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Semistable sheaves in positive characteristic

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Abstract

We prove Maruyama's conjecture on the boundedness of slope semistable sheaves on a projective variety defined over a noetherian ring. Our approach also gives a new proof of the boundedness for varieties defined over a characteristic zero field. This result implies that in mixed characteristic the moduli spaces of Gieseker semistable sheaves are projective schemes of finite type. The proof uses a new inequality bounding slopes of the restriction of a sheaf to a hypersurface in terms of its slope and the discriminant. This inequality also leads to effective restriction theorems in all characteristics, improving earlier results in characteristic zero.

0. Introduction

Let k be an algebraically closed field of any characteristic. Let X be a smooth n-dimensional projective variety over k with a very ample divisor H. If E is a torsion-free sheaf on X then one can define its slope by setting

$$\mu(E) = \frac{c_1 E \cdot H^{n-1}}{\operatorname{rk} E},$$

where $\operatorname{rk} E$ is the rank of E. Then E is semistable if for any nonzero subsheaf $F \subset E$ we have $\mu(F) \le \mu(E)$.

Semistability was introduced for bundles on curves by Mumford, and later generalized by Takemoto, Gieseker, Maruyama and Simpson. This notion was used to construct the moduli spaces parametrizing sheaves with fixed topological data. As for the construction of these moduli spaces the boundedness of semistable sheaves is a fundamental problem equivalent for these moduli spaces to be of finite type over the base field (see [Ma2, Th. 7.5]).

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In the curve case the problem is easy. In higher dimensions this problem was successfully treated in characteristic zero using the Grauert-Mülich theorem with important contributions by Barth, Spindler, Maruyama, Forster, Hirschowitz and Schneider. In positive characteristic Maruyama proved the boundedness of semistable sheaves on surfaces and the boundedness of sheaves of rank at most 3 in any dimension.

In another direction Mehta and Ramanathan proved their restriction theorem saying that the restriction of a semistable sheaf to a general hypersurface of a sufficiently large degree is still semistable. This theorem is valid in any characteristic but the result does not give any information on the degree of this hypersurface. It was well known that an effective restriction theorem would prove the boundedness. In the characteristic zero case such a theorem was proved by Flenner. Ein and Noma tried to use a similar approach in positive characteristic but they succeeded only for rank 2 bundles on surfaces.

About the same time as people were studying the boundedness of semistable sheaves, Bogomolov proved his famous inequality saying that

$$\Delta(E) = 2\,\mathrm{rk}\,E\,c_2E - (\mathrm{rk}\,E - 1)c_1^2E$$

is nonnegative if E is a semistable bundle on a surface over a characteristic zero base field. This result can easily be generalized to higher dimensions by the Mumford-Mehta-Ramanathan restriction theorem. Bogomolov's inequality was generalized by Shepherd-Barron [SB1], Moriwaki [Mo] and Megyesi [Me] to positive characteristic but only in the surface case. The higher dimensional version of this inequality follows only from the boundedness of semistable sheaves (see [Mo], the proof of Theorem 1), which is what we want to prove.

In this paper we prove the boundedness of semistable sheaves and Bogomolov's inequality in positive characteristic. Moreover, we prove effective restriction theorems. Our methods also give new proofs of these results in characteristic zero.

Our approach to these problems is through a theorem combining the Grauert-Mülich type theorem and Bogomolov's inequality at the same time. To explain the basic idea let us state a special case of our Theorems 3.1 and 3.2. We say that E is *strongly semistable* if either $\mathrm{char}\,k = 0$ or $\mathrm{char}\,k > 0$ and all the Frobenius pull backs of E are semistable.

THEOREM 0.1. *Assume that $n \geq 2$. Let E be a strongly semistable torsion-free sheaf. Let μ_i (r_i) denote slopes (respectively: ranks) of the Harder-Narasimhan filtration of the restriction of E to a general divisor $D \in |H|$. Then*

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq H^n \cdot \Delta(E) H^{n-2}.$$

In particular, $\Delta(E)H^{n-2} \geq 0$.

Let us note that theorems of this type do not immediately give even the usual Mumford-Mehta-Ramanathan theorem. However, together with Kleiman's criterion, this theorem gives the boundedness of semistable sheaves on surfaces. Later we will prove a much stronger theorem (see Section 3) implying the boundedness of all semistable pure sheaves with bounded slopes and fixed Hilbert polynomial in all dimensions and in any characteristic (see Theorem 4.1). In fact, we prove a stronger statement of boundedness in mixed characteristic, which was conjectured by Maruyama (see [Ma1, Question 7.18], [Ma2, Conj. 2.11]). Then a standard technique (see [HL, Ch. 4]; see also [Ma3]) implies the following corollary.

THEOREM 0.2. *Let R be a universally Japanese ring. Let $f : X \to S$ be a projective morphism of R-schemes of finite type with geometrically connected fibers and let $\mathcal{O}_X(1)$ be an f-ample line bundle. Then for a fixed polynomial P there exists a projective S-scheme $M_{X/S}(P)$ of finite type over S, which uniformly corepresents the functor*

$$\mathcal{M}_{X/S}(P) : \{\text{schemes over } S\}^o \to \{\text{sets}\}$$

defined by

$$(\mathcal{M}_{X/S}(P))(T) = \left\{ \begin{array}{l} S\text{-equivalence classes of families of pure semistable} \\ \text{sheaves on the fibres of } T \times_S X \to T \text{ which are} \\ \text{flat over } T \text{ and have Hilbert polynomial } P \end{array} \right\}.$$

Moreover, there is an open scheme $M^s_{X/S}(P) \subset M_{X/S}(P)$ that universally corepresents the subfunctor of families of geometrically stable sheaves.

Universally Japanese rings are also called Nagata rings. In the above theorem semistability is defined by means of the Hilbert polynomial. Apart from that exception semistability in this paper is always defined using the slope.

Let us also remark that quotients of semistable points in mixed characteristic are uniform categorical and universally closed but not necessarily universal. Therefore the moduli space $M_{X/S}(P)$ does not in general universally corepresent $\mathcal{M}_{X/S}(P)$ (but it does in characteristic 0). However, $M^s_{X/S}(P)$ universally corepresents the corresponding subfunctor, because in this case the corresponding quotient is in fact a $\mathrm{PGL}(m)$-principal bundle in fppf topology (but not in étale topology; see [Ma1, Cor. 6.4.1]).

As a final application of our theorems we give a new effective restriction theorem, which works in all characteristics (see Section 5). In characteristic zero our result is a stronger version of Bogomolov's restriction theorem (see [HL, Th. 7.3.5]). It has immediate applications to the study of moduli spaces of Gieseker semistable sheaves.

The paper is organized as follows. In Section 1 we recall some basic facts and prove some useful inequalities. In Section 2 we explain that Frobenius pull backs of semistable sheaves are semistable (although the notion of semistability has to be altered) and we use it to explain some basic properties of the Harder-Narasimhan filtrations in positive characteristic. Section 3 is the heart of the paper and it contains formulations and proofs of our restriction theorem and a few versions of Bogomolov's inequality. We prove our theorems by induction on the rank of a sheaf. In Section 4 we use these results to prove the boundedness of semistable sheaves. In Section 5 we prove effective restriction theorems in all characteristics. In Section 6 we further study semistable sheaves in positive characteristic.

Notation used in this paper is consistent with that in the literature. For basic notions, facts and history of the problems we refer the reader to the excellent book [HL] by Huybrechts and Lehn.

1. Preliminaries

Let X be a normal projective variety of dimension n and let $\mathcal{O}_X(1)$ be a very ample line bundle. Let $[x]_+ = \max(0, x)$ for any real number x. If E is a torsion-free sheaf then $\mu_{\max}(E)$ denotes the maximal slope in the Harder-Narasimhan filtration of E (counted with respect to the natural polarization).

THEOREM 1.1 (Kleiman's criterion; see [HL, Th. 1.7.8]). *Let $\{E_t\}$ be a family of coherent sheaves on X with the same Hilbert polynomial P. Then the family is bounded if and only if there are constants C_i, $i = 0, \dots, \deg P$, such that for every E_t there exists an E_t-regular sequence of hyperplane sections $H_1, \dots, H_{\deg P}$, such that $h^0(E_t|_{\bigcap_{j\le i} H_j}) \le C_i$.*

LEMMA 1.2 (see [HL, Lemma 3.3.2]). *Let E be a torsion-free sheaf of rank r. Then for any E-regular sequence of hyperplane sections $H_1, \dots, H_n$ the following inequality holds for $i = 1, \dots, n$:*

$$\frac{h^0(X_i, E|_{X_i})}{r \deg(X)} \le \frac{1}{i!} \left[\frac{\mu_{\max}(E|_{X_1})}{\deg(X)} + i \right]_+^i ,$$

where $X_i \in |H_1| \cap \dots \cap |H_{n-i}|$.

LEMMA 1.3. *Let r_i be positive real numbers and μ_i any real numbers for $i = 1, \dots, m$. Set $r = \sum r_i$. Then*

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \ge \frac{r_1 r_m}{r_1 + r_m} r(\mu_1 - \mu_m)^2.$$

Proof. For $m = 1, 2$ the inequality is easy to check. For $m = 3$ the required inequality is equivalent to

$$r_1(\mu_1 - \mu_2)^2 + r_3(\mu_2 - \mu_3)^2 \geq \frac{r_1 r_3}{r_1 + r_3}(\mu_1 - \mu_3)^2.$$

If we set $a = \mu_1 - \mu_2$ and $b = \mu_2 - \mu_3$ then this is equivalent to

$$\left(\frac{1}{r_1} + \frac{1}{r_3}\right)(r_1 a^2 + r_3 b^2) \geq (a + b)^2.$$

But this inequality follows from

$$\frac{r_1}{r_3} a^2 + \frac{r_3}{r_1} b^2 \geq 2\sqrt{\frac{r_1}{r_3} a^2 \cdot \frac{r_3}{r_1} b^2} = 2|ab|.$$

This proves the lemma for $m = 3$.

Now assume that $m \geq 3$. Set $r_1' = r_1$, $r_2' = \sum_{i=2}^{m-1} r_i$, $r_3' = r_m$, $\mu_1' = \mu_1$, $\mu_2' = \sum_{i=2}^{m-1} r_i \mu_i / (\sum_{i=2}^{m-1} r_i)$ and $\mu_3' = \mu_m$. Then using the inequality for $m = 3$ we get

$$\begin{aligned}\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 = r(\sum r_i \mu_i^2) - (\sum r_i \mu_i)^2 \geq r(\sum r_i' (\mu_i')^2) - (\sum r_i' \mu_i')^2 \\ = \sum_{i<j} r_i' r_j' (\mu_i' - \mu_j')^2 \geq \frac{r_1' r_3'}{r_1' + r_3'} r(\mu_1' - \mu_3')^2 \\ = \frac{r_1 r_m}{r_1 + r_m} r(\mu_1 - \mu_m)^2.\end{aligned}$$

□

Lemma 1.4. *Let r_i be positive real numbers and $\mu_1 > \mu_2 > \cdots > \mu_m$ real numbers. Set $r = \sum r_i$ and $r\mu = \sum r_i \mu_i$. Then*

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq r^2 (\mu_1 - \mu)(\mu - \mu_m).$$

Proof. Note that

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 = r \left(\sum_{i=1}^{m-1} \left(\sum_{j \leq i} r_j (\mu_j - \mu) \right) (\mu_i - \mu_{i+1}) \right).$$

Using $\sum_{j \leq i} r_j \mu_j \leq (\sum_{j \leq i} r_j)\mu_1$ and simplifying we obtain the required inequality. □

Let $p_i = (x_i, y_i)$, $i = 0, 1, \ldots, l$, be some points in the plane and assume that $x_0 < x_1 < \cdots < x_l$. Let us set $r_i = x_i - x_{i-1}$ and $\mu_i = (y_i - y_{i-1})/r_i$, and assume that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l$. Let P be the polygon obtained by joining p_i to p_{i+1} for $i = 0, \ldots, l-1$ and p_l to p_0. By assumption, P is the convex hull $\operatorname{conv}(p_0, \ldots, p_l)$ of points $p_0, \ldots, p_l$.

LEMMA 1.5. *Let P and P' be two such convex polygons (possibly degenerated) and assume that they have the same beginning and end points (i.e., $p_0 = p'_0$ and $p_l = p'_{l'}$). If P' is contained in P then*

$$\sum r_i \mu_i^2 \geq \sum r'_i (\mu'_i)^2.$$

Proof. We prove the lemma by induction on l'.

If $l' = 1$ then the inequality follows from

$$\sum r_i \mu_i^2 \geq \frac{(\sum r_i \mu_i)^2}{\sum r_i}.$$

Assume that $l' = k \geq 2$ and the lemma holds for all pairs of polygons with $l' < k$. In this case for each nonnegative number α let us set $p'_0(\alpha) = p'_0$, $p'_i(\alpha) = (x'_i, y'_i + \alpha)$ for $i = 1, \dots, l' - 1$ and $p'_l(\alpha) = p'_l$. Then the corresponding sequence $\mu'_i(\alpha)$ is still decreasing. Consider the largest nonnegative α such that the polygon $P'' = \operatorname{conv}(p'_0(\alpha), \dots, p'_{l'}(\alpha))$ is still contained in P. Then there exists a vertex $p'_j(\alpha)$, $j \neq 0, l'$, which lies on the (upper) boundary of P.

Now let us note that

$$\begin{aligned} \sum r'_i (\mu'_i(\alpha))^2 &= r'_1 \left(\mu'_1 + \frac{\alpha}{r'_1} \right)^2 + r'_2 (\mu'_2)^2 + \cdots \\ &\qquad \cdots + r'_{l'-1} (\mu'_{l'-1})^2 + r'_{l'} \left(\mu'_{l'} - \frac{\alpha}{r'_{l'}} \right)^2 \\ &\geq \sum r'_i (\mu'_i)^2 \end{aligned}$$

because $\mu'_1 \geq \mu'_{l'}$. Therefore the inequality for the pair P and P'' is stronger than the one for P and P'. But the inequality for P and P'' follows (by summing) from the inequalities for two pairs of smaller polygons, which hold by the induction assumption. □

2. Semistability of Frobenius pull backs

In this section we assume that X is a smooth n-dimensional projective variety defined over an algebraically closed field k of characteristic $p > 0$.

Let $X^{(i)} = X \times_{\operatorname{Spec} k} \operatorname{Spec} k$, where the product is taken over the i^{th} power of an absolute Frobenius map on $\operatorname{Spec} k$. Then the factorization of the absolute Frobenius morphism $F : X \to X$ gives the geometric Frobenius morphism $F_g : X \to X^{(1)}$.

If E is a coherent sheaf on X and $\nabla : E \to E \otimes \Omega_X$ is an integrable k-connection then one can define its *p-curvature* $\psi : \operatorname{Der}_k(X) \to \mathcal{E}nd_X(E)$ by $\psi(D) = (\nabla(D))^p - \nabla(D^p)$ (note that ψ is not an $\mathcal{O}_X$-homomorphism, but it is p-linear).

If E is a coherent sheaf on $X^{(1)}$ then one can construct a canonical connection ∇_{can} on F_g^*E (by using the usual differentiation in tangent directions). Now let us recall Cartier's theorem (see, e.g., [Ka, Th. 5.1]).

THEOREM 2.1 (Cartier). *There is an equivalence of categories between the category of quasi-coherent sheaves on $X^{(1)}$ and the category of quasi-coherent $\mathcal{O}_X$-modules with integrable k-connections, whose p-curvature is zero. This equivalence is given by $E \to (F_g^*E, \nabla_{\mathrm{can}})$ and $(G, \nabla) \to \ker \nabla$.*

Gieseker [Gi] gave examples of semistable bundles whose Frobenius pull backs are no longer semistable. However, Theorem 2.1 allows for inseparable descent and it allows us to explain the behaviour of semistable sheaves under Frobenius pull-backs.

Let us recall that a coherent $\mathcal{O}_X$-sheaf E with a W-valued operator $\eta : E \to E \otimes W$ is called *η-semistable* if the inequality on slopes is satisfied for all nonzero subsheaves of E preserved by η.

PROPOSITION 2.2. *A coherent sheaf E on $X^{(1)}$ is semistable with respect to H if and only if the sheaf F_g^*E is ∇_{can}-semistable with respect to F_g^*H.*

LEMMA 2.3. *Let E be a torsion-free sheaf with a k-connection ∇. Assume that E is ∇-semistable and let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$ be the usual Harder-Narasimhan filtration. Then the induced maps $E_i \to (E/E_i) \otimes \Omega_X$ are $\mathcal{O}_X$-homomorphisms and they are nonzero for $i = 1, \ldots, m-1$.*

Lemma 2.3 together with Proposition 2.2 lead to the following lemma proved by N. Shepherd-Barron (and many others).

COROLLARY 2.4 (see [SB2, Prop. 1]). *Let E be a semistable torsion-free sheaf such that F^*E is unstable. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = F^*E$ be the Harder-Narasimhan filtration. Then the $\mathcal{O}_X$-homomorphisms $E_i \to (E/E_i) \otimes \Omega_X$ induced by ∇_{can} are nontrivial.*

Note that Shepherd-Barron's proof is much less elementary and it uses Ekedahl's results on quotients by foliations in positive characteristic.

Let us fix a collection of nef divisors $D_1, \ldots, D_{n-1}$. The maximal (minimal) slope in the Harder-Narasimhan filtration of E (with respect to $(D_1, \ldots, D_{n-1})$) is denoted by $\mu_{\max}(E)$ ($\mu_{\min}(E)$, respectively). Since it will usually be clear which polarizations are used, we suppress $D_1, \ldots, D_{n-1}$ in the notation.

Set

$$L_{\max}(E) = \lim_{k \to \infty} \frac{\mu_{\max}((F^k)^*E)}{p^k}$$

and

$$L_{\min}(E) = \lim_{k\to\infty} \frac{\mu_{\min}((F^k)^*E)}{p^k}.$$

Note that the sequence $\frac{\mu_{\max}((F^k)^*\mathcal{E})}{p^k}$ $(\frac{\mu_{\min}((F^k)^*\mathcal{E})}{p^k})$ is weakly increasing (respectively: decreasing), so its limit exists (though we do not yet know if it is finite). Moreover, $L_{\max}(E) \geq \mu_{\max}(E)$ and $L_{\min}(E) \leq \mu_{\min}(E)$. Let us also remark that if E is semistable then $L_{\max}(E) = \mu(E)$ (or $L_{\min}(E) = \mu(E)$) if and only if E is strongly semistable.

Let us also set

$$\alpha(E) = \max(L_{\max}(E) - \mu_{\max}(E), \mu_{\min}(E) - L_{\min}(E)).$$

Corollary 2.5. *Let A be a* nef *divisor such that $T_X(A)$ is globally generated. Then for any torsion-free sheaf E of rank r*

$$\alpha(E) \leq \frac{r-1}{p-1} AD_1 \dots D_{n-1}.$$

Proof. First we prove that if E is semistable then $\mu_{\max}(F^*E) - \mu_{\min}(F^*E) \leq (r-1)AD_1 \dots D_{n-1}$ (cf. [SB2, Cor. 2]). To prove this take the Harder-Narasimhan filtration $0 = E_0 \subset E_1 \subset \dots \subset E_m = F^*E$. By Corollary 2.4 $\mu_{\min}(E_i) \leq \mu_{\max}((F^*E/E_i) \otimes \Omega_X)$. By assumption Ω_X embeds into a direct sum of copies of $\mathcal{O}_X(A)$, so that $\mu_{\max}((F^*E/E_i) \otimes \Omega_X) \leq \mu_{\max}((F^*E/E_i) \otimes \mathcal{O}_X(A))$. Summing inequalities $\mu(E_i/E_{i-1}) \leq \mu(E_{i+1}/E_i) + AD_1 \dots D_{n-1}$ yields the required inequality.

Now we get $\frac{\mu_{\max}((F^k)^*E)}{p^k} \leq \mu_{\max} + \frac{r-1}{p-1} AD_1 \dots D_{n-1}$ by simple induction. Passing to the limit yields the required inequality for $L_{\max}(E) - \mu_{\max}(E)$. Similarly one can show that the corresponding inequality holds for $\mu_{\min}(E) - L_{\min}(E)$. □

In Section 6 we prove a much stronger version of Corollary 2.5 (see Corollary 6.2).

2.6. Let E be a torsion-free sheaf. We say that E has an fdHN property ("finite determinacy of the Harder-Narasimhan filtration") if there exists k_0 such that all quotients in the Harder-Narasimhan filtration of $(F^{k_0})^*E$ are strongly semistable.

If E has an fdHN property (we say that "E is fdHN" for short) and $E_\bullet$ is the Harder-Narasimhan filtration of $(F^k)^*E$ for some $k \geq k_0$, then $F^*(E_\bullet)$ is the Harder-Narasimhan filtration of $(F^{k+1})^*E$.

Let E be a torsion-free sheaf and let $0 = E_0 \subset E_1 \subset \dots \subset E_m = E$ be the Harder-Narasimhan filtration of E. To any sheaf G we may associate the point $p(G) = (\operatorname{rk} G, \deg G)$ in the plane. Now consider the points $p(E_0), \dots, p(E_m)$

and connect them successively by line segments connecting the last point with the first one. The resulting polygon $\mathrm{HNP}(E)$ is called the *Harder-Narasimhan polygon* of E (see [Sh]).

Let us recall that $\mathrm{HNP}(E)$ lies above the corresponding polygon obtained from any other filtration of E with torsion free quotients (see, e.g., [Sh, Th. 2]).

If $\operatorname{char} k = p$ then we may associate to E a sequence of polygons $\mathrm{HNP}_k(E)$, where $\mathrm{HNP}_k(E)$ is defined by contracting $\mathrm{HNP}((F^k)^*E)$ along the degree axis by the factor $1/p^k$. By the above remark the polygon $\mathrm{HNP}_k(E)$ is contained in $\mathrm{HNP}_{k+1}(E)$. Moreover, all these polygons are bounded, by Corollary 2.5. Therefore there exists the limit polygon $\mathrm{HNP}_\infty(E)$. Using it one can define $\mu_{i\infty}(E)$ and $r_{i\infty}(E)$ in the obvious way.

Note that E is fdHN if and only if there exists k_0 such that $\mathrm{HNP}_{k_0}(E) = \mathrm{HNP}_\infty(E)$.

THEOREM 2.7. *Every torsion-free sheaf is* fdHN.

Proof. The proof is by induction on rank. For rank 1 the assertion is obvious. Assume that the theorem holds for every sheaf of rank less than r and let E be a rank r sheaf. Let $0 = p_{0\infty}, p_{1\infty}, \dots, p_{(l-1)\infty}, p_{l\infty} = (r, \deg E)$ be the vertices of $\mathrm{HNP}_\infty(E)$. Let $0 = E_{0k} \subset E_{1k} \subset \dots \subset E_{l_k k} = (F^k)^*E$ be the Harder-Narasimhan filtration of $(F^k)^*E$ and let p_{ik} denote the corresponding vertices of $\mathrm{HNP}_k(E)$. For every $j = 0, \dots, l$ there exists a sequence $\{p_{i_j k}\}$ which tends to $p_{j\infty}$.

CLAIM 2.7.1. *There exists k_0 such that $E_{i_1 k} = (F^{k-k_0})^*E_{i_1 k_0}$ for all $k \geq k_0$.*

Proof. First let us note that for every $\varepsilon > 0$ there exists $k(\varepsilon)$ such that $||p_{i_j k} - p_{j\infty}|| < \varepsilon$ for $k \geq k(\varepsilon)$. If we take $\varepsilon < 1$ then $\operatorname{rk} E_{i_1 k} = r_{1\infty}$ for $k \geq k(\varepsilon)$.

Let us take $k \geq k(\varepsilon)$ and consider $\mathrm{HNP}_\infty(E_{i_1 k})$. If the first line segment s of $\mathrm{HNP}_\infty(E_{i_1 k})$ lies on the line segment $\overline{p_{0\infty}p_{1\infty}}$ then by the induction assumption there exists l and a subsheaf G of $(F^l)^*E_{i_1 k} \subset (F^{k+l})^*E$ such that the point $p(G)$ lies on $\overline{p_{0\infty}p_{1\infty}}$. Then E is fdHN since $(F^{k+l})^*E/G$ is fdHN by the induction assumption.

Therefore we can assume that the segment s lies below $\overline{p_{0\infty}p_{1\infty}}$. In particular there exists l such that the line segment $\overline{p_{0\infty}p_{i_1(k+l)}}$ lies above s. Then there exists $j > i_1$ such that

$$\mu_{\max}((F^l)^*(E_{jk}/E_{(j-1)k})) > \mu_{\max}((F^l)^*E_{i_1 k}).$$

Otherwise, $\mu_{\max}((F^{k+l})^*E) \leq \mu_{\max}((F^l)^*E_{i_1 k})$, a contradiction.

There also exists a saturated subsheaf $G \subset (F^l)^*E_{jk}$ containing $(F^l)^*E_{(j-1)k}$ such that

$$\mu(G/(F^l)^*E_{(j-1)k}) = \mu_{\max}((F^l)^*(E_{jk}/E_{(j-1)k})).$$

Consider the point $p(G)=(\operatorname{rk} G, \frac{\deg G}{p^{k+1}})$. Then $\mathrm{HNP}_{k+l}(E)$ contains the smallest convex polygon W containing $\mathrm{HNP}_k(E)$ and $p(G)$. Here we again use the fact that any polygon whose vertices are saturated subsheaves of a fixed sheaf lies below the Harder-Narasimhan polygon.

But the difference of areas of W and $\mathrm{HNP}_k(E)$ is at least

$$\begin{aligned}\frac{1}{2}\left(\frac{\mu(G/(F^l)^*E_{(j-1)k})}{p^{k+l}}-\frac{\mu((F^l)^*(E_{jk}/E_{(j-1)k})}{p^{k+l}}\right)&>\frac{1}{2}\left(\frac{\mu(E_{i_1k})}{p^k}-\frac{\mu(E_{jk})}{p^k}\right)\\&\geq\frac{1}{2}(\mu_{1\infty}-\mu_{2\infty}-3\varepsilon).\end{aligned}$$

On the other hand, the difference of areas of $\mathrm{HNP}_\infty(E)$ and $\mathrm{HNP}_k(E)$ is at most $r\varepsilon$. So for sufficiently small ε we get a contradiction. $\square$

It is easy to make the above procedure more efficient.

By the claim, $p_{i_1k}=p_{1\infty}$ for $k\geq k_0$ and hence $E_{i_1k_0}$ is strongly semistable (since $p_{1\infty}$ is the first nonzero vertex of $\mathrm{HNP}_\infty(E)$ and $\mathrm{HNP}_{k_0}(E)$ is convex). Since $(F^{k_0})^*E/E_{i_1k_0}$ is fdHN by the induction assumption, the sheaf E is also fdHN. $\square$

3. Restriction to hypersurfaces and Bogomolov's inequality

Notation. Let k be an algebraically closed field of any characteristic. Let X be a smooth projective variety of dimension $n\geq 2$ over k and let $D_1,\dots,D_{n-1}$ be nef divisors on X such that the 1-cycle $D_1\dots D_{n-1}$ is numerically nontrivial. Set $d=D_1^2D_2\dots D_{n-1}\geq 0$.

Let E be a rank r torsion-free sheaf on X. Set $\Delta(E)=2rc_2(E)-(r-1)c_1(E)^2$, $\mu=\mu(E)$, $\mu_{\min}=\mu_{\min}(E)$ and $\mu_{\max}=\mu_{\max}(E)$. For simplicity we usually ignore the dependence of slopes on the collection $(D_1,\dots,D_{n-1})$.

In the following, F always denotes the absolute Frobenius morphism or identity if the characteristic is zero. If $\operatorname{char} k=p>0$ then we already defined $L_{\max}(E)$ and $L_{\min}(E)$ in Section 2. If $\operatorname{char} k=0$ then we set $L_{\max}(E)=\mu_{\max}(E)$ and $L_{\min}(E)=\mu_{\min}(E)$.

Corollary 2.5 and Theorem 2.7 imply that $L_{\max}(E)$ and $L_{\min}(E)$ are well defined rational numbers. For simplicity, we set $L_{\max}=L_{\max}(E)$ and $L_{\min}=L_{\min}(E)$.

For any pair G,G' of nontrivial torsion free sheaves we set

$$\xi_{G',G}=\frac{c_1(G')}{\operatorname{rk} G'}-\frac{c_1(G)}{\operatorname{rk} G}.$$

Now choose a nef divisor A on X such that $T_X(A)$ is globally generated. Then we set

$$\beta_r(A; D_1, \dots, D_{n-1}) = \begin{cases} 0 & \text{if } \operatorname{char} k = 0, \\ \left(\dfrac{r(r-1)}{p-1} AD_1 \dots D_{n-1}\right)^2 & \text{if } \operatorname{char} k = p. \end{cases}$$

To simplify notation, we usually write $\beta_r = \beta_r(A; D_1, \dots, D_{n-1})$.

Let $\operatorname{Num}(X) = \operatorname{Pic}(X) \otimes \mathbb{R}/\sim$, where $\sim$ is an equivalence relation defined by $L_1 \sim L_2$ if and only if $L_1 AD_2 \dots D_{n-1} = L_2 AD_2 \dots D_{n-1}$ for all divisors A. Then we define an open cone

$$K^+ = \{D \in \operatorname{Num}(X) : D^2 D_2 \dots D_{n-1} > 0 \\ \text{and } DD_1 \dots D_{n-1} \geq 0 \text{ for all nef } D_1\}.$$

As in the surface case, by the Hodge index theorem, this cone is "self-dual" in the following sense:

$$D \in K^+ \text{ if and only if } DLD_2 \dots D_{n-1} > 0 \text{ for all } L \in \overline{K^+} - \{0\}.$$

Theorem 3.1. *Assume that D_1 is very ample and the restriction of E to a general divisor $D \in |D_1|$ is not μ-semistable (with respect to $(D_2|_D, \dots, D_{n-1}|_D)$). Let μ_i (r_i) denote slopes (respectively: ranks) of the Harder-Narasimhan filtration of $E|_D$. Then*

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq d\Delta(E) D_2 \dots D_{n-1} + 2r^2 (L_{\max} - \mu)(\mu - L_{\min}). \tag{3.1.1}$$

The inequality in Theorem 3.1 is sharp for unstable sheaves. Equality holds, e.g., for $\mathcal{O}_{\mathbb{P}^n}(k) \oplus \mathcal{O}_{\mathbb{P}^n}(-k)$. For semistable sheaves of rank 2 it can be slightly improved (see (3.10.1)).

The following theorems generalize Bogomolov's instability theorem.

Theorem 3.2. *Let E be a strongly $(D_1, \dots, D_{n-1})$-semistable torsion-free sheaf. Then*

$$\Delta(E) D_2 \dots D_{n-1} \geq 0.$$

Theorem 3.3. *If E is $(D_1, \dots, D_{n-1})$-semistable then*

$$D_1^2 D_2 \dots D_{n-1} \cdot \Delta(E) D_2 \dots D_{n-1} + \beta_r \geq 0. \tag{3.3.1}$$

Theorem 3.4. *If $D_1^2 D_2 \dots D_{n-1} \cdot \Delta(E) D_2 \dots D_{n-1} + \beta_r < 0$ then there exists a saturated subsheaf $E' \subset E$ such that $\xi_{E',E} \in K^+$.*

Strategy of the proof. Let $T^i(r)$, $i = 1, \dots, 4$ denote the statement: Theorem 3.i holds for all sheaves of rank $\le r$ on any smooth variety. Let $T^5(r)$ denote the statement: Theorem 3.2 holds if $D_1, \dots, D_{n-1}$ are ample and $\mathrm{rk}\, E \le r$.

We will prove that $T^1(r)$ implies $T^5(r)$, $T^5(r)$ implies $T^3(r)$, $T^3(r)$ implies $T^4(r)$, $T^4(r)$ implies $T^2(r)$ and finally $T^2(r)$ implies $T^1(r+1)$. Since $T^1(1)$ is trivial this will prove all the theorems at the same time by simple induction.

Proofs.

3.5. $T^1(r)$ implies $T^5(r)$.

Let us assume that $D_1, \dots, D_{n-1}$ are very ample, E is strongly semistable and $\Delta(E)D_2 \dots D_{n-1} < 0$. Then $L_{\max} = L_{\min} = \mu$ and $T^1(r)$ implies that the restriction of E to D_1 is semistable. Since $(F^k)^*E$ is also strongly semistable the restriction of $(F^k)^*E$ to a general element of $|D_1|$ is also semistable. Therefore the restriction of E to a very general element of $|D_1|$ is strongly semistable. By induction, the restriction of $(F^l)^*E$ to a very general complete intersection $X_i = |D_1| \cap \dots \cap |D_i|$ is strongly semistable for $i = 1, \dots, n-1$.

Now without loss of generality we can assume that X is a surface, E is locally free (because $\Delta(E^{**}) \le \Delta(E)$ and E^{**} is locally free on a smooth surface) and the restriction of E to a very general curve $C \in |D_1|$ is strongly semistable. Then $T^5(r)$ follows from Bogomolov's inequality if $\mathrm{char}\, k = 0$ and from [Mo, Th. 1] if $\mathrm{char}\, k = p$. However, we prefer to give a different proof, which does not depend on the characteristic of the base field. We use the method of Y. Miyaoka in [Mi].

On a curve, bundles associated to representations of a strongly semistable bundle are strongly semistable (see [Mi, §§5 and 6]). Therefore $S^{kr}E|_C$ is strongly semistable. The standard short exact sequence

$$0 \to S^{kr}E(-kc_1E) \otimes \mathcal{O}_X(-H_1) \longrightarrow S^{kr}E(-kc_1E) \longrightarrow S^{kr}E(-kc_1E)|_C \to 0$$

gives

$$\begin{aligned} h^0(S^{kr}E(-kc_1E)) &\le h^0(S^{kr}E(-kc_1E - H_1)) + h^0(S^{kr}E(-kc_1E)|_C) \\ &= h^0(S^{kr}E(-kc_1E)|_C), \end{aligned}$$

where the last equality follows from the strong semistability of $S^{kr}E$ with respect to D_1. Recall that $h^0(G) \le [\deg G + \mathrm{rk}\, G]_+$ for any semistable vector bundle G over a curve. Hence $h^0(S^{kr}E(-kc_1E)) = \mathrm{O}(k^r)$. Similarly, by Serre duality $h^2(S^{kr}E(-kc_1E)) = h^0((S^{kr}E(-kc_1E))^* \otimes \omega_X) = \mathrm{O}(k^r)$.

On the other hand, the Riemann-Roch theorem gives

$$\chi(X, S^{kr}E(-kc_1E)) = -\frac{r^r\Delta(E)}{2(r+1)!}k^{r+1} + \mathrm{O}(k^r),$$

and we get a contradiction. $\square$

3.6. $T^5(r)$ implies $T^3(r)$.

First we need to prove the following:

Claim. If $D_1, \dots, D_{n-1}$ are ample then

$$D_1^2 D_2 \dots D_{n-1} \cdot \Delta(E) D_2 \dots D_{n-1} + r^2 (L_{\max} - \mu)(\mu - L_{\min}) \geq 0. \tag{3.6.1}$$

Proof. By Theorem 2.7 there exists k such that all the quotients in the Harder-Narasimhan filtration of $(F^k)^*E$ are strongly semistable. Let $0 = E_0 \subset E_1 \subset \dots \subset E_m = (F^k)^*E$ be the corresponding Harder-Narasimhan filtration. Set $F_i = E_i/E_{i-1}$, $r_i = \operatorname{rk} F_i$, $\mu_i = \mu(F_i)$. Then by the Hodge index theorem

$$\begin{aligned} \frac{\Delta((F^k)^*E) D_2 \dots D_{n-1}}{r} &= \sum \frac{\Delta(F_i) D_2 \dots D_{n-1}}{r_i} \\ &\quad - \frac{1}{r} \sum_{i<j} r_i r_j \left(\frac{c_1 F_i}{r_i} - \frac{c_1 F_j}{r_j} \right)^2 D_2 \dots D_{n-1} \\ &\geq \sum \frac{\Delta(F_i) D_2 \dots D_{n-1}}{r_i} - \frac{1}{rd} \sum_{i<j} r_i r_j (\mu_i - \mu_j)^2. \end{aligned}$$

By $T^5(r)$

$$\Delta(F_i) D_2 \dots D_{n-1} \geq 0,$$

and so combining the above inequality with Lemma 1.4, we get

$$\begin{aligned} &\frac{d\Delta((F^k)^*E) D_2 \dots D_{n-1}}{r} \\ &\qquad \geq -r(\mu_{\max}((F^k)^*E) - \mu((F^k)^*E))(\mu((F^k)^*E)) - \mu_{\min}((F^k)^*E))). \end{aligned}$$

Division by p^{2k} yields the required inequality (we do not need to pass with k to infinity since we used Theorem 2.7). □

Now assume that E is $(D_1, \dots, D_{n-1})$-semistable. Let us fix an ample divisor H and set $H_i(t) = D_i + tH$. Then the Harder-Narasimhan filtration of E with respect to $(H_1(t), \dots, H_{n-1}(t))$ is independent of t for small positive t.

To prove it let us consider the set S of slopes of all subsheaves of E considered as polynomials in t. The coefficients of these polynomials are the slopes of subsheaves of E with respect to some polarizations depending only on $D_1, \dots, D_{n-1}$ and H. Therefore they are bounded from the above (by some constant C) and there exists the maximal polynomial W_1 in S with respect to the lexicographic order on coefficients. Take any other polynomial $W_2 \in S$ and write $W_1(t) = a_0 + a_1 t + \dots + a_{n-1} t^{n-1}$, $W_2(t) = b_0 + b_1 t + \dots + b_{n-1} t^{n-1}$. By the choice of W_1 there exists i such that $a_j = b_j$ for $j < i$ and $a_i > b_i$. Then $a_i \geq b_i + \frac{1}{r}$ and

$$W_1(t) - W_2(t) = \sum_{j \geq i} (a_j - b_j) t^j \geq t^i (a_i - b_i + t \sum_{j \geq i+1} (a_j - C) t^{j-i-1})$$

for any positive t. Set $M_i = |\inf_{t\in[0,1]}(\sum_{j\geq i+1}(a_j - C)t^{j-i-1})|$ and $M = \max_i M_i$. Then

$$W_1(t) - W_2(t) \geq t^i\left(\frac{1}{r} - tM_i\right)$$

for $t \in (0,1)$. In particular, if $t \in (0, \frac{1}{Mr})$ then $W_1(t) > W_2(t)$ (note that $\frac{1}{Mr} \leq 1$). Therefore the sheaf of maximal rank among the sheaves corresponding to W_1 is the maximal $(H_1(t), \dots, H_{n-1}(t))$-destabilizing subsheaf of E for all $t \in (0, \frac{1}{Mr})$.

Now we can proceed by induction to prove the corresponding statement for the Harder-Narasimhan filtration.

Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$ be the corresponding filtration. Since E is $(D_1, \dots, D_{n-1})$-semistable, we have

$$\begin{aligned}\mu_{D_1,\dots,D_{n-1}}(E) &\geq \mu_{D_1,\dots,D_{n-1}}(E_1) = \lim_{t\to 0}\mu_{H_1(t),\dots,H_{n-1}(t)}(E_1)\\ &\geq \lim_{t\to 0}\mu_{H_1(t),\dots,H_{n-1}(t)}(E) = \mu_{D_1,\dots,D_{n-1}}(E).\end{aligned}$$

Therefore

$$\lim_{t\to 0}\mu_{\max,H_1(t),\dots,H_{n-1}(t)}(E) = \lim_{t\to 0}\mu_{H_1(t),\dots,H_{n-1}(t)}(E_1) = \mu_{D_1,\dots,D_{n-1}}(E).$$

Similarly,

$$\lim_{t\to 0}\mu_{\min,H_1(t),\dots,H_{n-1}(t)}(E) = \mu_{D_1,\dots,D_{n-1}}(E).$$

Applying inequality (3.6.1) and Corollary 2.5 to $(H_1(t), \dots, H_{n-1}(t))$ and passing with t to 0 yields the required inequality (3.3.1). □

3.7. $T^3(r)$ implies $T^4(r)$.

By $T^3(r)$ E is not $(D_1, \dots, D_{n-1})$-semistable. Let E' be the maximal destabilizing subsheaf of E (with respect to $(D_1, \dots, D_{n-1})$). Set $E'' = E/E'$, $r' = \operatorname{rk} E'$, $r'' = \operatorname{rk} E''$. Note that

$$\begin{aligned}&\frac{\Delta(E)D_2\dots D_{n-1}}{r} + \frac{rr'}{r''}\xi^2_{E',E}D_2\dots D_{n-1}\\ &\qquad = \frac{\Delta(E')D_2\dots D_{n-1}}{r'} + \frac{\Delta(E'')D_2\dots D_{n-1}}{r''}\end{aligned}$$

and

$$\frac{\beta_r}{r} \geq \frac{\beta_{r'}}{r'} + \frac{\beta_{r''}}{r''}.$$

Therefore either $\xi^2_{E',E}D_2\dots D_{n-1} > 0$ or one of the numbers $d\Delta(E')D_2\dots D_{n-1} + \beta_{r'}$ and $d\Delta(E'')D_2\dots D_{n-1} + \beta_{r''}$ is negative. Then we proceed by induction as in Bogomolov's proof of the same theorem in characteristic 0 (see the proof of Theorem 7.3.3, [HL]). □

3.8. $T^4(r)$ implies $T^2(r)$.

Assume that $\Delta(E)D_2 \dots D_{n-1} < 0$ and choose a nef (or ample) divisor H_1 such that $H_1^2 D_2 \dots D_{n-1} > 0$. Let us try to apply $T^4(r)$ to $(F^l)^*E$. The inequality

$$H_1^2 D_2 \dots D_{n-1} \cdot \Delta((F^l)^*E)D_2 \dots D_{n-1} + \beta_r < 0$$

is equivalent to

$$l > \frac{1}{2} \log_p \left(- \frac{\beta_r}{H_1^2 D_2 \dots D_{n-1} \, \Delta(E) D_2 \dots D_{n-1}} \right).$$

So for large l there exists a saturated torsion free subsheaf E' of $(F^l)^*E$ such that $\xi_{E',(F^l)^*E} \in K^+$. By the "self-duality" of K^+ mentioned before Theorem 3.1, $\xi_{E',(F^l)^*E} D_1 \dots D_{n-1} > 0$. Hence the sheaf E is not strongly $(D_1, \dots, D_{n-1})$-semistable. ☐

3.9. $T^2(r-1)$ implies $T^1(r)$.

Let Π denote the complete linear system $|D_1|$. Let $Z = \{(D, x) \in \Pi \times X : x \in D\}$ be the incidence variety with projections $p : Z \to \Pi$ and $q : Z \to X$. Let Z_s denote the scheme theoretic fibre of p over the point $s \in \Pi$. Let $0 \subset E_0 \subset E_1 \subset \dots \subset E_m = q^*E$ be the relative Harder-Narasimhan filtration with respect to p. By definition this means that there exists a nonempty open subset U of Π such that all factors $F_i = E_i/E_{i-1}$ are flat over U and such that for every $s \in U$ the fibres $(E_\bullet)_s$ form the Harder-Narasimhan filtration of $E_s = q^*E|_{Z_s}$.

Usually, the relative Harder-Narasimhan filtration is defined only with respect to one p-ample divisor but it is obvious that one can define it with respect to the collection of divisors if $D_2, \dots, D_{n-1}$ are ample (this case is already sufficient to prove our theorems). The usual construction fails for the collection of nef divisors but in this case we can use the same trick as in 3.6. Namely, the relative Harder-Narasimhan filtration of q^*E with respect to p and the collection $(q^*H_2(t), \dots q^*H_{n-1}(t))$ is independent of t for small positive t. Then grouping the quotients with the same slope with respect to $(q^*D_2, \dots, q^*D_{n-1})$ (on the fibres of p), we get the required filtration.

One can easily see that this relative Harder-Narasimhan filtration is the Harder-Narasimhan filtration of q^*E with respect to

$$(p^*\mathcal{O}_\Pi(1))^{\dim \Pi} q^*(D_2 \dots D_{n-1}).$$

Indeed, for any sheaf G the slope of $G|_{Z_s}$ on a general fibre Z_s of p is equal to the slope of G with respect to $(p^*\mathcal{O}_\Pi(1))^{\dim \Pi} q^*(D_2 \dots D_{n-1})$. Since by the construction the restriction $F_i|_{Z_s}$ is semistable for $s \in U$, the sheaves F_i are semistable with respect to $(p^*\mathcal{O}_\Pi(1))^{\dim \Pi} q^*(D_2 \dots D_{n-1})$. Now the required assertion follows from the fact that the corresponding sequence of slopes is strictly decreasing.

By Theorem 2.7 applied to q^*E with respect to

$$(p^*\mathcal{O}_\Pi(1))^{\dim \Pi} q^*(D_2 \dots D_{n-1})$$

there exists k such that all the quotients in the Harder-Narasimhan filtration of $(F^k)^*(q^*E) = q^*((F^k)^*E)$ are strongly semistable.

By Lemma 1.5 the inequality (3.1.1) applied to $(F^k)^*E$ implies the inequality (3.1.1) for E. Therefore we can assume that all the F_i's are strongly semistable with respect to $(p^*\mathcal{O}_\Pi(1))^{\dim \Pi} q^*(D_2 \dots D_{n-1})$.

Let $\Lambda \subset \Pi$ be a general pencil. Set $Y = p^{-1}(\Lambda)$. Then the restriction $q|_Y$ is the blow up of X along the base locus B of Λ. If $n \geq 3$ then B is a smooth, connected variety. Let N be the exceptional divisor of $q|_Y$. Then there exist integers b_i and divisors M_i such that $c_1(F_i|_Y) = (q|_Y)^* M_i + b_i N$. If $n = 2$ then B consists of d distinct points. Let $N_1, \dots, N_d$ be the exceptional divisors of $q|_Y$. As above there exist integers b_{ij} and divisors M_i such that $c_1(F_i|_Y) = (q|_Y)^* M_i + \sum_j b_{ij} N_j$. Set $b_i = \sum_j b_{ij}/d$. Then

$$\mu_i = \frac{c_1(F_i|_Y) p^*\mathcal{O}_\Lambda(1) q^*D_2 \dots q^*D_{n-1}}{r_i} = \frac{M_i D_1 \dots D_{n-1} + b_i d}{r_i}. \tag{3.9.1}$$

On the other hand $(q|_Y)_*(E_i|_Y) \subset E$, so that

$$\frac{\sum_{j \leq i} M_j D_1 \dots D_{n-1}}{\sum_{j \leq i} r_j} \leq \mu_{\max}.$$

Hence

$$\sum_{j \leq i} b_j d \geq \sum_{j \leq i} r_j (\mu_j - \mu_{\max}). \tag{3.9.2}$$

Since $(p^*\mathcal{O}_\Pi(1))^{\dim \Pi} q^*(D_2 \dots D_{n-1})$ is numerically nontrivial, $T^2(r-1)$ implies that $\Delta(F_j)(p^*\mathcal{O}_\Pi(1))^{\dim \Pi - 1} q^*(D_2 \dots D_{n-1}) \geq 0$ for every j. Therefore

$$\begin{aligned}
&\frac{d\Delta(E) D_2 \dots D_{n-1}}{r} \\
&\quad = \sum \frac{d\Delta(F_i|_Y)(q|_Y)^*(D_2 \dots D_{n-1})}{r_i} \\
&\qquad - \frac{d}{r} \sum_{i<j} r_i r_j \left(\frac{c_1(F_i|_Y)}{r_i} - \frac{c_1(F_j|_Y)}{r_j} \right)^2 (q|_Y)^*(D_2 \dots D_{n-1}) \\
&\quad \geq \frac{d}{r} \sum_{i<j} r_i r_j \left(d \left(\frac{b_i}{r_i} - \frac{b_j}{r_j} \right)^2 - \left(\frac{M_i}{r_i} - \frac{M_j}{r_j} \right)^2 D_2 \dots D_{n-1} \right) \\
&\quad \geq \frac{1}{r} \sum_{i<j} r_i r_j \left(d^2 \left(\frac{b_i}{r_i} - \frac{b_j}{r_j} \right)^2 - \left(\frac{M_i D_1 \dots D_{n-1}}{r_i} - \frac{M_j D_1 \dots D_{n-1}}{r_j} \right)^2 \right),
\end{aligned}$$

where the last inequality follows from the Hodge index theorem. Using (3.9.1) and simplifying one can see that the last expression in the above inequality is equal to

$$2\sum db_i\mu_i - \frac{1}{r}\sum_{i<j} r_ir_j(\mu_i-\mu_j)^2.$$

By (3.9.2)

$$\begin{aligned}\sum db_i\mu_i &= \sum_i(\sum_{j\le i} db_j)(\mu_i-\mu_{i+1}) \ge \sum_i(\sum_{j\le i} r_j(\mu_j-\mu_{max}))(\mu_i-\mu_{i+1})\\ &= \sum r_i\mu_i^2 - r\mu^2 + r(\mu-\mu_{\max})(\mu-\mu_{\min})\\ &= \sum_{i<j}\frac{r_ir_j}{r}(\mu_i-\mu_j)^2 + r(\mu-\mu_{\max})(\mu-\mu_{\min}).\end{aligned}$$

Therefore we obtain

$$\frac{d\Delta(E)D_2\dots D_{n-1}}{r} \ge \sum_{i<j}\frac{r_ir_j}{r}(\mu_i-\mu_j)^2 + 2r(\mu-\mu_{\max})(\mu-\mu_{\min}). \quad \square$$

Remarks 3.10.

(3.10.1) Let E be a rank 2 vector bundle on a surface. Assume that E is semistable and the restriction to a general curve $C\in|D_1|$ is not semistable. In this situation Ein [Ei] and Noma [No] proved that $(\mu_1-\mu_2)^2\le d\Delta(E)/3$. From our proof one can immediately get a slightly worse inequality $(\mu_1-\mu_2)^2\le d\Delta(E)D_2\dots D_{n-1}$ for n-dimensional variety. In this case it is also possible to improve slightly our method to get $(\mu_1-\mu_2)^2\le d\Delta(E)D_2\dots D_{n-1}/3$.

(3.10.2) If $\Delta(E)D_2\dots D_{n-1}=0$ then by Theorem 3.1 the restriction of a strongly semistable sheaf to a very general complete intersection in $|D_1|\cap\dots\cap|D_i|$ is strongly semistable for $i=1,\dots,n-1$. In the surface case this was also proved by Moriwaki (see [Mo, Cor. C.3]). This fact is interesting since semistable vector bundles with vanishing Chern classes play an important role in understanding algebraic varieties. In characteristic zero they correspond to flat bundles.

It is not clear if there exists a restriction theorem for strongly semistable sheaves if $\Delta(E)D_2\dots D_{n-1}>0$.

(3.10.3) There are several papers by N. Shepherd-Barron, A. Moriwaki and T. Nakashima exploring rank 2 and 3 vector bundles. (These ranks are always very special in proofs of the Bogomolov type inequalities.) They prove slightly more precise versions of Bogomolov's inequality in this case and use it to prove vanishing theorems and Reider-type theorems on adjoint linear systems in positive characteristic. Some of these results were proved earlier by T. Ekedahl, who used different methods.

Theorem 3.2 was conjectured by A. Moriwaki in [Mo] and proved in the surface case (using boundedness of semistable sheaves on surfaces). A special case of Theorem 3.4 (see 3.8) was proved in the surface case by G. Megyesi [Me], who used Moriwaki's result. The statement in 3.8 was also conjectured in the higher dimensional case. The papers [Mo] and [Me] were preceded by the paper [SB1] of N. Shepherd-Barron, who proved analogous results for rank 2 vector bundles on surfaces.

(3.10.4) Bogomolov proved his instability theorem only for surfaces in characteristic zero. The higher dimensional case can then be reduced to the surface case by the Mumford-Mehta-Ramanathan restriction theorem (see [Mi]). This reduction is no longer possible in positive characteristic.

Theorems 3.1, 3.2 and Lemma 1.3 imply the following corollary.

Corollary 3.11. *Assume that D_1 is very ample. Let D be a very general divisor in $|D_1|$. Then*

$$\frac{r}{2}(L_{\max}(E|_D) - L_{\min}(E|_D))^2 \leq d\Delta(E)D_2 \dots D_{n-1} + 2r^2(L_{\max} - \mu)(\mu - L_{\min}).$$

As in characteristic zero one can see that Theorem 3.4 implies the following stronger theorem (which in characteristic zero is due to Bogomolov; see Theorem 7.3.4, [HL]).

Theorem 3.12. *If*

$$\Delta(E)D_2 \dots D_{n-1} + \inf\left\{\frac{\beta_r(A; D, D_2, \dots D_{n-1})}{D^2 D_2 \dots D_{n-1}} : D \textit{ is } \text{nef} \textit{ and } D^2 D_2 \dots D_{n-1} > 0\right\} < 0$$

then there exists a saturated subsheaf $E' \subset E$ such that $\xi_{E',E} \in K^+$ and

$$\xi_{E',E}^2 D_2 \dots D_{n-1} \geq -\frac{\Delta(E)D_2 \dots D_{n-1}}{r^2(r-1)}.$$

Note that in the statement of the above theorem we do not use D_1.

4. Boundedness of pure sheaves

Let $H_1, \dots, H_{n-1}$ be very ample divisors and let A be a nef divisor such that $T_{X_l}(A)$ is globally generated for a very general complete intersection X_l in $|H_1| \cap \dots \cap |H_l|$ and all $0 \leq l \leq n-2$. It is easy to see that one can always find a divisor A satisfying these assumptions.

Set $\beta_r = \beta(r; A, H_1, \dots, H_{n-1})$ and let us recall that $[x]_+ = \max(0, x)$.

THEOREM 4.1. *Let $\mu_{\max,l}$ ($\mu_{\min,l}$) denote the maximal (respectively: minimal) slope of the restriction of E to a very general complete intersection in $|H_1|\cap\dots\cap|H_l|$, $1\le l\le n-1$. Then*

$$\mu_{\max,l}-\mu_{\min,l}\le \frac{r^{l/2}-1}{r-\sqrt{r}}\left(\sqrt{2[d\Delta(E)H_2\dots H_{n-1}]_+}+2\sqrt{\beta_r}\right)+r^{l/2}(\mu_{\max}-\mu_{\min}).$$

Proof. By Corollary 3.11,

$$\begin{aligned}\frac{r}{2}(L_{\max}(E|_{X_1})-L_{\min}(E|_{X_1}))^2 &\le d\Delta(E)H_2\dots H_{n-1}\\ &\quad + 2r^2(L_{\max}-\mu)(\mu-L_{\min})\\ &\le d\Delta(E)H_2\dots H_{n-1}+2r^2\left(\frac{L_{\max}-L_{\min}}{2}\right)^2.\end{aligned}$$

Since $L_{\max}(E|_{X_1})-L_{\min}(E|_{X_1})\ge \mu_{\max,1}-\mu_{\min,1}$ and $L_{\max}-L_{\min}\le\mu_{\max}-\mu_{\min}+2\sqrt{\beta_r}/r$ (by Corollary 2.5),

$$\begin{aligned}\mu_{\max,1}-\mu_{\min,1}&\le\sqrt{\frac{2}{r}d\Delta(E)H_2\dots H_{n-1}+4r\left(\frac{1}{2}(\mu_{\max}-\mu_{\min})+\frac{\sqrt{\beta_r}}{r}\right)^2}\\ &\le\sqrt{\frac{2}{r}d[\Delta(E)H_2\dots H_{n-1}]_+}+2\sqrt{\frac{\beta_r}{r}}+\sqrt{r}(\mu_{\max}-\mu_{\min}),\end{aligned}$$

where the last inequality follows from $\sqrt{a+b^2}\le\sqrt{[a]_+}+|b|$. The inequality in Theorem 4.1 is obtained by repetitive use of this inequality. If we pass to the hyperplane section we may need to change A required in Corollary 3.11, so we need to use assumptions appearing at the beginning of Section 4. □

THEOREM 4.2. *Let $f: X\to S$ be a projective morphism of schemes of finite type over an algebraically closed field k and let $\mathcal{O}_X(1)$ be an f-ample line bundle on X. Fix a degree d polynomial P and a real number μ_0. Then the family of purely d-dimensional sheaves on geometric fibers of f with Hilbert polynomial P and the maximal slope bounded by μ_0 is bounded.*

Proof. Boundedness of torsion-free sheaves follows from Theorem 1.1, Lemma 1.2 and Theorem 4.1. Then the proof given by Simpson in [Si] implies boundedness for pure sheaves (see [Si], proofs of Theorem 1.1 and Proposition 3.5). Caution: the proof of this implication given in [HL] does not work in positive characteristic. □

Theorem 4.2 can also be generalized to $\mathcal{O}_X$-coherent Λ-sheaves, where Λ is a sheaf of rings of differential operators (see [Si]). Below we prove a more refined version of this theorem working in mixed characteristic but without fixing Hilbert polynomial.

In the following we will use the following notation. Let X_k be an n-dimensional projective scheme over an algebraically closed field k and $H = \mathcal{O}_{X_k}(1)$ an ample divisor on X_k. Let E be a torsion-free sheaf of pure dimension d on X_k. Then there exist integers $a_0(E), \dots, a_d(E)$ such that

$$\chi(X_k, E(m)) = \sum_{i=0}^{d} a_i(E) \binom{m+d-i}{d-i}.$$

Definition 4.3 (Maruyama, [Ma3, Def. 1.6]). Let $f : X \to S$ be a projective morphism of noetherian schemes of relative dimension n and let $\mathcal{O}_{X/S}(1)$ be an f-very ample line bundle on X.

(1) Let $\mathcal{S}_{X/S}(d; r, a_1, \dots, a_d, \mu_{\max})$ be the family of the classes of coherent sheaves on the fibres of f such that E on a geometric fibre X_s is a member of the family if E is of pure dimension d, $\mu_{\max}(E) \leq \mu_{\max}$, $a_0(E) = r$, $a_1(E) = a_1$ and $a_i(E) \geq a_i$ for $i \geq 2$.

(2) Let $\mathcal{S}'_{X/S}(d; r, a_1, a_2, \mu_{\max})$ be the family of the classes of coherent sheaves on the fibres of f such that E on a geometric fibre X_s is a member of the family if E is reflexive of dimension d, $\mu_{\max}(E) \leq \mu_{\max}$, $a_0(E) = r$, $a_1(E) = a_1$ and $a_2(E) \geq a_2$.

Our definition is equivalent to Maruyama's definition, but for simplicity we replaced the condition on the type of E by $\mu_{\max}(E) \leq \mu_{\max}$. Our Theorem 4.1 allows us to prove Maruyama's conjecture on boundedness of sheaves in mixed characteristic.

THEOREM 4.4 ([Ma1, Question 7.18] and [Ma2, Conj. 2.11]). *The families $\mathcal{S}_{X/S}(d; r, a_1, \dots, a_d, \mu_{\max})$ and $\mathcal{S}'_{X/S}(d; r, a_1, a_2, \mu_{\max})$ are bounded.*

Proof. Using ideas of C. Simpson and J. Le Potier (see [Si] and [Ma3, Th. 1.8]) one can reduce to the case of a smooth morphism f and $d = n$ (i.e., the sheaves are torsion-free).

We can find a nonnegative integer a such that $T_{X/S} \otimes \mathcal{O}_{X/S}(a)$ is f-globally generated. Therefore β_r can be uniformly bounded for all geometric fibres of X (it is crucial that β_r not be increasing with characteristic p). The only thing we need to check is that $\Delta(E)H^{n-2}$ is bounded from the above for E in our families. Then we can use Theorem 4.1 to proceed by induction on n (see [Ma2, Prop. 2.5]).

Let X_s be a geometric fibre of f and set $H = \mathcal{O}_{X_s}(1)$. Let E be a rank r torsion-free sheaf on X_s. Using the Riemann-Roch theorem one can write $\frac{\Delta(E)H^{n-2}}{2r} + a_2(E)$ as a sum of $\frac{1}{2r}(c_1 E - \frac{r}{2} K_X)^2 H^{n-2}$ and some other terms depending only on r, $c_1 E \cdot H^{n-1}$ and numerical invariants of (X, H). Using the Hodge index theorem one can bound $\frac{\Delta(E)H^{n-2}}{2r} + a_2(E)$ by a polynomial in

$a_1(E)$, r and numerical invariants of X. Therefore the condition $a_2(E) \geq a_2$ can be replaced by $\Delta(E)H^{n-2} \leq C_X(r, a_1, a_2)$ for some function C (which one can write down explicitly). □

This theorem implies that the moduli spaces of semistable sheaves in mixed characteristic are projective (cf. Theorem 0.2 and [Ma3, Th. 7.6]). Another nontrivial corollary says that the number of different Hilbert polynomials for sheaves in $\mathcal{S}_{X/S}(d; r, a_1, \dots, a_d, \mu_{\max})$ and $\mathcal{S}'_{X/S}(d; r, a_1, a_2, \mu_{\max})$ is finite. Theorem 4.4 also implies existence of Bogomolov type inequality in mixed characteristic (cf. [Ma2, Cor. 2.10] and [Mo, Th. 1]).

5. Effective restriction theorems

The notation is as in Section 3. We will need the following strenghtening of Theorem 3.3, which also works for unstable sheaves.

THEOREM 5.1. *If E is a torsion-free sheaf then*

$$D_1^2 D_2 \dots D_{n-1} \cdot \Delta(E) D_2 \dots D_{n-1} + r^2 (L_{\max} - \mu)(\mu - L_{\min}) \geq 0 \tag{5.1.1}$$

and

$$D_1^2 D_2 \dots D_{n-1} \cdot \Delta(E) D_2 \dots D_{n-1} + r^2 (\mu_{\max} - \mu)(\mu - \mu_{\min}) + \beta_r \geq 0. \tag{5.1.2}$$

Proof. (5.1.1) follows from Theorem 3.2 by the same arguments as in the proof of Claim 3.6. The proof of (5.1.2) is similar. Namely, let $0 = E_0 \subset E_1 \subset \dots \subset E_m = E$ be the Harder-Narasimhan filtration. Set $F_i = E_i / E_{i-1}$, $r_i = \operatorname{rk} F_i$, $\mu_i = \mu(F_i)$. Then by the Hodge index theorem

$$\begin{aligned}
\frac{\Delta(E) D_2 \dots D_{n-1}}{r} &= \sum \frac{\Delta(F_i) D_2 \dots D_{n-1}}{r_i} \\
&\quad - \frac{1}{r} \sum_{i<j} r_i r_j \left(\frac{c_1 F_i}{r_i} - \frac{c_1 F_j}{r_j} \right)^2 D_2 \dots D_{n-1} \\
&\geq \sum \frac{\Delta(F_i) D_2 \dots D_{n-1}}{r_i} - \frac{1}{rd} \sum_{i<j} r_i r_j (\mu_i - \mu_j)^2.
\end{aligned}$$

By Theorem 3.3

$$d \Delta(F_i) D_2 \dots D_{n-1} \geq -\beta_{r_i}.$$

Therefore the required inequality follows from Lemma 1.4 and

$$\frac{\beta_r}{r} \geq \sum_i \frac{\beta_{r_i}}{r_i}.$$

□

As an application of Theorem 5.1 we get the following effective restriction theorem.

THEOREM 5.2. *Let E be a torsion-free sheaf of rank $r \geq 2$. Assume that E is μ-stable with respect to $(D_1, \dots, D_{n-1})$. Let $D \in |kD_1|$ be a normal divisor such that $E|_D$ has no torsion. If*

$$k > \left\lfloor \frac{r-1}{r}\Delta(E)D_2 \dots D_{n-1} + \frac{1}{dr(r-1)} + \frac{(r-1)\beta_r}{dr} \right\rfloor$$

then $E|_D$ is μ-stable with respect to $(D_2|_D, \dots, D_{n-1}|_D)$.

Proof. Suppose that $E|_D$ is not stable and let S be a saturated destabilizing subsheaf of rank ρ. Set $T = (E|_D)/S$. Let G be the kernel of the composition $E \to E|_D \to T$. Then we have two short exact sequences:

$$0 \to G \longrightarrow E \longrightarrow T \to 0$$

and

$$0 \to E(-D) \longrightarrow G \longrightarrow S \to 0.$$

Computing $\Delta(G)$ we get

$$\begin{aligned} \Delta(G)D_2 \dots D_{n-1} = \Delta(E)D_2 \dots D_{n-1} - \rho(r-\rho)D^2 D_2 \dots D_{n-1} \\ + 2(rc_1(T) - (r-\rho)Dc_1(E))D_2 \dots D_{n-1}. \end{aligned}$$

By assumption $(rc_1(T) - (r-\rho)Dc_1(E))D_2 \dots D_{n-1} \leq 0$, so that

$$\Delta(G)D_2 \dots D_{n-1} \leq \Delta(E)D_2 \dots D_{n-1} - \rho(r-\rho)D^2 D_2 \dots D_{n-1}.$$

Using the stability of E and $E(-D)$ we get

$$\mu_{\max}(G) - \mu(G) = \mu_{\max}(G) - \mu(E) + \frac{r-\rho}{r}DD_1 \dots D_{n-1} \leq \frac{r-\rho}{r}dk - \frac{1}{r(r-1)}$$

and

$$\mu(G) - \mu_{\min}(G) = \mu(E(-C)) - \mu_{\min}(G) + \frac{\rho}{r}DD_1 \dots D_{n-1} \leq \frac{\rho}{r}dk - \frac{1}{r(r-1)}.$$

Hence, application of (5.1.2) to G gives

$$\begin{aligned} -\beta_r &\leq d\Delta(G) + r^2(\mu_{\max}(G) - \mu(G))(\mu(G) - \mu_{\min}(G)) \\ &\leq d\Delta(E) - \rho(r-\rho)d^2k^2 + r^2\left(\frac{r-\rho}{r}dk - \frac{1}{r(r-1)}\right)\left(\frac{\rho}{r}dk - \frac{1}{r(r-1)}\right). \end{aligned}$$

Therefore

$$\frac{dr}{r-1}k \leq d\Delta(E) + \frac{1}{(r-1)^2} + \beta_r,$$

which contradicts our assumption on k. $\square$

Remarks 5.3.

(5.3.1) Note that if E is torsion free then the restriction $E|_D$ is also torsion free for a general divisor D in a base point-free linear system (see [HL, Cor. 1.1.14] for a precise statement).

(5.3.2) In characteristic zero if $r > 2$ or $d \geq 2$ it is sufficient to assume that $k > \frac{r-1}{r}\Delta(E)D_2 \dots D_{n-1}$. If $r = 2$ and $d = 1$ then we need to assume that $k > (\Delta(E)D_2 \dots D_{n-1} + 1)/2$. Looking at the proof one can see that Theorem 5.2 can be further improved at the cost of simplicity.

(5.3.3) The idea of proof of Theorem 5.2 is similar to that of Bogomolov's restriction theorem (see [HL, Th. 7.3.5]). However, the proof of Bogomolov's restriction theorem used the Kobayashi-Hitchin correspondence, a strong version of Bogomolov's instability theorem (see [HL, Th. 7.3.4]; cf. Theorem 3.12) and the semistability of representations of a semistable bundle. Each of these facts makes this proof impossible to follow in positive characteristic.

As a corollary to Theorem 5.2 we get an effective restriction theorem for semistable sheaves. It also explains the meaning of a "general" element of $|kD_1|$ in previously known restriction theorems.

Corollary 5.4. *Let E be a torsion-free sheaf of rank $r \geq 2$. Assume that E is μ-semistable with respect to $(D_1, \dots, D_{n-1})$ and let $0 = E_0 \subset E_1 \subset \dots \subset E_m = E$ be the corresponding Jordan-Hölder filtration of E. Set $F_i = E_i/E_{i-1}$, $r_i = \operatorname{rk} F_i$. Let $D \in |kD_1|$ be a normal divisor such that all the sheaves $F_i|_D$ have no torsion. If*

$$k > \left\lfloor \frac{r-1}{r}\Delta(E)D_2 \dots D_{n-1} + \frac{1}{dr(r-1)} + \frac{(r-1)\beta_r}{dr} \right\rfloor$$

then $E|_D$ is μ-semistable with respect to $(D_2|_D, \dots, D_{n-1}|_D)$.

Proof. The corollary follows from Theorem 5.2 and the following inequality

$$\frac{\Delta(E)D_2 \dots D_{n-1}}{r} \geq \sum \frac{\Delta(F_i)D_2 \dots D_{n-1}}{r_i}$$

(cf. the proof of Theorem 5.1). □

Theorem 5.2 and Corollary 5.4 show that for a sufficiently large k the rational map from the moduli space of Gieseker semistable sheaves on X with fixed rank and Chern classes to the moduli space of Gieseker semistable sheaves on a smooth divisor $D \in |kD_1|$ is an injective immersion on the open subset of μ-stable locally free sheaves.

6. Semistable sheaves in positive characteristic

In this section we assume that the base field k has a positive characteristic p. Let $H_1, \dots, H_{n-1}$ be ample divisors on a smooth n-dimensional variety X. Semistability in this section denotes μ-semistability with respect to $(H_1, \dots, H_{n-1})$.

The following theorem is a special case of a theorem proved by Ramanan and Ramanathan, [RR, Th. 3.23] (see the remark at the end of Section 4, [RR]). In the curve case this theorem was known earlier (see, e.g., [Mi, §5] and [Ba]).

Theorem 6.1. *A tensor product of strongly semistable sheaves is strongly semistable.*

In the curve case Ilangovan-Mehta-Parameswaran and Balaji-Parameswaran proved that if the characteristic of k is large with respect to ranks of semistable bundles having trivial determinants then their tensor product is also semistable. The author was informed by V. Balaji that all these bounds hold for any bundles. The precise statement of the Ilangovan-Mehta-Parameswaran theorem is: if E_1 and E_2 are two semistable bundles, and the sum of their ranks is less than $p+2$, then $E_1 \otimes E_2$ is again semistable. Using Corollary 5.4 one gets the same result for torsion free sheaves in higher dimensions (although it is also sufficient to use the Mumford-Mehta-Ramanathan restriction theorem). This shows that if p is large with respect to r and n then the inequality in Corollary 2.5 can be improved to

$$\alpha(E) \le \frac{r-1}{p}[\mu_{\max}(\Omega_X)]_+.$$

Below we prove an analogous result in all characteristics.

The next two corollaries also allow us to improve theorems in previous sections by improving the bound for β_r.

Corollary 6.2. *Let E be a torsion-free sheaf of rank r. Then*

$$\alpha(E) \le \frac{r-1}{p}[L_{\max}(\Omega_X)]_+.$$

If E is semistable then

$$L_{\max}(E) - L_{\min}(E) \le \frac{r-1}{p}[L_{\max}(\Omega_X)]_+.$$

Proof. It is sufficient to prove the second part of the corollary. If E is semistable then $(F^k)^*E$ is η-semistable for $\eta : E \to E \otimes (\Omega_X \oplus \dots \oplus (F^{k-1})^*\Omega_X)$ given by $\eta = (\nabla_{\mathrm{can}}, \dots, (F^{k-1})^*\nabla_{\mathrm{can}})$ (cf. Proposition 2.2). By Theorem 2.7 there exists k_0 such that all the factors in the Harder-Narasimhan filtrations of $(F^{k_0})^*E$ and $(F^{k_0})^*\Omega_X$ are strongly semistable. Let $0 = E_{0k} \subset E_{1k} \subset \cdots \subset E_{mk} = (F^k)^*E$ be the Harder-Narasimhan filtration of $(F^k)^*E$ for $k \geq k_0$. Let μ_{ik} be the corresponding slopes of quotients of this filtration. We fix a nef divisor A such that $T_X(A)$ is globally generated. Then by Lemma 2.3 and Theorem 6.1

$$\mu_{ik} \leq \mu_{(i+1)k} + \max(p^{k_0} AH_1 \dots H_{n-1}, \max\{\mu_{\max}((F^j)^*\Omega_X)) : k_0 \leq j \leq k-1\}).$$

Now the required inequality is obtained by summing all these inequalities, dividing by p^k and passing with k to infinity. □

Applying the above corollary to Ω_X, we get the following corollary.

COROLLARY 6.3. *If $p \geq n = \dim X$ and $L_{\max}(\Omega_X) \geq 0$ then*

$$L_{\max}(\Omega_X) \leq \frac{p}{p+1-n}\mu_{\max}(\Omega_X).$$

In particular, if $p \geq n$ and $\mu_{\max}(\Omega_X) \leq 0$ then all semistable sheaves are strongly semistable.

Note that by Corollary 2.4, if $\mu_{\max}(\Omega_X) \leq 0$ then all semistable sheaves are strongly semistable (Corollary 6.3 gives the same assertion for $p \geq n$). This should be compared with Mori's theorem saying that if $\mu(\Omega_X) < 0$ then X is uniruled.

One of the referees pointed out to the author that this fact was proved already in [MR, Th. 2.1]. This fact and Bogomolov's vanishing theorem immediately imply the following proposition.

PROPOSITION 6.4. *Let (X, H) be a smooth polarized complex projective variety and E a semistable sheaf on X. Assume that the set Σ of primes p modulo which E is not strongly semistable is infinite. Then Ω_X contains a subsheaf F such that $c_1F \cdot H^{n-1} > 0$ and the Kodaira dimension of c_1F is less than or equal to the rank of F.*

For rank 2 vector bundles this proposition is a special case of [SB2, Th. 5].

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