Annals of Mathematics, **157** (2003), 919–938

Large Riemannian manifolds which are flexible

By A. N. Dranishnikov, Steven C. Ferry, and Shmuel Weinberger*

Abstract

For each $k \in \mathbb{Z}$, we construct a uniformly contractible metric on Euclidean space which is not mod k hypereuclidean. We also construct a pair of uniformly contractible Riemannian metrics on $\mathbb{R}^n$, $n \geq 11$, so that the resulting manifolds Z and Z' are bounded homotopy equivalent by a homotopy equivalence which is not boundedly close to a homeomorphism. We show that for these spaces the C^*-algebra assembly map $K_*^{lf}(Z) \to K_*(C^*(Z))$ from locally finite K-homology to the K-theory of the bounded propagation algebra is not a monomorphism. This shows that an integral version of the coarse Novikov conjecture fails for real operator algebras. If we allow a single cone-like singularity, a similar construction yields a counterexample for complex C^*-algebras.

1. Introduction

This paper is a contribution to the collection of problems that surrounds positive scalar curvature, topological rigidity (a.k.a. the Borel conjecture), the Novikov, and Baum-Connes conjectures. Much work in this area (see e.g. [14], [4], [3], [15]) has focused attention on bounded and controlled analogues of these problems, which analogues often imply the originals. Recently, success in attacks on the Novikov and Gromov-Lawson conjectures has been achieved along these lines by proving the coarse Baum-Connes conjecture for certain classes of groups [23], [27], [28]. A form of the coarse Baum-Connes conjecture states that the C^*-algebra assembly map $\mu : K_*^{lf}(X) \to K_*(C^*(X))$ is an isomorphism for uniformly contractible metric spaces X with bounded geometry [21].

Using work of Gromov on embedding of expanding graphs in groups Γ with $B\Gamma$ a finite complex [16], the epimorphism part of the coarse Baum-Connes con-

*The authors are partially supported by NSF grants. The second author would like to thank the University of Chicago for its hospitality during numerous visits.

1991 *Mathematics Subject Classification.* 53C23, 53C20, 57R65, 57N60.

Key words and phrases. uniformly contractible.

jecture was disproved [17]. In this paper we will show that the monomorphism part of the coarse Baum-Connes conjecture (i.e. the coarse Novikov conjecture) does not hold true without the bounded geometry condition. We will construct a uniformly contractible metric on $\mathbb{R}^8$ for which μ is not a monomorphism. Thus, a coarse form of the integral Novikov conjecture fails even for finite-dimensional uniformly contractible manifolds. In fact we will prove more: our uniformly contractible $\mathbb{R}^8$ is not integrally hypereuclidean, which is to say that it does not admit a degree one coarse Lipschitz map to euclidean space. Also in this paper, we will produce a uniformly contractible Riemannian manifold, abstractly homeomorphic to $\mathbb{R}^n$, $n \geq 11$, which is boundedly homotopy equivalent to another such manifold, but not boundedly homeomorphic to it. This disproves one coarse analog of the rigidity conjecture for closed aspherical manifolds. We will also show that for each $k \in \mathbb{Z}$ some of these manifolds are not mod k hypereuclidean.

Our construction is ultimately based on examples of Dranishnikov [5], [6] of spaces for which cohomological dimension disagrees with covering dimension, and the consequent phenomenon, using a theorem of Edwards (see [25]), of cell-like maps which raise dimension.

Definition 1.1. We will use the notation $B_r(x)$ to denote the ball of radius r centered at x. A metric space (X, d) is *uniformly contractible* if for every r there is an $R \geq r$ so that for every $x \in X$, $B_r(x)$ contracts to a point in $B_R(x)$. The main examples of this are the universal cover of a compact aspherical polyhedron and the open cone in $\mathbb{R}^n$ of a finite subpolyhedron of the boundary of the unit cube. There is a similar notion of *uniformly n-connected* which says that any map of an n-dimensional CW complex into $B_r(x)$ is nullhomotopic in $B_R(x)$.

Definition 1.2. We will say that a Riemannian manifold M^n is *integrally* (mod k, *or rationally*) *hypereuclidean* if there is a coarsely proper coarse Lipschitz map $f : M \to \mathbb{R}^n$ which is of degree 1 (of degree $\equiv 1$ mod k, or of nonzero integral degree, respectively). See Section 4 for definitions and elaborations.

Here are our main results:

Theorem A. *For any given k and $n \geq 8$, there is a Riemannian manifold Z which is diffeomorphic to $\mathbb{R}^n$ such that Z is uniformly contractible and rationally hypereuclidean but is not mod k* (*or integrally*) *hypereuclidean.*

Definition 1.3. (i) A map $f : X \to Y$ is a *coarse isometry* if there is a K so that $|d_Y(f(x), f(x')) - d_X(x, x')| < K$ for all $x, x' \in X$ and so that for each $y \in Y$ there is an $x \in X$ with $d_Y(y, f(x)) < K$.

(ii) We will say that uniformly contractible Riemannian manifolds Z and Z' are *boundedly homeomorphic* if there is a homeomorphism $f : Z \to Z'$ which is a coarse isometry.

Theorem B. *There is a coarse isometry between uniformly contractible Riemannian manifolds Z and Z' which is not boundedly close to a homeomorphism.*

An easy inductive argument shows that a coarse isometry of uniformly contractible Riemannian manifolds is a bounded homotopy equivalence, so this gives a counterexample to a coarse form of the Borel conjecture.

Theorem C. *There is a uniformly contractible singular Riemannian manifold Z such that the assembly map* (*see* [20])

$$K_*^{\ell f}(Z) \to K_*(C^*(Z))$$

fails to be an integral monomorphism. Z is diffeomorphic away from one point to the open cone on a differentiable manifold M.

It was shown in [8] that Z has infinite asymptotic dimension in the sense of Gromov. This fact cannot be derived from [27] since Z does not have bounded geometry.

When we first discovered these results, we thought a way around these problems might be to use a large scale version of K-theory in place of the K-theory of the uniformly contractible manifold. Yu has observed that even that version of the (C^*-analytic) Novikov conjecture fails in general (see [28]), although not for any examples that arise from finite dimensional uniformly contractible manifolds. On the other hand, bounded geometry does suffice to eliminate both sets of examples.

In the past year, motivated by Gromov's observation that spaces which contain expander graphs cannot embed in Hilbert space, several researchers (see [17] and the references contained therein), have given examples of various sorts of counterexamples to general forms of the Baum-Connes conjecture. Using the methodolgy of Farrell and Jones, Kevin Whyte and the last author have observed that some of these are not counterexamples to the corresponding topological statements. Thus the examples of this paper remain the only counterexamples to the topological problems.

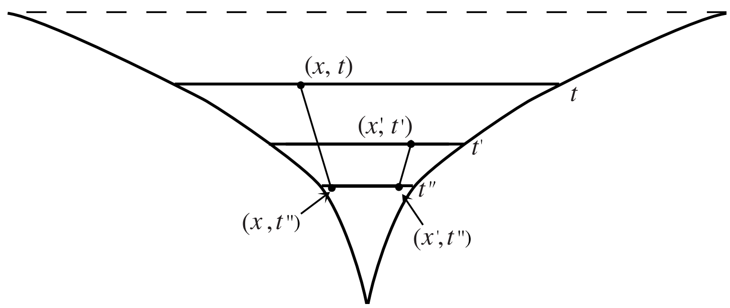
2. Weighted cones on uniformly k-connected spaces

The *open cone* on a topological space X is the topological space $OX = X \times [0,\infty)/X \times 0$.

Definition 2.1. A compact metric space (X, d) is *locally k-connected* if for every $\epsilon > 0$ there is a $\delta > 0$ such that for each k-dimensional simplicial complex K^k and each map $\alpha : K^k \to X$ with $\operatorname{diam}(\alpha(K^k)) < \delta$ there is a map $\bar{\alpha} : \operatorname{Cone}(K) \to X$ extending α with $\operatorname{diam}(\bar{\alpha}(\operatorname{Cone}(K))) < \epsilon$. Here, $\operatorname{Cone}(K)$ denotes the ordinary closed cone.

LEMMA 2.2. *Let (X, d) be a compact metric space which is locally k-connected for all k. For each n, the open cone on X has a complete uniformly n-connected metric. We will denote any such metric space by cX.*[1]

Proof. We will even produce a metric which has a linear contraction function. Its construction is based on the *weighted cone* often used in differential geometry. Draw the cone vertically, so that horizontal slices are copies of X.



Choose a continuous strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$. Let d be the original metric on X and define a function ρ' by

(i) $\rho'((x, t), (x', t)) = \phi(t)d(x, x')$.

(ii) $\rho'((x, t), (x, t')) = |t - t'|$.

We then define $\rho : OX \times OX \to [0, \infty)$ to be

$$\rho((x, t), (x', t')) = \inf \sum_{i=1}^{\ell} \rho'((x_i, t_i), (x_{i-1}, t_{i-1}))$$

where the sum is over all chains

$$(x, t) = (x_0, t_0), (x_1, t_1), \ldots, (x_\ell, t_\ell) = (x', t')$$

and each segment is either horizontal or vertical. It pays to move towards 0 before moving in the X-direction, so chains of shortest length have the form pictured above. The function ρ is a metric on OX. The natural projection $OX \to [0, \infty)$ decreases distances, so Cauchy sequences are bounded in the $[0, \infty)$-direction. It follows that the metric on OX is complete. We write cX for the metric space (OX, ρ).

It remains to define ϕ so that cX is uniformly n-contractible. We will define $\phi(1) = 1$ and $\phi(i + 1) = N_{i+1}\phi(i)$ for $i \in \mathbb{Z}$, where the sequence $\{N_i\}$ will be specified below. For nonintegral values of t, we set

$$\phi(t) = \phi([t]) + (t - [t])\phi([t] + 1).$$

[1]The "c" notation in cX refers to a specific choice of weights. There probably should be an "n" in our notation, but we leave it out for simplicity.

Since X is locally n-connected, there is an infinite decreasing positive sequence $\{r_i\}$ such that for every x the inclusions $\ldots \subset B^d_{r_{i+1}}(x) \subset B^d_{r_i}(x) \subset B^d_{r_{i-1}}(x)$ are nullhomotopic on n-skeleta. Refine the sequence so that actually inclusions $B^d_{ir_i}(x) \subset B^d_{r_{i-1}}(x)$ are nullhomotopic on n-skeleta. We set $N_i = \frac{r_{i-1}}{r_i}$.

Now consider the ball $B^\rho_1(x,i) \subset cX$. First, we note that $B^\rho_1(x,i) \subset B^d_{\frac{1}{N_{i-1}}}(x) \times [i-1, i+1]$ and that $B^\rho_1(x,i)$ contracts in itself to $B^\rho_1(x,i) \cap (X \times [i-1,i]) \subset B^d_{\frac{1}{N_{i-1}}}(x) \times [i-1,i]$. But $B^\rho_3(x,i) \supset B^d_{\frac{1}{N_{i-2}}}(x) \times \{i-2\}$ so $B^\rho_1(x,i)$ n-contracts in $B^\rho_3\,(x,i)$ by pushing down to the $(i-2)$-level and performing the n-contraction there.

For balls of radius 2 the same reasoning applies if the center is at least 3 away from the vertex. We continue in this way and observe that for any given size ball, centered sufficiently far out, one obtains a n-contractibility function of $f(r) = r + 2$ as required. The whole space is therefore uniformly contractible. □

3. Designer compacta

Definition 3.1. A map $f : M \to X$ from a closed manifold onto a compact metric space is *cell-like or* CE if for each $x \in X$ and neighborhood U of $f^{-1}(x)$ there is a neighborhood V of $f^{-1}(x)$ in U so that V contracts to a point in U.

The purpose of this section is to give examples of CE maps $f : M \to X$ so that $f_* : H_n(M; \mathbb{L}(e)) \to H_n(X; \mathbb{L}(e))$ has nontrivial kernel. The argument given below is a modification of the first author's construction of infinite-dimensional compacta with finite cohomological dimension. Here is the result which we will use in proving Theorems A, B, and C of the introduction.

THEOREM 3.2. *Let M^n be a* 2*-connected n-manifold, $n \geq 7$, and let α be an element of $\widetilde{KO}_*(M; \mathbb{Z}_m)$. Then there is a* CE *map $q : M \to X$ with $\alpha \in \ker(q_* : \widetilde{KO}_*(M; \mathbb{Z}_m) \to \widetilde{KO}_*(X; \mathbb{Z}_m))$. It follows that if $\alpha \in H_*(M; \mathbb{L}(e))$ is an element of order m, m odd, then there is a* CE *map $c : M \to X$ so that $c_*(\alpha) = 0$ in $H_*(X; \mathbb{L}(e))$.*

We begin the proof of this theorem by recalling the statement of a major step in the construction of infinite-dimensional compacta with finite cohomological dimension.

THEOREM 3.3. *Suppose that $\tilde{h}_*(K(\mathbb{Z}, n)) = 0$ for some generalized homology theory h_*. Then for any finite polyhedron L and any element $\alpha \in \tilde{h}_*(L)$ there exist a compactum Y and a map $f : Y \to L$ so that*

(1)c-$\dim_{\mathbb{Z}} Y \leq n$.

(2) $\alpha \in \operatorname{Im}(f_*)$.

Remark 3.4. In [5], [6] the analogous result was proven for cohomology theory. The proof is similar for homology theory. See [9].

Theorem 3.3 also has a relative version:

THEOREM 3.3′. *Suppose that* $\tilde{h}_*(K(\mathbb{Z},n)) = 0$. *Then for any finite polyhedral pair* (K,L) *and any element* $\alpha \in \tilde{h}_*(K,L)$ *there exist a compactum* Y *and a map* $f:(Y,L)\to(K,L)$ *so that*

(i) $c\text{-}\dim_{\mathbb{Z}}(Y-L)\le n$.

(ii) $\alpha\in \operatorname{Im}(f_*)$.

(iii) $f|_L=\mathrm{id}_L$.

The proof is essentially the same. Here is the key lemma in the proof of Theorem 3.2. In what follows, $\widetilde{K}_*$ will refer to reduced complex K-homology and $\widetilde{KO}_*$ will refer to reduced real K-homology.

LEMMA 3.5. *Let* M^n *be a* 2*-connected* n*-manifold,* $n\ge 7$*, and let* α *be an element in* $\widetilde{KO}_*(M;\mathbb{Z}_m)$, $m\in\mathbb{Z}$. *Then there exist compacta* $Z\supset M$ *and* $Y\supset M$ *along with a* CE *map* $g:(Z,M)\to(Y,M)$ *so that*

(1) $g|M=\mathrm{id}_M$.

(2) $\dim(Z-M)=3$.

(3) $j_*(\alpha)=0$, *where* $j:M\to Y$ *is the inclusion.*

Proof. By [26], $\widetilde{KO}_*(K(\mathbb{Z}^k,n);\mathbb{Z}_m)=0$ for $n\ge 3$. We can now apply Theorem 3.3′ to the pair $(\mathrm{Cone}(M),M)$ and the element $\bar\alpha\in\widetilde{KO}_{*+1}(\mathrm{Cone}(M),M)$ with $\partial\bar\alpha=\alpha$ in the long exact sequence of $(\mathrm{Cone}(M),M)$, obtaining a space $Y\supset M$ with $c\dim(Y-M)=3$ so that there is a class $\bar\alpha'\in\widetilde{KO}_{*+1}(Y,M)$ with $\partial\bar\alpha'=\alpha$ and a CE map $g:(Z,M)\to(Y,M)$ with $\dim(Z-M)=3$. The exact sequence:

$$\widetilde{KO}_{*+1}(Y,M)\xrightarrow{\partial}\widetilde{KO}_*(M)\xrightarrow{j_*}\widetilde{KO}_*(Y)$$

shows that $j_*(\alpha)=0$. □

Next, we construct a particularly nice retraction $Z\to M$.

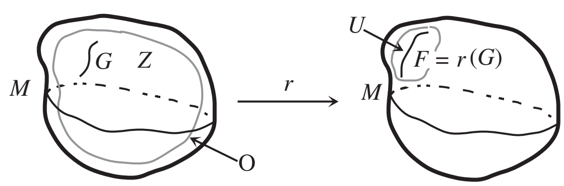
LEMMA 3.6. *Let* (Z,M) *be a compact pair with* $\dim(Z-M)=3$ *and* M *a* 2*-connected* n*-manifold,* $n\ge 7$. *Then there is a retraction* $r:Z\to M$ *with* $r|(Z-M)$ *one-to-one.*

Proof. The existence of the retraction follows from obstruction theory applied to the nerve of a fine cover of Z. The rest is standard dimension theory using the Baire category theorem. □

LEMMA 3.7. *Let $r : Z \to M$ be a retraction which is one-to-one on $(Z - M)$ and let $g : (Z, M) \to (Y, M)$ be a* CE *map which is the identity over M. Then the decomposition of M whose nondegenerate elements are $r(g^{-1}(y))$ is upper semicontinuous.*

Proof. We need to show that if F is an element of this decomposition and $U \supset F$ then there is a V with $F \subset V \subset U$ such that if F' is a decomposition element with $F' \cap V = \emptyset$, then $F' \subset U$.

Case I. $F = r(G)$, $G = g^{-1}(y)$. Then $G \cap M = \emptyset$. For $U \supset F$, let $d = \mathrm{dist}(F, M - U)$. Since r is a retraction, there is an open neighborhood $O \subset Z$ of M so that for all G' such that $G' \cap \bar{O} \neq \emptyset$, diam $z(G') < \frac{d}{2}$. We may assume that O has been chosen so small that $\bar{O} \cap G = \emptyset$. By continuity of g, there is an open V' with $G \subset V' \subset (Z - \bar{O}) \cap Z^{-1}(U)$. Since r is one-to-one and $Z - O$ is compact, $r(V')$ is open in $r(Z - O)$. This means that there is an open $W \subset M$ so that $W \cap r(Z - O) = r(V')$. Let $V = W \cap)_{\frac{d}{2}}(F) \subset U$. If $F' \cap V \neq \emptyset$ then $F' \subset U$, since F' is either a singleton, a set with diameter $< \frac{d}{2}$, or $r(G')$ with $G \subset Z - \bar{O}$, and all three cases are accounted for above.



Case II. F is a singleton, $F = \{x\}$ with $F \notin z(Z - M)$. Let $x \in U$ and let $d = \mathrm{dist}(x, M - U)$. By continuity of g, there is a compact $C \subset Z - M$ so that if $G \subset C$, then $\mathrm{diam}(Z(G)) < \frac{d}{2}$. Let $\rho = \mathrm{dist}(x, r(C))$ and define $V = B_\tau(x)$ where $\tau = \min\{\rho, \frac{d}{2}\}$. □

Proof of Theorem 3.2. Consider the coefficient sequence

$$\to \widetilde{KO}_{*+1}(M; \mathbb{Z}_m) \xrightarrow{\partial} \widetilde{KO}_*(M) \xrightarrow{\times m} \widetilde{KO}_*(M) \to .$$

If $\alpha \in \widetilde{KO}_*(M)$ is of order m, then $\alpha = \partial\bar{\alpha}$, where $\bar{\alpha} \in \widetilde{KO}_{*+1}(M; \mathbb{Z}_m)$. We choose $g : Z \to Y$ as in Lemma 3.5 so that $j_*\bar{\alpha} = 0$. This gives us a diagram

$$\begin{array}{ccccc} M & \xleftarrow{r} & Z & \xleftarrow{i} & M \\ \downarrow f & & \downarrow g & & \downarrow \mathrm{id} \\ X & \xleftarrow{r'} & Y & \xleftarrow{j} & M \end{array}$$

where $f : M \to X$ is the CE map induced by the decomposition $\{r(G) | G = g^{-1}(y),\ y \in Y\}$ and r' is the induced map from Y to X. It follows immediately from this diagram that $f_*(\bar{\alpha}) = 0$. It then follows from the ladder of coefficient sequences

$$\begin{array}{ccccc} \widetilde{KO}_{*+1}(M;\mathbb{Z}_m) & \longrightarrow & \widetilde{KO}_*(M) & \longrightarrow & \widetilde{KO}_*(M) \\ {\scriptstyle f_*}\big\downarrow & & {\scriptstyle f_*}\big\downarrow & & {\scriptstyle f_*}\big\downarrow \\ \widetilde{KO}_{*+1}(X;\mathbb{Z}_m) & \longrightarrow & \widetilde{KO}_*(X) & \longrightarrow & \widetilde{KO}_*(X) \end{array}$$

that $f_*(\alpha) = 0$. The L-theory statement in Theorem 2 now follows from the fact that $KO[\frac{1}{2}] = \mathbb{L}(e)[\frac{1}{2}]$. □

4. The proof of Theorem A

We begin by stating some definitions.

Definition 4.1.

(i) A map $f : X \to Y$ between metric spaces is said to be *coarse Lipschitz* if there are constants C and D so that $d_Y(f(x), f(x')) < Cd_X(x, x')$ whenever $d_X(x, x') > D$. Notice that coarse Lipschitz maps are not necessarily continuous. In fact, if diam $X < \infty$, every map defined on X is coarse Lipschitz.

(ii) A map $f : X \to Y$ is *coarsely proper* if for each bounded set $B \subset Y$, $f^{-1}(B)$ has compact closure in X.

The following corollary constructs the Riemannian manifolds appearing in all of our main theorems.

Proposition 4.2. *If X is the cell-like image of a compact manifold and n is given, then for some suitable choice of weights, cX is uniformly n-connected.*

Proof. The CE image of any compact ANR (absolute neighborhood retract) is locally n-connected for all n, so the proposition follows from Lemma 2.2. See [19] for references. □

Corollary 4.3. *Let $f : S^{k-1} \to X$ be a cell-like map. Then $\mathbb{R}^k$ has a uniformly contractible Riemannian metric which is coarsely equivalent to cX, where the cone is weighted as in Proposition* 4.2.

Proof. Consider $cf : cS^{k-1} \to cX$. This induces a pseudometric on $\mathbb{R}^k$. The basic lifting property for cell-like maps (see [19]) shows that $\mathbb{R}^k$ with this pseudometric is uniformly n-connected if and only if cX is. If $n \geq k-1$, this means that the induced pseudometric on $\mathbb{R}^k$ is uniformly contractible. Adding any sufficiently small metric to this pseudometric — the metric from $\mathbb{R}^k \cong \mathring{D}^k$

will do — produces a uniformly contractible metric on $\mathbb{R}^k$ which is quasi-equivalent to cX. Since X is locally connected, a theorem of Bing [1] says that X has a path metric. If we start with a path metric on X, the metric on cX is also a path metric and the results of [11] allow us to construct a Riemannian metric on cS^{k-1} which is uniformly contractible and coarse Lipschitz equivalent to cX. □

We have constructed a Riemannian manifold Z^n homeomorphic to $\mathbb{R}^n$ so that Z is coarsely isometric to a weighted open cone on a "Dranishnikov space" X. By Theorem 3.2, we can choose $c : S^{n-1} \to X$ so that c does not induce a monomorphism in $K(\quad;\mathbb{Z}_k)$-homology and such that the map $c \times \mathrm{id} : \mathbb{R}^n \to cX$ is a coarse isometry, where we are using polar coordinates to think of $\mathbb{R}^n$ as the cone on S^{n-1}. In this notation, "$c \times \mathrm{id}$" refers to a map which preserves levels in the cone structure and which is equal to c on each level.

We need to see that Z is not hypereuclidean. The next lemma should be comforting to readers who find themselves wondering about the "degree" of a map which is not required to be continuous.

LEMMA 4.4. *If Z is any metric space and $f : Z \to \mathbb{R}^n$ (with the euclidean metric) is coarse Lipschitz, then there is a continuous map $\bar{f} : Z \to \mathbb{R}^n$ which is boundedly close to f. If f is continuous on a closed $Y \subset Z$, then we can choose $\bar{f}|Y = f|Y$.*

Proof. Choose an open cover $\mathcal{U}$ of X by sets of diameter < 1. For each $U \in \mathcal{U}$, choose $x_U \in U$. Let $\{\phi_U\}$ be a partition of unity subordinate to $\mathcal{U}$ and let

$$\bar{f}(x) = \sum_{U \in \mathcal{U}} \phi_U(x) f(x_u).$$

By the coarse Lipschitz condition, there is a K such that

$$d(x, x') < 1 \Rightarrow d(f(x), f(x')) < K.$$

Since $d(x_U, x) < 1$ for all U with $\phi_U(x) \neq 0$, $\bar{f}(x) \in B_K(f(x))$, so $d(f, \bar{f}) < K$. □

Continuing with the proof of Theorem A, let $f' : Z \to \mathbb{R}^n$ be a coarsely proper coarse Lipschitz map. Since Z is coarsely isomorphic to cX, there is a coarse Lipschitz map $f : cX \to \mathbb{R}^n$. By the above, we may assume that f is continuous.

Since f is coarsely proper, $f^{-1}(B)$ is a compact subset of cX, where B is the unit ball in $\mathbb{R}^n$. Choose T so large that

$$(X \times [T, \infty)) \cap f^{-1}(B) = \emptyset.$$

Now consider the composition

$$S^{n-1} \xrightarrow{(c\times \mathrm{id})|} X \times \{T\} \longrightarrow (\mathbb{R}^n - B) \xrightarrow{\frac{x}{|x|}} S^{n-1} = \partial B.$$

This composition is degree one because the original coarse Lipschitz map $W \to \mathbb{R}^n$ was degree one. This is a contradiction, since degree one maps of spheres are homotopy equivalences and the first map $S^{n-1} \to X \times \{T\}$ has kernel in $\widetilde{KO}(\ \ ;\mathbb{Z}_k)$-homology. We conclude that W is not mod k (or integrally) hypereuclidean. □

Proposition 4.5. *If $f : S^n \to X$ is a cell-like map, there is a map $g : X \to S^n$ such that the composition $g \circ f$ has nonzero degree.*

Proof. By [10], there exist a finite polyhedron Q and a map $p : X \to Q$ so that the composition $p \circ f : S^n \to Q$ is $(2n+3)$-connected. Since Q is a rational homology sphere, there is an essential map $q : Q \to K(\mathbb{Q}, n)$. If n is odd, this finishes the proof, since $K(\mathbb{Q}, n)$ is a telescope of maps between spheres and compactness implies that the image of X lies in a finite subtelescope, which is homotopy equivalent to S^n.

If $n = 2k$ is even, there is a fibration sequence $T \to K(\mathbb{Q}, n) \to K(\mathbb{Q}, 2n)$, where T is a rational sphere (and therefore a telescope) and the map $K(\mathbb{Q}, n) \to K(\mathbb{Q}, 2n)$ is induced by squaring in cohomology. Since the square of the generator of $H^n(Q)$ is zero, the essential map $Q \to K(\mathbb{Q}, n)$ lifts to an essential map $Q \to T$ and the argument from the odd-dimensional case completes the proof. □

Theorem 4.6. *Let $f : S^n \to X$ be a cell-like map and let $\mathbb{R}^{n+1}_\Phi$ be diffeomorphic to $\mathbb{R}^{n+1}$ with a Riemannian metric quasi-isometric to cX, where c is a weight function tending to infinity. Then $\mathbb{R}^{n+1}_\Phi$ is rationally hypereuclidean.*

Proof. We will denote the Higson-Roe compactification of a proper metric space Y by $\overline{Y}$ and we will let $\nu(Y)$ denote the remainder, $\overline{Y} - Y$, which is called the Higson corona. For general results about the Higson-Roe compactification, see chapter 5 of [20]. By results of Roe as modified in Lemma 3.4 of [7], it suffices to produce a map $\nu(\mathbb{R}^{n+1}_\Phi) \to S^n$ which has nonzero degree in the sense that the composite

$$H^n(S^n; \mathbb{Q}) \to H^n(\nu\mathbb{R}^{n+1}_\Phi); \mathbb{Q}) \to H^{n+1}(\overline{\mathbb{R}^{n+1}_\Phi}, \nu(\mathbb{R}^{n+1}_\Phi); \mathbb{Q})$$

is nonzero. The Higson corona is a coarse invariant, so the map $cf : \mathbb{R}^{n+1}_\Phi \to cX$ induced by f extends to a map

$$(\overline{\mathbb{R}^{n+1}_\Phi}, \nu(\mathbb{R}^{n+1}_\Phi)) \to (\overline{cX}, \nu(cX))$$

which is a homeomorphism on the Higson coronas. This gives us a diagram

$$\begin{array}{ccccc} H^n(\overline{cX};\mathbb{Q}) & \longrightarrow & H^n(\nu(cX);\mathbb{Q}) & \longrightarrow & H^{n+1}(\overline{cX},\nu(cX);\mathbb{Q}) \\ \downarrow{\scriptstyle cf^*} & & \text{homeo}\downarrow{\scriptstyle cf^*} & & \cong\downarrow{\scriptstyle cf^*} \\ H^n(\overline{\mathbb{R}^{n+1}_\Phi};\mathbb{Q}) & \longrightarrow & H^n(\nu(\mathbb{R}^{n+1}_\Phi);\mathbb{Q}) & \longrightarrow & H^{n+1}(\overline{\mathbb{R}^{n+1}_\Phi}),\nu(\mathbb{R}^{n+1}_\Phi);\mathbb{Q}) \end{array}$$

in which the rightmost vertical map is an isomorphism by the Vietoris-Begle theorem and the leftmost vertical map is an isomorphism by the five lemma. Let the map $g : X \to S^n$ be constructed as in Proposition 4.5. The diagram shows that it suffices to find a map $\nu(cX) \to X$ so that the composition

$$H^n(S^n;\mathbb{Q}) \xrightarrow{g^*} H^n(X,\mathbb{Q}) \longrightarrow H^n(\nu(cX);\mathbb{Q}) \longrightarrow H^{n+1}(\overline{cX},\nu(cX);\mathbb{Q})$$

is nonzero.

By the universal property of the Higson-Roe compactification, there is a map of pairs $(\overline{cX},\nu(cX)) \to (\text{Cone}(X), X)$, where $\text{Cone}(X)$ is the closed cone on X. Compare with example 5.28 of [20]. As in that example, elementary algebraic topology gives us a diagram

$$\begin{array}{ccccc} H^n(S^n;\mathbb{Q}) & \xrightarrow{g^*} & H^n(X;\mathbb{Q}) & \xrightarrow{\cong} & H^{n+1}(\text{Cone}(X),X;\mathbb{Q}) \\ & & \downarrow & & \downarrow{\scriptstyle\cong} \\ H^n(\overline{cX};\mathbb{Q}) & \longrightarrow & H^n(\nu(cX);\mathbb{Q}) & \longrightarrow & H^n(\overline{cX},\nu(cX);\mathbb{Q}). \end{array}$$

Since $f^* \circ g^* \neq 0$, $g^*(1) \neq 0$ in $H^n(X;\mathbb{Q})$, so the proof is complete. ∎

5. The proof of Theorem B

In this section, we will exploit properties of a CE map $f : S^{k-1} \to X$ which does not induce a *surjection* on periodic $KO[\frac{1}{2}]$-homology. The manifold $Z \cong \mathbb{R}^k$ will be constructed as above to be coarsely equivalent to cX.

We will produce Z' coarsely equivalent to Z by using the bounded version of the Sullivan-Wall surgery exact sequence [24] which is established in [12]. A *structure* on a closed manifold M is a pair (N, f) where $f : N \to M$ is a simple homotopy equivalence. Two structures (N, f) and (N', f') are *equivalent* if there is a homeomorphism $\phi : N \to N'$ so that $f' \circ \phi \sim f$. For $n \geq 5$, the classical surgery exact sequence studies $\mathcal{S}(M)$, the collection of equivalence classes of structures on M. A functorial version of the sequence in the topological category is

$$\begin{aligned} \ldots \longrightarrow H_{n+1}(M;\mathbb{L}(e)) \longrightarrow L_{n+1}(\mathbb{Z}\pi) \longrightarrow \mathcal{S}^{\text{Top}}(M^n) \\ \longrightarrow H_n(M;\mathbb{L}(e)) \longrightarrow L_n(\mathbb{Z}\pi) \end{aligned}$$

where π is the fundamental group of M, $\mathbb{L}(e)$ is the 4-periodic surgery spectrum, which is isomorphic to BO away from 2, and the 4-periodic groups $L_*(\mathbb{Z}\pi)$ are Wall's surgery obstruction groups ([24]). The structure set in this functorial version of the surgery sequence is bigger by a $\mathbb{Z}$ or less than the geometric structure set described above. As shown in [2], the structure set in this stabilized surgery sequence corresponds geometrically to a structure set which contains certain nonmanifolds.

For manifolds bounded over a space X, there is a similar sequence with the L-group replaced by a bounded L-group. (In full generality, one has to also take into account the fundamental group of M over X. In this paper, though, we will always be dealing with bounded surgery which is "simply connected" in the fiber direction.) The appropriate bounded Wall groups were described in [12]. Here is a piece of the bounded surgery exact sequence from [12].

$$0 = H^{\ell f}_{k+1}(Z;\mathbb{L}(e)) \longrightarrow L^{bdd}_{k+1,\, cX}(e) \longrightarrow \mathcal{S}^{bdd}\begin{pmatrix} Z \\ \downarrow \\ cX \end{pmatrix}$$

It follows immediately from this sequence that $\mathcal{S}^{bdd}\begin{pmatrix} Z \\ \downarrow \\ cX \end{pmatrix}$ is nonzero if $L^{bdd}_{k+1,\, cX}(e)$ is nonzero. Such a structure gives us the desired manifold Z' and a bounded homotopy equivalence $Z' \to Z$ which is not boundedly homotopic over cX to a homeomorphism. The structures arising from this construction are manifolds because they come from our original manifold via Wall realization.

Proposition 5.1. *For $k \geq 11$ and an appropriate choice of X, $L^{bdd}_{k+1,\, cX}(e)$ is not* 0.

Proof. Let $1 \in \widetilde{KO}_r(S^r) \cong \mathbb{Z}$ be a generator, $r \geq 7$, and let 1 also denote the corresponding generator of $\widetilde{KO}_r(S^r;\mathbb{Z}_p)$, with p odd. As in Section 3, we can construct a cell-like map $f : S^r \to Z$ so that $1 \in \ker(f_* : \widetilde{KO}_r(S^r;\mathbb{Z}_p) \to \widetilde{KO}_r(Z;\mathbb{Z}_p))$. By Proposition 4.5, there is a map $g : Z \to S^r$ so that $(g \circ f)_*$ has degree $\ell \neq 0$. We have a commuting diagram:

$$\begin{array}{ccccccc} \longrightarrow & \widetilde{KO}_r(S^r) & \xrightarrow{f_*} & \widetilde{KO}_r(Z) & \longrightarrow & \widetilde{KO}_r(C_f) & \longrightarrow \\ & \downarrow\cong & \searrow^{\times\ell} & \downarrow & & \downarrow & \\ \longrightarrow & \widetilde{KO}_r(S^r) & \xrightarrow{(g\circ f)_*} & \widetilde{KO}_r(S^r) & \longrightarrow & \widetilde{KO}_r(C_{g\circ f}) & \longrightarrow \end{array}\quad .$$

By the condition on $\widetilde{KO}_r(S^r;\mathbb{Z}_p)$, $f_*(1) = p\alpha$ for some $\alpha \in \widetilde{KO}_r(Z)$. Here, 1 is the generator of $\widetilde{KO}_r(S^r) \cong \mathbb{Z}$. Since $(g \circ f)_*(1) = \ell \cdot 1$, we have $g_*(\alpha) \neq 0$. Moreover, α projects to an odd torsion element $[\alpha]$ in $\widetilde{KO}_r(C_f)$. The image of $[\alpha]$ in $\widetilde{KO}_r(C_{g\circ f})$ is nontrivial – the image of $p\alpha$ is ℓ times the generator of $\widetilde{KO}_r(S^r) \cong \mathbb{Z}$ and the image of α is therefore $\ell/q \cdot 1$, which is not in the image of the previous term in the lower exact sequence.

Next, we consider $S^r \subset S^{k-1}$ and form a cell-like map $\bar{f} : S^k \to X = S^{k-1} \cup_f Z$. Let $q : S^n \to Q = S^{k-1} \cup_{g\circ\bar{f}} S^r$. We have a similar-looking diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & \widetilde{KO}_r(S^{k-1}) & \xrightarrow{\bar{f}_*} & \widetilde{KO}_r(X) & \longrightarrow & \widetilde{KO}_r(C_{\bar{f}}) & \longrightarrow & \widetilde{KO}_{r-1}(S^{k-1}) & \longrightarrow \\
& \downarrow\cong & \searrow & \downarrow & & \downarrow & & \downarrow\cong & \\
\longrightarrow & \widetilde{KO}_r(S^{k-1}) & \xrightarrow{q_*} & \widetilde{KO}_r(Q) & \longrightarrow & \widetilde{KO}_r(C_q) & \longrightarrow & \widetilde{KO}_{r-1}(S^{k-1}) & \longrightarrow
\end{array} .$$

The inclusion maps $C_f \hookrightarrow C_{\bar{f}}$ and $C_{g\circ f} \hookrightarrow C_{g\circ\bar{f}} = C_q$ induce isomorphisms on $\widetilde{KO}_*$-homology, so there is an odd torsion element in $\widetilde{KO}_r(C_{\bar{f}})$ which maps to a nontrivial odd torsion element of $\widetilde{KO}_r(C_q)$.

$$\begin{array}{ccccccccc}
\longrightarrow & \widetilde{KO}_r(S^{k-1}) & \xrightarrow{\bar{f}_*} & \widetilde{KO}_r(X) & \longrightarrow & \widetilde{KO}_r(C_{\bar{f}}) & \longrightarrow & \widetilde{KO}_{r-1}(S^{k-1}) & \longrightarrow \\
& \downarrow\cong & \searrow & \downarrow & & \downarrow & & \downarrow\cong & \\
\longrightarrow & \widetilde{KO}_r(S^{k-1}) & \xrightarrow{q_*} & \widetilde{KO}_r(Q) & \longrightarrow & \widetilde{KO}_r(C_q) & \longrightarrow & \widetilde{KO}_{r-1}(S^{k-1}) & \longrightarrow
\end{array} .$$

Now, set $k = r + 4$. This is what forces us to take $k \geq 11$. Away from 2, $\widetilde{KO}_*$ is 4-periodic, so $\widetilde{KO}_r(Y) \cong \widetilde{KO}_k(Y)$ for any space Y. This gives us a diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & \widetilde{KO}_k(S^{k-1}) & \xrightarrow{\bar{f}_*} & \widetilde{KO}_k(X) & \longrightarrow & \widetilde{KO}_k(C_{\bar{f}}) & \longrightarrow & \widetilde{KO}_{k-1}(S^{k-1}) & \longrightarrow \\
& \downarrow\cong & \searrow & \downarrow & & \downarrow & & \downarrow\cong & \\
\longrightarrow & \widetilde{KO}_k(S^{k-1}) & \xrightarrow{q_*} & \widetilde{KO}_k(Q) & \longrightarrow & \widetilde{KO}_k(C_q) & \longrightarrow & \widetilde{KO}_{k-1}(S^{k-1}) & \longrightarrow
\end{array} .$$

Since $\widetilde{KO}_k(S^{k-1}) = 0$ and $\widetilde{KO}_{k-1}(S^{k-1}) \cong \mathbb{Z}$, it follows that the induced homomorphism $\widetilde{KO}_k(X) \to \widetilde{KO}_k(Q)$ is nontrivial at p.

We have a commuting diagram of assembly maps

$$\begin{array}{ccc}
H^{\ell f}_{k+1}(cX; \mathbb{L}(e)) & \longrightarrow & L^{bdd}_{k+1,cX}(e) \\
\downarrow & & \downarrow \\
H^{\ell f}_{k+1}(cQ; \mathbb{L}(e)) & \xrightarrow{\cong} & L^{bdd}_{k+1,cQ}(e)
\end{array}$$

which shows that $L^{bdd}_{k+1,cX}(e)$ is nontrivial, where the isomorphism on the bottom follows from [12]. (Strictly speaking, we should be working with standard cones and coarse Lipschitz maps to cite [12], but one can use coarse homotopies as in [21] to extend the theory to warped cones and arbitrary continuous maps.) $\square$

6. The proof of Theorem C

Choose a 2-connected $(n-1)$-manifold M^n, $n \geq 7$, and a CE map $\rho : M \xrightarrow{\text{CE}} X$ which is not injective on K-homology. Such maps were constructed in Section 3. Construct a uniformly n-connected weighted cone cX as above and let $\bar{M}$ be a uniformly contractible singular Riemannian manifold coarsely isomorphic to cX. The assembly map

$$K_*^{lf}(\bar{M}) \to K_*(C^*(\bar{M}))$$

factors through[2] a natural map

$$K_*^{lf}(\bar{M}) \to KX_*(\bar{M}) \cong KX_*(cX).$$

By the main theorem of Section 7, $KX_*(cX) \cong K_{*-1}(X)$, so the assembly map $K_*^{\ell f}(\bar{M}) \to KX_*(\bar{M})$ has kernel. □

Remark 6.1. In particular, Conjecture 6.28 of [20] is incorrect. The map c does induce rational isomorphisms, so the rational version of the conjecture is still open.

The same procedure shows that both injectivity and surjectivity assertions in a bounded analog (for uniformly contractible spaces) of the Generalized Borel Conjecture of Ferry-Rosenberg-Weinberger [13] are false. A regular neighborhood of a suitable suspension of a Moore space provides a manifold with boundary which has odd torsion in its $\mathbb{L}(e)$-homology. This can be killed by a CE map as in Section 3 and the rest of the construction proceeds as above. We could get a counterexample to the integral isomorphism conjecture on a manifold diffeomorphic to euclidean space by performing our construction starting with a CE map $\rho : S^n \to X$ where the induced map on K_* was multiplication by k. Such a CE map would be the result of applying the procedure of Section 3 to a kill the mod k reduction of an integral class in $K_*(S^n)$. Using real, rather than complex K-theory, we could get counterexamples of the same sort to the analogous injectivity conjecture.

Remark 6.2. All of our examples are based on the difference between K^{lf} and KX. Consequently, if one is careful to assert all conjectures for general metric spaces in terms of KX rather than K^{lf}, one obtains statements which are not contradicted by these examples. In case the manifold Z has bounded geometry — in particular, if Z is the universal cover of closed manifold — it is not difficult to show that $K_*^{\ell f}(Z) \to KX_*(Z)$ is a monomorphism, so no contradiction to Conjecture 6.28 arises from our construction in that case.

[2]See [18].

7. KX_* of weighted open cones

John Roe [20] has introduced the following notion of *coarse homology*:

Definition 7.1. If X is a complete locally compact metric space, a sequence $\{\mathcal{U}_i\}$ of locally finite covers of X by relatively compact open sets is called an *Anti-Čech system* if there are numbers $R_i \to \infty$ such that

(i) $\operatorname{diam}(U) < R_i$ for all $U \in \mathcal{U}_i$.

(ii) R_i is a Lebesgue number for $\mathcal{U}_i$.

The *coarse homology of X with coefficients in $\mathcal{S}$* is

$$HX_*(X;\mathcal{S}) = \varinjlim H_*^{lf}(N(\mathcal{U}_i);\mathcal{S}),$$

where $N(\mathcal{U}_i)$ is the nerve of the open cover $\mathcal{U}_i$ and $H_*^{lf}(P;\mathcal{S})$ is the Steenrod $\mathcal{S}$-homology of the 1-point compactification of P, rel infinity. $\mathcal{S}$, of course, is a spectrum.

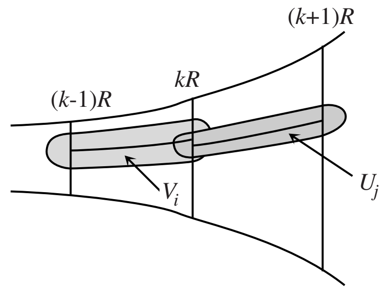
It is not difficult to construct anti-Čech systems of covers, at least when X is a complete locally compact metric space. For some $\epsilon > 0$, choose a maximal collection of disjoint open ϵ balls in X and consider the collection of R-balls on the same centers. For $R > 2\epsilon$, this is a cover with Lebesgue number at least $R - 2\epsilon$. If $\{R_i\}$ is any monotone sequence approaching infinity, this allows us to construct a sequence of coarse covers $\{\mathcal{U}_i\}$ with diameters $< R_i$. Any anti-Čech system is cofinal, so HX_* is well-defined.

An interesting question in metric topology is to find conditions under which $HX_*(X;\mathcal{S})$ is equal to the $\mathcal{S}$-homology of X at infinity. In such cases HX_* is a topological invariant, rather than a metric invariant. The usual sort of nerve argument gives a proper map $X \to N(\mathcal{U}_i)$ for each i and therefore a map $H_*^{lf}(X;\mathcal{S}) \to HX_*(X;\mathcal{S})$. Even when X is uniformly contractible, the results of this paper show that this map need be neither an integral monomorphism nor an integral epimorphism. We do, however, have the following:

THEOREM 7.2. *If X is compact metric and cX is a weighted cone on X, then $H_*^{lf}(cX;\mathcal{S}) \to HX_*(cX;\mathcal{S})$ is an isomorphism for any spectrum $\mathcal{S}$.*

The analogous result has been proven by Higson and/or Roe for many uniformly contractible spaces. We will give a proof for weighted cones which includes the infinite-dimensional case.

Proof. Let R be given and consider levels kR in the weighted cone cX. Pack each of these levels with ϵ_k-balls on centers c_{ki} as above, ϵ_k small, and draw radial arcs from each c_{ki} to the point below it in level $(k-1)R$. Now take $\frac{1}{3}(1+\frac{1}{k})R$-neighborhoods of the arcs.



This gives a cover $\mathcal{O}_R$ of cX by open sets of diameter $\simeq \frac{5}{3}R$ with Lebesgue number $\simeq \frac{1}{3}R$. Consider the nerve of this cover, restricting attention for the moment to two consecutive levels as in the picture above. We will call the open cover corresponding to points in the $(k+1)R$-level $\mathcal{U}$ and the cover corresponding to points at the kR-level $\mathcal{V}$.

For $U \in \mathcal{U}$ and $V \in \mathcal{V}$, we will let U_- and V_- denote the intersections of U and V with the part of cX between the cone point and the kR-level.

Lemma 7.3. *If* $U_1 \cap \ldots \cap U_k \cap V_1 \cap \ldots \cap V_\ell \neq \emptyset$, *then* $U_{1-} \cap \ldots \cap U_{k-} \cap V_{1-} \cap \ldots \cap V_{\ell-} \neq \emptyset$.

Proof. This is clear, since if a point x above the kR-level is in the first intersection, then following the cone lines back to the kR-level gives a point in the second intersection. □

For $\{\epsilon_k\}$ small, there is a map $\phi : \mathcal{U} \to \mathcal{V}$ so that

$$(*) \qquad U_- \subset \phi(U)_- \text{ for all } U \in \mathcal{U}$$

which is defined by choosing $\phi(U)$ to be an element $V \in \mathcal{V}$ so that $U \cap kR \subset V \cap kR$. Define a map $\rho : \mathcal{N}(\mathcal{U} \cup \mathcal{V}) \to \mathcal{N}(\mathcal{V})$ by $\rho(V) = V$ and $\rho(U) = \phi(U)$.

Proposition 7.4. *The map ρ defines a simplicial strong deformation retraction.*

Proof. To see that ρ is simplicial, suppose that we have a simplex

$$\langle U_1, \ldots, U_n, V_1, \ldots, V_k \rangle \in \mathcal{N}(\mathcal{U} \cup \mathcal{V}).$$

By definition,

$$U_1 \cap \ldots \cap U_n \cap V_1 \cap \ldots \cap V_k \neq \emptyset,$$

so

$$U_{1-} \cap \ldots \cap U_{n-} \cap V_{1-} \cap \ldots \cap V_{k-} \neq \emptyset;$$

whence $(*)$ guarantees that

$$\phi(U_1) \cap \ldots \cap \phi(U_n) \cap V_1 \cap \ldots \cap V_k \neq \emptyset,$$

so ρ defines a simplicial map.

To see that ρ is a strong deformation retraction, we begin by noting that $\mathcal{N}(\mathcal{U} \cup \mathcal{V}) = \mathcal{N}(\mathcal{U}_- \cup \mathcal{V}_-)$, where $\mathcal{U}_- = \{U_- \mid U \in \mathcal{U}\}$ and $\mathcal{V}_- = \{V_- \mid V \in \mathcal{V}\}$. Since $U_- \subset \phi(U)_-$, the situation is not difficult to analyze.

LEMMA 7.5. *If $\mathcal{V}$ is a finite open cover of a space X and $U \subset X$ is an open set such that $U \subset V$ for some $V \in \mathcal{V}$, then $\mathcal{N}(\mathcal{V})$ is a strong deformation retract of $\mathcal{N}(\mathcal{V} \cup \{U\})$.*

Proof. Consider the link of $\langle U \rangle$ in $\mathcal{N}(\mathcal{V} \cup \{U\})$. This consists of simplices $\langle V_0, \ldots, V_n \rangle$ such that

$$U \cap \left(\bigcap_{i=0}^{n} V_i \right) \neq \emptyset.$$

If V is a particular element of $\mathcal{V}$ containing U, this means that

$$V \cap \left(\bigcap_{i=0}^{n} V_i \right) \neq \emptyset,$$

which means that the link is contractible (even collapsible!), since it is a cone from $\langle V \rangle$. But then $\mathcal{N}(\mathcal{V} \cup \{U\})$ collapses to $\mathcal{N}(\mathcal{V})$, since the cone on a collapsible complex collapses to its base.This completes the proof of the lemma. □

This collapse sends $\langle U \rangle$ to $\langle V \rangle$, so removing elements of $\mathcal{U}_-$ one at a time gives a collapse from $\mathcal{N}(\mathcal{U}_- \cup \mathcal{V}_-)$ to $\mathcal{N}(\mathcal{U}_-)$, completing the proof that ρ is a strong deformation retraction.

We will need the following well-known lemma.

LEMMA 7.6. *If $f : (X, X_0) \to (Y, Y_0)$ is a map of* CW *pairs so that f and $f|X_0$ are homotopy equivalences, then f is a homotopy equivalence of pairs.*

Proof. Form the mapping cylinder and construct strong deformation retraction of pairs by retracting the smaller mapping cylinder first and then retracting the larger one. □

LEMMA 7.7. *$\mathcal{N}(\mathcal{U} \cup \mathcal{V})$ is homotopy equivalent* rel $\mathcal{N}(\mathcal{U}) \cup \mathcal{N}(\mathcal{V})$ *to the mapping cylinder of ρ.*

Proof. Let $M(\rho)$ be the mapping cylinder of ρ. We have an "identity" map $\mathcal{N}(\mathcal{U}) \cup \mathcal{N}(\mathcal{V}) \to \mathcal{N}(\mathcal{U} \cup \mathcal{V})$. By the lemma above, it suffices to show that this map extends to all of $M(\rho)$. But this map is homotopic to the map which is the identity on $\mathcal{N}(\mathcal{V})$ and which sends $\mathcal{N}(\mathcal{U})$ to $\mathcal{N}(\mathcal{V})$ via ρ. Collapsing $M(\rho)$ to its base and including into $\mathcal{N}(\mathcal{U} \cup \mathcal{V})$ give an extension of this map to all of $M(\rho)$, so by the homotopy extension theorem, the original map extends to all of $M(\rho)$. □

It follows from all of this that the nerve of the cover $\mathcal{O}_R$ constructed above is proper homotopy equivalent to the mapping telescope of the nerves of the $\mathcal{U}_k$ which correspond to arcs connecting levels kR and $(k+1)R$ in cX. We will show that for all R, this telescope is proper homotopy equivalent to the complement of X in the Hilbert cube. It follows that for any locally finite Steenrod homology theory, the locally finite homology of the telescope is equal to the homology of X with a dimension shift, as desired.

LEMMA 7.8. *If X is a compact metric space, and $\bar{\mathcal{U}}_i$ is a sequence of open covers of X such that $\bar{\mathcal{U}}_0 = \{X\}$, $\bar{\mathcal{U}}_{i+1}$ refines $\bar{\mathcal{U}}_i$, and* $\mathrm{mesh}(\bar{\mathcal{U}}_i) \to 0$, *then the mapping telescope of the nerves $\mathcal{N}(\bar{\mathcal{U}}_i)$ is proper homotopy equivalent to $Q - X$, where X is embedded in the Hilbert cube Q as a Z-set.*

Proof. This is a form of Chapman's Complement Theorem [22], which says that the homeomorphism type of the complement of a Z-embedded compactum X in the Hilbert cube depends only on the shape of X. The point is that the mapping telescope can be completed to a contractible ANR by adding a copy of the inverse limit of the $\mathcal{N}(\bar{\mathcal{U}}_i)$'s at infinity. Crossing with Q gives a copy of Q containing a Z-set X' which is shape equivalent to X. The complement of X' is the product of the telescope with Q.

One should be careful here, since a little bit of thought gives examples where X is the unit interval and X' is the Hilbert cube. The argument of [22] shows that if $\{K_i, \alpha_i\}$ is an inverse system with $K_0 = pt$, then the mapping telescope of $\{K_i, \alpha_i\}$ is proper homotopy equivalent (even infinite simple equivalent!) to the mapping telescope of any system $\{L_i, \beta_i\}$ equivalent to $\{K_i, \alpha_i\}$ in pro-homotopy. If $X = \varinjlim\{K_i, \alpha_i\}$, it is easy to construct a sequence of covers $\mathcal{U}_i$ of X so that $\mathcal{N}(\mathcal{U}_i)$ is PL homeomorphic to K_i and so that the maps induced by refinement are homotopic to the α_i's. Since all such sequences are easily seen to be pro-equivalent, Lemma 7.8 follows. □

Finally, we note that the sequence of nerves $\mathcal{N}(\mathcal{U}_k)$ and bonding maps above is cofinal with a sequence as in the statement of Lemma 7.8. This completes the proof of Theorem 7.2.

RUTGERS UNIVERSITY, PISCATAWAY, NJ
E-mail address: sferry@math.rutgers.edu

UNIVERSITY OF FLORIDA, GAINESVILLE, FL
E-mail address: dranishn@ufl.edu

THE UNIVERSITY OF CHICAGO, CHICAGO, IL
E-mail address: shmuel@math.uchicago.edu

References

[1] R. H. Bing, A convex metric with unique segments, *Proc. Amer. Math. Soc.* **4** (1953), 167–174.

[2] J. Bryant, S. Ferry, W. Mio, and S. Weinberger, Topology of homology manifolds, *Ann. of Math.* **143** (1996), 435–467.

[3] G. Carlsson and E. K. Pedersen, Controlled algebra and the Novikov conjectures for K- and L-theory, *Topology* **34** (1995), 731–758.

[4] A. Connes, M. Gromov, and H. Moscovici, Group cohomology with Lipschitz control and higher signatures, *Geom. Funct. Anal.* **3** (1993), 1–78.

[5] A. N. Dranishnikov, Homological dimension theory, *Uspekhi Mat. Nauk* **43** (1988), 11–55, 255.

[6] ———, On a problem of P. S. Aleksandrov, *Mat. Sb.* (N.S.) **135** (1988), 551–557, 560.

[7] A. N. Dranishnikov and S. Ferry, On the Higson-Roe corona, *Uspekhi Mat. Nauk* **52** (1997), 133–146.

[8] A. N. Dranishnikov, J. Keesling, and V. V. Uspenskij, On the Higson corona of uniformly contractible spaces, *Topology* **37** (1998), 791–803.

[9] J. Dydak, Realizing dimension functions via homology, *Topology Appl.* **65** (1995), 1–7.

[10] S. Ferry, A stable converse to the Vietoris-Smale theorem with applications to shape theory, *Trans. Amer. Math. Soc.*. **261** (1980), 369–386.

[11] S. C. Ferry and B. L. Okun, Approximating topological metrics by Riemannian metrics, *Proc. Amer. Math. Soc.* **123** (1995), 1865–1872.

[12] S. C. Ferry and E. K. Pedersen, Epsilon surgery theory, in *Novikov Conjectures, Index Theorems and Rigidity*, Vol. 2 (Oberwolfach, 1993), Cambridge Univ. Press, Cambridge, 1995, pp. 167–226.

[13] S. Ferry, J. Rosenberg, and S. Weinberger, Phénomènes de rigidité topologique équivariante, *C. R. Acad. Sci. Paris Sér. Math.* **306** (1988), 777–782.

[14] S. C. Ferry and S. Weinberger, A coarse approach to the Novikov conjecture, in *Novikov Conjectures, Index Theorems and Rigidity*, Vol. 1 (Oberwolfach, 1993), Cambridge Univ. Press, Cambridge, 1995, pp. 147–163.

[15] M. Gromov, Positive curvature, macroscopic dimension, spectral gaps and higher signatures, in *Functional Analysis on the Eve of the 21st Century*, Vol. II (New Brunswick, NJ, 1993), Birkhäuser Boston, Boston, MA, 1996, pp. 1–213.

[16] ———, Spaces and questions, *Geom. Funct. Anal. GAFA* (2000) (Special Volume, Part I) (Tel Aviv, 1999), pp. 118–161.

[17] N. Higson, V. Lafforgue, and G. Skandalis, Counterexamples to the Baum-Connes conjecture, *Geom. Funct. Anal.* **12** (2000), 330–354.

[18] N. Higson and J. Roe, On the coarse Baum-Connes conjecture, in *Novikov Conjectures, Index Theorems and Rigidity*, Vol. 2 (Oberwolfach, 1993), Cambridge Univ. Press, Cambridge, 1995, pp. 227–254.

[19] R. C. Lacher, Cell-like mappings and their generalizations, *Bull. Amer. Math. Soc.* **83** (1977), 495–552.

[20] J. Roe, *Coarse Cohomology and Index Theory on Complete Riemannian Manifolds*, *Mem. Amer. Math. Soc.* **104** (1993), A. M. S., Providence, RI.

[21] ———, Index theory, coarse geometry, and topology of manifolds, *CBMS Reg. Conf. Series in Math.* **90**, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996.

[22] L. Siebenmann, Chapman's classification of shapes: a proof using collapsing, *Manuscripta Math.* **16** (1975), 373–384.

[23] J.-L. Tu, La conjecture de Baum-Connes pour les feuilletages moyennables, *K-Theory* **17** (1999), 215–264.

[24] C. T. C. Wall, *Surgery on Compact Manifolds*, *London Math. Soc. Monographs* **1**, Academic Press, New York, 1970.

[25] J. J. Walsh, Dimension, cohomological dimension, and cell-like mappings, in *Shape Theory and Geometric Topology* (Dubrovnik, 1981), Springer-Verlag, New York, 1981, pp. 105–118.

[26] Z. Yosimura, A note on complex K-theory of infinite CW-complexes, *J. Math. Soc. Japan* **26** (1974), 289–295.

[27] G. Yu, The Novikov conjecture for groups with finite asymptotic dimension, *Ann. of Math.* **147** (1998), 325–355.

[28] ———, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, *Invent. Math.* **139** (2000), 201–240.

(Received October 18, 2001)