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Y-systems and generalized associahedra

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To the memory of Rodica Simion

The goals of this paper are two-fold. First, we prove, for an arbitrary finite root system Φ, the periodicity conjecture of Al. B. Zamolodchikov [24] that concerns Y*-systems,* a particular class of functional relations playing an important role in the theory of thermodynamic Bethe ansatz. Algebraically, Y-systems can be viewed as families of rational functions defined by certain birational recurrences formulated in terms of the root system Φ. We obtain explicit formulas for these rational functions, which always turn out to be Laurent polynomials, and prove that they exhibit the periodicity property conjectured by Zamolodchikov.

In a closely related development, we introduce and study a simplicial complex $\Delta(\Phi)$, which can be viewed as a generalization of the Stasheff polytope (also known as *associahedron*) for an arbitrary root system Φ. In type A, this complex is the face complex of the ordinary associahedron, whereas in type B, our construction produces the Bott-Taubes polytope, or cyclohedron. We enumerate the faces of the complex $\Delta(\Phi)$, prove that its geometric realization is always a sphere, and describe it in concrete combinatorial terms for the classical types $ABCD$.

The primary motivation for this investigation came from the theory of *cluster algebras,* introduced in [9] as a device for studying dual canonical bases and total positivity in semisimple Lie groups. This connection remains behind the scenes in the text of this paper, and will be brought to light in a forthcoming sequel[1] to [9].

Contents

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[1] *Added in proof.* See S. Fomin and A. Zelevinsky, Cluster algebras II: Finite type classification, *Invent. Math.*, to appear.

1. Main results

Throughout this paper, I is an n-element set of indices, and $A = (a_{ij})_{i,j\in I}$ an indecomposable Cartan matrix of finite type; in other words, A is of one of the types $A_n, B_n, \ldots, G_2$ on the Cartan-Killing list. Let Φ be the corresponding root system (of rank n), and h the Coxeter number.

The first main result of this paper is the following theorem.

Theorem 1.1 (Zamolodchikov's conjecture). *A family $(Y_i(t))_{i\in I, t\in\mathbb{Z}}$ of commuting variables satisfying the recurrence relations*

$$Y_i(t+1)Y_i(t-1) = \prod_{j\neq i}(Y_j(t)+1)^{-a_{ij}} \tag{1.1}$$

is periodic with period $2(h+2)$; i.e., $Y_i(t+2(h+2)) = Y_i(t)$ for all i and t.

We refer to the relations (1.1) as the *Y-system* associated with the matrix A (or with the root system Φ). Y-systems arise in the theory of thermodynamic Bethe ansatz, as first shown by Al. B. Zamolodchikov [24]. The periodicity in Theorem 1.1 also was conjectured by Zamolodchikov [24] in the simply-laced case, i.e., when the product in the right-hand-side of (1.1) is square-free. The type A case of Zamolodchikov's conjecture was proved independently by E. Frenkel and A. Szenes [12] and by F. Gliozzi and R. Tateo [14]; the type D case was considered in [6]. This paper does not deal with Y-systems more general than (1.1), defined by pairs of Dynkin diagrams (see [19], [16], and [15]).

Our proof of Theorem 1.1 is based on the following reformulation. Recall that the Coxeter graph associated to a Cartan matrix A has the indices in I as vertices, with $i, j \in I$ joined by an edge whenever $a_{ij}a_{ji} > 0$. This graph is a tree, hence is bipartite. We denote the two parts of I by I_+ and I_-, and write $\varepsilon(i) = \varepsilon$ for $i \in I_\varepsilon$. Let $\mathbb{Q}(u)$ be the field of rational functions in the variables $u_i\,(i \in I)$. We introduce the involutive automorphisms τ_+ and τ_- of $\mathbb{Q}(u)$ by setting

$$\tau_\varepsilon(u_i) = \begin{cases} \dfrac{\prod_{j\neq i}(u_j+1)^{-a_{ij}}}{u_i} & \text{if } \varepsilon(i)=\varepsilon; \\ u_i & \text{otherwise.} \end{cases} \tag{1.2}$$

THEOREM 1.2. *The automorphism $\tau_-\tau_+$ of $\mathbb{Q}(u)$ is of finite order. More precisely, let $w_\circ$ denote the longest element in the Weyl group associated to A. Then the order of $\tau_-\tau_+$ is equal to $(h+2)/2$ if $w_\circ = -1$, and is equal to $h+2$ otherwise.*

Theorem 1.2 is essentially equivalent to Zamolodchikov's conjecture; here is why. First, we note that each equation (1.1) only involves the variables $Y_i(k)$ with a fixed "parity" $\varepsilon(i)\cdot(-1)^k$. We may therefore assume, without loss of generality, that our Y-system satisfies the condition

$$Y_i(k) = Y_i(k+1) \quad \text{whenever} \quad \varepsilon(i) = (-1)^k. \tag{1.3}$$

Combine (1.1) and (1.3) into

$$Y_i(k+1) = \begin{cases} \dfrac{\prod_{j\neq i}(Y_j(k)+1)^{-a_{ij}}}{Y_i(k)} & \text{if } \varepsilon(i)=(-1)^{k+1}; \\ Y_i(k) & \text{if } \varepsilon(i)=(-1)^k. \end{cases} \tag{1.4}$$

Then set $u_i = Y_i(0)$ for $i\in I$ and compare (1.2) with (1.4). By induction on k, we obtain $Y_i(k) = (\underbrace{\tau_-\tau_+\cdots\tau_\pm}_{k \text{ times}})(u_i)$ for all $k\in\mathbb{Z}_{\geq 0}$ and $i\in I$, establishing the claim. (Informally, the map $(\tau_-\tau_+)^m$ can be computed either by iterations "from within," i.e, by repeating the substitution of variables $\tau_-\tau_+$, or by iterations "from the outside," via the recursion (1.4).)

Example 1.3. *Type* A_2. Let Φ be the root system of type A_2, with $I = \{1,2\}$. Set $I_+ = \{1\}$ and $I_- = \{2\}$. Then

$$\tau_+(u_1) = \frac{u_2+1}{u_1}, \quad \tau_-\tau_+(u_1) = \frac{\frac{u_1+1}{u_2}+1}{u_1} = \frac{u_1+u_2+1}{u_1u_2},$$

etc. Continuing these calculations, we obtain the following diagram:

$$\begin{array}{ccccccccc} u_1 & \overset{\tau_+}{\longleftrightarrow} & \dfrac{u_2+1}{u_1} & \overset{\tau_-}{\longleftrightarrow} & \dfrac{u_1+u_2+1}{u_1u_2} & \overset{\tau_+}{\longleftrightarrow} & \dfrac{u_1+1}{u_2} & \overset{\tau_-}{\longleftrightarrow} & u_2\,. \\ \circlearrowleft & & & & & & & & \circlearrowleft \\ \tau_- & & & & & & & & \tau_+ \end{array} \tag{1.5}$$

Thus the map $\tau_-\tau_+$ acts by

$$\begin{array}{ccccc} u_1 & \longrightarrow & \dfrac{u_1+u_2+1}{u_1u_2} & \longrightarrow & u_2 \\ \uparrow & & & & \downarrow \\ \dfrac{u_2+1}{u_1} & & \longleftarrow & & \dfrac{u_1+1}{u_2} \end{array} \tag{1.6}$$

and has period $5 = h + 2$, as prescribed by Theorem 1.2. To compare, the Y-system recurrence (1.4) (which incorporates the convention (1.3)) has period $10 = 2(h + 2)$:

	$Y_i(0)$	$Y_i(1)$	$Y_i(2)$	$Y_i(3)$	$Y_i(4)$	$Y_i(5)$	$\cdots$	$Y_i(10)$
$i=1$	u_1	$\dfrac{u_2+1}{u_1}$	$\dfrac{u_2+1}{u_1}$	$\dfrac{u_1+1}{u_2}$	$\dfrac{u_1+1}{u_2}$	u_2	$\cdots$	u_1
$i=2$	u_2	u_2	$\dfrac{u_1+u_2+1}{u_1u_2}$	$\dfrac{u_1+u_2+1}{u_1u_2}$	u_1	u_1	$\cdots$	u_2

Let $\mathcal{Y}$ denote the smallest set of rational functions that contains all coordinate functions u_i and is stable under τ_+ and τ_-. (This set can be viewed as the collection of all distinct variables in a Y-system of the corresponding type.) For example, in type A_2,

$$\mathcal{Y} = \left\{u_1, u_2, \frac{u_2+1}{u_1}, \frac{u_1+1}{u_2}, \frac{u_1+u_2+1}{u_1u_2}\right\}$$

(see (1.5) and (1.6)). Our proof of Theorem 1.2 is based on establishing a bijective correspondence between the set $\mathcal{Y}$ and a certain subset $\Phi_{\geq -1}$ of the root system Φ; under this bijection, the involutions τ_+ and τ_- correspond to some piecewise-linear automorphisms of the ambient vector space of Φ, which exhibit the desired periodicity properties. To be more precise, let us define

$$\Phi_{\geq -1} = \Phi_{>0} \cup (-\Pi) \ ,$$

where $\Pi = \{\alpha_i : i \in I\} \subset \Phi$ is the set of simple roots, and $\Phi_{>0}$ the set of positive roots of Φ. The case A_2 of this definition is illustrated in Figure 1.

Let $Q = \mathbb{Z}\Pi$ be the root lattice, and $Q_\mathbb{R}$ its ambient real vector space. For $\alpha \in Q_\mathbb{R}$, we denote by $[\alpha : \alpha_i]$ the coefficient of α_i in the expansion of α in the basis Π. Let τ_+ and τ_- denote the piecewise-linear automorphisms of $Q_\mathbb{R}$ given by

$$[\tau_\varepsilon \alpha : \alpha_i] = \begin{cases} -[\alpha : \alpha_i] - \sum_{j \neq i} a_{ij} \max([\alpha : \alpha_j], 0) & \text{if } \varepsilon(i) = \varepsilon; \\ [\alpha : \alpha_i] & \text{otherwise.} \end{cases} \tag{1.7}$$

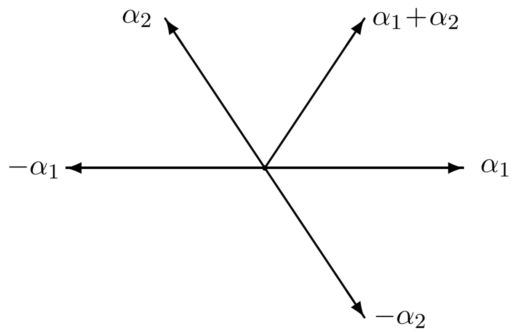


Figure 1. The set $\Phi_{\geq -1}$ in type A_2

The reason we use the same symbols for the birational transformations (1.2) and the piecewise-linear transformations (1.7) is that the latter can be viewed as the *tropical specialization* of the former. This means replacing the usual addition and multiplication by their tropical versions

$$a \oplus b = \max(a, b)\ , \quad a \odot b = a + b\ , \tag{1.8}$$

and replacing the multiplicative unit 1 by 0.

It is easy to show (see Proposition 2.4) that each of the maps $\tau_\pm$ defined by (1.7) preserves the subset $\Phi_{\geq -1}$.

THEOREM 1.4. *There exists a unique bijection $\alpha \mapsto Y[\alpha]$ between $\Phi_{\geq -1}$ and $\mathcal{Y}$ such that $Y[-\alpha_i] = u_i$ for all $i \in I$, and $\tau_\pm(Y[\alpha]) = Y[\tau_\pm(\alpha)]$ for all $\alpha \in \Phi_{\geq -1}$.*

Passing from $\mathcal{Y}$ to $\Phi_{\geq -1}$ and from (1.2) to (1.7) can be viewed as a kind of "linearization," with the important distinction that the action of $\tau_\pm$ in $Q_\mathbb{R}$ given by (1.7) is piecewise-linear rather than linear. This "tropicalization" procedure appeared in some of our previous work [2], [3], [9], although there it was the birational version that shed the light on the piecewise-linear one. In the present context, we go in the opposite direction: we first prove the tropical version of Theorem 1.2 (see Theorem 2.6), and then obtain the original version by combining the tropical one with Theorem 1.4.

In the process of proving Theorem 1.4, we find explicit expressions for the rational functions $Y[\alpha]$. It turns out that these functions exhibit the *Laurent phenomenon* (cf. [10]), that is, all of them are Laurent polynomials in the variables u_i. Furthermore, the denominators of these Laurent polynomials are all distinct, and are canonically in bijection with the elements of the set $\Phi_{\geq -1}$. More precisely, let $\alpha \mapsto \alpha^\vee$ denote the natural bijection between Φ and the dual root system $\Phi^\vee$, and let us abbreviate

$$u^{\alpha^\vee} = \prod_{i\in I} u_i^{[\alpha^\vee:\alpha_i^\vee]} .$$

THEOREM 1.5. *For every root* $\alpha \in \Phi_{\geq -1}$,

$$Y[\alpha] = \frac{N[\alpha]}{u^{\alpha^\vee}}, \tag{1.9}$$

where $N[\alpha]$ *is a polynomial in the* u_i *with positive integral coefficients and constant term* 1*.*

To illustrate Theorem 1.5: in type A_2, we have

$$\begin{array}{lll} Y[-\alpha_1] = u_1 = \dfrac{1}{u_1^{-1}}, & Y[\alpha_1] = \dfrac{u_2+1}{u_1}, & \\ & & Y[\alpha_1+\alpha_2] = \dfrac{u_1+u_2+1}{u_1u_2}. \\ Y[-\alpha_2] = u_2 = \dfrac{1}{u_2^{-1}}, & Y[\alpha_2] = \dfrac{u_1+1}{u_2}, & \end{array}$$

In any type, we have

$$Y[-\alpha_i] = u_i, \quad N[-\alpha_i] = 1,$$

$$Y[\alpha_i] = \tau_{\varepsilon(i)} u_i = \frac{\prod_{j\neq i}(u_j+1)^{-a_{ij}}}{u_i} , \quad N[\alpha_i] = \prod_{j\neq i}(u_j+1)^{-a_{ij}} .$$

Each numerator $N[\alpha]$ in (1.9) can be expressed as a product of "smaller" polynomials, which are also labeled by roots from $\Phi_{\geq -1}$. These polynomials are defined as follows.

THEOREM 1.6. *There exists a unique family* $(F[\alpha])_{\alpha\in\Phi_{\geq -1}}$ *of polynomials in the variables* $u_i (i \in I)$ *such that*

(i) $F[-\alpha_i] = 1$ *for all* $i \in I$;

(ii) *for any* $\alpha \in \Phi_{\geq -1}$ *and any* $\varepsilon \in \{+, -\}$,

$$\tau_\varepsilon(F[\alpha]) = \frac{\prod_{\varepsilon(i)=-\varepsilon}(u_i+1)^{[\alpha^\vee:\alpha_i^\vee]}}{\prod_{\varepsilon(i)=\varepsilon} u_i^{\max([\alpha^\vee:\alpha_i^\vee],0)}} \cdot F[\tau_{-\varepsilon}(\alpha)]. \tag{1.10}$$

Furthermore, each $F[\alpha]$ *is a polynomial in the* u_i *with positive integral coefficients and constant term* 1*.*

We call the polynomials $F[\alpha]$ described in Theorem 1.6 the *Fibonacci polynomials* of type Φ. The terminology comes from the fact that in the type A case, each of these polynomials is a sum of a Fibonacci number of monomials; cf. Example 2.15.

In view of Theorem 1.4, every root $\alpha \in \Phi_{\geq -1}$ can be written as

$$\alpha = \alpha(k;i) \stackrel{\text{def}}{=} (\tau_-\tau_+)^k(-\alpha_i) \tag{1.11}$$

for some $k \in \mathbb{Z}$ and $i \in I$.

THEOREM 1.7. *For* $\alpha = \alpha(k;i) \in \Phi_{\geq -1}$,

$$N[\alpha] = \prod_{j \neq i} F[\alpha(-k;j)]^{-a_{ij}} \ . \tag{1.12}$$

We conjecture that all polynomials $F[\alpha]$ are irreducible, so that (1.12) provides the irreducible factorization of $N[\alpha]$.

Among the theorems stated above, the core result, which implies the rest (see Section 2.3), is Theorem 1.6. This theorem is proved in Section 2.4 according to the following plan. We begin by reducing the problem to the simply-laced case by a standard "folding" argument. In the *ADE* case, the proof is obtained by explicitly writing the monomial expansions of the polynomials $F[\alpha]$ and checking that the polynomials thus defined satisfy the conditions in Theorem 1.6. This is done in two steps. First, we give a uniform formula for the monomial expansion of $F[\alpha]$ whenever $\alpha = \alpha^\vee$ is a positive root of "classical type," i.e., all the coefficients $[\alpha : \alpha_i]$ are equal to 0, 1, or 2 (see (2.21)). This in particular covers the A and D series of root systems. We compute the rest of the Fibonacci polynomials for the exceptional types E_6, E_7, and E_8 using `Maple` (see the last part of Section 2.4). In fact, the computational resources of `Maple` (on a 16-bit processor) turned out to be barely sufficient for handling the case of E_8; it seems that for this type, it would be next to impossible to prove Zamolodchikov's conjecture by direct calculations based on iterations of the recurrence (1.1).

We next turn to the second group of our results, which concern a particular simplicial complex $\Delta(\Phi)$ associated to the root system Φ. This complex has $\Phi_{\geq -1}$ as the set of vertices. To describe the faces of $\Delta(\Phi)$, we will need the notion of a *compatibility degree* $(\alpha \| \beta)$ of two roots $\alpha, \beta \in \Phi_{\geq -1}$. We define

$$(\alpha \| \beta) = [Y[\alpha] + 1]_{\text{trop}}(\beta), \tag{1.13}$$

where $[Y[\alpha]+1]_{\text{trop}}$ denotes the tropical specialization (cf. (1.8)) of the Laurent polynomial $Y[\alpha] + 1$, which is then evaluated at the n-tuple $(u_i = [\beta : \alpha_i])_{i \in I}$.

We say that two vertices α and β are *compatible* if $(\alpha \| \beta) = 0$. The compatibility degree can be given a simple alternative definition (see Proposition 3.1), which implies, somewhat surprisingly, that the condition $(\alpha \| \beta) = 0$ is symmetric in α and β (see Proposition 3.3). We then define the simplices of $\Delta(\Phi)$ as mutually compatible subsets of $\Phi_{\geq -1}$. The maximal simplices of $\Delta(\Phi)$ are called the *clusters* associated to Φ.

To illustrate, in type A_2, the values of $(\alpha\|\beta)$ are given by the table

	$-\alpha_1$	$-\alpha_2$	α_1	α_2	$\alpha_1+\alpha_2$
$-\alpha_1$	0	0	1	0	1
$-\alpha_2$	0	0	0	1	1
α_1	1	0	0	1	0
α_2	0	1	1	0	0
$\alpha_1+\alpha_2$	1	1	0	0	0

The clusters of type A_2 are thus given by the list

$$\{-\alpha_1, \alpha_2\},\ \{\alpha_2, \alpha_1+\alpha_2\},\ \{\alpha_1+\alpha_2, \alpha_1\},\ \{\alpha_1, -\alpha_2\}, \{-\alpha_2, -\alpha_1\}.$$

Note that these are exactly the pairs of roots represented by adjacent vectors in Figure 1.6.

THEOREM 1.8. *The complex $\Delta(\Phi)$ is pure of dimension $n-1$. In other words, all clusters are of the same size n. Moreover, each cluster is a $\mathbb{Z}$-basis of the root lattice Q.*

We obtain recurrence relations for the face numbers of $\Delta(\Phi)$, which enumerate simplices of any given dimension (see Proposition 3.7). In particular, we compute explicitly the total number of clusters.

THEOREM 1.9. *For a root system Φ of a Cartan-Killing type X_n, the total number of clusters is given by the formula*

$$N(X_n) = \prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1}\ , \tag{1.14}$$

where $e_1, \ldots, e_n$ are the exponents of Φ, and h is the Coxeter number.

Explicit expressions for the numbers $N(X_n)$ for all Cartan-Killing types X_n are given in Table 3 (Section 3). We are grateful to Frédéric Chapoton who observed that these expressions, which we obtained on a case by case basis, can be replaced by the unifying formula (1.14). F. Chapoton also brought to our attention that the numbers in (1.14) appear in the study of noncrossing and nonnesting partitions[2] by V. Reiner, C. Athanasiadis, and A. Postnikov [20], [1]. For the classical types A_n and B_n, a bijection between clusters and noncrossing partitions is established in Section 3.5.

We next turn to the geometric realization of $\Delta(\Phi)$. The reader is referred to [25] for terminology and basic background on convex polytopes.

[2] *Added in proof.* For a review of several other contexts in which these numbers arise, see C. A. Athanasiadis, On a refinement of the Catalan numbers for Weyl groups, preprint, March 2003.

THEOREM 1.10. *The simplicial cones* $\mathbb{R}_{\geq 0}C$ *generated by all clusters* C *form a complete simplicial fan in the ambient real vector space* $Q_{\mathbb{R}}$; *the interiors of these cones are mutually disjoint, and the union of these cones is the entire space* $Q_{\mathbb{R}}$.

COROLLARY 1.11. *The geometric realization of the complex* $\Delta(\Phi)$ *is an* $(n-1)$*-dimensional sphere.*

CONJECTURE 1.12.[3] *The simplicial fan in Theorem* 1.10 *is the normal fan of a simple* n*-dimensional convex polytope* $P(\Phi)$.

The type A_2 case is illustrated in Figure 2.

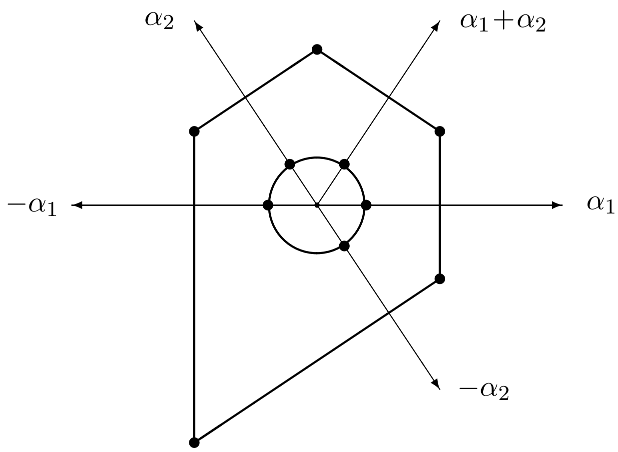


Figure 2. The complex $\Delta(\Phi)$ and the polytope $P(\Phi)$ in type A_2

The following is a weaker version of Conjecture 1.12.

CONJECTURE 1.13. *The complex* $\Delta(\Phi)$ *viewed as a poset under reverse inclusion is the face lattice of a simple* n*-dimensional convex polytope* $P(\Phi)$.

By the Blind-Mani theorem (see, e.g., [25, Section 3.4]), the face lattice of a simple polytope P is uniquely determined by the 1-skeleton (the edge graph) of P. In our situation, the edge graph $E(\Phi)$ of the (conjectural) polytope $P(\Phi)$ can be described as follows.

Definition 1.14. The *exchange graph* $E(\Phi)$ is an (unoriented) graph whose vertices are the clusters for the root system Φ, with two clusters joined by an edge whenever their intersection is of cardinality $n-1$.

[3] *Note added in revision.* This conjecture has been proved in [7].

The following theorem is a corollary of Theorem 1.10.

THEOREM 1.15. *For every cluster C and every element $\alpha \in C$, there is a unique cluster C' such that $C \cap C' = C - \{\alpha\}$. Thus, the exchange graph $E(\Phi)$ is regular of degree n: every vertex in $E(\Phi)$ is incident to precisely n edges.*

We describe the poset $\Delta(\Phi)$ and the exchange graph $E(\Phi)$ in concrete combinatorial terms for all classical types. This description in particular implies Conjecture 1.13 for types A_n and B_n; the posets $\Delta(\Phi)$ and $\Delta(\Phi^\vee)$ are canonically isomorphic, so that the statement for type C_n follows as well. For type A_n, the corresponding poset $\Delta(A_n)$ can be identified with the poset of polygonal subdivisions of a regular convex $(n+3)$-gon by noncrossing diagonals. This is known to be the face lattice of the *Stasheff polytope,* or *associahedron* (see [23], [17], [13, Ch. 7]). For type B_n, we identify $\Delta(B_n)$ with the sublattice of $\Delta(A_{2n-1})$ that consists of centrally symmetric polygonal subdivisions of a regular convex $2(n+1)$-gon by noncrossing diagonals. This is the face lattice of type B associahedron introduced by R. Simion (see [21, §5.2] and [22]). Simion's construction is combinatorially equivalent [8] to the "cyclohedron" complex of R. Bott and C. Taubes [4]. Polytopal realizations of the cyclohedron were constructed explicitly by M. Markl [18] and R. Simion [22].

Associahedra of types A and B have a number of remarkable connections with algebraic geometry [13], topology [23], knots and operads [4], [8], combinatorics [20], etc. It would be interesting to extend these connections to type D and the exceptional types.

The primary motivation for this investigation came from the theory of *cluster algebras,* which we introduced in [9] as a device for studying dual canonical bases and total positivity in semisimple Lie groups. This connection remains behind the scene in the text of this paper, and will be brought to light in a forthcoming sequel to [9].

The general layout of the paper is as follows. The material related to Y-systems is treated in Section 2; in particular, Theorems 1.2, 1.4, 1.5, 1.6, and 1.7 are proved there. Section 3 is devoted to the study of the complexes $\Delta(\Phi)$, including the proofs of Theorems 1.8, 1.9, and 1.10.

Acknowledgments. We are grateful to András Szenes for introducing us to Y-systems; to Alexander Barvinok, Satyan Devadoss, Mikhail Kapranov, Victor Reiner, John Stembridge, and Roberto Tateo for bibliographical guidance; and to Frédéric Chapoton for pointing out the numerological connection between $\Delta(\Phi)$ and noncrossing/nonnesting partitions.

Our work on the complexes $\Delta(\Phi)$ was influenced by Rodica Simion's beautiful construction [21], [22] of type B associahedra (see §3.5). We dedicate this paper to Rodica's memory.

2. Y-systems

2.1. *Root system preliminaries.* We start by laying out the basic terminology and notation related to root systems, to be used throughout the paper; some of it has already appeared in the introduction. In what follows, $A = (a_{ij})_{i,j \in I}$ is an indecomposable $n \times n$ Cartan matrix of finite type, i.e., one of the matrices $A_n, B_n, \ldots, G_2$ in the Cartan-Killing classification. Let Φ be the corresponding rank n root system with the set of simple roots $\Pi = \{\alpha_i : i \in I\}$. Let W be the Weyl group of Φ, and $w_\circ$ the longest element of W.

We denote by $\Phi^\vee$ the dual root system with the set of simple coroots $\Pi^\vee = \{\alpha_i^\vee : i \in I\}$. The correspondence $\alpha_i \mapsto \alpha_i^\vee$ extends uniquely to a W-equivariant bijection $\alpha \mapsto \alpha^\vee$ between Φ and $\Phi^\vee$. Let $\langle \alpha^\vee, \beta \rangle$ denote the natural pairing $\Phi^\vee \times \Phi \to \mathbb{Z}$. We adopt the convention $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$.

Let $Q = \mathbb{Z}\Pi$ denote the root lattice, $Q_+ = \mathbb{Z}_{\geq 0}\Pi \subset Q$ the additive semigroup generated by Π, and $Q_\mathbb{R}$ the ambient real vector space. For every $\alpha \in Q_\mathbb{R}$, we denote by $[\alpha : \alpha_i]$ the coefficient of α_i in the expansion of α in the basis of simple roots. In this notation, the action of simple reflections $s_i \in W$ in Q is given as follows:

$$[s_i\alpha : \alpha_{i'}] = \begin{cases} [\alpha : \alpha_{i'}] & \text{if } i' \neq i; \\ -[\alpha : \alpha_i] - \sum_{j \neq i} a_{ij}[\alpha : \alpha_j] & \text{if } i' = i. \end{cases} \tag{2.1}$$

The *Coxeter graph* associated to Φ has the index set I as the set of vertices, with i and j joined by an edge whenever $a_{ij}a_{ji} > 0$. Since we assume that A is indecomposable, the root system Φ is irreducible, and the Coxeter graph I is a tree. Therefore, I is a bipartite graph. Let I_+ and I_- be the two parts of I; they are determined uniquely up to renaming. We write $\varepsilon(i) = \varepsilon$ for $i \in I_\varepsilon$.

Let h denote the Coxeter number of Φ, i.e., the order of any Coxeter element in W. Recall that a Coxeter element is the product of all simple reflections s_i (for $i \in I$) taken in an arbitrary order. Our favorite choice of a Coxeter element t will be the following: take $t = t_- t_+$, where

$$t_\pm = \prod_{i \in I_\pm} s_i\,. \tag{2.2}$$

Note that the order of factors in (2.2) does not matter because s_i and s_j commute whenever $\varepsilon(i) = \varepsilon(j)$.

Let us fix some reduced words $\mathbf{i}_-$ and $\mathbf{i}_+$ for the elements t_- and t_+. (Recall that $\mathbf{i} = (\mathbf{i_1}, \ldots, \mathbf{i_l})$ is called a reduced word for $w \in W$ if $w = s_{i_1} \cdots s_{i_l}$ is a shortest-length factorization of w into simple reflections.)

Lemma 2.1 ([5, Exercise V.§6.2]). *The word*

$$\o ii_\circ \stackrel{\text{def}}{=} \underbrace{\mathbf{i}_-\mathbf{i}_+\mathbf{i}_- \cdots \mathbf{i}_\mp\mathbf{i}_\pm}_{h} \tag{2.3}$$

(*concatenation of* h *segments*) *is a reduced word for* $w_\circ$.

Regarding Lemma 2.1, recall that h is even for all types except A_n with n even; in the exceptional case of type A_{2e}, we have $h = 2e + 1$.

We denote by $\Phi_{>0}$ the set of positive roots of Φ, and let

$$\Phi_{\geq -1} = \Phi_{>0} \cup (-\Pi) \ .$$

2.2. *Piecewise-linear version of a Y-system.* For every $i \in I$, we define a *piecewise-linear modification* $\sigma_i : Q \to Q$ of a simple reflection s_i by setting

$$[\sigma_i \alpha : \alpha_{i'}] = \begin{cases} [\alpha : \alpha_{i'}] & \text{if } i' \neq i; \\ -[\alpha : \alpha_i] - \sum_{j \neq i} a_{ij} \max([\alpha : \alpha_j], 0) & \text{if } i' = i. \end{cases} \tag{2.4}$$

PROPOSITION 2.2. (1) *Each map $\sigma_i : Q \to Q$ is an involution.*

(2) *If i and j are not adjacent in the Coxeter graph, then σ_i and σ_j commute with each other. In particular, this is the case whenever $\varepsilon(i) = \varepsilon(j)$.*

(3) *Each map σ_i preserves the set $\Phi_{\geq -1}$.*

Proof. Parts 1 and 2 are immediate from the definition. To prove Part 3, notice that for every $i \in I$ and $\alpha \in \Phi_{\geq -1}$,

$$\sigma_i(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_j \neq -\alpha_i; \\ s_i(\alpha) & \text{otherwise.} \end{cases} \tag{2.5}$$

□

Example 2.3. In type A_2 (cf. Example 1.3), the actions of σ_1 and σ_2 on $\Phi_{\geq -1}$ are given by

$$\begin{array}{ccccccccc} -\alpha_1 & \overset{\sigma_1}{\longleftrightarrow} & \alpha_1 & \overset{\sigma_2}{\longleftrightarrow} & \alpha_1+\alpha_2 & \overset{\sigma_1}{\longleftrightarrow} & \alpha_2 & \overset{\sigma_2}{\longleftrightarrow} & -\alpha_2 \, . \\ \circlearrowright & & & & & & & & \circlearrowright \\ \sigma_2 & & & & & & & & \sigma_1 \end{array} \tag{2.6}$$

By analogy with (2.2), we introduce the piecewise-linear transformations τ_+ and τ_- of Q by setting

$$\tau_\pm = \prod_{i \in I_\pm} \sigma_i \, ; \tag{2.7}$$

this is well-defined in view of Proposition 2.2.2. This definition is of course equivalent to (1.7). The following properties are easily checked.

PROPOSITION 2.4. (1) *Both transformations τ_+ and τ_- are involutions and preserve $\Phi_{\geq -1}$.*

(2) *$\tau_\pm(\alpha) = t_\pm(\alpha)$ for any $\alpha \in Q_+$.*

(3) *The bijection $\alpha \mapsto \alpha^\vee$ between $\Phi_{\geq -1}$ and $\Phi^\vee_{\geq -1}$ is $\tau_\pm$-equivariant.*

It would be interesting to study the group of piecewise-linear transformations of $Q_{\mathbb{R}}$ generated by all the σ_i. In this paper, we focus our attention on the subgroup of this group generated by the involutions τ_- and τ_+. For $k \in \mathbb{Z}$ and $i \in I$, we abbreviate

$$\alpha(k;i) = (\tau_-\tau_+)^k(-\alpha_i)$$

(cf. (1.11)). In particular, $\alpha(0;i) = -\alpha_i$ for all i and $\alpha(\pm 1;i) = \alpha_i$ for $i \in I_\mp$.

Let $i \mapsto i^*$ denote the involution on I defined by $w_\circ(\alpha_i) = -\alpha_{i^*}$. It is known that this involution preserves each of the sets I_+ and I_- when h is even, and interchanges them when h is odd.

PROPOSITION 2.5. (1) *Suppose $h = 2e$ is even. Then the map $(k,i) \mapsto \alpha(k;i)$ restricts to a bijection*

$$[0,e] \times I \to \Phi_{\geq -1} .$$

Furthermore, $\alpha(e+1;i) = -\alpha_{i^*}$ for any i.

(2) *Suppose $h = 2e+1$ is odd. Then the map $(k,i) \mapsto \alpha(k;i)$ restricts to a bijection*

$$([0,e+1] \times I_-) \bigcup ([0,e] \times I_+) \to \Phi_{\geq -1} .$$

Furthermore, $\alpha(e+2;i) = -\alpha_{i^*}$ for $i \in I_-$, and $\alpha(e+1;i) = -\alpha_{i^*}$ for $i \in I_+$.

To illustrate Part 2 of Proposition 2.5, consider type A_2 (cf. (2.6)). Then $\tau_-\tau_+ = \sigma_2\sigma_1$ acts on $\Phi_{\geq -1}$ by

$$\begin{array}{ccccc} -\alpha_1 & \longrightarrow & \alpha_1+\alpha_2 & \longrightarrow & -\alpha_2 \\ \uparrow & & & & \downarrow \\ \alpha_1 & \longleftarrow & \longleftarrow & \longleftarrow & \alpha_2 . \end{array} \tag{2.8}$$

We thus have

$$\begin{array}{ll} \alpha(0;1) = -\alpha_1 & \alpha(0;2) = -\alpha_2 \\ \alpha(1;1) = \alpha_1+\alpha_2 & \alpha(1;2) = \alpha_2 \\ \alpha(2;1) = -\alpha_2 & \alpha(2;2) = \alpha_1 \\ \alpha(3;1) = \alpha_2 & \alpha(3;2) = -\alpha_1. \end{array} \tag{2.9}$$

To illustrate Part 1 of Proposition 2.5: in type A_3, with the standard numbering of roots, we have

$$\begin{array}{lll} \alpha(0;1) = -\alpha_1 & \alpha(0;2) = -\alpha_2 & \alpha(0;3) = -\alpha_3 \\ \alpha(1;1) = \alpha_1+\alpha_2 & \alpha(1;2) = \alpha_2 & \alpha(1;3) = \alpha_2+\alpha_3 \\ \alpha(2;1) = \alpha_3 & \alpha(2;2) = \alpha_1+\alpha_2+\alpha_3 & \alpha(2;3) = \alpha_1 \\ \alpha(3;1) = -\alpha_3 & \alpha(3;2) = -\alpha_2 & \alpha(3;3) = -\alpha_1. \end{array} \tag{2.10}$$

Proof. We shall use the following well-known fact: for every reduced word $\mathbf{i} = (\mathbf{i_1}, \ldots, \mathbf{i_m})$ of $w_\circ$, the sequence of roots $\alpha^{(k)} = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$, for $k = 1, 2, \ldots, m$, is a permutation of $\Phi_{>0}$ (in particular, $m = |\Phi_{>0}|$). Let $\mathbf{i} = \mathbf{i}_\circ$ be the reduced word defined in (2.3). Direct check shows that in the case when $h = 2e$ is even, the corresponding sequence of positive roots $\alpha^{(k)}$ has the form

$$\alpha(1;1), \ldots, \alpha(1;n), \alpha(2;1), \ldots, \alpha(2;n), \ldots, \alpha(e;1), \ldots, \alpha(e;n).$$

This implies the first statement in Part 1, and also shows that

$$\alpha(e;i) = \begin{cases} t_-(t_+t_-)^{e-1}(\alpha_i) & \text{if } i \in I_+; \\ (t_-t_+)^{e-1}(\alpha_i) & \text{if } i \in I_- \end{cases}$$

(recall that t_- and t_+ are defined by (2.2)). Then, for $i \in I_+$,

$$\begin{aligned} \alpha(e+1;i) &= \tau_-\tau_+t_-(t_+t_-)^{e-1}(\alpha_i) = \tau_-\tau_+t_-(t_+t_-)^{e-1}w_\circ(-\alpha_{i^*}) \\ &= \tau_-\tau_+t_+(-\alpha_{i^*}) = -\alpha_{i^*}\,, \end{aligned}$$

whereas for $i \in I_-$,

$$\begin{aligned} \alpha(e+1;i) &= \tau_-\tau_+(t_-t_+)^{e-1}(\alpha_i) = \tau_-\tau_+(t_-t_+)^{e-1}w_\circ(-\alpha_{i^*}) \\ &= \tau_-\tau_+t_+t_-(-\alpha_{i^*}) = -\alpha_{i^*}\,. \end{aligned}$$

This proves the second statement in Part 1.

The proof of Part 2 is similar. □

As an immediate corollary of Proposition 2.5, we obtain the following tropical version of Theorem 1.2. Let D denote the group of permutations of $\Phi_{\geq -1}$ generated by τ_- and τ_+.

THEOREM 2.6. (1) *Every D-orbit in $\Phi_{\geq -1}$ has a nonempty intersection with $-\Pi$. More specifically, the correspondence $\Omega \mapsto \Omega \cap (-\Pi)$ is a bijection between the D-orbits in $\Phi_{\geq -1}$ and the $\langle -w_\circ \rangle$-orbits in $(-\Pi)$.*

(2) *The order of $\tau_-\tau_+$ in D is equal to $(h+2)/2$ if $w_\circ = -1$, and is equal to $h+2$ otherwise. Accordingly, D is the dihedral group of order $(h+2)$ or $2(h+2)$.*

To illustrate, consider the case of type A_2 (cf. (2.6), (2.8)). Then D is the dihedral group of order 10, given by

$$D = \langle \tau_+, \tau_- : \tau_-^2 = \tau_+^2 = (\tau_-\tau_+)^5 = 1 \rangle\,.$$

2.3. *Theorem 1.6 implies Zamolodchikov's conjecture.* In this section, we show that Theorem 1.6 implies Theorems 1.1, 1.2, 1.4, 1.5, and 1.7. Thus, we assume the existence of a family of *Fibonacci polynomials* $(F[\alpha])_{\alpha \in \Phi_{\geq -1}}$ in the variables $u_i (i \in I)$ satisfying the conditions in Theorem 1.6.

As explained in the introduction, Theorem 1.1 is a corollary of Theorem 1.2. In turn, Theorem 1.2 is obtained by combining Theorem 1.4 with Theorem 2.6.

As for Theorems 1.4, 1.5 and 1.7, we are going to obtain them simultaneously, as parts of a single package. Namely, we will define the polynomials $N[\alpha]$ by (1.12), then define the Laurent polynomials $Y[\alpha]$ by (1.9), and then show that these $Y[\alpha]$ satisfy the conditions in Theorems 1.4.

Our first task is to prove that the correspondence $\alpha \mapsto N[\alpha]$ is well-defined, i.e., the right-hand side of (1.12) depends only on α, not on the particular choice of k and i such that $\alpha = \alpha(k;i)$. To this end, for every $k \in \mathbb{Z}$ and $i \in I$, let us denote

$$\Psi(k;i) = \{(\alpha(-k;j), -a_{ij}) : j \in I - \{i\}, a_{ij} \neq 0\} \subset \Phi_{\geq -1} \times \mathbb{Z}_{>0} \ .$$

This definition is given with (1.12) in view: note that the latter can be restated as

$$N[\alpha(k;i)] = \prod_{(\beta,d)\in\Psi(k;i)} F[\beta]^d. \tag{2.11}$$

Lemma 2.7. (1) *The set* $\Psi(k;i)$ *depends only on the root* $\alpha = \alpha(k;i)$; *hence it can and will be denoted by* $\Psi(\alpha)$.

(2) *For every* $\alpha \in \Phi_{\geq -1}$ *and every sign* ε,

$$\Psi(\tau_\varepsilon \alpha) = \{(\tau_{-\varepsilon}\beta, d) : (\beta, d) \in \Psi(\alpha)\} \ . \tag{2.12}$$

(3) *For every* $\alpha \in \Phi_{\geq -1}$,

$$\sum_{(\beta,d)\in\Psi(\alpha)} d\beta^\vee = t_-\alpha^\vee + t_+\alpha^\vee \ . \tag{2.13}$$

Proof. Parts 1 and 2 follow by routine inspection from Proposition 2.5. To prove Part 3, we first check that it holds for $\alpha = \mp\alpha_i$ for some $i \in I$. Indeed, we have

$$\Psi(\mp\alpha_i) = \{(\mp\alpha_j, -a_{ij}) : j \in I - \{i\}, a_{ij} \neq 0\} \ .$$

Therefore

$$t_-(\mp\alpha_i^\vee) + t_+(\mp\alpha_i^\vee) = \pm\alpha_i^\vee \mp (\alpha_i^\vee - \sum_{j\neq i} a_{ij}\alpha_j^\vee) = \sum_{(\beta,d)\in\Psi(\mp\alpha_i)} d\beta^\vee \ ,$$

as claimed. It remains to show that if (2.13) holds for some positive root α, then it also holds for $\tau_\pm\alpha$. To see this, we notice that, by Proposition 2.4.2, we have $t_\pm\beta^\vee = \tau_\pm\beta^\vee$ for $\beta \in \Phi_{\geq 0}$. Using (2.12), we then obtain

$$\sum_{(\beta,d)\in\Psi(\tau_\varepsilon\alpha)} d\beta^\vee = t_{-\varepsilon} \sum_{(\beta,d)\in\Psi(\alpha)} d\beta^\vee = \alpha^\vee + t_{-\varepsilon}t_\varepsilon\alpha^\vee = (t_- + t_+)\tau_\varepsilon\alpha^\vee\ ,$$

as desired. □

By Lemma 2.7.1, the polynomials $N[\alpha]$ are well defined by the formula

$$N[\alpha] = \prod_{(\beta,d)\in\Psi(\alpha)} F[\beta]^d$$

(cf. (2.11)), for every $\alpha \in \Phi_{\geq -1}$. We then set

$$Y[\alpha] = \frac{\prod_{(\beta,d)\in\Psi(\alpha)} F[\beta]^d}{u^{\alpha^\vee}}\ . \tag{2.14}$$

In particular, $Y[-\alpha_i] = u_i$ for all i. Since all the Laurent polynomials $Y[\alpha]$ defined by (2.14) have different denominators, we conclude that the correspondence $\alpha \mapsto Y[\alpha]$ is injective. To complete the proof of Theorems 1.4, 1.5 and 1.7, it remains to verify the relation $\tau_\pm(Y[\alpha]) = Y[\tau_\pm(\alpha)]$ for $\alpha \in \Phi_{\geq -1}$.

For any sign ε, we introduce the notation

$$C_\varepsilon(\beta) = \frac{\prod_{j\in I_{-\varepsilon}} (u_j+1)^{[\beta^\vee:\alpha_j^\vee]}}{\prod_{i\in I_\varepsilon} u_i^{\max([\beta^\vee:\alpha_i^\vee],0)}}$$

and use it to rewrite (1.10) as

$$\tau_\varepsilon(F[\alpha]) = C_\varepsilon(\alpha)F[\tau_{-\varepsilon}(\alpha)]\,.$$

Together with (2.14) and (2.12), this implies

$$\tau_\varepsilon(Y[\alpha]) = \frac{\prod_{(\beta,d)\in\Psi(\alpha)} C_\varepsilon(\beta)^d F[\tau_{-\varepsilon}(\beta)]^d}{\tau_\varepsilon(u^{\alpha^\vee})} = \frac{N[\tau_\varepsilon\alpha]}{\tau_\varepsilon(u^{\alpha^\vee})} \prod_{(\beta,d)\in\Psi(\alpha)} C_\varepsilon(\beta)^d.$$

Thus, it remains to verify the identity

$$\prod_{(\beta,d)\in\Psi(\alpha)} C_\varepsilon(\beta)^d = \frac{\tau_\varepsilon(u^{\alpha^\vee})}{u^{\tau_\varepsilon\alpha^\vee}}\ . \tag{2.15}$$

Using (1.2), (1.7), (2.1), and (2.2), we calculate the right-hand side of (2.15) as follows:

$$\begin{aligned}\frac{\tau_\varepsilon(u^{\alpha^\vee})}{u^{\tau_\varepsilon\alpha^\vee}} &= \frac{\prod_{i\in I_{-\varepsilon}} u_i^{[\alpha^\vee:\alpha_i^\vee]} \prod_{j\in I_{-\varepsilon}} (u_j+1)^{-\sum_{i\neq j} a_{ij}[\alpha^\vee:\alpha_i^\vee]}}{\prod_{i\in I} u_i^{[\tau_\varepsilon\alpha^\vee:\alpha_i^\vee]} \prod_{i\in I_\varepsilon} u_i^{[\alpha^\vee:\alpha_i^\vee]}} \\ &= \frac{\prod_{j\in I_{-\varepsilon}} (u_j+1)^{[t_-\alpha^\vee+t_+\alpha^\vee:\alpha_j^\vee]}}{\prod_{i\in I_\varepsilon} u_i^{[\alpha^\vee+\tau_\varepsilon\alpha^\vee:\alpha_i^\vee]}}\ .\end{aligned} \tag{2.16}$$

On the other hand, the left-hand side of (2.15) is given by

$$\prod_{(\beta,d)\in\Psi(\alpha)} C_\varepsilon(\beta)^d = \frac{\prod_{j\in I_{-\varepsilon}}(u_j+1)^{\sum_{(\beta,d)\in\Psi(\alpha)} d\,[\beta^\vee:\alpha_j^\vee]}}{\prod_{i\in I_\varepsilon} u_i^{\sum_{(\beta,d)\in\Psi(\alpha)} d\max([\beta^\vee:\alpha_i^\vee],0)}}\,. \tag{2.17}$$

The expressions (2.16) and (2.17) are indeed equal, for the following reasons. Their numerators are equal by virtue of (2.13). If α is a positive root, then the equality of denominators follows again from (2.13) (note that all the roots β are positive as well), whereas if $\alpha \in -\Pi$, then both denominators are equal to 1.

This completes the derivation of Theorems 1.4, 1.5 and 1.7 (which in turn imply Theorems 1.1 and 1.2) from Theorem 1.6.

Remark 2.8. The Laurent polynomial $Y[\alpha]+1$ has a factorization similar to the factorization of $Y[\alpha]$ given by (2.14):

$$Y[\alpha]+1 = \frac{F[\tau_+\alpha]F[\tau_-\alpha]}{\prod_{i\in I} u_i^{\max([\alpha^\vee:\alpha_i^\vee],0)}}\,. \tag{2.18}$$

This can be deduced from Theorem 1.6 by an argument similar to the one given above.

2.4. *Fibonacci polynomials.* In this section we prove Theorem 1.6, proceeding in three steps.

Step 1. *Reduction to the simply-laced case.* This is done by means of the well-known *folding* procedure—cf., e.g., [11, 1.87], although we use a different convention (see (2.20) below). Let $\tilde\Phi$ be a simply laced irreducible root system (i.e., one of type A_n, D_n, E_6, E_7, or E_8) with the index set $\tilde I$, the set of simple roots $\tilde\Pi$, etc. Suppose ρ is an automorphism of the Coxeter graph $\tilde I$ that preserves the parts $\tilde I_+$ and $\tilde I_-$. Let $I = \tilde I/\langle\rho\rangle$ be the set of ρ-orbits in $\tilde I$, and let $\pi : \tilde I \to I$ be the canonical projection. We denote by the same symbol π the projection of polynomial rings

$$\begin{array}{ccc} \mathbb{Z}[u_{\tilde i} : \tilde i \in \tilde I] & \longrightarrow & \mathbb{Z}[u_i : i \in I] \\ u_{\tilde i} & \longmapsto & u_{\pi(\tilde i)}. \end{array} \tag{2.19}$$

The "folded" Cartan matrix $A = (a_{ij})_{i,j\in I}$ is defined as follows: for $i \in I$, pick some $\tilde i \in \tilde I$ such that $\pi(\tilde i) = i$, and set $(-a_{ij})$ for $j \neq i$ to be the number of indices $\tilde j \in \tilde I$ such that $\pi(\tilde j) = j$, and $\tilde j$ is adjacent to $\tilde i$ in $\tilde I$. It is known (and easy to check) that A is of finite type, and that all non-simply laced indecomposable Cartan matrices can be obtained this way:

$$A_{2n-1} \to B_n\,, \quad D_{n+1} \to C_n\,, \quad E_6 \to F_4\,, \quad D_4 \to G_2\,. \tag{2.20}$$

The mapping $\tilde\Pi^\vee \to \Pi^\vee$ sending each $\alpha_{\tilde i}^\vee$ to $\alpha_{\pi(\tilde i)}^\vee$ extends by linearity to a surjection $\tilde\Phi^\vee \to \Phi^\vee$, which we will also denote by π. With even more abuse of notation, we also denote by π the surjection $\tilde{\Phi} \to \Phi$ such that $(\pi(\tilde{\alpha}))^\vee = \pi(\tilde{\alpha}^\vee)$. Note that ρ extends naturally to an automorphism of the root system $\tilde{\Phi}$, and the fibers of the projection $\pi : \tilde{\Phi} \to \Phi$ are the ρ-orbits on $\tilde{\Phi}$. Also, π restricts to a surjection $\tilde{\Phi}_{\geq -1} \to \Phi_{\geq -1}$, and we have $\pi \circ \tilde{\tau}_\pm = \tau_\pm \circ \pi$.

The following proposition follows at once from the above description.

PROPOSITION 2.9. *Suppose that a family of polynomials* $(F[\tilde{\alpha}])_{\tilde{\alpha} \in \tilde{\Phi}_{\geq -1}}$ *in the variables* $u_{\tilde{i}}\,(\tilde{i} \in \tilde{I})$ *satisfies the conditions in Theorem* 1.6 *for a simply laced root system* $\tilde{\Phi}$. *Let* Φ *be the "folding" of* $\tilde{\Phi}$, *as described above. Then the polynomials* $(F[\alpha])_{\alpha \in \Phi_{\geq -1}}$ *in the variables* $u_i\,(i \in I)$ *given by* $F[\alpha] = \pi(F[\tilde{\alpha}])$ (*cf.* (2.19)), *where* $\tilde{\alpha} \in \tilde{\Phi}_{\geq -1}$ *is any root such that* $\pi(\tilde{\alpha}) = \alpha$, *are well-defined, and satisfy the conditions in Theorem* 1.6.

Thus, it is enough to calculate the Fibonacci polynomials of types ADE, and verify that they have the desired properties. For the other types, these polynomials can be obtained by simply identifying the variables $u_{\tilde{i}}$ which fold into the same variable u_i.

Step 2. *Types A and D.* We will now give an explicit formula for the Fibonacci polynomials $F[\alpha]$ in the case when Φ is the root system of type A_n or D_n. Recall that these systems have the property that $[\alpha : \alpha_i] \leq 2$ for every $\alpha \in \Phi_{>0}$ and every $i \in I$. Let us fix a positive root α and abbreviate $a_i = [\alpha : \alpha_i]$. We call a vector $\gamma = \sum_i c_i \alpha_i$ of the root lattice α-*acceptable* if it satisfies the following three conditions:

(1) $0 \leq c_i \leq a_i$ for all i;

(2) $c_i + c_j \leq 2$ for any adjacent vertices i and j;

(3) there is no simple path (ordinary or closed) $(i_0, i_1, \cdots, i_m)$, $m \geq 1$, with $c_0 = c_1 = \ldots = c_m = 1$ and $a_0 = a_m = 1$.

In condition 3 above, by a simple path we mean any path in the Coxeter graph whose all vertices are distinct, except that we allow for $i_0 = i_m$.

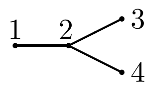
As before, we abbreviate $u^\gamma = \prod_i u_i^{c_i}$.

PROPOSITION 2.10. *Theorem* 1.6 *holds when* Φ *is of type* A_n *or* D_n. *In this case, for every positive root* $\alpha = \sum a_i \alpha_i$,

$$F[\alpha] = \sum_\gamma 2^{e(\gamma;\alpha)} u^\gamma \ , \tag{2.21}$$

where the sum is over all α-*acceptable* $\gamma \in Q$, *and* $e(\gamma;\alpha)$ *is the number of connected components of the set* $\{i \in I : c_i = 1\}$ *that are contained in* $\{i \in I : a_i = 2\}$.

To give one example, in type D_4 with the labeling



,

we have

$$F(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4) = u_1 u_3 u_4 + u_2^2 + \sum_{1 \le i < j \le 4} u_i u_j + \sum_{i \ne 2} u_i + 2u_2 + 1.$$

Proof. All we need to do is to verify that the polynomials given by (2.21) (together with $F[-\alpha_i] = 1$ for all $i \in I$) satisfy the relation (1.10) in Theorem 1.6.

Let us consider a more general situation. Let I be the vertex set of an arbitrary finite bipartite graph (without loops and multiple edges); we will write $i \leftrightarrow j$ to denote that two vertices $i, j \in I$ are adjacent to each other. Let Q be a free $\mathbb{Z}$-module with a chosen basis $(\alpha_i)_{i \in I}$. A vector $\alpha = \sum_{i \in I} a_i \alpha_i \in Q$ is called 2-*restricted* if $0 \le a_i \le 2$ for all $i \in I$.

LEMMA 2.11. *Let α be a* 2*-restricted vector, and let $F[\alpha]$ denote the polynomial in the variables u_i $(i \in I)$ defined by* (2.21)*. Then*

$$F[\alpha] = \sum_{\gamma} 2^{e(\gamma;\alpha)} u^{\gamma} \prod_{j \in I_-} (u_j + 1)^{\max(a_j - \sum_{i \leftrightarrow j} c_i, 0)} , \tag{2.22}$$

where the sum is over all α-acceptable integer vectors $\gamma = \sum c_i \alpha_i$ such that

$$c_j = 0 \text{ whenever } j \in I_- \text{ and } a_j > \textstyle\sum_{i \leftrightarrow j} c_i . \tag{2.23}$$

Proof. The proof is based on regrouping the summands in (2.21) according to the projection that is defined on the set of α-acceptable vectors γ as follows: it replaces each coordinate c_j that violates the condition (2.23) by 0.

The equivalence of (2.21) and (2.22) is then verified as follows. Suppose that $\gamma = \sum c_i \alpha_i$ is an α-acceptable integer vector. Suppose furthermore that $j \in I_-$ is such that $a_j > \sum_{i \leftrightarrow j} c_i$. It is easy to check that, once the values of a_j and $\sum_{i \leftrightarrow j} c_i$ have been fixed, the possible choices of c_j are determined as shown in the first three columns of Table 1. Comparison of the last two columns completes the verification. □

a_j	$\sum_{i\leftrightarrow j} c_i$	c_j	replacing c_j by 0 results in dividing $2^{e(\gamma;\alpha)}u^\gamma$ by:	$(u_j+1)^{\max(a_j-\sum_{i\leftrightarrow j} c_i,0)}$
1	0	0	1	$1+u_j$
1	0	1	u_j	
2	0	0	1	
2	0	1	$2u_j$	$(1+u_j)^2$
2	0	2	u_j^2	
2	1	0	1	$1+u_j$
2	1	1	u_j	

Table 1. Proof of Lemma 2.11

It will be convenient to restate Lemma 2.11 as follows. For an integer vector $\gamma_+ = \sum_{i\in I_+} c_i\alpha_i$ satisfying the condition

$$0 \le c_i \le a_i \text{ for all } i \in I_+\,, \tag{2.24}$$

we define the polynomial $H[\alpha : \gamma_+]$ in the variables u_j ($j \in I_-$) by

$$H[\alpha : \gamma_+] = \sum_{\gamma_-} 2^{e(\gamma_+ + \gamma_-;\alpha)} u^{\gamma_-}\ , \tag{2.25}$$

where the sum is over all vectors $\gamma_- = \sum_{j\in I_-} c_j\alpha_j$ such that $(\gamma_+ + \gamma_-)$ is α-acceptable, and $c_j = 0$ whenever $a_j > \sum_{i\leftrightarrow j} c_i$. Then

$$F[\alpha] = \sum_{\gamma_+} u^{\gamma_+} H[\alpha : \gamma_+] \prod_{j\in I_-} (u_j+1)^{\max(a_j - \sum_{i\leftrightarrow j} c_i, 0)}\ , \tag{2.26}$$

where the sum is over all integral vectors $\gamma_+ = \sum_{i\in I_+} c_i\alpha_i$ satisfying (2.24).

Lemma 2.12. *Suppose that both $\alpha = \sum_{i\in I} a_i\alpha_i$ and $\tau_-\alpha$ are 2-restricted. Denote $\alpha_+ = \sum_{i\in I_+} a_i\alpha_i$. Then $H[\alpha : \gamma_+] = H[\tau_-\alpha : \alpha_+ - \gamma_+]$ for any integral vector γ_+ satisfying* (2.24).

Proof. We will first prove that the sets of monomials u^{γ_-} that contribute to $H[\alpha : \gamma_+]$ and $H[\tau_-\alpha : \alpha_+ - \gamma_+]$ with positive coefficients are the same, and then check the equality of their coefficients. For the first task, we need to show that if integral vectors $\alpha = \sum_{i\in I} a_i\alpha_i$ and $\gamma = \sum_{i\in I} c_i\alpha_i$ satisfy the conditions

(0) $0 \le a_i \le 2$ for $i \in I$, and $0 \le -a_j + \sum_{i\leftrightarrow j} a_i \le 2$ for $j \in I_-$;

(1) $0 \le c_i \le a_i$ for $i \in I$;

(2) $c_i + c_j \le 2$ for any adjacent i and j;

(3) there is no simple path $(i_0, \dots, i_m)$, $m \ge 1$, with $c_0 = \cdots = c_m = 1$ and $a_0 = a_m = 1$;

(4) if $c_j > 0$ and $j \in I_-$, then $a_j \le \sum_{i\leftrightarrow j} c_i$,

then the same conditions are satisfied for the vectors $\tilde{\alpha} = \tau_- \alpha = \sum_{i \in I} \tilde{a}_i \alpha_i$ and $\tilde{\gamma} = \sum_{i \in I} \tilde{c}_i \alpha_i$, where

$$\tilde{a}_i = a_i \quad \text{for } i \in I_+; \qquad \tilde{c}_i = a_i - c_i \ \ \text{for } i \in I_+; \tag{2.27}$$
$$\tilde{a}_j = -a_j + \textstyle\sum_{i \leftrightarrow j} a_i \ \ \text{for } j \in I_-; \qquad \tilde{c}_j = c_j \ \ \text{for } j \in I_-.$$

We do not have to prove the converse implication since the map $(\alpha, \gamma) \mapsto (\tilde{\alpha}, \tilde{\gamma})$ is involutive.

Here is the crucial part of the required verification: we claim that conditions (0)–(4) leave only the five possibilities shown in Table 2 for the vicinity of a point $j \in I_-$ such that $c_j > 0$. (We omit the pairs of the form $(a_i, c_i) = (0, 0)$ since they will be of no relevance in the arguments below.) To prove this, we first note that by (4), $c_i \geq 1$ for some $i \leftrightarrow j$. It then follows from (2) that $c_j = 1$, and furthermore $c_i \leq 1$ for all $i \leftrightarrow j$. By (3), if $a_j = 1$, then $(a_i, c_i) \neq (1, 1)$ for $i \leftrightarrow j$; and if $a_j = 2$, then there is at most one index i such that $i \leftrightarrow j$ and $(a_i, c_i) = (1, 1)$. Combining (0), (1) and (4), we obtain the chain of inequalities

$$1 \leq c_j \leq a_j \leq \textstyle\sum_{i \leftrightarrow j} c_i \leq \sum_{i \leftrightarrow j} a_i \leq a_j + 2$$

which in particular implies that there can be at most two indices i such that $i \leftrightarrow j$ and $a_i > c_i$. An easy inspection now shows that all these restrictions combined do indeed leave only the five possibilities in Table 2.

(a_j, c_j)	$\big((a_i, c_i)\big)_{i \leftrightarrow j}$	$(\tilde{a}_j, \tilde{c}_j)$	$\big((\tilde{a}_i, \tilde{c}_i)\big)_{i \leftrightarrow j}$
(1,1)	(2,1)	(1,1)	(2,1)
(1,1)	(2,1), (1,0)	(2,1)	(2,1), (1,1)
(2,1)	(2,1), (1,1)	(1,1)	(2,1), (1,0)
(2,1)	(2,1), (2,1)	(2,1)	(2,1), (2,1)
(2,1)	(2,1), (1,1), (1,0)	(2,1)	(2,1), (1,0), (1,1)

Table 2. Proof of Lemma 2.12

Conversely, if we assume that α and γ satisfy (0) and $0 \leq c_i \leq a_i$ for $i \in I_+$, then the restrictions in Table 2 imply the rest of (1) as well as (2) and (4). These restrictions are evidently preserved under the transformation (2.27): compare the left and the right halves of Table 2. It only remains to show the following: if (α, γ) and $(\tilde{\alpha}, \tilde{\gamma})$ satisfy (0), (1), (2), (4) and the restrictions in Table 2 but $(\tilde{\alpha}, \tilde{\gamma})$ violates the only nonlocal condition (3), then (α, γ) must also violate (3).

Note that in each of the five cases, condition (3) is satisfied by $(\tilde{\alpha}, \tilde{\gamma})$ in the immediate vicinity of the vertex j. It follows that (3) could only be violated by a path that has at least one nonterminal vertex $i \in I_+$ (then necessarily $\tilde{a}_i = 2$

and $\tilde{c}_i = 1$). It is immediate from Table 2 that the segments of this path that lie between such vertices remain intact under the involution $(\alpha, \gamma) \leftrightarrow (\tilde{\alpha}, \tilde{\gamma})$; i.e., $a_i = \tilde{a}_i = 2$ and $c_i = \tilde{c}_i = 1$ holds throughout these segments. The possibilities at the ends of the path are then examined one by one with the help of Table 2; in each case, we verify that condition (3) is violated by (α, γ), and we are done.

To complete the proof of Lemma 2.12, we need to check the equality of the coefficients in the polynomials $H[\alpha : \gamma_+]$ and $H[\tau_-\alpha : \alpha_+ - \gamma_+]$. In other words, we need to show that (0)–(4) imply $e(\gamma; \alpha) = e(\tilde{\gamma}; \tilde{\alpha})$. (Recall that $e(\gamma; \alpha)$ was defined in Proposition 2.10.) For this, we note that $a_i = 2, c_i = 1$ is equivalent to $\tilde{a}_i = 2, \tilde{c}_i = 1$ for $i \in I_+$; and if a vertex $j \in I_-$ belongs to a connected component in question, then we must be in the situation described in row 4 of Table 2, so that the transformation $(\alpha, \gamma) \mapsto (\tilde{\alpha}, \tilde{\gamma})$ does not change the vicinity of j. □

With Lemmas 2.11 and 2.12 under our belt, the proof of Proposition 2.10 is now completed as follows. Interchanging I_+ and I_- if necessary, we find it suffices to check (1.10) with $\varepsilon = +$. Since (2.21) gives in particular $F[\alpha_i] = u_i + 1$, (1.10) checks for $\alpha = \pm\alpha_i$, i.e., when one of α or $\tau_-\alpha$ may be negative. It remains to verify (1.10) when $\varepsilon = +$, both roots α and $\tau_-\alpha$ are positive, and the polynomials $F[\alpha]$ and $F[\tau_-\alpha]$ are given by (2.21). Note also that when Φ is simply laced, the dual root system is canonically identified with Φ by a linear isomorphism of ambient spaces; thus we can replace each coefficient $[\alpha^\vee : \alpha_i^\vee]$ appearing in (1.10) by $[\alpha : \alpha_i] = a_i$. Using (2.26) and Lemma 2.12, we obtain:

$$\begin{aligned}
&\Big(\prod_{i\in I_+} u_i^{a_i}\Big)\Big(\prod_{j\in I_-}(u_j+1)^{-a_j}\Big)\,\tau_+(F[\alpha])\\
&\qquad = \sum_{\gamma_+} H[\alpha:\gamma_+]\Big(\prod_{i\in I_+} u_i^{a_i-c_i}\Big)\prod_{j\in I_-}(u_j+1)^{\max(0,-a_j+\sum_{i\leftrightarrow j} c_i)}\\
&\qquad = \sum_{\gamma_+} H[\tau_-\alpha:\alpha_+-\gamma_+]\,u^{\alpha_+-\gamma_+}\prod_{j\in I_-}(u_j+1)^{\max(\tilde{a}_j-\sum_{i\leftrightarrow j}\tilde{c}_i,0)}\\
&\qquad = F[\tau_-\alpha]
\end{aligned}$$

(here we used the notation of (2.27)). Proposition 2.10 is proved.

Remark 2.13. We note that formula (2.21) also holds for the Fibonacci polynomial $F[\alpha]$ of an arbitrary 2-restricted root α in an exceptional root system of type E_6, E_7, or E_8. The proof remains unchanged; the only additional ingredient is the statement, easily verifiable by a direct computation, that any such root can be obtained from a root of the form $-\alpha_i$, $i \in I$, by a sequence of transformations $\tau_\pm$, so that all intermediate roots are also 2-restricted.

Formula (2.21) becomes much simpler in the special case when a positive root α is *multiplicity-free*, i.e., $[\alpha : \alpha_i] \le 1$ for all i. Let us denote

$$\mathrm{Supp}(\alpha) = \{i \in I : [\alpha : \alpha_i] \ne 0\} \ . \tag{2.28}$$

We call a subset $\Omega \subset I$ *totally disconnected* if Ω contains no two indices that are adjacent in the Coxeter graph. As a special case of (2.21), we obtain the following.

PROPOSITION 2.14. *For a multiplicity-free positive root* α,

$$F[\alpha] = \sum_{\Omega} \prod_{i \in \Omega} u_i \ , \tag{2.29}$$

where the sum is over all totally disconnected subsets $\Omega \subset \mathrm{Supp}(\alpha)$.

Example 2.15. In the type A_n case, we can take $I = [1, n] = \{1, \dots, n\}$, with i and j adjacent whenever $|i - j| = 1$. Every positive root is multiplicity-free, and their supports are all intervals $[a, b] \subset [1, n]$. Thus, we have

$$F[\sum_{i \in [a,b]} \alpha_i] = F[a, b] = \sum_{\Omega \subset [a,b]} \prod_{i \in \Omega} u_i \ , \tag{2.30}$$

the sum over totally disconnected subsets $\Omega \subset [a, b]$. For example, for $n = 3$, the Fibonacci polynomials are given by

$$\begin{matrix} F[1,1] = u_1{+}1, \\ F[2,2] = u_2{+}1, \\ F[3,3] = u_3{+}1, \end{matrix} \quad \begin{matrix} F[1,2] = u_1{+}u_2{+}1, \\ F[2,3] = u_2{+}u_3{+}1, \end{matrix} \quad F[1,3] = u_1u_3{+}u_1{+}u_2{+}u_3{+}1.$$

When all the u_i are set to 1, the polynomials $F[a, b]$ specialize to Fibonacci numbers, which explains our choice of the name.

Step 3. *Exceptional types.* To complete the proof of Theorem 1.6, it remains to consider the types E_6, E_7 and E_8. In each of these cases we used (1.10) to recursively compute all Fibonacci polynomials $F[\alpha]$ with the help of a `Maple` program. Since some of the expressions involved are very large (for example, for the highest root $\alpha_{\max}$ in type E_8, the polynomial $F[\alpha_{\max}](u_1, \dots, u_8)$ has 26908 terms in its monomial expansion, and its largest coefficient is 3396), one needs an efficient way to organize these computations.

We introduce the variables $v_i = u_i + 1$, for $i \in I$. Suppose that the polynomial $F[\alpha]$, for some root $\alpha \in \Phi_{\ge -1}$, has already been computed. (As initial values, we can take $\alpha = -\alpha_i$, $i \in I$, with $F[-\alpha_i] = 1$.) For a sign ε, let us express $F[\alpha]$ as a polynomial in the variables

$$(u_i : i \in I_\varepsilon) \cup (v_i : i \in I_{-\varepsilon}). \tag{2.31}$$

In these variables, τ_ϵ becomes a (Laurent) monomial transformation; in particular, it does not change the number of terms in the monomial expansion of $F[\alpha]$. Using the recursion (1.10), rewritten in the form

$$F[\tau_{-\varepsilon}(\alpha)] = \frac{\prod_{\varepsilon(i)=\varepsilon} u_i^{[\alpha^\vee:\alpha_i^\vee]}}{\prod_{\varepsilon(i)=-\varepsilon} v_i^{[\alpha^\vee:\alpha_i^\vee]}} \cdot \tau_\varepsilon(F[\alpha]), \tag{2.32}$$

we compute $F[\tau_{-\varepsilon}(\alpha)]$ as a function, indeed a polynomial, in the variables (2.31). We then make the substitution $v_i \leftarrow u_i + 1$ for all $i \in I_{-\varepsilon}$ to express $F[\tau_{-\varepsilon}(\alpha)]$ in terms of the original variables $(u_i)_{i\in I}$, and record the result in our files. Next, we substitute $u_i \leftarrow v_i - 1$ for all $i \in I_\varepsilon$, thus expressing $F[\tau_{-\varepsilon}(\alpha)]$ as a polynomial in the variables

$$(u_i : i \in I_{-\varepsilon}) \cup (v_i : i \in I_\varepsilon).$$

We then reset $\varepsilon := -\varepsilon$ and $\alpha := \tau_{-\varepsilon}(\alpha)$, completing the loop. The steps described above in this paragraph are repeated until we arrive at $\alpha = -\alpha_i$, for some $i \in I$. Taking as initial values for α all possible negative simple roots, we can compute all polynomials $F[\alpha]$, $\alpha \in \Phi_{\geq -1}$.

To check the validity of Theorem 1.6 for a given root system Φ, we need to verify that

- every $F[\alpha]$ is a polynomial;
- when expressed in the variables $(u_i)_{i\in I}$, each $F[\alpha]$ has nonnegative integral coefficients and constant term 1;
- each time the process arrives at $\alpha = -\alpha_i$, it returns $F[\alpha] = 1$.

The algorithm described above does indeed verify these properties for the types E_6, E_7 and E_8. This completes the proof of Theorem 1.6.

3. Generalized associahedra

Throughout this section, we retain the terminology and notation from the previous sections, with one important exception: we drop the assumption that the Cartan matrix A is indecomposable. Thus, the corresponding (reduced finite) root system Φ is no longer assumed to be irreducible, and its Coxeter graph can be a forest, rather than a tree. We are in fact forced to pass to this more general setting because most of the proofs in this section are based on passing from Φ to a proper subsystem of Φ which may not be irreducible even if Φ is. For every subset $J \subset I$, let $\Phi(J)$ denote the root subsystem in Φ spanned by the set of simple roots $\{\alpha_i : i \in J\}$. If $I_1, \ldots, I_r$ are the connected components of I, then Φ is the disjoint union of irreducible root systems $\Phi(I_1), \ldots, \Phi(I_r)$, and all results of the previous sections extend in an obvious way to this more general setting. In particular, we can still subdivide I into the disjoint union of two totally disconnected subsets I_+ and I_- (by doing this independently for each connected component of I), and consider the corresponding piecewise-linear involutions τ_+ and τ_- of the set $\Phi_{\geq -1}$. Theorem 1.4 holds verbatim. We note that if $\alpha \in \Phi_{\geq -1}$ belongs to an irreducible subsystem $\Phi(I_k)$, then the corresponding Laurent polynomial $Y[\alpha]$ involves only variables u_i for $i \in I_k$.

3.1. *The compatibility degree.* We define the function

$$\begin{array}{ccc} \Phi_{\geq -1} \times \Phi_{\geq -1} & \rightarrow & \mathbb{Z}_{\geq 0} \\ (\alpha, \beta) & \mapsto & (\alpha \| \beta)\,, \end{array}$$

called the *compatibility degree* (of α and β), by

$$(\alpha\|\beta) = [Y[\alpha] + 1]_{\mathrm{trop}}(\beta)$$

(cf. (1.13)). As before, this notation means evaluating the "tropical specialization" of the Laurent polynomial $Y[\alpha]+1$ (obtained by replacing ordinary addition and multiplication by the operations (1.8)) at the tuple $(u_i = [\beta : \alpha_i])_{i\in I}$. By virtue of its definition, this function is uniquely characterized by the following two properties:

$$(-\alpha_i\|\beta) = \max\,([\beta : \alpha_i], 0), \tag{3.1}$$

$$(\tau_\varepsilon\alpha\|\tau_\varepsilon\beta) = (\alpha\|\beta), \tag{3.2}$$

for any $\alpha, \beta \in \Phi_{\geq -1}$, any $i \in I$, and any sign ε.

The next proposition gives an unexpectedly simple explicit formula for the compatibility degree. This formula involves the perfect bilinear pairing

$$\begin{array}{ccc} Q^\vee \times Q & \rightarrow & \mathbb{Z} \\ (\xi, \gamma) & \mapsto & \{\xi, \gamma\} \end{array}$$

(recall that Q is the root lattice, and $Q^\vee$ is the root lattice for the dual root system) defined by

$$\{\xi, \gamma\} = \sum_{i\in I} \varepsilon(i)[\xi : \alpha_i^\vee][\gamma : \alpha_i]\ . \tag{3.3}$$

Proposition 3.1. *The compatibility degree $(\alpha\|\beta)$ is given by*

$$(\alpha\|\beta) = \max(\{\alpha^\vee, \tau_+\beta\}, \{\tau_+\alpha^\vee, \beta\}, 0)\,. \tag{3.4}$$

Alternatively,

$$(\alpha\|\beta) = \max(-\{\tau_-\alpha^\vee, \beta\}, -\{\alpha^\vee, \tau_-\beta\}, 0)\ . \tag{3.5}$$

Proof. First let us show that (3.4) and (3.5) agree with each other, i.e., define the same function $\Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$. To do this, we note that the pairing $\{\cdot, \cdot\}$ satisfies the identity

$$\{\xi, t_\varepsilon \gamma\} = -\varepsilon \Big(\sum_{i \in I} \xi_i \gamma_i + \sum_{\varepsilon(i)=\varepsilon=-\varepsilon(j)} a_{ij} \xi_i \gamma_j \Big) = -\{t_{-\varepsilon}\xi, \gamma\} \tag{3.6}$$

for any sign ε, any $\xi \in Q^\vee$ and any $\gamma \in Q$, where we abbreviate $\xi_i = [\xi : \alpha_i^\vee]$ and $\gamma_i = [\gamma : \alpha_i]$ (this follows from (3.3) and (1.7)). Since $t_\pm$ agrees with $\tau_\pm$ on positive roots and coroots (see Proposition 2.4.2), we conclude that (3.4) and (3.5) agree with each other on $\Phi_{>0} \times \Phi_{>0}$. It remains to treat the case when at least one of α and β belongs to $-\Pi$. If say $\alpha = -\alpha_i$ with $i \in I_+$ then we have

$$\{\alpha^\vee, \tau_+\beta\} = [\beta : \alpha_i] + \sum_{j \neq i} a_{ij} \max([\beta : \alpha_j], 0) \leq [\beta : \alpha_i] = \{\tau_+\alpha^\vee, \beta\}$$

and

$$-\{\alpha^\vee, \tau_-\beta\} = [\beta : \alpha_i] = -\{\tau_-\alpha^\vee, \beta\};$$

thus, (3.4) and (3.5) agree in this case as well. The cases when $\alpha = -\alpha_i$ with $i \in I_-$, or $\beta \in -\Pi$ are handled in the same way.

To complete the proof of Proposition 3.1, note first that both (3.4) and (3.5) agree with (3.1) (we just demonstrated this for $\alpha = -\alpha_i$ with $i \in I_+$; the case $i \in I_-$ is similar). On the other hand, (3.4) (resp., (3.5)) makes it obvious that $(\alpha\|\beta)$ is τ_+-invariant (resp., τ_--invariant), and we are done. □

Remark 3.2. Comparing (3.4) and (3.5), we see that the compatibility degree $(\alpha\|\beta)$ does not depend on the choice of the sign function $\varepsilon : I \to \{\pm 1\}$.

The following proposition summarizes some properties of $(\alpha\|\beta)$.

PROPOSITION 3.3. (1) *$(\alpha\|\beta) = (\beta^\vee\|\alpha^\vee)$ for every $\alpha, \beta \in \Phi_{\geq -1}$. In particular, if Φ is simply laced, then $(\alpha\|\beta) = (\beta\|\alpha)$.*

(2) *If $(\alpha\|\beta) = 0$, then $(\beta\|\alpha) = 0$.*

(3) *If α and β belong to $\Phi(J)_{\geq -1}$ for some proper subset $J \subset I$ then their compatibility degree with respect to the root subsystem $\Phi(J)$ is equal to $(\alpha\|\beta)$.*

Proof. Parts (1) and (3) are immediate from (3.4); to verify (3) in the only nontrivial case where both α and β are positive roots, expand the terms in (3.4) using (3.6). To show (2), recall the following well-known property of root systems: there is a linear isomorphism between the dual root lattices Q and $Q^\vee$ under which every coroot $\alpha^\vee$ becomes a positive rational multiple of the corresponding root α. The definition (3.3) implies that under this identification, $\{\cdot, \cdot\}$ becomes a *symmetric* bilinear form on Q. It follows that $\{\alpha^\vee, \tau_+\beta\}$

and $\{\tau_+\beta^\vee, \alpha\}$ (resp., $\{\tau_+\alpha^\vee, \beta\}$ and $\{\beta^\vee, \tau_+\alpha\}$) are of the same sign. In view of (3.4), we conclude that

$$\begin{aligned}(\alpha\|\beta) = 0 \quad &\Leftrightarrow \quad \max(\{\alpha^\vee, \tau_+\beta\}, \{\tau_+\alpha^\vee, \beta\}) \le 0 \\ &\Leftrightarrow \quad \max(\{\tau_+\beta^\vee, \alpha\}, \{\beta^\vee, \tau_+\alpha\}) \le 0 \Leftrightarrow (\beta\|\alpha) = 0,\end{aligned}$$

as claimed. □

3.2. *Compatible subsets and clusters.* We say that two roots $\alpha, \beta \in \Phi_{\ge -1}$ are *compatible* if $(\alpha\|\beta) = 0$. In view of Proposition 3.3.2, the compatibility relation is symmetric. By (3.2), both τ_+ and τ_- preserve compatibility.

Definition 3.4. A subset of $\Phi_{\ge -1}$ is called *compatible* if it consists of mutually compatible elements. The (root) *clusters* associated to a root system Φ are the maximal (by inclusion) compatible subsets of $\Phi_{\ge -1}$.

The following proposition will allow us to establish properties of compatible subsets and clusters using induction on the rank $n = |I|$ of the root system.

PROPOSITION 3.5. (1) *Both τ_+ and τ_- take compatible subsets to compatible subsets and clusters to clusters.*

(2) *If $I_1, \ldots, I_r \subset I$ are the connected components of the Coxeter graph, then the compatible subsets* (*resp.*, *clusters*) *for Φ are the disjoint unions $C_1 \bigsqcup \cdots \bigsqcup C_r$, where each C_k is a compatible subset* (*resp.*, *cluster*) *for $\Phi(I_k)$.*

(3) *For every $i \in I$, the correspondence $C \mapsto C - \{-\alpha_i\}$ is a bijection between the set of all compatible subsets* (*resp.*, *clusters*) *for Φ that contain $-\alpha_i$ and the set of all compatible subsets* (*resp.*, *clusters*) *for $\Phi(I - \{i\})$.*

Proof. Part 1 follows from the fact that τ_+ and τ_- preserve compatibility. Part 2 follows from the fact that τ_+ and τ_- preserve each set $\Phi(I_k)_{\ge -1}$. Part 3 follows from (3.1). □

For a compatible subset C, we set

$$S_-(C) = \{i \in I : -\alpha_i \in C\} \ ,$$

and call $S_-(C)$ the *negative support* of C. We say that C is *positive* if $S_-(C) = \emptyset$, i.e., if C consists of positive roots only. The following proposition is obtained by iterating Proposition 3.5.3.

PROPOSITION 3.6. *For every subset $J \subset I$, the correspondence $C \mapsto C - \{-\alpha_i : i \in J\}$ is a bijection between the set of all compatible subsets* (*resp.*, *clusters*) *for Φ with negative support J and the set of all positive compatible subsets* (*resp.*, *clusters*) *for $\Phi(I - J)$.*

We are now ready to prove the purity property for clusters.

Proof of Theorem 1.8. We need to show that every cluster C for Φ is a $\mathbb{Z}$-basis of the root lattice Q. In view of Proposition 3.6, it suffices to prove this in the case when C is positive. Combining Propositions 3.5.1 and 2.4.2, we see that both collections $t_+(C)$ and $t_-(C)$ are also clusters. Obviously, each of them is a $\mathbb{Z}$-basis of Q if and only if this is true for C. Iterating this construction if necessary, we will arrive at a cluster C' which is no longer positive; this follows from Theorem 2.6.1. Now it suffices to prove that C' is a $\mathbb{Z}$-basis of Q. Again using Proposition 3.6, it is enough to show this for the positive part of C', which is a cluster for a root subsystem of smaller rank. Induction on the rank completes the proof. □

3.3. *Counting compatible subsets and clusters.* Let Φ be a root system of rank n. For $k = 0, \ldots, n$, let $f_k(\Phi)$ denote the number of compatible k-subsets of $\Phi_{\geq -1}$; in particular, $f_n(\Phi)$ is the number of clusters associated to Φ. We have $f_0(\Phi) = 1$, and $f_1(\Phi) = |\Phi_{\geq -1}|$; if Φ is irreducible, then the latter number is equal to $|\Phi_{\geq -1}| = n(h+2)/2$, where h is the Coxeter number of Φ. Let

$$f(\Phi) = \sum_{0 \leq k \leq n} f_k(\Phi) x^k$$

be the corresponding generating function.

Proposition 3.7. (1) $f(\Phi_1 \times \Phi_2) = f(\Phi_1) f(\Phi_2)$.

(2) *If* Φ *is irreducible, then, for every* $k \geq 1$,

$$f_k(\Phi) = \frac{h+2}{2k} \sum_{i \in I} f_{k-1}(\Phi(I - \{i\}))\,.$$

Equivalently,

$$\frac{df(\Phi)}{dx} = \frac{h+2}{2} \sum_{i \in I} f(\Phi(I - \{i\}))\ .$$

Proof. Part 1 follows at once from Proposition 3.5.2. To prove Part 2, we count in two different ways the number of pairs (α, S), where $S \subset \Phi_{\geq -1}$ is a compatible k-subset, and $\alpha \in S$. On one hand, the number of pairs in question is $k f_k(\Phi)$. On the other hand, combining Proposition 2.5 and Theorem 2.6, formula (3.1), and Proposition 3.5 (Parts 1 and 3), we conclude that the roots α belonging to each D-orbit Ω in $\Phi_{\geq -1}$ contribute

$$\frac{h+2}{2} \sum_{i \in I : -\alpha_i \in \Omega} f_{k-1}(\Phi(I - \{i\}))$$

to the count, implying the claim. □

Proposition 3.7 provides a recursive way to compute the numbers $f_k(\Phi)$. It can also be used to obtain explicit formulas for $f_n(\Phi)$, the total number of clusters. When Φ is of some Cartan-Killing type X_n, we shall write $N(X_n)$ instead of $f_n(\Phi)$.

PROPOSITION 3.8. *For every irreducible root system* Φ, *say of type* X_n, *the corresponding number of clusters* $N(X_n)$ *is as in Table* 3.

X_n	A_n	B_n, C_n	D_n	E_6	E_7	E_8	F_4	G_2
$N(X_n)$	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	833	4160	25080	105	8

Table 3. Counting the clusters

Since the number of clusters in type A_n is a Catalan number, the numbers $N(X_n)$ can be regarded as generalizations of the Catalan numbers to arbitrary Dynkin diagrams. Another generalization appears in Proposition 3.9, Table 4 below.

Proof. We will present the proof for the type A_n; other types are treated in a similar way. Let us abbreviate $a_n = N(A_n)$. We need to show that $a_n = c_{n+1}$, where $c_k = \frac{1}{k+1}\binom{2k}{k}$ denotes the kth Catalan number.

Proposition 3.7 produces the recursion

$$a_n = \frac{n+3}{2n}\sum_{i=1}^{n} a_{i-1}a_{n-i}\,, \tag{3.7}$$

for $n \geq 1$. All we need to do is to check that the sequence $a_n = c_{n+1}$ satisfies (3.7), together with the initial condition $a_0 = 1$. In other words, we need to show that the Catalan numbers satisfy

$$c_{n+1} = \frac{n+3}{2n}\sum_{i=1}^{n} c_i c_{n+1-i}\,.$$

With the help of the well-known identity $\sum_{i=0}^{n+1} c_i c_{n+1-i} = c_{n+2}$, this is transformed into $c_{n+1} = \frac{n+3}{2n}(c_{n+2}-2c_{n+1})$, which is easily checked using the formula for c_n. □

Proof of Theorem 1.9. Formula (1.14) follows from Proposition 3.8 by a case by case inspection (the definition of the exponents e_i and their values for all irreducible root systems can be found in [5]). □

The appearance of exponents in the formula (1.14) for the number of clusters is a mystery to us at the moment. To add to this mystery, a similar expression can be given for the number $N^+(X_n)$ of *positive* clusters.

PROPOSITION 3.9. *For every Cartan-Killing type X_n, the number of positive clusters is given by*

$$N^+(X_n) = \prod_{i=1}^{n} \frac{e_i + h - 1}{e_i + 1} \ , \tag{3.8}$$

where $e_1, \ldots, e_n$ are the exponents of the root system of type X_n, and h is the Coxeter number. Explicit formulas are given in Table 4.

X_n	A_n	B_n, C_n	D_n	E_6	E_7	E_8	F_4	G_2
$N^+(X_n)$	$\frac{1}{n+1}\binom{2n}{n}$	$\binom{2n-1}{n}$	$\frac{3n-4}{n}\binom{2n-3}{n-1}$	418	2431	17342	66	5

Table 4. Counting the positive clusters

Proof. For each subset $J \subset I$, let $N(J)$ (resp., $N^+(J)$) denote the number of clusters (resp., positive clusters) for the root system $\Phi(J)$. By Proposition 3.6, we have

$$N(I) = \sum_{J \subset I} N^+(J) \, . \tag{3.9}$$

By the inclusion-exclusion principle, this implies

$$N^+(I) = \sum_{J \subset I} (-1)^{|I-J|} N(J) \, . \tag{3.10}$$

Substituting into the right-hand side the data from Table 3, we can calculate $N^+(X_n)$ for all types, and verify the formula (3.8) by a case by case inspection. To illustrate, consider the case of type A_n (other cases are treated in a similar way). In that case, we can take $I = [1, n]$, and (3.10) becomes

$$N^+(I) = \sum_{J \subset [1,n]} (-1)^{|I-J|} c_J \, , \tag{3.11}$$

where c_J denotes the product of the Catalan numbers $c_{|J_i|+1}$ over all connected components J_i of J. Let us encode each $J \subset [1, n]$, say of cardinality $n - k$, by a sequence $(j_0, \ldots, j_k)$ of positive integers adding up to $n + 1$, as follows:

$$[1, n] - J = \{ \, j_0 \, , \, j_0 + j_1 \, , \, \ldots \, , \, j_0 + \cdots + j_{k-1} \, \}.$$

Then $c_J = c_{j_0} \cdots c_{j_k}$, and therefore (3.11) can be rewritten as

$$N^+([1, n]) = \sum_{k=0}^{n} (-1)^k \sum_{\substack{j_0 + \ldots + j_k = n+1 \\ j_i > 0}} c_{j_0} \cdots c_{j_k} \, .$$

Let $C = C(t) = \sum_{j\geq 0} c_j t^j = 1 + t + 2t^2 + \cdots$ be the generating function for the Catalan numbers; it is uniquely determined by the equation $C = 1 + tC^2$. We have

$$\sum_{n\geq 0} N^+([1,n])t^n = -t^{-1}\sum_{k\geq 0}(1-C)^{k+1} = \frac{C-1}{tC} = C\,;$$

so that $N^+([1,n]) = c_n$, as needed. □

Needless to say, it would be nice to find a unified explanation of (1.14) and (3.8).

3.4. *Cluster expansions.*

Definition 3.10. A *cluster expansion* of a vector γ in the root lattice Q is a way to express γ as

$$\gamma = \sum_{\alpha\in\Phi_{\geq -1}} m_\alpha \alpha\,, \tag{3.12}$$

where all m_α are nonnegative integers, and $m_\alpha m_\beta = 0$ whenever α and β are not compatible. In other words, a cluster expansion is an expansion into a sum of pairwise compatible roots in $\Phi_{\geq -1}$.

Theorem 3.11. *Every element of the root lattice has a unique cluster expansion.*

Proof. Our proof follows the same strategy as the proof of Theorem 1.8 given in Section 3.2, although this time, we need a little bit more preparation. For $\gamma \in Q$, set

$$S_+(\gamma) = \{i \in I : [\gamma : \alpha_i] > 0\},$$
$$S_-(\gamma) = \{i \in I : [\gamma : \alpha_i] < 0\}\,.$$

In particular, for a positive root α, we have $S_+(\alpha) = \mathrm{Supp}(\alpha)$ (cf. (2.28)).

The following lemma is an easy consequence of (3.1).

Lemma 3.12. *Suppose that $\alpha \in \Phi_{\geq -1}$ occurs in a cluster expansion of γ; that is, $m_\alpha > 0$ in* (3.12). *Then either α is a positive root with* $\mathrm{Supp}(\alpha) \subset S_+(\gamma)$, *or else $\alpha = -\alpha_i$ for some $i \in S_-(\gamma)$. In particular, if $\gamma \in Q_+$, then a cluster expansion of γ may only involve positive roots.*

Let us denote

$$\gamma^{(+)} = \sum_{i\in I} \max([\gamma:\alpha_i],0)\alpha_i = \sum_{i\in S_+(\gamma)} [\gamma:\alpha_i]\alpha_i\ .$$

The next lemma follows at once from Lemma 3.12.

LEMMA 3.13. *A vector $\gamma \in Q$ has a unique cluster expansion if and only if $\gamma^{(+)}$ has a unique cluster expansion with respect to the root system $\Phi(S_+(\gamma))$.*

In view of Lemma 3.13, it suffices to prove Theorem 3.11 for $\gamma \in Q_+$. Lemma 3.12 implies in particular that the statement holds for $\gamma = 0$, and so we can assume that $\gamma \neq 0$. We can also assume without loss of generality that Φ is irreducible (cf. Proposition 3.5.2). Let $\varepsilon \in \{+,-\}$. Combining Propositions 2.4.2 and 3.5.1, we conclude that

$$\gamma = \sum_{\alpha \in \Phi_{\geq -1}} m_\alpha \alpha$$

is a cluster expansion of γ if and only if

$$t_\varepsilon \gamma = \sum_{\alpha \in \Phi_{\geq -1}} m_\alpha \tau_\varepsilon \alpha$$

is a cluster expansion of $t_\varepsilon \gamma$. Thus, γ has a unique cluster expansion if and only if $t_\varepsilon \gamma$ has.

To complete the proof, note that some product of the transformations t_+ and t_- must move γ outside Q_+. Indeed, $w_\circ \gamma \in -Q_+$, and $w_\circ$ can be written as such a product by Lemma 2.1. Using this fact, we can assume without loss of generality that already say $t_+\gamma \notin Q_+$ (while $\gamma \in Q_+$). By Lemma 3.13, the statement that γ has a unique cluster expansion follows from the same property for the vector $(t_+\gamma)^{(+)}$, which lies in a root lattice of smaller rank. The proof of Theorem 3.11 is now completed by induction on the rank. □

Proof of Theorem 1.10. Theorem 1.10 is essentially a direct corollary of Theorem 3.11. We need to show two things:

(i) No two of the cones $\mathbb{R}_{\geq 0} C$ generated by clusters have a common interior point;

(ii) The union of these cones is the entire space $Q_\mathbb{R}$.

To show (i), assume on the contrary that two of the cones $\mathbb{R}_{\geq 0} C$ have a common interior point. Since the rational vector space $Q_\mathbb{Q}$ is dense in $Q_\mathbb{R}$, it follows that there is a common interior point in $Q_\mathbb{Q}$. Multiplying if necessary by a suitable positive integer, we conclude that there is also a common interior point in Q. Since we have already proved that every cluster is a $\mathbb{Z}$-basis of Q, the latter statement contradicts the uniqueness of a cluster expansion in Theorem 3.11.

To show (ii), note that the existence of a cluster expansion in Theorem 3.11 implies that the union of the cones $\mathbb{R}_{\geq 0} C$ contains Q. Since this union is closed in $Q_\mathbb{R}$ and is stable under multiplication by positive real numbers, it is the entire space $Q_\mathbb{R}$, and we are done. □

3.5. *Compatible subsets and clusters for the classical types.*

Type A_n. We use the standard labeling of the simple roots by the set $I = [1, n] = \{1, \dots, n\}$. Thus, the Coxeter graph is the chain with the vertices $1, \dots, n$, and the positive roots are given by $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$, for $1 \le i \le j \le n$. For $i \in [1, n]$, we set $\varepsilon(i) = (-1)^{i-1}$.

The cardinality of the set $\Phi_{\ge -1}$ is equal to

$$\ell(w_\circ) + n = \frac{n(n+1)}{2} + n = \frac{n(n+3)}{2} .$$

This number is also the cardinality of the set of diagonals in a convex $(n+3)$-gon. We shall identify these two sets as follows.

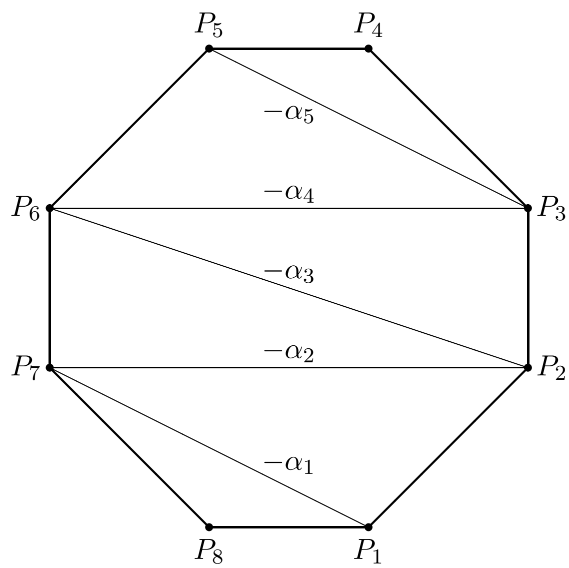


Figure 3. The "snake" in type A_5

Let $P_1, \dots, P_{n+3}$ be the vertices of a regular convex $(n+3)$-gon, labeled counterclockwise. For $1 \le i \le \frac{n+1}{2}$, we identify the root $-\alpha_{2i-1} \in \Phi_{\ge -1}$ with the diagonal $[P_i, P_{n+3-i}]$; for $1 \le i \le \frac{n}{2}$, we identify the root $-\alpha_{2i}$ with the diagonal $[P_{i+1}, P_{n+3-i}]$. These diagonals form a "snake" shown in Figure 3. To identify the remaining diagonals (not belonging to the snake) with positive roots, we associate each α_{ij} with the unique diagonal that crosses the diagonals $-\alpha_i, -\alpha_{i+1}, \dots, -\alpha_j$ and does not cross any other diagonal $-\alpha_k$ on the snake. (From this point on, we shall use the term "cross" to mean "intersect inside the polygon.") See Figure 4.

Under the identification described above, all notions related to compatible subsets and clusters of type A_n can be translated into the language of plane geometry. The following proposition is checked by direct inspection.

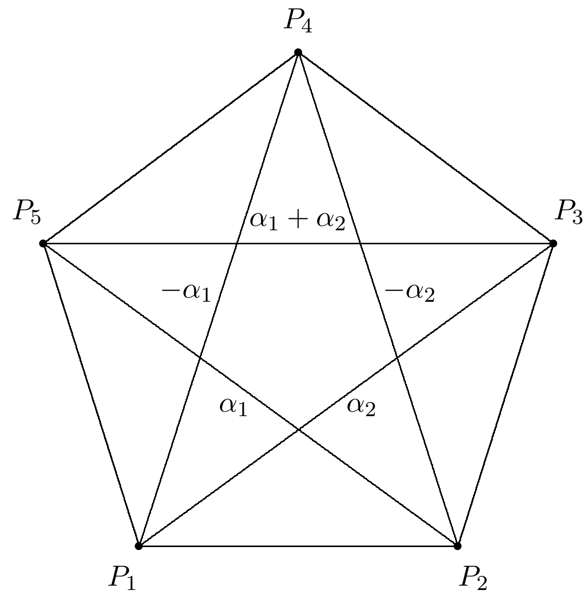


Figure 4. Labelling of diagonals in type A_2

PROPOSITION 3.14. *Let Φ be a root system of type A_n.*

(1) *The transformation τ_+ (resp., τ_-) acts in $\Phi_{\geq -1}$ by the orthogonal reflection of the $(n+3)$-gon that sends each vertex P_i to P_{n+4-i} (resp., to P_{n+3-i}, with the convention $P_0 = P_{n+3}$). Thus, $\tau_-\tau_+$ (resp., $\tau_+\tau_-$) acts by clockwise (resp., counter-clockwise) rotation by $\frac{2\pi}{n+3}$.*

(2) *For $\alpha, \beta \in \Phi_{\geq -1}$, the compatibility degree $(\alpha\|\beta)$ is equal to 1 if the diagonals α and β cross each other, and 0 otherwise.*

(3) *Compatible sets are collections of mutually noncrossing diagonals. Thus, clusters are in bijection with triangulations of the $(n+3)$-gon by noncrossing diagonals (and therefore with noncrossing partitions of $[1, n{+}1]$; see, e.g., [21, 5.1]).*

(4) *Two triangulations are joined by an edge in the exchange graph if and only if they are obtained from each other by a flip which replaces a diagonal in a quadrilateral formed by two triangles of the triangulation by another diagonal of the same quadrilateral.*

(Recall that according to Definition 1.14, the vertices of the exchange graph are the clusters, and two of them are connected by an edge if they intersect by $n-1$ elements.)

The description of the exchange graph in Proposition 3.14 implies Conjecture 1.13 for the type A_n. It shows that the polytope in question is the *Stasheff polytope,* or *associahedron* (see [23], [17], [13, Ch. 7]).

Types B_n and C_n. Let Φ be a root system of type B_n, and $\Phi^\vee$ the dual root system of type C_n. To describe compatible subsets for Φ and $\Phi^\vee$, we employ the folding procedure $A_{2n-1} \to B_n$ discussed at the beginning of Section 2.4. Let $\tilde\Phi$ be a root system of type A_{2n-1}, and let ρ be the automorphism of $\tilde\Phi$ that sends each simple root $\tilde\alpha_i$ to $\tilde\alpha_{2n-i}$. Then each of the sets $\Phi_{\geq -1}$ and $\Phi^\vee_{\geq -1}$ can be identified with the set of ρ-orbits in $\tilde\Phi_{\geq -1}$. This identification induces the labeling of simple roots of type B_n by $[1,n]$ and also the choice of a sign function ε: thus, a simple root α_i in Φ corresponds to the ρ-orbit of $\tilde\alpha_i$, and $\varepsilon(i) = (-1)^{i-1}$.

We represent the elements of $\tilde\Phi_{\geq -1}$ as diagonals of the regular $(2n+2)$-gon, as described above in our discussion of the type A case. Then ρ is geometrically represented by the *central symmetry* (or, equivalently, the 180° rotation) of the polygon, which sends each vertex $P = P_{\tilde i}$ to the antipodal vertex $-P \stackrel{\text{def}}{=} P_{\langle n+1+\tilde i\rangle}$; here $\langle m\rangle$ denotes the element of $[1, 2n+2]$ congruent to m modulo $2n+2$. We shall refer to the diagonals that join antipodal vertices as *diameters*. It follows that one can represent an element of $\Phi_{\geq -1}$ (resp., $\Phi^\vee_{\geq -1}$) either as a diameter $[P,-P]$, or as an unordered pair of centrally symmetric nondiameter diagonals $\{[P,Q],[-P,-Q]\}$ of the $(2n+2)$-gon. In particular, each of the roots $-\alpha_i$ (resp., $-\alpha_i^\vee$) for $i = 1,\dots,n-1$ is identified with the pair of diagonals representing $-\tilde\alpha_i$ and $-\tilde\alpha_{2n-i}$, whereas $-\alpha_n$ (resp., $-\alpha_n^\vee$) is identified with the diameter representing $-\tilde\alpha_n$.

The case $n = 3$ of this construction is illustrated in Figure 5 (next page).

Proposition 3.14 implies the following.

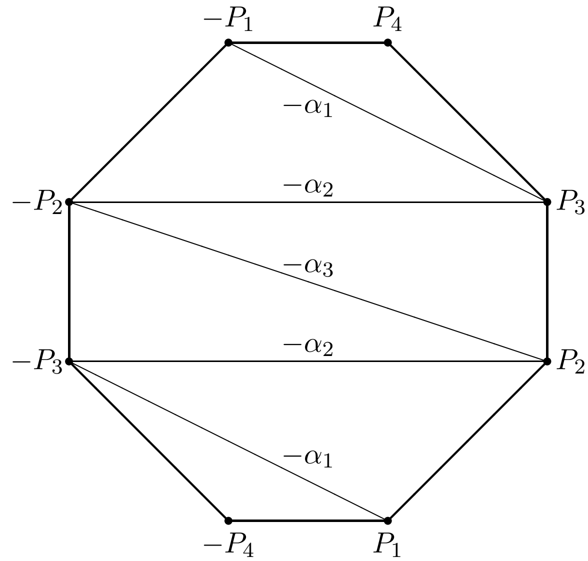
Proposition 3.15. *Let Φ be a root system of type B_n, and $\Phi^\vee$ the dual root system of type C_n.*

(1) *The transformations τ_+, τ_- and their compositions have the same geometric meaning as in Proposition* 3.14.1.

(2) *For $\alpha,\beta \in \Phi_{\geq -1}$, the compatibility degree $(\alpha\|\beta) = (\beta^\vee\|\alpha^\vee)$ (cf. Proposition* 3.3.1) *has the following geometric meaning*: *if $[P,Q]$ is one of the diagonals representing α, then $(\alpha\|\beta)$ is equal to the number of crossings of $[P,Q]$ by the diagonals representing β.*

(3) *The clusters for type B_n or C_n are in bijection with centrally symmetric triangulations of a regular $(2n+2)$-gon by noncrossing diagonals.*

(4) *Two centrally symmetric triangulations are joined by an edge in the exchange graph $E(\Phi)$ (or $E(\Phi^\vee)$) if and only if they are obtained from each other either by a flip involving two diameters (see Proposition* 3.14.4), *or by a pair of centrally symmetricflips.*



type B root	type C root	diameter or pair of diagonals
α_1	$\alpha_1^\vee$	$[\pm P_2, \mp P_4]$
α_2	$\alpha_2^\vee$	$[\pm P_1, \mp P_2]$
$\alpha_1+\alpha_2$	$\alpha_1^\vee+\alpha_2^\vee$	$[\pm P_2, \pm P_4]$
$\alpha_2+2\alpha_3$	$\alpha_2^\vee+\alpha_3^\vee$	$[\pm P_1, \pm P_3]$
$\alpha_1+\alpha_2+2\alpha_3$	$\alpha_1^\vee+\alpha_2^\vee+\alpha_3^\vee$	$[\pm P_3, \mp P_4]$
$\alpha_1+2\alpha_2+2\alpha_3$	$\alpha_1^\vee+2\alpha_2^\vee+\alpha_3^\vee$	$[\pm P_1, \pm P_4]$
α_3	$\alpha_3^\vee$	$[P_3, -P_3]$
$\alpha_2+\alpha_3$	$2\alpha_2^\vee+\alpha_3^\vee$	$[P_1, -P_1]$
$\alpha_1+\alpha_2+\alpha_3$	$2\alpha_1^\vee+2\alpha_2^\vee+\alpha_3^\vee$	$[P_4, -P_4]$

Figure 5. Representing the roots in $\Phi_{\geq -1}$ for the types B_3 and C_3

The above description of the exchange graph of type B shows that the corresponding simplicial complex $\Delta(\Phi)$ is identical to Simion's type B associahedron (see [21, §5.2] and [22]). The centrally symmetric triangulations that label its vertices (and our clusters) are in bijection with noncrossing partitions of type B defined by V. Reiner [20]. As shown by S. Devadoss [8], Simion's construction is combinatorially equivalent to the "cyclohedron" complex of R. Bott and C. Taubes [4]. This implies Conjecture 1.13 for the types B_n and C_n, since the cyclohedron can be realized as a convex polytope (see M. Markl [18] or R. Simion [22]).

The number of centrally symmetric triangulations of a regular $(2n+2)$-gon by noncrossing diagonals is equal to $(n+1)c_n = \binom{2n}{n}$, in agreement with the type B_n entry in Table 3. Indeed, any such triangulation involves precisely one diameter; we have $n+1$ choices for it, and c_n ways to complete a triangulation thereafter.

Type D_n. Let Φ be a root system of type D_n. We use the following numbering of simple roots: the indices $1,\dots,n-2$ form a chain in the Coxeter graph I, while both $n-1$ and n are adjacent to $n-2$. (See Figure 6.) We also use the following sign function: $\varepsilon(i) = (-1)^{i-1}$ for $i \in [1, n-1]$, and $\varepsilon(n) = (-1)^n$.

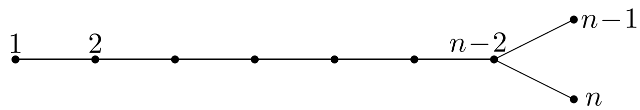


Figure 6. Coxeter graph of type D_n

The folding $D_n \to C_{n-1}$ motivates our choice of a geometric realization of $\Phi_{\geq -1}$. Thus, we have the canonical surjection $\pi : \Phi_{\geq -1} \to \Phi'_{\geq -1}$, where Φ' is the root system of type B_{n-1}. Let $\alpha_1,\dots,\alpha_n$ be the simple roots of Φ, and $\alpha'_1,\dots,\alpha'_{n-1}$ the simple roots of Φ'. The projection π is two-to-one over the n-element subset

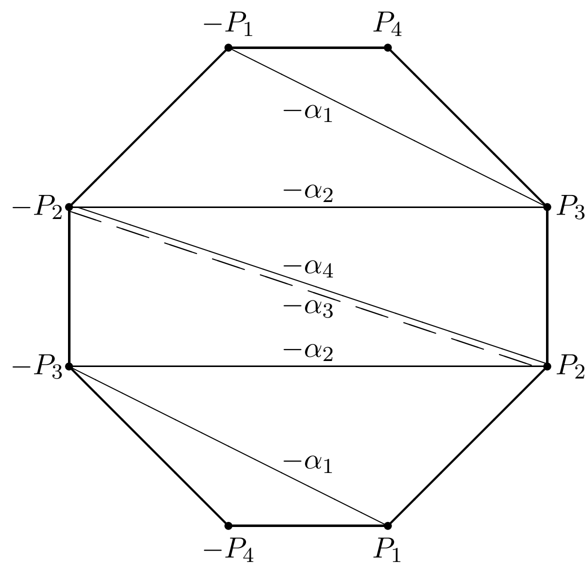
$$D = \{-\alpha'_{n-1}\,,\ \textstyle\sum_{j=i}^{n-1} \alpha'_j\ (1 \leq i \leq n-1)\} \subset \Phi'_{\geq -1}\,,$$

and one-to-one over its complement $\Phi'_{\geq -1} - D$.

As before, we identify $\Phi'_{\geq -1}$ with the set of diameters $[P, -P]$ and unordered pairs of centrally symmetric nondiameter diagonals $\{[P,Q], [-P,-Q]\}$ in a regular $2n$-gon. Under this identification, D becomes the set of diameters. Motivated by the above description of the projection $\pi : \Phi_{\geq -1} \to \Phi'_{\geq -1}$, we shall identify $\Phi_{\geq -1}$ with the disjoint union $\Phi'_{\geq -1} \cup D'$, where D' is an additional set of n elements $[P,-P]'$ associated with the diameters of the $2n$-gon. It is convenient to think that each diameter in $\Phi_{\geq -1}$ is colored in one of two different colors.

To specify the bijection between $\Phi_{\geq -1}$ and $\Phi'_{\geq -1} \cup D'$, we have to make a choice: for every $\alpha' \in D$, we must decide which of the two roots α in $\pi^{-1}(\alpha')$ is to be identified with the diameter $[P,-P]$ representing α'; the remaining element of $\pi^{-1}(\alpha')$ will then be identified with $[P,-P]' \in D'$. Our choice is the following: take $\alpha = -\alpha_n$ for $\alpha' = -\alpha'_{n-1}$, and $\alpha = \alpha_i + \cdots + \alpha_{n-1}$ for $\alpha' = \alpha'_i + \cdots + \alpha'_{n-1}$.

The case $n = 4$ of this construction is illustrated in Figure 7 below.



type B root	type D root	diameter or pair of diagonals
$-\alpha_1$	$-\alpha_1$	$[\pm P_1, \mp P_3]$
$-\alpha_2$	$-\alpha_2$	$[\pm P_2, \mp P_3]$
α_1	α_1	$[\pm P_2, \mp P_4]$
α_2	α_2	$[\pm P_1, \mp P_2]$
$\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2$	$[\pm P_2, \pm P_4]$
$\alpha_2 + 2\alpha_3$	$\alpha_2 + \alpha_3 + \alpha_4$	$[\pm P_1, \pm P_3]$
$\alpha_1 + \alpha_2 + 2\alpha_3$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$[\pm P_3, \mp P_4]$
$\alpha_1 + 2\alpha_2 + 2\alpha_3$	$\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$	$[\pm P_1, \pm P_4]$
$-\alpha_3$	$-\alpha_4$	$[P_2, -P_2]$
	$-\alpha_3$	$[P_2, -P_2]'$
α_3	α_3	$[P_3, -P_3]$
	α_4	$[P_3, -P_3]'$
$\alpha_2 + \alpha_3$	$\alpha_2 + \alpha_3$	$[P_1, -P_1]$
	$\alpha_2 + \alpha_4$	$[P_1, -P_1]'$
$\alpha_1 + \alpha_2 + \alpha_3$	$\alpha_1 + \alpha_2 + \alpha_3$	$[P_4, -P_4]$
	$\alpha_1 + \alpha_2 + \alpha_4$	$[P_4, -P_4]'$

Figure 7. Representing the roots in $\Phi_{\geq -1}$ for the type D_4

The following type D_n counterpart of Propositions 3.14–3.15 is verified by a direct inspection.

PROPOSITION 3.16. *Let* Φ *be a root system of type* D_n *.*

(1) *The transformations* τ_+ *and* τ_- *of* $\Phi_{\geq -1}$ *are realized by the same reflections as in Propositions* 3.14.1 *and* 3.15.1, *with the following modification*: $\tau_{(-1)^n}$ *involves changing the colors of all diameters.*

(2) *For* $\alpha, \beta \in \Phi_{\geq -1}$, *the compatibility degree* $(\alpha\|\beta) = (\beta\|\alpha)$ *has the following geometric meaning*:

- *If* $\alpha \notin D \cup D'$, $\beta \notin D \cup D'$, *then* $(\alpha\|\beta)$ *is the same as in Proposition* 3.15.2;
- *If, say,* $\alpha \in D \cup D'$, $\beta \notin D \cup D'$, *then* $(\alpha\|\beta) = 1$ *if the diameter representing* α *crosses both diagonals representing* β; *otherwise,* $(\alpha\|\beta) = 0$;
- *If* $\alpha, \beta \in D$ *or* $\alpha, \beta \in D'$, *then* $(\alpha\|\beta) = 0$;
- *If* $\alpha \in D$ *and* $\beta \in D'$, *then* $(\alpha\|\beta) = 1$ *if the diameters representing* α *and* β *are different; otherwise,* $(\alpha\|\beta) = 0$.

(3) *The clusters for type* D_n *are in bijection with centrally symmetric colored triangulations of the regular* $2n$*-gon which fit the following description* (*see Figure* 8):

- *Each triangulation is made of nondiameter diagonals together with at least two colored diameters* (*that is, elements of either* D *or* D');
- *The diagonals making up a triangulation do not have common internal points, except for diameters of the same color crossing at the center, or diameters of different color connecting the same antipodal points.*

(4) *Two triangulations of the kind described in Part* 3 *above are joined by an edge in the exchange graph* $E(\Phi)$ *if and only if they are obtained from each other by one of the "type* D *flips" shown in Figure* 9:

(a) *a pair of centrally symmetric flips* (*cf. Proposition* 3.14.4);

(b) *a flip involving two diameters of different colors*;

(c) *a "hexagonal flip" exchanging a diameter with a pair of diagonals.*

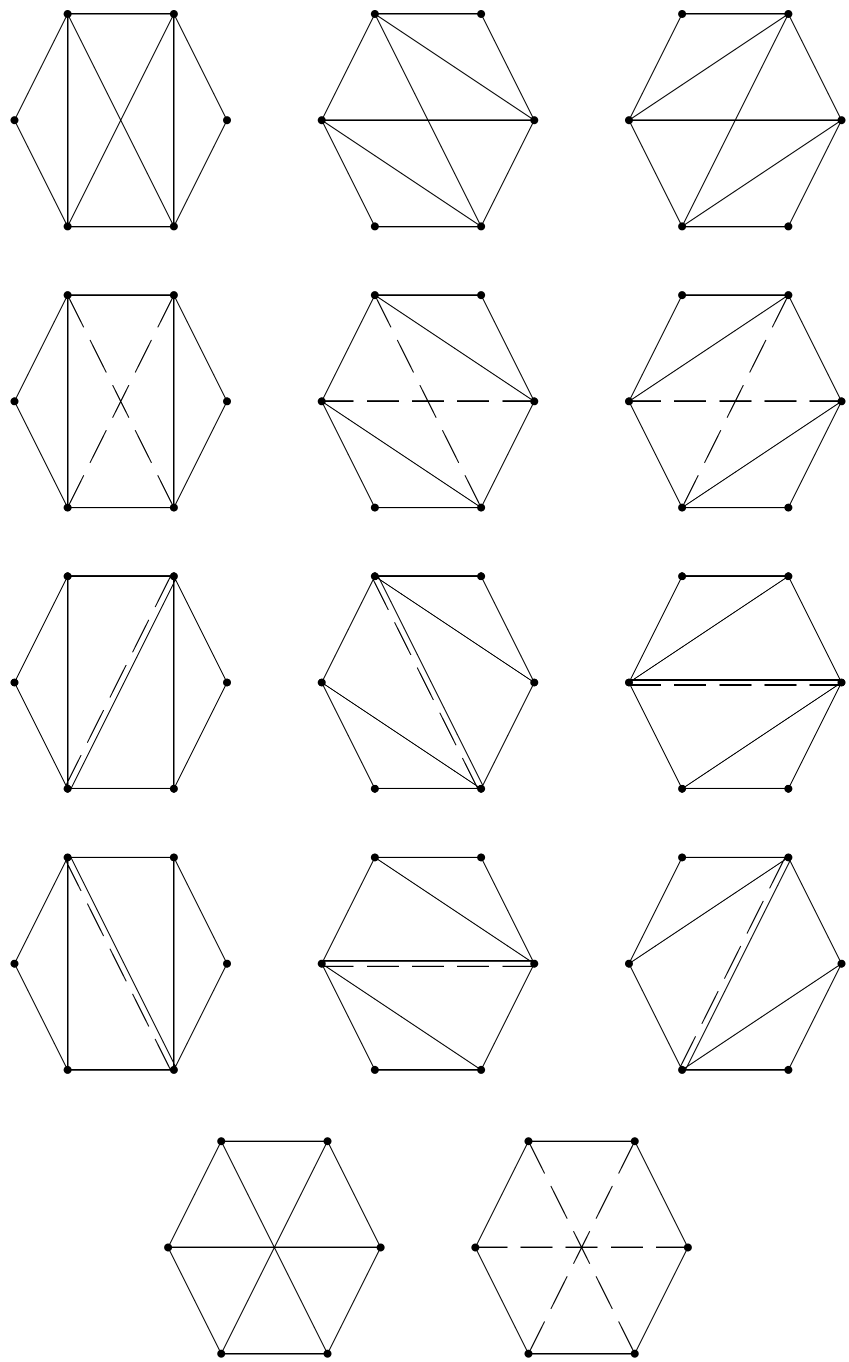


Figure 8. Triangulations representing the clusters of type D_3

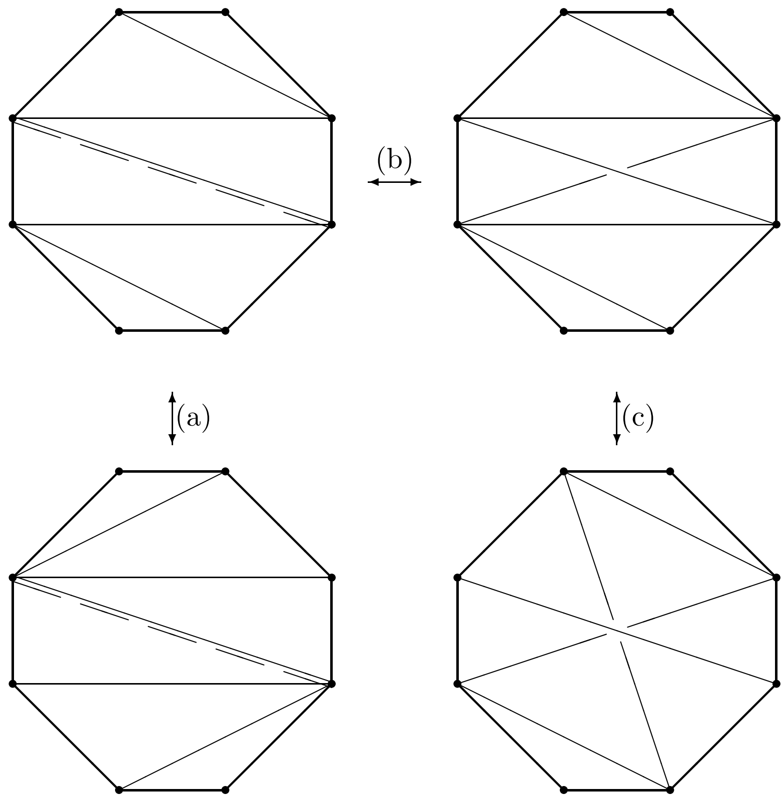


Figure 9. Flips of type D

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