambda+\rho)$ correspond to all subsets of Θ in the second case and all subsets of even cardinality in the first and third cases. Set $\Theta = \Phi \cup \Phi^\vee$ and

(5.2.5)
$$\lambda_\Phi + \rho = \begin{cases} (\Phi^\vee, \Theta^s, 0, -\Theta^s, -\Phi, \Psi)^+, & \text{if zero is a coordinate of } \lambda+\rho \\ (\Phi^\vee, \Theta^s, -\Theta^s, -\Phi, \Psi)^+, & \text{if zero is not a coordinate of } \lambda+\rho\ . \end{cases}$$

As in the previous example, we have equality of sets:

$$\{\lambda_\Phi\} = \{s \circledast \lambda | s \in \mathcal{W}_\lambda^{\mathfrak{k}}\}\ , \tag{5.2.6}$$

where Φ is any subset of Θ in the second case and any subset of even cardinality in the first and third cases. If $\lambda_\Phi = s \circledast \lambda$ then define ε_Φ to be 0 or 1 depending on whether the parity of the length of s in $\mathcal{W}_\lambda$ is even or odd. So in all cases ε_Φ has the same parity as the cardinality of $s\Delta_\lambda^+ \cap -\Delta_\lambda^+$. Note that when 0 occurs as a coordinate (the first case), 0 is not in Θ. However in the $\mathrm{so}^*(2n)$ case $0 \in \Theta$. So the descriptions vary accordingly in the $\mathrm{so}^*(2n)$ and the first case here.

(5.3) *The* $\mathrm{u}(p,q)$ *case.* Let $n = p+q$ and let $\lambda+\rho = (a_1, \dots, a_p; b_1, \dots, b_q) = (\underline{a}; \underline{b})$ be the highest weight plus ρ of a unitarizable highest weight representation of $\mathrm{u}(p,q)$. Let Θ^s be the intersection of the coordinates of $\underline{a}$ and coordinates of $\underline{b}$. Now let Θ_1 be the coordinates a_i which are not in Θ^s and for which $a_i > b_j$ for some b_j not in Θ^s. Similarly let Θ_2 be the coordinates b_j which are not in Θ^s and for which $a_i > b_j$ for some a_i not in Θ^s. Finally let Ψ_i , $i = 1, 2$ be the complementing coordinates for which

$$\lambda + \rho = ((\Theta_1, \Theta^s, \Psi_1)^+;\ (\Psi_2, \Theta^s, \Theta_2)^+)\ . \tag{5.3.1}$$

Choose indices $i_1 < \cdots < i_{p'}$ and $j_1 < \cdots < j_{q'}$ so that

$$\Theta_1 = \{(\lambda+\rho)_{i_1}, \ldots, (\lambda+\rho)_{i_{p'}}\}, \qquad \Theta_2 = \{(\lambda+\rho)_{j_1}, \ldots, (\lambda+\rho)_{j_{q'}}\}\ . \tag{5.3.2}$$

From (4.10.3), for some half integers c and d with $d \geq c$,
(5.3.3)

$$\lambda + \rho = (\underbrace{x + c, \ldots, 1 + c}_{x}, \hat{c}, \cdots ; \cdots, 1 \overset{\wedge}{+} d, \underbrace{d, -1 + d, \ldots, -y + 1 + d}_{y})\ ,$$

where $l = d - c + 1$ equals the level of reduction of λ.

Lemma. *Suppose λ is a reduction point. Then $1 \leq i_1 < \cdots < i_{p'} \leq x < n + 1 - y \leq j_1 < \cdots < j_{q'} \leq n$. Moreover*

$$\Delta_\lambda^+ = \{e_j - e_k | 1 \leq j < k \leq n \text{ and } j, k \in \{i_1 \cdots i_{p'}\} \cup \{j_1 \cdots j_{q'}\}\}$$

which is of type $A_{p'+q'-1}$ and $[\mathfrak{g}_\lambda, \mathfrak{g}_\lambda] \cong \mathrm{su}(p', q')$.

Proof. The form (5.3.3) implies that $d \geq c$ and so any element of Ψ_1 is less than $d - y + 1$ and any element of Ψ_2 is greater than $x + c$. Then for $a \in \Psi_1$, $a < d - y + 1 \leq d - l + 1 = c$. Similarly, for $b \in \Psi_2$, $b > x + c \geq l + c = d + 1$.

Suppose $i \in \Theta_1$ and $i > x$. Then for some j, $q + 1 \leq j \leq n$, $(\lambda+\rho)_i > (\lambda+\rho)_j$. So $(\lambda+\rho)_j < (\lambda+\rho)_i < c \leq d$. Since the last y coordinates of $\lambda+\rho$ are a consecutive set of integers or odd half integers, $(\lambda+\rho)_i$ occurs twice as a coordinate of $\lambda+\rho$. This implies $(\lambda+\rho)_i \in \Theta^s$ which is a contradiction and proves $i \leq x$. Similarly suppose $j \in \Theta_2$ and $j < n + 1 - y$. Then for some i, $1 \leq i \leq p$, $(\lambda+\rho)_i > (\lambda+\rho)_j$. So $(\lambda+\rho)_i > (\lambda+\rho)_j \geq d + 2 \geq c + 2$. Since the first x coordinates of $\lambda+\rho$ are a consecutive set of integers or odd half integers, $(\lambda+\rho)_j$ occurs twice as a coordinate of $\lambda+\rho$. This implies $(\lambda+\rho)_j \in \Theta^s$ which is a contradiction and proves $j \geq n + 1 - y$. It follows that $1 \leq i_1 < \cdots < i_{p'} \leq x < n + 1 - y \leq j_1 < \cdots < j_{q'} \leq n$. Then $\Delta_\lambda^+ = \{e_j - e_k | 1 \leq j < k \leq n \text{ and } j, k \in \{i_1 \cdots i_{p'}\} \cup \{j_1 \cdots j_{q'}\}\}$ which is of type $A_{p'+q'-1}$ and $[\mathfrak{g}_\lambda, \mathfrak{g}_\lambda] \cong \mathrm{su}(p', q')$.

The terms in the resolution of $L(\lambda+\rho)$ are parametrized by all pairs of subsets $\Phi = (\Phi_1, \Phi_2)$ with $\Phi_i \subset \Theta_i$, $i = 1, 2$ and card Φ_1 = card Φ_2 as follows. Define $\Phi_i^\vee$ and λ_Φ by the identities:

$$\Theta_i = \Phi_i \cup \Phi_i^\vee \text{ and } \quad \lambda_\Phi + \rho = ((\Phi_2, \Phi_1^\vee, \Theta^s, \Psi_1)^+;\ (\Psi_2, \Theta^s, \Phi_1, \Phi_2^\vee)^+)\ . \tag{5.3.4}$$

As in the previous examples, we have equality of sets:

$$\{\lambda_\Phi\} = \{s \circledast \lambda | s \in \mathcal{W}_\lambda^{\mathfrak{k}}\} \ , \tag{5.3.5}$$

where $\Phi = (\Phi_1, \Phi_2)$ with $\Phi_i \subset \Theta_i$, $i = 1, 2$ and card $\Phi_1 =$ card Φ_2. If $\lambda_\Phi = s \circledast \lambda$ then define ε_Φ to be 0 or 1 depending as the parity of the length of s in $\mathcal{W}_\lambda$ is even or odd. So in all cases ε_Φ has the same parity as the cardinality of $s\Delta_\lambda^+ \cap -\Delta_\lambda^+$.

(5.4) Let F_Φ denote the finite dimensional $\mathfrak{k}$-module with highest weight λ_Φ. Set $D_m = \prod_{1\le j\le m-1} j!$. For any r-tuple $\underline{a}$, let $\prod \underline{a} = \prod_{1\le i<j\le r}(a_i - a_j)$.

LEMMA. (i) *Suppose* $\mathfrak{k} \cong \mathrm{u}(n)$ *and* $\lambda_\Phi + \rho = (a_1, \dots , a_n)$. *Then*

$$\dim F_\Phi = \frac{1}{D_n} \prod_{1\le i<j\le n} (a_i - a_j) \ .$$

(ii) *Suppose* $\mathfrak{k} \cong \mathrm{u}(p) \times \mathrm{u}(q)$ *and* $\lambda_\Phi + \rho = (a_1, \dots , a_p; b_1, \dots , b_q)$. *Then*

$$\dim F_\Phi = \frac{1}{D_p D_q} \prod_{1\le i<j\le p} (a_i - a_j) \prod_{1\le i<j\le q} (b_i - b_j) \ .$$

(iii) *Suppose* (5.1.3) *holds and* $\Psi = \emptyset$. *Then*

$$\dim F_\Phi = \dim F_\emptyset \frac{\prod(-\Phi, \Phi^\vee)^+}{\prod(\Phi, \Phi^\vee)^+} \ .$$

(iv) *Suppose* (5.2.5) *holds and* $\Psi = \emptyset$. *Then*

$$\dim F_\Phi = \dim F_\emptyset \frac{\prod(-\Phi, \Phi^\vee)^+}{\prod(\Phi, \Phi^\vee)^+} \ .$$

(v) *Suppose* (5.3.4) *holds and* $\Psi_1 = \Psi_2 = \emptyset$. *Then*

$$\dim F_\Phi = \dim F_\emptyset \frac{\prod(\Phi_2, \Phi_1^\vee)^+ \ \prod(\Phi_1, \Phi_2^\vee)^+}{\prod(\Phi_1, \Phi_1^\vee)^+ \ \prod(\Phi_2, \Phi_2^\vee)^+} \ .$$

Proof. The formulas in (i) and (ii) are coordinate versions of the Weyl character formula. To verify (iii) , note that (i) gives:

$$\frac{\dim F_\Phi}{\dim F_\emptyset} = \frac{\prod(-\Phi, \Phi^\vee, \Theta^s, -\Theta^s)^+}{\prod(\Phi, \Phi^\vee, \Theta^s, -\Theta^s)^+} \ .$$

Since the set $\Theta^s \cup -\Theta^s$ is stable under multiplication by -1, all terms in this ratio cancel except those in

$$\frac{\prod(-\Phi, \Phi^\vee)^+}{\prod(\Phi, \Phi^\vee)^+} \ .$$

This gives (iii). The verification of (iv) is similar. To verify (v), we begin with (ii). Then

$$\frac{\dim F_\Phi}{\dim F_\emptyset} = \frac{\prod(\Phi_2, \Phi_1^\vee, \Theta^s)^+ \ \prod(\Phi_1, \Phi_2^\vee, \Theta^s)^+}{\prod(\Phi_1, \Phi_1^\vee, \Theta^s)^+ \ \prod(\Phi_2, \Phi_2^\vee, \Theta^s)^+} .$$

Cancelling factors of the form $a - b$ with $b \in \Theta^s$ gives (v).

(5.5) For any weight μ which is dominant for $\mathfrak{k}_\lambda$, let E_μ denote the finite dimensional $\mathfrak{k}_\lambda$-module with highest weight μ. For each Φ as above, let E_Φ denote the finite dimensional irreducible $\mathfrak{k}_\lambda$-module with highest weight λ_Φ.

Corollary. *Suppose* $\Psi = \Psi_1 = \Psi_2 = \emptyset$. *Then in all cases*:

$$\dim F_\Phi = \frac{\dim F_\emptyset}{\dim E_\emptyset} \ \dim E_\Phi .$$

Proof. First suppose we are in the cases so$^*(2n)$ and sp(n), and $\Delta^+_{\mathfrak{k},\lambda} = \{e_{i_j} - e_{i_k} | 1 \le j < k \le m\}$ where $\Theta = \{(\lambda+\rho)_{i_1}, \ldots, (\lambda+\rho)_{i_m}\}$. From Lemma 5.4 (i), $\dim E_\Phi = \frac{1}{D_c} \prod(-\Phi, \Phi^\vee)^+$ where c equals the cardinality of Θ. Combining this with (iii) and (iv) gives the result. By Lemmas 5.1 and 5.2 in all the remaining cases $\mathfrak{k}_\lambda = \mathfrak{h}$, $\mathcal{W}^{\mathfrak{k}}_\lambda$ contains only two elements and the corresponding weights in the $\mathcal{W}^{\mathfrak{k}}_\lambda$ orbit are $\{(\pm(\lambda+\rho))^+\}$. The two representations of $\mathfrak{k}$ with these highest weights are dual to each other and so have the same dimension. So the formula holds in these degenerate cases as well.

Now consider the u(p, q) case. Here by Lemma 5.3,

$$\Delta^+_{\mathfrak{k},\lambda} = \{e_{i_j} - e_{i_k} | 1 \le j < k \le p'\} \cup \{e_{j_s} - e_{j_t} | 1 \le s < t \le q'\} .$$

Then $\dim E_\Phi = \frac{1}{D_{p'} \, D_{q'}} \prod(\Phi_2, \Phi_1^\vee)^+ \ \prod(\Phi_1, \Phi_2^\vee)^+$. Therefore identity (v) in Lemma 5.4 implies the formula.

(5.6) For any $\mathfrak{g}_\lambda$-dominant integral μ we let B_μ denote the irreducible finite dimensional $\mathfrak{g}_\lambda$-module with highest weight μ. Let B^i_μ denote the grading of B_μ as a $\mathfrak{g}_\lambda \cap \mathfrak{p}^-$-module. Define the Hilbert series by:

(5.6.1) $$P(q) = P_\lambda(q) = \sum \dim B^i_\lambda \ q^i .$$

Theorem. *Set* $e = \dim \mathfrak{p}^+$ *and* $e' = \dim \mathfrak{p}^+ \cap \mathfrak{g}_\lambda$. *Suppose that* $L = L(\lambda+\rho)$ *is unitarizable and* λ *is a quasi-dominant reduction point and that in the coordinate form either* (5.1.3), (5.2.5) *or* (5.3.4) *holds. Then* $\Psi = \Psi_1 = \Psi_2 = \emptyset$. *Set* d *equal to the Gelfand-Kirillov dimension of* L *given by Theorem* 6. *Then* $d = e - e'$ *and the Hilbert series in reduced form for* L *is*:

(5.6.2) $$\mathrm{H}_L(q) = \frac{\dim F_\emptyset}{\dim E_\emptyset} \ \frac{P(q)}{(1-q)^d} .$$

Proof. In this setting (4.1.3) becomes:

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^e} \sum_{\Phi} (-1)^{\varepsilon_\Phi} \dim F_\Phi \quad q^{\langle\Phi\rangle} \ , \tag{5.6.3}$$

where $\langle\Phi\rangle = (\lambda - \lambda_\Phi)(z_0)$. By (5.5), this becomes:

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^e} \frac{\dim F_\emptyset}{\dim E_\emptyset} \sum_{\Phi} (-1)^{\varepsilon_\Phi} \dim E_\Phi \quad q^{\langle\Phi\rangle} \ . \tag{5.6.4}$$

Now using the generalized BGG resolution [L] for finite dimensional representations, this becomes:

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^{e-e'}} \frac{\dim F_\emptyset}{\dim E_\emptyset} \ P(q). \tag{5.6.5}$$

Since the polynomial $P(q)$ is finite nonzero at $q = 1$, the exponent $e - e'$ must equal the Gelfand-Kirillov dimension d. Alternatively we could verify this formula directly from the expressions in Theorem 6. This completes the proof.

COROLLARY. *The Bernstein degree of* $L(\lambda+\rho)$ *is* $\frac{\dim F_\emptyset}{\dim E_\emptyset}\dim B_\lambda$.

Proof. This follows from (5.6.2) and the identity: $P(1) = \dim B_\lambda$.

6. Examples of Hilbert series

(6.1) Suppose $L = L(\lambda+\rho)$ is a unitarizable highest weight representation occurring in a dual pair setting. We say L is minimal if its Gelfand-Kirillov dimension is positive and minimal in the set of Gelfand-Kirillov dimensions of unitarizable highest weight representations occurring in the dual pair settings. For the cases $\mathrm{so}^*(2n), \mathrm{sp}(n)$ and $\mathrm{u}(p,q)$ we now describe quite explicitly the Hilbert series of all minimal L with singular $\lambda+\rho$. These results are shown to be a consequence of Theorem 5.6. A calculation using Theorem 6 shows that the minimal L are those with $\lambda+\rho$ given by (4.6.2), (4.7.2) and (4.10.3) with $k = 1$.

(6.2) *The* $\mathrm{sp}(n)$ *case.* This is the easiest of the three cases. Here for $k = 1$ we obtain only two highest weights, the two components of the Weil representation. The two parameters are: $\lambda+\rho = (n-1/2, \dots, 1/2)$ and $\lambda+\rho = (n-1/2, \dots, 3/2, -1/2)$. Then the B^λ are the representations of type D_n with highest weight $(1/2, \dots, 1/2)$ and $(1/2, \dots, 1/2, -1/2)$ which are the two spin representations each of dimension 2^{n-1}. The weight spaces are all one dimensional with weights having coordinates of $\pm 1/2$ where the number of negatives is even (resp. odd). A quick calculation shows that the Hilbert series for the spin representations are:

$$P(q) = \sum_{0\le j\le \frac{n}{2}} \binom{n}{2j} q^j \tag{6.2.1}$$

and

$$P(q) = \sum_{0\le j\le \frac{n}{2}} \binom{n}{2j+1} q^j \ . \tag{6.2.2}$$

Then by (5.6), the Hilbert series for the minimal representations of sp(n) are:

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^n} \sum_{0\le j\le \frac{n}{2}} \binom{n}{2j} q^j \tag{6.2.3}$$

and

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^n} \sum_{0\le j\le \frac{n}{2}} \binom{n}{2j+1} q^j \ . \tag{6.2.4}$$

The Bernstein degree of these two representations is 2^{n-1}.

(6.3) *The* so$^*(2n)$ *case.* From (4.6.2) the $\lambda+\rho$ for minimal L in this case are:

$$\lambda+\rho = (n-2, n-3, \dots, 0, -1-w), \tag{6.3.1}$$

where w is a nonnegative integer. If $\lambda+\rho$ is singular then $0 \le w \le n-3$. Then $\mathfrak{g}_\lambda$ is of type D_{n-2} and B^λ is isomorphic to $\wedge^{n-3-w}E$ tensored with a central character of $\mathfrak{g}_\lambda$ where E is the first fundamental representation of $\mathfrak{g}_\lambda$ of dimension $2n-4$. Here the zero exponent denotes the trivial representation. As a $\mathfrak{k}_\lambda$ module $E = E_+ \oplus E_-$ where E_+ (resp. E_-) has all weights with 1 (resp. -1) as one coordinate and all others zero. Then

$$\wedge^{n-3-w}E \cong \sum_{0\le j\le n-3-w} \wedge^{n-3-w-j}E_+ \otimes \wedge^j E_-. \tag{6.3.2}$$

From this isomorphism the Hilbert series of the finite dimensional $\mathfrak{g}_\lambda$ module B_λ is:

$$P(q) = \sum_{0\le j\le n-3-w} \binom{n-2}{n-3-w-j}\binom{n-2}{j} q^j. \tag{6.3.3}$$

If $\lambda+\rho$ is singular and given by (6.3.1) then its Hilbert series is:
(6.3.4)

$$\mathrm{H}_L(q) = \frac{1}{(1-q)^{2n-3}} \frac{\binom{n-1+w}{n-1}}{\binom{n-2}{n-3-w}} \sum_{0\le j\le n-3-w} \binom{n-2}{n-3-w-j}\binom{n-2}{j} q^j.$$

The Bernstein degree of this representation is:

$$\frac{\binom{n-1+w}{n-1}}{\binom{n-2}{n-3-w}}\binom{2n-4}{n-3-w}\ . \tag{6.3.5}$$

(6.4) *The* u(p,q) *case.* Suppose $\mathfrak{g}$ is of type u(p,q), $k=1$ and $\lambda+\rho$ is singular. If either p or q equals one then $N(\lambda+\rho)$ is irreducible and thus $H(q)=\frac{1}{(1-q)^{pq}}$. So we assume $p,q\geq 2$. From (4.10) the possibilities are:

$$\lambda+\rho=(n-1,n-2,\dots,q-w;q,q-1,\dots,1) \tag{6.4.1a}$$

or

$$\lambda+\rho=(n-1,n-2,\dots,q;u+q,q-1,\dots,1)\ . \tag{6.4.1b}$$

where $0\leq w\leq q-1$, $0\leq u\leq p-1$ and for convenience we introduce a shift to ρ and write $\rho=(n,n-1,\dots,1)$. For all of these values of w and u, $\mathfrak{g}_\lambda\cong$ u$(p-1,q-1)$. The representation B_λ equals a one dimensional representation for the extremes $w=0,q-1$ and $u=0,p-1$ and otherwise B_λ is the fundamental representation with highest weight ω_{p-1+w} or ω_{p-1-u} plus a central character of $\mathfrak{g}_\lambda$. Let E denote the first fundamental representation of u$(p-1,q-1)$ and write $E=E_{p-1}\oplus E_{q-1}$ with the components the $\mathfrak{k}$ submodules of dimension $p-1$ and $q-1$ respectively.

The decompositions we want are:

$$\bigwedge^{p-1+w}\mathbb{C}^{n-2}\cong\sum_{0\leq j\leq p-1}\bigwedge^{p-1-j}\mathbb{C}^{p-1}\otimes\bigwedge^{w+j}\mathbb{C}^{q-1}\quad\text{and} \tag{6.4.2a}$$

$$\bigwedge^{p-1-u}\mathbb{C}^{n-2}\cong\sum_{0\leq j\leq q-1}\bigwedge^{p-1-u-j}\mathbb{C}^{p-1}\otimes\bigwedge^{j}\mathbb{C}^{q-1} \tag{6.4.2b}$$

This leads to the Hilbert series for B_λ;
(6.4.3)

$$P(q)=\sum_{0\leq j\leq p-1}\binom{p-1}{j}\binom{q-1}{w+j}q^j\ \text{and}\ P(q)=\sum_{0\leq j\leq q-1}\binom{p-1}{u+j}\binom{q-1}{j}q^j\ ,$$

which leads to the Hilbert series for $L=L(\lambda+\rho)$;

$$\mathrm{H}_L(q)=\frac{1}{(1-q)^{n-1}}\frac{\binom{p-1+w}{p-1}}{\binom{q-1}{w}}\sum_{0\leq j\leq p-1}\binom{p-1}{j}\binom{q-1}{w+j}q^j \tag{6.4.4a}$$

and

$$\mathrm{H}_L(q)=\frac{1}{(1-q)^{n-1}}\frac{\binom{q-1+u}{q-1}}{\binom{p-1}{u}}\sum_{0\leq j\leq q-1}\binom{p-1}{u+j}\binom{q-1}{j}q^j\ . \tag{6.4.4b}$$

The Bernstein degrees of these representations are respectively:

$$\frac{\binom{p-1+w}{p-1}}{\binom{q-1}{w}}\binom{n-2}{p-1+w} \quad \text{and} \quad \frac{\binom{q-1+u}{q-1}}{\binom{p-1}{u}}\binom{n-2}{p-1-u}\,. \tag{6.4.5}$$

This completes the singular cases of minimal Gelfand-Kirillov dimension.

7. Branching rules

(7.1) By a partition λ we mean a finite sequence of weakly decreasing positive integers, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. The number of terms in the sequence λ will be called the length of λ and be denoted $\ell(\lambda)$. Partitions will always be denoted by lower case Greek letters. Let $|\lambda| = \sum_i \lambda_i$ denote the size of λ. Given a partition λ we denote the conjugate partition to λ by λ'. That is, the partition obtained by flipping the Young diagram of λ over the main diagonal. Equivalently, $(\lambda')_i = |\{j : \lambda_j \geq i\}|$. Note that $|\lambda| = |\lambda'|$ and $\ell(\lambda) = (\lambda')_1$. Let P denote the set of partitions. Define

$$\begin{aligned} P_R &= \{\lambda \in P : \lambda_i \in 2\mathbb{N} \text{ for all } i\}\,, \\ P_C &= \left\{\lambda \in P : (\lambda')_i \in 2\mathbb{N} \text{ for all } i\right\}. \end{aligned} \tag{7.1.1}$$

The set P_R (resp. P_C) consists of partitions whose Young diagrams have even rows (resp. columns).

For each partition λ such that $\ell(\lambda) \leq m$ let $F^\lambda_{(m)}$ denote the irreducible (finite dimensional) representation of $\mathrm{GL}(m)$ with highest weight $\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_m\epsilon_m$. Similarly, let $V^\lambda_{(m)}$ be the irreducible $\mathrm{Sp}(m)$ representation indexed by λ ; and, for λ with $(\lambda')_1 + (\lambda')_2 \leq m$ let $E^\lambda_{(m)}$ be the irreducible $\mathrm{O}(m)$ representation indexed by λ.

(7.2) Given nonnegative integer partitions, μ, σ and ν, each with at most m parts, define the classical Littlewood-Richardson coefficients $c^\sigma_{\mu\nu}$ by,

$$c^\sigma_{\mu\nu} = \dim \mathrm{Hom}_{\mathrm{GL}(m)}\left(F^\sigma_{(m)}, F^\mu_{(m)} \otimes F^\nu_{(m)}\right). \tag{7.2.1}$$

Given σ, μ, and ν such that $\ell(\sigma)$, $\ell(\mu)$, $\ell(\nu) \leq n_0$, for all $n \geq n_0$,

$$\dim \mathrm{Hom}_{\mathrm{GL}(n)}(F^\sigma_{(n)}, F^\mu_{(n)} \otimes F^\nu_{(n)}) = \dim \mathrm{Hom}_{\mathrm{GL}(n_0)}(F^\sigma_{(n_0)}, F^\mu_{(n_0)} \otimes F^\nu_{(n_0)}).$$

In this sense $c^\sigma_{\mu\nu}$ is independent of n.

For partitions σ and μ, we define the following sums of Littlewood-Richardson coefficients,

$$C^\sigma_\mu := \textstyle\sum_{\nu\in P_R} c^\sigma_{\nu\mu} \quad \text{and} \quad D^\sigma_\mu := \sum_{\nu\in P_C} c^\sigma_{\nu\mu}. \tag{7.2.2}$$

THEOREM (Littlewood Restriction Formula (LRF), [Lit1]). (i) (LRF *for* $O(k) \subseteq GL(k)$). *Set* $r = \lfloor \frac{k}{2} \rfloor$. *Let* σ *and* μ *be partitions with at most* r *parts. Then*,

$$C_\mu^\sigma = \dim \mathrm{Hom}_{O(k)}(E_{(k)}^\mu, F_{(k)}^\sigma). \tag{7.2.3}$$

(ii) (LRF *for* $\mathrm{Sp}(k) \subseteq GL(2k)$). *For partitions* σ *and* μ, *with at most* k *parts*,

$$D_\mu^\sigma = \dim \mathrm{Hom}_{\mathrm{Sp}(k)}(V_{(k)}^\mu, F_{(2k)}^\sigma). \tag{7.2.4}$$

The above theorem does not answer the branching question in general because the length of σ is restricted. In the following, we provide a description for general σ.

(7.3) Theorem 4 provides a resolution where each term is of the Littlewood form. If n is large, ($n \geq k$ for $O(k)$ and $n \geq 2k$ for $\mathrm{Sp}(k)$) then σ indexes a general irreducible $GL(n)$ representation. In particular, the above is a general solution to the branching problem in the sense of Littlewood if we take $n = k$ (resp. $n = 2k$) for $O(n)$ (resp. $\mathrm{Sp}(k)$).

LEMMA. *Under the Littlewood hypotheses*, *the general branching formulas* (1.2.9) *and* (1.2.10) *imply the formulas of Littlewood*, (7.2.3) *and* (7.2.4).

Proof. In the first case we suppose that σ and μ have at most $[\frac{k}{2}]$ parts. Set $n = [\frac{k}{2}]$. Then from (4.7.1) we observe that $\mu^\sharp + \rho$ has all nonpositive coordinates and thus the generalized Verma module $N(\mu^\sharp + \rho)$ is irreducible. So the sum in formula (1.2.9) degenerates to one term, giving the Littlewood formula.

In the second case we suppose that σ and μ have at most k parts. Set $n = k$. Then from (4.6.2) we observe that $\mu^\sharp + \rho$ has all nonpositive coordinates and thus the generalized Verma module $N(\mu^\sharp + \rho)$ is irreducible. Now, the sum in formula (1.2.10) degenerates to one term, again giving the Littlewood formula. In both cases, $\mathcal{W}_{\mu^\sharp}^{\mathfrak{k}}$ is the trivial group.

(7.4) The proof of Theorem 4 will depend on several applications of Howe duality. Next we review the three cases needed. Let $M = M_{m \times n}$ denote the $m \times n$ matrices and let $G = \mathrm{Gl}(m) \times \mathrm{Gl}(n)$. Let θ denote the automorphism of $\mathrm{Gl}(m)$ given by the composite of inverse and transpose.Then there are two natural group actions on M. Let ϕ denote the first and τ the second. For $a \in M, (g, h) \in G$,

$$\phi(g,h)a = gah^{-1} \quad \tau(g,h)a = \theta(g)ah^{-1} \ .$$

These actions also induce representations on $\mathcal{P}(M_{m\times n})$ which we also denote by ϕ and τ respectively. For $f \in \mathcal{P}(M_{m\times n})$,

$$\begin{aligned}\phi(g,h)f(a) &= f(\phi(g^{-1},h^{-1})a) = f(g^{-1}ah),\\ \tau(g,h)f(a) &= f(\tau(g^{-1},h^{-1})a) = f(g^T ah).\end{aligned} \tag{7.4.1}$$

Let D_m and D_n be the invertible diagonal matrices in $\mathrm{Gl}(m)$ and $\mathrm{Gl}(n)$. Let N_m and N_n be the upper triangular unipotent matrices in $\mathrm{Gl}(m)$ and $\mathrm{Gl}(n)$ and let $\overline{N}_m$ and $\overline{N}_n$ be the lower triangular unipotent matrices. A set of generators for the polynomial algebra $\mathcal{P}(M_{m\times n})$ is the set of $x_{i,j}, 1 \le i \le m$, $1 \le j \le n$, where $x_{i,j}$ is the functional with $x_{i,j}(a)$ equaling the i,j^{th} entry of a. For $a \in D_m$ and $b \in D_n$ let $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_n)$ denote the diagonal entries. Then the weight structure is determined by:

$$\phi(a,b)x_{i,j}^s = a_i^{-s} b_j^s x_{i,j}^s, \quad \tau(a,b)x_{i,j}^s = a_i^s b_j^s x_{i,j}^s\ . \tag{7.4.2}$$

Theorem (Howe duality $\mathrm{GL}(m,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$). *The two actions of* $\mathrm{GL}(m) \times \mathrm{GL}(n)$ *lead to two multiplicity-free decompositions*: *Under the action of* ϕ,

$$\mathcal{P}(M_{m\times n}) = \bigoplus_\sigma (F^\sigma_{(m)})^* \otimes F^\sigma_{(n)}, \tag{7.4.3}$$

and under the action τ,

$$\mathcal{P}(M_{m\times n}) = \bigoplus_\sigma F^\sigma_{(m)} \otimes F^\sigma_{(n)}, \tag{7.4.4}$$

where both sums are over all partitions σ *with at most* $\min(m,n)$ *parts.*

The form 7.4.3 of duality can be found in [GW, Th. 5.2.7] and 7.4.4 is easily obtained from it.

In order to compare these actions with those of other Howe dual pairs, we now express ϕ and τ in terms of Euler type operators. Choose the standard basis of $\mathrm{gl}(n)$ of elementary matrices $\{e_{i,j}\}_{1\le i,j\le n}$ where $e_{i,j}$ has 1 in the ij^{th} entry and zeros elsewhere. Then

$$\begin{aligned}(\phi(0,e_{i,j})x_{a,b})(m) &= x_{a,b}(me_{i,j}) = \delta_{b,j}\ x_{a,i}(m)\ ,\\ (\phi(e_{i,j},0)x_{a,b})(m) &= x_{a,b}(-e_{i,j}m) = -\delta_{a,i}\ x_{j,b}(m).\end{aligned} \tag{7.4.5}$$

Define operators on $\mathcal{P}(M_{m\times n})$ for $1 \le i,j \le n, 1 \le s,t \le m$,

$$E_{i,j} = \sum_{1\le p\le m} x_{p,i} \frac{\partial}{\partial x_{p,j}}\ , \quad E'_{s,t} = \sum_{1\le q\le n} x_{s,q} \frac{\partial}{\partial x_{t,q}}\ . \tag{7.4.6}$$

These definitions and the identities above give:

$$\phi(0,e_{i,j}) = \tau(0,e_{i,j}) = E_{i,j}\ , \quad \phi(e_{i,j},0) = -E'_{j,i}, \quad \tau(e_{i,j},0) = E'_{i,j}\ . \tag{7.4.7}$$

Let $\mathfrak{n}$ denote the upper triangular matrices in $\mathrm{gl}(m) \times \mathrm{gl}(n)$ and for $0 \leq i \leq \min\{m,n\}$, let

$$\Delta_i = \det \begin{pmatrix} x_{11} & \cdots & x_{1i} \\ \vdots & & \vdots \\ x_{i1} & \cdots & x_{ii} \end{pmatrix}$$

be the i^{th} principal minor. These polynomials generate the algebra of $\mathfrak{n}$ highest weight vectors under the action of τ and establish the isomorphism (7.4.4).

(7.5) Next we consider the dual pair $\mathrm{O}(k) \times \mathrm{sp}(n)$ following the notation in [GW, §4.5.5]. Let π_1 denote the action of this pair on $M_{k\times n}$. Then the action of the left factor $\mathrm{O}(k)$ is the restriction of the representation ϕ from $\mathrm{Gl}(k)$ to $\mathrm{O}(k)$. The action of the right factor $\mathrm{sp}(n)$ is given by certain polynomial differential operators. The compactly embedded subalgebra $\mathfrak{k}$ is isomorphic to $\mathrm{gl}(n)$ and this right action is given by:

$$\pi_1(0, e_{i,j}) = E_{i,j} + \frac{k}{2}\delta_{ij} \ . \tag{7.5.1}$$

Now consider the dual pair $\mathrm{Sp}(k) \times \mathrm{so}^*(2n)$ again following the notation in [GW, §4.5.5]. Let π_2 denote the action of this pair on $M_{2k\times n}$. Then the action of the left factor $\mathrm{Sp}(k)$ is the restriction of the representation ϕ from $\mathrm{Gl}(2k)$ to $\mathrm{Sp}(k)$. The action of the right factor $\mathrm{so}^*(2n)$ is given by certain polynomial differential operators. The compactly embedded subalgebra $\mathfrak{k}$ is isomorphic to $\mathrm{gl}(n)$ and this right action is given by:

$$\pi_2(0, e_{i,j}) = E_{i,j} + k\delta_{ij} \ . \tag{7.5.2}$$

In both of these cases the positive noncompact root spaces are identified with certain upper triangular matrices which act as multiplication operators on polynomials. So in order to obtain a decomposition in terms of highest rather than lowest weight representations we compose the action of the right factor with the involution $\theta(x) = -X^T$. To conveniently compare the decompositions with those in Theorem 7.4 we also compose the action of the left factor with $\Theta(g) = (g^{-1})^T$. Define

$$\overline{\pi}_1(g, X) = \pi_1(\Theta(g), \theta(X)) \ , \quad \overline{\pi}_2(g, X) = \pi_2(\Theta(g), \theta(X)) \ . \tag{7.5.3}$$

The dual pair decompositions now take the form:

(7.6) Theorem (Howe Duality $\mathrm{O}(k,\mathbb{C}) \times \mathrm{sp}(n)$, [H]). *Under the representation* $\overline{\pi}_1$ of $\mathrm{O}(k) \times \mathrm{sp}(n)$ *the space* $\mathcal{P}(M_{k\times n})$ *has a multiplicity free decomposition*:

$$\mathcal{P}(M_{k\times n}) = \bigoplus_{\mu} E^{\mu}_{(k)} \otimes E_{\mu}, \tag{7.6.1}$$

where the above direct sum is over all partitions μ *with at most* $r = \min(k,n)$ *parts and subject to the conditions that* $(\mu')_1 + (\mu')_2 \leq k$. *Furthermore,* E_μ *is the irreducible highest weight representation of* $\mathrm{sp}(n)$ *with highest weight* $\mu^\sharp$.

(7.7) THEOREM (Howe duality $\mathrm{Sp}(k,\mathbb{C}) \times \mathrm{so}^*(2n)$, [H]). *Under the representation* $\overline{\pi}_2$ *of* $\mathrm{Sp}(k) \times \mathrm{so}^*(2n)$ *the space* $\mathcal{P}(M_{2k\times n})$ *has a multiplicity free decomposition*:

$$\mathcal{P}(M_{2k\times n}) = \bigoplus_{\mu} V^{\mu}_{(k)} \otimes V_{\mu} \tag{7.7.1}$$

where the above summation is over all partitions μ *with at most* $\min(k,n)$ *parts. Furthermore*, V_μ *is the irreducible highest weight representation of* $\mathrm{so}^*(2n)$ *with highest weight* $\mu^\sharp$.

(7.8) We now study the multiplicities of various isotypic subspaces of these polynomial spaces.

LEMMA. *Suppose* W *is a subspace of* $\mathcal{P}(M_{k\times n})$ *(resp.* $\mathcal{P}(M_{2k\times n})$*. If* W *is stable under the action of* $\overline{\pi}_1$ *restricted to* $\mathrm{O}(k) \times \mathrm{gl}(n)$ (resp. $\overline{\pi}_2$ *restricted to* $\mathrm{Sp}(k) \times \mathrm{gl}(n)$) *then it is also stable under the restriction of* τ *(or* ϕ*) and vice versa. If* W *is irreducible under one action then it is irreducible under the other. Finally if* W *is irreducible under the restricted* τ *action of type* $\mu \otimes \nu$, *then under the restricted action* $\overline{\pi}_1$ *(resp.* $\overline{\pi}_2$*),* W *has type* $\mu \otimes \nu^\sharp$.

Proof. The actions of the left factor are all equal and the actions of the right are given by $e_{i,j} \to E_{i,j}$, $e_{i,j} \to -E_{j,i} - k\delta_{i,j}$ and $e_{i,j} \to -E_{j,i} - \frac{k}{2}\delta_{i,j}$.

COROLLARY. *Suppose* W *is the isotypic subspace of* $\mathcal{P}(M_{k\times n})$ *(resp.* $\mathcal{P}(M_{2k\times n})$ *for type* $\mu \otimes \nu$ *for the action* τ *restricted to* $\mathrm{O}(k) \times \mathrm{gl}(n)$ *(resp.* $\mathrm{Sp}(k) \times \mathrm{gl}(n)$*). Then under the action of* $\overline{\pi}_1$ *restricted to* $\mathrm{O}(k) \times \mathrm{gl}(n)$ *(resp.* $\overline{\pi}_2$ *restricted to* $\mathrm{Sp}(k) \times \mathrm{gl}(n)$*),* W *is the isotypic subspace of type* $\mu \otimes \nu^\sharp$.

Finally the multiplicity of this isotypic space W *is equal to both the multiplicity of the* $\mathrm{O}(k)$ *(resp.* $\mathrm{Sp}(k)$*) representation* $E^{\nu}_{(k)}$ *(resp.* $V^{\nu}_{(k)}$*) in the* $\mathrm{Gl}(k)$ *(resp.* $\mathrm{Gl}(2k)$*) representation* $F^{\mu}_{(k)}$ *(resp.* $F^{\mu}_{(2k)}$*) as well as the multiplicity of the* $\mathrm{gl}(n)$ *representation* $F^{\mu^\sharp}_{(n)}$ *in the highest weight representation of* $\mathrm{sp}(n)$ *(resp.* $\mathrm{so}^*(2n)$*) with highest weight* $\nu^\sharp$.

Proof. The first assertions are clear. As for the multiplicities, the first comes from the decomposition in Theorem 7.4 while the second and third come from the decompositions Theorems 7.6 and 7.7.

(7.9) LEMMA. (i) *As a representation of* $\mathrm{GL}(n)$, *the space of symmetric* $n\times n$ *matrices* (*under the action* $(g,X) \mapsto gXg^T$) *is equivalent to the symmetric square of the standard representation, denoted* $S^2(\mathbb{C}^n)$. *Furthermore, if* $\mathfrak{g} \cong \mathrm{sp}(n)$,

$$S(\mathfrak{p}^+) \cong S\left(S^2(\mathbb{C}^n)\right) \cong \bigoplus F^{\sigma}_{(n)}, \tag{7.9.1}$$

where the sum is over $\sigma \in P_R$ *such that* $\ell(\sigma) \leq n$.

(ii) *As a representation of* $\mathrm{GL}(n)$, *the space of skew symmetric* $n \times n$ *matrices* (*under the action* $(g, X) \mapsto gXg^T$) *is equivalent to the exterior square of the standard representation*, $\wedge^2(\mathbb{C}^n)$. *Furthermore, if* $\mathfrak{g} \cong \mathrm{so}^*(2n)$,

$$S(\mathfrak{p}^+) \cong S\left(\wedge^2(\mathbb{C}^n)\right) \cong \bigoplus F^{\sigma}_{(n)}, \tag{7.9.2}$$

where the sum is over $\sigma \in P_C$ *such that* $\ell(\sigma) \leq n$.

Proof. See [GW, p. 257, §5.2.5] and [GW, p. 258, §5.2.6].

These identities lead to the $\mathfrak{k}$ multiplicity formulas for generalized Verma modules in the next corollary.

COROLLARY. (i) *In the* $(\mathrm{O}(k), \mathrm{sp}(n))$ *case*, $\mathfrak{g} \cong \mathrm{sp}(n)$ *and*

$$C^{\sigma}_{\mu} = \dim \mathrm{Hom}_{\mathfrak{k}}\left(F^{\sigma^\sharp}_{(n)},\ N(\mu^\sharp)\right). \tag{7.9.3}$$

(ii) *In the* $(\mathrm{Sp}(k), \mathrm{so}^*(2n))$ *case*, $\mathfrak{g} \cong \mathrm{so}^*(2n)$ *and*

$$D^{\sigma}_{\mu} = \dim \mathrm{Hom}_{\mathfrak{k}}\left(F^{\sigma^\sharp}_{(n)},\ N(\mu^\sharp)\right). \tag{7.9.4}$$

Proof. As a $\mathfrak{k}$ ($\cong \mathrm{gl}(n)$) representation,

$$\mathfrak{p}^- \cong \begin{cases} S^2(\mathbb{C}^n)^* & \text{for } \mathrm{sp}(n), \\ \wedge^2(\mathbb{C}^n)^* & \text{for } \mathrm{so}^*(2n), \end{cases} \qquad \mathfrak{p}^+ \cong \begin{cases} S^2(\mathbb{C}^n) & \text{for } \mathrm{sp}(n), \\ \wedge^2(\mathbb{C}^n) & \text{for } \mathrm{so}^*(2n). \end{cases} \tag{7.9.5}$$

By Lemma 7.9, $S(\mathfrak{p}^+)$ has a multiplicity-free decomposition involving partitions from either P_R or P_C depending on which case we are in ($\mathrm{sp}(n)$ or $\mathrm{so}^*(n)$). By definition of C^{σ}_{μ} (resp. D^{σ}_{μ}) we have,

$$S(\mathfrak{p}^+) \otimes F^{\mu}_{(n)} \cong \begin{cases} \bigoplus C^{\sigma}_{\mu}\ F^{\sigma}_{(n)} & \text{for } \mathrm{sp}(n), \\ \bigoplus D^{\sigma}_{\mu}\ F^{\sigma}_{(n)} & \text{for } \mathrm{so}^*(2n). \end{cases}$$

As a $\mathfrak{k}$ representation, for any partition ν,

$$F^{\nu^\sharp}_{(n)} \cong \begin{cases} \left(F^{\nu}_{(n)} \otimes F^{\frac{k}{2}\zeta}_{(n)}\right)^* & \text{for } \mathrm{sp}(n), \\ \left(F^{\nu}_{(n)} \otimes F^{k\zeta}_{(n)}\right)^* & \text{for } \mathrm{so}^*(2n). \end{cases}$$

For any half-integer m, $F^{m\zeta}_{(n)}$ is one dimensional and therefore for any highest weights ν_1 and ν_2,

$$\mathrm{Hom}_{\mathrm{gl}(n)}\left(F^{\nu_1}_{(n)} \otimes F^{m\zeta}_{(n)}, F^{\nu_2}_{(n)} \otimes F^{m\zeta}_{(n)}\right) \cong \mathrm{Hom}_{\mathrm{gl}(n)}\left(F^{\nu_1}_{(n)}, F^{\nu_2}_{(n)}\right).$$

And this implies,

$$\mathrm{Hom}_{\mathrm{gl}(n)}\left(F^{\sigma}_{(n)}, S(\mathfrak{p}^+) \otimes F^{\mu}_{n}\right) \cong \mathrm{Hom}_{\mathrm{gl}(n)}\left(F^{\sigma^\sharp}_{(n)}, S(\mathfrak{p}^-) \otimes F^{\mu^\sharp}_{(n)})\right) .$$

Since as a $\mathfrak{k}$ representation $N(\mu^\sharp) \cong S(\mathfrak{p}^-) \otimes F^{\mu^\sharp}_{(n)}$, this completes the proof.

(7.10) *Proof of Theorem* 4. Set $m = \dim \mathrm{Hom}_{\mathrm{O}(k)}(E^{\mu}_{(k)}, F^{\sigma}_{(k)})$. Then by Corollary 7.8, m equals the multiplicity under $\overline{\pi}_1$ of $F^{\sigma^\sharp}_{(n)}$ in the irreducible highest weight representation $L = L(\mu^\sharp + \rho)$ of $\mathrm{sp}(n)$. By Theorem 2, L has a resolution in terms of generalized Verma modules. So combining this resolution with the identities in Corollary 7.9, we obtain the formula (1.2.9). The proof of the other formula (1.2.10) is essentially the same when we replace the identities (7.9.3) by (7.9.4). This completes the proof of Theorem 4.

(7.11) Set $n = k$ and consider the dual pair $\mathrm{O}(k) \times \mathrm{sp}(n)$. For positive integers a, b and c, let μ be the partition:

$$\mu = \underbrace{(d, \overbrace{2, \cdots, 2}^{a}, \overbrace{1, \cdots, 1}^{b}, \overbrace{0, \cdots, 0}^{c})}_{k} \tag{7.11.1}$$

with: $d \geq 2$ and $(\mu^c)_1 + (\mu^c)_2 \leq k$ (i.e. $2 + 2a + b \leq k$). Similarly define μ' to be the k tuple:

$$\mu' = (d, \overbrace{2, \cdots, 2}^{c}, \overbrace{1, \cdots, 1}^{b}, \overbrace{0, \cdots, 0}^{a}). \tag{7.11.2}$$

A short calculation shows that $\mathfrak{g}_{\mu^\sharp} \cong \mathrm{sl}(2, \mathbb{R})$ and thus there are exactly two Littlewood coefficients in the sum in Theorem 4. The theorem gives: for all ν,

$$\dim \mathrm{Hom}_{\mathrm{O}(k)}(E^{\mu}, F^{\nu}) = C^{\nu}_{\mu} - C^{\nu}_{\mu'}.$$

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References

[A] J. Adams, Unitary highest weight modules, *Adv. in Math.* **63** (1987), 113–137.

[BGG] I. N. Bernštein, I. M. Gel′fand, and S. I. Gel′fand, Structure of representations that are generated by vectors of higher weight, *Funckcional Anal. i Priložen* **5** (1971), 1–9.

[BK] W. Borho and H. Kraft, Über die Gelfand-Kirillov-Dimension, *Math. Ann.* **220** (1976), 1–24.

[C] D. Collingwood, The $\mathfrak{n}$-cohomology of Harish-Chandra modules: generalizing a theorem of Kostant, *Math. Ann.* **272** (1985), 161–187.

[DES] M. G. Davidson, T. J. Enright, and R. J. Stanke, *Differential Operators and Highest Weight Representations*, *Mem. Amer. Math. Soc.* **94**, A. M. S., Providence, RI, 1991.

[E] T. J. Enright, Analogues of Kostant's $\mathfrak{u}$-cohomology formulas for unitary highest weight modules, *J. Reine Angew. Math.* **392** (1988), 27–36.

[EHW] T. J. Enright, R. Howe, and N. R. Wallach, A classification of unitary highest weight modules, *Representation Theory of Reductive Groups* (Park City, Utah, 1982) *Progr. Math.* **40**, 97–143, Birkhäuser Boston, 1983.

[EJ] T. J. Enright and A. Joseph, An intrinsic analysis of unitarizable highest weight modules, *Math. Ann.* **288** (1990), 571–594.

[ES] T. J. Enright and B. Shelton, *Categories of Highest Weight Modules: Applications to Classical Hermitian Symmetric Pairs*, *Mem. Amer. Math. Soc.* **67**, A. M. S., Providence, RI, 1987.

[ES2] ———, Highest weight modules for Hermitian symmetric pairs of exceptional type, *Proc. Amer. Math. Soc.* **106** (1989), 807–819.

[EW] T. J. Enright and J. F. Willenbring, Hilbert series, Howe duality and branching rules, *Proc. Natl. Acad. Sci. USA* **100** (2003), 434–437.

[G] F. Gavarini, A Brauer algebra-theoretic proof of Littlewood's restriction rules, *J. Algebra* **212** (1999), 240–271.

[GP] F. Gavarini and P. Papi, Representations of the Brauer algebra and Littlewood's restriction rules, *J. Algebra* **194** (1997), 275–298.

[GW] R. Goodman and N. Wallach, *Representations and Invariants of the Classical Groups*, *Encyc. Math. and its Appl.* **68**, 406–463, Cambridge Univ. Press, Cambridge, 1998.

[H] R. Howe, Remarks on classical invariant theory. *Trans. Amer. Math. Soc.* **313** (1989), 539–570.

[J] H. P. Jakobsen, Hermitian symmetric spaces and their unitary highest weight modules, *J. Funct. Anal.* **52** (1983), 385–412.

[KV] M. Kashiwara and M. Vergne, On the Segal-Shale-Weil representations and harmonic polynomials, *Invent. Math.* **44** (1978), 1–47.

[L] J. Lepowsky, A generalization of the Bernstein-Gelfand-Gelfand resolution, *J. Algebra* **49** (1977), 496–511.

[Lit1] D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, Oxford Univ. Press, New York, 1940.

[Lit2] ———, On invariant theory under restricted groups, *Philos. Trans. Roy. Soc. London Ser. A* **239** (1944), 387–417.

[N] M. J. Newell, Modification rules for the orthogonal and symplectic groups, *Proc. Roy. Irish Acad. Sect. A* **54** (1951), 153–163.

[NOTYK] K. Nishiyama, H. Ochiai, K. Taniguchi, H. Yamashita, and S. Kato, Nilpotent orbits, associated cycles and Whittaker models for highest weight representations, *Astérisque* **273** (2001), 1–163.

[S] S. Sundaram, Tableaux in the representation theory of the classical Lie Groups, *Invariant Theory and Tableaux* (Minneapolis, MN, 1988) *IMA Vol. Math. Appl.* **19**, 191–225, Springer-Verlag, New York, 1990.

[W] N. R. Wallach, The analytic continuation of the discrete series. I, II, *Trans. Amer. Math. Soc.* **251** (1979), 1–17, 19–37.

[V] D. A. Vogan, Gelfand-Kirillov dimension for Harish-Chandra modules, *Invent. Math.* **48** (1978), 75–98.

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