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# Poincaré inequalities in punctured domains

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**Abstract**

The classic Poincaré inequality bounds the $L^q$-norm of a function $f$ in a bounded domain $\Omega \subset \mathbb{R}^n$ in terms of some $L^p$-norm of its gradient in $\Omega$. We generalize this in two ways: In the first generalization we remove a set $\Gamma$ from $\Omega$ and concentrate our attention on $\Lambda = \Omega \setminus \Gamma$. This new domain might not even be connected and hence no Poincaré inequality can generally hold for it, or if it does hold it might have a very bad constant. This is so even if the volume of $\Gamma$ is arbitrarily small. A Poincaré inequality *does hold*, however, if one makes the additional assumption that $f$ has a finite $L^p$ gradient norm on the whole of $\Omega$, not just on $\Lambda$. The important point is that the Poincaré inequality thus obtained bounds the $L^q$-norm of $f$ in terms of the $L^p$ gradient norm on $\Lambda$ (not $\Omega$) plus an additional term that goes to zero as the volume of $\Gamma$ goes to zero. This error term depends on $\Gamma$ only through its volume. Apart from this additive error term, the constant in the inequality remains that of the 'nice' domain $\Omega$. In the second generalization we are given a vector field $A$ and replace $\nabla$ by $\nabla + iA(x)$ (geometrically, a connection on a $U(1)$ bundle). Unlike the $A = 0$ case, the infimum of $\|(\nabla + iA)f\|_p$ over all $f$ with a given $\|f\|_q$ is in general not zero. This permits an improvement of the inequality by the addition of a term whose sharp value we derive. We describe some open problems that arise from these generalizations.

## 1. Introduction

The simplest Poincaré inequality refers to a bounded, connected domain $\Omega \subset \mathbb{R}^n$, and a function $f \in L^2(\Omega)$ whose distributional gradient is also in $L^2(\Omega)$ (namely, $f \in W^{1,2}(\Omega)$). While it is false that there is a finite constant $S$,

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depending only on $\Omega$, such that

$$\int_\Omega |f|^2 \leq S \int_\Omega |\nabla f|^2 \tag{1}$$

for all $f$, such an inequality *does* hold if we impose the additional condition that $\int_\Omega f = 0$. The constant $S$ depends on $\Omega$, but it is independent of $f$. In fact, $1/S$ is the second eigenvalue of the Laplacian in $\Omega$ with Neumann boundary conditions. This is merely a consequence of Bessel's inequality.

Some simple generalizations of (1) are well known. One involves replacing the condition $\int_\Omega f = 0$ by the condition $\int_\Omega fg = 0$, where $g$ is any $L^2(\Omega)$ function that is not orthogonal to the lowest Neumann eigenfunction of the Laplacian, i.e., $\int_\Omega g \neq 0$. Another involves replacing $L^2$ by $L^p$ for $1 \leq p \leq \infty$ on both sides of (1) . Finally, by Sobolev's inequality, the $L^p$-norm of $f$ can be replaced by a suitable $L^q$-norm with $q > p$. The situation is summarized in (see, e.g., [1, Thms. 8.11 and 8.12]) the following statement:

THEOREM 1 (Standard Poincaré inequalities for $W^{1,p}(\Omega)$). *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set with the cone property. Let $1 \leq q \leq \infty$ and let $\max\{1, qn/(n+q)\} \leq p \leq \infty$ if $q < \infty$ and $n < p \leq \infty$ if $q = \infty$. Let $g$ be a function in $L^{p'}(\Omega)$, $p' = p/(p-1)$, such that $\int_\Omega g = 1$. Then there is a constant $S_{p,q} > 0$, which depends on $\Omega$, $g$, $p$, $q$, such that for any $f \in W^{1,p}(\Omega)$,*

$$\left\| f - \int_\Omega fg \right\|_{L^q(\Omega)} \leq S_{p,q} \|\nabla f\|_{L^p(\Omega)} \ . \tag{2}$$

*Remarks.* The case $q = np/(n-p)$ requires the Sobolev inequality explicitly for the proof, and thus the inequality can be called the Poincaré-Sobolev inequality in this case. The domain $\Omega$ is required to have the "cone property" (see, e.g., [2]); i.e., each point of $\Omega$ is the vertex of a spherical cone with fixed height and angle, which is situated in $\Omega$. Note that the constants $S_{p,q}$ depend not only on the volume of $\Omega$, but also on its shape. The Poincaré inequality is usually presented as (2) with $q = p$ and with $g = 1/|\Omega|$, where, in general, $|\cdot|$ denotes the Lebesgue measure of a set in $\mathbb{R}^n$. A generalization to $W^{m,p}(\Omega)$ for $m > 1$ is possible (see, e.g., [1]).

Now we turn to the two generalizations that concern us in this paper. They were motivated by a treatment of the quantum mechanical many-body problem, specifically, the proof of Bose-Einstein condensation in a physically realistic model [3]. Further developments required a version of the Poincaré inequality in which $\nabla$ is replaced by a connection on a $U(1)$ bundle, namely $\nabla \to \nabla + iA$, where $A$ is a vector field. The generalization to this situation leads to a proof of superfluidity for the same quantum-mechanical system [4]. Note that for $n = 1$ the bundle is trivial, since the vector field $A$ can be eliminated by a unitary transformation, namely $f(x) \to f(x)\exp(-i\int_z^x A(y)dy)$ for some $z \in \Omega$.

Our main result is Theorem 3, which contains the two generalizations, and which we now describe in detail.

The first generalization concerns the following obstruction to the use of the Poincaré inequality (2): Let us remove a small set $\Gamma$ from $\Omega$ and concentrate our attention on $\Lambda = \Omega \setminus \Gamma$. This new domain might not even be connected and hence no Poincaré inequality can generally hold for $\Lambda$, no matter how small $|\Gamma|$ might be. Even if $\Lambda$ is connected, the constant $S_{p,q}$ could be very large, or even infinite.

A trivial example is to let $\Omega$ be a unit square in $\mathbb{R}^2$ and to let $\Gamma$ be a thin annulus in $\Omega$ of outer radius $1/2$ and inner radius $1/2 - \varepsilon$. Take $g(x) = 1$. We can take $f = 1$ inside the disk of radius $1/2 - \varepsilon$ and $f = 0$ elsewhere. Thus, regardless of how small $\varepsilon$ may be, the right side of (2), with $\Omega$ replaced by $\Lambda$, will be zero while the left side is positive. Another, perhaps more interesting example is one in which $\Lambda$ is connected but fails to satisfy Theorem 1 because the cone property is absent. This can be accomplished with a small $\Gamma$ that is topologically a ball, but which has a sufficiently rough surface (see, e.g., [1, §8.7]).

The smallness of $|\Gamma|$ cannot restore the Poincaré inequality in $\Lambda$. A generalized Poincaré inequality *does hold*, however, if one makes the additional assumption that $f$ has an extension to a function with a finite $L^p$ gradient norm on the whole of $\Omega$, not just on $\Lambda$. The important point is that the Poincaré inequality thus obtained bounds the $L^q$-norm of $f$ in terms of the $L^p$ gradient norm on $\Lambda$ (not $\Omega$) plus an additional error term. Furthermore – and this is also important – the effective Poincaré inequality that holds for $\Lambda$ approaches that given in (2) as the volume of $\Gamma$ tends to zero — in a manner that depends only on the fixed $L^p$ gradient norm on $\Omega$ and on $|\Gamma|$, but not on its shape. There is no regularity assumption on $\Gamma$.

In the second generalization $\nabla$ is replaced by $\nabla + iA(x)$ on the right side of (2), where $A : \Omega \to \mathbb{R}^n$ is some given vector field. (For simplicity we assume that $A$ is a bounded, measurable function, but a weaker condition will certainly suffice (see, e.g., [1, §7.20]).) We observe that hidden in (2) is a (nonlinear) 'eigenvalue' that happens to be zero, namely the smallest value of $\|\nabla f\|_p$ given the value of $\|f\|_q$. Thus, (2) states that if the right side of (2) is small then $f$ must be close to the 'lowest' eigenfunction, which happens to be the constant function. Our goal is to do something similar when $A(x) \neq 0$, i.e., to show that if $\|(\nabla + iA(x))f\|_p$ is close to the lowest $L^p \to L^q$ 'eigenvalue' defined in (3) then $f$ must be close to the corresponding 'eigenfunction'.

For $1 \le q \le \infty$ and $\max\{1, qn/(n+q)\} < p \le \infty$ the 'energy' $E_A^{p,q}$ is defined by

$$E_A^{p,q} = \inf \left\{ \frac{\|(\nabla + iA)f\|_{L^p(\Omega)}}{\|f\|_{L^q(\Omega)}} \, : \, f \in W^{1,p}(\Omega), \, f \neq 0 \right\} . \tag{3}$$

The *ground state manifold* $\mathcal{M}_A^{p,q}$ is given by the set of minimizers of (3), which is nonempty as a consequence of Theorem 2 below. Note that, in general, this will not be a linear space. Also its dimension can be greater than one, as we show in the appendix for the case $p = q = 2$.

When $A = 0$ the ground state manifold $\mathcal{M}_A^{p,q}$ is one-dimensional, spanned by the constant function. In this case the replacement of $f$ by $f - \int fg$ as in (2) has the same qualitative effect as restricting the inequality to functions $f$ whose $L^q(\Omega)$ distance to the constant function is bounded below by a fixed multiple of the $L^q(\Omega)$ norm of $f$. The fixed multiple depends on $g$, of course, but so does the constant $S_{p,q}$ appearing in (2).

For $A \neq 0$ the dimension of $\mathcal{M}_A^{p,q}$ can be greater than 1, and we adopt the second viewpoint in this case. We obtain an inequality for functions whose distance $d_A^q$ to $\mathcal{M}_A^{p,q}$ exceeds a certain value $\delta > 0$, i.e.,

$$d_A^q(f) := \inf_{\phi \in \mathcal{M}_A^{p,q}} \|f - \phi\|_{L^q(\Omega)} \ge \delta \, \|f\|_{L^q(\Omega)} . \tag{4}$$

Before giving our main Theorem 3 on punctured domains with $A$ fields it may be useful to state the following theorem, which generalizes Theorem 1 to the case of $A$ fields alone. We consider here only the case $p > \max\{1, qn/(n+q)\}$, and leave the case of the 'critical' $p = qn/(n+q) \ge 1$ as well as the case $p = 1$ as open problems.

THEOREM 2 (Poincaré inequalities with vector fields). *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set with the cone property. Let $1 \le q \le \infty$, $\max\{1, qn/(n+q)\} < p \le \infty$, and let $0 < \delta \le 1$. Then there is a constant $S_\delta^{p,q} > 0$, which depends on $\Omega$, $\delta$, $p$, $q$, such that for any $f \in W^{1,p}(\Omega)$ with $d_A^q(f) \ge \delta \|f\|_{L^q(\Omega)}$,*

$$\|f\|_{L^q(\Omega)} \le S_\delta^{p,q} \left[ \|(\nabla + iA)f\|_{L^p(\Omega)} - E_A^{p,q} \|f\|_{L^q(\Omega)} \right] . \tag{5}$$

Note that Theorem 2 implies in particular that $\mathcal{M}_A^{p,q}$ is not the empty set.

*Remark.* If $\mathcal{M}_A^{p,q}$ is one-dimensional, spanned by the minimizer $\phi_A$, which we assume to be normalized by $\|\phi_A\|_{L^q(\Omega)} = 1$, we can go back to our original formulation and take $g$ to be an $L^{q'}(\Omega)$ function satisfying $\int_\Omega g\phi_A = 1$ (compare with Theorem 1). The corresponding generalization of (2) is then

$$\left\| f(\cdot) - \phi_A(\cdot)[\textstyle\int_\Omega fg] \right\|_{L^q(\Omega)} \le \widetilde{S}_g^{p,g} \left[ \|(\nabla + iA)f\|_{L^p(\Omega)} - E_A^{p,q} \|f\|_{L^q(\Omega)} \right] , \tag{6}$$

which now holds for all $f \in W^{1,p}(\Omega)$. Here $\widetilde{S}_g^{p,q}$ is some constant depending on $g$ (besides $p$, $q$ and $\Omega$). For $q < \infty$ a possible choice for $g$ would be $g(x) = \overline{\phi_A(x)}|\phi_A(x)|^{q-2}$ if $\phi_A(x) \neq 0$, and $g(x) = 0$ otherwise.

The generalization to punctured domains is the following, which is our main theorem.

Theorem 3 (Poincaré inequalities in punctured domains). *Let* $1 \leq q \leq \infty$ *and* $\max\{1, qn/(n+q)\} < p \leq \infty$. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded, connected, open set with the cone property, and let* $E_A^{p,q}$ *and* $\mathcal{M}_A^{p,q}$ *be as explained above. Let* $\Lambda \subset \Omega$ *be a measurable subset of* $\Omega$, $\Gamma = \Omega \setminus \Lambda$, *and let* $0 < \delta \leq 1$. *For any* $\varepsilon > 0$ *there exists a positive constant* $C$, *depending on* $\Omega$, $A$, $p$, $q$, $\delta$ *and* $\varepsilon$ (*but not on* $\Lambda$ *and* $\Gamma$) *such that, for every* $f \in W^{1,p}(\Omega)$ *satisfying* $d_A^q(f) \geq \delta \, \|f\|_{L^q(\Omega)}$

$$\|(\nabla + iA)f\|_{L^p(\Lambda)} + C\,\|(\nabla + iA)f\|_{L^r(\Gamma)} \geq \left(\frac{1}{S_\delta^{p,q} + \varepsilon} + E_A^{p,q}\right)\|f\|_{L^q(\Omega)}\ , \tag{7}$$

*where* $r = \max\{1, qn/(n+q)\}$ *if* $1 \leq q < \infty$, *and* $S_\delta^{p,q}$ *is the optimal constant in* (5). *For* $q = \infty$ (7) *holds for any* $r > n$, *and* $C$ *will also depend on* $r$.

The crucial points to note about (7) are the constant 1 in front of the first term on the left side and the constants $E_A^{p,q}$ and $S_\delta^{p,q}$ on the right side – which are clearly optimal. The only unknown constant is $C$. Note that $p > r$ by assumption.

The reader might justly wonder how the volume of $\Gamma$ plays a role in the error term, as claimed above. The following corollary displays this dependence; its proof consists just of applying Hölder's inequality to the second term in (7).

Corollary 1 (Explicit volume dependence). *Under the same assumptions as in Theorem 3*,

$$\|(\nabla + iA)f\|_{L^p(\Lambda)} + C\,|\Gamma|^{1/r-1/p}\|(\nabla + iA)f\|_{L^p(\Gamma)} \geq \left(\frac{1}{S_\delta^{p,q} + \varepsilon} + E_A^{p,q}\right)\|f\|_{L^q(\Omega)}\ . \tag{8}$$

*Remarks.* 1. A weaker inequality is obtained by substituting

$$\|(\nabla + iA)f\|_{L^p(\Omega)}$$

for $\|(\nabla + iA)f\|_{L^p(\Gamma)}$ in the second term of (8). In this way the dependence on $\Gamma$ is solely through its volume (for any given value of $\|(\nabla + iA)f\|_{L^p(\Omega)}$).

2. For $1 \leq qn/(q+n) < n$ the exponents of $|\Gamma|$ appearing in Corollary 2 are optimal. This can be seen as follows. If $f$ is supported in a small ball of volume $|\Gamma|$, the corresponding minimal 'energy' $\|(\nabla + iA)f\|_{L^p}/\|f\|_{L^q}$ is of the order $|\Gamma|^{1/p-1/q-1/n}$. Inequality (8) cannot hold for a larger exponent since an $f$ supported on several disjoint small balls of volume $|\Gamma|$ can be chosen so that

$d_A^q(f) \geq \delta \|f\|_{L^q(\Omega)}$. This would violate (8) for small enough $|\Gamma|$. Note that in order to obtain the optimal exponent it is necessary to have Theorem 3 in the critical case $r = qn/(n+q) \geq 1$.

If $q = \infty$ or $q < n/(n-1)$ the optimal dependence on the volume of $\Gamma$ remains an open problem. In particular this is the case for $n = 1$.

3. It is clear that (7) cannot hold for $\varepsilon = 0$. The constant $C$ has to go to infinity as $\varepsilon \to 0$. Otherwise, the inequality would be violated by an $f$ that yields equality in (5).

The proofs of Theorems 2 and 3 will be given in the next section. In Section 3 we consider the special case $A = 0$, and in Section 4 we comment on the $L^2$-case $p = q = 2$. Section 5 contains some open problems.

The inequalities in this paper can obviously be extended in various ways, e.g., to smooth compact manifolds [5], weighted Sobolev spaces [6], or $W^{m,p}(\Omega)$ for $m > 1$, but we resist the temptation to do so here. In fact, in the physics application [4], Theorem 3 is needed for a cube in $\mathbb{R}^3$ with a pair of opposite faces identified.

## 2. Proof of Theorems 2 and 3

*Proof of Theorem 2.* As in the proof of Theorem 1 (see [1]) we use a compactness argument. Suppose (5) is false. Then there exists a sequence of functions $f_j \in W^{1,p}(\Omega)$, with $\|f_j\|_{L^q(\Omega)} = 1$, such that $d_A^q(f_j) \geq \delta$ and

$$\lim_{j\to\infty} \|(\nabla + iA)f_j\|_{L^p(\Omega)} = E_A^{p,q} \ . \tag{9}$$

The sequence $f_j$ is bounded in $L^q(\Omega)$, and it follows from Theorem 1 and (9) that $f_j$ is actually bounded in $L^p(\Omega)$. Since $A$ is bounded by assumption, $f_j$ is also bounded in $W^{1,p}(\Omega)$. Hence there exists a subsequence, still denoted by $f_j$, and a function $f \in W^{1,p}(\Omega)$ such that $\nabla f_j \rightharpoonup \nabla f$ weakly in $L^p(\Omega)$. (Note that $p > 1$ is important here; for $p = \infty$, weak convergence has to be replaced by weak-* convergence.) The Rellich-Kondrashov theorem [7, Thm. 6.2] implies that $f_j \to f$ strongly in $L^r(\Omega)$ for all $1 \leq r < np/(n-p)$ for $p \leq n$, and for all $1 \leq r \leq \infty$ if $p > n$. Hence, by our assumptions on $q$, $\|f\|_{L^q(\Omega)} = 1$ and, by weak lower semicontinuity,

$$\lim_{j\to\infty} \|(\nabla + iA)f_j\|_{L^p(\Omega)} \geq \|(\nabla + iA)f\|_{L^p(\Omega)} \geq E_A^{p,q}\|f\|_{L^q(\Omega)} = E_A^{p,q} \ . \tag{10}$$

This shows that $f \in \mathcal{M}_A^{p,q}$ and hence contradicts the fact, which follows from strong convergence, that $d_A^q(f) \geq \delta$. □

Before giving the proof of Theorem 3, we state the following lemma, which is needed to prove Theorem 3 in the critical case $r = qn/(n+q) \geq 1$. It establishes the Poincaré inequalities for functions that vanish on a set of positive measure.

LEMMA 1 (Poincaré inequalities for functions with small support). *Let $\Omega$, $p$ and $q$ be as in Theorem 1, and let $0 < \delta < 1$. Then there is a finite number $\widetilde{S}_{p,q} > 0$, which depends on $\Omega$, $\delta$, $p$, $q$, such that for any $f \in W^{1,p}(\Omega)$ with $|\{x : f(x) \neq 0\}| \leq |\Omega|(1-\delta)$*

$$\|f\|_{L^q(\Omega)} \leq \widetilde{S}_{p,q} \|\nabla f\|_{L^p(\Omega)} \ . \tag{11}$$

*Proof.* Since $\Omega$ is bounded it suffices to prove this lemma for the largest possible $q$, given $p$. In particular, it is sufficient to consider the case $q > 1$. From Theorem 1 we know that

$$\left\| f - \frac{1}{|\Omega|} \int_\Omega f \right\|_{L^q(\Omega)} \leq S_{p,q} \|\nabla f\|_{L^p(\Omega)} \tag{12}$$

for the $p$'s and $q$'s in question. By the triangle inequality

$$\|f\|_{L^q(\Omega)} \leq \left\| f - \frac{1}{|\Omega|} \int_\Omega f \right\|_{L^q(\Omega)} + |\Omega|^{1/q-1} \left| \int_\Omega f \right| \ . \tag{13}$$

By Hölder's inequality and the assumption on the support of $f$

$$\left| \int_\Omega f \right| \leq \|f\|_{L^q(\Omega)} \Big| \{x : f(x) \neq 0\} \Big|^{1-1/q} \leq \|f\|_{L^q(\Omega)} \Big[ (1-\delta)|\Omega| \Big]^{1-1/q} \ . \tag{14}$$

Inserting (13) and (14) in (12) we arrive at

$$\|f\|_{L^q(\Omega)} \leq S_{p,q} \left( 1 - (1-\delta)^{1-1/q} \right)^{-1} \|\nabla f\|_{L^p(\Omega)} \ . \tag{15}$$

□

*Proof of Theorem 3.* Assume that the assertion of the theorem is false. Then there exists a sequence of triples $(C_j, f_j, \Gamma_j)$, with $\|f_j\|_{L^q(\Omega)} = 1$, $d_A^q(f_j) \geq \delta$ and $\lim_{j\to\infty} C_j = \infty$, such that

$$\lim_{j\to\infty} \Big( \|(\nabla + iA)f_j\|_{L^p(\Lambda_j)} + C_j \|(\nabla + iA)f_j\|_{L^r(\Gamma_j)} \Big) < 1/S_\delta^{p,q} + E_A^{p,q} \ , \tag{16}$$

where we set $\Lambda_j = \Omega \backslash \Gamma_j$. This implies in particular that $\|(\nabla + iA)f_j\|_{L^r(\Gamma_j)} \to 0$ as $j \to \infty$.

We claim that it is no restriction to assume that $\lim_{j\to\infty} |\Gamma_j| = 0$. If this is not the case, define $\gamma_j \subset \Gamma_j$ by

$$\gamma_j = \left\{ x \in \Gamma_j : |(\nabla + iA(x))f_j(x)| \geq \|(\nabla + iA)f_j\|_{L^r(\Gamma_j)}^{1/2} \right\} \ . \tag{17}$$

Note that $|\gamma_j| \leq \|(\nabla + iA)f_j\|_{L^r(\Gamma_j)}^{r/2} \to 0$ as $j \to \infty$. Moreover,

$$\|(\nabla + iA)f_j\|_{L^p(\Gamma_j \backslash \gamma_j)} \leq |\Omega|^{1/p} \|(\nabla + iA)f_j\|_{L^r(\Gamma_j)}^{1/2} \ , \tag{18}$$

which also goes to zero as $j \to \infty$. Therefore (16) holds with $\Lambda_j$ and $\Gamma_j$ replaced by $\Lambda_j \cup (\Gamma_j \setminus \gamma_j)$ and $\gamma_j$, respectively.

It suffices, therefore, to consider the case $\lim_{j\to\infty} |\Gamma_j| = 0$. By passing to a subsequence we can assume that $\sum_j |\Gamma_j|$ is finite.

The sequence $f_j$ is bounded in $L^q(\Omega)$, and also $(\nabla + iA)f_j$ is bounded in $L^r(\Omega)$. From Theorem 1 we see that $f_j$ is actually bounded in $L^{\tilde{q}}(\Omega)$ for $\tilde{q} = \max\{q, n/(n-1)\} > 1$, and hence there is a subsequence, still denoted by $f_j$, and an $f \in L^q(\Omega)$, such that $f_j \rightharpoonup f$ weakly in $L^q(\Omega)$. (If $q = \infty$ weak convergence has to be replaced by weak-* convergence.)

For some fixed $N$ let $\Sigma_N = \Omega \setminus \bigcup_{j\geq N} \Gamma_j$. Note that $(\nabla + iA)f_j$ is bounded in $L^p(\Sigma_N)$ and, therefore, we can choose a subsequence such that $(\nabla + iA)f_j \rightharpoonup (\nabla + iA)f$ weakly in $L^p(\Sigma_N)$. (Again, replace weak by weak-* if $p = \infty$.) By weak lower semicontinuity of norms and the fact that $\Sigma_N \subset \Lambda_j$ for $j \geq N$,

$$\liminf_{j\to\infty} \|(\nabla + iA)f_j\|_{L^p(\Lambda_j)} \geq \liminf_{j\to\infty} \|(\nabla + iA)f_j\|_{L^p(\Sigma_N)} \geq \|(\nabla + iA)f\|_{L^p(\Sigma_N)} \ . \tag{19}$$

This holds for all $N$ and, since $\Sigma_N \subset \Sigma_{N+1}$ and $|\bigcup_N \Sigma_N| = |\Omega|$,

$$\liminf_{j\to\infty} \|(\nabla + iA)f_j\|_{L^p(\Lambda_j)} \geq \|(\nabla + iA)f\|_{L^p(\Omega)} \ . \tag{20}$$

Suppose that we knew that $f_j \to f$ *strongly* in $L^q(\Omega)$. Then clearly $d_A^q(f) \geq \delta \|f\|_{L^q(\Omega)} = \delta$, so the right side of (20) would be $\geq 1/S_\delta^{p,q} + E_A^{p,q}$ by (5), and thereby contradict (16) and establish (7).

In the following, we will show that $f_j \to f$ *strongly* in $L^q(\Omega)$. Note that for $q < n/(n-1)$ and for $q = \infty$ this follows immediately from the Rellich-Kondrashov theorem [7, Thm. 6.2], so we can restrict ourselves to the case $1 < q < \infty$. For $M > 0$, define

$$f_j^M(x) = \min\{M, |f_j(x)|\} \tag{21}$$

and

$$h_j^M(x) = |f_j(x)| - f_j^M(x) \ . \tag{22}$$

Note that both $f_j^M$ and $h_j^M$ are in $W^{1,p}(\Omega)$. Moreover,

$$\left|\left\{x \,:\, h_j^M(x) \neq 0\right\}\right| = \left|\left\{x \in \Omega \,:\, |f_j(x)| > M\right\}\right| \leq \frac{\|f_j\|_{L^q(\Omega)}^q}{M^q} = \frac{1}{M^q} \ . \tag{23}$$

By choosing $M$ larger that $2|\Omega|^{-1/q}$ we can use Lemma 1 to conclude that

$$\|h_j^M\|_{L^q(\Omega)} \leq S \|\nabla h_j^M\|_{L^r(\Omega)} \tag{24}$$

for some constant $S$ independent of $M$ and $j$. Note that the intersection of the two sets $\alpha_j := \{x \,:\, \nabla f_j^M(x) \neq 0\}$ and $\beta_j := \{x \,:\, \nabla h_j^M(x) \neq 0\}$ has measure zero. Therefore

$$\|\nabla h_j^M\|_{L^r(\Omega)} = \|\nabla |f_j|\|_{L^r(\beta_j)} \leq \|(\nabla + iA)f_j\|_{L^p(\Lambda_j)} |\beta_j|^{1/r - 1/p} + \|(\nabla + iA)f_j\|_{L^r(\Gamma_j)} \ , \tag{25}$$

where we used again Hölder's inequality and also the diamagnetic inequality $|\nabla|f|(x)| \leq |(\nabla+iA(x))f(x)|$ (see [1, Thm. 7.21]). By (23), $|\beta_j| \leq 1/M^q$. This fact, together with (24), (25), and (16), implies that

$$\limsup_{j\to\infty} \|h_j^M\|_{L^q(\Omega)} \leq S(E_A^{p,q} + 1/S_\delta^{p,q}) M^{(r-p)/prq} \ . \tag{26}$$

(If $p = \infty$, the exponent in the last term has to be replaced by $-1/rq$.)

From (16) we see that $(\nabla + iA)f_j$ is a bounded sequence in $L^r(\Omega)$ and, since $A$ is bounded by assumption, the same is true for $\nabla f_j$. Hence we can apply the Rellich-Kondrashov theorem (see, e.g., [7, Thm. 6.2]) to conclude that, modulo choice of a subsequence, $f_j \to f$ *strongly* in $L^{q-\nu}(\Omega)$ for any $0 < \nu \leq q-1$, and therefore

$$\int_\Omega |f|^{q-\nu} = \lim_{j\to\infty} \int_\Omega |f_j|^{q-\nu} \ . \tag{27}$$

By definition of $f_j^M$,

$$\int_\Omega |f_j|^{q-\nu} \geq \int_\Omega |f_j^M|^{q-\nu} \geq \frac{1}{M^\nu} \int_\Omega |f_j^M|^q \ . \tag{28}$$

Using (26) we therefore obtain

$$\int_\Omega |f|^{q-\nu} \geq \frac{1}{M^\nu} \left(1 - \left[S(E_A^{p,q} + 1/S_\delta^{p,q})\right]^q M^{(r-p)/pr}\right) \ , \tag{29}$$

and hence

$$\int_\Omega |f|^q = \lim_{\nu\to 0} \int_\Omega |f|^{q-\nu} \geq 1 - \left[S(E_A^{p,q} + 1/S_\delta^{p,q})\right]^q M^{(r-p)/pr} \ . \tag{30}$$

Since $M$ can be chosen arbitrarily large, and $p > r$, this shows that $\|f\|_{L^q(\Omega)} = 1$, implying strong convergence and finishing the proof. □

As might be expected, the proof of (7) can be simplified if one is not interested in the optimal $r$, but rather $r > \max\{1, qn/(n+q)\}$.

## 3. The special case $A = 0$

In the case of vanishing magnetic field $A = 0$, there is a much simpler proof of Theorem 3. In fact this theorem follows easily from Theorem 1, as we now show. However, this simple proof has the disadvantage of not yielding any information about the optimal constants.

THEOREM 4 (Generalized Poincaré inequalities for $A = 0$). *Let* $\Omega$, $g$, $p$, $q$ *be as in Theorem 1, and let* $\tilde{q}_n = \max\{1, qn/(n+q)\}$. *Let* $\Lambda \subset \Omega$ *be a measurable subset of* $\Omega$ *and let* $\Gamma = \Omega \setminus \Lambda$.

*There are constants* $S^{p,q}$ *(generally different from* $S_{p,q}$*), depending only on* $\Omega$, $g$, $p$ *and* $q$, *but not on* $\Lambda$, *such that for all* $f \in W^{1,p}(\Omega)$

$$\left\| f - \int_\Omega fg \right\|_{L^q(\Omega)} \leq S^{p,q} \Big[ \|\nabla f\|_{L^p(\Lambda)} + |\Omega|^{1/p - 1/\tilde{q}_n} \|\nabla f\|_{L^{\tilde{q}_n}(\Gamma)} \Big] \tag{31}$$

*if* $1 \leq q < \infty$ *and* $\tilde{q}_n \leq p \leq \infty$. *One can take* $S^{p,q} = S_{\tilde{q}_n,q} |\Omega|^{1/\tilde{q}_n - 1/p}$.

*For* $q = \infty$, *there exist constants* $\widehat{S}^{p,r}$ *such that*

$$\left\| f - \int_\Omega fg \right\|_{L^\infty(\Omega)} \leq \widehat{S}^{p,r} \Big[ \|\nabla f\|_{L^p(\Lambda)} + |\Omega|^{1/p - 1/r} \|\nabla f\|_{L^r(\Gamma)} \Big] \tag{32}$$

*for all* $n < r \leq p \leq \infty$. *Now,* $\widehat{S}^{p,r} = S_{p,\infty} |\Omega|^{1/r - 1/p}$.

As in Corollary 1, the application of Hölder's inequality to the rightmost norms in (31) and (32) displays the dependence on the volume of $|\Gamma|$. We obtain

Corollary 2 (Explicit volume dependence). *Under the assumptions of Theorem 4*

$$\left\| f - \int_\Omega fg \right\|_{L^q(\Omega)} \leq S^{p,q} \left[ \|\nabla f\|_{L^p(\Lambda)} + \left( \frac{|\Gamma|}{|\Omega|} \right)^{1/\tilde{q}_n - 1/p} \|\nabla f\|_{L^p(\Gamma)} \right] \tag{33}$$

*if* $1 \leq q < \infty$ *and* $\tilde{q}_n \leq p \leq \infty$.

*For* $q = \infty$,

$$\left\| f - \int_\Omega fg \right\|_{L^\infty(\Omega)} \leq \widehat{S}^{p,r} \left[ \|\nabla f\|_{L^p(\Lambda)} + \left( \frac{|\Gamma|}{|\Omega|} \right)^{1/r - 1/p} \|\nabla f\|_{L^p(\Gamma)} \right] \tag{34}$$

*for all* $n < r \leq p \leq \infty$.

*Remarks.* 1. As a special case, we can assume that $g(x) = 0$ for $x \in \Gamma$ in Corollary 2, and use the simple fact that $\|\cdot\|_{L^q(\Omega)} \geq \|\cdot\|_{L^q(\Lambda)}$ to obtain

$$\left\| f - \int_\Lambda fg \right\|_{L^q(\Lambda)} \leq S^{p,q} \left[ \|\nabla f\|_{L^p(\Lambda)} + \left( \frac{|\Gamma|}{|\Omega|} \right)^{1/\tilde{q}_n - 1/p} \|\nabla f\|_{L^p(\Gamma)} \right] \tag{35}$$

when $q < \infty$, and similarly for (32). The virtue of (35) is that it is an inequality that depends only on $\Lambda$, except for an error term. We emphasize again that the constants $S^{p,q}$ do not depend on $\Lambda$, but only on $\Omega$ and $g$.

2. Theorem 4 is a corollary of Theorem 1. This is in contrast to our general result, Theorem 3, which does not appear to follow easily from Theorem 1.

3. The optimal constant $S^{p,q}$ in Theorem 4 is left unspecified. This is in contrast to Theorem 3, where the constant appearing on the right side of (7) is optimal, up to an $\varepsilon$. The simple proof we shall give of Theorem 4, as a corollary of Theorem 1, does not allow us to relate $S^{p,q}$ to the optimal constant

for the usual Poincaré inequality for $\Lambda = \Omega$ (although we can relate it to $S_{\tilde{q}_n,q}$). Thus, even in the $A = 0$ case, the more complicated proof of Theorem 3 has the advantage of yielding information about the sharp constant.

4. In contrast to Theorem 3, Theorem 4 includes the critical case $p = \tilde{q}_n$. Note, however, that in this case Theorem 4 does not represent any improvement over Theorem 1.

*Proof of Theorem 4.* By Theorem 1 with $1 \le q < \infty$,

$$\left\| f - \int_\Omega fg \right\|_{L^q(\Omega)} \le S_{\tilde{q}_n,q} \|\nabla f\|_{L^{\tilde{q}_n}(\Omega)} \ . \tag{36}$$

We estimate the right side by the triangle inequality

$$\|\nabla f\|_{L^{\tilde{q}_n}(\Omega)} \le \|\nabla f\|_{L^{\tilde{q}_n}(\Lambda)} + \|\nabla f\|_{L^{\tilde{q}_n}(\Gamma)} \ . \tag{37}$$

Hölder's inequality implies that, for any $p \ge \tilde{q}_n$,

$$\|\nabla f\|_{L^{\tilde{q}_n}(\Lambda)} \le \|\nabla f\|_{L^p(\Lambda)} |\Lambda|^{1/\tilde{q}_n - 1/p} \ . \tag{38}$$

By the fact that $|\Lambda| \le |\Omega|$, this proves (31), with $S^{p,q} = S_{\tilde{q}_n,q}|\Omega|^{1/\tilde{q}_n - 1/p}$. The same proof works for $q = \infty$, with $\widehat{S}^{p,r} = S_{p,\infty}|\Omega|^{1/r-1/p}$. □

## 4. The special case $p = q = 2$

In the case $p = q = 2$, the ground state manifold $\mathcal{M}_A^{2,2}$ is a linear subspace of $L^2(\Omega)$, spanned by the eigenfunctions corresponding to the lowest (Neumann) eigenvalue of the operator $H = -(\nabla + iA)^2$. And Theorem 2 is just the statement that there is a gap above the lowest eigenvalue, which follows from the discreteness of the spectrum of $H$. The dimension of $\mathcal{M}_A^{2,2}$ is finite, but it can be strictly greater than one. These properties are shown in the appendix. This is in contrast to the case $A = 0$, where $\mathcal{M}_A^{2,2}$ is one-dimensional, spanned by the constant function.

We can use Theorem 3 in the case $p = q = 2$ to get an inequality of the form (6) that holds for all $f \in W^{1,2}(\Omega)$. For simplicity we state it for the case of $\mathcal{M}_A^{2,2}$ being one-dimensional. The proof is obtained by replacing $f$ in (8) by $f - \phi_A \int_\Omega fg$ and using the Cauchy-Schwarz inequality.

Corollary 3 (analogue of (6) for punctured domains). *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set that has the cone property, and let $E_A^{2,2}$ and $\mathcal{M}_A^{2,2}$ be as explained above. Let $\Lambda \subset \Omega$ be a measurable subset of $\Omega$, $\Gamma = \Omega \backslash \Lambda$, and let $0 < \delta \le 1$. Suppose that $\mathcal{M}_A^{2,2}$ is one-dimensional, spanned by the normalized eigenfunction $\phi_A$ corresponding to $E_A^{2,2}$. Let $g$ be an $L^2(\Omega)$ function satisfying*

$\int_\Omega g\phi_A = 1$. *For any* $\varepsilon > 0$ *there exists a positive constant* $C'$ *depending on* $\Omega$, $A$, $g$ *and* $\varepsilon$ (*but not on* $\Lambda$ *and* $\Gamma$) *such that, for every* $f \in W^{1,2}(\Omega)$,

$$\begin{aligned}(39)\quad \|(\nabla + iA)f\|^2_{L^2(\Lambda)} &+ C'\,|\Gamma|^{\min\{1,2/n\}}\|(\nabla + iA)f\|^2_{L^2(\Gamma)} \\ &+C'\left|\int_\Omega fg\right| \|(\nabla + iA)f\|_{L^2(\Gamma)}\|(\nabla + iA)\phi_A\|_{L^2(\Gamma)} \\ &\geq \left(E_A^{2,2}\right)^2\|f\|^2_{L^2(\Omega)} + \frac{1}{S^g+\varepsilon}\left\|f - \phi_A[\textstyle\int_\Omega fg]\right\|^2_{L^2(\Omega)},\end{aligned}$$

*with* $S^g$ *related to the optimal constant* $\widetilde{S}_g^{2,2}$ *in* (6) *by*

$$S^g = \left(S_g^{2,2}\right)^2\left(1 + 2E_A^{2,2}S_g^{2,2}\right)^{-1}.$$

*Remarks.* 1. By the same argument as in Remark 2 after Theorem 3 the exponent $2/n$ in (39) is sharp for $n \geq 2$; i.e., it cannot be increased. It is natural to conjecture that (39) holds with exponent $2 = 2/n$ also for $n = 1$. Unfortunately, the method of proof presented here does not allow for this generalization.

2. For regular enough boundary of $\Omega$ it follows from elliptic regularity that $(\nabla + iA)\phi_A$ is in fact a bounded function [8]. This allows us to replace $\|(\nabla + iA)\phi_A\|_{L^2(\Gamma)}$ by const. $|\Gamma|^{1/2}$. In any case, $\|(\nabla + iA)\phi_A\|_{L^2(\Gamma)}$ goes to zero as $|\Gamma| \to 0$.

3. As in Remark 1 after Theorem 4 one can consider the special case where $g$ vanishes on $\Gamma$ to obtain an inequality that depends on $\Gamma$ only via its volume. It has to be noted, however, that $E_A^{2,2}$ is defined on the whole of $\Omega$ and not just on $\Lambda$.

4. Strictly speaking, Corollary 3 is not really a corollary of Theorem 3 because of the optimal constant $S^g$ appearing in (39). From (8) we can only infer (39) with $S^g$ replaced by $S_\delta$, for some $\delta$ depending on $g$. However, by imitating the proof of Theorem 3 one can show that (39) holds.

## 5. Some open problems

- In Theorems 2 and 3 we have excluded the 'critical' case $p = \max\{1, qn/(n+q)\}$. In this case, the existence of minimizers of (3) and hence the nonemptiness of $\mathcal{M}_A^{p,q}$ is *a priori* not clear, except for the case $A = 0$, where $\mathcal{M}_A^{p,q}$ is trivially just the one-dimensional space spanned by the constant function.

- The optimal exponent in the dependence on the volume of $\Gamma$ in (8) (see Remark 2 after Corollary 1) is open for the cases $q = \infty$ and $q < n/(n-1)$. This comes from the fact that (8) is obtained as a corollary of (7), where necessarily $r \geq 1$.

• Since Theorem 3 is proved by a compactness argument, the constant $C$ appearing in (7) is left unspecified. It would be desirable to obtain a decent upper bound for this value. Also the optimal values of the constants $S_\delta^{p,q}$ appearing in Theorem 2 are in general unknown. Indeed, decent estimates of $S_{p,q}$ in Theorem 1 are not readily available when $p, q \neq 2$.

• The dimension of the ground state manifold $\mathcal{M}_A^{p,q}$ can be bigger that one. In the appendix we give an example for the case $p = q = 2$ where its dimension is two. It would be interesting to construct examples (or prove their existence) where the dimension can be arbitrarily large.

## Appendix: Spectrum of $-(\nabla + iA)^2$

Here we prove two facts about $-(\nabla + iA)^2$ which were used in the text. As before, we have a bounded, connected domain $\Omega$ in $\mathbb{R}^n$ with the cone property. Connectedness is not really necessary here, but the number of connected components should be finite. The vector field $A$ is bounded and measurable. The boundedness is not crucial but we assume it for simplicity.

We define the eigenvalues (spectrum) $E_k$ and eigenfunctions $\phi_k$ of $-(\nabla + iA)^2$ in $L^2(\Omega)$ by means of quadratic forms as in [1], i.e., $E_{k+1}$ is defined by

(39)
$$E_{k+1} = \inf \left\{ \|(\nabla + iA)f\|^2_{L^2(\Omega)} \,:\, \|f\|^2_{L^2(\Omega)} = 1, \int_\Omega f\overline{\phi_j} = 0 \text{ for } j = 1, \dots, k \right\} .$$

Then, by standard methods (using the Rellich-Kondrashov theorem), one shows that there is a minimizer for $E_{k+1}$, which is called $\phi_{k+1}$. In the text, $\sqrt{E_1}$ was called $E_A^{2,2}$.

The two facts are the following.

LEMMA 2 (Spectrum of $-(\nabla + iA)^2$).

A. *The spectrum is discrete*; *i.e., the number of eigenvalues less than any number $E$ is finite.*

B. *The multiplicity of $E_1$ can be greater than one.*

*Proof.* To prove A we suppose that there are infinitely many eigenvalues below $E$. If $\psi \in W^{1,2}(\Omega)$, with $\|\psi\|_{L^2(\Omega)} = 1$ and $\|(\nabla + iA)\psi\|_{L^2(\Omega)} \leq E^{1/2}$, then, since $A$ is bounded, $\|\nabla\psi\|_{L^2(\Omega)} \leq E^{1/2} + \|A\|_{L^\infty(\Omega)}$. Thus, the infinite sequence of functions $\phi_1$, $\phi_2, \dots$ is bounded in $W^{1,2}(\Omega)$. By the Rellich-Kondrashov theorem this sequence has a subsequence that converges strongly in $L^2(\Omega)$. This is impossible since the $\phi_i$'s are orthonormal (and hence $\|\phi_i - \phi_j\|_{L^2(\Omega)} = \sqrt{2}$ for $i \neq j$).

To prove B, consider the case in which $\Omega \subset \mathbb{R}^2$ is an annulus centered at the origin (or a cylinder in $\mathbb{R}^n$ based on an annulus in $\mathbb{R}^2$). We take $A$ to be (in polar coordinates) $A(r,\theta) = \lambda r^{-1}\widehat{e}_\theta$, where $\widehat{e}_\theta$ is the unit vector in the $\theta$ direction. We shall show that for suitable values of $\lambda$ the multiplicity of $E_1$ is two.

The Hilbert space $W^{1,2}(\Omega)$ is the direct sum of Hilbert spaces $W^{1,2}_l(\Omega)$, $l \in \mathbb{Z}$, consisting of functions of the form $\exp(-il\theta)g(r)$, and these subspaces continue to be orthogonal under the action of $\nabla + iA$. Thus, the eigenvectors in our case can be chosen to belong to exactly one of these subspaces. Thus, we can define $E_A(l)$ to be the lowest eigenvalue in $W^{1,2}_l(\Omega)$, and $E_1$ is then the minimum among the numbers $E_A(l)$. Since

$$-(\nabla + iA)^2 = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\left(\frac{\partial}{\partial\theta} + i\lambda\right)^2 \ , \tag{40}$$

$E_1 = E_A(l_0)$, where $l_0$ is the integer closest to $\lambda$. Therefore, for $\lambda \in \mathbb{Z}+\frac{1}{2}$, there are two eigenfunctions with the same eigenvalue $E_1 = E_A(\lambda-\frac{1}{2}) = E_A(\lambda+\frac{1}{2})$. □

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