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The homotopy type of the matroid grassmannian

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1. Introduction

Characteristic cohomology classes, defined in modulo 2 coefficients by Stiefel [26] and Whitney [28] and with integral coefficients by Pontrjagin [24], make up the primary source of first-order invariants of smooth manifolds. When their utility was first recognized, it became an obvious goal to study the ways in which they admitted extensions to other categories, such as the categories of topological or PL manifolds; perhaps a clean description of characteristic classes for simplicial complexes could even give useful computational techniques. Modulo 2, this hope was realized rather quickly: it is not hard to see that the Stiefel-Whitney classes are PL invariants. Moreover, Whitney was able to produce a simple explicit formula for the class in codimension i in terms of the i-skeleton of the barycentric subdivision of a triangulated manifold (for a proof of this result, see [13]).

One would like to find an analogue of these results for the Pontrjagin classes. However, such a naive goal is entirely out of reach; indeed, Milnor's use of the Pontrjagin classes to construct an invariant which distinguishes between nondiffeomorphic manifolds which are homeomorphic and PL isomorphic to S^7 suggested that they cannot possibly be topological or PL invariants [19]. Milnor was in fact later able to construct explicit examples of homeomorphic smooth 8-manifolds with distinct Pontrjagin classes [20]. On the other hand, Thom [27] constructed rational characteristic classes for PL manifolds which agreed with the Pontrjagin classes, and Novikov [23] was able to show that, rationally, the Pontrjagin classes of a smooth manifold were topological invariants. This led to a surge of effort to find an explicit combinatorial expression for the rational Pontrjagin classes analogous to Whitney's formula for the Stiefel-Whitney classes. This arc of research, represented in part by the work of Miller [18], Levitt-Rourke [15], Cheeger [8], and Gabrièlov-Gelfand-Losik [10], culminated with the discovery by Gelfand and MacPherson [12] of a formula built on the language of oriented matroids.

Their construction makes use of an auxiliary simplicial complex on which certain universal rational cohomology classes lie; this simplicial complex can be thought of as a combinatorial approximation to BO_k. Our main result is that this complex is in fact homotopy equivalent to BO_k, so that the Gelfand-MacPherson techniques can actually be used to locate the integral Pontrjagin classes as well. Equivalently, the oriented matroids on which their formula rests entirely determine the tangent bundle up to isomorphism.

A closer examination of these ideas led MacPherson [16] to realize that they actually amounted to the construction of characteristic classes for a new, purely combinatorial type of geometric object. These objects, which he called combinatorial differential (CD) manifolds, are simplicial complexes furnished with some extra combinatorial data that attempt to behave like smooth structures. The additional combinatorial data come in the form of a number of oriented matroids; in the case that we begin with a smooth triangulation of a differentiable manifold, these oriented matroids can be recovered by playing the linear structure of the simplices and the smooth structure of the manifold off of one another. For a somewhat more precise discussion of this relationship, see Section 3.

The world of CD manifolds admits a purely combinatorial notion of bundles, called matroid bundles. As one would expect, a k-dimensional CD manifold comes equipped with a rank k tangent matroid bundle; moreover, matroid bundles admit familiar operations such as pullback and Whitney sum. There is a classifying space for rank k matroid bundles, namely the geometric realization of an infinite partially ordered set (poset) called the *MacPhersonian* $\mathrm{MacP}(k,\infty)$; this is the "combinatorial approximation to BO_k" alluded to above. The MacPhersonian is the colimit of a collection of finite posets $\mathrm{MacP}(k,n)$, which can be viewed as combinatorial analogues of the Grassmannians $\mathrm{G}(k,n)$ of k-planes in $\mathbb{R}^n$. In fact, there exist maps

$$\pi : \mathrm{G}(k,n) \longrightarrow \|\,\mathrm{MacP}(k,n)\|$$

compatible with the inclusions $\mathrm{G}(k,n) \hookrightarrow \mathrm{G}(k,n+1)$ and $\mathrm{MacP}(k,n) \hookrightarrow \mathrm{MacP}(k,n+1)$, as well as $\mathrm{G}(k,n) \hookrightarrow \mathrm{G}(k+1,n+1)$ and $\mathrm{MacP}(k,n) \hookrightarrow \mathrm{MacP}(k+1,n+1)$, and therefore giving rise to maps

$$\pi : BO_k = \mathrm{G}(k,\infty) \longrightarrow \|\,\mathrm{MacP}(k,\infty)\|$$

and

$$\pi : BO \longrightarrow \|\,\mathrm{MacP}(\infty,\infty)\|.$$

The first complete construction of the maps π was given in [4]; for earlier related work, see [11] or [16]. Because it will always be clear from the context what k and n are, the use of the symbol π to denote each of these maps should cause no confusion.

In view of this recasting of the Gelfand-MacPherson construction, one would expect the map $\pi : BO_k \to \|\operatorname{MacP}(k,\infty)\|$ to induce a surjection on rational cohomology. This turns out to be the case; for a detailed discussion of this point of view; see [3]. Of course, when appropriately reinterpreted in this language, the Gelfand-MacPherson result is stronger: it actually provides explicit formulas for elements $p_i \in H^{4i}(\|\operatorname{MacP}(k,\infty)\|, \mathbb{Q})$ such that $\pi^*(p_i)$ is the ith rational Pontrjagin class. Nonetheless, this work indicates that further understanding the cohomology of the MacPhersonian would have two benefits. First of all, it would constitute a foothold from which to begin a systematic study of CD manifolds; indeed, the first step in the standard approach to the study of any category in topology or geometry is an analysis of the homotopy type of the classifying space of the accompanying bundle theory. Secondly, it might point the direction for possible further results concerning the application of oriented matroids to computation of characteristic classes.

Accordingly, the MacPhersonian has been the object of much study (see, for example, [1], [5], or [22]). Most recently, Anderson and Davis [4] have been able to show that the maps π induce split surjections in cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients; thus, one can define Stiefel-Whitney classes for CD manifolds. However, none of these results establishes whether the CD world manages to capture any purely local phenomena of smooth manifolds, that is, whether it can see more than the PL structure. The aim of this article is to prove the following theorem.

THEOREM 1.1. *For every positive integer n or for $n = \infty$, and for any $k \leq n$, the map*

$$\pi : \mathrm{G}(k,n) \to \|\operatorname{MacP}(k,n)\|$$

is a homotopy equivalence.

Of course, in the case $n = \infty$, this result implies that the theory of matroid bundles is the same as the theory of vector bundles. This gives substantial evidence that a CD manifold has the capacity to model many properties of smooth manifolds. To make this connection more precise, we give in [6] a definition of morphisms that makes CD manifolds into a category admitting a functor from the category of smoothly triangulated manifolds. Furthermore, these morphisms have appropriate naturality properties for matroid bundles and hence characteristic classes, so many maneuvers in differential topology carry over verbatim to the CD setting. This represents the first demonstration that the CD category succeeds in capturing structures contained in the smooth but absent in the topological and PL categories, and suggests that it might be possible to develop a purely combinatorial approach to smooth manifold topology.

Furthermore, our result tells us that the integral Pontrjagin classes lie in the cohomology of the MacPhersonian; thus, it ought to be possible to find extensions of the Gelfand-MacPherson formula that hold over $\mathbb{Z}$. That is, the integral Pontrjagin classes of a triangulated manifold depend only on the PL isomorphism class of the manifold enriched with some extra combinatorial data, or, equivalently, on the CD isomorphism class of the manifold.

COROLLARY 1.2. *Given a matroid bundle E over a cell complex B, there are combinatorially defined classes $p_i(E) \in H^{4i}(B, \mathbb{Z})$, functorial in B, which satisfy the usual axioms for Pontrjagin classes* (*see, for example*, [21]). *Furthermore, when M is a smoothly triangulated manifold, the underlying simplicial complex of M accordingly enjoys the structure of a* CD *manifold, whose tangent matroid bundle is denoted by T. Then*

$$p_i(M) = p_i(T).$$

We have not been able to find an especially illuminating explicit formulation of this result, which would of course be extremely appealing. It is also interesting to note that it is does not seem clear that this combinatorial description of the Pontrjagin classes is rationally independent of the CD structure.

The plan of our proof is very simple. First of all, the compatibility of the various maps π implies that it suffices to check our result for finite n and k. We then stratify the spaces $\|\operatorname{MacP}(k,n)\|$ into pieces corresponding to the Schubert cells in the ordinary Grassmannian. It can be shown that these open strata are actually contractible, and furthermore that $\|\operatorname{MacP}(k,n)\|$ is constructed inductively by forming a series of mapping cones. Moreover, it is not too hard to see that the map from $\mathrm{G}(k,n)$ to $\|\operatorname{MacP}(k,n)\|$ takes open cells to open strata. Thus, to complete the argument, all we need to do is show that the open strata are actually "homotopy cells," that is, that they are cones on homotopy spheres of the appropriate dimension. This forms the technical heart of the proof.

Because the idea of applying oriented matroids to differential topology is a relatively new one, it is instinctual to reinvent the wheel and introduce from scratch all necessary preliminaries from combinatorics. Since this has already been done more than adequately, we try to shy away from this tendency; however, our techniques rely on some subtle combinatorial results that have not been used before in the study of CD manifolds, and accordingly we provide a brief introduction to oriented matroids in Section 2. Armed with these definitions, we give in Section 3 a motivational sketch of the general theory of CD manifolds and matroid bundles. Then, in Section 4, we describe the combinatorial analogue of the Schubert cell decomposition, and explain why in order to complete the proof, it suffices to show that certain spaces are homotopy equivalent to spheres and sit inside $\|\operatorname{MacP}(k,n)\|$ in a particular way. Finally, in Section 5, we actually prove these facts.

2. Combinatorial preliminaries

In this section, we provide a brief introduction to the ideas we will use from the theory of oriented matroids. For a more a comprehensive survey of the combinatorial side of the study of CD manifolds, see [2] or [4]; for complete details of the constructions and theorems we describe, [7] is the standard reference. Probably the best summary of the basic definitions concerning CD manifolds can be found in MacPherson's original exposition [16].

An oriented matroid is a combinatorial model for a finite arrangement of vectors in a vector space. To motivate the definition, first suppose we are furnished with a finite set S and a map $\rho : S \to V$ to a vector space V over $\mathbb{R}$ such that the set $\rho(S)$ spans V. We may then consider the set M of all maps $S \to \{+, -, 0\}$ obtained as compositions

$$S \xrightarrow{\rho} V \xrightarrow{\ell} \mathbb{R} \xrightarrow{\mathrm{sgn}} \{+, -, 0\}$$

where $\ell : V \to \mathbb{R}$ is any linear map. In general, an oriented matroid is an abstraction of this setting: it remembers the information (S, M) without assuming the existence of an ambient vector space V.

The data encoded by the pair (S, M) can be reinterpreted in the following way. A (nonzero) linear map $\ell : V \to \mathbb{R}$ divides V into three components: a hyperplane $\ell^{-1}(0)$, the "positive" side $\ell^{-1}(\mathbb{R}_+)$ of the hyperplane, and the "negative" side $\ell^{-1}(\mathbb{R}_-)$. The oriented matroid simply keeps track of what partition of S is induced by this stratification of V. Thus, roughly speaking, the information contained in (S, M) allows us to read off two types of information about S. First of all, since we are able to see which subsets of S lie in a hyperplane in V, we can tell which subsets of S are dependent. Secondly, because we can see on which side of any hyperplane a vector lies, given two ordered bases of V contained in S, we can determine whether they have equal or opposite orientations. Incidentally, the presence of the word "oriented" in the term "oriented matroid" refers to the latter: an ordinary matroid is more or less an oriented matroid which has forgotten how to see whether two bases carry the same orientation, or, equivalently, on which side of the hyperplane $\ell^{-1}(0)$ an element lies.

Definition 2.1. An *oriented matroid* on a finite set S is a subset $M \subset \{+, -, 0\}^S$ satisfying the following axioms:

1. The constant zero function is an element of M.

2. If $X \in M$, then $-X \in M$.

3. If X and Y are in M, then so is the function $X \circ Y$ defined by

$$(X \circ Y)(s) = \begin{cases} X(s) & \text{if } X(s) \neq 0 \\ Y(s) & \text{otherwise.} \end{cases}$$

4. If $X, Y \in M$ and s_0 is an element of S with $X(s_0) = +$ and $Y(s_0) = -$, then there is a $Z \in M$ with $Z(s_0) = 0$ and for all $s \in S$ with $\{X(s), Y(s)\} \neq \{+, -\}$, we have $Z(s) = (X \circ Y)(s)$.

Elements of the set M are referred to as *covectors*.

These four axioms all correspond to familiar maneuvers on vector spaces; indeed, suppose that S is actually a subset of a vector space V and that X and Y arise from linear maps $\ell_X, \ell_Y : V \to \mathbb{R}$. The first axiom simply states that the zero map $V \to \mathbb{R}$ is linear. The second axiom means that $-\ell_X : V \to \mathbb{R}$ is linear. The element $X \circ Y \in M$ from the third axiom is induced by the map $A\ell_X + \ell_Y$, for any A large enough that $A\ell_X$ dominates ℓ_Y, that is, for any

$$A > \max \left\{ \frac{|\ell_Y(s)|}{|\ell_X(s)|}, s \in S, \ell_X(s) \neq 0 \right\}.$$

Lastly, the element Z of the fourth axiom is induced by the linear map $\ell_Z = -\ell_Y(s_0)\ell_X + \ell_X(s_0)\ell_Y$.

An oriented matroid arising from a map $\rho : S \to V$ as above is said to be *realizable*. Not all oriented matroids are realizable, but many constructions that are familiar in the realizable setting have analogues for arbitrary oriented matroids. In particular, there is a well-defined notion of the rank of an oriented matroid, and we may form the convex hull of a subset of an oriented matroid.

Definition 2.2. Let M be an oriented matroid on the set S. A subset $\{s_1, \ldots, s_k\} \subset S$ is said to be *independent* if there exist covectors $X_1, \ldots, X_k \in M$ with $X_i(s_j) = \delta_{ij}$. Here, δ_{ij} denotes the Kronecker delta:

$$\delta_{ij} = \begin{cases} + & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

The *rank* of an oriented matroid is the size of any maximal independent subset (one can show that this is well-defined). An element $s \in S$ is said to be in the *convex hull* of a subset $S' \subset S$ if for every covector $X \in M$ with $X(S') \subset \{+, 0\}$, we have $X(s) \in \{+, 0\}$. An element $s \in S$ is said to be a *loop* of M if for every covector $X \in M$, we have $X(s) = 0$. In the case that M is realized by the map $\rho : S \to V$, this is equivalent to the condition that $\rho(s) = 0$. An element $s \in S$ is said to be a *coloop* of M if there is a covector $X \in M$ with $X(s) = +$ and $X(s') = 0$ for all $s' \neq s$. In the realizable case, this is equivalent to the condition that the set $\rho(S \backslash \{s\})$ lies in a hyperplane of V.

There is one slightly more subtle concept that will be the basis of all our work.

Definition 2.3. Let M and M' be two oriented matroids on the same set S. Then M' is said to be a *specialization* or *weak map image* of M (denoted $M \rightsquigarrow M'$) if for every $X' \in M'$ there is an $X \in M$ with $X(s) = X'(s)$ whenever $X'(s) \neq 0$.

This is the case, for example, if M and M' are both realizable oriented matroids, and if the vector arrangement M' is in more "special position" than that of M; that is, if one can produce a realization of M' from a realization of M by forcing additional dependencies. For example, in Figure 1 below, the oriented matroid realized by the left-hand arrangement of vectors specializes to the oriented matroid realized by the right-hand arrangement. More precisely,

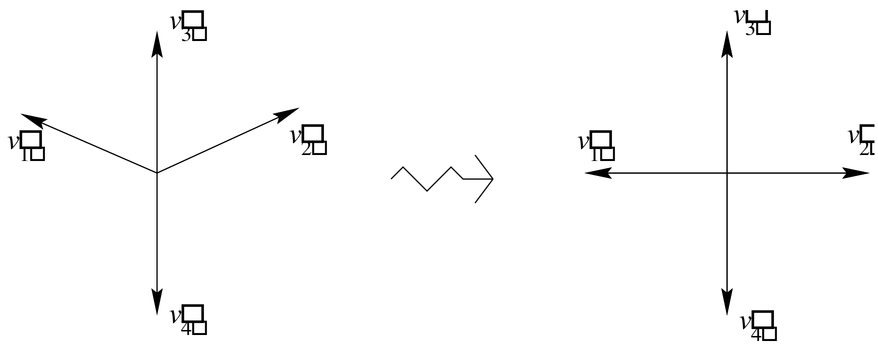

Figure 1: A specialization of realizable oriented matroids

this relation holds whenever the space of realizations $\rho : S \to V$ of M', which can be viewed as a subspace of V^S, and accordingly comes with a natural topology, intersects the closure of the space of realizations of M.

We are now ready to define the basic objects of study, the MacPhersonians.

Definition 2.4. The *MacPhersonian* $\mathrm{MacP}(k, n)$ is the poset of rank k oriented matroids on the set $\{1, \dots, n\}$, where the order is given by $M \geq M'$ if and only if $M \rightsquigarrow M'$. $\mathrm{MacP}(k, \infty)$ is the colimit over all n of the maps $\mathrm{MacP}(k, n) \hookrightarrow \mathrm{MacP}(k, n+1)$, defined by taking a rank k oriented matroid on $\{1, \dots, n\}$ and producing one on $\{1, \dots, n+1\}$ by declaring $n+1$ to be a loop. Moreover, the maps $\mathrm{MacP}(k, n) \hookrightarrow \mathrm{MacP}(k+1, n+1)$ defined by declaring $n+1$ to be a coloop induce maps $\mathrm{MacP}(k, \infty) \to \mathrm{MacP}(k+1, \infty)$. The colimit over all k is denoted $\mathrm{MacP}(\infty, \infty)$.

Since the content of this article consists in a study of the homotopy type of the MacPhersonian and related posets, we will need to establish some general facts about the topology of posets. As in the introduction, for any poset P, we denote by $\|P\|$ its nerve. This is the simplicial complex whose vertices are the elements of P, and whose k-simplices are chains $p_k > p_{k-1} > \cdots > p_0$ in P.

Definition 2.5. Let $f : P \to Q$ be a map of posets, and fix $q \in Q$. Denote by P^l_q the subset of $\|P\|$ consisting of the interior of each simplex whose maximal vertex lies in $f^{-1}(q)$, and let P^u_q denote the subset made up of the interior of each simplex whose minimal vertex lies in $f^{-1}(q)$.

PROPOSITION 2.6. *Let $f : P \to Q$ be any map of posets, and q any element of Q. Then there are deformation retractions $P^l_q \to \|f^{-1}(q)\|$ and $P^u_q \to \|f^{-1}(q)\|$.*

Proof. We carry out the argument only for the case of P^l_q; the analogous statement for P^u_q follows by reversing the orders of both P and Q. The basic strategy of the proof is to collapse P^l_q onto $\|f^{-1}(q)\|$ one cell at a time. More precisely, given a maximal open cell C of $P^l_q \backslash \|f^{-1}(q)\|$, its closure in $\|P\|$ is an α-simplex which corresponds to some saturated chain $p_0 > \cdots > p_\alpha$ in P, with $f(p_0) = q$. Then since by assumption C is not contained in $\|f^{-1}(q)\|$, it must be the case that for some $\beta \leq \alpha$, we have $f(p_{\beta-1}) = q$ and $f(p_\beta) < q$. In particular, the $(\alpha - \beta)$-simplex $p_\beta > \cdots > p_\alpha$ is precisely $\bar{C} \backslash P^l_q$; denote its interior by C'. The result follows from an induction on cells along with the fact that $\bar{C} \backslash (C \cup C') = \partial \bar{C} \backslash C'$ is a deformation retract of $\bar{C} \backslash C'$, and thus that $P^l_q \backslash C$ is a deformation retract of P^l_q. $\square$

Lastly, recall the following basic result of Quillen [25].

PROPOSITION 2.7 (Quillen's Theorem A). *Let $f : P \to Q$ be a poset map. If for each $q \in Q$, the space*

$$\left\| f^{-1}(\{q' \in Q | q' \geq q\}) \right\|$$

is contractible, then $\|f\|$ is a homotopy equivalence.

We now present an alternate characterization of oriented matroids that is especially well-suited to analysis of the MacPhersonian. If $\rho : S \to V$ is a realization of a rank k oriented matroid, then for either orientation of V, we obtain a map $\chi : S^k \to \{+, -, 0\}$ by defining $\chi(s_1, \ldots, s_k) = \mathrm{sgn}(\det(\rho(s_1), \ldots, \rho(s_k)))$; that is, although the determinant itself depends on a choice of basis (or, more precisely, on an identification of $\bigwedge^k V$ with $\mathbb{R}$), its sign depends only an an orientation of V (or, equivalently, on an orientation of $\bigwedge^k V$). Moreover, it is easy to see that the map χ depends only on the oriented matroid determined by ρ. The following definition generalizes this to the setting of arbitrary oriented matroids.

Definition 2.8. A *chirotope* of rank k on a set S is a map $\chi : S^k \to \{+, -, 0\}$ such that:

1. χ is not identically zero.

2. For all $s_1, \dots, s_k \in S$ and $\sigma \in \Sigma_k$, we have

$$\chi(s_{\sigma(1)}, \dots, s_{\sigma(k)}) = \operatorname{sgn}(\sigma)\chi(s_1, \dots, s_k).$$

3. For all $s_1, \dots, s_k, t_1, \dots, t_k \in S$ such that $\chi(s_1, \dots, s_k) \cdot \chi(t_1, \dots, t_k) \neq 0$, there exists an i such that $\chi(t_i, s_2, \dots, s_k) \cdot \chi(t_1, \dots, t_{i-1}, s_1, t_{i+1}, \dots, t_k)$ $= \chi(s_1, \dots, s_k) \cdot \chi(t_1, \dots, t_k)$.

Of course, to every realization of an oriented matroid in a vector space V, one can associate two (opposite) chirotopes, one for each orientation of V. The analogous statement can also be shown for general oriented matroids.

PROPOSITION 2.9 (See [7]). *There is a two-to-one correspondence between the set of rank k chirotopes on the set S and the set of rank k oriented matroids on S. For any oriented matroid M, the two chirotopes χ_M^1 and χ_M^2 corresponding to it satisfy*

$$\chi_M^1(s_1, \dots, s_k) = -\chi_M^2(s_1, \dots, s_k)$$

for all $(s_1, \dots, s_k) \in S^k$. Moreover, if M and M' both have rank k, then $M \rightsquigarrow M'$ if and only if M and M' admit chirotopes χ_M and $\chi_{M'}$ satisfying $\chi_M(s_1, \dots, s_k) = \chi_{M'}(s_1, \dots, s_k)$ whenever $\chi_{M'}(s_1, \dots, s_k) \neq 0$.

Finally (recall [4]) there exists a map $\pi : \mathrm{G}(k,n) \to \|\operatorname{MacP}(k,n)\|$. Although in Section 3, we will indicate a conceptual proof of the existence of such a map, for our purposes it will be necessary to have a much more hands-on approach, which we outline here. Let $\{\zeta_1, \dots, \zeta_n\}$ denote the standard basis of $\mathbb{R}^n$; it is orthonormal in the standard inner product. The first step in the construction of π is the observation that for any k-plane $V \subset \mathbb{R}^n$, the standard inner product on $\mathbb{R}^n$ defines an orthogonal projection $\wp : \mathbb{R}^n \to V$. This, in turn, gives rise to a rank k (realizable) oriented matroid on n elements—namely, the one realized by the images $\{\wp(\zeta_1), \dots, \wp(\zeta_n)\}$ of the n standard basis vectors in $\mathbb{R}^n$. In this way, we produce a (lower semi-continuous) map of sets $\mu : \mathrm{G}(k,n) \to \operatorname{MacP}(k,n)$. Our goal is to use this map to produce the continuous map π.

In [4], this is done by constructing a simplicial subdivision of $\mathrm{G}(k,n)$ refining the decomposition of $\mathrm{G}(k,n)$ into the fibers $\mu^{-1}(M)$, using the fact that the fibers are semi-algebraic and a result of Hironaka [14], and then defining a map from the barycentric subdivision of this simplicial structure to $\|\operatorname{MacP}(k,n)\|$ by taking nerves. There is, however, a less technologically intensive argument.

Let P be a poset, and fix a point $p \in \|P\|$; it is in the interior of a unique simplex, which corresponds to some chain in P, whose maximal element is an element $\nu(p)$ of P. This defines a map $\nu : \|P\| \to P$. So far, we have only been considering the discrete topology on our posets; however, the lowersemicontinuity of the map $\mu : \mathrm{G}(k,n) \to \mathrm{MacP}(k,n)$ suggests that this map might have more geometric meaning in some other setting. In fact, there is another topology on P, which better captures the homotopy type of $\|P\|$.

Definition 2.10. Let P be a poset. An *order ideal* in P is a subset $Q \subset P$ with the property that if $q \in Q$ and $p \le q$ then $p \in Q$. The *order topology* on P is the topology generated by declaring that each order ideal in P be closed.

In this topology, the map ν is always continuous; moreover, the following result is not hard to verify.

PROPOSITION 2.11 (See [17]). *For any poset P endowed with the order topology, the map $\nu : \|P\| \to P$ is a weak homotopy equivalence.*

Furthermore, the definitions have been rigged in such a way that the map $\mu : \mathrm{G}(k,n) \to \mathrm{MacP}(k,n)$ is continuous when we endow $\mathrm{MacP}(k,n)$ with the order topology. We now have the following diagram

$$\begin{tikzcd} \mathrm{G}(k,n) \arrow[r, dashed] \arrow[dr, "\mu"'] & \|\mathrm{MacP}(k,n)\| \arrow[d, "\nu", "\sim"'] \\ & \mathrm{MacP}(k,n) \end{tikzcd}$$

in which the right-hand map is a weak homotopy equivalence and the dashed map is the map π we would like to construct. However, it is well-known that if X is a CW complex and $f : Y \to Z$ is a weak homotopy equivalence, then the map $f_* : [X,Y] \to [X,Z]$ is a bijection; here $[A,B]$ denotes the set of homotopy classes of maps from A to B. Thus, we can make the following definition.

Definition 2.12. The map $\pi : \mathrm{G}(k,n) \to \|\mathrm{MacP}(k,n)\|$ is a map chosen to make the above diagram homotopy commutative. There is precisely one such map up to homotopy.

This approach fails to give one essential property of π which follows directly from Hironaka's result, namely, the fact that π can be chosen so that the above triangle commutes on the nose. We will use this fact heavily throughout, so without further mention, we always assume

$$\nu \circ \pi = \mu.$$

The image of the map μ is precisely the subposet $\mathrm{MacP}_{\mathrm{real}}(k,n)$ consisting of all realizable oriented matroids. Thus, by our conventions, the image of the map π is contained in the space $\|\mathrm{MacP}_{\mathrm{real}}(k,n)\|$. Our techniques actually give us the following result.

Corollary 2.13. *Both maps in the composition*

$$\mathrm{G}(k,n) \to \|\,\mathrm{MacP}_{\mathrm{real}}(k,n)\| \hookrightarrow \|\,\mathrm{MacP}(k,n)\|$$

are homotopy equivalences.

Our proof of Theorem 1.1 carries over verbatim to show that the map $\mathrm{G}(k,n) \to \|\,\mathrm{MacP}_{\mathrm{real}}(k,n)\|$ is a homotopy equivalence as well; we leave it to the reader to verify this. The reader should be warned that there are two different conceivable definitions of $\mathrm{MacP}_{\mathrm{real}}(k,n)$. That is, it must certainly be some partial order on the set of realizable rank k oriented matroids on $\{1,\dots,n\}$. However, for two realizable oriented matroids M and M', we could either say $M \geq M'$ if $M \rightsquigarrow M'$ as in Definition 2.3, or we could say that $M \geq M'$ if the space of realizations of M' is contained in the closure of the space of realizations of M. These two definitions are not the same: the second is strictly more restrictive than the first. We use the symbol $\mathrm{MacP}_{\mathrm{real}}(k,n)$ to denote the former definition, that is, a full subposet of $\mathrm{MacP}(k,n)$; Corollary 2.13 is true for either definition, but is substantially more useful for future purposes in the definition we use, so we will henceforth ignore the smaller poset.

3. Interlude: the connection with vector bundles over smooth manifolds

We now provide a brief motivation for the definitions given in the previous section. Suppose B is a finite simplicial complex, and $\xi : E \to B$ is a rank k vector bundle over B. Then for n sufficiently large, we can find global sections $s_1,\dots,s_n$ of ξ such that for every point $p \in B$, the vectors $s_1(p),\dots,s_n(p)$ span $E_p = \xi^{-1}(p)$. This setup therefore gives rise to a rank k realizable oriented matroid on the set $S = \{s_1,\dots,s_n\}$ for every $p \in B$. Moreover, one can check that we could have actually chosen the sections s_i in such a way that the matroid stratification of B they determine is a simplicial subdivision of B. In this situation, if Δ and Δ' are two simplices in this subdivision with $\Delta' \subset \partial\Delta$, and if M and M' are the corresponding oriented matroids, then we have $M \rightsquigarrow M'$. That is, we obtain a map of posets from the set of simplices of B to $\mathrm{MacP}(k,n)$. Taking geometric realizations gives a map $B \to \|\,\mathrm{MacP}(k,n)\|$, since the nerve of the poset of simplices of B is simply the barycentric subdivision of B itself.

One can show (see [4]) that the homotopy class of the composition

$$B \longrightarrow \|\,\mathrm{MacP}(k,n)\| \hookrightarrow \|\,\mathrm{MacP}(k,\infty)\|$$

depends only on the isomorphism class of the vector bundle ξ, and not on the choice of sections. This gives us a map, natural in B,

$$\left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of rank } k\\ \text{vector bundles over } B\end{array}\right\} \longrightarrow [B, \|\,\mathrm{MacP}(k,\infty)\|]\,.$$

Incidentally, we accordingly obtain an alternate, more functorial construction of the map $\pi : \mathrm{G}(k,\infty) \to \|\,\mathrm{MacP}(k,\infty)\|$ as the colimit over all sufficiently large n of the maps $\mathrm{G}(k,n) \to \|\,\mathrm{MacP}(k,\infty)\|$ induced by the tautological k-plane bundle γ_k over $\mathrm{G}(k,n)$. (It is not hard to see that they are compatible up to homotopy.) Thus, it is reasonable to think of $\|\,\mathrm{MacP}(k,\infty)\|$ as the representing object of a theory of combinatorial vector bundles, usually referred to as *matroid bundles*. Theorem 1.1 tells us that the theory of matroid bundles is actually the same as the theory of ordinary vector bundles.

The natural source for matroid bundles lies in the world of CD manifolds. To appropriately situate these ideas, we provide a brief sketch of the theory of CD manifolds. This is not intended to be comprehensive, and has no direct mathematical bearing on the proof of our main theorem, but will, we hope, be of some motivational value. Again, for a more complete discussion, see [16]. Consider, then, a simplicial complex B, a smooth k-manifold M, and a smooth triangulation $\eta : B \to M$. This means that η is a homeomorphism which is smooth on closed simplices. In other words, for any l-simplex Δ of B, there are a linear embedding $\iota : \Delta \hookrightarrow \mathbb{R}^l$, an open neighborhood $U \subset \mathbb{R}^l$ of $\iota(\Delta)$, and a smooth immersion $\tilde{\eta} : U \to M$ with $\tilde{\eta} \circ \iota = \eta|_\Delta$.

Now, pick a point $p \in B$, and let Δ be the unique simplex whose interior contains p. Recall that $\mathrm{Star}(\Delta)$ is the subcomplex of B generated by the closed simplices which contain Δ. Let Δ' be a maximal simplex of $\mathrm{Star}(\Delta)$, so that p is contained in the closure $\bar{\Delta}'$ of Δ'. By the definition of a smooth triangulation, we can differentiate the restriction of η to $\bar{\Delta}'$ at p to obtain a linear map

$$d(\eta|_{\bar{\Delta}'})_p : T_p\bar{\Delta}' \to T_{\eta(p)}M.$$

There is then a unique linear map $f : \bar{\Delta}' \to T_{\eta(p)}M$ with $f(p) = 0$ and $df_p = d(\eta|_{\bar{\Delta}'})_p$; piecing these together gives a simplex-wise linear map $F_\Delta : \mathrm{Star}(\Delta) \to T_{\eta(p)}M$. The images under F_Δ of the vertices of $\mathrm{Star}(\Delta)$ give rise to a realizable oriented matroid of rank k, and the basic philosophy of the subject is that these oriented matroids alone carry a great deal of information about the smooth structure of the manifold.

So far, of course, we have only managed to produce a family of oriented matroids parametrized by the points p of M, which can hardly be described as a purely combinatorial object. However, much as in the construction of a matroid bundle from a vector bundle already discussed, if the initial smooth triangulation is "suitably generic," the corresponding matroid stratification of M turns out to be a cell complex refining B. Furthermore, as one would expect, if σ and σ' are two cells satisfying $\sigma' \subset \partial\sigma$ and with corresponding matroids M and M', then we have $M \rightsquigarrow M'$.

A CD manifold structure of dimension k on a simplicial complex B with n vertices is a generalization of this: it is a cell complex $\hat{B}$ refining B and a map from the poset of cells of $\hat{B}$ to $\mathrm{MacP}(k,n)$ satisfying certain additional axioms that obviously hold in the setting outlined above. It then becomes apparent that every CD manifold gives rise to an associated tangent matroid bundle; one of the primary benefits of Theorem 1.1 is the fact that this matroid bundle actually arises from a vector bundle, and therefore that all the information (most notably, all characteristic classes) encoded in the tangent bundle of a smooth manifold can be extracted from the combinatorial remnants of the smooth structure provided by the oriented matroids.

4. Schubert stratification and its consequences

We first briefly recall the definition of the Schubert cells in the Grassmannian of k-planes in $\mathbb{R}^n$. Here, when we write $\mathbb{R}^n$, a choice of coordinates is implicit, and these are used in defining the cells. Of course, all that is actually needed to define the Schubert cells is a complete flag of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{R}^n$ with $\dim V_i = i$; the standard basis $\{\zeta_1, \ldots, \zeta_n\}$ determines these data by setting V_i to be the span of the set $\{\zeta_1, \ldots, \zeta_i\}$. Now, for any k-dimensional subspace V of $\mathbb{R}^n$, we have an associated sequence of integers $0 = \dim(V_0 \cap V) \leq \dim(V_1 \cap V) \leq \cdots \leq \dim(V_{n-1} \cap V) \leq \dim(V_n \cap V) = k$. The Schubert stratification of $\mathrm{G}(k,n)$ is obtained by letting each stratum consist of all k-planes whose corresponding sequence of integers is some fixed sequence. More formally, let $1 \leq d_1 < d_2 < \cdots < d_k \leq n$ be a sequence of integers. The corresponding open Schubert cell $D_{d_1,\ldots,d_k} \subset \mathrm{G}(k,n)$ is then the set of all k-planes $V \subset \mathbb{R}^n$ with

$$\dim(V_{d_i} \cap V) = i$$

and

$$\dim(V_{d_i - 1} \cap V) = i - 1$$

for all $i = 1, \ldots, k$. One can verify that $D_{d_1,\ldots,d_k}$ is an open cell of dimension $\sum_i (d_i - i)$.

Our analysis of the geometry of $\|\operatorname{MacP}(k,n)\|$ proceeds by analogy with the Schubert stratification of $\mathrm{G}(k,n)$. We first define a stratification of the poset $\operatorname{MacP}(k,n)$, and later explain how to lift it to a stratification of the geometric realization.

Definition 4.1. Fix a sequence of integers $1 \leq d_1 < \cdots < d_k \leq n$. We let $E_{d_1,\ldots,d_k}$ denote the stratum of rank k oriented matroids M on the set $\{a_1, \ldots, a_n\}$ having the property that for all i,

$$\operatorname{rank}_M\{a_{d_i}, a_{d_i+1}, \ldots, a_n\} = k - i + 1$$

and

$$\operatorname{rank}_M\{a_{d_i+1}, a_{d_i+2}, \ldots, a_n\} = k - i.$$

These sets $E_{d_1,\ldots,d_k}$ give us a stratification of $\mathrm{MacP}(k,n)$ whose pieces are indexed by k-element subsets of $\{1,\ldots,n\}$. Let us examine the stratification more carefully. Consider the Gale order [9], or majorization order, on the set of k-element subsets of $\{1,\ldots,n\}$: we say that $\{d'_1,\ldots,d'_k\} \le \{d_1,\ldots,d_k\}$ in this order if $d'_1 < \cdots < d'_k$ and $d_1 < \cdots < d_k$ and $d'_i \le d_i$ for all i. It is a basic fact that for any rank k oriented matroid M on the set $\{a_1,\ldots,a_n\}$, the set of independent k-element subsets of $\{a_1,\ldots,a_n\}$ has a unique maximal element in this order; this can be seen easily by using an induction argument and the chirotope axioms. From this fact, we can reinterpret the Schubert stratification of the MacPhersonian.

Proposition 4.2. *Let $M \in \mathrm{MacP}(k,n)$ and suppose that $\{a_{d_1},\ldots,a_{d_k}\}$ is the maximal basis for M in the Gale order. We then have $M \in E_{d_1,\ldots,d_k}$.*

Proof. Since $\{a_{d_i},\ldots,a_{d_k}\}$ and $\{a_{d_{i+1}},\ldots,a_{d_k}\}$ are independent for M, it must certainly be the case that $\mathrm{rank}_M\{a_{d_i},a_{d_i+1},\ldots,a_n\} \ge k-i+1$ and $\mathrm{rank}_M\{a_{d_i+1},a_{d_i+2},\ldots,a_n\} \ge k-i$. On the other hand, if $\mathrm{rank}_M\{a_{d_i},a_{d_i+1},\ldots,a_n\} > k-i+1$, then there must be an independent set $\{a_{d'_{i-1}},\ldots,a_{d'_k}\} \subset \{a_{d_i},\ldots,a_n\}$ which could be completed to a basis $\{a_{d'_1},\ldots,a_{d'_k}\}$. This basis would then not precede $\{a_{d_1},\ldots,a_{d_k}\}$ in the Gale order, which is a contradiction. Hence, $\mathrm{rank}_M\{a_{d_i},a_{d_i+1},\ldots,a_n\} = k-i+1$ and, by a similar argument $\mathrm{rank}_M\{a_{d_i+1},a_{d_i+2},\ldots,a_n\} = k-i$. □

As a result, if $M \rightsquigarrow M'$ and $M \in E_{d_1,\ldots,d_k}$ and $M' \in E_{d'_1,\ldots,d'_k}$, then it must be the case that $\{a_{d'_1},\ldots,a_{d'_k}\}$ is a basis for M' and hence also for M, and thus that $\{d'_1,\ldots,d'_k\} \le \{d_1,\ldots,d_k\}$. In other words, the Schubert stratification gives rise to a poset map $\mathrm{MacP}(k,n) \to \binom{n}{k}$, where $\binom{n}{k}$ denotes the Gale-ordered poset of k-element subsets of $\{1,\ldots,n\}$.

We are now ready to define the stratification of the geometric realization of $\mathrm{MacP}(k,n)$; the strata will, again, be indexed by k-element subsets of $\{1,\ldots,n\}$.

Definition 4.3. Recall that there is a map $\nu : \|\mathrm{MacP}(k,n)\| \to \mathrm{MacP}(k,n)$. Let $e_{d_1,\ldots,d_k} \subset \|\mathrm{MacP}(k,n)\|$ denote the space $\nu^{-1}(E_{d_1,\ldots,d_k})$.

We must now check that if we have a k-plane $V \subset \mathbb{R}^n$ which is an element of $D_{d_1,\ldots,d_k}$, then its corresponding oriented matroid is in $E_{d_1,\ldots,d_k}$. Indeed, let W_i denote the span of $\{\zeta_{i+1},\ldots,\zeta_n\}$, that is, the orthogonal complement in $\mathbb{R}^n$ of V_i. Then we need only show that for all i, we have $\dim(\mathrm{im}(W_i \to V)) = k - \dim(V_i \cap V)$. By restricting to V the orthogonal projection of $\mathbb{R}^n$ onto W_i, we obtain a short exact sequence

$$0 \longrightarrow V_i \cap V \longrightarrow V \longrightarrow \mathrm{im}(V \to W_i) \longrightarrow 0.$$

The result then follows from the fact that $\dim(\operatorname{im}(V \to W_i)) = \dim(\operatorname{im}(W_i \to V))$, which holds because if we fix choices of bases for V and W_i, then the matrices of these two linear transformations are transpose to one another. This allows us to make the following observation.

Proposition 4.4. *The map* $\mu : \mathrm{G}(k,n) \to \mathrm{MacP}(k,n)$ *sends the open Schubert cell* $D_{d_1,\dots,d_k}$ *to* $E_{d_1,\dots,d_k}$. *Consequently,* $\pi : \mathrm{G}(k,n) \to \|\mathrm{MacP}(k,n)\|$ *sends* $D_{d_1,\dots,d_k}$ *to the open stratum* $e_{d_1,\dots,d_k}$.

Proof. The first sentence follows directly from the discussion above. As for the second sentence, let $p \in D_{d_1,\dots,d_k}$. Notice that by Proposition 4.2, the matroid stratification given by the fibers of μ is subordinate to the Schubert stratification of $\mathrm{G}(k,n)$ into the $D_{d_1,\dots,d_k}$. The map π strictly speaking takes as its domain the barycentric subdivision of a simplicial decomposition of $\mathrm{G}(k,n)$ subordinate to the stratification of $\mathrm{G}(k,n)$ provided by the map μ, as explained in Section 2. Denote by ΔG the poset of simplices of $\mathrm{G}(k,n)$ in this decomposition. The point p is in the interior of some simplex in $\|\Delta G\|$, which corresponds to a chain in ΔG. But the open simplex corresponding to the maximal element of this chain is certainly contained in $D_{d_1,\dots,d_k}$, and so the maximal element of the chain it is mapped to in $\mathrm{MacP}(k,n)$ is contained in $E_{d_1,\dots,d_k}$. Therefore, $\pi(p) \in e_{d_1,\dots,d_k}$. □

Now, we would like to study the geometric structure of the stratum $e_{d_1,\dots,d_k}$. Consider the oriented matroid $M_{d_1,\dots,d_k}$ on the set $\{a_1,\dots,a_n\}$ whose only basis is the set $\{a_{d_1},\dots,a_{d_k}\}$. In other words, $M_{d_1,\dots,d_k}$ is realized by any map $\rho : \{a_1,\dots,a_n\} \to V$ to a rank k vector space taking $\{a_{d_1},\dots,a_{d_k}\}$ to a basis and a_c to the zero vector for every $a_c \notin \{a_{d_1},\dots,a_{d_k}\}$. One easily sees by examining chirotopes that $M_{d_1,\dots,d_k}$ is a specialization of every other element of $E_{d_1,\dots,d_k}$. Therefore, the space $\|E_{d_1,\dots,d_k}\|$ is contractible, because it is a cone over the space $\|E_{d_1,\dots,d_k} \backslash \{M_{d_1,\dots,d_k}\}\|$.

Moreover, recall that the Schubert stratification gives rise to a poset map $\mathrm{MacP}(k,n) \to \binom{n}{k}$. In this language, we may identify $e_{d_1,\dots,d_k}$ with the space $P^l_{\{d_1,\dots,d_k\}}$. Thus, by Proposition 2.6, $\|E_{d_1,\dots,d_k}\|$ is a deformation retract of $e_{d_1,\dots,d_k}$. The above observation then tells us that $e_{d_1,\dots,d_k}$ is contractible.

We now must analyze the geometric procedure of attaching the stratum $e_{d_1,\dots,d_k}$ to smaller strata. Let S denote the poset $E_{d_1,\dots,d_k} \backslash \{M_{d_1,\dots,d_k}\}$, and, as usual, $\|S\|$ its geometric realization. Also, let

$$X = \bigcup_{\{d'_1,\dots,d'_k\} < \{d_1,\dots,d_k\}} E_{d'_1,\dots,d'_k}$$

and $Y = X \cup E_{d_1,\dots,d_k}$. Recall that $\|E_{d_1,\dots,d_k}\|$ is a cone over $\|S\|$. This of course immediately tells us the homotopy type of $\|E_{d_1,\dots,d_k}\|$ and $e_{d_1,\dots,d_k}$, but in order to better understand their geometry, we need to come to grips with the homotopy type of $\|S\|$ as well.

Let $N = \sum_{i=1}^{k}(d_i - i)$. The open Schubert cell $D_{d_1,\ldots,d_k}$ is then an N-dimensional real affine space. Now, recall that we constructed the map π by first defining a map $\mu : \mathrm{G}(k,n) \to \mathrm{MacP}(k,n)$ and then finding a simplicial decomposition of $\mathrm{G}(k,n)$ subordinate to the stratification by fibers of μ. The composition

$$\mathrm{G}(k,n) \xrightarrow{\mu} \mathrm{MacP}(k,n) \longrightarrow \binom{n}{k}$$

defines a map from the poset ΔG of simplices of $\mathrm{G}(k,n)$ in this decomposition to $\binom{n}{k}$. Then application of Proposition 2.6 to the space $\Delta G^l_{\{d_1,\ldots,d_k\}}$ tells us that the open cell $D_{d_1,\ldots,d_k} = \pi^{-1}(e_{d_1,\ldots,d_k})$ deformation retracts onto $\pi^{-1}(\|E_{d_1,\ldots,d_k}\|)$.

Furthermore, $\mu^{-1}(M_{d_1,\ldots,d_k})$ is a single vertex $p_{d_1,\ldots,d_k}$, corresponding to the plane spanned by the basis vectors $\zeta_{d_1},\ldots,\zeta_{d_k}$. Let $\mathbf{1}$ denote the two element poset $\{0 < 1\}$. Then we get a map from the poset $\Delta D \subset \Delta G$ of simplices of $\pi^{-1}(\|E_{d_1,\ldots,d_k}\|)$ to $\mathbf{1}$ by sending $p_{d_1,\ldots,d_k}$ to 0 and every other simplex to 1. Now, we can apply Proposition 2.6 to ΔD^l_1 to see that the space $\pi^{-1}(\|S\|)$ is a deformation retract of $\pi^{-1}(\|E_{d_1,\ldots,d_k}\|\backslash\{p_{d_1,\ldots,d_k}\})$. Thus, $\pi^{-1}(\|S\|)$ is a homotopy S^{N-1}; hereafter, we will refer to it simply as $\tilde{S}^{N-1}$.

Proposition 4.5. *The map* $\pi|_{\tilde{S}^{N-1}} : \tilde{S}^{N-1} \to \|S\|$ *is a homotopy equivalence.*

We postpone the proof to Section 5. Assuming this result, we have a complete homotopy-theoretic charactarization of the pair $(\|E_{d_1,\ldots,d_k}\|, \|S\|)$. What remains to be established is the relationship between the homotopy type of $\|Y\|$ and those of $\|X\|$, $\|E_{d_1,\ldots,d_k}\|$, and $\|S\|$. Our intuition, in analogy with the case of the Grassmannian, is that $\|Y\|$ ought to be recovered as the mapping cone of a map $\|S\| \to \|X\|$, obtained by "sliding" $\|S\|$ toward $\|X\|$ in the space $\|X \cup S\|$. In other words, we would like for the deformation retraction $\|X \cup S\|\backslash\|S\| \to \|X\|$ whose existence is guaranteed by Proposition 2.6 to extend to a retraction $\|X \cup S\| \to \|X\|$. We do not actually construct such a deformation retraction. However, the following result is enough to give us what we need.

Proposition 4.6. *The inclusion* $X \hookrightarrow X \cup S$ *induces a homotopy equivalence on geometric realizations.*

Again, we postpone the proof to Section 5. Using this result, we can easily explain the proof of Theorem 1.1.

Proof of Theorem 1.1. This is an induction on the (Gale-ordered) cells. Indeed, let $A \subset \mathrm{G}(k,n)$ denote $\pi^{-1}(\|X\|)$, the union of the open cells $D_{d'_1,\ldots,d'_k}$ with $\{d'_1,\ldots,d'_k\} < \{d_1,\ldots,d_k\}$. We may assume by induction that the map

$\pi|_A : A \to \|X\|$ is a homotopy equivalence; the base case of this induction is simply the fact that $D_{1,\ldots,k} \cong \|E_{1,\ldots,k}\| \cong *$, the one-point space. For the inductive step, we need to verify that π is still a homotopy equivalence after we attach the cell $D_{d_1,\ldots,d_k}$ to A to obtain $B \subset \mathrm{G}(k,n)$ and the stratum $e_{d_1,\ldots,d_k}$ to $\|X\|$ to obtain $\|Y\|$.

In other words, we have a map of pairs $\pi : (B, A) \to (\|Y\|, \|X\|)$, and we know that $\pi|_A$ is a homotopy equivalence. We build on this fact by covering $\|Y\|$ by spaces we can understand using Propositions 4.5 and 4.6 and then studying the behavior of π on the pullback of this cover.

We cover $\|Y\|$ by two subsets, U and L, defined by

$$U := \|Y\| \backslash \|X\|$$

and

$$L := \|Y\| \backslash \|M_{d_1,\ldots,d_k}\|.$$

By Proposition 2.6, the inclusion $\|E_{d_1,\ldots,d_k}\| \hookrightarrow U$ is a homotopy equivalence, and so U is contractible. Moreover, $\pi^{-1}(U) = D_{d_1,\ldots,d_k}$ which is of course also contractible. Thus, the map $\pi|_{\pi^{-1}(U)}$ is a homotopy equivalence.

Secondly, again by Proposition 2.6, the inclusion $\|X \cup S\| \hookrightarrow L$ is an equivalence. Moreover, by Proposition 4.6, the inclusion $\|X\| \hookrightarrow L$ is a homotopy equivalence. Furthermore, $\pi^{-1}(L) = B \backslash \{\pi^{-1}(p_{d_1,\ldots,d_k})\}$ and so the inclusion $A \hookrightarrow \pi^{-1}(L)$ induces an equivalence. Therefore, we have a commutative square

$$\begin{tikzcd} A \arrow[r, "\pi|_A"] \arrow[d, hook] & \|X\| \arrow[d, hook] \\ \pi^{-1}(L) \arrow[r, "\pi|_{\pi^{-1}(L)}"] & L \end{tikzcd}$$

where the two vertical maps are equivalences, and the top map is an equivalence by induction, so the bottom map must be a homotopy equivalence as well.

Lastly, $\pi^{-1}(U \cap L) = \tilde{S}^{N-1}$, and, by Proposition 2.6, the inclusion $\|S\| \hookrightarrow U \cap L$ is an equivalence. Also, by Proposition 4.5, the map $\pi|_{\pi^{-1}(U\cap L)}$ is a homotopy equivalence. Therefore, the map $\pi|_B : B \to \|Y\|$ induces an equivalence on all the opens and intersections of a cover of the target space, and is therefore an equivalence itself. This completes the inductive step and the proof. $\square$

5. Proofs of Propositions 4.5 and 4.6

Throughout, we study a fixed stratum $E_{d_1,\ldots,d_k}$. As in the previous section, denote the poset $E_{d_1,\ldots,d_k} \backslash \{M_{d_1,\ldots,d_k}\}$ by S and its geometric realization by $\|S\|$. To complete the proof of Theorem 1.1, it suffices to prove Propositions 4.5

and 4.6. We begin with Proposition 4.5, that is, the statement that the map $\pi|_{\tilde{S}^{N-1}} : \tilde{S}^{N-1} \to \|S\|$ is a homotopy equivalence.

To address this issue, we will need to introduce more notation. Suppose we are given a rank k oriented matroid M on the set $\{a_1, \ldots, a_n\}$ for which $\{a_{d_1}, \ldots, a_{d_k}\}$ is a basis; let χ be a chirotope for M, normalized by the assumption $\chi(a_{d_1}, \ldots, a_{d_k}) = +$. We can then associate a k-tuple $(\tau_c^1, \ldots, \tau_c^k)$ of elements of $\{+, -, 0\}$ to each a_c, by defining $\tau_c^j = \chi(a_{d_1}, \ldots, a_{d_{j-1}}, a_c, a_{d_{j+1}}, \ldots, a_{d_k})$. (Thus, we have $\tau_{d_i}^j = \delta_i^j$, the Kronecker delta.) In the realizable case, these k-tuples are very easy to understand: if ρ is any realization of the oriented matroid (or, in fact, of the oriented matroid obtained by deleting all elements except $\{a_c, a_{d_1}, \ldots, a_{d_k}\}$), then in the inner product for which the basis $\{\rho(a_1), \ldots, \rho(a_d)\}$ is orthonormal, we have $\tau_c^j = \operatorname{sgn}(\langle \rho(a_c), \rho(a_{d_j}) \rangle)$.

Now, arrange these k-tuples of τ_c's into an $n \times k$ matrix $\tau(M)$ whose (c, j)-entry is τ_c^j. The effect of a specialization $M \rightsquigarrow M'$ on these matrices is obviously to set some (possibly empty) collection of nonzero entries to zero. That is, if we partially order the set of all $n \times k$ matrices over the set $\{+, -, 0\}$ by setting $A \geq A'$ whenever A and A' agree in all nonzero entries of A', then the map τ from oriented matroids to matrices becomes a poset map.

Proof of Proposition 4.5. The proof proceeds by induction on the cells in the Gale order. The first two cases are $\{d_1, \ldots, d_k\} = \{1, \ldots, k\}$, for which $\tilde{S}^{N-1} = \|S\| = S = \emptyset$, and $\{d_1, \ldots, d_k\} = \{1, 2, \ldots, k-1, k+1\}$. In this second case, $S = \{M^+, M^-\}$, where for M^+ (resp. M^-), the only elements that are not loops are $a_{d_1}, \ldots, a_{d_k}, a_c$, and a_c is parallel (resp. antiparallel) to a_{d_k}. Of course $\pi|_{\tilde{S}^{N-1}}$ is then a homotopy equivalence.

For the inductive step, our strategy is to cover $\|S\|$ by two spaces whose homotopy types we can identify (in fact, they will be contractible) and the homotopy type of whose intersection we can also understand, by induction. The behavior of π with respect to this cover will then be almost obvious.

So, pick a pair (c, d_i) such that $c = d_i - 1$ and $c \notin \{d_1, \ldots, d_k\}$. Such a pair will exist since $\{d_1, \ldots, d_k\} \neq \{1, \ldots, k\}$. Let $S^+ \subset S$ denote the set consisting of all oriented matroids M such that $\tau_c^i(M) \geq 0$ and $S^- \subset S$ the set of all M with $\tau_c^i(M) \leq 0$. Equivalently, let A^+ (resp. A^-) denote the $n \times k$ matrix whose only nonzero element other than the obligatory $+$ in the (d_j, j) slot for all $j = 1, \ldots, k$ is a $+$ (resp. $-$) in the (c, i) slot. Then

$$S^+ = \bigcup_{A \geq A^+} \tau^{-1}(A)$$

and

$$S^- = \bigcup_{A \geq A^-} \tau^{-1}(A).$$

The induction tells us precisely that $\pi|_{\pi^{-1}(S^+\cap S^-)}$ is a homotopy equivalence. Indeed, consider the permutation $\sigma \in \Sigma_n$ that transposes c and d_i. It acts on both $\mathrm{MacP}(k,n)$ and $\mathrm{G}(k,n)$, and because the action permutes fibers of the map $\mu : \mathrm{G}(k,n) \to \mathrm{MacP}(k,n)$, we get the following commutative square:

$$\begin{tikzcd} \mathrm{G}(k,n) \arrow[r, "\sigma"] \arrow[d, "\pi"] & \mathrm{G}(k,n) \arrow[d, "\pi"] \\ \|\mathrm{MacP}(k,n)\| \arrow[r, "\sigma"] & \|\mathrm{MacP}(k,n)\|. \end{tikzcd}$$

Now,

$$\sigma\left(S^+ \cap S^-\right) = E_{d_1,\ldots,d_{i-1},c,d_i,\ldots,d_k} \backslash \{M_{d_1,\ldots,d_{i-1},c,d_{i+1},\ldots,d_k}\}\ .$$

and

$$\sigma\left(\pi^{-1}(S^+ \cap S^-)\right) = D_{d_1,\ldots,d_{i-1},c,d_{i+1},\ldots,d_k} \backslash \pi^{-1}(M_{d_1,\ldots,d_{i-1},c,d_{i+1},\ldots,d_k}).$$

This is because if $M \in S$, then $\sigma(M)$ contains $\{a_{d_1},\ldots,a_{d_{i-1}},a_c,a_{d_{i+1}},\ldots,a_{d_k}\}$ as a basis, and it is the maximal basis of $\sigma(M)$ if and only if $\{a_{d_1},\ldots,a_{d_k}\}$ is not a basis. But $\{a_{d_1},\ldots,a_{d_k}\}$ is not a basis of $\sigma(M)$ if and only if $\{a_{d_1},\ldots,a_{d_{i-1}}$, $a_c,a_{d_{i+1}},\ldots,a_{d_k}\}$ is not a basis of M, i.e., if and only if $M \in S^+ \cap S^-$. Thus $\sigma^{-1} \circ \left(\pi|_{\pi^{-1}(S^+\cap S^-)}\right) \circ \sigma$ is a homotopy equivalence by induction, and therefore $\pi|_{\pi^{-1}(S^+\cap S^-)}$ is as well.

Now, $D_{d_1,\ldots,d_k}$ admits a canonical coordinate system, whose axes are parametrized by the pairs (c',d_j) with $c' \in \{1,2,\ldots,d_j\}\backslash\{d_1,\ldots,d_j\}$ and $j \in \{1,\ldots,k\}$. We can then describe $\pi^{-1}(\|S^+\|)$ as the subspace of $D_{d_1,\ldots,d_k}$ consisting of all elements other than the origin for which the (c,d_i)-coordinate is nonnegative. Thus, $\pi^{-1}(\|S^+\|)$ is contractible, and similarly, $\pi^{-1}(\|S^-\|)$ is as well.

We must now show that $\|S^+\|$ and $\|S^-\|$ are contractible. This will again be an induction, this time on d_k. The base case is $d_k = k+1$. In this situation, the fact is essentially obvious, and can in any case be seen from the fact that all such oriented matroids are realizable (for a proof of this, see [7]).

To see the general case, pick any element $c' \in \{1,2,\ldots,d_k\}\backslash\{d_1,d_2,\ldots, d_k,c\}$. Let $S_0^+ \subset S^+$ denote the subposet consisting of all oriented matroids for which $a_{c'}$ is a loop. By induction, $\left\|S_0^+\right\|$ is contractible. Let $P_0^+ \subset S^+$ denote the subposet consisting of all oriented matroids specializing to some element of S_0^+, that is, all oriented matroids for which $\{a_{d_1},\ldots,a_{d_k},a_{c'}\}$ are not the only elements that are not loops. We can easily see by Quillen's Theorem A that the inclusion $S_0^+ \hookrightarrow P_0^+$ induces an equivalence, that is, that $\left\|P_0^+\right\|$ is contractible. To check this, we must show that for all $M \in P_0^+$, the poset

$$\{M' \in S_0^+ | M \rightsquigarrow M'\}$$

is contractible. But this poset has a unique maximal element, namely the oriented matroid obtained by replacing with $a_{c'}$ a loop in M. Thus, $\left\|P_0^+\right\|$ is contractible.

We are now prepared to show that the entirety of $\|S^+\|$ is contractible. To see this, set $A := S^+ \backslash P_0^+$. That is, A consists of all the oriented matroids in S^+ for which $\{a_{d_1}, \ldots, a_{d_k}, c'\}$ are the only elements which are not loops. Then by Proposition 2.6, the space $\|S^+\| \backslash \|A\|$ deformation retracts onto $\|P_0^+\|$ and is hence contractible. Let $B \subset S^+$ denote the poset consisting of oriented matroids for which $a_{c'}$ is not a loop, and the only element outside of $\{a_{d_1}, \ldots, a_{d_k}, a_{c'}\}$ which permitted to not be a loop is a_c; moreover, if a_c is not a loop then it is requited to be parallel to a_{d_i}. Thus, $B \cong A \times \mathbf{1}$. We now cover $\|S^+\|$ by two subsets, $\|S^+\| \backslash \|A\|$ and $\|B\|$. The intersection is $\|B\| \backslash \|A\|$ which is homotopy equivalent to $\|B\|$ by Proposition 2.6, and thus the inclusion $\|S^+\| \backslash \|A\| \hookrightarrow \|S^+\|$ induces a homotopy equivalence, and $\|S^+\|$ is therefore contractible.

The proof that $\|S^-\|$ is contractible is identical; equivalently, one can simply observe that the map $f : S^+ \to S^-$ which replaces a_c by an element antiparallel to it is an isomorphism of posets and thus induces a homeomorphism on nerves. □

So, to obtain our main result, we need only prove Proposition 4.6. The proof comes in several steps; the first is essentially a simplified version of the argument used to demonstrate Proposition 4.5.

PROPOSITION 5.1. *Fix $M \in \mathrm{MacP}(k, n)$. The poset*

$$S^M := \{M' \in S | M \rightsquigarrow M'\}$$

is either empty or contractible.

Proof. Throughout the argument, we assume that S^M is nonempty. We perform an induction which will follow just like the proof of Proposition 4.5, only this time, rather than inducting on d_k, we induct on the number of elements of $\{a_1, \ldots, a_n\}$ which are not loops in all of S^M. In the base case, there are $k+1$ such elements; k of them are the $a_{d_1}, \ldots, a_{d_k}$, and we denote the $(k+1)$st by $a_{c'}$. Denote by M_0 the oriented matroid obtained by replacing all the elements other than the $\{a_{d_1}, \ldots, a_{d_k}, a_{c'}\}$ in M by loops. Then M_0 admits a realization $\rho_0 : \{a_1, \ldots, a_n\} \to \mathbb{R}^k$ for which $\rho_0(a_{d_i}) = \zeta_i$, the ith standard basis vector, each coordinate of $\rho_0(a_{c'})$ comes from the set $\{1, -1, 0\}$, and of course we have $\rho_0(a_c) = 0$ for all the rest of the a_c. Now, let $\rho_1 : \{a_1, \ldots, a_n\} \to \mathbb{R}^k$ be the map defined by $\rho_1(a_c) = \rho_0(a_c)$ if $c \neq c'$ and

$$\langle \rho_1(c'), \zeta_i \rangle = \begin{cases} \langle \rho_0(c'), \zeta_i \rangle & \text{if } d_i > c' \\ 0 & \text{if } d_i < c', \end{cases}$$

where $\langle , \rangle$ as usual denotes the standard inner product. Denote by M_1 the oriented matroid realized by ρ_1. It is easy to see that M_1 is the unique maximal element of S^M, and so $\|S^M\|$ is contractible as desired.

In the inductive step, we fix two elements $c', c'' \in \{1, 2, \ldots, d_k\} \backslash \{d_1, d_2, \ldots, d_k\}$ such that neither $a_{c'}$ nor $a_{c''}$ is always a loop in S^M. Furthermore, assume that $c' > c''$. Now, let $S_0^M \subset S^M$ denote the set of oriented matroids for which $a_{c'}$ is a loop. Then, by induction, $\|S_0^M\|$ is contractible. Letting $P_0^M \subset S^M$ denote the set of all oriented matroids in S^M specializing to an element of S_0^M, we find just as before that the inclusion $S_0^M \hookrightarrow P_0^M$ induces an equivalence. This is because, once again, for all $N \in P_0^M$, the set

$$\{M' \in S_0^M | N \rightsquigarrow M'\}$$

has a unique maximal element, namely the matroid obtained by replacing c' in N by a loop.

Now, let $A = S^M \backslash P_0^M$. Let $N_0 \in \mathrm{MacP}(k, n)$ denote the oriented matroid obtained by replacing every element of M other than the $\{a_{d_1}, \ldots, a_{d_k}, a_{c'}, a_{c''}\}$ by a loop. It is easy to check that there is an oriented matroid $N_1 \in S$ satisfying $N_0 \rightsquigarrow N_1$ and

$$(N_1)\Big|_{\{a_{d_1}, \ldots, a_{d_k}, a_{c'}\}} = (N_0)\Big|_{\{a_{d_1}, \ldots, a_{d_k}, a_{c'}\}},$$

and for which there is an $x \in \{d_1, \ldots, d_k, c'\}$ with $x > c''$ such that $a_{c''}$ is either parallel or antiparallel to a_x. We then let $B \subset S^M$ denote the poset consisting of the oriented matroids for which all elements other than the $\{a_{d_1}, \ldots, a_{d_k}, a_{c'}, a_{c''}\}$ are loops and for which $a_{c''}$ is either parallel or antiparallel to a_x. Then once again $B \cong A \times \mathbf{1}$ and so the inclusion of $\|B\| \backslash \|A\|$ in $\|B\|$ is an equivalence. Therefore, the space $\|S^M\| = (\|S^M\| \backslash \|A\|) \cup \|B\|$ is homotopy equivalent to $(\|S^M\| \backslash \|A\|)$ which is homotopy equivalent to $\|P_0^M\|$ and is therefore contractible. This completes the proof. $\square$

For the next step we need to introduce more notation. Fix some subset

$$\underline{d} := \{\{d_1^1, \ldots, d_k^1\}, \ldots, \{d_1^r, \ldots, d_k^r\}\} \subset \binom{n}{k}$$

and suppose it is an order ideal. Then set

$$E_{\underline{d}} := \bigcup_{i=1}^{r} E_{d_1^i, \ldots, d_k^i}.$$

Proposition 5.2. *Suppose as above that* $\underline{d} \subset \binom{n}{k}$ *is an order ideal. Then for all* $M \in \mathrm{MacP}(k, n)$, *the poset*

$$E_{\underline{d}}^M := \{M' \in E_{\underline{d}} | M \rightsquigarrow M'\}$$

is either contractible or empty.

Proof. We assume throughout that $E_{\underline{d}}^M$ is nonempty and argue by induction on $\underline{d}$, ordered by inclusion. The base case is the example $\underline{d} = \{\{1, 2, \ldots, k\}\}$ which is, of course, trivial.

We now carry out the inductive step. Suppose that $\{d_1^r, \ldots, d_k^r\}$ is a maximal element of $\underline{d}$ in the Gale order. We may then assume that $E_{\underline{d}}^M \cap E_{d_1^r,\ldots,d_k^r} \neq \emptyset$, for otherwise we are finished by induction. We cover $E_{\underline{d}}^M$ by two sets,

$$E_{d_1^r,\ldots,d_k^r}^M := E_{\underline{d}}^M \cap E_{d_1^r,\ldots,d_k^r}$$

and

$$E_0^M := E_{\underline{d}}^M \backslash \{M_{d_1^r,\ldots,d_k^r}\}.$$

First of all, $\left\|E_{d_1^r,\ldots,d_k^r}^M\right\|$ is contractible because $E_{d_1^r,\ldots,d_k^r}^M$ contains a unique minimal element $M_{d_1^r,\ldots,d_k^r}$. Secondly, $\left\|E_{d_1^r,\ldots,d_k^r}^M \cap E_0^M\right\|$ is either empty or contractible by Proposition 5.1. If it is empty, then we have $E_{\underline{d}}^M = \{M_{d_1^r,\ldots,d_k^r}\}$ which is obviously contractible, and we are done.

Thus, it remains only to address the case in which $\left\|E_{d_1^r,\ldots,d_k^r}^M \cap E_0^M\right\|$ is contractible by showing that $\left\|E_0^M\right\|$ is contractible as well. Let $\underline{d}' = \underline{d}\backslash\{d_1^r, \ldots, d_k^r\}$. Then $\left\|E_{\underline{d}'}^M\right\|$ is contractible by induction, and we have an inclusion $E_{\underline{d}'}^M \hookrightarrow E_0^M$. We will use Quillen's Theorem A to show that this inclusion induces an equivalence. Fix $N \in E_0^M$. We need only see that the poset

$$\left\{M' \in E_{\underline{d}'}^M | N \rightsquigarrow M'\right\}$$

is contractible. But this is precisely the poset $E_{\underline{d}'}^N$, which is contractible by induction, and we are done. □

We now have all the necessary ingredients to finish our argument.

Proof of Proposition 4.6. This will follow, as usual, from Quillen's Theorem A. Recall that we need to show that the inclusion $X \hookrightarrow X \cup S$ induces an equivalence. By Theorem A, it suffices to check that for all $M \in S$, the set

$$\{M' \in X | M \rightsquigarrow M'\}$$

is contractible. Proposition 5.2 guarantees that this space must either be empty or contractible. By the definition of S, there must be some basis $\{d_1', \ldots, d_k'\}$ of M which strictly precedes $\{d_1, \ldots, d_k\}$ in the Gale order. Then $M_{d_1',\ldots,d_k'} \in X$ and $M \rightsquigarrow M_{d_1',\ldots,d_k'}$. Thus, the space in question is nonempty and the proof is complete. □

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