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A proof of Kirillov's conjecture

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Dedicated to Ilya Piatetski-Shapiro

1. Introduction

Let $G = \mathrm{GL}_n(K)$ where K is either $\mathbf{R}$ or $\mathbf{C}$ and let $P = P_n(K)$ be the subgroup of matrices in $\mathrm{GL}_n(K)$ consisting of matrices whose last row is $(0, 0, \ldots, 0, 1)$. Let π be an irreducible unitary representation of G. Gelfand and Neumark [Gel-Neu] proved that if $K = \mathbf{C}$ and π is in the Gelfand-Neumark series of irreducible unitary representations of G then the restriction of π to P remains irreducible. Kirillov [Kir] conjectured that this should be true for all irreducible unitary representations π of $\mathrm{GL}_n(K)$, where K is $\mathbf{R}$ or $\mathbf{C}$:

Conjecture 1.1. *If π is an irreducible unitary representations of G on a Hilbert space H then $\pi|P$ is irreducible.*

Bernstein [Ber] proved Conjecture 1.1 for the case where K is a p-adic field. Sahi [Sah] proved Conjecture 1.1 for the case where $K = \mathbf{C}$ or where π is a tempered unitary representation of G. Sahi and Stein [Sah-Ste] proved Conjecture 1.1 for Speh's representations of $\mathrm{GL}_n(\mathbf{R})$ leaving the case of Speh's complementary series unsettled. Sahi [Sah] showed that Conjecture 1.1 has important applications to the description of the unitary dual of G. In particular, Sahi showed how to use the Kirillov conjecture to give a simple proof for the following theorem:

Theorem 1.2 ([Vog]). *Every representation of G which is parabolically induced from an irreducible unitary representation of a Levi subgroup is irreducible.*

Tadić [Tad] showed that Theorem 1.2 together with some known representation theoretic results can be used to give a complete (external) description of the unitary dual of G. Here "external" is used by Tadić to distinguish this approach from the "internal" approach of Vogan [Vog] who was the first to determine the unitary dual of G.

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For a proof of his conjecture, Kirillov suggested the following line of attack: Fix a Haar measure dg on G. Let π be an irreducible unitary representation of G on a Hilbert space H. Let $f \in C_c^\infty(G)$ and set $\pi(f) : H \to H$ to be

$$\pi(f)v = \int_G f(g)\pi(g)v dg.$$

Let $R : H \to H$ be a bounded linear operator which commutes with all the operators $\pi(p), p \in P$. Then it is enough to prove that R is a scalar multiple of the identity operator. Since π is irreducible, it is enough to prove that R commutes with all the operators $\pi(g)$, $g \in G$. Consider the distribution

$$\Lambda_R(f) = \mathrm{trace}(R\pi(f)), \quad f \in C_c^\infty(G).$$

Then Λ_R is P invariant under conjugation. Kirillov conjectured that

CONJECTURE 1.3. *Λ_R is G invariant under conjugation.*

Kirillov (see also Tadić [Tad], p.247) proved that Conjecture 1.3 implies Conjecture 1.1 as follows. Fix $g \in G$. Since Λ_R is G invariant it follows that

$$\begin{aligned}\Lambda_R(f) = \Lambda_R(\pi(g)\pi(f)\pi(g)^{-1}) &= \mathrm{trace}(R\pi(g)\pi(f)\pi(g)^{-1}) \\ &= \mathrm{trace}(\pi(g)^{-1}R\pi(g)\pi(f)).\end{aligned}$$

Hence

$$\mathrm{trace}((\pi(g)^{-1}R\pi(g) - R)\pi(f)) = 0$$

for all $f \in C_c^\infty(G)$. Since π is irreducible it follows that $\pi(g)^{-1}R\pi(g) - R = 0$ and we are done.

It is easy to see that Λ_R is an eigendistribution with respect to the center of the universal enveloping algebra associated to G. Hence, to prove Conjecture 1.3 we shall prove the following theorem which is the main theorem of this paper:

THEOREM 1.4. *Let T be a P invariant distribution on G which is an eigendistribution with respect to the center of the universal enveloping algebra associated with G. Then there exists a locally integrable function, F, on G which is G invariant and real analytic on the regular set G', such that $T = F$. In particular, T is G invariant.*

Bernstein [Ber] proved that every $P_n(K)$ invariant distribution, T, on $\mathrm{GL}_n(K)$ where K is a p-adic field is $\mathrm{GL}_n(K)$ invariant under conjugation. Since he does not assume any analog for T being an eigendistribution, his result requires a different approach and a different proof. In particular, the distributions that he considers are not necessarily functions. However, for all known applications, the P invariant p-adic distributions in use will be admissible, hence, by Harish-Chandra's theory, are functions. Bernstein obtained many representation theoretic applications for his theorem. We are in particular interested in his result that every P invariant pairing between the smooth space of an irreducible admissible representation of G and its dual is G invariant. He also constructed this bilinear form in the Whittaker or Kirillov model of π. This formula is very useful for the theory of automorphic forms where it is sometimes essential to normalize various local and global data using such bilinear forms ([Bar-Mao]). We shall obtain analogous results and formulas for the archimedean case using Theorem 1.4.

Theorem 1.4 is a regularity theorem in the spirit of Harish-Chandra. Since we only assume that our distribution is P invariant, this theorem in the case of $\mathrm{GL}(n)$ is stronger than Harish-Chandra's regularity theorem. This means that several new ideas and techniques are needed. Some of the ideas can be found in [Ber] and [Ral]. We shall also use extensively a stronger version of the regularity theorem due to Wallach [Wal]. Before going into the details of the proof we would like to mention two key parts of the proof which are new. We believe that these results and ideas will turn out to be very useful in the study of certain Gelfand-Graev models. These models were studied in the p-adic case by Steve Rallis.

The starting point for the proof is the following proposition. For a proof see step A in Section 2.1 or Proposition 8.2.

Key Proposition. *Let T be a P invariant distribution on the regular set G'. Then T is G invariant.*

Notice that we do not assume that T is an eigendistribution. Now it follows from Harish-Chandra's theory that if T as above is also an eigendistribution for the center of the universal enveloping algebra then it is given on G' by a G invariant function F_T which is locally integrable on G. Starting with a P invariant eigendistribution T on G we can now form the distribution $Q = T - F_T$ which vanishes on G'. We proceed to show that $Q = 0$. For a more detailed sketch of the proof see Section 2.1 .

The strategy is to prove an analogous result for the Lie algebra case. After proving an analog of the "Key Proposition" for the Lie algebra case we proceed by induction on centralizers of semisimple elements to show that Q is supported on the set of nilpotent elements times the center. Next we prove that every P invariant distribution which is finite under the "Casimir" and supported on such a set is identically zero. Here lies the heart of the proof. The main difficulty is to study P conjugacy classes of nilpotent elements, their tangent spaces and the transversals to these tangent spaces. We recall some of the results:

Let X be a nilpotent element in $\mathfrak{g}$, the Lie algebra of G. We can identify $\mathfrak{g}$ with $M_n(K)$ and X with an $n \times n$ nilpotent matrix with complex or real entries. We let $O_P(X)$ be the P conjugacy class of X, that is $O_P(X) = \{pXp^{-1} : p \in P\}$.

Lemma 1.5. *Let X' be a nilpotent element. Then there exist $X \in O_P(X')$ with real entries such that $X, Y = X^t, H = [X, Y]$ form an $\mathfrak{sl}(2)$.*

For a proof see Lemma 6.2. Using this lemma we can study the tangent space of $O_P(X)$. Let $\mathfrak{p}$ be the Lie algebra of P. Then $[\mathfrak{p}, X]$ can be identified with the tangent space of $O_P(X)$ at X. We proceed to find a complement (transversal) to $[\mathfrak{p}, X]$. Let $X, Y = X^t$ be as in Lemma 1.5. Let $\mathfrak{p}^c$ be the Lie subalgebra of matrices whose first $n-1$ rows are zero. Let $\mathfrak{g}^{Y,\mathfrak{p}^c} = \{Z \in \mathfrak{g} : [Z, Y] \in \mathfrak{p}^c\}$.

Lemma 1.6.

$$\mathfrak{g} = [\mathfrak{p}, X] \oplus \mathfrak{g}^{Y,\mathfrak{p}^c}.$$

For a proof see Lemma 6.1. One should compare this decomposition with the decomposition

$$\mathfrak{g} = [\mathfrak{g}, X] \oplus \mathfrak{g}^Y$$

where $\mathfrak{g}^Y$ is the centralizer of Y. Harish-Chandra proved that if X, Y, H form an $\mathfrak{sl}(2)$ then $\mathrm{ad}H$ stabilizes $\mathfrak{g}^Y$. Moreover, $\mathrm{ad}H$ has nonpositive eigenvalues on $\mathfrak{g}^Y$ and the sum of these eigenvalues is $\dim(\mathfrak{g}^Y) - \dim(\mathfrak{g})$. This result was crucial in studying the G invariant distributions with nilpotent support. The difficulty for us lies in the fact that $\mathrm{ad}H$ does not stabilize $\mathfrak{g}^{Y,\mathfrak{p}^c}$ in general and might have positive eigenvalues on this space. Moreover, we would need H to be in $\mathfrak{p}$ which is not true in general. To overcome this difficulty we prove the following theorem which is one of the main theorems of this paper.

Theorem 1.7. *Assume that $X, Y = X^t$ and $H = [X, Y]$ are as in Lemma 1.5. Then there exists $H' \in \mathfrak{g}$ such that*

(1) $H' \in \mathfrak{p}$.

(2) $[H', X] = 2X, \ [H', Y] = -2Y$.

(3) $\mathrm{ad}(H')$ *acts semisimply on* $\mathfrak{g}^{Y,\mathfrak{p}^c}$ *with nonpositive eigenvalues* $\{\mu_1, \mu_2, \ldots, \mu_k\}$.

(4) $\mu_1 + \mu_2 + ... + \mu_k \leq k - \dim(\mathfrak{g})$.

It will follow from the proof that H' is determined uniquely by these properties in most cases. The proof of this theorem requires a careful analysis of nilpotent P conjugacy classes including a parametrization of these conjugacy classes. We also need to give a more explicit description of the space $\mathfrak{g}^{Y,\mathfrak{p}^c}$. We do that in Sections 5 and 6.

The paper is organized as follows. In Section 2 we introduce some notation and prove some auxiliary lemmas which are needed for the proof of our "Key Proposition" above. We also sketch the proof of Theorem 1.4. In Section 3 we recall some facts about distributions. In Section 4 we reformulate Theorem 1.4 following [Ber] and formulate the analogous statement for the Lie algebra case. In Sections 5 and 6 we prove the results mentioned above. Section 7 treats the case of P invariant distributions with nilpotent support on the Lie algebra. In Section 8 we prove the general Lie algebra statement and in Section 9 we prove the general group statement by lifting the Lie algebra result with the use of the exponential map. Sections 8 and 9 are standard and follow almost line by line the arguments given in [Wal]. In Section 10 we give another proof of Conjecture 1.1 and give the bilinear form in the Whittaker model mentioned above.

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2. Preliminaries and notation

Let $K = \mathbf{R}$ or $K = \mathbf{C}$. Let $G = \mathrm{GL}_n(K)$ and $\mathfrak{g}$ be the Lie algebra of G. That is, $\mathfrak{g} = M_n(K)$, viewed as a real Lie algebra. Let $\mathfrak{g}_{\mathbf{C}}$ be the complexified Lie algebra and let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$. Let $\mathcal{S}(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}_{\mathbf{C}}$. $\mathcal{S}(\mathfrak{g})$ is identified with the algebra of constant coefficients differential operators on $\mathfrak{g}$ in the usual way. That is, if $X \in \mathfrak{g}$ and $f \in C^\infty(\mathfrak{g})$ then we define

$$X(f)(A) = \frac{d}{dt} f(A + tX)_{|t=0}, \quad A \in \mathfrak{g},$$

and extend this action to $\mathcal{S}(\mathfrak{g})$. We identify $\mathcal{U}(\mathfrak{g})$ with left invariant differential operators on G in the usual way. That is, if $X \in \mathfrak{g}$ and $f \in C^\infty(G)$ then we define

$$X(f)(g) = \frac{d}{dt} f(g \exp(tX))_{|t=0}, \quad g \in G,$$

where exp is the exponential map from $\mathfrak{g}$ to G. This action extends in a natural way to $\mathcal{U}(\mathfrak{g})$.

We view $G = \mathrm{GL}_n(K)$ and $\mathfrak{g} = \mathfrak{g}l_n(K)$ as groups of linear transformations on a real vector space $\mathcal{V} = \mathcal{V}(K)$. If we think of G and $\mathfrak{g}$ as groups of matrices (under multiplication or addition respectively) then $\mathcal{V}$ is identified with the row vector space K^n. Note that G acts on $\mathcal{V}$ in a natural way. Let P be the subgroup fixing the row vector

$$v_0 = \begin{pmatrix} 0 & 0 & . & . & 0 & 1 \end{pmatrix}. \tag{2.1}$$

Let $\mathfrak{p}$ be the Lie algebra of P. Then $\mathfrak{p}$ is the set of matrices which send v_0 to 0. In matrix notation,

$$\begin{aligned} P &= \left\{ \begin{pmatrix} h & u \\ 0 & 1 \end{pmatrix} : h \in \mathrm{GL}_{n-1}(K), u \in M_{n-1,1}(K) \right\}, \\ \mathfrak{p} &= \left\{ \begin{pmatrix} A \\ 0 \end{pmatrix} : A \in M_{n-1,n}(K), 0 \in M_{1,n}(K) \right\}. \end{aligned} \tag{2.2}$$

The Lie algebra $\mathfrak{g} = M_n(K)$ acts on $C_c^\infty(\mathcal{V})$ by the differential operators

$$Xf(v) = \frac{d}{dt} f(v \exp(tX))_{|t=0}, \quad X \in \mathfrak{g}, v \in \mathcal{V}. \tag{2.3}$$

This action extends in a natural way to $\mathfrak{g}_{\mathbf{C}}$ and to $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$. We shall need the following lemma later.

LEMMA 2.1. *Let $\mathfrak{b}$ be a maximal Cartan subalgebra in $\mathfrak{g}_{\mathbf{C}}$ and let α be a root of $\mathfrak{b}$. Let $X_\alpha, X_{-\alpha} \in \mathfrak{g}_{\mathbf{C}}$ be nontrivial root vectors for α and $-\alpha$ respectively. Then there exists $D \in \mathcal{U}(\mathfrak{b})$ such that D and $X_\alpha X_{-\alpha}$ are the same as differential operators on $\mathcal{V}$.*

Proof. The action of $\mathfrak{g}$ defined in (2.3) induces a homomorphism from $\mathcal{U}(\mathfrak{g})$ to $\mathrm{DO}(\mathcal{V})$, the algebra of differential operators on $\mathcal{V}$. We need to find a $D \in \mathcal{U}(\mathfrak{b})$ such that $D - X_\alpha X_{-\alpha}$ is in the kernel of this homomorphism. Since this kernel is stable under the "Ad' action of $G_{\mathbf{C}}$, the complex group associated to $\mathfrak{g}_{\mathbf{C}}$, we can conjugate $\mathfrak{b}$ to the diagonal Cartan in $M_n(K)$. Hence, we can assume that $X_\alpha = X_{i,j}$, a matrix with 1 in the (i,j) entry, $i \neq j$ and zeroes elsewhere and that $X_{-\alpha} = X_{j,i}$. Let $y_1, \ldots, y_n$ be standard coordinates on $\mathcal{V}$. Then the mapping above sends

$$X_{i,j} \mapsto y_i \frac{\partial}{y_j}.$$

It follows that $X_\alpha X_{-\alpha} = X_{i,j} X_{j,i} = X_{i,i} X_{j,j} + X_{i,i} = D$ as differential operators on $\mathcal{V}$. □

The following lemmas are well known and we include them here for the sake of completeness. Let $\alpha \in \mathbf{R}^* = \mathbf{R} - \{0\}$ or $\alpha \in C^*$. For a function $f : \mathbf{R} \to \mathbf{C}$ or $f : \mathbf{C} \to \mathbf{C}$ define $f_\alpha(x) = f(\alpha x)$. We let $|\alpha|_{\mathbf{R}}$ be the usual absolute value of α and $|\alpha|_{\mathbf{C}}$ be the square of the usual absolute value on $\mathbf{C}$.

LEMMA 2.2. *Let T be a distribution on $\mathbf{R}^*$ satisfying $(\alpha T)(f) = T(f_\alpha) = |\alpha|_{\mathbf{R}}^{-1} T(f)$ for every $\alpha \in \mathbf{R}^*$ and $f \in C_c^\infty(\mathbf{R}^*)$. Then there exist $\lambda \in \mathbf{C}$ such that*

$$T(f) = \lambda \int_{\mathbf{R}^*} f(x)dx$$

where dx is the standard Lebesgue measure on $\mathbf{R}$.

Proof. Define $Hf(x) = \frac{d}{dt} f(e^t x)|_{t=0}$. Then $T(Hf) = T(f)$ for all f. Thus, $H^2T - T = 0$; that is, T satisfies an elliptic differential equation. It follows that there exists a real analytic function $p : \mathbf{R}^* \to \mathbf{C}$ such that

$$T(f) = \int_{\mathbf{R}^*} p(x)f(x)dx.$$

It is easy to see that $p(x)$ is constant. □

LEMMA 2.3. *Let T be a distribution on $\mathbf{R}$ satisfying $T(f_\alpha) = |\alpha|_{\mathbf{R}}^{-1} T(f)$ for every $\alpha \in \mathbf{R}^*$ and $f \in C_c^\infty(\mathbf{R})$. Then there exists $\lambda \in \mathbf{C}$ such that*

$$T(f) = \lambda \int_{\mathbf{R}} f(x)dx.$$

Proof. We restrict T to $\mathbf{R}^*$. By the above Lemma $T = \lambda dx$ on $\mathbf{R}^*$. Hence $Q = T - \lambda dx$ has the same invariance conditions as T and is supported at 0. It follows that there exist constants c_i, $i = 0, 1, \ldots,$ (all but a finite number of them are zero), such that

$$Q = c_0\delta_0 + \sum c_i \frac{\partial^i}{\partial x^i}|_{x=0}.$$

Thus

$$\alpha Q = c_0\delta_0 + \sum c_i \alpha^i \frac{\partial^i}{\partial x^i}|_{x=0}. \tag{2.4}$$

On the other hand, $\alpha Q = |\alpha|^{-1}Q$. Now the uniqueness of (2.4) forces $c_i = 0$, $i = 0, 1 \ldots$, hence $Q = 0$. □

LEMMA 2.4. *Let T be a distribution on $\mathbf{C}^*$ satisfying $T(f_\alpha) = |\alpha|_{\mathbf{C}}^{-1} T(f)$ for every $\alpha \in \mathbf{C}^*$ and $f \in C_c^\infty(\mathbf{C}^*)$. Then there exists $\lambda \in \mathbf{C}$ such that $T = \lambda dz$ where dz is the standard Lebesgue measure on $\mathbf{C}$.*

Proof. The proof is the same as in Lemma 2.2. It is easy to construct an elliptic differential operator on $\mathbf{C}$ which annihilates T. □

LEMMA 2.5. *Let T be a distribution on $\mathbf{C}$ satisfying $T(f_\alpha) = |\alpha|_{\mathbf{C}}^{-1} T(f)$ for every $\alpha \in \mathbf{C}^*$ and $f \in C_c^\infty(\mathbf{C})$. Then there exists $\lambda \in \mathbf{C}$ such that $T = \lambda dz$.*

Proof. The proof is the same as in Lemma 2.3. It is based on the form of distributions on $\mathbf{C} \cong \mathbf{R}^2$ which are supported on $\{0\}$. □

Let $V_1, \dots V_k$, be one-dimensional real vector spaces and $V_{k+1}, \dots, V_r$, be one-dimensional complex vector spaces. Let $V = V_1 \oplus \cdots \oplus V_r$ and $H = (R^*)^k \times (C^*)^{r-k}$. Then H acts naturally (component by component) on V. Let dv be the usual Lebesgue measure on V. For $\alpha = (\alpha_1, \dots, \alpha_r)$ we define

$$|\alpha| = |\alpha_1|_{\mathbf{R}} \cdots |\alpha_k|_{\mathbf{R}} |\alpha_{k+1}|_{\mathbf{C}} \cdots |\alpha_r|_{\mathbf{C}}.$$

For $i = 1, \dots, r$, let $\mathcal{X}_i$ be V_i or V_i^* (arbitrarily depending on i) and set $\mathcal{X} = \prod \mathcal{X}_i$. Then H acts on $\mathcal{X}$, hence on functions on $\mathcal{X}$ and on distributions on $\mathcal{X}$.

Lemma 2.6. *Let T be a distribution on $\mathcal{X}$ satisfying $\alpha T = |\alpha|^{-1} T$ for every $\alpha \in H$. Then there exists a constant λ such that $T = \lambda dv$.*

Proof. The proof follows the same ideas as in Lemma 2.3. We first restrict T to the open set $\mathcal{X}^0 = \prod V_i^*$. It is easy to construct an elliptic differential operator that annihilates T on $\mathcal{X}^0$. Thus $T = \lambda dv$ on $\mathcal{X}^0$ for some $\lambda \in \mathbf{C}$. We now consider the distribution $Q = T - \lambda dv$. It is possible to restrict Q inductively to larger and larger open sets in $\mathcal{X}$ such that the support of Q will be at $\{0\}$ at least in one coordinate. Now using the form of such distributions we can show that the invariance condition implies that they vanish. □

2.1. *A sketch of the proof of the main theorem.* We can use the above lemma to give a rough sketch of the proof. We are given a distribution T on $G = \mathrm{GL}_n(K)$, $K = \mathbf{R}$ or $\mathbf{C}$ which is invariant under conjugation by $P = P_n(K)$ and is an eigendistribution for the center of the universal enveloping algebra. We would like to show that it is given by a G invariant function. There are basically three steps to the proof:

A. We show that every P invariant distribution T is G invariant on the regular set. This is our “Key Proposition” from the introduction. Hence the distribution T is G invariant on the regular set. Since it is an eigendistribution, it follows from Harish-Chandra’s proof of the Regularity Theorem that it is given by a locally integrable function F on the regular set.

   Consider the distribution $Q = T - F$.

B. Using a descent method on centralizers of semisimple elements we show that Q is supported on the unipotent set times center. In practice we consider distributions on the Lie algebra and repeat the above process to get a distribution Q which is supported on the nilpotent set times center and is finite under the Casimir element.

C. We show that every distribution Q which is P invariant, supported on the nilpotent set times center and is finite under the Casimir element vanishes identically. Hence, our distribution $Q = T - F$ vanishes and we are done.

Remarks on each step:

Step A. Consider a Cartan subgroup H in G. Then the G conjugates of the regular part of H, H' give an open set in the regular set G'. Using the submersion principle we can induce the restriction of T to this set to get a distribution $\tilde{T}$ on $G \times H'$. In Harish-Chandra's case, where our original distribution T is G invariant this distribution is right invariant by G in the G component, hence induces a distribution σ_T on H'. In our case, the distribution $\tilde{T}$ is only right P invariant in the G component, hence induces a distribution σ_T on $P \setminus G \times H'$. However, σ_T is H equivariant under the diagonal action of H which acts by conjugation in the H' coordinate and by right translation in the $P \setminus G$ coordinate. Since H is commutative it acts only on the $P \setminus G$ coordinate. Now $P \setminus G$ is isomorphic to $V^* = V - \{0\}$ for an appropriate vector space V and the action of H on V decomposes into one-dimensional components as in Lemma 2.6. It follows from Lemma 2.6 that $\sigma_T = dv \otimes T'$ for a distribution T' on H'. It is now easy to see that T is G invariant on the open set conjugated from H'. Proceeding this way on all the nonconjugate Cartans we get statement **A**. In practice it will be more convenient to replace our distribution on G with a distribution on $G \times V^*$ without losing any information. We shall carry out an analogous process in that case for the set $G' \times V^*$. (See Proposition 8.2 and Step B below.)

Step B. Induction on semisimple elements and their centralizers: As in Harish-Chandra's case we would like to use the descent method to go from G to a smaller group, namely a centralizer of a semi-simple element. Let $s \in G$ be semisimple and let $H = G^s$ (similarly in the Lie algebra case). As in Harish-Chandra's proof we can define an open set H'' in H such that the conjugates of H'' in G produce an open set around s and such that it is possible to use the submersion principle. This will produce a distribution σ_T on $P \setminus G \times H''$ which is equivariant under the diagonal action of H. The problem here is that we are not in the induction assumption situation. To rectify this we will start with a situation similar to the one that we obtained, namely our distribution will be on $H \times V$ where H is now a product of GLs and $V = \bigoplus V_i$ where each V_i is the standard representation of the appropriate $\mathrm{GL}(k_i)$. Now the submersion principle will lead us to a similar lower dimensional situation and we will be able to use the induction hypothesis (see the reformulation of our main theorem in Section 4).

Step C. Once Step A and Step B are completed, we are left with a P invariant distribution T with nilpotent support and finite under $\square$, the Casimir element. As in Harish-Chandra's proof, we add two differential operators to the Casimir, an Euler operator E and a multiplication operator Q so that the triple $\{\square, Q, E - rI\}$ generates an $\mathfrak{sl}(2)$. To show that T vanishes it is enough to show that $E - rI$ is of finite order on the space of distributions with nilpotent support and that the eigenvalues of $E - rI$ are all negative on this space. (See [Wal, 8.A.5.1].) This process involves a careful study of P nilpotent orbits and the Jacobson-Morosov triples associated with them.

3. Distributions

We denote the space of distributions on a manifold M by $D'(M)$. An action of a Lie group G on M induces an action of G on $C_c^\infty(M)$ and an action of G on $D'(M)$. We denote by $D'(M)^G$ the set of distributions in $D'(M)$ which are invariant under G. If χ is a character of G then we denote by $D'(M)^{G,\chi}$ the set of distributions $T \in D'(M)$ satisfying

$$gT = \chi(g)T, \quad g \in G. \tag{3.1}$$

If T satisfies (3.1) then we say that T is (G, χ) invariant.

3.1. *Harish-Chandra's submersion principle and radial components.* We shall describe Harish-Chandra's submersion principle in the following context. Let G be a Lie group acting on a manifold M. Let U be a submanifold of M and $\Psi : G \times U \to M$ be a submersion onto an open set W of M. Let dm be a volume form on M, dg be a left invariant Haar measure on G and du be a volume form on U. By Harish-Chandra's submersion principle (see [Wal, 8.A.2.5]), there exists a mapping from $C_c^\infty(G) \otimes C_c^\infty(U) \to C_c^\infty(W)$ such that if $\alpha \in C_c^\infty(G)$ and $\beta \in C_c^\infty(U)$ then $\alpha \otimes \beta \mapsto f_{\alpha\otimes\beta}$ where $f_{\alpha\otimes\beta}$ satisfies

$$\int_W f_{\alpha\otimes\beta}(m)F(m)dm = \int_{G\times U} \alpha(g)\beta(u)F(\Psi(g,u))dgdu.$$

This mapping induces a mapping on distributions. If $T \in D'(W)$ then we define the distribution $\Psi'(T) \in D'(G \times U)$ by

$$\Psi'(T)(\alpha \otimes \beta) = T(f_{\alpha\otimes\beta}). \tag{3.2}$$

For $g \in G$, let l_g be the left action of G on the G component of $G \times U$. Then l_g induces an action of G on $D'(G \times U)$ which we denote again by l_g. If T is (G, χ) invariant for a character χ of G then $\Psi'(T)$ satisfies $l_g(\Psi'(T)) = \chi(g)\Psi'(T)$ for every $g \in G$. It follows from [Wal, 8.A.2.9] that there exist a distribution $\Psi^0(T)$ on U such that

$$\Psi'(T) = \chi\, dg \otimes \Psi^0(T). \tag{3.3}$$

Here dg is a left invariant Haar measure on G and $\chi\, dg \otimes \Psi^0(T)$ is a distribution of the form

$$(\chi\, dg \otimes \Psi^0(T))(\alpha \otimes \beta) = \left(\int_G \alpha(g)\chi(g)dg\right) \Psi^0(T)(\beta). \tag{3.4}$$

We shall be interested in the following examples.

3.2. *Example.* Let $M = M_n(K)$, $P = P_n(K)$, $\mathfrak{p}$ be the Lie algebra of P and X be a nilpotent element in M. Let $V = [\mathfrak{p}, X]$ and U' be a subspace in M such that $M = V \oplus U'$. Let U be an open set in U' and assume that the map $\Psi(p,u) = p(x+u)p^{-1}$ from $P \times U$ onto an open set W of M is submersive. If $T \in D'(W)^G$ then $\Psi'(T)$ on $P \times U$ is left P invariant in the P component (i.e. $\chi = 1$ in the discussion above), hence we can define $\Psi^0(T)$ as above. If E is a $G = \mathrm{GL}_n(K)$ invariant differential operator on M and $\alpha, \beta, f_{\alpha\otimes\beta}$ are as above then we have

$$(3.5)\quad \begin{aligned} \int_W Ef_{\alpha\otimes\beta}(m)F(m)dm &= \int_W f_{\alpha\otimes\beta}(m)(E^T F)(m) \\ &= \int_{P\times U} \alpha(p)\beta(u)(E^T F)(p(X+u)p^{-1})dpdu \\ &= \int_{P\times U} \alpha(p)\beta(u)E^T(F^p)(X+u)dpdu, \end{aligned}$$

where $F^p(m) = F(\mathrm{Ad}(p)(m))$. Hence, if we can find $H \in \mathfrak{p}$ and a differential operator E' on U such that the function $H\alpha$ on $\mathfrak{p}$ defined by $H\alpha(p) = \frac{d}{dt}(\alpha(p\exp(tH))\Delta_P(tH))|_{t=0}$ and $E'\beta$ satisfy

$$\int_{P\times U} H\alpha(p)E'\beta(u)F^p(X+u)dpdu = \int_{P\times U} \alpha(p)\beta(u)E^T(F^p)(X+u)dpdu,$$

for every $\alpha \in C_c^\infty(P)$ and $\beta \in C_c^\infty(U)$ then we have

$$(3.6)\qquad \Psi^0(ET) = E'\Psi^0(T).$$

3.3. *Example.* Here we follow [Wal, 8.A.3 and 7.A.2]. Let G be a real reductive group and $\mathfrak{g}$ the Lie algebra of G. Assume that G acts on a finite-dimensional real vector space $\mathcal{V}$. Then G acts on $\mathfrak{g} \times \mathcal{V}$ by

$$g(A,v) = (\mathrm{Ad}(g)A, gv), \quad g \in G, A \in \mathfrak{g}, v \in \mathcal{V}.$$

This action induces an action of G on $C_c^\infty(\mathfrak{g} \times \mathcal{V})$ and on $D'(\mathfrak{g} \times \mathcal{V})$, the space of distributions on $\mathfrak{g} \times \mathcal{V}$. Let χ be a character of G and let $D'(\mathfrak{g} \times \mathcal{V})^{G,\chi}$ be the space of distributions $T \in D'(\mathfrak{g} \times \mathcal{V})$ satisfying $gT = \chi(g)T$ for all $g \in G$. Let H be a closed subgroup of G and assume that $\mathfrak{g} = \mathfrak{h} \oplus V$ for some subspace V of $\mathfrak{g}$ which is stable under $\mathrm{Ad}(H)$. We also assume that $\mathfrak{h}'' = \{A \in \mathfrak{h} \mid \det(\mathrm{ad}A)|_V \neq 0\}$ is nonempty. As in Lemma 8.A.3.3 in [Wal], we have that the map $\tilde{\Psi}(g,A,v) = g(A,v)$ from $G \times \mathfrak{h}'' \times \mathcal{V}$ into $\mathfrak{g} \times \mathcal{V}$ is a submersion onto an open set W. Hence if $T \in D'(W)^{G,\chi}$ we can define the distribution $\Psi'(T)$ on $G \times \mathfrak{h}'' \times \mathcal{V}$ as above. It is easy to see that $l_g\Psi'(T) = \chi(g)\Psi'(T)$ for all $g \in G$. Here l_g is left translation in the G component as above. Hence $\Psi'(T) = \chi\, dg \otimes \tilde{\Psi}^0(T)$ for $\tilde{\Psi}^0(T) \in D'(\mathfrak{h}'' \times \mathcal{V})^H$. We would like to compute the radial component of this mapping.

Set $L = G \times \mathfrak{g} \times \mathcal{V}$ which we look upon as a Lie group with multiplication given as follows:

$$(g_1, X_1, v_1)(g_2, X_2, v_2) = (g_1 g_2, \mathrm{Ad}(g_2^{-1})X_1 + X_2, g_2^{-1}v_1 + v_2).$$

The Lie algebra $\mathfrak{l}$ of L is $\mathfrak{g} \times \mathfrak{g} \times \mathcal{V}$ with bracket given by

$$[(X_1, Y_1, v_1), (X_2, Y_2, v_2)] = ([x_1, x_2], [Y_1, X_2] + [X_1, Y_2], Y_1 v_2 - Y_2 v_1)$$

where $\mathfrak{g}$ acts on $\mathcal{V}$ by the derived action. L acts on $\mathfrak{g} \times \mathcal{V}$ by $(g, X, v) \cdot (Y, u) = (\mathrm{Ad}(g)(Y + x), g(u + v))$. This makes $\mathfrak{g} \times \mathcal{V}$ into an L space. Let $\mathrm{DO}(\mathfrak{g} \times \mathcal{V})$ be the algebra of all differential operators on $\mathfrak{g}$ with smooth coefficients. If $X \in \mathfrak{l}$ set $T(X)f(Y) = \frac{d}{dt} f(\exp(-tX)Y)|_{t=0}$ for $f \in C^\infty(\mathfrak{g} \times \mathcal{V})$. Then T is a Lie algebra homomorphism of $\mathfrak{l}$ into $\mathrm{DO}(\mathfrak{g} \times \mathcal{V})$. Hence T extends to an algebra homomorphism of $\mathcal{U}(\mathfrak{l}_{\mathbf{C}})$ into $\mathrm{DO}(\mathfrak{g} \times \mathcal{V})$.

In $\mathfrak{l}$, $\mathfrak{g} \times 0$ is a Lie subalgebra isomorphic with $\mathfrak{g}$, and $0 \times \mathfrak{g} \times \mathcal{V}$ is a Lie subalgebra with 0 bracket operation. Thus

$$\mathcal{U}(\mathfrak{l}_{\mathbf{C}}) = \mathcal{U}(\mathfrak{g}_{\mathbf{C}}) \otimes S(\mathfrak{g}_{\mathbf{C}} \times \mathcal{V}_{\mathbf{C}}).$$

The discussion now follows [Wal, 7.A.2.2, 7.A.2.3, 7.A.2.4 and 7.A.2.5]. In particular we define $R(x \otimes y) = T(1 \otimes y)T(x \otimes 1)$ for $x \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ and $y \in S(\mathfrak{g}_{\mathbf{C}} \times \mathcal{V}_{\mathbf{C}})$. For $A \in \mathfrak{h}''$, $v \in \mathcal{V}$, we define $\Gamma_{A,v}$, $\delta_{A,v}$ and δ analogous to their definition in [Wal, 7.A.2.4].

Remark 3.1. The definition of δ above is slightly twisted from the definition in [Wal, 7.A.2.4]. This twist is caused by the existence of the character χ. In particular, if

$$\Gamma_{A,v}(D_1 \otimes 1 + 1 \otimes D_2) = D, \tag{3.7}$$

for $A \in \mathfrak{h}''$, $v \in V$, $D_1 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$, $D_2 \in S(\mathfrak{h}_{\mathbf{C}} \times \mathcal{V}_{\mathbf{C}})$ and $D \in S(\mathfrak{g}_{\mathbf{C}} \times \mathcal{V}_{\mathbf{C}})$ then

$$\delta(D)_{A,v} = \delta_{A,v}(D) = d\chi(D_1)\mathrm{Id} + D_2.$$

Here $d\chi$ is the differential of χ viewed as a linear functional on $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$.

We now assume that H is reductive and that we have an invariant nondegenerate symmetric bilinear form, B, on $\mathfrak{g}$ such that B restricted to $\mathfrak{h}$ is nondegenerate. We first observe that if $\alpha \in C_c^\infty(G)$ and $D_1 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ then

$$\int_G D_1 \alpha(g) \chi(g) dg = d\chi(D_1) \int \alpha(g) dg.$$

Using this and applying the same arguments as in [Wal, 7.A.2.5] and [Wal, 8.A.3.4] we have

Lemma 3.2. *If $D \in \mathrm{DO}(\mathfrak{g} \times \mathcal{V})^G$ and if $T \in D'(\mathfrak{g} \times \mathcal{V})^{G,\chi}$ then*

$$\tilde{\Psi}^0(DT) = \delta(D)\tilde{\Psi}^0(T).$$

3.4. *Example*: *Frobenius reciprocity.* Let G be a Lie group acting by ρ on a manifold M . Let H be a closed subgroup of of G. We shall assume that G is unimodular and that there exists a character χ of G such that $\chi|_H = \Delta_H$ where Δ_H is the modular function of H. (For a more general situation see [Ber, 1.5].) Then G acts naturally on the space $M \times (H \setminus G)$ by

$$g(m, v) = (\rho(g)m, vg^{-1}), \quad g \in G, m \in M, v \in H \setminus G.$$

This action induces an action of G on $C_c^\infty(M \times H \setminus G)$ and on $D'(M \times H \setminus G)$. Define $\Psi : G \times M \to M \times H \setminus G$ by

$$\Psi(g, m) = (\rho(g)m, Hg^{-1}), \quad m \in M, g \in G.$$

It is easy to see that Ψ is a submersive map at every point (g, m). Hence, by (3.2) and (3.3) there exists a mapping Ψ^0 from $D'(M \times H \setminus G)^{G,\chi}$ to $D'(M)^H$. In the generality of (3.3), Ψ^0 is a one-to-one mapping but not onto. However, in the case at hand, Bernstein ([Ber, 1.5]) constructed an inverse map which we now describe. Let dg be a (G, χ) quasi-invariant measure on $H \setminus G$. If $\phi \in C_c^\infty(M \times H \setminus G)$, $g \in G$, $v \in H \setminus G$ we define a function $\phi(\rho(g)(\cdot), v) \in C_c^\infty(M)$ by $\phi(\rho(g)(\cdot), v)(m) = \phi(\rho(g)(m), v)$. If $T \in D'(M)^H$ then we define a distribution $\mathrm{Fr}(T) \in D'(M \times H \setminus G)^{G,\chi}$ by

$$\mathrm{Fr}(T)(\phi) = \int_{H\setminus G} T(\phi(\rho(g)(\cdot), Hg)dg. \tag{3.8}$$

Since Fr is the inverse map to Ψ^0 we get the following Frobenius reciprocity theorem:

Theorem 3.3 ([Ber]). *The map $T \mapsto \mathrm{Fr}(T)$ given by* (3.8) *is a vector space isomorphism between $D'(M)^H$ and $D'(M \times H \setminus G)^{G,\chi}$. Moreover, if E is a G invariant differential operator on M then* $\mathrm{Fr}(ET) = (E \otimes 1)\mathrm{Fr}(T)$ *for every $T \in D'(M)^H$.*

The second part of the theorem is a simple computation using formula (3.8).

4. Statement of the main results

It will be useful to formulate an equivalent statement for our main result and an analogue for the Lie algebra case. For similar statements in the p-adic case see [Ber].

Let K be $\mathbf{R}$ or $\mathbf{C}$. For $\alpha \in K$ we denote by $|\alpha|$ the standard absolute value of α if $K = \mathbf{R}$ and the square of the standard absolute value if $F = \mathbf{C}$. Let $G = \mathrm{GL}_n(K)$ and $\mathcal{V} = K^n$. Then G acts on the row space $\mathcal{V}$ by $\rho(g)v = vg^{-1}$, and on $G \times \mathcal{V}$ by

$$g_1(g, v) = (g_1 g g_1^{-1}, \rho(g)v), \quad g, g_1 \in G, v \in \mathcal{V}.$$

Let $v_0 \in \mathcal{V}^* = \mathcal{V} - \{0\}$ and let $P \subset G$ be the stabilizer of v_0. For each $v \in \mathcal{V}^*$ we fix $g_v \in G$ such that $\rho(g_v)v_0 = v$. Then g_v is determined up to a right translate by an element of P. Frobenius reciprocity (Theorem 3.3) gives an isomorphism between $D'(G)^P$ and $D'(G \times \mathcal{V}^*)^{G,|\det|}$. The map is given by $T \mapsto \mathrm{Fr}(T)$ where T is a P invariant distribution on G and $\mathrm{Fr}(T)$ is given by

$$\mathrm{Fr}(T)(f) = \int_{\mathcal{V}^*} T(\mathrm{Ad}(g_v)f(.,v))dv, \quad f \in C_c^\infty(G \times \mathcal{V}^*). \tag{4.1}$$

Here dv is a Haar measure on $\mathcal{V}$, and the integral does not depend on the choice of g_v since T is P invariant. It is easy to see that $\mathrm{Fr}(T)$ is $(G, |\det|)$ invariant (see (3.1)), and that for $z \in \mathcal{Z}(\mathfrak{g})$ we have $\mathrm{Fr}(zT) = (z \otimes 1)\mathrm{Fr}(T)$. Hence, if T is finite under $\mathcal{Z}(\mathfrak{g})$ then $\mathrm{Fr}(T)$ is finite under $\mathcal{Z}(\mathfrak{g}) \otimes 1$. If T is G invariant then it is easy to see that $\mathrm{Fr}(T) = T \otimes dv$. Conversely, if $\mathrm{Fr}(T) = R \otimes dv$ for some distribution R on G then R is G invariant and $T = R$. Hence, Theorem 1.4 will follow from

THEOREM 4.1. *Let T be in $D'(G \times \mathcal{V}^*)^{G,|\det|}$ and assume that*

$$\dim(\mathcal{Z}(\mathfrak{g}) \otimes 1)T < \infty.$$

Then there exists a locally integrable, G invariant function, F, on G which is real analytic on G' such that $T = F \otimes dv$. That is

$$T(f) = \int_{G \times \mathcal{V}^*} f(g,v)F(g)dgdv$$

for every $f \in C_c^\infty(G \times \mathcal{V}^*)$.

The proof of Theorem 4.1 will follow from an analogous theorem for the Lie algebra. We let $\mathfrak{g} = M_n(K)$ be as above. Then $G = \mathrm{GL}_n(K)$ acts on $\mathfrak{g} \times \mathcal{V}$ by

$$g(A, v) = (\mathrm{Ad}(g)A, \rho(g)v), \quad g \in G, A \in \mathfrak{g}, v \in \mathcal{V}. \tag{4.2}$$

Let $D'(\mathfrak{g} \times \mathcal{V}^*)^{G,|\det|}$ be the space of distributions defined in (3.1). Let $I(\mathfrak{g}) = S(\mathfrak{g}_{\mathbf{C}})^G$.

THEOREM 4.2. *Let T be in $D'(\mathfrak{g} \times \mathcal{V}^*)^{G,|\det|}$ and assume that*

$$\dim(I(\mathfrak{g}) \otimes 1)T < \infty.$$

Then there exists a locally integrable, G invariant function, F, on $\mathfrak{g}$ which is real analytic on $\mathfrak{g}'$ such that $T = F \otimes dv$.

As in Harish-Chandra's work, it is necessary to generalize Theorem 4.2 to certain G invariant subsets in $\mathfrak{g} \times \mathcal{V}^*$. It is also necessary to consider the case where we replace $\mathfrak{g} \times \mathcal{V}^*$ with $\mathfrak{g} \times \mathcal{V}$. (Here and throughout, $\mathcal{V}^* = \mathcal{V} - \{0\}$.)

We also need to prepare for an induction argument using centralizers of semisimple elements in $\mathfrak{g}$. These centralizers will have the form of a product of $\mathfrak{gl}_n$s. We shall formulate all these generalizations at once. Let t be a positive integer and for each $1 \le i \le t$ fix $\mathfrak{g}_i = M_{k_i}(K_i)$ and $\mathcal{V}_i = (K_i)^{k_i}$. Here K_i is $\mathbf{R}$ or $\mathbf{C}$ and can change with i. Let $\mathcal{X}_i$, $1 \le i \le r$, be $\mathcal{V}_i$ or $\mathcal{V}_i^*$ and $G_i = \mathrm{GL}_{k_i}(K_i)$. Set $\mathfrak{g} = \prod_{i=1}^t g_i$, $\mathcal{X} = \prod \mathcal{X}_i$ and $G = \prod G_i$. Then $\mathfrak{g}$ is the Lie algebra of G, G acts on $\mathcal{X}$ in a natural way and G acts on $\mathfrak{g} \times \mathcal{X}$ by an action which extends (4.2). The character $|\det|$ is also extended in a natural way to G. We let $dx = dv_1 \cdots dv_t$ where dv_i is a translation invariant measure on $\mathcal{V}_i$.

Let Ω be an open G invariant subset of $\mathfrak{g}$ of the type described in [Wal, 8.3.3]. That is, there exist homogeneous $\mathrm{Ad}(G)$-invariant polynomials $\phi_1, \dots, \phi_d$ on $[\mathfrak{g}, \mathfrak{g}]$ and $r > 0$ such that

$$\Omega = \Omega(\phi_1, \dots, \phi_d, r) + U \tag{4.3}$$

where U is an open, connected subset of $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}$ and

$$\Omega(\phi_1, \dots, \phi_d, r) = \{X \in [\mathfrak{g}, \mathfrak{g}] \,|\, |\phi_i(X)| < r, i = 1, \dots, d\}.$$

Denote by $D'(\Omega \times \mathcal{X})^{G,|\det|}$ the space of $(G, |\det|)$ invariant distributions on $\Omega \times \mathcal{X}$ as in (3.1). We shall prove the following theorem:

THEOREM 4.3. *Let $T \in D'(\Omega \times \mathcal{X})^{G,|\det|}$ be such that*

$$\dim(I(\mathfrak{g}) \otimes 1)T < \infty.$$

Then there exists a locally integrable, G invariant function, F, on Ω which is real analytic on $\Omega' = \Omega \cap \mathfrak{g}'$ such that $T = F \otimes dx$.

As in [Wal, Th. 8.3.5], it is convenient to strengthen this theorem somewhat. Let B be a symmetric invariant nondegenerate bilinear form on $\mathfrak{g}$ (see (6.1) and [Wal, 0.2.2)]). Let $X_1, \dots, X_l$ be a basis of $\mathfrak{g}$. Define X^j by the equations $B(X_i, X^j) = \delta_{i,j}$. Put

$$\square = \sum X_i X^i. \tag{4.4}$$

Then $\square \in I(\mathfrak{g})$.

THEOREM 4.4. *Let $T \in D'(\Omega \times \mathcal{X})^{G,|\det|}$ be such that*

$$\dim(\mathbf{C}[\square \otimes 1]T) < \infty \quad \textit{on } \Omega \times \mathcal{X}$$

and such that

$$\dim(I(\mathfrak{g}) \otimes 1)T < \infty, \quad \textit{on } \Omega' \times \mathcal{X}.$$

Then there exists a locally integrable, G invariant function, F, on Ω which is real analytic on $\Omega' = \Omega \cap \mathfrak{g}'$ such that $T = F \otimes dx$.

5. Nilpotent conjugacy classes and P orbits

In this section we describe the $P = P_n(K)$ conjugacy classes of nilpotent elements in $\mathfrak{g} = \mathfrak{gl}_n(K)$. We shall also describe certain Jacobson-Morosov triples that are associate to these conjugacy classes.

Every nilpotent matrix A in $\mathfrak{gl}_n(K)$ is conjugate to a unique matrix of the form

$$A_{\underline{r}} = \begin{pmatrix} A_{r_1} & & & & \\ & A_{r_2} & & & \\ & & . & & \\ & & & . & \\ & & & & A_{r_k} \end{pmatrix}$$

where A_{r_i} is an $r_i \times r_i$ matrix of the form

$$\begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & & . & & \\ & & & & . & \\ & & & & & 1 \\ & & & & & 0 \end{pmatrix}$$

and

$$\underline{r} = (r_1, \ldots, r_l), \text{ with } 1 \le r_1 \le r_2 \le ... \le r_k \le n,\ r_1 + r_2 + ... + r_k = n. \tag{5.1}$$

A Jacobson-Morosov triple in $\mathfrak{g}$ is a triple of elements X, Y, H satisfying

$$[X, Y] = H,\ [H, X] = 2X,\ [H, Y] = -2Y.$$

We can change the nonzero entries in $A_{\underline{r}}$ to positive entries such that the new matrix $X_{\underline{r}}$, its transpose $Y_{\underline{r}} = (X_{\underline{r}})^t$ and the diagonal matrix $H_{\underline{r}} = [X_{\underline{r}}, Y_{\underline{r}}] = X_{\underline{r}}Y_{\underline{r}} - Y_{\underline{r}}X_{\underline{r}}$ form a Jacobson-Morosov triple. $X_{\underline{r}}$ and $H_{\underline{r}}$ are block diagonal:
(5.2)

$$X_{\underline{r}} = \begin{pmatrix} X_{r_1} & & & & \\ & X_{r_2} & & & \\ & & . & & \\ & & & . & \\ & & & & X_{r_l} \end{pmatrix}, \quad H_{\underline{r}} = \begin{pmatrix} H_{r_1} & & & & \\ & H_{r_2} & & & \\ & & . & & \\ & & & . & \\ & & & & H_{r_l} \end{pmatrix}.$$

Here H_{r_i} is an $r_i \times r_i$ diagonal matrix of the form

$$(5.3) \qquad \begin{pmatrix} r_i - 1 & & & & & \\ & r_i - 3 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & -r_i + 3 & \\ & & & & & -r_i + 1 \end{pmatrix}$$

and X_{r_i} is an $r_i \times r_i$ matrix whose $(j, j+1)$ entry is $\sqrt{jr_i - j^2}$ for $1 \leq j \leq r_i - 1$, and all other entries are zero. We summarize this in the following lemma.

LEMMA 5.1. *Let $X \in \mathfrak{g}$ be a nonzero nilpotent element. Then there exist a unique partition $\underline{r}$ of n as in* (5.1) *and a unique matrix $X_{\underline{r}}$ as above in the G conjugacy class of X such that the triple $X = X_{\underline{r}}$, $Y = (X_{\underline{r}})^t$ and $H = H_{\underline{r}} = [X, Y]$ forms a Jacobson-Morosov triple.*

Set $G = \mathrm{GL}_n(K)$. The G conjugacy class of $X \in \mathfrak{g}$ is the set of elements of $\mathfrak{g}$ of the form $O_G(X) = \{gXg^{-1} \mid g \in G\}$. The G conjugacy class of X is partitioned into P conjugacy classes $O_P(\tilde{X}) = \{p\tilde{X}p^{-1} \mid p \in P\}$ where $\tilde{X}$ is in $O_G(X)$. It is well known [Ber] that there are only a finite number of P conjugacy classes in a given G conjugacy class. We now recall how to parametrize these P conjugacy classes and how to find nice representatives for them.

Let X be a nilpotent element in $\mathfrak{g}$. Without changing the G conjugacy class of X, we can assume that $X = X_{\underline{r}}$ for some partition $\underline{r}$. Let $C = C_{X_{\underline{r}}}$ be the centralizer of X in G. There is a canonical bijection between G/C and $O_G(X)$ given by $gC \mapsto gXg^{-1}$, $g \in G$. The action of P on $O_G(X)$ induces the left action of P on G/C. Hence P orbits in G/C are in bijection with P conjugacy classes in $O_G(X)$. Since P orbits in G/C are in bijection with $P \setminus G/C$ double cosets, and since these are in bijection with C orbits in $P \setminus G$ we get a bijection from C orbits in $P \setminus G$ to P conjugacy classes in $O_G(X)$. We shall now describe this bijection explicitly.

Let $\mathcal{V} = \mathcal{V}(K)$ be the vector space of row vectors as defined in Section 2 and let $v_0 \in \mathcal{V}$ be as defined in (2.1). Then $P \setminus G$ is isomorphic to $\mathcal{V}^* = \mathcal{V} - \{0\}$ via the map $Pg \mapsto \rho(g^{-1})v_0$. (Here g is a matrix, v_0 a row vector and $\rho(g^{-1})v_0 = v_0 g$.) To every vector $v \in \mathcal{V}^*$, we associate a P conjugacy class as follows. Let $g \in G$ be such that $\rho(g)v = v_0$. Set $v \mapsto O_P(gXg^{-1})$. It is easy to check that this map is constant on C orbits in $\mathcal{V}^*$ and induces a bijection between such orbits and P conjugacy classes in $O_G(X)$.

Following Rallis [Ral] we shall now give nice representatives for each C orbit in $\mathcal{V}^*$.

Let $C' = C_{A_{\underline{r}}}$ be the centralizer of $A_{\underline{r}}$ in G. We decompose $\mathcal{V}$ according to the diagonal blocks of $A_{\underline{r}}$, $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_k$ where $\mathcal{V}_i$, $1 \leq i \leq k$, is the space of row vectors of the form

$$\begin{pmatrix} 0 & 0 & . & . & 0 & v_i & 0 & . & . & 0 \end{pmatrix}$$

with v_i as an arbitrary vector of length r_i. We identify the subspace $\mathcal{V}_i$ with the space of row vectors of length r_i. Let e_t be a row vector such that the t^{th} entry of e_t is one and all other entries are zero. If $t = 0$ then we set $e_t = 0$, the zero vector of the appropriate size. For a sequence of nonnegative integers $\alpha = (t_1, t_2, \dots, t_k)$ such that $t_i \le r_i$, $i = 1, \dots, k$ we let $v_\alpha \in \mathcal{V}$ be the vector given by

$$v_\alpha = \begin{pmatrix} e_{t_1} & e_{t_2} & . & . & e_{t_k} \end{pmatrix} \tag{5.4}$$

where $e_{t_i} \in \mathcal{V}_i$ is an r_i row vector. Set $S(\alpha)$ to be

$$S(\alpha) = \{i \in \{1, 2, \dots, k\} \mid t_i \neq 0\}. \tag{5.5}$$

The following lemma asserts the existence of nice representatives for the C' orbits in $\mathcal{V}^*$. (Uniqueness may not be true.)

LEMMA 5.2 ([Ral]). *Let $\rho(C')v$, $v \in \mathcal{V}^*$, be a C' orbit in $\mathcal{V}^*$. Then there exist a sequence α as above and a vector $v_\alpha \in \rho(C')v$ such that α satisfies the following conditions*:

$$\begin{aligned} &0 \le t_i \le r_i, \quad i = 1, \dots, k. \\ &\textit{If } i, j \in S(\alpha) \textit{ and } i < j \textit{ then } t_i \le t_j \textit{ and } t_i - t_j \ge r_i - r_j. \end{aligned} \tag{5.6}$$

COROLLARY 5.3. *There are only a finite number of C' orbits in $\mathcal{V}^*$.*

COROLLARY 5.4. *Let $C = C_{X_{\underline{r}}}$. Then each C orbit in $\mathcal{V}^*$ contains an element v_α where α satisfies* (5.6).

Proof. There exists a diagonal element $d \in G$ such that $X_{\underline{r}} = dA_{\underline{r}}d^{-1}$. Thus $C = dC'd^{-1}$. Let $\rho(C)v$ be a C orbit in $\mathcal{V}^*$. By the above lemma, The C' orbit $\rho(C')\rho(d^{-1})v$ contains an element of the form v_α satisfying (5.6). It follows that $\rho(C)v$ contains the element $\rho(d)v_\alpha$. Since d is diagonal we get that $\rho(d)v_\alpha$ has nonzero entries in the same positions as v_α. Since v_α has at most one nonzero entry in each component $\mathcal{V}_i$ it follows that we can change $\rho(d)v_\alpha$ to v_α by a diagonal matrix of the form

$$d(c_1, c_2, \dots, c_k) = \begin{pmatrix} c_1 I_{r_1} & & & & \\ & c_2 I_{r_2} & & & \\ & & . & & \\ & & & . & \\ & & & & c_k I_{r_k} \end{pmatrix} \tag{5.7}$$

where c_j, $1 \le j \le k$, is a nonzero scalar and I_{r_j} is the identity matrix of order r_j. Since the above matrix is clearly in C we get our result. $\square$

The proof of Lemma 5.2 is an easy consequence of the description of C' given in [Ral]. We recall it now. C' is the set of invertible elements h of the block form $h = (Q_{i,j})$. Here $1 \le i, j \le k$, and $Q_{i,j}$ is an $r_i \times r_j$ matrix satisfying the following conditions. Set $A = Q_{i,j}$ and $A = (a_{p,q})$. Then

1) $a_{p_1,q_1} = a_{p_2,q_2}$ if $q_1 - p_1 = q_2 - p_2$.

2) If $r_j \ge r_i$ then $a_{p,q} = 0$ for $q - p < r_j - r_i$ and if $r_j \le r_i$ then $a_{r,s} = 0$ for $s - r < 0$.

In matrix form we have

$$Q_{i,j} = \begin{pmatrix} 0 & . & . & 0 & c_1 & c_2 & . & . & c_{r_i} \\ 0 & . & . & . & 0 & c_1 & . & . & c_{r_i-1} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & 0 & c_1 \end{pmatrix} \text{ if } r_j \ge r_i$$

and

$$Q_{i,j} = \begin{pmatrix} c_1 & c_2 & . & . & c_{r_j} \\ 0 & c_1 & . & . & c_{r_j-1} \\ 0 & 0 & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & c_1 \\ 0 & 0 & . & . & 0 \\ . & . & . & . & . \\ 0 & 0 & . & . & 0 \end{pmatrix} \text{ if } r_j \le r_i.$$

Using this description we can see that given a vector in a C' orbit we can eliminate all but one entry in each block using the diagonal blocks $Q_{i,i}$. Now, given two nonzero entries, one in the ${t_i}^{\text{th}}$ entry of a block of order r_i and one in the ${t_j}^{\text{th}}$ entry of a block of order r_j such that $i < j$, $r_i \le r_j$ and $t_i > t_j$ then we can use the matrix $Q_{j,i}$ to eliminate the ${t_j}^{\text{th}}$ entry in the r_i block. Hence we get the first condition of (5.6). For the second condition we use the block $Q_{i,j}$.

6. Jacobson-Morosov triples in P conjugacy classes

In this section we shall associate with every nilpotent P conjugacy class a triple X, Y, H' which is almost a Jacobson-Morosov triple. X will be in the given conjugacy class and the triple will satisfy the relations $[H', X] = 2X$, $[H', Y] = 2Y$. We will also require that $H' \in \mathfrak{p}$ and that we can make $\mathrm{ad}H'$ act with nonpositive eigenvalues on a certain subspace of $\mathfrak{g}$.

Recall first that if $X, Y, H \in \mathfrak{g}$ form a Jacobson-Morosov triple then $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = [\mathfrak{g}, X] \oplus \mathfrak{g}^Y.$$

(See [Wal, 8.3.6].) Here we can identify $[\mathfrak{g}, X]$ with the tangent space to $O_G(X)$ at X and $\mathfrak{g}^Y$ with the transversal to $O_G(X)$ at X. We would like first to find an analogous decomposition for the P conjugacy class of a nilpotent element X.

For $A, C \in \mathfrak{g} = M_n(K)$, define

$$\langle A, C\rangle = B(A, C) = \mathrm{real}(\mathrm{Trace}(AC)). \tag{6.1}$$

This defines a real symmetric invariant and nondegenerate bilinear form on $\mathfrak{g}$. It is a form of the type which is introduced in [Wal, 0.2.2]. It is clear that the restriction of B to $[\mathfrak{g}, \mathfrak{g}]$ is a scalar multiple of the killing form. For a subspace $\mathfrak{q}$ of $\mathfrak{g}$ and an element $Y \in \mathfrak{g}$ we define

$$\mathfrak{g}^{Y,\mathfrak{q}} = \{A \in \mathfrak{g} \,:\, [A, Y] \in \mathfrak{q}\}. \tag{6.2}$$

Let

$$\mathfrak{p}^c = \left\{ \begin{pmatrix} 0 \\ b \end{pmatrix} \mid b \in M_{n,1}(K) \right\}$$

be the Lie subalgebra of $\mathfrak{g}$ which is the complement to $\mathfrak{p}$. Define $\mathfrak{g}^{Y,\mathfrak{p}^c}$ as in (6.2).

Lemma 6.1. *Let $X \in \mathfrak{g}$ be a matrix with real entries and set $Y = X^t$. Then*

$$\mathfrak{g} = [\mathfrak{p}, X] \oplus \mathfrak{g}^{Y,\mathfrak{p}^c}.$$

Proof. We first show that $([\mathfrak{p}^t, Y])^\perp = \mathfrak{g}^{Y,\mathfrak{p}^c}$ with respect to $<,>$. Here $\mathfrak{p}^t = \{A^t \,:\, A \in \mathfrak{p}\}$. Let $B \in \mathfrak{g}^{Y,\mathfrak{p}^c}$ and $C \in \mathfrak{p}^t$. Then

$$\langle [C, Y], B\rangle = \langle C, [Y, B]\rangle = \langle C, D\rangle = 0 \tag{6.3}$$

where $D = [Y, B] \in \mathfrak{p}^c$. On the other hand, assume $B \in ([\mathfrak{p}^t, Y])^\perp$. Then it follows from (6.3) that $D = [Y, B]$ is perpendicular to every $C \in \mathfrak{p}^t$. But this means that $D \in \mathfrak{p}^c$ and we are done.

Since $Y = X^t$ and X has real entries, it follows that $[\mathfrak{p}^t, Y] = \overline{([\mathfrak{p}, X])}^t$. Hence, $[\mathfrak{p}^t, Y]$ and $[\mathfrak{p}, X]$ are nondegenerately paired. It follows that $[\mathfrak{p}, X]$ and $\mathfrak{g}^{Y,\mathfrak{p}^c}$ intersect only at 0 and that the sum of the dimensions is the right one; hence the lemma is proved. $\square$

Lemma 6.2. *Let X' be a nilpotent element. Then there exists $X \in O_P(X')$ such that X has real entries and such that $X, Y = X^t, H = [X, Y]$ form a Jacobson-Morosov triple.*

Proof. Let $X_{\underline{r}}$ be the unique representative of $O_G(X)$ as defined in Lemma 5.1. From Section 5 we know that $O_P(X)$ is matched with a $C = C_{X_{\underline{r}}}$ orbit O_C in $\mathcal{V}^*$ such that if $v \in O_C$ and if $g \in G$ satisfies $\rho(g)v = v_0$ then $gX_{\underline{r}}g^{-1} \in O_P(X)$. Pick $v \in O_C$ to be a unit vector with real entries. This is possible by Lemma 5.2. It is easy to see that every invertible matrix g

whose last row is v will satisfy $\rho(g^{-1})v_0 = v$. Hence we can find a real orthogonal matrix g such that $\rho(g)v = v_0$ and such that $g^{-1} = g^t$. Now the triple $X = gX_{\underline{r}}g^{-1}, Y = gY_{\underline{r}}g^{-1}, H = gH_{\underline{r}}g^{-1}$ satisfies the requirements of the lemma. $\square$

Combining Lemma 6.1 and Lemma 6.2 we can associate to each nilpotent P conjugacy class a Jacobson-Morosov triple (X, Y, H) such that X is in the given conjugacy class and such that $\mathfrak{g} = [\mathfrak{p}, X] \oplus \mathfrak{g}^{Y,\mathfrak{p}^c}$. The problem with this triple X, Y, H is that ad$H$ in general does not stabilize $\mathfrak{g}^{Y,\mathfrak{p}^c}$ and that even if it does, the eigenvalues of ad$H$ on that space are not always nonpositive. (Compare it with the fact that ad$H$ always stabilizes $\mathfrak{g}^Y$ when H and Y are part of a Jacobson-Morosov triple, and that the eigenvalues of H on $\mathfrak{g}^Y$ are always nonpositive.)

The main difficulty is to "adjust" H in a "nice" way so that the "new" H will stabilize $\mathfrak{g}^{Y,\mathfrak{p}^c}$, and that it will have nonpositive eigenvalues on $\mathfrak{g}^{Y,\mathfrak{p}^c}$. However, we still want the new H to act the same on X and Y. To do that we must translate $H = [X, Y]$ by an element of $\mathfrak{g}^Y \cap \mathfrak{g}^X$. This is the content of the following theorem which is the main result of this section and is one of the key results in this paper. It was stated in the introduction as Theorem 1.7.

THEOREM 6.3. *Let X' be a nonzero nilpotent element. Then there exist $X \in O_P(X')$ and $H' \in \mathfrak{p} \cap \mathfrak{p}^t$ such that*

(1) $X, Y = X^t, H = [X, Y]$ *form a Jacobson-Morosov triple.*

(2) $(H' - H) \in \mathfrak{g}^Y \cap \mathfrak{g}^X$ *and in particular* $[H', X] = 2X$ *and* $[H', Y] = -2Y$.

(3) $\mathrm{ad}(H')$ *is semisimple, with integer eigenvalues, and stabilizes* $\mathfrak{g}^{Y,\mathfrak{p}^c}$.

(4) *The eigenvalues of* $\mathrm{ad}(H')$ *on* $\mathfrak{g}^{Y,\mathfrak{p}^c}$ *are all nonpositive and their sum is less than or equal to* $\dim_{\mathbf{R}}(\mathfrak{g}^{Y,\mathfrak{p}^c}) - \dim_{\mathbf{R}}(\mathfrak{g})$. *That is,*

$$\mathrm{Trace}(\mathrm{ad}(H')_{|\mathfrak{g}^{Y,\mathfrak{p}^c}}) \leq \dim_{\mathbf{R}}(\mathfrak{g}^{Y,\mathfrak{p}^c}) - \dim_{\mathbf{R}}(\mathfrak{g}).$$

The proof of Theorem 6.3 is quite technical and will take the rest of this section. We recommend that the reader skip it on the first reading and go on to Section 7.

Set $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$. It is easy to see that $\mathfrak{g}^{Y,\mathfrak{p}^c} = \mathfrak{z} \oplus \mathfrak{s}^{Y,\mathfrak{p}^c}$ and that the sum of the eigenvalues of $\mathrm{ad}(H')$ on $\mathfrak{s}^{Y,\mathfrak{p}^c}$ is the same as the sum of the eigenvalues of $\mathrm{ad}(H')$ on $\mathfrak{g}^{Y,\mathfrak{p}^c}$. Since $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{s}$ we get the following corollary from Theorem 6.3.

COROLLARY 6.4. *Let X, Y and H' be as in Theorem* 6.3. *Then the eigenvalues of* $\mathrm{ad}(H')$ *on* $\mathfrak{s}^{Y,\mathfrak{p}^c}$ *are all nonpositive and*

$$\mathrm{Trace}(\mathrm{ad}(H')_{|\mathfrak{s}^{Y,\mathfrak{p}^c}}) \leq \dim_{\mathbf{R}}(\mathfrak{s}^{Y,\mathfrak{p}^c}) - \dim_{\mathbf{R}}(\mathfrak{s}).$$

Remark 6.5. From the assumption that $H' \in \mathfrak{p}^t$ and from $[H', Y] = -2Y$, we immediately get that $\mathrm{ad}(H')$ stabilizes $\mathfrak{g}^{Y, \mathfrak{p}^c}$. To see this, let $A \in \mathfrak{g}^{Y, \mathfrak{p}^c}$. Then

$$[[H', A], Y] = [H', [A, Y]] + [[H', Y], A] = [H', C] + [-2Y, A]$$

for some $C \in \mathfrak{p}^c$. It is clear that both summands are in $\mathfrak{p}^c$.

It follows from the remark that in order to achieve (1) (2) and (3) it is enough to do the following. Let $X_{\underline{r}}$ be the special representative of $O_G(X')$. Choose a "nice" representative v of the $C_{X_{\underline{r}}}$ orbit in $\mathcal{V}^*$ corresponding to $O_P(X')$. Choose an orthogonal matrix g such that $\rho(g)v_0 = v$. (As above, this will be achieved by forcing the last row of g to be v.) Translate $H_{\underline{r}}$ by an integral diagonal element $d \in \mathfrak{g}^{Y_{\underline{r}}} \cap \mathfrak{g}^{X_{\underline{r}}}$ such that the resulting diagonal matrix $H_v = H_{\underline{r}} + d$ satisfies $H' = g_v H g^{-1} \in \mathfrak{p}$. (This will be achieved if the diagonal entries of H_v corresponding to the nonzero elements of v are all zero.) Since H_v is diagonal we get that $H' = (H')^t$ and the triple $X = gX_{\underline{r}}g^{-1}, Y = gY_{\underline{r}}g^{-1}, H' = g_v H g^{-1}$ satisfies (1),(2) and (3).

We chose a sequence $\alpha = (t_1, \ldots, t_k)$ satisfying (5.6) and such that $v = v_\alpha$ (see (5.4)) is in our given C orbit. Chose $g = g_\alpha$ such that $\rho(g_\alpha)v_\alpha = v_0$. Let $S(\alpha) \subseteq \{1, 2, \ldots, k\}$ be the set defined in (5.5) and set

$$c_i = -H_{r_i}(t_i) = -r_i - 1 + 2t_i, \quad i \in S(\alpha). \tag{6.4}$$

Here H_{r_i} is the diagonal matrix defined in (5.3) and $H_{r_i}(t_i)$ is the (t_i, t_i) entry of H_{r_i}. We let c_i be an arbitrary nonzero integer if $i \notin S$. Set

$$H_\alpha = H_v = H_{\underline{r}} + d(c_1, c_2, \ldots, c_k). \tag{6.5}$$

Here $d(c_1, c_2, \ldots, c_k)$ is as defined in (5.7). In block form $H_\alpha = H_\alpha^1 \oplus H_\alpha^2 \oplus \cdots \oplus H_\alpha^k$. where $H_\alpha^i = H_{r_i} + c_i I_{r_i}$. If $t_i \neq 0$ then the (t_i, t_i) entry of H_α^i is zero. Since the last row of $g = g_\alpha$ is the vector $v = v_\alpha$, it is easy to see that $H' = gH_\alpha g^{-1} = gH_\alpha g^t \in \mathfrak{p} \cap \mathfrak{p}^t$. Hence, the triple $X = gX_{\underline{r}}g^{-1}, Y = gY_{\underline{r}}g^{-1}, H' = gH_\alpha g^{-1}$ satisfies (1),(2) and (3).

Remark 6.6. It is clear that $d(c_1, c_2, \ldots, c_k) \in \mathfrak{g}^{Y_{\underline{r}}} \cap \mathfrak{g}^{X_{\underline{r}}}$. It remains for us to show that we can choose the c_j, $j \notin S(\alpha)$, in such a way that the action of $\mathrm{ad}H'$ on $\mathfrak{g}^{Y, \mathfrak{p}^c}$ will satisfy (4) of Theorem 6.3. It might be that $S(\alpha) = \{1, 2, \ldots, k\}$ in which case all the c_js are already determined. In that case H' is now fixed and we have to show that it satisfies (4).

It will be convenient to replace the action of $\mathrm{ad}H'$ on $\mathfrak{g}^{Y, \mathfrak{p}^c}$ with the action of $\mathrm{ad}H_\alpha$ on an appropriate space. This is the content of the following lemma:

LEMMA 6.7. *$\mathrm{ad}(H_\alpha)$ stabilizes $\mathfrak{g}^{Y_{\underline{r}}, g_\alpha^{-1}\mathfrak{p}^c g_\alpha}$ and the action of $\mathrm{ad}H'$ on $\mathfrak{g}^{Y, \mathfrak{p}^c}$ is equivalent to the action of $\mathrm{ad}H_\alpha$ on $\mathfrak{g}^{Y_{\underline{r}}, g_\alpha^{-1}\mathfrak{p}^c g_\alpha}$.*

Proof. If $A \in \mathfrak{g}^{Y,\mathfrak{p}^c}$ then

$$[H', A] = [gH_\alpha g^{-1}, A] = g[H_\alpha, g^{-1}Ag]g^{-1}. \tag{6.6}$$

Hence we can replace the action of adH' on $\mathfrak{g}^{Y,\mathfrak{p}^c}$ by the action of adH_α on $Ad(g^{-1})\mathfrak{g}^{Y,\mathfrak{p}^c}$. Now

$$\begin{aligned} A \in \mathfrak{g}^{Y,\mathfrak{p}^c} \quad &\Leftrightarrow \quad [A, Y] \in \mathfrak{p}^c \\ &\Leftrightarrow \quad [A, gY_{\underline{r}}g^{-1}] \in \mathfrak{p}^c \\ &\Leftrightarrow \quad g[g^{-1}Ag, Y_{\underline{r}}]g^{-1} \in \mathfrak{p}^c \\ &\Leftrightarrow \quad [gAg^{-1}, Y_{\underline{r}}] \in g^{-1}\mathfrak{p}^c g. \end{aligned} \tag{6.7}$$

Hence $Ad(g^{-1})\mathfrak{g}^{Y,\mathfrak{p}^c} = \mathfrak{g}^{Y_{\underline{r}},g^{-1}\mathfrak{p}^c g}$. $\square$

To analyze the action of adH_α on $\mathfrak{g}^{Y_{\underline{r}},g_\alpha^{-1}\mathfrak{p}^c g_\alpha}$ we need to give a simple description of $g_\alpha^{-1}\mathfrak{p}^c g_\alpha$. Set $\mathfrak{x}_\alpha = g_\alpha^{-1}\mathfrak{p}^c g_\alpha$. Then $\mathfrak{x}_\alpha$ is the set of matrices such that the rows corresponding to the zero entries of v_α are zero and the rows corresponding to the nonzero elements of v_α are all the same. In other words, let S be the set of indices j such that the entry $v_j \neq 0$. For $A \in \mathfrak{g}l_n(F)$ let A_i be the i^{th} row of A. Then $A \in \mathfrak{x}_\alpha$ if and only if $A_i = 0$ for $i \notin S$ and $A_p = A_q$ for every $p, q \in S$. In matrix form we have that $\mathfrak{x}_\alpha$ is the set of matrices A of the form

$$A = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ u \\ 0 \\ . \\ 0 \\ u \\ 0 \\ . \\ . \\ 0 \end{pmatrix} \tag{6.8}$$

where the row vector u appears at all the rows indexed by the set S.

Let $B_{\underline{r}} = (A_{\underline{r}})^t$. In matrix notation we have that $B_{\underline{r}} = \mathrm{diag}(B_{r_1}, B_{r_2}, \ldots, B_{r_k})$ where B_{r_i} is an $r_i \times r_i$ matrix of the form

$$B_{r_i} = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 1 & 0 \end{pmatrix}. \tag{6.9}$$

It is easier to make computations with $B_{\underline{r}}$ than with $Y_{\underline{r}}$. Therefor we shall prove:

Lemma 6.8. *$\mathrm{ad}H_\alpha$ stabilizes $\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}$ and the action of $\mathrm{ad}H_\alpha$ on $\mathfrak{g}^{Y_{\underline{r}},\mathfrak{x}_\alpha}$ is equivalent to the action of $\mathrm{ad}H_\alpha$ on $\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}$.*

Proof. There exists an invertible diagonal matrix d such that $Y_{\underline{r}} = dB_{\underline{r}}d^{-1}$. Moreover, we can choose d so that the (t_i, t_i) element of the i^{th} block of d is one for every $i \in \alpha$. It is easy to check that $\mathrm{Ad}(d)(\mathfrak{x}_\alpha) = \mathfrak{x}_\alpha$. Now

$$\begin{aligned} A \in \mathfrak{g}^{Y_{\underline{r}},\mathfrak{x}_\alpha} \quad &\Leftrightarrow \quad [A, Y_{\underline{r}}] \in \mathfrak{x}_\alpha \\ &\Leftrightarrow \quad [A, dB_{\underline{r}}d^{-1}] \in \mathfrak{x}_\alpha \\ &\Leftrightarrow \quad d[d^{-1}Ad, B_{\underline{r}}]d^{-1} \in \mathfrak{p}^c \\ &\Leftrightarrow \quad [d^{-1}Ad, B_{\underline{r}}] \in d^{-1}\mathfrak{x}_\alpha d = \mathfrak{x}_\alpha. \end{aligned} \tag{6.10}$$

Hence the map $A \mapsto d^{-1}Ad$ is an isomorphism between $\mathfrak{g}^{Y_{\underline{r}},\mathfrak{x}_\alpha}$ and $\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}$. Since H_α and d are both diagonal they commute, hence, if A is an eigenmatrix of $\mathrm{ad}H_\alpha$ with eigenvalue λ_A then dAd^{-1} is an eigenmatrix of $\mathrm{ad}H_\alpha$ with eigenvalue λ_A. $\square$

It follows from Remark 6.6, Lemma 6.7 and Lemma 6.8 that Theorem 6.3 is reduced to analyzing the action of $\mathrm{ad}H_\alpha$ on $\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}$. We summarize what we need to prove to complete the proof of Theorem 6.3 in the following proposition:

Proposition 6.9. *Let $\underline{r} = (r_1, \ldots, r_k)$ be a partition of n as in* (5.1) *and let $\alpha = (t_1, \ldots, t_k)$ be a sequence of nonnegative integers such that $t_i \leq r_i$ for all i and such that α satisfies* (5.6)*. Let $S(\alpha)$ be as in* (5.5) *and assume that $S(\alpha) \neq \emptyset$. Define H_α as in* (6.5) *with integers c_j, $j \in S(\alpha)$ defined by* (6.4)*. Let $B_{\underline{r}}$ be the matrix defined in* (6.9)*. Then there exist integers c_j, $j \notin S(\alpha)$ such that the eigenvalues of $\mathrm{ad}(H_\alpha)$ on $\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}$ are all nonnegative integers, such that their sum is less than or equal to $\dim_{\mathbf{R}}(\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}) - \dim_{\mathbf{R}}(\mathfrak{g})$.*

6.1. *The space $\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}$.* Let $0 \leq t \leq q$ and define $\mathfrak{x}_t \subseteq M_{p,q}(K)$ as the space of $p \times q$ matrices such that all rows other than the t^{th} row are zero. If $t = 0$ then we let $\mathfrak{x}_t = \{0\}$.

LEMMA 6.10. *Let $q \geq p$ be positive integers. Let B_p and B_q be the matrices defined in* (6.9). *Let $A \in M_{p,q}(K)$ and set $M = AB_q - B_pA$. Let $A = (a_{i,j})$ and $M = (m_{i,j})$. We use the convention that $m_{i,j} = a_{i,j} = 0$ for $(i,j) \notin \{1,\ldots,p\} \times \{1,\ldots,q\}$. Fix t such that $0 \leq t \leq q$. If $M \in \mathfrak{x}_t$ then*

(a) *$a_{i,j} = 0$ for $j - i > q - p$.*

(b) *$m_{t,j} = 0$ for $j \geq t + (q - p)$.*

(c) *If there exists an integer l, $l \geq t$, such that $m_{t,j} = 0$ for all $j \geq l$ then $a_{i,j} = 0$ for $j - i > l - t$.*

(d) *If $t = 0$ then $a_{i,j} = 0$ for $j - i > 0$.*

Proof. The proof is a simple computation. We have $m_{i,j} = a_{i,j+1} - a_{i-1,j}$ where we have used the convention above. We look at the block of M which starts from the first row and ends with the $t-1$ row. This block is zero; hence going down the rows of this block we get that $a_{i,j} = 0$ for $j - i > 0$ and $i < t$. Looking at the block of M that starts with the $t + 1$ row and ends with the last row and going from the last column backward, we get that $a_{i,j} = 0$ for $j - i > q - p$ and $i \geq t$. Combining the two we get (a).

Now $m_{t,j} = a_{t,j+1} - a_{t-1,j}$. If $j \geq t + (q - p)$ then $j - t + 1 > q - p$ and by (a), $a_{t,j+1} = 0$, $a_{j-1,t} = 0$ and (b) follows.

To prove (c), fix $r \geq t$ and assume that $m_{t,r} = 0$. Let $f = r - t$. Since $M \in \mathfrak{x}_t$ we have that $m_{i,j} = 0$ for $j - i = f$, hence $a_{i,j} = 0$ for $j - i = f + 1$. Since we assumed in (b) that $m_{t,l} = 0$ for all $l \geq t$ we get the required result and (d) follows in the same way. □

LEMMA 6.11. *Let $q \leq p$ be positive integers. Let $A \in M_{p,q}(K)$ and let $M = AB_q - B_pA$. Let $A = (a_{i,j})$ and $M = (m_{i,j})$. We use the convention that $m_{i,j} = a_{i,j} = 0$ for $(i,j) \notin \{1,\ldots,p\} \times \{1,\ldots,q\}$. Fix t such that $0 \leq t \leq q$. If $M \in \mathfrak{x}_t$ then*

(a) *$a_{i,j} = 0$ for $j - i > 0$.*

(b) *$m_{t,j} = 0$ for $j \geq t$.*

(c) *If there exists an integer l, $l \geq t + q - p$, such that $m_{t,j} = 0$ for all $j \geq l$ then $a_{i,j} = 0$ for $j - i > l - t$.*

(d) *If $t = 0$ then $a_{i,j} = 0$ for $j - i > q - p$.*

The proof of this lemma is the same as that of the previous lemma and is omitted.

We can now study further the space $\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}$. Let $\underline{r} = (r_1, \ldots, r_k)$ be a partition of n. Let $\alpha = (t_1, \ldots, t_k)$ be a tuple of integers satisfying (5.6) and

$S(\alpha) = \{i \,|\, t_i \neq 0\}$. Assume $S(\alpha) \neq \emptyset$. Fix $j \notin S(\alpha)$ and define integers $q_{i,j}$, $i \in S(\alpha)$ by

$$q_{i,j} = \begin{cases} t_i + r_j - r_i, & \text{if } i < j,\ i \in S(\alpha); \\ t_i & \text{if } i > j,\ i \in S(\alpha). \end{cases} \tag{6.11}$$

We also define the integers p_j, $j \notin S(\alpha)$, by

$$p_j = \min_{i \notin S(\alpha)} q_{i,j}. \tag{6.12}$$

Let $\mathfrak{x}_\alpha \subset \mathfrak{g} = M_n(F)$ be as in (6.8).

Corollary 6.12. *Assume that $A \in \mathfrak{g}^{B_{\underline{r}}, \mathfrak{x}_\alpha}$. That is,*

$$AB_{\underline{r}} - B_{\underline{r}}A \in \mathfrak{x}_\alpha. \tag{6.13}$$

Write $A = (A_{i,j})$ in block form where $A_{i,j}$ is an $r_i \times r_j$ block of A. Fix $i_0 \in S(\alpha)$ and $j_0 \notin S(\alpha)$. Then $A_{i_0,j_0} = (\tilde{a}_{l,s})$ satisfies

$$\tilde{a}_{l,s} = 0 \ \ \textit{if } s - l > p_{j_0} - t_{i_0}.$$

Proof. The proof is a straightforward application of Lemma 6.10 and Lemma 6.11. Let $M = AB_{\underline{r}} - B_{\underline{r}}A$ and write $M = (M_{i,j})$ in block form. By (6.13) we have that

$$M_{i,j} = A_{i,j}B_j - B_iA_{i,j} \in \mathfrak{x}_{t_i}.$$

Set $M_{i,j} = (m^{i,j}_{l,s})$. Assume $i \in S(\alpha)$ and $j \notin S(\alpha)$. Then by Lemma 6.10 (b) and Lemma 6.11 (b) we have that $m^{i,j}_{t_i,s} = 0$ for $s \geq q_{i,j}$. Since $M \in \mathfrak{x}_\alpha$ we have that $m^{i_1,j_0}_{t_{i_1},s} = m^{i_2,j_0}_{t_{i_2},s}$ for every $i_1, i_2 \in S(\alpha)$. Hence,

$$m^{i_0,j_0}_{t_{i_0},s} = 0 \ \text{ for } s \geq p_{j_0}. \tag{6.14}$$

We would now like to apply part (c) of the above lemmas for $A_{i_0,j_0} = (\tilde{a}_{l,s})$. Assume first that $j_0 > i_0$. We must show that $p_{j_0} \geq t_{i_0}$. If $k > j_0$ and $k \in S(\alpha)$ then $q_{k,j_0} = t_k$ and $t_k \geq t_{i_0}$ by (5.6). If $k < j_0$ then $q_{k,j_0} = t_k + r_{j_0} - r_k$. So $q_{k,j_0} - t_{i_0} = t_k - t_{i_0} + r_{j_0} - r_{i_0}$. If $k \geq i_0$ then $t_k \geq t_{i_0}$ and $r_{j_0} \geq r_{i_0}$. If $k < i_0$ then by (5.6), $t_k - t_{i_0} + r_{j_0} - r_{i_0} \geq t_k - t_{i_0} + r_k - r_{i_0} \geq 0$.

In all cases $q_{k,j_0} \geq t_{i_0}$, hence $p_{j_0} \geq t_{i_0}$ and we can apply (6.14) with Lemma 6.10 (c) to get the required result. A similar argument for the case $j_0 < i_0$ will conclude the proof. □

6.2. *Proof of Proposition* 6.9. We shall divide the proof into two parts. We shall first show that for a given choice of c_js in H_α the eigenvalues are all nonpositive. In the second part we shall estimate the sum of the eigenvalues.

Recall that we are given a partition $\underline{r}$ of n such that $\underline{r} = (r_1, \ldots, r_k)$, with $r_i \le r_j$ for $i \le j$ and a matrix

$$H_\alpha = \begin{pmatrix} H_{r_1} & & & & \\ & H_{r_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & H_{r_k} \end{pmatrix} + \begin{pmatrix} c_1 I_{r_1} & & & & \\ & c_2 I_{r_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & c_k I_{r_k} \end{pmatrix} = H_{\underline{r}} + d(c_1, \ldots, c_k).$$

We are given a sequence $\alpha = (t_1, \ldots, t_k)$ satisfying (5.6) and we let $S(\alpha)$ be the set of indices i where $t_i \ne 0$. Assume that $S(\alpha) \ne \emptyset$. Define the integers p_j, $j \notin S(\alpha)$, as in (6.12). We define the integers c_i, $i = 1, \ldots k$, by

$$c_i = \begin{cases} -r_i - 1 + 2t_i, & \text{if } i \in S(\alpha); \\ -r_i - 1 + 2p_i, & \text{if } i \notin S(\alpha). \end{cases} \tag{6.15}$$

Proposition 6.13. *The eigenvalues of* $\mathrm{ad}(H_\alpha)$ *on* $\mathfrak{g}^{B_{\underline{r}}, \mathfrak{x}_\alpha}$ *are all nonpositive.*

Proof. We shall start with two special cases. This will give the reader a chance to consider a simple case as well as provide a tool for the general case.

Case I: $k = 1$. We assume that there is only one block. That is, $k = 1$, $\underline{r} = (n)$, $\alpha = (t)$ with $t > 0$. Now, $\mathfrak{x}_\alpha = \mathfrak{x}_t$ is the set of $n \times n$ matrices whose rows are all zero except possibly the t^{th} row. This case corresponds to the largest nilpotent G conjugacy class in $\mathfrak{g} = \mathfrak{g}l_n(K)$ and it is easy to see that the number of P conjugacy classes in this G conjugacy class is exactly n, corresponding to the possible values of t. In this case $H_\alpha = H_n - cI_n$ where H_n is defined in (5.3) and $c = -n - 1 + 2t$. Since $\mathrm{ad}(I_n) = 0$, the eigenvalues of $\mathrm{ad}(H_\alpha)$ on $\mathfrak{g}^{B_{\underline{r}}, \mathfrak{x}_t}$ are the same as the eigenvalues of $\mathrm{ad}(H_n)$ on this space. The eigenvalues of $\mathrm{ad}(H_n)$ depend only on the difference $j - i$ of a nonzero entry $a_{i,j}$ in an eigenmatrix A. Moreover they are increasing with respect to this difference. If $A = (a_{i,j}) \in \mathfrak{g}^{B_{\underline{r}}, \mathfrak{x}_t}$ then A satisfies condition (a) of Lemma 6.11; hence $a_{i,j} = 0$ for $j > i$. If A is an eigenvector of $\mathrm{ad}(H_n)$ with eigenvalue λ_A then λ_A will be maximal if there exists $i > 0$ such that $a_{i,i} \ne 0$. In that case $\lambda_A = 0$ and we are done.

Case II: $k = 2$. $S(\alpha) = \{1,2\}$. We assume that there are two blocks. That is, $k = 2$, $\underline{r} = (r_1, r_2)$ with $r_1 \leq r_2$, $r_1 + r_2 = n$. We let $\alpha = (t_1, t_2)$ with $0 \leq t_i \leq r_i$, $i = 1, 2$. We shall further assume that $t_i > 0$, $i = 1, 2$, that is $S(\alpha) = \{1, 2\}$. Since α is assumed to satisfy (5.6) we have that

$$t_1 \leq t_2, \;\; t_1 - t_2 \geq r_1 - r_2. \tag{6.16}$$

$\mathfrak{x}_\alpha$ is the set of $n \times n$ matrices such that the ${t_1}^{\text{th}}$ row and the $r_1 + {t_2}^{\text{th}}$ row are the same and all other rows are zero. Finally we have

$$H_\alpha = \begin{pmatrix} H_{r_1} & \\ & H_{r_2} \end{pmatrix} + \begin{pmatrix} c_1 I_{r_1} & \\ & c_2 I_{r_2} \end{pmatrix} = H_{\underline{r}} + d(c_1, c_2),$$

where $c_i = -r_i - 1 + 2t_i$. Let $A = (a_{i,j}) \in \mathfrak{g}^{B_{\underline{r}}, \mathfrak{x}_\alpha}$ be an eigenmatrix of H_α with eigenvalue λ_A. Write

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

where $A_{i,j}$ is an $r_i \times r_j$ matrix. Write $M = AB_{\underline{r}} - B_{\underline{r}} A$ and set $M = (M_{i,j})$. Then $M \in \mathfrak{x}_\alpha$; hence $A_{i,j}$ satisfies

$$A_{i,j} B_j - B_i A_{i,j} \in \mathfrak{x}_{t_j}, \quad i, j = 1, 2. \tag{6.17}$$

If $A_{1,1}$ or $A_{2,2}$ have nonzero entries then the eigenvalue λ_A is determined by the action of $\mathrm{ad}(H_{\underline{r}})$ and since $A_{i,i}$ satisfy (6.17) we get from the proof of Case I that $\lambda_A \leq 0$. Hence we can assume that $A_{i,i} = 0$, $i = 1, 2$. This means that $M_{i,i} = A_{i,i} B_i - B_i A_{i,i} = 0$. But since $M \in \mathfrak{x}_\alpha$, it follows that $M_{i,j} = 0, i \neq j$ (that is, $M = 0$). So $A_{i,j}$ satisfy a stronger condition than (6.17) namely

$$B_i A_{i,j} - A_{i,j} B_j = 0, \;\; i = 1, 2, \;\; j = 1, 2. \tag{6.18}$$

Set $A_{1,2} = (\tilde{a}_{i,j})$. By Lemma 6.10 (d), $\tilde{a}_{i,j} = 0$ for $j \geq i$. If $A_{1,2} \neq 0$ then λ_A is maximal if there exists i such that $\tilde{a}_{i,i} \neq 0$. Without loss of generality we can assume that $\tilde{a}_{1,1} \neq 0$. Then

$$\lambda_A = (r_1 - 1) + (r_2 + 1 - 2t_2) - (r_1 + 1 - 2t_1) - (r_2 - 1) = 2(t_1 - t_2) \leq 0.$$

Let $A_{2,1} = (\hat{a}_{i,j})$. Then by Lemma 6.11 we have $\hat{a}_{i,j} = 0$ for $j - i > r_1 - r_2$. If $A_{2,1} \neq 0$ then λ_A is maximal if there exist i, j such that $\hat{a}_{i,j} \neq 0$ with $j - i = r_1 - r_2$. Without loss of generality we can assume that $\hat{a}_{r_2, r_1} \neq 0$. Then

$$\lambda_A = (r_1 + 1 - 2t_1) + (r_2 - 1) - (-r_1 + 1) - (r_2 + 1 - 2t_2) = 2(r_1 - r_2) + 2(t_2 - t_1).$$

By (6.16) $r_1 - r_2 \leq t_1 - t_2$; hence $\lambda_A \leq 0$ and Case II is completed.

The general case. Let $A \in \mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}$ and assume that A is an eigenvector for $\mathrm{ad}(H_\alpha)$ with eigenvalue λ_A. Write $A = (A_{i,j})$ where $A_{i,j}$ is an $r_i \times r_j$ block of A. We shall assume that one of these blocks is nonzero and compute how large λ_A can get. If $A_{i,i} \neq 0$ for some i then the proof of Case I shows that $\lambda_A \leq 0$. If $A_{i,j} \neq 0$, $i \neq j$ and $i, j \in S(\alpha)$ then the proof of Case II gives that $\lambda_A \leq 0$.

We now consider the remaining cases where an off-diagonal block, $A_{i,j}$ is nonzero and at most one of i and j is in $S(\alpha)$. Again, assume that $A \in \mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}$ is an eigenmatrix of $\mathrm{ad}(H_\alpha)$ with eigenvalue λ_A and that $A_{i,j} \neq 0$.

1. $j > i$. We consider the submatrix of A of the form

$$\begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{j,i} & A_{j,j} \end{pmatrix}.$$

The corresponding submatrix of H_α is of the form

$$\begin{pmatrix} H_{r_i} & \\ & H_{r_j} \end{pmatrix} + \begin{pmatrix} c_i I_{r_1} & \\ & c_j I_{r_j} \end{pmatrix},$$

(a) $A_{i,j} \neq 0$, $i \in S(\alpha)$, $j \notin S(\alpha)$. In this case we have $c_i = -r_i - 1 + 2t_i$ and $c_j = -r_j - 1 + 2p_j$. Set $A_{i,j} = (\tilde{a}_{m,l})$. By Corollary 6.12, $\tilde{a}_{l,s} = 0$ if $s - l > p_j - t_i$. It follows that λ_A will be maximal if $\tilde{a}_{s,l} \neq 0$ for some s, l satisfying $s - l = p_j - t_i$. Without loss of generality we can assume that $l = t_i$ and $s = p_j$. Then

$$\lambda_A = (r_i + 1 - 2t_i) + (-r_i - 1 + 2t_i) - (r_j + 1 - 2p_j) - (-r_j - 1 + 2p_j) = 0.$$

(b) $A_{i,j} \neq 0$, $i \notin S(\alpha)$, $j \in S(\alpha)$. In this case we have $c_i = -r_i - 1 + 2p_i$ and $c_j = -r_j - 1 + 2t_j$.

Set $A_{i,j} = (\tilde{a}_{l,s})$. Then by Lemma 6.10 part (d) we have that $\tilde{a}_{l,s} = 0$ for $s - l > 0$. It follows that λ_A will be maximal if $\tilde{a}_{s,l} \neq 0$ for some s, l satisfying $s - l = 0$. Without loss of generality we can assume that $\tilde{a}_{1,1} \neq 0$. Then

$$\lambda_A = (r_i - 1) + (-r_i + 1 + 2p_i) - (r_j - 1) - (-r_j - 1 + 2t_j) = 2(p_i - t_j).$$

Since $q_{j,i} = t_j$ and $p_i = \min_{k \notin S(\alpha)} q_{k,i}$ it follows that $p_i \leq t_j$ and $\lambda_A \leq 0$.

(c) $A_{i,j} \neq 0$, $i \notin S(\alpha)$, $j \notin S(\alpha)$. In this case we have $c_i = -r_i - 1 + 2p_i$ and $c_j = -r_j - 1 + 2p_j$.

Set $A_{i,j} = (\tilde{a}_{m,l})$. By Lemma 6.10, $\tilde{a}_{l,s} = 0$ if $s - l > 0$. It follows that λ_A will be maximal if $\tilde{a}_{s,l} \neq 0$ for some s, l satisfying $s - l = 0$. Without loss of generality we can assume that $l = 1$ and $s = 1$. Then

$$\lambda_A = (r_i - 1) + (-r_i - 1 + 2p_i) - (r_j - 1) + -(-r_j - 1 + 2p_j) = 2(p_i - p_j).$$

Fix $k \in S(\alpha)$. If $k > j$ then $q_{k,i} = t_k$ and $q_{k,j} = t_k$. If $i < k < j$ then $q_{k,i} = t_k$ and $q_{k,j} = t_k + r_j - r_k$. If $k < i$ then $q_{k,i} = t_k + r_i - r_k$ and $q_{k,j} = t_k + r_j - r_k$. In all cases we have $q_{k,i} \leq q_{k,j}$ hence $p_i \leq p_j$ and we conclude that $\lambda_A \leq 0$.

2. $i < j$. This case again splits into three sub-cases. Sub-case (a) in which $i \in S(\alpha)$ and $j \notin S(\alpha)$ and sub-case (b) in which $i \notin S(\alpha)$ and $j \in S(\alpha)$ are similar to the ones above. We conclude with the last case:

(c) $A_{i,j} \neq 0$, $i \notin S(\alpha)$, $j \notin S(\alpha)$. In this case we have $c_i = -r_i - 1 + 2p_i$ and $c_j = -r_j - 1 + 2p_j$.

Set $A_{i,j} = (\tilde{a}_{m,l})$.

By Lemma 6.11, $\tilde{a}_{s,l} = 0$ for $l - s > r_i - r_j$. It follows that λ_A will be maximal if $\tilde{a}_{s,l} \neq 0$ for some s, l satisfying $s - l = r_j - r_i$. Without loss of generality we can assume that $l = r_i$ and $s = r_j$. Then

$$\lambda_A = (1-r_i)+(-r_i-1+2p_i)-(1-r_j)-(-r_j-1+2p_j) = 2(r_j-r_i)+2(p_i-p_j).$$

We shall now show that $2(r_j - r_i) + 2(q_{k,i} - q_{k,j}) \leq 0$ for all $k \in S(\alpha)$. If $k > i$ then $q_{k,i} = q_{k,j} = t_k$ and since $r_i \geq r_j$ we are done. If $j < k < i$ then

$$2(r_j - r_i) + 2(q_{k,j} - q_{k,i}) = 2(r_j - r_i) + 2(t_k + r_i - r_k - t_k) = 2(r_j - r_k) \leq 0.$$

If $k < i$ then

$$2(r_j - r_i) + 2(q_{k,i} - q_{k,j}) = 2(r_j - r_i) + 2(t_k + r_i - r_k - (t_k + r_j - r_k)) = 0.$$

Now, there exists $k_0 \in S(\alpha)$ such that $p_j = q_{k_0,j}$. It is clear that $2(p_i - p_j) \leq 2(q_{k_0,i} - q_{k_0,j})$; hence, $\lambda_A = 2(r_j - r_i) + 2(p_i - p_j) \leq 0$. $\square$

We are left to prove that the sum of the eigenvalues of $\mathrm{ad}(H_\alpha)$ on $\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}$ is less than $\dim_{\mathbf{R}}(\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}) - \dim_{\mathbf{R}}(\mathfrak{g})$. That is,

$$\mathrm{Trace}(\mathrm{ad}(H_\alpha)_{|\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}}) \leq \dim_{\mathbf{R}}(\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}) - \dim_{\mathbf{R}}(\mathfrak{g}).$$

Since $\mathfrak{g}^{B_{\underline{r}}} \subseteq \mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}$ and since the eigenvalues of $\mathrm{ad}(H_\alpha)$ are nonpositive we have

$$\mathrm{Trace}(\mathrm{ad}(H_\alpha)_{|\mathfrak{g}^{B_{\underline{r}}}}) \geq \mathrm{Trace}(\mathrm{ad}(H_\alpha)_{|\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}}).$$

Hence, it is enough to prove the inequality for the smaller space $\mathfrak{g}^{B_{\underline{r}}}$.

From Section 5 we have that there exists $X \in \mathfrak{g}$ such that $(X, B_{\underline{r}}, H_{\underline{r}} = [X, B_{\underline{r}}])$ form a Jacobson-Morosov triple. (Here, $X = dX_{\underline{r}}d^{-1}$ for some diagonal matrix d. See Lemma 5.1). Hence, by [Wal, 8.3.6] we have that

$$\mathrm{Trace}(\mathrm{ad}(H_{\underline{r}})_{|\mathfrak{g}^{B_{\underline{r}}}}) \leq \dim_{\mathbf{R}}(\mathfrak{g}^{B_{\underline{r}},\mathfrak{x}_\alpha}) - \dim_{\mathbf{R}}(\mathfrak{g}).$$

Thus, to conclude the proof of Proposition 8.2, it is enough to prove that $\mathrm{Trace}(\mathrm{ad}(d(c_1, \ldots, c_k))_{|\mathfrak{g}^{B_{\underline{r}}}}) = 0$. This is the content of the following lemma:

LEMMA 6.14.

$$\mathrm{Trace}(\mathrm{ad}(d(c_1, \ldots, c_k))_{|\mathfrak{g}^{B_{\underline{r}}}}) = 0.$$

Proof. Let $s = \mathrm{Trace}(\mathrm{ad}(d(c_1, \dots, c_k))_{|\mathfrak{g}^{B_{\underline{r}}}})$. There exists $w \in G = \mathrm{GL}(n)$ such that $wA_{\underline{r}}w^{-1} = B_{\underline{r}}$ and $wd(c_1, c_2, \dots, c_k)w^{-1} = d(c_1, c_2, \dots, c_k)$. It follows that

$$\mathrm{Trace}(\mathrm{ad}(d(c_1, \dots, c_k))_{|\mathfrak{g}^{B_{\underline{r}}}}) = \mathrm{Trace}(\mathrm{ad}(d(c_1, \dots, c_k))_{|\mathfrak{g}^{A_{\underline{r}}}}). \tag{6.19}$$

However, if $A \in \mathfrak{g}^{B_{\underline{r}}}$ then $A^t \in \mathfrak{g}^{A_{\underline{r}}}$, and if A is an eigenvector of $\mathrm{ad}(d(c_1, c_2, \dots, c_k)$ with eigenvalue λ_A then A^t is an eigenvector of $\mathrm{ad}(d(c_1, c_2, \dots, c_k))$ with eigenvalue $-\lambda_A$. Thus, it follows from (6.19) that $s = -s$, hence $s = 0$ and we are done. $\square$

7. Invariant distributions with nilpotent support

In this section we prove that P invariant distributions which are of finite order under $\square$ and have nilpotent support (up to the center) vanish. The analogous result for G invariant distributions is a famous result of Harish-Chandra which is key to his general regularity theorem. More so in our situation where the general case borrows from Harish-Chandra's results and does not differ much from them. For the nilpotent support case we will adapt his proof for our special case. Our presentation will follow [Wal, §§8.3.5–8.3.10]. Wallach, in fact, proves a slightly stronger result than Harish-Chandra, and so shall we. Since we will use many of the results in [Wal] it will be helpful for the reader to have [Wal] at hand. We shall try to conform our notation and style of proof to [Wal] as much as possible.

We start by adapting the arguments in [Wal, 8.3.6] to the case in hand. Let $K = \mathbf{R}$ or $\mathbf{C}$. Let $G = \mathrm{GL}_n(K)$ and $\mathfrak{g} = \mathfrak{g}l_n(K)$. Let $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$. Then G acts on $\mathfrak{g}$ and $\mathfrak{s}$ by the Adjoint action. For $A \in \mathfrak{g}$ and $g \in G$ we denote $gA = \mathrm{Ad}(g)A = gAg^{-1}$. Let $\Omega = \Omega_{\mathfrak{s}}$ be a fixed open set in $\mathfrak{s}$ of the type described in (4.3). Let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{s}$. By [Wal, 8.3.6], $\mathcal{N} = O_1 \cup O_2 \cup \cdots \cup O_r$ with $O_j = GX_j$ and O_1 open in $\mathcal{N}$, O_2 open in $\mathcal{N} - O_1$ etc. By [Ber] (see also Corollary 5.3), $O_j = \tilde{O}_{1,j} \cup \tilde{O}_{2,j} \cup \cdots \cup \tilde{O}_{l_j,j}$ where $O_{i,j} = PX_{i,j}$ and $O_{1,j}$ is open in O_j, $O_{2,j}$ is open in $O_j - O_{1,j}$ etc. After re-indexing we can write $\mathcal{N} = \tilde{O}_1 \cup \tilde{O}_2 \cup \cdots \cup \tilde{O}_l$ with $\tilde{O}_j = PX_j$, $\tilde{O}_1$ open in $\mathcal{N}$, etc. Moreover, if $X_j \neq 0$, we can assume that X_j, $Y = (X_j)^t$ and $H_j = [X_j, Y_j]$ form a Jacobson-Morosov triple and we can assume the existence of H_j' satisfying the conditions of Theorem 6.3. Set $\mathcal{N}_p = \bigcup_{i \geq p} \tilde{O}_i$. Then $\mathcal{N}_p$ is closed in $\mathfrak{s}$. By Lemma 6.1 we have

$$\mathfrak{g} = [\mathfrak{p}, X_j] \oplus \mathfrak{g}^{Y_j, \mathfrak{p}^c}.$$

Hence

$$\mathfrak{s} = [\mathfrak{p}, X_j] \oplus \mathfrak{s}^{Y_j, \mathfrak{p}^c}.$$

Fix j and set $V = V_j = \mathfrak{s}^{Y_j, \mathfrak{p}^c}$, $X = X_j$ and $Y = Y_j$. If $p \in P$ and if $Z \in V$ then set $\Phi(p, Z) = \Phi_j(p, Z) = p(X + Z)$. Then $d\Phi_{g,0}(P, V_j) = g(V + [\mathfrak{p}, X]) = \mathfrak{s}$. This implies that there exist an open neighborhood $\tilde{V}$ of 0 in V such that $X + \tilde{V} \subset \Omega_{\mathfrak{s}}$ and Φ_j restricted to $P \times \tilde{V}$ is a submersion onto its image which is contained in $\Omega_{\mathfrak{s}}$.

Let Q be an open P-invariant subset of $\mathfrak{s}$ such that $Q \bigcap \mathcal{N}_j = \tilde{O}_j$. Let $\hat{V} = \{Z \in \tilde{V} \, : \, \Phi_j(p, Z) \in Q \text{ for all } p \in P\}$. Then $\hat{V}$ is an open neighborhood of 0 in $\tilde{V}$ and $\Phi_j(P \times \hat{V}) \cap \mathcal{N}_j = \mathcal{O}_j$.

It is easy to show that the map $\mathrm{ad}(X)$ is a linear isomorphism from $[\mathfrak{s}, Y]$ onto $[\mathfrak{s}, X]$ (see [Wal, p. 300]). Hence there exists a subspace, W, of $[\mathfrak{s}, Y]$ such that $\mathrm{ad}(X)$ maps W onto $[\mathfrak{p}, X]$. Thus there exist a neighborhood, W_0, of 0 in W and a neighborhood U' of 0 in $\hat{V}$ such that

$$x, Z \to \Phi_j(\exp x, Z) \tag{7.1}$$

is a diffeomorphism of $W_0 \times U'$ onto an open neighborhood of $X \in \mathfrak{s}$. Let W_1 be an open neighborhood of 0 in W_0 such that $e^{\mathrm{ad}(W_1)}X$ is a neighborhood of X in $\mathcal{N}_j$. If we shrink W_0 and U' we may assume that $\Phi_j(\exp W_0, U') \cap \mathcal{N}_j \subset e^{\mathrm{ad}(W_1)}X$. Suppose that $Z \in U'$ and that $X + Z \in \tilde{O}_j$. Then $X + Z \in \tilde{O}_j \cap \Phi_j(\exp W_0, U')$. Thus $X + Z = e^{\mathrm{ad}(v)}X$ with $v \in W_1$. Since the map in (7.1) is a homeomorphism we get that $v = Z = 0$. Thus, taking $U_j = U'$, $\Omega = \Omega_{\mathfrak{s}}$ and summarizing the above discussion we have that Φ_j on $P \times U_j$ satisfies:

(7.2)
- (i) Φ_j is a submersion onto a P invariant open neighborhood Ω_j, of X_j in Ω.
- (ii) $\Omega_j \cap \mathcal{N}_j = \tilde{O}_j$.
- (iii) $(X_j + U_j) \cap \tilde{O}_j = \{X_j\}$.

Replacing $U_j = U'$ with $U_j = U' \oplus U$ where U is the open set in $\mathfrak{z}$ defined in (4.3) and replacing $\Omega_{\mathfrak{s}}$ with $\Omega = \Omega_{\mathfrak{s}} \oplus U$ we have that Φ_j is a map from $P \times U_j$ onto a P invariant open subset Ω_j of $\mathfrak{g}$ and that (7.2) still holds.

7.1. *Vector fields and invariant distributions.* We now proceed as in Sections 8.3.7 and 8.3.8 in [Wal]. Let $q = \dim_{\mathbf{R}}(\mathfrak{s}) = \dim_{\mathbf{R}}([\mathfrak{g}, \mathfrak{g}])$. Let E be the vector field on $\mathfrak{g}$ defined by

$$Ef(x + y) = \frac{d}{dt}(f(x + ty))_{t=1}, \quad x \in \mathfrak{z}, y \in [\mathfrak{g}, \mathfrak{g}]. \tag{7.3}$$

If $x_1, \ldots, x_s$ are linear coordinates on $\mathfrak{g}$ such that $\{x_i\}_{i \leq q}$ are linear coordinates on $[\mathfrak{g}, \mathfrak{g}]$ and $\{x_i\}_{i > q}$ are coordinates on $\mathfrak{z}$ then

$$E = \sum_{i \leq q} x_i \partial / \partial x_i.$$

LEMMA 7.1. *Let F be the space of all P invariant distributions on Ω supported on $(\mathfrak{z} \oplus \mathcal{N}) \cap \Omega$. If $T \in F$ then* $\dim(\mathbf{C}[E]T) < \infty$ *and the eigenvalues of E on F are all real and strictly less than $-q/2$.*

Proof. Let j be fixed and assume that $\tilde{O}_j \neq 0$. Let $X = X_j$, $H' = H'_j$, $Y = Y_j$ as defined above and $V = V_j = \mathfrak{g}^{Y_j, \mathfrak{p}^c}$. Write $V = V_1 \oplus V_2 \oplus \cdots \oplus V_{d_j} \oplus \mathfrak{z}$. Here $V_1 \oplus V_2 \oplus \cdots \oplus V_{d_j} = V \cap [\mathfrak{g}, \mathfrak{g}]$ and V_t, $1 \leq t \leq d_j$, is a one-dimensional eigenspace of $\mathrm{ad}H'$ corresponding to an eigenvalue μ_t. Since the eigenvalues are not necessarily distinct, it follows that this decomposition is not necessarily unique. However, we can and do fix one such decomposition. Let $y_1, \ldots, y_{d_j}$, be linear coordinates on V_j corresponding to this decomposition, such that $y_k(V_m) = 0$ for $k \neq m$ and such that $y_k(\mathfrak{z}) = 0$ for all k. If $Z \in V$, write $Z = \zeta + \sum Z_t$ where $Z_t \in V_t$ and $\zeta \in \mathfrak{z}$ according to the decomposition above. We have

$$(d\Phi_j)_{p,Z}\left(\frac{1}{2}H', \zeta + \sum(\frac{1}{2}\mu_t + 1)Z_t\right) = p(X+Z) = \Phi_j(p, Z) \tag{7.4}$$

for $p \in P$ and $Z \in U_j$. Since Φ_j is a submersion, we may define the map on distributions Φ_j^0 as in Example 3.2 (see also [Wal, 8.A.3.2(2)]). Also, Φ_j^0 takes P invariant distribution T on $\mathfrak{g}$ (or, more precisely, on Ω_j) and produces a distribution $\Phi_j^0(T)$ on $U_j = U \oplus U'$ where U' is the open set of $\mathfrak{s}^{Y_j, \mathfrak{p}^c}$ constructed above. Now (7.4) and (3.6) imply that

$$\Phi_j^0(ET) = \left(\sum(\frac{1}{2}\mu_t + 1)y_t\partial/\partial y_t\right)\Phi_j^0(T). \tag{7.5}$$

The choices in (7.2) imply that if $\mathrm{supp}(T) \subset (\mathfrak{z} \oplus \mathcal{N}_j) \cap \Omega$ then $\mathrm{supp}(\Phi_j^0(T)) \subset U \times \{0\}$. Let F_j denote the subspace of those elements of F with support contained in $(\mathfrak{z} \oplus \mathcal{N}_j) \cap \Omega$. We prove by downward induction that if $T \in F_j$ then $\dim(\mathbf{C}[E]T) < \infty$ and that the eigenvalues of E on F_j are strictly less than $-q/2$. If $j = l$ then $\tilde{O}_l = \{0\}$ is the smallest P orbit. Then 8.A.5.4 of [Wal] implies that E acts semisimply on F_l with eigenvalues less than or equal to $-q < -q/2$. We assume the result for F_{j+1} and prove it for F_j. Let $T \in F_j$. Then $\Phi_j^0(T)$ has support in $U \times \{0\}$. By our assumption on $H' = H'_j$ (Theorem 6.3, (4)) $\mu_t \leq 0$ for all t. Hence by (7.5) and by [Wal, 8.A.5.4] there exist $a_1, \ldots, a_s \in \mathbf{R}$ such that $-a_i > d_j + \frac{1}{2}\sum -\mu_t$ and such that $\Phi_j(\prod(E - a_i)T) = 0$. Now by our assumption (Theorem 6.3, (4) and Corollary 6.4), $\sum -\mu_t > q - d_j$. Thus $-a_i \geq \frac{1}{2}(d_j + q)$. By the above, $\mathrm{supp}(\prod(E - a_i)T) \subset (\mathfrak{z} \oplus \mathcal{N}_{j+1}) \cap \Omega$. Hence $\prod(E - a_i)T \in F_{j+1}$ and we can use the induction assumption to conclude the proof of the lemma. $\square$

Let B be the bilinear form on $\mathfrak{g}$ defined by (6.1) (or by [Wal], 0.2.2) and let $\square = \sum X_i Y_i \in S(\mathfrak{g})$ where $\{X_i\}$ is a basis of $\mathfrak{g}$ and $Y_i \in \mathfrak{g}$ satisfy $B(X_j, Y_i) = \delta_{i,j}$. Exactly as in [Wal, 8.3.9] the above lemma implies the main result of this section:

Theorem 7.2. *If $T \in F$ and if p is a nonzero polynomial in one variable such that $p(\Box)T = 0$ then $T = 0$.*

We shall need to consider some other distributions with nilpotent support. By Theorem 3.3 there is an isomorphism from the space of P invariant distributions on $\mathfrak{g}$ and $(G, |\det|)$ invariant distributions on $\mathfrak{g} \times \mathcal{V}^*$. Here the G action on $\mathfrak{g} \times \mathcal{V}^*$ is as defined in (4.2). If $T \mapsto \tilde{T}$ by this map, then $\Box T \mapsto (\Box \otimes 1)\tilde{T}$. Hence, from Theorem 7.2 we have:

Corollary 7.3. *If $\tilde{T}$ is a $(G, |\det|)$ invariant distribution on $\Omega \times \mathcal{V}^*$ with support in $((\mathfrak{z} \oplus \mathcal{N}) \cap \Omega) \times \mathcal{V}^*$ and if p is a nonzero polynomial in one variable such that $p(\Box \otimes 1)\tilde{T} = 0$ then $\tilde{T} = 0$.*

Needing also to consider distributions on $\Omega\times\mathcal{V}$, we first prove the following. Let $\tilde{F}$ be the space of $(G, |\det|)$ invariant distributions on $\Omega \times \mathcal{V}$ with support in $((\mathfrak{z} \oplus \mathcal{N}) \cap \Omega) \times \{0\}$. Let E be the vector field on $\mathfrak{g}$ defined in (7.3) and let $\tilde{E} = E \otimes 1$.

Lemma 7.4. *If $\tilde{T} \in \tilde{F}$ then $\dim(\mathbf{C}[\tilde{E}]\tilde{T}) < \infty$ and the eigenvalues of $\tilde{E}$ on $\tilde{F}$ are all real and strictly less than $-q/2$.*

Proof. The proof follows the same steps as in [Wal, 8.3.6 and 8.3.7]. We shall keep his notation and discard the above similar notation that was used for the proof of Lemma 7.1. In particular, O_j, $X = X_j$, Y, H and V are the same as in [Wal, 8.3.6]. So are $\mathcal{N}_j$, Ω_j and U_j. We replace the map Φ_j of [Wal, 8.3.6] with the map $\tilde{\Phi}_j$ on $G \times U_j \times \mathcal{V}$ defined by

$$\tilde{\Phi}_j(g, Z, v) = (g(X + Z), \rho(g)v).$$

It is easy to check that $\tilde{\Phi}_j$ is a submersion. Let $y_1, \ldots, y_d$ be linear coordinates on V as in [Wal, 8.3.7]. By Lemma 5.1 we can assume that $H = H_{\underline{r}}$ for some partition $\underline{r}$ corresponding to $X = X_j$. Let $u_1, \ldots, u_l$ be the standard coordinates on the row space $\mathcal{V} = K^n$. If $K = \mathbf{R}$ then $l = n$ and u_j corresponds to the standard vector e_j whose entries are all zeroes except the j^{th} entry which is one. If $K = \mathbf{C}$ then $l = 2n$, u_1 corresponds to the vector e_1, u_2 corresponds to the vector ie_1 where $i = \sqrt{-1}$ and so on. For $v \in \mathcal{V}$ we write $v = \sum v_j$ where v_j is in the span of e_k (or ie_j) for an appropriate k. We write α_j for the entry in H corresponding to u_j. (That is, if $K = \mathbf{R}$ then α_j is the (j, j) entry in H. If $K = \mathbf{C}$ and $u_j = e_k$ or $u_j = ie_k$ then α_j is the (k, k) entry in H.) For $Z \in V$ we write $Z = \sum Z_m$ as in [Wal, 8.3.7], with $\operatorname{ad}(H)Z_m = -\mu_m Z_m$ for the nonnegative integers μ_m defined in [Wal, 8.3.6]. We therefore get

(7.6)
$$(d\tilde{\Phi}_j)_{g,Z,v}(\frac{1}{2}H, \sum(\frac{1}{2}\mu_m + 1)Z_m, \sum \alpha_j u_j) = (g(X + Z), \rho(g)v) = \tilde{\Phi}_j(g, Z, v)$$

for $g \in G$, $Z \in U_j$ and $v \in \mathcal{V}$. Since $\tilde{\Phi}_j$ is a submersion, we can define the distribution map $\tilde{\Phi}_j^0$. From (7.6) we get

(7.7)

$$\tilde{\Phi}_j^0(\tilde{E}\tilde{T}) = \left(\sum_{m=1}^{d}\left(\frac{1}{2}\mu_m + 1\right) y_m\partial/\partial y_m \otimes 1 + 1 \otimes \sum \alpha_j u_j \partial/\partial u_j\right)\tilde{\Phi}_j^0(\tilde{T}).$$

The fact that some entries α_j of $H_{\underline{r}}$ are negative will pose a problem when we will try to control the eigenvalues (See also [Wal, 8.A.5.3 and 8.A.5.4].) To rectify that we can use the following trick. Change $\frac{1}{2}H$ in (7.6) to $H' = \frac{1}{2}H + \lambda I_n$ for a positive λ such that H' has nonnegative entries α'_j. Equation (7.6) now holds with H' replacing H and α'_j replacing α_j. since trace$(H') = n\lambda \neq 0$ we shall get an extra summand in the radial component (see Example 3.3). Hence we get that (7.7) becomes

(7.8)

$$\begin{aligned}&\tilde{\Phi}_j^0(\tilde{E}\tilde{T})\\ &\quad = \left(\sum\left(\frac{1}{2}\mu_m + 1\right) y_m\partial/\partial y_m) \otimes 1 + 1 \otimes \sum(\alpha'_j u_j \partial/\partial u_j) + n_K\lambda I\right)\tilde{\Phi}_j^0(\tilde{T})\end{aligned}$$

where $n_K = \dim_{\mathbf{R}}(\mathcal{V})$. We now continue as in [Wal, 8.3.7]. Define $\tilde{F}_j$ as the subspace of the elements of F with support in $((\mathfrak{z} \oplus \mathcal{N}_j) \cap \Omega) \times \{0\}$. We prove by downward induction that if $\tilde{T} \in \tilde{F}_j$ then $\dim(\mathbf{C}[\tilde{E}]\tilde{T}) < \infty$ and the eigenvalues of $\tilde{E}$ on $\tilde{F}_j$ are strictly less than $-q/2$. Let

$$D = \left(\sum\left(\frac{1}{2}\mu_m + 1\right) y_m\partial/\partial y_m) \otimes 1 + 1 \otimes \sum(\alpha'_j u_j \partial/\partial u_j)\right).$$

By (7.7) we have that $\tilde{\Phi}_j^0(\tilde{E}\tilde{T}) = (D + n_K\lambda)\tilde{\Phi}_j^0(\tilde{T})$. By [Wal, 8.3.6], if $\tilde{T} \in \tilde{F}_j$ then $\tilde{\Phi}_j^0(\tilde{T})$ is supported on $(U \oplus \{0\}) \times \{0\}$. By [Wal, 8.A.5.4], D is of finite order on the distributions with support on $(U \oplus \{0\}) \times \{0\}$ and the eigenvalues of D are less than or equal to $-(\sum(\frac{1}{2}\mu_m + 1)) - \sum \alpha'_j$. Since $\sum \alpha'_j = n_K\lambda$ we get that $D + n_K\lambda$ is also of finite order and has eigenvalues less than or equal to $-(\sum(\frac{1}{2}\mu_m + 1)$. By [Wal, 8.3.7], $-(\sum(\frac{1}{2}\mu_m + 1) < -q/2$ and we can proceed exactly as in [Wal, 8.3.7]. □

As in [Wal, 8.3.9] we get the following theorem:

THEOREM 7.5. *If $\tilde{T} \in \tilde{F}$ and if p is a nonzero polynomial in one variable such that $p(\Box \otimes 1)\tilde{T} = 0$ then $\tilde{T} = 0$.*

Now from Corollary 7.3 and Theorem 7.5 we get

THEOREM 7.6. *If $\tilde{T}$ is a $(G, |\det|)$ invariant distribution on $\Omega \times \mathcal{V}$ with support in $((\mathfrak{z} \oplus \mathcal{N}) \cap \Omega) \times \mathcal{V}$ and if p is a nonzero polynomial in one variable such that $p(\Box \otimes 1)\tilde{T} = 0$ then $\tilde{T} = 0$.*

Proof. Restrict $\tilde{T}$ to the open set $\Omega \times \mathcal{V}^*$. Then by Corollary 7.3 we have that this restriction is zero. Hence $\tilde{T}$ is supported on $(\mathfrak{z} \oplus \mathcal{N}) \cap \Omega \times \{0\}$. By Theorem 7.5, $\tilde{T} = 0$. ☐

In a similar fashion we can prove a generalization of Corollary 7.3 and Theorem 7.6. Let $G = \prod G_i$ where $G_i = \mathrm{GL}_{k_i}(K_i)$ and K_i is $\mathbf{R}$ or $\mathbf{C}$. Let $\mathfrak{g} = \prod \mathfrak{g}_i$ where $\mathfrak{g}_i = M_{n_i}(K_i)$ and let $\mathcal{X}_i$ be $\mathcal{V}_i$ or $\mathcal{V}_i^*$ where $\mathcal{V}_i = (K_i)^{k_i}$ is the row vector space which G_i acts on. Let $\mathcal{X} = \prod \mathcal{X}_i$. Then G acts naturally on $G \times \mathcal{X}$. Extend $|\det|$ to G. Let $\Box$ be the appropriate element of $I(\mathfrak{g}) = S(\mathfrak{g}_{\mathbf{C}})^G$ and let $\tilde{\Box} = \Box \otimes 1$. Let $\mathcal{N}$, $\mathfrak{z}$ and $\Omega \subset \mathfrak{g}$ be as before.

THEOREM 7.7. *If $\tilde{T}$ is a $(G, |\det|)$ invariant distribution on $\Omega \times \mathcal{X}$ with support in $((\mathfrak{z} \oplus \mathcal{N}) \cap \Omega) \times \mathcal{X}$ and if p is a nonzero polynomial in one variable such that $p(\tilde{\Box})\tilde{T} = 0$ then $\tilde{T} = 0$.*

8. The Lie algebra case

In this section we will prove Theorem 4.4. Our proof will follow closely [Wal, §§8.3.10, 8.3.11, 8.3.12, 8.3.13].

The difference from [Wal] is that we consider distributions on $\mathfrak{g} \times \mathcal{V}$ where $\mathfrak{g}$ is the Lie algebra of a group G and $\mathcal{V}$ is a finite dimensional vector space which comes with an action ρ of G. Now, G acts on $\mathfrak{g} \times \mathcal{V}$ by

$$g(A, v) = (\mathrm{Ad}(g)A, \rho(g)v), \quad g \in G, A \in \mathfrak{g}, v \in \mathcal{V}. \tag{8.1}$$

Our distributions will be G equivariant under this action.

We are interested in the case where

$$\begin{aligned} G &= \prod_{i=1}^{r} G_i = \prod \mathrm{GL}_{k_i}(K_i). \\ \mathfrak{g} &= \prod \mathfrak{g}l_{k_i}(K_i) = \prod M_{k_i}(K_i). \\ \mathcal{V} &= \prod \mathcal{V}_i = \prod K_i^{k_i}. \end{aligned} \tag{8.2}$$

Here K_i is $\mathbf{R}$ or $\mathbf{C}$ and can change with i.

The group $G_i = \mathrm{GL}_{k_i}(K_i)$ acts on the row space $\mathcal{V}_i$ in the usual way: $\rho(g)v = vg^{-1}$ where v is row vector of order k_i and g is a $k_i \times k_i$ invertible matrix. This action extends naturally to an action ρ of G on $\mathcal{V}$. Now using (8.1) we get an action of G on $\mathfrak{g} \times \mathcal{V}$.

For $\alpha \in K_i$ where $K_i = \mathbf{R}$ we let $|\alpha|_i$ be the usual absolute value on $\mathbf{R}$. For $\beta \in K_j$ where $K_j = \mathbf{C}$ we let $|\beta|_j$ be the square of the usual absolute value on $\mathbf{C}$. For $g \in G$, $G = (g_1, \ldots, g_r)$ we set $|\det(g)| = \prod |\det(g_i)|_i$. We are interested in $(G, |\det|)$ invariant distributions on $\mathfrak{g} \times \mathcal{V}$ (see (3.1)). To prove our main theorem we shall use induction on centralizers of semisimple elements.

Our choice of the group G and its Lie algebra $\mathfrak{g}$ is dictated by the following observation:

Remark 8.1. Let X be a semisimple element in $\mathfrak{g}$. Then its centralizer $\mathfrak{m} = \mathfrak{g}^X$ is of the same form as $\mathfrak{g}$, that is, a product of $\mathfrak{gl}$s (some real and some complex).

Our distributions are defined on certain G invariant subsets of $\mathfrak{g} \times \mathcal{V}$ which we now define. Let $\mathcal{V}_i$ be a vector space as above and $\mathcal{V}_i^* = \mathcal{V}_i - \{0\}$. For each i, $i = 1, \ldots, r$, let $\mathcal{X}_i$ be $\mathcal{V}_i$ or $\mathcal{V}_i^*$ (changing with i). Set

$$\mathcal{X} = \prod \mathcal{X}_i. \tag{8.3}$$

Then $\mathcal{X}$ is a G invariant subset of $\mathcal{V}$ under the ρ action and $\mathfrak{g} \times \mathcal{X}$ is invariant under the action of G defined in (8.1). We let $dx = dx_1 dx_2 \cdots dx_k$ be a measure on $\mathcal{X}$ where dx_i are the standard Lebesgue measures on $\mathcal{X}_i$. It is easy to see that dx is a $(G, |\det|)$ invariant measure (see (3.1)).

Let T be a distribution on $\mathfrak{g} \times \mathcal{X}$ and let F be a function on $\mathfrak{g}$. We say that $T = T_F = F \otimes dx$ if for every $\phi \in C_c^\infty(\mathfrak{g} \times \mathcal{X})$,

$$T(\phi) = \int_{\mathfrak{h} \times \mathcal{X}} \phi(A, x) F(A) dA dx.$$

where dA is a fixed Haar measure on $\mathfrak{g}$ and dx a measure on $\mathcal{X}$ as above. In fact, it is enough that for every $\phi \in C_c^\infty(\mathfrak{g} \times \mathcal{X})$ of the form $\phi(A, x) = \phi_1(A)\phi_2(x)$ where $\phi_1 \in C_c^\infty(\mathfrak{g})$ and $\phi_2 \in C_c^\infty(\mathcal{X})$ we have

$$T(\phi) = \int_{\mathfrak{g}} \phi_1(A) F(A) dA \int_{\mathcal{X}} \phi_2(x) dx.$$

Let $\Omega \subset \mathfrak{g}$ be as defined in (4.3) and Ω' be the set of regular elements in Ω. Let $I(\mathfrak{g}_{\mathbf{C}}) = S(\mathfrak{g}_{\mathbf{C}})^G$. The following proposition is analogous to our "Key Proposition" which was stated in the introduction. Using the Frobenius map we may deduce the "Key Proposition" from the following proposition:

PROPOSITION 8.2. *Let $T \in D'(\Omega' \times \mathcal{X})^{G,|\det|}$. Then $T = T' \otimes dx$ where T' is a G invariant distribution on Ω'.*

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let H be the corresponding Lie subgroup of G and $\Omega'_{\mathfrak{h}} = G(\Omega' \cap \mathfrak{h})$. If $g \in G$, $A \in \Omega' \cap \mathfrak{h}$ and $x \in \mathcal{X}$, set $\tilde{\Psi}(g, A, x) = (gAg^{-1}, \rho(g)x)$. By [Wal, 8.3.1 and 8.A.3.3], $\tilde{\Psi}$ is a submersion of $G \times (\Omega' \cap \mathfrak{h}) \times \mathcal{X}$ onto $\Omega'_{\mathfrak{h}} \times \mathcal{X}$. The distribution $\tilde{\Psi}^0(T)$ defined in Example 3.3 on $(\Omega' \cap \mathfrak{h}) \times \mathcal{X}$ is $(H, |\det|)$ invariant. Since H acts trivially on $\mathfrak{h}$ it follows from Lemma 2.6 that $\tilde{\Psi}^0(T) = T_{\mathfrak{h}} \otimes dx$ for some distribution $T_{\mathfrak{h}}$ on $\Omega' \cap \mathfrak{h}$. Hence $T = T^{\mathfrak{h}} \otimes 1$ on $\Omega'_{\mathfrak{h}} \times \mathcal{X}$. Since $\Omega' = \bigcup_{\mathfrak{h}} \Omega'_{\mathfrak{h}}$ where $\mathfrak{h}$ runs over the nonconjugate Cartan subalgebras of $\mathfrak{g}$ we get that $T = T' \otimes dx$ on $\Omega' \times \mathcal{X}$ for some G invariant distribution T' on Ω'. $\square$

The following theorem is the analog of Theorem 8.3.4 in [Wal].

THEOREM 8.3. *Let* $T \in D'(\Omega \times \mathcal{X})^{G,|\det|}$ *be such that* $\dim(I(\mathfrak{g}_{\mathbf{C}}) \otimes 1)T < \infty$ *on* $\Omega' \times \mathcal{X}$. *Then there exists an analytic function* $F_T = F$ *on* Ω' *such that*

$$T = F \otimes dx \quad \text{on } \Omega' \times \mathcal{X}. \tag{8.4}$$

Furthermore, if F *is extended to* Ω *when* $F = 0$ *on* $\Omega - \Omega'$ *then* F *is locally integrable on* Ω.

Proof. By Proposition 8.2, $T = T' \otimes dx$ on $\Omega' \times \mathcal{X}$. Since $\dim(I(\mathfrak{g}_{\mathbf{C}}) \otimes 1)T < \infty$ on $\Omega' \times \mathcal{X}$, we get that $\dim(I(\mathfrak{g}_{\mathbf{C}}))T' < \infty$ on Ω'. Hence, by [Wal, Th. 8.3.4], $T' = F$ on Ω' for a real analytic function F which is locally integrable on Ω. $\square$

Let $\square = \square_{\mathfrak{g}} = \sum X_i Y_i \in S(\mathfrak{g})$ where $\{X_i\}$ is a basis of $\mathfrak{g}$ and $Y_i \in \mathfrak{g}$ satisfies $B(X_j, Y_i) = \delta_{i,j}$. (See (6.1) or [Wal, 0.2.2] for the definition of B.)

Our main result of this section is Theorem 4.4 which we state again now. (See [Wal, Th. 8.3.5] for an analog.)

THEOREM 8.4. *Let* Ω *be as above and let* $T \in D'(\Omega \times \mathcal{X})^{G,|\det|}$ *be such that* $\dim(I(\mathfrak{g}_{\mathbf{C}}) \otimes 1)T < \infty$ *on* $\Omega' \times \mathcal{X}$ *and such that*

$$\dim(\mathbf{C}[\square \otimes 1]T) < \infty \quad \text{on } \Omega \times \mathcal{X}.$$

Then $T = T_F = F \otimes dx$ *for some* G *invariant function* F *on* $\mathfrak{g}$ *which is real analytic on* Ω'.

We begin with two lemmas that are necessary for the proof. Our proof will proceed by induction on centralizers of semisimple elements in the Lie algebra $\mathfrak{g}$. Hence we will have to compute the radial component of $\square \otimes 1$ when we descend to such centralizers. This is the content of the first lemma. We shall descend to a set in $\mathfrak{m} \times \mathcal{X}$ where $\mathfrak{m}$ is a centralizer of a semisimple element. However, $\mathfrak{m} \times \mathcal{X}$ is not of the type we started with. In the second lemma we will show that $\mathcal{X}$ can be covered by a finite number of open sets $\mathcal{Y}_j$ such that the sets $\mathfrak{m} \times \mathcal{Y}_j$ are of the type in our induction hypothesis.

Let $S \in \mathfrak{g}$ be a semisimple element and let $\mathfrak{m} = \mathfrak{g}^S$, $\mathfrak{q} = \mathfrak{m}^{\perp}$ and $\mathfrak{m}'' = \{Y \in \mathfrak{m} : \det(\operatorname{ad} Y|_{\mathfrak{q}}) \neq 0\}$. Then

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{q}.$$

(Converting to the notation of [Wal, 8.A.3.3] we would have $\mathfrak{m} = \mathfrak{h}$ and $\mathfrak{q} = V$.)

Let $M = \{g \in G : \operatorname{Ad}(g)X_s = X_s\}$. If $g \in G$, $Y \in \mathfrak{m}''$ and $x \in X$ set

$$\tilde{\Psi}(g, Y, x) = (gYg^{-1}, \rho(g)x).$$

Let $W = G\mathfrak{m}''$. Then $\tilde{\Psi}$ is a submersion onto the open set $W \times \mathcal{X}$. Let $\tilde{\Psi}^0$ be the map on distributions defined in Example 3.3. For $X \in \mathfrak{m}$ we set

$$\eta(X) = \eta_{\mathfrak{g}/\mathfrak{m}}(X) = |\det(\operatorname{ad} X|_{\mathfrak{q}})|.$$

LEMMA 8.5. *Let T be a G invariant distribution on $W \times \mathcal{X}$. Then there exist a scalar λ such that*

$$\tilde{\Psi}^0((\Box \otimes 1)T) = (\eta^{-1/2} \otimes 1)(\Box_{\mathfrak{m}} \otimes 1 + \lambda)(\eta^{1/2} \otimes 1)\tilde{\Psi}^0(T). \tag{8.5}$$

Proof. Our proof follows the same lines as in [Wal, 8.A.3.5]. Fix $X \in \mathfrak{m}'$ and set $\mathfrak{b} = \{Y \in \mathfrak{g} : [X, Y] = 0\}$. Then $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{m}$. Let $\Phi = \Phi(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$ be the set of roots. Set $\Phi_{\mathfrak{m}} = \Phi(\mathfrak{m}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$ and $\Phi_{\mathfrak{q}} = \Phi_{\mathfrak{g}} - \Phi_{\mathfrak{m}}$. Fix Φ^+, a system of positive roots for Φ and set $\Phi_{\mathfrak{m}}^+ = \Phi_{\mathfrak{m}} \cap \Phi^+$ and $\Phi_{\mathfrak{q}}^+ = \Phi_{\mathfrak{q}} \cap \Phi^+$. Put $\Pi = \prod_{\alpha \in \Phi^+} \alpha$, $\Pi_{\mathfrak{m}} = \prod_{\alpha \in \Phi_{\mathfrak{m}}^+} \alpha$, and $\Pi_{\mathfrak{q}} = \prod_{\alpha \in \Phi_{\mathfrak{q}}^+} \alpha$. Then $\Pi = \Pi_{\mathfrak{m}} \Pi_{\mathfrak{q}}$. We chose Φ^+ such that for each connected component C of $\mathfrak{b}'$ there exists a complex number, μ_C such that

$$\eta^{1/2} = \mu_C \Pi_{\mathfrak{q}} \text{ on } C.$$

Let $H_1, \ldots, H_r$ be an orthonormal basis for $\mathfrak{b}$. We write

$$\Box_{\mathfrak{m}} = \sum H_j^2 + 2 \sum_{\alpha \in \Phi_{\mathfrak{m}}^+} E_\alpha E_{-\alpha}$$

and

$$\Box = \Box_{\mathfrak{m}} + 2 \sum_{\alpha \in \Phi_{\mathfrak{q}}^+} E_\alpha E_{-\alpha}.$$

Let $\Gamma_{X,v}$ and δ be as in Example 3.3. From the computations in [Wal, 7.A.2.8 and 8.A.3.5], we have that

$$\begin{aligned}\Gamma_{X,v} \Bigg(& -2 \sum_{\alpha \in \Phi_{\mathfrak{q}}^+} \alpha(X)^{-2} \operatorname{symm}(E_\alpha E_{-\alpha}) \otimes 1 \otimes 1 \\ & + 1 \otimes \Box_{\mathfrak{m}} \otimes 1 + 1 \otimes 2 \sum_{\alpha \in \Phi_{\mathfrak{q}}^+} \alpha(X)^{-1} H_\alpha \\ & + 2\alpha(X)^{-2} 1 \otimes 1 \otimes \sum_{\alpha \in \Phi_{\mathfrak{q}}^+} \operatorname{symm}(E_\alpha E_{-\alpha}) \Bigg) = \Box \otimes 1.\end{aligned} \tag{8.6}$$

By Lemma 2.1 there exists $D \in \mathcal{U}(\mathfrak{b})$ such that $\sum_{\alpha \in \Phi_{\mathfrak{q}}^+} \operatorname{symm}(E_\alpha E_{-\alpha}) = D$ as differential operators on $\mathcal{X}$, so we can replace the term $2\alpha(X)^{-2} 1 \otimes 1 \otimes \sum_{\alpha \in \Phi_{\mathfrak{q}}^+} \operatorname{symm}(E_\alpha E_{-\alpha})$ in the above sum by $D \otimes 1 \otimes 1$. Hence, by Lemma 3.2

and Remark 3.1 we get that the radial component $\delta(\Box\otimes 1)_{X,v}$ is given by

$$\delta(\Box\otimes 1)_{X,v}=\lambda\mathrm{Id}+\Box_{\mathfrak{m}}\otimes 1+2\sum_{\alpha\in\Phi_{\mathfrak{q}}^{+}}\alpha(X)^{-1}H_{\alpha}\otimes 1.$$

Now using the argument in [Wal, 8.A.3.5] we get

$$\delta(\Box_{\mathfrak{g}}\otimes 1)=\delta_{G,M}(\Box_{\mathfrak{g}}\otimes 1)=\eta^{-1/2}\otimes 1(\Box_{\mathfrak{m}}+\lambda)\eta^{1/2}\otimes 1. \qquad \Box$$

LEMMA 8.6. *Let V be a finite-dimensional vector space and write $V=\oplus_{i=1}^{r}W_i$ where W_i, $i=1,\dots,r$, is a subspace of V. Then $V^*=V-\{0\}$ can be written in the form $V^*=\bigcup_{j=1}^{s}Y_j$ where $Y_j=\oplus_{k=1}^{r}Y_j^k$ and Y_j^k is either W_k or $W_k^*=W_k-\{0\}$.*

The proof is immediate.

Proof of Theorem 8.4. We shall follow the same steps and use the same notation as in [Wal, 8.3.12]. We start by induction on $\dim([\mathfrak{g},\mathfrak{g}])$. If $\mathfrak{g}$ and G are abelian then $\Omega=\Omega'$ and the proof of Theorem 8.4 reduces to Theorem 8.3. Assume that $\mathfrak{g}$ is not abelian. Let F be the function on $\mathfrak{g}$ whose existence is guaranteed by Theorem 8.3 and set $T_F=F\otimes dx$. Our goal is to prove that

$$\operatorname{supp}(T-T_F)\subset(\mathcal{N}\oplus U)\times\mathcal{X}, \tag{8.7}$$

where $\mathcal{N}$ is the nilpotent cone and $U=\Omega\cap\mathfrak{z}$. Suppose that $(X,x)\in\operatorname{supp}(T-T_F)$ and that the semisimple part of X, X_s is not in $\mathfrak{z}$. Set $\mathfrak{m}=\mathfrak{g}^{X_s}$. Then $\dim([\mathfrak{m},\mathfrak{m}])<\dim([\mathfrak{g},\mathfrak{g}])$. Let $\mathfrak{q}=\mathfrak{m}^{\perp}$ with respect to the killing form and $\mathfrak{m}''=\{Y\in\mathfrak{m}\,:\,\det(\operatorname{ad}Y|_{\mathfrak{q}})\neq 0\}$. Let $M=\{g\in G\,:\,\mathrm{Ad}(g)X_s=X_s\}$. Let $\Omega_{\mathfrak{m}}$ be the neighborhood of X in $\mathfrak{m}''\cap\Omega$ constructed in [Wal, 8.3.12]. If $g\in G$, $Y\in\Omega_{\mathfrak{m}}$ and $x\in\mathcal{X}$ set

$$\Psi(g,Y)=gYg^{-1}$$

and

$$\tilde{\Psi}(g,Y,x)=(gYg^{-1},\rho(g)x).$$

Let $W=G\Omega_{\mathfrak{m}}$. Then Ψ is a submersion onto W and $\tilde{\Psi}$ is a submersion onto $W\times\mathcal{X}$. Since $T=F\otimes dx$ on $\Omega'\times\mathcal{X}$ it follows that $\tilde{\Psi}^0(T)=\Psi^0(F)\otimes dx$ on $\Omega'_{\mathfrak{m}}\times\mathcal{X}$. Hence, if D is a differential operator on $\mathfrak{g}$ then

$$\tilde{\Psi}^0((D\otimes 1)T)=\Psi^0(DF)\otimes dx$$

Hence, [Wal, 8.A.3.5] implies that

$$\dim(I(\mathfrak{m}_{\mathbf{C}})\otimes 1)|(\eta|^{1/2}\otimes 1)\tilde{\Phi}^0(T)<\infty\ \text{ on }\Omega'_{\mathfrak{m}}\times\mathcal{X}.$$

For $\Box$ we use Lemma 8.5 to conclude that

$$\dim(\mathbf{C}[\Box_{\mathfrak{m}}\otimes 1])|(\eta|^{1/2}\otimes 1)\tilde{\Phi}^0(T)<\infty\ \text{ on }\Omega_{\mathfrak{m}}\times\mathcal{X}.$$

Set $F_{\mathfrak{m}} = F|_{\mathfrak{m}}$. We have that $\mathfrak{m} = \prod \mathfrak{m}_i$ and $M = \prod M_i$ where M_i is a GL_{r_i} factor and $\mathfrak{m}_i$ is the Lie algebra of M_i. If $\mathcal{V}$ is the standard (row space) representation of G then the above decomposition of M gives rise to a decomposition of $\mathcal{V} = \oplus \mathcal{W}_i$ where $\mathcal{W}_i$ can be viewed as the standard (row) representation of M_i. By Lemma 8.6 we can write $\Omega_{\mathfrak{m}} \times \mathcal{X}$ as a finite union of the open sets of the form $\Omega_{\mathfrak{m}} \times \mathcal{Y}$ where $\mathcal{Y} = \oplus \mathcal{Y}_j$ and $\mathcal{Y}_j$ is either $\mathcal{W}_i$ or $\mathcal{W}_i^*$.

Then by the induction hypothesis $\tilde{\Psi}^0(T) = F_{\mathfrak{m}} \otimes dx$ on each such open set; hence $\tilde{\Psi}^0(T) = F_{\mathfrak{m}} \otimes dx$ on $\Omega_{\mathfrak{m}} \times \mathcal{X}$. Hence $T = T_F$ on $W \times \mathcal{X}$ which is a contradiction. Thus we have established (8.7).

To summarize, we have currently proved, using the induction hypothesis, that if T satisfies the assumptions of Theorem 4.4 stated above then T satisfies (8.7).

The next step is to show that $T - T_F$ is finite under $\square \otimes 1$. The arguments follow [Wal, 8.3.11, 8.3.12 and 8.3.13].

More precisely, consider the distribution $(\square \otimes 1)T$. It satisfies the assumptions of Theorem 4.4 and $F_{(\square\otimes 1)T} = \square F$ (recall that F is really $F \otimes 1$ on $\Omega' \times \mathcal{X}$). It follows that $(\square \otimes 1)T - T_{\square F}$ is supported on $(\mathcal{N} \oplus U) \times \mathcal{X}$. On the other hand $(\square\otimes 1)(T - T_F) = (\square\otimes 1)T - (\square\otimes 1)T_F$ is supported on $(\mathcal{N} \oplus U) \times \mathcal{X}$ since $(\square \otimes 1)$ cannot increase the support. Hence $(\square \otimes 1)T_F - T_{\square F} = \mu \otimes 1$ is supported on $(\mathcal{N} \oplus U) \times \mathcal{X}$.

Now the explicit form of μ in [Wal, 8.3.11] for the case $\mathfrak{g} = \mathfrak{g}l_2(\mathbf{R})$ and in [Wal, 8.3.13] for $\dim[\mathfrak{g}, \mathfrak{g}] > 3$ remains valid here and the argument for $\mu = 0$ in [Wal, 8.3.13] for $\dim[\mathfrak{g}, \mathfrak{g}] > 3$ is also valid. Hence, in this case, $(\square \otimes 1)T_F = T_{\square F}$ and $p(\square \otimes 1)T_F = T_{p(\square)F}$ for every polynomial p. There exists a nonzero polynomial p_0 such that $p_0(\square \otimes 1)T = 0$ and consequently $p_0(\square)F = 0$. Thus $p_0(\square \otimes 1)T_F = 0$ and $p_0(\square \otimes 1)(T - T_F) = 0$ as asserted.

Now assume that $\mathfrak{g} = \mathfrak{g}l_2(\mathbf{R})$. By Lemma 7.1, Frobenius reciprocity and Lemma 7.4 it follows that the Euler differential operator $\tilde{E} = E \otimes 1$ acts finitely on $\mu \otimes 1 = (\square\otimes 1)T_F - T_{\square F}$ with real eigenvalues strictly less than $-3/2$. (That is, $\mu \otimes 1$ can be written as a finite sum of generalized eigenvectors for $\tilde{E}$ each of them with eigenvalues less than $-3/2$. See (3) in [Wal], p. 304). Now the argument continues word for word as in [Wal, 8.3.11] and the conclusion of the existence of a nonzero polynomial p_0 such that $p_0(\square \otimes 1)(T - T_F) = 0$.

Hence for both cases we can apply Theorem 7.7, to conclude that $T - T_F = 0$. $\square$

9. The group case

In this section we shall prove the main result of this paper, Theorem 4.1. The method follows Harish-Chandra's proof by lifting the Lie algebra result to the group with the use of the exponential. Our proof here is the same as in [Wal, 8.4] and we include it for the sake of completeness. We shall use the same notations as in [Wal, 8.4]. Whenever our notation is not self-explanatory we urge the reader to look it up in [Wal].

Let T be a $(G, |\det|)$ invariant distribution on $G \times \mathcal{V}^*$ which is $\mathcal{Z}(\mathfrak{g}) \otimes 1$ finite. Let H be a Cartan subgroup of G and $\mathfrak{h}$ be the Lie algebra of H. Set $H' = G' \cap H$. Let $\tilde{\psi}(g, h, v) = (ghg^{-1}, \rho(g)v)$ for $g \in G$, $h \in H'$ and $v \in \mathcal{V}^*$. Then $\tilde{\psi}$ is a submersion of $G \times H' \times \mathcal{V}^*$ onto an open set $U \times \mathcal{V}^*$ in $G \times \mathcal{V}^*$. Let $\Delta = \Delta_{G,H}$ and $\delta = \delta_{G,H}$ be as in [Wal, 7.A.3.6]. If $z \in \mathcal{Z}(\mathfrak{g})$ then

$$\tilde{\psi}^0((z \otimes 1)T) = (\delta(z) \otimes 1 + 1 \otimes \beta(z))\tilde{\psi}^0(T)$$

for some differential operator $\beta(z)$ on $\mathcal{V}^*$. Since $\tilde{\psi}^0(T)$ is H invariant on $H' \times \mathcal{V}^*$ and since H is abelian it follows from Lemma 2.6 that $\tilde{\psi}^0(T) = \tilde{T} \otimes dv$ for some distribution $\tilde{T}$ on H'. Thus, the differential operator $1 \otimes \beta(z)$ on $\tilde{\psi}^0(T)$ is given by

$$(1 \otimes \beta(z))\tilde{\psi}^0(T) = \beta_z \tilde{\psi}^0(T)$$

for some scalar β_z. It follows that $\tilde{T}$ is finite under $\delta(\mathcal{Z}(\mathfrak{g}))$ and that $\Delta\tilde{T}$ is finite under $\gamma(\mathcal{Z}(\mathfrak{g}))$ where γ is the Harish Chandra homomorphism defined in [Wal, 3.2.2]. Since $\mathcal{U}(\mathfrak{h})$ is finitely generated as a $\gamma(\mathcal{Z}(\mathfrak{g}))$ module it follows that there exists a real analytic function ζ on H' such that $\tilde{\psi}^0(T) = T_\zeta \otimes dv$ on $H' \times \mathcal{V}^*$. Continuing this way on a set of nonconjugate Cartans we get that there exists a real analytic function F on G' such that $T = F \otimes dv$ on $G' \times \mathcal{V}^*$. By [Wal, 8.4.1], F is locally integrable on G. Define $T_F = F \otimes dv$ on $G \times \mathcal{V}^*$. In order to complete the proof we must show that $T = T_F$.

Let x be a semisimple element of G. Let $M = \{g \in G \mid gx = xg\}$. Then $M = \prod_i \mathrm{GL}_{r_i}(K_i)$ where K_i is either $\mathbf{R}$ or $\mathbf{C}$. We can write $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{q}$ with $\mathfrak{q}$ an $\mathrm{Ad}(M)$ invariant subspace of $\mathfrak{g}$. Set $M'' = \{m \in M \mid \det((Ad(m) - I)|_{\mathfrak{q}}) \neq 0\}$. Set $\psi(g, m) = gmg^{-1}$ and $\tilde{\psi}^0(g, m, v) = (gmg^{-1}, \rho(g)v)$ for $g \in G$, $m \in M''$, $v \in \mathcal{V}^*$. Then $\tilde{\psi}$ is a submersion of $G \times M'' \times \mathcal{V}^*$ onto an open subset of $G \times \mathcal{V}^*$. By [Wal, 8.4.2] there exists an open neighborhood of 0, U_0, in $\mathfrak{m}$ of the type described in (4.3) such that exp restricted to U_0 is a diffeomorphism onto an open neighborhood U_1 of 1 in M and xU_1 is a neighborhood of x in M''.

Set $\Omega = \psi(G \times xU_1)$. Let j be the "j function for M" (see [Wal, 8.A.3.6]). Set $\tilde{\zeta}(g, u, v) = \tilde{\psi}(g, xu, v)$ for $g \in G$ and $u \in U_1$. Let $\tilde{\zeta}^0(T) \in D'(U_1 \times \mathcal{V}^*)$. We note that $M'' \cap G' = M'' \cap M'$. Since $T = F \otimes dv$ on G' it follows that

$$\tilde{\zeta}^0((z \otimes 1)T) = (\delta_{G,M}(z) \otimes 1)\tilde{\zeta}^0(T) \text{ on } (U_1 \cap M') \times \mathcal{V}^*.$$

for all $z \in \mathcal{Z}(\mathfrak{g})$. Thus

$$\dim(\delta_{G,M}(\mathcal{Z}(\mathfrak{g})) \otimes 1)\tilde{\zeta}^0(T) < \infty \text{ on } (U_1 \cap M') \times \mathcal{V}^*.$$

Since $\delta_{G,M} = (\Delta_{G,M})^{-1}\gamma_{G,M}\Delta_{G,M}$, and since $\mathcal{Z}(\mathfrak{m})$ is finitely generated over $\gamma_{G,M}(\mathcal{Z}(\mathfrak{g}))$ it follows that

$$\dim(\mathcal{Z}(\mathfrak{m}) \otimes 1)(\Delta_{G,M} \otimes 1)\tilde{\zeta}^0(T) < \infty \text{ on } (U_1 \cap M') \times \mathcal{V}^*. \tag{9.1}$$

If $C_{\mathfrak{m}}$ is the Casimir operator of $\mathfrak{m}$ then $\gamma_{G,M}(C) = C_{\mathfrak{m}} + \lambda I$ with $\lambda \in \mathbf{C}$. A computation similar to the one in Lemma 8.5 will give

$$\dim(\mathbf{C}[C_{\mathfrak{m}}] \otimes 1)(\Delta_{G,M} \otimes 1)\tilde{\zeta}^0(T) < \infty \text{ on } U_1 \times \mathcal{V}^*. \tag{9.2}$$

Let $v \in \mathcal{V}^*$. By Lemma 8.6 there exists an open set $\mathcal{X}_v \subset \mathcal{V}^*$ containing v such that $\mathcal{X}_v = \oplus \mathcal{X}_i$ and $\mathcal{X}_i = \mathcal{V}_i$ or $\mathcal{X}_i = \mathcal{V}_i^*$ corresponding to the decomposition $M = \prod G_i$ and $G_i = \mathrm{GL}(\mathcal{V}_i)$ where $\mathcal{V}_i$ is either a complex or a real vector space. Also, $\mathcal{X}_v$ is M invariant and we can define $Q = \exp^* \otimes 1(\Delta_{G,M}\tilde{\zeta}^0(T))$ in $D'(U_0 \times \mathcal{X}_v)^M$. By [Wal, 8.A.3.6] and by (9.1) we have

$$\dim(I(\mathfrak{m})j^{1/2} \otimes 1)Q < \infty \text{ on } (U_0 \cap \mathfrak{m}') \times \mathcal{X}_v.$$

By [Wal, 8.A.3.7] and by (9.2) we have that

$$\dim(\mathbf{C}[\Box_{\mathfrak{m}}]j^{1/2} \otimes 1)Q < \infty \text{ on } U_0 \times \mathcal{X}_v.$$

Hence by Theorem 4.4 there exists a G invariant locally integrable function, μ, on U_0 which is real analytic on $(U_0 \cap \mathfrak{m}') \times \mathcal{X}_v$ such that

$$(j^{1/2} \otimes 1)Q = \mu \otimes 1.$$

Since this is true for all $v \in \mathcal{V}^*$ we get that $\tilde{\zeta}^0(T) = \tilde{\zeta}^0(F \otimes 1)$. Hence $T = T_F$ on $\Omega \times \mathcal{V}^*$. When we vary our semisimple element s the sets Ω cover all of G; hence $T = T_F$ on all of G and our proof is complete. □

10. P invariant forms

In this section we use our main theorem, Theorem 1.4, to give another proof of Conjecture 1.1 and to construct an inner product formula for the Whittaker model. Such a proof and an inner product construction were given by Bernstein [Ber, 5.1, 5.4, 6.4] in the p-adic case. We follow Bernstein's proofs closely. In what follows we set $G = \mathrm{GL}_n(\mathbf{R})$ or $G = \mathrm{GL}_n(\mathbf{C})$ and set P to be $P_n(\mathbf{R})$ or $P_n(\mathbf{C})$ respectively.

10.1. *P invariant pairings on representations of G.* Here we prove the analog of [Ber, Th. A] for the archimedean case.

Let (π, H) be an irreducible admissible representation of G on a Hilbert space H. For $f \in C_c^\infty(G)$ we define the bounded linear operator $\pi(f) : H \to H$ by

$$\pi(f)v = \int_G f(g)\pi(g)v\,dg.$$

Let $H_\infty = \{\pi(f)v \mid f \in C_c^\infty(G), v \in H\}$. We topologize H_∞ in the usual way. Let (π^*, H^*) be the contragredient representation on the dual space and $H_\infty^* \subset H^*$ as above.

THEOREM 10.1. *Let $B : H_\infty \times H_\infty^* \to \mathbf{C}$ be a continuous P invariant bilinear form. Then B is G invariant, hence is a scalar multiple of the canonical pairing $B_0(\xi, \tilde{\xi}) = \left\langle \xi, \tilde{\xi} \right\rangle$.*

Proof. Let $M = G \times G$, $Q = G^\triangle$, the diagonal embedding of G into M and $P_Q = P^\triangle$, the diagonal embedding of P into M. Let $V = H_\infty \otimes H_\infty^*$ and $\Pi = \pi \otimes \pi^*$. Then (Π, V) is an irreducible admissible representation of M. The bilinear form B induces a continuous functional λ_B on V. Since B is P invariant, it follows that λ_B is P_Q invariant. Let λ_G be a nonzero Q invariant continuous functional on V. (It is induced from the canonical G invariant form on $H_\infty \times H_\infty^*$.) For a function $f \in C_c^\infty(M)$ we define the distribution

$$\Theta(f) = \langle \lambda_B, \Pi(f)\lambda_G \rangle .$$

Here λ_G is in the dual of $H \otimes H^*$ and $\Pi(f)\lambda_G$ is in the smooth part of $H \otimes H^*$. Since $H \otimes H^*$ is a self-dual representation of M we can identify the smooth part of $H \otimes H^*$ with V. Hence, $\Pi(f)\lambda_G$ is identified with a vector v_f in V and $\Theta(f) = \lambda_P(v_f)$. It is easy to see that $\Theta(f)$ is right Q invariant and left P_Q invariant. It follows that there exists a distribution T on G such that if $f_1, f_2 \in C_c^\infty(G)$ then

$$\Theta(f_1 \otimes f_2) = T(f_1 * \hat{f}_2).$$

Here $\hat{f}(g) = f(g^{-1})$. It is easy to see that T is P invariant under conjugation and is an eigendistribution for $\mathcal{Z}(\mathfrak{g})$ (see Shalika [Sha, p. 184] for a similar argument). Hence, by Theorem 1.4, T is G invariant. It follows that Θ is left Q invariant. Fix $h = (g, g) \in H$. Then

$$\langle \Pi(h)\lambda_B - \lambda_B, \Pi(f)\lambda_G \rangle = 0$$

for all $f \in C_c^\infty(M)$. Since V is irreducible it follows that $\Pi(h)\lambda_B - \lambda_B = 0$, that is, B is G invariant. $\square$

Remark 10.2. As a corollary to Theorem 10.1 we obtain another proof of Conjecture 1.1. The proof follows word for word the proof in [Ber, 5.4], and is omitted.

10.2. *Scalar product in the Whittaker model.* Here we follow [Ber, 6.3, 6.4]. Let $U = U_n$ be the upper triangular matrices in G and let ψ be a nondegenerate character of U. Let (π, H) be an irreducible admissible representation of G. We say that π is generic if there exists a continuous nonzero linear functional $l : H_\infty \to \mathbf{C}$ such that

$$l(\pi(n)v) = \psi(n)l(v), \quad n \in U, v \in H_\infty.$$

Let $\mathcal{W}(\pi, \psi)$ be the space of functions

$$W_v(g) = l(\pi(g)v), \quad g \in G, v \in H_\infty.$$

$\mathcal{W}(\pi,\psi)$ is called the ψ Whittaker model of π. It is well known that the Whittaker model is unique and that if π has a nonzero ψ Whittaker model then π^* has a nonzero ψ^{-1} Whittaker model $\mathcal{W}(\pi^*,\psi^{-1})$. For $W \in \mathcal{W}(\pi,\psi)$ and $\tilde{W} \in \mathcal{W}(\pi^*,\psi^{-1})$ we define the integral

$$Z(W,\tilde{W},s) = \int_{U_{n-1}\backslash \mathrm{GL}(n-1)} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \tilde{W}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} |\det(g)|^s dg.$$

By [Jac-Sha], $Z(W,\tilde{W},s)$ converges absolutely in a right half-plane and can be meromorphically continued to the whole complex plane. For every real number α there exists a polynomial $P_\alpha(s)$ such that $P_\alpha(s)Z(W,\tilde{W},s)$ is analytic for $\mathrm{re}(s) \geq \alpha$. Let k be the maximal order of the poles of all functions $Z(W,\tilde{W},s)$ at $s = 0$. Then Theorem 10.1 gives the following:

Theorem 10.3. *For $v \in H_\infty$ and $\tilde{v} \in H^*_\infty$ set*

$$B(v,\tilde{v}) = (s^k Z(W_v,\tilde{W}_{\tilde{v}},s)|_{s=0}.$$

Then B is a nonzero scalar multiple of the canonical pairing on $H_\infty \times H^*_\infty$.

The following corollary was obtained independently by Jacquet for the case where π is a component of a cuspidal automorphic representation:

Corollary 10.4. *Let π, H be a unitary representation of G. Let k be an integer as above. Then $B(v_1,v_2) = (s^k Z(W_{v_1},\overline{W_{v_2}},s)|_{s=0}$ is a nonzero scalar multiple of the canonical G invariant inner form on H_∞.*

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