

The local converse theorem for $\mathrm{SO}(2n+1)$ and applications

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Abstract

In this paper we characterize irreducible generic representations of $\mathrm{SO}_{2n+1}(k)$ (where k is a p -adic field) by means of twisted local gamma factors (the *Local Converse Theorem*). As applications, we prove that two irreducible generic cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ (where \mathbb{A} is the ring of adèles of a number field) are equivalent if their local components are equivalent at almost all local places (the *Rigidity Theorem*); and prove the *Local Langlands Reciprocity Conjecture* for generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$.

1. Introduction

In the theory of admissible representations of p -adic reductive groups, one of the basic problems is to characterize an irreducible admissible representation up to isomorphism. Keeping in mind the link of the theory of admissible representations of p -adic reductive groups to the modern theory of automorphic forms, we consider in this paper the characterization of irreducible admissible representations by the local gamma factors and their twisted versions. Such a characterization is traditionally called the *Local Converse Theorem*, and is the local analogue of the (global) *Converse Theorem* for $\mathrm{GL}(n)$. We refer to [CP-S1] and [CP-S2] for detailed explanation of converse theorems.

The local converse theorem for the general linear group, $\mathrm{GL}(n)$, was first formulated by I. Piatetski-Shapiro in his unpublished Maryland notes (1976) with his idea of deducing the local converse theorem from his (global) converse

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theorem. It was first proved by G. Henniart in [Hn2] using a local approach. The local converse theorem is a basic ingredient in the recent proof of the local Langlands conjecture for $\mathrm{GL}(n)$ by M. Harris and R. Taylor [HT] and by G. Henniart [Hn3].

The formulation of the local converse theorem in this case is as follows. Let τ and τ' be irreducible admissible generic representations of $\mathrm{GL}_n(k)$, where k is a p -adic field (non-archimedean local field of characteristics zero). Following [JP-SS], one defines the twisted local gamma factors $\gamma(\tau \times \varrho, s, \psi)$ and $\gamma(\tau' \times \varrho, s, \psi)$, where ϱ is an irreducible admissible generic representation of $\mathrm{GL}_l(k)$ and ψ is a given nontrivial additive character of k .

THEOREM 1.1 (Henniart, [Hn2]). *Let τ and τ' be irreducible admissible generic representations of $\mathrm{GL}_n(k)$ with the same central character. If the twisted local gamma factors are the same, i.e.*

$$\gamma(\tau \times \varrho, s, \psi) = \gamma(\tau' \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ with $l = 1, 2, \dots, n - 1$, then the representation τ is isomorphic to the representation τ' .

This theorem has been refined by J. Chen in [Ch] (unpublished) so that the twisting condition on l reduces from $n - 1$ to $n - 2$ (using a local approach) and by J. Cogdell and I. Piatetski-Shapiro in [CP-S1] (using a global approach and assuming both τ and τ' are supercuspidal). It is expected (as a conjecture of H. Jacquet, §8 in [CP-S1]) that the twisting condition on l should be reduced from $n - 1$ to $\lfloor \frac{n}{2} \rfloor$. We note also that the local converse theorem for generic representations of $\mathrm{U}(2, 1)$ and for $\mathrm{GSp}(4)$ was established by E. M. Baruch in [B1] and [B2].

The objective of this paper is to prove the local converse theorem for irreducible admissible generic representations of $\mathrm{SO}_{2n+1}(k)$.

THEOREM 1.2 (The Local Converse Theorem). *Let σ and σ' be irreducible admissible generic representations of $\mathrm{SO}_{2n+1}(k)$. If the twisted local gamma factors $\gamma(\sigma \times \varrho, s, \psi)$ and $\gamma(\sigma' \times \varrho, s, \psi)$ are the same, i.e.*

$$\gamma(\sigma \times \varrho, s, \psi) = \gamma(\sigma' \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ with $l = 1, 2, \dots, 2n - 1$, then the representations σ and σ' are isomorphic.

Note that the twisted local gamma factors used here are the ones studied either by F. Shahidi in [Sh1] and [Sh2] or by D. Soudry in [S1] and [S2]. It was proved by Soudry that the twisted local gamma factors defined by these two different methods are in fact the same. It is expected that the local converse theorem (Theorem 1.2) should be refined so that it is enough to twist the local gamma factors in Theorem 1.2 by the irreducible supercuspidal

representations ρ of $GL_l(k)$ for $l = 1, 2, \dots, n$. This is compatible with the conjecture of Jacquet as mentioned above. In a forthcoming paper of the authors, we shall prove the finite field analogue of Jacquet’s conjecture and provide strong evidence for the refined local converse theorem.

The local converse theorem for $SO(2n + 1)$ has many significant applications to both the local and global theory of representations of $SO(2n + 1)$. For the global theory, we can prove that the weak Langlands functorial lift from irreducible generic cuspidal automorphic representations of $SO(2n + 1)$ to irreducible automorphic representations of $GL(2n)$ is injective up to isomorphism (Theorem 5.2) (The weak Langlands functorial lift in this case was recently established in [CKP-SS].); that the image of the backward lift from irreducible generic self-dual automorphic representations of $GL(2n)$ to $SO(2n + 1)$ is irreducible, which was conjectured in [GRS1] (The details of this application will be given in [GRS5].); and that the *Rigidity Theorem* holds for irreducible generic cuspidal automorphic representations of $SO(2n + 1)$ (Theorem 5.3).

Two important applications of the local converse theorem to the theory of admissible representations of $SO_{2n+1}(k)$ are included in this paper. The first one is the explicit local Langlands functorial lifting taking irreducible generic supercuspidal representations of $SO_{2n+1}(k)$ to $GL_{2n}(k)$ (Theorem 6.1). Since the Langlands dual group of $SO_{2n+1}(k)$ is $Sp_{2n}(\mathbb{C})$, the Langlands functorial lift conjecture asserts that the natural embedding of $Sp_{2n}(\mathbb{C})$ into $GL_{2n}(\mathbb{C})$ yields a lift of irreducible admissible representations of $SO_{2n+1}(k)$ to $GL_{2n}(k)$. Let $\mathcal{GL}_{2n}^{\text{ifl}}(k)$ (‘ifl’ denotes the image of the functorial lifting) be the set of all equivalence classes of irreducible admissible generic representations of $GL_{2n}(k)$ of the form

$$\tau = \eta_1 \times \eta_2 \times \dots \times \eta_t,$$

where η_i are irreducible unitary supercuspidal self-dual representations of $GL_{2n_j}(k)$ with $j = 1, 2, \dots, t$ and $\sum_{j=1}^t n_j = n$, such that

- (1) $\eta_i \not\cong \eta_j$ if $i \neq j$, and
- (2) the local L -function $L(\eta_j, \Lambda^2, s)$ has a pole at $s = 0$ for $j = 1, 2, \dots, t$.

We denote by $\mathcal{SO}_{2n+1}^{\text{igsc}}(k)$ the set of all equivalence classes of irreducible generic supercuspidal representations of $SO_{2n+1}(k)$. We prove the local Langlands functorial conjecture for $\mathcal{SO}_{2n+1}^{\text{igsc}}(k)$ in this paper.

THEOREM 1.3. *There exists a unique bijective map*

$$\ell : \sigma \mapsto \tau = \ell(\sigma)$$

from $\mathcal{SO}_{2n+1}^{\text{igsc}}(k)$ to $\mathcal{GL}_{2n}^{\text{ifl}}(k)$, which preserves the twisted local L -factors, ϵ -factors and gamma factors, i.e.

$$\begin{aligned} L(\sigma \times \varrho, s) &= L(\tau \times \varrho, s), \\ \epsilon(\sigma \times \varrho, s, \psi) &= \epsilon(\tau \times \varrho, s, \psi) \end{aligned}$$

and

$$\gamma(\sigma \times \varrho, s, \psi) = \gamma(\tau \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ with l being any positive integer.

The second application is the local Langlands reciprocity conjecture for irreducible generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$ (Theorem 6.4). Let W_k be the Weil group associated to the local field k . We take

$$W_k \times \mathrm{SL}_2(\mathbb{C})$$

as the Weil-Deligne group ([M] and [Kn]). Let $\mathcal{G}_{2n}^{\mathrm{ah}}(k)$ be the set of conjugacy classes of admissible homomorphisms ρ from $W_k \times \mathrm{SL}_2(\mathbb{C})$ to $\mathrm{Sp}_{2n}(\mathbb{C})$. If we write

$$\rho = \oplus_i \rho_i^0 \otimes \lambda_i^0,$$

then the admissibility of ρ means that ρ_i^0 's are continuous complex representations of W_k with $\rho_i^0(W_k)$ semi-simple and λ_i^0 's are algebraic complex representations of $\mathrm{SL}_2(\mathbb{C})$. The local Langlands reciprocity conjecture for $\mathrm{SO}_{2n+1}(k)$ asserts that for each local Langlands parameter ρ in $\mathcal{G}_{2n}^{\mathrm{ah}}(k)$, there is a finite set $\Pi(\rho)$ (called the local L -packet associated to ρ) of equivalence classes of irreducible admissible representations of $\mathrm{SO}_{2n+1}(k)$, such that the union $\cup_{\rho} \Pi(\rho)$ gives a partition of the set of equivalence classes of irreducible admissible representations of $\mathrm{SO}_{2n+1}(k)$ and the reciprocity map taking ρ to $\Pi(\rho)$ is compatible with various local factors attached to ρ and $\Pi(\rho)$, respectively.

Let $\mathcal{G}_{2n}^0(k)$ be the set of conjugacy classes of all $2n$ -dimensional, admissible, completely reducible, multiplicity-free, symplectic complex representations ρ^0 of the Weil group W_k . Then we prove the following theorem.

THEOREM 1.4 (Local Langlands Reciprocity Law (Theorem 6.4)). *There exists a unique bijection*

$$\mathfrak{R}_{2n} : \rho_{2n}^0 \mapsto \mathfrak{R}_{2n}(\rho_{2n}^0)$$

from the set $\mathcal{G}_{2n}^0(k)$ onto the set $\mathcal{SO}_{2n+1}^{\mathrm{igsc}}(k)$ such that

- (L) $L(\rho_{2n}^0 \otimes \rho_l^0, s) = L(\mathfrak{R}_{2n}(\rho_{2n}^0) \times \mathfrak{r}_l(\rho_l^0), s),$
- (ϵ) $\epsilon(\rho_{2n}^0 \otimes \rho_l^0, s, \psi) = \epsilon(\mathfrak{R}_{2n}(\rho_{2n}^0) \times \mathfrak{r}_l(\rho_l^0), s, \psi),$ and
- (γ) $\gamma(\rho_{2n}^0 \otimes \rho_l^0, s, \psi) = \gamma(\mathfrak{R}_{2n}(\rho_{2n}^0) \times \mathfrak{r}_l(\rho_l^0), s, \psi)$

for all irreducible continuous representations ρ_l^0 of W_k of dimension l . Here τ is the reciprocity map to $\mathrm{GL}_l(k)$, obtained by [HT], [Hn3] (see Theorem 6.2).

Note that by Theorem 1.2, each local L -packet $\Pi(\rho)$ has at most one generic member. Theorem 1.4 establishes the Langlands conjecture in this case up to the explicit construction of the relevant L -packets, which is a very interesting and difficult problem. We shall consider the local Langlands conjectures for general generic representations of $\mathrm{SO}_{2n+1}(k)$ and other related problems in a forthcoming work ([JngS]).

Our proof of the local converse theorem goes as follows. Based on the basic properties of twisted local gamma factors established by D. Soudry in [S1] and [S2] and by F. Shahidi [Sh1] and [Sh2], we study the existence of poles of twisted local gamma factors and related properties. This leads us to reduce the proof of Theorem 1.2 to the case where both σ and σ' are supercuspidal (Theorem 5.1). To prove the local converse theorem for the case of supercuspidal representations (Theorem 4.1), we must combine the local method with the global method. More precisely, we first develop the explicit local Howe duality for irreducible generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$ and $\widetilde{\mathrm{Sp}}_{2n}(k)$, the metaplectic (double) cover of $\mathrm{Sp}_{2n}(k)$ (Theorem 2.2), which is more or less the local version of the global results of M. Furusawa [F]. Then, using the global weak Langlands functorial lifting from $\mathrm{SO}(2n+1)$ to $\mathrm{GL}(2n)$ [CKP-SS] and the local backward lifting from $\mathrm{GL}_{2n}(k)$ to $\widetilde{\mathrm{Sp}}_{2n}(k)$ [GRS2] and [GRS6], we can basically relate our local converse theorem for $\mathrm{SO}(2n+1)$ to that for $\mathrm{GL}(2n)$. See the proof of Theorem 4.1 for details. The point here is to use preservation properties of twisted local gamma factors under various liftings (Propositions 3.3 and 3.4). It is worthwhile to mention here that the ideas and the methods used in this paper are applicable to other classical groups.

This paper is organized as follows. In Section 2, we work out some explicit properties of local Howe duality for irreducible generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$ and $\widetilde{\mathrm{Sp}}_{2n}(k)$. The preservation property of (the pole at $s = 1$ of) twisted local gamma factors under various liftings will be discussed in Section 3. In Section 4, we prove the local converse theorem for supercuspidal representations and in Section 5, we prove the theorem in the general case. The global applications mentioned above will be discussed at the end of Section 5. We determine in Section 6 the explicit structure of the image of the local Langlands functorial lifting from irreducible generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$ to $\mathrm{GL}_{2n}(k)$ and prove the local Langlands reciprocity law for irreducible generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$.

Since $\mathrm{SO}(3) \cong \mathrm{PGL}(2)$, the main theorems in this paper are known in the case of $n = 1$. Note that the theories of twisted local gamma factors for $\mathrm{SO}(3) \times \mathrm{GL}(r)$ via [S1,2], or via Shahidi's method, or via [JP-SS], for $\mathrm{PGL}(2) \times \mathrm{GL}(r)$ are all the same. The reason for this is the multiplicativity property of gamma factors (which is known in all cases above). This reduces comparison of gamma factors to supercuspidal representations. Such representations can be embedded as components at one place of (irreducible)

automorphic cuspidal representations, unramified at all remaining finite places. Since gamma factors are "globally 1" (this is a restatement of the functional equation for the global L function), we get the identity of the gamma factors for supercuspidal representations. From now on we assume that $n \geq 2$ (this will be helpful for one technical reason concerning the theta lifting).

Our project on this topic was started when we attended the conference on Automorphic Forms and Representations at Oberwolfach (March 2000) organized by Professors S. Kudla and J. Schwermer. The main results of this paper were obtained when we participated at the Automorphic Forms Semester at Institut Henri Poincaré (Paris, Spring 2000) organized by Professors H. Carayol, M. Harris, J. Tilouine, and M.-F. Vignéras. This paper was finished when the first named author was a member of the Institute for Advanced Study (Princeton, Fall 2000). We would like to thank all the organizers of the above two research activities and the Institutes for providing a stimulating research environment. We would like to thank D. Ginzburg and S. Rallis for their encouragement during our work on this project. Our discussion with G. Henniart was very important for the proof of Theorem 6.4. We are grateful to him for providing us the proof of Theorem 6.3 [Hn1]. We thank the referee for his careful reading, and for his valuable comments, questions and suggestions.

2. Howe duality for $\mathrm{SO}(2n+1)$ and $\widetilde{\mathrm{Sp}}(2n)$

In this section, we prove certain properties of the local Howe duality between $\mathrm{SO}_{2n+1}(k)$ and $\widetilde{\mathrm{Sp}}_{2n}(k)$, applied to irreducible, generic, supercuspidal representations, and then we discuss relevant aspects of the global theta correspondence for irreducible, automorphic, cuspidal representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ and $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. Here $\widetilde{\mathrm{Sp}}_{2n}$ denotes the metaplectic (double) cover of Sp_{2n} over both the local field k and the ring of adèles \mathbb{A} ([Mt]).

2.1. Local Howe duality. Let k be a non-archimedean local field of characteristic zero. Let V be a $(2n+1)$ -dimensional vector space over k , equipped with a nondegenerate symmetric form $(\cdot, \cdot)_V$ of Witt index n . Let W be a $2m$ -dimensional vector space over k , equipped with a nondegenerate symplectic form $(\cdot, \cdot)_W$. We fix a basis

$$\{e_1, \dots, e_n, e, e_{-n}, \dots, e_{-1}\}$$

of V over k , such that $(e_i, e_j)_V = (e_{-i}, e_{-j})_V = 0$, $(e_i, e_{-j})_V = \delta_{ij}$, for $i, j = 1, \dots, n$, and we may assume that $(e, e)_V = 1$. Thus

$$V^+ = \mathrm{Span}_k\{e_1, \dots, e_n\}, \quad V^- = \mathrm{Span}_k\{e_{-1}, \dots, e_{-n}\}$$

are dual maximal totally isotropic subspaces of V , and we get a polarization of V ,

$$V = V^+ + ke + V^- .$$

Similarly, we fix a basis

$$\{f_1, \dots, f_m, f_{-m}, \dots, f_{-1}\}$$

of W over k , such that $(f_i, f_j)_W = (f_{-i}, f_{-j})_W = 0$ and $(f_i, f_{-j})_W = \delta_{ij}$, for $i, j = 1, \dots, m$. Thus,

$$W^+ = \text{Span}_k\{f_1, \dots, f_m\} \quad , \quad W^- = \text{Span}_k\{f_{-1}, \dots, f_{-m}\}$$

are dual maximal isotropic subspaces of W , and we get the polarization

$$W = W^+ + W^- .$$

Consider the tensor product $V \otimes W$ of V and W . It is a symplectic space of dimension $2m(2n + 1)$, equipped with the symplectic form $(,)_V \otimes (,)_W$. With the chosen bases, we may identify V with k^{2n+1} (column vectors) and W with k_{2m} (row vectors). Then we have $O_{2n+1}(k) \cong O(V)$, acting from the left on V , and $Sp_{2m}(k) \cong Sp(W)$, acting from the right on W . We let $Sp(V \otimes W) \cong Sp_{2m(2n+1)}(k)$ act from the right on $V \otimes W$. Then $O(V) \times Sp(W)$ is naturally embedded in $Sp(V \otimes W)$ by means of the following action

$$(v \otimes w)(g, h) = g^{-1} \cdot v \otimes w \cdot h.$$

Let ψ be a nontrivial character of k . The Weil representation ω_ψ of the metaplectic group $\widetilde{Sp}_{2m(2n+1)}(k)$ can be realized in the space of Bruhat-Schwartz functions $\mathcal{S}(V^m)$, where $V^m = V \times \dots \times V$ (m copies). We restrict ω_ψ to the image of the natural embedding of $O_{2n+1}(k) \times \widetilde{Sp}_{2m}(k)$ inside $\widetilde{Sp}_{2m(2n+1)}(k)$, in order to study the local Howe duality between representations of $O_{2n+1}(k)$ and $\widetilde{Sp}_{2m}(k)$.

In the following we identify

$$(2.1) \quad V^m \cong V \otimes W^+ = V \otimes f_1 \oplus \dots \oplus V \otimes f_m.$$

We restrict ω_ψ to the image of the embedding of $O_{2n+1}(k) \times \widetilde{Sp}_{2m}(k)$ inside $\widetilde{Sp}_{2m(2n+1)}(k)$. Here are some formulae. Let $\varphi \in \mathcal{S}(V^m)$. Then

$$\omega_\psi(g, 1)\varphi(v_1, \dots, v_m) = \varphi(g^{-1} \cdot v_1, \dots, g^{-1} \cdot v_m)$$

for $g \in O_{2n+1}(k)$ and $(v_1, \dots, v_m) \in V^m$. Next, let $\tilde{P}_m = \tilde{M}_m \tilde{N}_m$ be the inverse image in $\widetilde{Sp}_{2m}(k)$ of the Siegel parabolic subgroup P_m of $Sp_{2m}(k)$. Thus,

$$\tilde{M}_m = \left\{ (\tilde{m}(a), \varepsilon) : \tilde{m}(a) = \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix} \in Sp_{2m}(k), a \in GL_m(k), \varepsilon = \pm 1 \right\}$$

which is a semi-direct product of $\mathrm{GL}_m(k)$ and $\{\pm 1\}$. Note that \widetilde{N}_m is the direct product of N_m and $\{\pm 1\}$, since the double cover splits over unipotent subgroups. (See [Mt].) Here

$$N_m = \left\{ n(X) = \begin{pmatrix} I_m & X \\ 0 & I_m \end{pmatrix} \in \mathrm{Sp}_{2m}(k) \right\} .$$

We will identify N_m with $N_m \times \{1\}$.

From the definition of the Weil (or Oscillator) representation ω_ψ , we have that for $(m(a), \varepsilon) \in \widetilde{M}_m$,

$$(2.2) \quad \omega_\psi(1, (\widetilde{m}(a), \varepsilon))\varphi(v_1, \dots, v_m) = \chi_\psi((\det a)^m) |\det a|^{\frac{m}{2}} \varphi((v_1, \dots, v_m)a)$$

where χ_ψ is the character of the two-fold cover of k associated to ψ (through the Weil factor); and for $n(X) \in N_m$,

$$(2.3) \quad \omega_\psi(1, n(X))\varphi(v_1, \dots, v_m) = \psi \left(\frac{1}{2} \mathrm{tr}[\mathrm{Gr}(v_1, \dots, v_m)Xw_m] \right) \varphi(v_1, \dots, v_m)$$

where $\mathrm{tr}(\cdot)$ is the usual trace of a matrix, w_m is the $m \times m$ matrix, whose entries are all zero except these along the second diagonal, which are all one, and finally

$$(2.4) \quad \mathrm{Gr}(v_1, \dots, v_m) = \left((v_i, v_j)_V \right)_{m \times m} ,$$

is the Gram matrix. (See (2.9) in [GRS4] for more formulas.)

Let σ be an irreducible admissible representation of $\mathrm{O}_{2n+1}(k)$, acting on a space V_σ . Consider, as in p. 47 of [MVW],

$$S(\sigma) := \bigcap_{\alpha} \ker(\alpha) ,$$

where α runs over all elements of $\mathrm{Hom}_{\mathrm{O}_{2n+1}(k)}(S, V_\sigma)$, $S = S(V^m)$. Define

$$(2.5) \quad S[\sigma] := S/S(\sigma)$$

It is clear that $S[\sigma]$ affords a representation of $\mathrm{O}_{2n+1}(k) \times \widetilde{\mathrm{Sp}}_{2m}(k)$. According to page 47 of [MVW], $S[\sigma]$ has the form

$$\sigma \otimes \Theta_\psi^{n,m}(\sigma)$$

where $\Theta_\psi^{n,m}(\sigma)$ is a smooth representation of $\widetilde{\mathrm{Sp}}_{2m}(k)$. Assume that

$$\mathrm{Hom}_{\mathrm{O}_{2n+1}(k)}(S, V_\sigma) \neq 0.$$

Then the Howe duality conjecture states that $\Theta_\psi^{n,m}(\sigma)$ has a unique subrepresentation $\Theta_\psi^{n,m}(\sigma)^0$, such that the quotient representation

$$(2.6) \quad \theta_\psi^{n,m}(\sigma) := \Theta_\psi^{n,m}(\sigma) / \Theta_\psi^{n,m}(\sigma)^0$$

is irreducible. The map taking σ to $\theta_\psi^{n,m}(\sigma)$ is called the local ψ -Howe lift from $O_{2n+1}(k)$ to $\widetilde{Sp}_{2m}(k)$. Similarly, in the reverse direction, given an irreducible, admissible representation π of $\widetilde{Sp}_{2m}(k)$, such that $\text{Hom}_{\widetilde{Sp}_{2m}(k)}(S, V_\pi) \neq 0$, we have the spaces $S(\pi), S[\pi], \Theta_{m,n}^\psi(\pi)$, such that

$$S[\pi] \cong \Theta_{m,n}^\psi(\pi) \otimes \pi$$

over $O_{2n+1}(k) \times \widetilde{Sp}_{2m}(k)$. The Howe duality conjecture states that $\Theta_{m,n}^\psi(\pi)$ has a unique sub-representation $\Theta_{m,n}^\psi(\pi)^0$, such that the quotient

$$\theta_{m,n}^\psi(\pi) := \Theta_{m,n}^\psi(\pi) / \Theta_{m,n}^\psi(\pi)^0$$

is irreducible. We will say in such a case that $\theta_{m,n}^\psi(\pi)$ is the local ψ -Howe lift of π to $O_{2n+1}(k)$.

In general, if σ and π are irreducible admissible representations of $O_{2n+1}(k)$ and $\widetilde{Sp}_{2m}(k)$ respectively such that

$$\text{Hom}_{O_{2n+1}(k) \times \widetilde{Sp}_{2m}(k)}(\omega_\psi, \sigma \otimes \pi) \neq 0,$$

then we say that π is a local ψ -Howe lift of σ , and σ is a local ψ -Howe lift of π (without assuming the existence of the local Howe duality conjecture). The local Howe duality conjecture was proved by Waldspurger [W], when the residual characteristic of k is odd. In particular, in such a case, if π is a ψ -local Howe lift of σ (notations as above) then $\pi = \theta_\psi^{n,m}(\sigma)$ and $\sigma = \theta_{m,n}^\psi(\pi)$. The following theorem of Kudla, concerning local Howe duality for supercuspidal representations is free from the restriction on the residual characteristic.

THEOREM 2.1 ([K1, Th. 2.1] or [MVW, §VI.4, Chap. 3]). *Let σ and π be irreducible, supercuspidal representations of $O_{2n+1}(k)$ and $\widetilde{Sp}_{2m}(k)$ respectively.*

(1) *There is a positive integer $m_0 = m_0(\sigma)$, such that for any integer $1 \leq m < m_0$, $\text{Hom}_{O_{2n+1}(k)}(S, V_\sigma) = 0$, and for any integer $m \geq m_0$, $(\text{Hom}_{O_{2n+1}(k)}(S, V_\sigma) \neq 0$, and hence) $\Theta_\psi^{n,m}(\sigma) \neq 0$. Moreover, if $m = m_0$ then $\Theta_\psi^{n,m}(\sigma)$ is irreducible and supercuspidal. In particular,*

$$\Theta_\psi^{n,m_0}(\sigma) = \theta_\psi^{n,m_0}(\sigma).$$

If $m > m_0$, then $\Theta_\psi^{n,m}(\sigma)$ is of finite length and is not supercuspidal.

Similar results hold for π (denote $n_0 = n_0(\pi)$).

(2) *We have,*

$$\theta_\psi^{n_0,m}(\theta_{m,n_0}^\psi(\pi)) = \pi$$

and

$$\theta_{m_0,n}^\psi(\theta_\psi^{n,m_0}(\sigma)) = \sigma.$$

(We use the Weil representation as in Remark 2.3 of [K1].)

Remark 2.1. Since $O_{2n+1}(k) = \{\pm I_{2n+1}\} \times SO_{2n+1}(k)$, every irreducible representation of $O_{2n+1}(k)$ remains irreducible upon restriction to $SO_{2n+1}(k)$. Conversely, let $\beta = -I_{2n+1}$. Then for every irreducible representation σ of $SO_{2n+1}(k)$, $\sigma = \sigma^\beta$, so that σ extends to an irreducible representation of $O_{2n+1}(k)$. It extends in two ways, σ^+ and σ^- , to $O_{2n+1}(k)$, where $\sigma^+(\beta) = id_{V_\sigma}$ and $\sigma^-(\beta) = -id_{V_\sigma}$. Clearly,

$$\begin{aligned} & \text{Hom}_{SO_{2n+1}(k) \times \widetilde{Sp}_{2m}(k)}(\omega_\psi, \sigma \otimes \pi) \\ = & \text{Hom}_{O_{2n+1}(k) \times \widetilde{Sp}_{2m}(k)}(\omega_\psi, \sigma^+ \otimes \pi) + \text{Hom}_{O_{2n+1}(k) \times \widetilde{Sp}_{2m}(k)}(\omega_\psi, \sigma^- \otimes \pi). \end{aligned}$$

In cases where the Howe duality conjecture holds (e.g. when k has odd residual characteristic) if

$$\text{Hom}_{SO_{2n+1}(k) \times \widetilde{Sp}_{2m}(k)}(\omega_\psi, \sigma \otimes \pi) \neq 0,$$

then exactly one of the spaces

$$\text{Hom}_{O_{2n+1}(k) \times \widetilde{Sp}_{2m}(k)}(\omega_\psi, \sigma^\pm \otimes \pi)$$

is nonzero.

Let σ and π be irreducible admissible representations of $SO_{2n+1}(k)$ and $\widetilde{Sp}_{2m}(k)$ respectively. Assume that

$$\text{Hom}_{SO_{2n+1}(k) \times \widetilde{Sp}_{2m}(k)}(\omega_\psi, \sigma \otimes \pi) \neq 0.$$

Then we say that σ is a local ψ -Howe lift of π , and that π is a local ψ -Howe lift of σ . (There shouldn't be confusion with the similar notion for $O_{2n+1}(k) \times \widetilde{Sp}_{2m}(k)$. (The groups are different.) Again, if the last condition holds and the Howe duality conjecture is valid, then the local ψ -Howe lift of π to $O_{2n+1}(k)$ is one of the representations σ^\pm , denote it by σ^ε , and then the local ψ -Howe lift of σ^ε to $\widetilde{Sp}_{2m}(k)$ is π . In general, if π is a local ψ -Howe lift of σ , then we can assert that at least one of the representations σ^\pm is a local ψ -Howe lift of π .

One of our main goals in this section is to show, for irreducible, generic, supercuspidal representations σ, π of $SO_{2n+1}(k)$ and $\widetilde{Sp}_{2m}(k)$ respectively, that $n_0(\pi) = m$ and for exactly one of the representations σ^\pm , denote it by σ^ε , $m_0(\sigma^\varepsilon) = n$. (In the first case π has to have a Whittaker model compatible with ψ .)

Let U_n (resp. \widetilde{U}_m) be the standard maximal unipotent subgroup of $SO_{2n+1}(k)$ (resp. $\widetilde{Sp}_{2m}(k)$); here \widetilde{U}_m is the image of the embedding of the standard maximal unipotent subgroup of $Sp_{2m}(k)$ inside $\widetilde{Sp}_{2m}(k)$. Let Z_l be the standard maximal unipotent subgroup of $GL_l(k)$. Then, since the covering of $\widetilde{Sp}_{2m}(k)$ splits over unipotent subgroups ([Mt]),

$$U_n = m(Z_n) \cdot V_n, \quad \widetilde{U}_m = \widetilde{m}(Z_m)N_m \times 1$$

where

$$\begin{aligned}
 m(Z_n) &= \left\{ m(z) = \begin{pmatrix} z & & \\ & 1 & \\ & & z^* \end{pmatrix} \in \mathrm{SO}_{2n+1}(k) : z \in Z_n \right\} \\
 \tilde{m}(Z_m) &= \left\{ \tilde{m}(z) = \begin{pmatrix} z & & \\ & & \\ & & z^* \end{pmatrix} \in \mathrm{Sp}_{2m}(k) : z \in Z_m \right\} \\
 V_n &= \left\{ v(y, z) = \begin{pmatrix} I_n & y & z \\ & 1 & y' \\ & & I_n \end{pmatrix} \in \mathrm{SO}_{2n+1}(k) \right\}.
 \end{aligned}$$

We will identify $\tilde{m}(Z_m)N_m$ with \tilde{U}_m .

Let ψ be a nontrivial character of k . Denote by ψ_U the following nondegenerate character of U_n :

$$(2.7) \quad \psi_U(m(z)v(y, e)) := \psi(z_{12} + \cdots + z_{n-1,n})\psi(y_n) := \psi_n(z)\psi_U(v(y, e))$$

where $m(z)v(y, e) \in m(Z_n) \cdot V_n = U_n$. For $\lambda \in k^*$, denote by $\psi_{\tilde{U},\lambda}$ the nondegenerate character of \tilde{U}_m , which corresponds to ψ and λ :

$$(2.8) \quad \psi_{\tilde{U},\lambda}(\tilde{m}(z)n(X)) := \psi(z_{12} + \cdots + z_{m-1,m})\psi\left(\frac{\lambda}{2}X_{m1}\right) := \psi_m(z)\psi_{\tilde{U},\lambda}(n(X))$$

where $\tilde{m}(z)n(X) \in \tilde{m}(Z_m)N_m = \tilde{U}_m$.

An irreducible admissible representation σ (resp. π) of $\mathrm{SO}_{2n+1}(k)$ (resp. $\widetilde{\mathrm{Sp}}_{2m}(k)$) is called ψ_U -generic (resp. $\psi_{\tilde{U},\lambda}$ -generic) if σ (resp. π) admits a nonzero ψ_U (resp. $\psi_{\tilde{U},\lambda}$) Whittaker functional, i.e. a nonzero element of $\mathrm{Hom}_{U_n}(\sigma, \psi_U)$ (resp. $\mathrm{Hom}_{\tilde{U}_m}(\pi, \psi_{\tilde{U},\lambda})$). Note that if a representation of $\mathrm{SO}_{2n+1}(k)$ has a Whittaker model with respect to one nondegenerate character, then it has a Whittaker model with respect to any nondegenerate character, since the maximal split torus of $\mathrm{SO}_{2n+1}(k)$ acts transitively on the set of all generic characters of U_n . This is not necessarily the case for representations of $\widetilde{\mathrm{Sp}}_{2m}(k)$.

PROPOSITION 2.1. *Let σ be an irreducible generic representation of $\mathrm{SO}_{2n+1}(k)$. Let $1 \leq m < n$ be an integer. Then σ has no nonzero local ψ -Howe lifts to $\widetilde{\mathrm{Sp}}_{2m}(k)$ (and thus, each of the representations σ^\pm has no nonzero local ψ -Howe lifts to $\widetilde{\mathrm{Sp}}_{2m}(k)$.)*

Proof. This is the local version of Proposition 2 in [F]. The proof is the appropriate analog of the proof in [F]. Let $m < n$, and assume that there is an irreducible admissible representation π_m of $\widetilde{\mathrm{Sp}}_{2m}(k)$, acting in a (nontrivial)

space V_{π_m} , which is a local ψ -Howe lift of σ . This means that there is a nontrivial $\mathrm{SO}_{2n+1}(k)$ -intertwining and $\widetilde{\mathrm{Sp}}_{2m}(k)$ -equivariant map

$$\rho : \mathcal{S}(V^m) \otimes V_{\pi_m^\vee} \longrightarrow V_\sigma.$$

Here π_m^\vee denotes the representation contragredient to π_m (acting in $V_{\pi_m^\vee}$.) Let η_{ψ_U} be a (nontrivial) ψ_U -Whittaker functional on V_σ . Consider

$$b_{\psi_U} := \eta_{\psi_U} \circ \rho : \mathcal{S}(V^m) \otimes V_{\pi_m^\vee} \longrightarrow \mathbb{C}$$

which is a nontrivial bilinear form satisfying

$$(2.9) \quad b_{\psi_U}(\omega_\psi(u, h)\varphi, \pi_m^\vee(h)\xi) = \psi_U(u)b_{\psi_U}(\varphi, \xi)$$

for $u \in U_n, h \in \widetilde{\mathrm{Sp}}_{2m}(k), \varphi \in \mathcal{S}(V^m), \xi \in V_{\pi_m^\vee}$. We will show that, for $m < n$, the space of bilinear forms, satisfying the equivariance property (2.9), is zero, and this will be a contradiction. To this end, we pass to a realization of ω_ψ in a mixed model

$$\mathcal{S}(W^n \times W^+) \cong \mathcal{S}(V^m)$$

where $W^n \times W^+$ is the direct product of the spaces W^n and W^+ (§II.7, Chapter 2 in [MVW]). More precisely,

$$[V^+ \otimes W + e \otimes W^-] + [V^- \otimes W + e \otimes W^+]$$

is a polarization of $V \otimes W$ (with respect to the symplectic form $(\cdot, \cdot)_V \otimes (\cdot, \cdot)_W$). We may realize ω_ψ in $S[V^- \otimes W + e \otimes W^+] \cong \mathcal{S}(W^n \times W^+) \cong \mathcal{S}(W^n) \otimes \mathcal{S}(W^+)$. We identify

$$\begin{aligned} (y_1, \dots, y_n) &\longleftrightarrow e_{-n} \otimes y_1 + \dots + e_{-1} \otimes y_n \quad , \quad y_i \in W \\ y^+ &\longleftrightarrow e \otimes y^+ \quad , \quad y^+ \in W^+ . \end{aligned}$$

We keep denoting the Weil representation by ω_ψ (in the mixed model as well). Let $\varphi \in \mathcal{S}(W^n \times W^+)$, and consider an element $v(0, z)$ in the center of V_n . We have, from the definition of the mixed model of the Weil representations (§II.7, Chapter 2 in [MVW]),

$$(2.10) \quad \begin{aligned} &(\omega_\psi(v(0, z), 1)\varphi)(y_1, \dots, y_n; y^+) \\ &= \psi \left(\frac{1}{2} \mathrm{tr}(\mathrm{Gr}(y_1, \dots, y_n)w_n z) \right) \varphi(y_1, \dots, y_n; y^+) \end{aligned}$$

where $\mathrm{Gr}(y_1, \dots, y_n) = ((y_i, y_j)_W)_{n \times n}$. Let $V_n(0, Z) = \{v(0, z) \in V_n \subset \mathrm{SO}_{2n+1}(k)\}$. Denote by $J_{V_n(0, Z)}$ the Jacquet functor along $V_n(0, Z)$ (with respect to the trivial character). We view b_{ψ_U} first as a bilinear form on $J_{V_n(0, Z)}(\mathcal{S}(W^n \times W^+)) \times V_{\pi_m^\vee}$, satisfying (2.9). Let

$$\begin{aligned} C_0 &= \{(y_1, \dots, y_n; y^+) \in W^n \times W^+ | (y_i, y_j)_W = 0, \forall i, j \leq n\} \\ C &= W^n \times W^+ \setminus C_0. \end{aligned}$$

It is clear that C_0 is closed in $W^n \times W^+$, and as the complement of C_0 , C is open in $W^n \times W^+$. We claim that

$$(2.11) \quad J_{V_n(0,Z)}(\mathcal{S}(W^n \times W^+)) \cong \mathcal{S}(C_0).$$

Indeed, by [BZ], we have an exact sequence

$$0 \rightarrow \mathcal{S}(C) \xrightarrow{i} \mathcal{S}(W^n \times W^+) \xrightarrow{r} \mathcal{S}(C_0) \rightarrow 0,$$

where r is the restriction to C_0 , and i is the embedding which takes a function supported in C , and extends it by zero to the whole of $W^n \times W^+$. Using the exactness of Jacquet functors, we get

$$0 \rightarrow J_{V_n(0,Z)}(\mathcal{S}(C)) \xrightarrow{i^*} J_{V_n(0,Z)}(\mathcal{S}(W^n \times W^+)) \xrightarrow{r^*} J_{V_n(0,Z)}(\mathcal{S}(C_0)) \rightarrow 0.$$

From (2.10), it follows that $J_{V_n(0,Z)}(\mathcal{S}(C)) = 0$. Note that for $(y_1, \dots, y_n; y^+) \in C$, the character

$$v(0, z) \mapsto \psi \left(\frac{1}{2} \text{tr}(\text{Gr}(y_1, \dots, y_n)w_n z) \right)$$

is nontrivial, and hence, for $\varphi \in \mathcal{S}(C)$, there is a large enough compact subgroup Ω_φ of $V_n(0, Z)$, such that

$$\begin{aligned} \int_{\Omega_\varphi} (\omega_\psi(1, v(0, z))\varphi)(y_1, \dots, y_n; y^+) dz \\ = \int_{\Omega_\varphi} \psi \left(\frac{1}{2} \text{tr}(\text{Gr}(y_1, \dots, y_n)w_n z) \right) \varphi(y_1, \dots, y_n; y^+) dz \\ = 0 \end{aligned}$$

(by (2.10)). We conclude that

$$J_{V_n(0,Z)}(\mathcal{S}(W^n \times W^+)) \cong J_{V_n(0,Z)}(\mathcal{S}(C_0)) \cong \mathcal{S}(C_0) .$$

With $\mathcal{S}(C_0)$ as a $U_n \times \widetilde{\text{Sp}}_{2m}(k)$ -module, we now view b_{ψ_U} as a bilinear form on $J_{m(Z_n), \psi_n}(\mathcal{S}(C_0)) \times V_{\pi_m^\vee}$, satisfying (2.9), where $J_{m(Z_n), \psi_n}$ denotes the Jacquet functor along $m(Z_n)$, with respect to the nondegenerate character $\psi_n = \psi_U \Big|_{m(Z_n)}$.

Note that from the definition of the Weil representations on the mixed model (§II.7, Chapter 2 in [MVW]), the action of

$$m(z) = \begin{pmatrix} z & & \\ & 1 & \\ & & z^* \end{pmatrix} \in m(Z_n) ,$$

induced by ω_ψ , in $\mathcal{S}(C_0)$ is given by

$$(2.12) \quad \omega_\psi(m(z), 1)\varphi(y_1, \dots, y_n; y^+) = \varphi((y_1, \dots, y_n)w_n z w_n; y^+)$$

where we still use ω_ψ to denote action on $\mathcal{S}(C_0)$. Consider the orbits of the action of $w_n Z_n w_n$ on $\{(y_1, \dots, y_n) \in W^n \mid \text{Gr}(y_1, \dots, y_n) = 0\}$. They have the form

$$(2.13) \quad (0 \cdots 0x_1 0 \cdots 0x_2 0 \cdots 0x_{j-1} 0 \cdots 0x_j 0 \cdots 0)w_n Z_n w_n$$

where $\{x_1, \dots, x_j\}$ are linearly independent, span a totally isotropic subspace of W , and the spaces of zeros in (2.13) are of given sizes. Let us write $C_0 = \bigcup_{0 \leq j} C_0(j)$, where

$$C_0(j) = \{(y_1, \dots, y_n; y^+) \in C_0 \mid \dim \text{Span}\{y_1, \dots, y_n\} \leq j\}.$$

Note that $C_0(j) = C_0$, if $j \geq n$. We let $C_0(-1)$ be the empty set.

By [BZ], we have the exact sequences

$$(2.14) \quad \begin{aligned} 0 \rightarrow J_{m(Z_n), \psi_n}(\mathcal{S}(C_0 \setminus C_0(j))) &\rightarrow J_{m(Z_n), \psi_n}(\mathcal{S}(C_0 \setminus C_0(j-1))) \\ &\rightarrow J_{m(Z_n), \psi_n}(\mathcal{S}(C_0(j) \setminus C_0(j-1))) \rightarrow 0 \end{aligned}$$

for $j = 0, 1, \dots, n$. We define the following subsets

$$(2.15) \quad \Omega_{j,e} = \bigcup [(0 \cdots 0x_1 0 \cdots 0x_2 0 \cdots 0x_{j-1} 0 \cdots 0x_j 0 \cdots 0)w_n Z_n w_n] \times W^+,$$

where the union is taken over all the representative sets (as in (2.13)) $\{x_1, \dots, x_j\}$ which span a j -dimensional, totally isotropic subspace of W ; e stands for an injective map from the index set $\{1, \dots, j\}$ of y_1, \dots, y_j into the whole index set $\{1, \dots, n\}$. Then we have

$$(2.16) \quad C_0(j) \setminus C_0(j-1) = \bigcup_e \Omega_{j,e}.$$

It is clear that in (2.16), there are $\binom{n}{j}$ different terms. We order them as: $e_1, e_2, \dots, e_{\binom{n}{j}}$. Since the orbits Ω_{j,e_i} have the same dimension, they are both

open and closed in $\bigcup_{i=1}^{\binom{n}{j}} \Omega_{j,e_i}$, and hence, we have

$$(2.17) \quad J_{m(Z_n), \psi_n}(\mathcal{S}(\cup_{i \geq k} \Omega_{j,e_i})) = J_{m(Z_n), \psi_n}(\mathcal{S}(\cup_{i \geq k+1} \Omega_{j,e_i})) \oplus J_{m(Z_n), \psi_n}(\mathcal{S}(\Omega_{j,e_k})).$$

Let us now show that, for $j < n$, and any k ,

$$(2.18) \quad \text{Hom}_{m(Z_n) \times \widetilde{\text{Sp}}_{2m}(k)} \left(J_{m(Z_n), \psi_n}(\mathcal{S}(\Omega_{j,e_k})) \otimes V_{\pi_m^\vee}, \mathbb{C} \right) = 0$$

where $m(Z_n)$ acts on \mathbb{C} according to ψ_n . For this, write $\mathcal{S}(\Omega_{j,e_k}) = \mathcal{S}(\Omega'_{j,e_k}) \otimes \mathcal{S}(W^+)$, where Ω'_{j,e_k} is the first factor of Ω_{j,e_k} in (2.15). Denote by ω'_ψ the standard Weil representation of $\widetilde{\text{Sp}}_{2m}(k)$ in $\mathcal{S}(W^+)$. Let $\phi_1 \in \mathcal{S}(\Omega'_{j,e_k})$, and $\phi_2 \in \mathcal{S}(W^+)$. Then the action of $m(Z_n) \times \widetilde{\text{Sp}}_{2m}(k)$, induced by ω_ψ , on $\mathcal{S}(\Omega'_{j,e_k}) \otimes \mathcal{S}(W^+)$ is given by

$$\omega_\psi(m(z), h)(\phi_1 \otimes \phi_2)(y_1, \dots, y_n; y^+) = \phi_1((y_1 \bar{h}, \dots, y_n \bar{h}) \cdot w_n z w_n)(\omega'_\psi(h)\phi_2)(y^+)$$

where \bar{h} is the projection of $h \in \widetilde{\mathrm{Sp}}_{2m}(k)$ to $\mathrm{Sp}_{2m}(k)$. Let R_{j,e_k} be the stabilizer, in $[m(Z_n) \times \widetilde{\mathrm{Sp}}_{2m}](k)$, of $(0 \cdots 0f_{-j}0 \cdots 0f_{-(j-1)}0 \cdots 0f_{-2}0 \cdots 0f_{-1}0 \cdots 0)$, where the ordering is given by e_k , and the action is given by

$$(y_1, \dots, y_n) \cdot (m(z), \bar{h}) = (y_1\bar{h}, \dots, y_n\bar{h}) \cdot w_n z w_n .$$

Then by Witt’s theorem, it follows that Ω'_{j,e_k} is an $[m(Z_n) \times \widetilde{\mathrm{Sp}}_{2m}](k)$ -orbit and hence $\mathcal{S}(\Omega'_{j,e_k})$ is isomorphic to the compactly induced representation

$$c - \mathrm{Ind}_{R_{j,e_k}}^{m(Z_n) \times \widetilde{\mathrm{Sp}}_{2m}}(1)$$

as a representation of $m(Z_n) \times \mathrm{Sp}_{2m}(k)$. Thus, the left-hand side of (2.18) is isomorphic to

$$\mathrm{Hom}_{m(Z_n) \times \widetilde{\mathrm{Sp}}_{2m}(k)} \left(c - \mathrm{Ind}_{R_{j,e_k}}^{m(Z_n) \times \widetilde{\mathrm{Sp}}_{2m}}(1) \otimes \omega'_\psi \otimes \pi_m^\vee, \psi_n \right)$$

which is, by Frobenius reciprocity, isomorphic to

$$(2.19) \quad \mathrm{Hom}_{R_{j,e_k}} \left(\psi_n^{-1} \otimes \omega'_\psi \otimes \pi_m^\vee, 1 \right) .$$

The space (2.19) is zero for $j < n$, since then R_{j,e_k} contains a subgroup of the form $L \times 1$, where $L \subset m(Z_n)$ contains a simple root subgroup, and hence $\psi_n|_L \neq 1$. This proves (2.18). It follows, from (2.9), (2.11), (2.14)–(2.18), that the space of b_{ψ_U} in (2.9) is isomorphic to

$$(2.20) \quad \mathrm{Hom}_{U_n \times \widetilde{\mathrm{Sp}}_{2m}(k)} \left(J_{U,\psi_U}(\mathcal{S}(\Omega_n)) \otimes V_{\pi_m^\vee}, \mathbb{C} \right) .$$

This space is zero if $m < n$, since then Ω_n is clearly empty. This completes the proof of Proposition 2.1. \square

Let us continue the line of argument of the proof for Proposition 2.1 in case $m = n$. Now re-denote $\pi = \pi_n$. Let T be an element of the space (2.20), which we view now as

$$\mathrm{Hom}_{U_n \times \widetilde{\mathrm{Sp}}_{2n}(k)} (\mathcal{S}(\Omega'_n) \otimes \mathcal{S}(W^+) \otimes V_{\pi^\vee}, \mathbb{C})$$

where U_n acts on \mathbb{C} according to ψ_U . Thus, we may think of T as a trilinear form $T(\phi_1, \phi_2, \xi)$. Fixing $\phi_2 \in \mathcal{S}(W^+)$ and $\xi \in V_{\pi^\vee}$, we obtain a map

$$\phi_1 \mapsto T(\phi_1, \phi_2, \xi)$$

which is a smooth distribution on the $m(Z_n) \times \mathrm{Sp}_{2n}(k)$ orbit Ω'_n , and hence can be written uniquely in the form

$$(2.21) \quad T(\phi_1, \phi_2, \xi) = \int_{R \backslash Z_n \times \mathrm{Sp}_{2n}(k)} \phi_1((f_{-n}h, \dots, f_{-1}h)w_n z w_n) \Phi_{\phi_2, \xi}(z, h) d(z, h)$$

where $d(z, h)$ is a right $Z_n \times \mathrm{Sp}_{2n}(k)$ -invariant measure on $R \backslash Z_n \times \mathrm{Sp}_{2n}(k)$, and $\Phi_{\phi_2, \xi}$ is a (right) smooth function on $Z_n \times \mathrm{Sp}_{2n}(k)$ and is left R -invariant.

Here R is the stabilizer (R_n) in $Z_n \times \mathrm{Sp}_{2n}(k)$ of (f_{-n}, \dots, f_{-1}) under the action (on Ω'_n)

$$(x_1, \dots, x_n) \cdot (z, h) = (x_1 h, \dots, x_n h) w_n z w_n.$$

Using the equivariance properties (with respect to $m(Z_n) \times \mathrm{Sp}_{2n}(k)$) we find that

$$(2.22) \quad \Phi_{\phi_2, \xi}(z, h) = \psi_n^{-1}(z) \Phi_{\omega'_\psi(\tilde{h})\phi_2, \pi^\vee(\tilde{h})\xi}(1, 1)$$

where \tilde{h} is a pre-image in $\widetilde{\mathrm{Sp}}_{2n}(k)$ of h . Note again that

$$\begin{aligned} \phi_1(f_{-n} \cdot h, \dots, f_{-1} \cdot h) w_n z w_n \omega'_\psi(\tilde{h}) \phi_2(y^+) \\ = \omega_\psi(m(z), \tilde{h})(\phi_1 \otimes \phi_2)(f_{-n}, \dots, f_{-1}; y^+). \end{aligned}$$

Thus, the integrand of (2.21) is (using (2.22))

$$\psi_n^{-1}(z) \Phi_{[\phi_1(f_{-n} \cdot h, \dots, f_{-1} \cdot h) w_n z w_n \omega'_\psi(\tilde{h}) \phi_2], \pi^\vee(\tilde{h})\xi}(1, 1).$$

The function in square brackets in the first index of Φ is (by the last equality)

$$y^+ \rightarrow \omega_\psi(m(z), \tilde{h})(\phi_1 \otimes \phi_2)(f_{-n}, \dots, f_{-1}; y^+).$$

Substituting in (2.21), we get

$$(2.23) \quad \begin{aligned} T(\phi_1, \phi_2, \xi) \\ = \int_{R \backslash Z_n \times \mathrm{Sp}_{2n}(k)} \psi_n^{-1}(z) \Phi_{\omega_\psi(m(z), \tilde{h})(\phi_1 \otimes \phi_2)(f_{-n}, \dots, f_{-1}; *), \pi^\vee(\tilde{h})\xi}(1, 1) d(z, h). \end{aligned}$$

We have not yet used the property

$$(2.24) \quad T(\omega_\psi(1, v_n(t, x))(\phi_1 \otimes \phi_2), \xi) = \psi(t_n) T(\phi_1, \phi_2, \xi).$$

It follows from the definition of the mixed model of the Weil representations (§II.7, Chapter 2 in [MVW]) that

$$(2.25) \quad \omega_\psi(1, v(t, x)) \varphi(f_{-n}, \dots, f_{-1}; y^+) = \psi \left(\sum_{i=1}^n t_i(y^+, f_{-i})_W \right) \varphi(f_{-n}, \dots, f_{-1}; y^+).$$

Using (2.22)–(2.25), we conclude that

$$\psi(t_n) \Phi_{\phi_2, \xi}(z, h) = \Phi_{\psi(\sum_{i=1}^n t_i(*, f_{-i})_W) \phi_2(*), \xi}(z, h)$$

for all $z \in Z_n$, $h \in \mathrm{Sp}_{2m}(k)$, and in particular, for $z = 1$,

$$(2.26) \quad \Phi_{\psi(\sum_{i=1}^n t_i(*, f_{-i})_W) \phi_2(*), \xi}(1, 1) = \psi(t_n) \Phi_{\phi_2, \xi}(1, 1)$$

Regarding, for fixed ξ , $\phi_2 \mapsto \Phi_{\phi_2, \xi}(1, 1)$ as a distribution on W^+ , (2.26) implies that it is supported at $y^+ = f_n$. Thus

$$(2.27) \quad \Phi_{\phi, \xi}(1, 1) = W(\xi) \phi_2(f_n)$$

for some $W(\xi) \in \mathbb{C}$. This implies that the trilinear functional $T(\phi_1, \phi_2, \xi)$ is in fact given by the following integral

$$(2.28) \quad \int_{R \backslash Z_n \times \mathrm{Sp}_{2n}(k)} \omega_\psi(m(z), \tilde{h})(\phi_1 \otimes \phi_2)(f_{-n}, \dots, f_{-1}; f_n) \psi_n^{-1}(z) W(\pi^\vee(\tilde{h})\xi) d(z, k)$$

where \tilde{h} is any pre-image (in $\widetilde{\mathrm{Sp}}_{2n}(k)$) of h . Finally, note that

$$R = \left\{ \left(z, \begin{pmatrix} z & x \\ 0 & z^* \end{pmatrix} \right) \in Z_n \times \mathrm{Sp}_{2n}(k) \right\}.$$

Using the left R -invariance of $\Phi_{\phi_2, \xi}(z, h)$, (2.22) and (2.27) we find that

$$W \left(\pi^\vee \begin{pmatrix} z & x \\ 0 & z^* \end{pmatrix} \xi \right) = \psi_n(z) \psi \left(\frac{1}{2} x_{n1} \right) W(\xi);$$

i.e., W is a (necessarily nontrivial, if T is nontrivial) $\psi_{\widetilde{U}, 1}$ -Whittaker functional on V_{π^\vee} . Put

$$W_{\pi^\vee}^{\psi_{\widetilde{U}, 1}}(\xi)(\tilde{h}) = W(\pi^\vee(\tilde{h})\xi).$$

This is the corresponding $\psi_{\widetilde{U}, 1}$ -Whittaker function. Now we can rewrite (2.28), for $\varphi = \phi_1 \otimes \phi_2$, as

$$(2.29) \quad \int_{\mu_2 \cdot \tilde{N}_n \backslash \widetilde{\mathrm{Sp}}_{2n}(k)} \omega_\psi(1, \tilde{h}) \varphi(f_{-n}, \dots, f_{-1}; f_n) W_{\pi^\vee}^{\psi_{\widetilde{U}, 1}}(\xi)(\tilde{h}) d\tilde{h}.$$

Here $\mu_2 = \{\pm 1\}$ is the kernel of the projection $\widetilde{\mathrm{Sp}}_{2n}(k) \rightarrow \mathrm{Sp}_{2n}(k)$. Note that this is the local version of [F, formula (12)]. Note that the integral (2.29) converges absolutely. Indeed the integrand has compact support. To see this, we may assume that $\varphi = \phi_1 \otimes \phi_2$, as before, and due to the Iwasawa decomposition, it is enough to note that $\phi_1(f_n \cdot za', \dots, f_1 \cdot za') W_{\pi^\vee}^{\psi_{\widetilde{U}, 1}}(\xi)(a', 1)$, has compact support, where $z \in \tilde{m}(Z_n)(k)$, $a = \mathrm{diag}(a_1, \dots, a_n)$, and $a' = \mathrm{diag}(a, a^*)$. Recall that a Whittaker function, restricted to the diagonal subgroup is "vanishing at infinity", meaning that if $\max_{1 \leq i \leq n} \{|a_i|\}$ is large, then $W_{\pi^\vee}^{\psi_{\widetilde{U}, 1}}(\xi)(a', 1)$ vanishes. Clearly $\phi_1(f_n \cdot za', \dots, f_1 \cdot za')$ vanishes if $\max_{1 \leq i \leq n} \{|a_i|^{-1}\}$ is large. We conclude that a' has to lie in a compact set of the diagonal subgroup, and hence also z has to lie in a compact set of $\tilde{m}(Z_n)(k)$.

Let us summarize what we have shown in case $m = n$.

COROLLARY 2.1. (1) *Let σ be an irreducible generic representation of $\mathrm{SO}_{2n+1}(k)$. Assume that π is an irreducible representation of $\widetilde{\mathrm{Sp}}_{2n}(k)$, which is a local ψ -Howe lift of σ . Then π is $\psi_{\widetilde{U}, 1}^{-1}$ -generic. Moreover, the functional b_{ψ_U} , viewed as a bilinear form on $\omega_\psi \otimes \pi^\vee$, equals, up to scalars, to (2.29) (with a fixed $\psi_{\widetilde{U}, 1}$ -Whittaker model on π^\vee). The ψ_U -Whittaker model of σ is spanned*

by the functions

$$(2.30) \quad g \mapsto \int_{\mu_2 \cdot \tilde{N}_n \backslash \tilde{\mathrm{Sp}}_{2n}(k)} \omega_\psi(g, \tilde{h}) \varphi(f_{-n}, \dots, f_{-1}; f_n) W_{\pi^\vee}^{\psi_{\tilde{U},1}}(\xi)(\tilde{h}) d\tilde{h}.$$

(2) Let π be an irreducible, supercuspidal, $\psi_{\tilde{U},1}^{-1}$ -generic representation of $\tilde{\mathrm{Sp}}_{2n}(k)$. Then π has a nontrivial local ψ -Howe lift to $\mathrm{SO}_{2n+1}(k)$. Moreover, there is a nontrivial space $t_\psi(\pi)$ of ψ -Whittaker functions on $\mathrm{SO}_{2n+1}(k)$, invariant to right translations, such that

$$(2.31) \quad \mathrm{Hom}_{\mathrm{SO}_{2n+1}(k) \tilde{\mathrm{Sp}}_{2n}(k)}(\omega_\psi \otimes \pi^\vee, t_\psi(\pi)) \neq 0$$

and $t_\psi(\pi)$ is spanned by the functions (2.30).

Proof. We have already shown part (1). We now prove part (2). Since π^\vee is $\psi_{\tilde{U},1}$ -generic, we may define the integrals (2.30), which are absolutely convergent (explained just before the statement of Cor. 2.1). It is easily seen that these integrals are not identically zero as (φ, ξ) varies. Let $t_\psi(\pi)$ be the space of functions on $\mathrm{SO}_{2n+1}(k)$, spanned by the integrals (2.30). Note that these are ψ_U -Whittaker functions on $\mathrm{SO}_{2n+1}(k)$, and that $t_\psi(\pi)$ affords a smooth representation, by right translations, of $\mathrm{SO}_{2n+1}(k)$. By construction, we clearly have (2.31). We may, of course, substitute in (2.30) any g in $\mathrm{O}_{2n+1}(k)$. Denote by $t'_\psi(\pi)$ the space of functions on $\mathrm{O}_{2n+1}(k)$ thus obtained; it affords, as before, a smooth representation by right translations of $\mathrm{O}_{2n+1}(k)$. We have

$$\mathrm{Hom}_{\mathrm{O}_{2n+1}(k) \tilde{\mathrm{Sp}}_{2n}(k)}(\omega_\psi \otimes \pi^\vee, t'_\psi(\pi)) \neq 0.$$

This implies that

$$\mathrm{Hom}_{\mathrm{O}_{2n+1}(k) \tilde{\mathrm{Sp}}_{2n}(k)}(\Theta_{n,n}^\psi(\pi) \otimes \pi, t'_\psi(\pi) \otimes \pi) \neq 0.$$

In particular, $\Theta_{n,n}^\psi(\pi) \neq 0$. Since π is supercuspidal, $\Theta_{n,n}^\psi(\pi)$ is of finite length as a representation of $\mathrm{O}_{2n+1}(k)$ (Theorem 2.1) and hence has an irreducible quotient; call it σ' . We have nontrivial maps

$$\omega_\psi \longrightarrow S[\pi] = \Theta_{n,n}^\psi(\pi) \otimes \pi \longrightarrow \sigma' \otimes \pi$$

and hence σ' is a local ψ -Howe lift of π to $\mathrm{O}_{2n+1}(k)$. Let σ be the restriction of σ' to $\mathrm{SO}_{2n+1}(k)$. Then σ is a local ψ -Howe lift of π to $\mathrm{SO}_{2n+1}(k)$. \square

To continue, we introduce the notion of a Bessel model of special type for representations of $\mathrm{SO}_{2n+1}(k)$. Bessel models for representations of orthogonal groups are discussed in general in [GP-SR].

Let $Q_{n-1} = M_{n-1}V_{n-1}$ be the standard maximal parabolic subgroup of $\mathrm{SO}_{2n+1}(k)$, with Levi subgroup isomorphic to $\mathrm{GL}_{n-1}(k) \times \mathrm{SO}_3(k)$, and unipo-

tent radical

$$V_{n-1} = \left\{ v'(y, z) = \begin{pmatrix} I_{n-1} & y & z \\ & I_3 & y' \\ & & I_{n-1} \end{pmatrix} \in \text{SO}_{2n+1}(k) \right\}.$$

Let $V_3 = \text{Span}_k\{e_n, e, e_{-n}\}$. We choose, for $\lambda \in k^*$, a vector $e_\lambda \in V_3$, such that $(e_\lambda, e_\lambda)_V = \lambda$. If λ is a square α^2 , we choose $e_{\alpha^2} = \alpha e$. Define a character χ_λ of V_{n-1} by

$$\chi_\lambda(v'(y, z)) = \psi\left((y \cdot e_\lambda, e_{-(n-1)})_V\right)$$

where we view y as the linear map which takes $x_1e_n + x_2e + x_3e_{-n}$ in V_3 to $\sum_{i=1}^{n-1}(y_{i1}x_1 + y_{i2}x_2 + y_{i3}x_3)e_i$. It follows that the connected component of the stabilizer of χ_λ in M_{n-1} is the subgroup

$$S_{M_{n-1}}(\chi_\lambda) = \left\{ s(p, d) = \begin{pmatrix} p & & \\ & d & \\ & & p^* \end{pmatrix} \right. \\ \left. \in \text{SO}_{2n+1}(k) : p \in P_{n-1}, d \in \text{SO}_3(k), d \cdot e_\lambda = e_\lambda \right\},$$

where $P_{n-1} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \text{GL}_{n-1}(k) \right\}$. Let

$$D_\lambda := \{s(I_{n-1}, d) \in S_{M_{n-1}}(\lambda)\}.$$

Note that D_λ is abelian and

$$D_1 = \left\{ s\left(I_{n-1}, \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix}\right) : a \in k^* \right\}.$$

For $g \in \text{GL}_{n-1}(k)$, denote $m'(g) = \begin{pmatrix} g & & \\ & I_3 & \\ & & g^* \end{pmatrix} \in M_{n-1}$. Put

$$R_\lambda := D_\lambda m'(Z_{n-1})V_{n-1}.$$

Let ν be a character of D_λ . D_λ is isomorphic to the special orthogonal group of the orthocomplement of e_λ in V_3 . Define a character of R_λ by

$$b_{(\nu, \psi, \lambda)}(d \cdot m'(z)v'(y, x)) = \nu(d)\psi_{n-1}(z)\chi_\lambda(v'(y, x))$$

for $d \in D_\lambda, z \in Z_{n-1}, v'(y, x) \in V_{n-1}$. We say that an irreducible, admissible representation σ of $\text{SO}_{2n+1}(k)$ has a (nontrivial) *Bessel model of type* (R_λ, ν) ,

if

$$\text{Hom}_{R_\lambda}(\sigma, b_{(\nu, \psi, \lambda)}) \neq 0 .$$

If $\nu = 1$, we say that the Bessel model (of type $(R_\lambda, 1)$) is *special*.

PROPOSITION 2.2. *If σ is an irreducible, supercuspidal, generic representation of $\text{SO}_{2n+1}(k)$, then σ has a nontrivial Bessel model of special type $(R_1, 1)$.*

Proof. Let σ be an irreducible, supercuspidal, generic representation of $\text{SO}_{2n+1}(k)$, acting in a space V_σ . Let η_{ψ_U} be a Whittaker functional on V_σ , with respect to (U_n, ψ_U) , i.e.

$$\eta_{\psi_U}(\sigma(u)\xi) = \psi_U(u)\eta_{\psi_U}(\xi)$$

for $u \in U_n$, $\xi \in V_\sigma$. For $\xi \in V_\sigma$, let $W_\xi(g) = \eta_{\psi_U}(\sigma(g)\xi)$ be the corresponding Whittaker function. Since σ is supercuspidal, W_ξ is compactly supported modulo U_n (on the left). Now consider

$$\beta(\xi) := \int_{k^{n-1} \times k^*} W_\xi \left(m \left(\begin{pmatrix} & & & 1 \\ & & & \\ & & I_{n-1} & \\ & & & \end{pmatrix} \begin{pmatrix} I_{n-1} & y \\ & t \end{pmatrix} \right) \right) dy |t|^{-n+1} d^*t$$

where for $a \in \text{GL}_n(k)$, $m(a) = \begin{pmatrix} a & & & \\ & 1 & & \\ & & & \\ & & & a^* \end{pmatrix} \in \text{SO}_{2n+1}(k)$. It follows from

the definition of $\beta(\xi)$ and the supercuspidality of σ that the integral (which is a Mellin transform) is absolutely convergent, and we can choose ξ so that $\beta(\xi) \neq 0$. By direct verification, one can check that β is a Bessel functional of special type $(R_1, 1)$ attached to σ . \square

PROPOSITION 2.3. *Let σ be an irreducible admissible representation of $\text{SO}_{2n+1}(k)$. Let π be an irreducible admissible $\psi_{\tilde{U}, \lambda}$ -generic representation of $\tilde{\text{Sp}}_{2n}(k)$, such that π is a local ψ -Howe lift of σ . Then σ has a nontrivial Bessel model of special type $(R_\lambda, 1)$.*

Proof. The idea of the proof is similar to that of the corresponding global statement (Prop. 1 in [F]). For later needs, we consider a slightly more general situation. Let σ be an irreducible, admissible representation of $\text{SO}_{2r+1}(k)$, where $r \leq n$. Let π be an irreducible, admissible $\psi_{\tilde{U}, \lambda}$ -generic representations of $\tilde{\text{Sp}}_{2n}(k)$ acting in a space V_π . Assume that π is a local ψ -Howe lift of σ . Then there is a nontrivial $\tilde{\text{Sp}}_{2n}(k)$ -intertwining and $\text{SO}_{2r+1}(k)$ -equivariant map

$$\rho : \mathcal{S}(V^n) \otimes V_{\sigma^\vee} \longrightarrow V_\pi$$

(V , as before, is the vector space, of dimension $2r + 1$, over k , on which $\text{SO}_{2r+1}(k)$ acts from the left, preserving $(\cdot, \cdot)_V$. Also, V_{σ^\vee} is a realization

of σ^\vee .) Let $\eta_{\psi_{\tilde{U},\lambda}}$ be a (nontrivial) Whittaker functional on V_π , with respect to $(\tilde{U}_n, \psi_{\tilde{U},\lambda})$. As in the proof of Proposition 2.1, consider the composition $b_{\psi_{\tilde{U},\lambda}} = \eta_{\psi_{\tilde{U},\lambda}} \circ \rho$. We view $b_{\psi_{\tilde{U},\lambda}}$, as a (nontrivial) bilinear form on $\mathcal{S}(V^n) \times V_{\sigma^\vee}$ satisfying the quasi-invariance property

$$(2.32) \quad b_{\psi_{\tilde{U},\lambda}}(\omega_\psi(g, u)\varphi, \sigma^\vee(g)\xi) = \psi_{\tilde{U},\lambda}(u)b_{\psi_{U,\lambda}}(\varphi, \xi)$$

for $u \in \tilde{U}_n$, $g \in \text{SO}_{2r+1}(k)$, $\varphi \in \mathcal{S}(V^n)$, $\xi \in V_{\sigma^\vee}$. Let $J_{N_n, \psi_{\tilde{U},\lambda}}$ denote the Jacquet functor with respect to N_n and $\psi_{\tilde{U},\lambda}|_{N_n}$. Then we may first view $b_{\psi_{\tilde{U},\lambda}}$ as a bilinear form on $J_{N_n, \psi_{\tilde{U},\lambda}}(\mathcal{S}(V^n)) \times V_{\sigma^\vee}$, satisfying

$$(2.33) \quad b_{\psi_{\tilde{U},\lambda}}(\omega_\psi(g, (\tilde{m}(z), 1))\varphi, \sigma^\vee(g)\xi) = \psi_n(z)b_{\psi_{U,\lambda}}(\varphi, \xi)$$

for $z \in Z_n$. We continue denoting by ω_ψ the action of $\tilde{m}(Z_n) \times \text{SO}_{2r+1}(k)$ on φ in $J_{N_n, \psi_{\tilde{U},\lambda}}(\mathcal{S}(V^n))$. Let

$$V_\lambda^n = \left\{ (v_1, \dots, v_n) \in V^n : \text{Gr}(v_1, \dots, v_n) = \begin{pmatrix} 0_{n-1} & 0 \\ 0 & \lambda \end{pmatrix} \right\}$$

where $\text{Gr}(v_1, \dots, v_n) = ((v_1, v_j)_V)_{n \times n}$. Now,

$$(2.34) \quad J_{N_n, \psi_{\tilde{U},\lambda}}(\mathcal{S}(V^n)) \cong \mathcal{S}(V_\lambda^n) .$$

This follows as in (2.11). We have the exact sequence

$$0 \rightarrow J_{N_n, \psi_{\tilde{U},\lambda}}(\mathcal{S}(V^n \setminus V_\lambda^n)) \xrightarrow{i} J_{N_n, \psi_{\tilde{U},\lambda}}(\mathcal{S}(V^n)) \xrightarrow{r} J_{N_n, \psi_{\tilde{U},\lambda}}(\mathcal{S}(V_\lambda^n)) \rightarrow 0$$

where r is induced by restriction of functions on V^n to V_λ^n , and i is induced by extending functions on $V^n \setminus V_\lambda^n$ by zero. By (2.32), we find, as in the proof of Proposition 2.1, that $J_{N_n, \psi_{\tilde{U},\lambda}}(\mathcal{S}(V^n \setminus V_\lambda^n)) = 0$. This proves (2.34). Note that N_n acts on $\mathcal{S}(V_\lambda^n)$ by $\psi_{\tilde{U},\lambda}$. Thus, the space of bilinear forms (2.33) is isomorphic to

$$(2.35) \quad \text{Hom}_{\text{SO}_{2r+1}(k)} \left(J_{\tilde{m}(Z_n), \psi_n}(\mathcal{S}(V_\lambda^n)) \otimes V_{\sigma^\vee}, \mathbb{C} \right)$$

where $\text{SO}_{2r+1}(k)$ acts trivially on \mathbb{C} . By Witt's theorem, the orbits of $\text{SO}_{2r+1}(k) \times \tilde{m}(Z_n)$ in V_λ^n have the form

$$(2.36) \quad \text{SO}_{2r+1}(k) \cdot (0 \cdots 0e_1 0 \cdots 0e_2 0 \cdots 0e_j 0 \cdots 0e_\lambda)Z_n .$$

Here j and the location of e_1, \dots, e_j among the zeroes determine the orbit. As in the proof of Proposition 2.1, the orbit (2.36) contributes zero to (2.35), as long as there are zeroes in the representative of (2.36). In particular, the space (2.35) is zero, if $r < n - 1$, and hence σ cannot have a (nontrivial) ψ -Howe lift to $\text{Sp}_{2n}(k)$, which is $\psi_{\tilde{U},\lambda}$ -generic.

We go back to the case of the proposition, $r = n$. As we just explained, the space (2.35) is isomorphic to

$$(2.37) \quad \text{Hom}_{\text{SO}_{2n+1}(k) \times \tilde{m}(Z_n)}(\mathcal{S}(\Omega_n) \otimes V_{\sigma^\vee}, \mathbb{C})$$

where $\tilde{m}(Z_n)$ acts on \mathbb{C} by ψ_n , and

$$\Omega_n = \text{SO}_{2n+1}(k)(e_1, e_2, \dots, e_{n-1}, e_\lambda)Z_n .$$

Note that the space $\mathcal{S}(\Omega_n)$ is isomorphic to the compactly induced representation

$$c - \text{Ind}_{R'_\lambda}^{\text{SO}_{2n+1}(k) \times \tilde{m}(Z_n)}(1),$$

where R'_λ is the stabilizer in $\text{SO}_{2n+1}(k) \times \tilde{m}(Z_n)$ of $(e_1, \dots, e_{n-1}, e_\lambda)$, consisting of elements of following type:

$$(2.38) \quad \left\{ \left(\left(\begin{pmatrix} \zeta & y & b \\ & d & y' \\ & & \zeta^* \end{pmatrix}, \tilde{m} \left(\begin{pmatrix} \zeta & x \\ 0 & 1 \end{pmatrix} \right) \right) \in \text{SO}_{2n+1}(k) \times \tilde{m}(Z_n) \mid \begin{array}{l} \zeta \in Z_{n-1}, de_\lambda = e_\lambda \\ y \cdot e_\lambda = \sum_{i=1}^{n-1} x_i e_i \end{array} \right\} .$$

Here, we view d as an element of $\text{SO}(V_3)$ and y as an element of

$$\text{Hom}_k(V_3, \text{Span}_k\{e_1, \dots, e_{n-1}\}).$$

What we proved, so far, is that the space of bilinear forms (2.32) is isomorphic to

$$\text{Hom}_{\text{SO}_{2n+1}(k) \times \tilde{m}(Z_n)} \left(c - \text{Ind}_{R'_\lambda}^{\text{SO}_{2n+1}(k) \times \tilde{m}(Z_n)}(1) \otimes \sigma^\vee, \psi_n \right),$$

which, by Frobenius reciprocity is isomorphic to $\text{Hom}_{R'_\lambda}(\text{res}_{R'_\lambda}(\psi_n^{-1} \otimes \sigma^\vee), 1)$. When we consider (2.38), it is easy to see that the last space is isomorphic to

$$(2.39) \quad \text{Hom}_{R_\lambda}(\text{res}_{R_\lambda}(\sigma), b_{(1, \psi, \lambda)}).$$

We used the fact that σ is self-dual (see [MVW, p. 91]). This means that σ has a (nontrivial) Bessel model of type $(R_\lambda, 1)$. □

Let us continue the line of proof of Proposition 2.3 and consider the case $r = n - 1$. We will keep the same notation. Since in this case λ must be a square (so take $\lambda = 1$), the space (2.35) (with $r = n - 1$) is isomorphic to

$$\text{Hom}_{\text{SO}_{2n-1}(k) \times \tilde{m}(Z_n)}(\mathcal{S}(\Omega'_n) \otimes V_{\sigma^\vee}, \mathbb{C}_{\psi_n})$$

where

$$\Omega'_n = \text{SO}_{2n-1}(k)(e_1, e_2, \dots, e_{n-1}, e)Z_n .$$

Again the space $\mathcal{S}(\Omega'_n)$ can be written as a compactly induced representation

$$c - \text{Ind}_{S_1}^{\text{SO}_{2n-1}(k) \times \tilde{m}(Z_n)}(1),$$

where

$$S_1 = \left\{ \left(\begin{pmatrix} \zeta & y & b \\ & 1 & y' \\ & & \zeta^* \end{pmatrix}, \tilde{m} \begin{pmatrix} \zeta & y \\ 0 & 1 \end{pmatrix} \right) \in \mathrm{SO}_{2n-1}(k) \times \tilde{m}(Z_n) \right\}.$$

Thus, the space of bilinear forms (2.32) is isomorphic to

$$\mathrm{Hom}_{\mathrm{SO}_{2n-1}(k) \times \tilde{m}(Z_n)} \left(\mathrm{c} - \mathrm{Ind}_{S_1}^{\mathrm{SO}_{2n-1}(k) \times \tilde{m}(Z_n)} (1) \otimes \sigma^\vee, \psi_n \right),$$

which is, by Frobenius reciprocity, isomorphic to

$$(2.40) \quad \mathrm{Hom}_{S_1}(\mathrm{res}_{S_1}(\psi_n^{-1} \otimes \sigma^\vee), 1) \cong \mathrm{Hom}_{U'}(\sigma^\vee, \psi_{U'}) \cong \mathrm{Hom}_{U'}(\sigma, \psi_{U'}).$$

Here U' is the standard maximal unipotent subgroup of $\mathrm{SO}_{2n-1}(k)$ and $\psi_{U'}$ is its standard nondegenerate character defined by ψ . Since the last space is nontrivial, we conclude that σ is generic.

As in the proof of Proposition 2.1, where we obtained (2.29), we may view $b_{\psi_{\tilde{U},\lambda}}(\varphi, \xi)$, satisfying (2.32), as a distribution on V^n , for fixed ξ ; then the content of the proof of the isomorphism of the space (2.32) with (2.39) is that $b_{\psi_{\tilde{U},\lambda}}(\varphi, \xi)$ has the form, for $r = n$,

$$(2.41) \quad b_{\psi_{\tilde{U},\lambda}}(\varphi, \xi) = \int_{S_\lambda \backslash \mathrm{SO}_{2n+1}(k)} \omega_\psi(g, 1) \varphi(e_1, e_2, \dots, e_{n-1}, e_\lambda) \beta(\xi)(g) dg$$

where β is a nonzero Bessel functional on V_σ , of type $(R_{\lambda,1})$, and S_λ is the stabilizer in $\mathrm{SO}_{2n+1}(k)$ of $(e_1, \dots, e_{n-1}, e_\lambda)$. Similarly, in case $r = n - 1, \lambda = 1$, the content of the isomorphism of the space (2.32) and the space (2.40) is that $b_{\psi_{\tilde{U},1}}(\varphi, \xi)$ has the form

$$(2.42) \quad b_{\psi_{\tilde{U},1}}(\varphi, \xi) = \int_{C' \backslash \mathrm{SO}_{2n-1}(k)} \omega_\psi(g, 1) \varphi(e_1, e_2, \dots, e_{n-1}, e) W(\sigma(g)\xi) dg$$

where W is a $\psi_{U'}$ -Whittaker functional on V_σ , and C' is the stabilizer in $\mathrm{SO}_{2n-1}(k)$ of (e_1, \dots, e_{n-1}, e) .

Note that the integrals in (2.41) and (2.42) converge absolutely. This is shown as for the integral in (2.29). For example, let us sketch the convergence of (2.41) in case $\lambda = 1$. By the Iwasawa decomposition, it is enough to show that the integration along $S_1 \backslash B$ is carried in a compact support, where B denotes the Borel subgroup of $\mathrm{SO}_{2n+1}(k)$. Thus it is enough to show that the following function has compact support

$$\varphi \left(z^{-1} a^{-1} e_1, \dots, z^{-1} a^{-1} e_{n-1}, e + \sum_{i=1}^n x_i e_i \right) \beta(\xi)(\mathrm{diag}(a, u(x_n), a^*))$$

where $a = \mathrm{diag}(a_1, \dots, a_{n-1})$, $z \in Z_{n-1}$, and

$$u(x_n) = \begin{pmatrix} 1 & -x_n & -1/2x_n^2 \\ & 1 & x_n \\ & & 1 \end{pmatrix}.$$

Since $\varphi \in S(V^n)$, looking at its last coordinate, we see that the support in $(x_1, \dots, x_n) \in k^n$ is compact. Next, the function $\beta(\xi)(\text{diag}(a, u(x_n), a^*))$ vanishes for $a \in (k^*)^{n-1}$, if $\max_{1 \leq i \leq n-1} \{|a_i|\}$ is large, and x_n remains in a compact set of k . The proof for this is the same as for Whittaker functions. Denote the last function by $f(a, x_n)$. We can find a unipotent element u in $\text{SO}_{2n+1}(k)$ close to the identity, such that it fixes $\beta(\xi)$. We then get $f(a, x_n) = b_{(1, \psi, 1)}(u^a)f(a, x_n)$ (see the paragraph before Prop. 2.2 for notation), where u^a denotes conjugation of u by $\text{diag}(a, I_3, a^*)$. From the last equality for any u close enough to the identity, we conclude that if $f(a, x_n)$ is nonzero, then the coordinates of a are bounded (above). Finally, if $\varphi(z^{-1}a^{-1}e_1, \dots, z^{-1}a^{-1}e_{n-1}, e + \sum_{i=1}^n x_i e_i)$ is nonzero, then $\max_{1 \leq i \leq n-1} \{|a_i|^{-1}\}$ is bounded, and then z must lie in a compact set as well.

We summarize.

COROLLARY 2.2. *Let π be an irreducible $\psi_{\widetilde{U}, \lambda}$ -generic representation of $\widetilde{\text{Sp}}_{2n}(k)$.*

(1) *Assume that σ is an irreducible representation of $\text{SO}_{2n+1}(k)$, which is a local ψ -Howe lift of π . Then σ has a Bessel model of special type $(R_{\lambda, 1})$. Moreover, the functional $b_{\psi_{\widetilde{U}, \lambda}}$, viewed as a bilinear form on $\omega_\psi \otimes \sigma \cong \omega_\psi \otimes \sigma^\vee$ has the form (2.41), where β is a Bessel functional on V_σ , of type $(R_\lambda, 1)$. The $\psi_{\widetilde{U}, \lambda}$ -Whittaker model of π is spanned by the functions*

$$(2.43) \quad h \mapsto \int_{S_\lambda \backslash \text{SO}_{2n+1}(k)} \omega_\psi(g, h) \varphi(e_1, e_2, \dots, e_{n-1}, e_\lambda) \beta(\xi)(g) dg .$$

(2) *The representation π has no nontrivial local ψ -Howe lifts to $\text{SO}_{2r+1}(k)$, for $r < n - 1$.*

(3) *Assume that σ is an irreducible representation of $\text{SO}_{2n-1}(k)$, which is a local ψ -Howe lift of π . Then λ is a square (take $\lambda = 1$) and σ is generic. Moreover, the functional $b_{\psi_{\widetilde{U}, 1}}$ has the form (2.42). The $\psi_{\widetilde{U}, 1}$ -Whittaker model of π is spanned by the functions*

$$(2.44) \quad h \mapsto \int_{C' \backslash \text{SO}_{2n-1}(k)} \omega_\psi(g, h) \varphi(e_1, e_2, \dots, e_{n-1}, e) W(\sigma(g)\xi) dg .$$

PROPOSITION 2.4. *Let σ be an irreducible, generic, supercuspidal representation of $\text{SO}_{2n+1}(k)$. Then σ has a nontrivial local ψ -Howe lift to $\widetilde{\text{Sp}}_{2n}(k)$. Moreover, there is a nontrivial space $t_\psi(\sigma)$ of $\psi_{\widetilde{U}, 1}$ -Whittaker functions on $\widetilde{\text{Sp}}_{2n}(k)$, which is invariant to right translations and is spanned by the functions (2.43) with β a Bessel functional on V_σ of special type $(R_1, 1)$, such that*

$$(2.45) \quad \text{Hom}_{\text{SO}_{2n+1}(k) \times \widetilde{\text{Sp}}_{2n}(k)} (\omega_\psi \otimes \sigma, t_\psi(\sigma)) \neq 0 .$$

Proof. By Proposition 2.2, σ has a nontrivial Bessel functional β of special type $(R_1, 1)$. Consider the integrals (2.43) (with this β and $\lambda = 1$). They converge absolutely (as shown before Cor. 2.2). Since the space consisting of the functions $\varphi(g^{-1}(e_1, \dots, e_{n-1}, e_n))$ contains the space $\mathcal{S}(S_1 \backslash SO_{2n+1}(k))$, it follows that integrals in (2.43) as the (φ, ξ) vary cannot be identically zero for any given $\beta(\xi)(g)$, by means of the usual density argument.

Let $t_\psi(\sigma)$ be the space of functions on $\widetilde{Sp}_{2n}(k)$ spanned by the integrals (2.43). Then $t_\psi(\sigma)$ consists of $\psi_{\widetilde{U},1}$ -Whittaker functions and affords a smooth representation, by right translations of $\widetilde{Sp}_{2n}(k)$. By construction, we clearly have (2.45). Write (2.43) as a sum of two terms $W_{\varphi,\xi}^+(h) + W_{\varphi,\xi}^-(h)$, where

$$W_{\varphi,\xi}^\pm(h) = \frac{1}{2} \int_{S_\lambda \backslash SO_{2n+1}(k)} (\omega_\psi(g, h)\varphi \pm \omega_\psi(-g, h)\varphi)(e_1, e_2, \dots, e_{n-1}, e_n)\beta(\xi)(g)dg.$$

Denote by $t_\psi^\pm(\sigma)$ the space spanned by the functions $W_{\varphi,\xi}^\pm$ as φ varies in S and ξ varies in V_σ . Since $t_\psi(\sigma)$ is nontrivial, one of the spaces $t_\psi^\pm(\sigma)$ say $t_\psi^\varepsilon(\sigma)$, is nontrivial. Of course $t_\psi^\varepsilon(\sigma)$, is a nontrivial $\widetilde{Sp}_{2n}(k)$ - submodule of $t_\psi(\sigma)$ and

$$\text{Hom}_{\widetilde{Sp}_{2n}(k)O_{2n+1}(k)}(\omega_\psi \otimes \sigma^\varepsilon, t_\psi^\varepsilon(\sigma)) \neq 0$$

and hence

$$\text{Hom}_{\widetilde{Sp}_{2n}(k)O_{2n+1}(k)}\left(\Theta_\psi^{n,n}(\sigma^\varepsilon) \otimes \sigma^\varepsilon, t_\psi^\varepsilon(\sigma) \otimes \sigma^\varepsilon\right) \neq 0.$$

In particular, $\Theta_\psi^{n,n}(\sigma^\varepsilon) \neq 0$, and since σ^ε is supercuspidal, we conclude, as in the proof of Corollary 2.1(2), that σ^ε has a nontrivial local ψ -Howe lift to $\widetilde{Sp}_{2n}(k)$. In particular, σ has a nontrivial local ψ -Howe lift to $\widetilde{Sp}_{2n}(k)$. \square

The main theorem of this section is:

THEOREM 2.2. *Let σ and π be irreducible, supercuspidal representations of $SO_{2n+1}(k)$ and $\widetilde{Sp}_{2n}(k)$ respectively. Assume that σ is generic and that π is $\psi_{\widetilde{U},1}$ -generic. Then*

- (1) σ has a unique nontrivial local ψ -Howe lift to $\widetilde{Sp}_{2n}(k)$. This lift is supercuspidal and $\psi_{\widetilde{U},1}$ -generic.
- (2) For $n \geq 2$, π has a unique nontrivial local ψ -Howe lift to $SO_{2n+1}(k)$. This lift is supercuspidal and generic.

Proof. Let σ be an irreducible, supercuspidal, generic representation of $SO_{2n+1}(k)$. By Proposition 2.4, σ has a nontrivial local ψ -Howe lift to $\widetilde{Sp}_{2n}(k)$. Let σ^ε be as in the proof of Prop. 2.4. By Proposition 2.1, $n = n_0(\sigma^\varepsilon)$, and

hence, by Theorem 2.1, $\Theta_\psi^{n,n}(\sigma^\varepsilon) = \theta_\psi^{n,n}(\sigma^\varepsilon)$ is irreducible and supercuspidal. By the proof of Proposition 2.4, we have

$$\text{Hom}_{\widetilde{\text{Sp}}_{2n}(k)\text{O}_{2n+1}(k)}\left(\theta_\psi^{n,n}(\sigma^\varepsilon) \otimes \sigma^\varepsilon, t_\psi^\varepsilon(\sigma) \otimes \sigma^\varepsilon\right) \neq 0.$$

This implies that $\theta_\psi^{n,n}(\sigma^\varepsilon)$ is a sub-representation of $t_\psi^\varepsilon(\sigma)$. Since $t_\psi^\varepsilon(\sigma)$ is realized in a space of $\psi_{\widetilde{U},1}$ -Whittaker functions, $\theta_\psi^{n,n}(\sigma^\varepsilon)$ is also $\psi_{\widetilde{U},1}$ -generic. Put $\pi_\varepsilon = \theta_\psi^{n,n}(\sigma^\varepsilon)$. Of course π_ε is a nontrivial local ψ -Howe lift of σ to $\widetilde{\text{Sp}}_{2n}(k)$. Note now that it is impossible to have both π_+ and π_- nontrivial, since in such a case, we will get that both π_\pm are $\psi_{\widetilde{U},1}$ -generic local ψ -Howe lifts of σ . By Corollary 2.2(1), both $\psi_{\widetilde{U},1}$ -Whittaker models of π_\pm are given by the spans of the integrals (2.43). This implies that $\pi_+ \cong \pi_-$, and hence, by Theorem 2.1, $\sigma^+ \cong \theta_{n,n}^\psi(\pi_+) \cong \theta_{n,n}^\psi(\pi_-) \cong \sigma^-$. This is impossible since σ^+ and σ^- are not isomorphic. Thus, if π is a local ψ -Howe lift of σ to $\widetilde{\text{Sp}}_{2n}(k)$, then π is a local ψ -Howe lift of one of the representations σ^\pm , say $\sigma^{\varepsilon'}$, and then it follows, by the above and Theorem 2.1 that $\pi_{\varepsilon'} \cong \theta_\psi^{n,n}(\sigma^{\varepsilon'}) \cong \pi$. This forces $\varepsilon' = \varepsilon$ and $\pi \cong \pi_\varepsilon$. This proves part (1). Note that $\theta_\psi^{n,n}(\sigma^\varepsilon) \cong t_\psi(\sigma)$. This follows from Corollary 2.2(1). Indeed, since $\theta_\psi^{n,n}(\sigma^\varepsilon)$ is a local ψ -Howe lift of σ , and it is $\psi_{\widetilde{U},1}$ -generic, then its $\psi_{\widetilde{U},1}$ -Whittaker model is spanned by the functions (2.43), i.e. the spanning set of $t_\psi(\sigma)$.

Let π be an irreducible, supercuspidal, $\psi_{\widetilde{U},\lambda}$ -generic representation of $\widetilde{\text{Sp}}_{2n}(k)$. Assume that $n \geq 2$. We claim that π has no nontrivial local ψ -Howe lift to $\text{SO}_{2n-1}(k)$. Otherwise, if σ' is an irreducible representation of $\text{SO}_{2n-1}(k)$, which is a local ψ -Howe lift of π , then by Corollary 2.2, parts (2), (3), σ' is supercuspidal and generic. By part (1) of Theorem 2.2 (just proved), σ' has a nontrivial, supercuspidal ($\psi_{\widetilde{U}}$ -generic) ψ -Howe lift to $\widetilde{\text{Sp}}_{2n-2}(k)$. This contradicts the tower principle of Theorem 2.1. (The ψ -Howe lifts of the supercuspidal representation σ' of $\text{SO}_{2n-1}(k)$ to both $\widetilde{\text{Sp}}_{2n-2}(k)$ and $\widetilde{\text{Sp}}_{2n}(k)$ are nontrivial and supercuspidal.) Note that supercuspidal Weil representations of $\widetilde{\text{SL}}_2(k)$ do lift to $\text{SO}_1(k)$. We conclude from this, Corollary 2.2(2) and Cor. 2.1(2) that $n_0(\pi) = n$, and hence, by Theorem 2.1, $\Theta_{n,n}^\psi(\pi) = \theta_{n,n}^\psi(\pi)$ is irreducible and supercuspidal. From the proof of Corollary 2.1(2), using the same notation, it follows that

$$\text{Hom}_{\widetilde{\text{Sp}}_{2n}(k)\text{O}_{2n+1}(k)}\left(\theta_{n,n}^\psi(\pi) \otimes \pi, t'_\psi(\pi) \otimes \pi\right) \neq 0.$$

This implies that $\theta_{n,n}^\psi(\pi)$ is a sub-representation of $t'_\psi(\pi)$. Since $t'_\psi(\pi)$ is realized in a space of $\psi_{\widetilde{U},1}$ -Whittaker functions, $\theta_{n,n}^\psi(\pi)$ is also $\psi_{\widetilde{U},1}$ -generic. (Note again that $\theta_{n,n}^\psi(\pi) \cong t'_\psi(\pi)$, as follows from Corollary 2.1(1).) This completes the proof of Theorem 2.2. \square

2.2. *Relation to global Howe duality.* In this subsection, we realize an irreducible, generic, supercuspidal representation as a local component of an irreducible, automorphic, cuspidal, generic representation and discuss the local-global relation.

Let F be a number field and ν_0 be a finite place of F , such that $F_{\nu_0} = k$. Let σ be an irreducible, supercuspidal, generic representation of $\mathrm{SO}_{2n+1}(k)$, and let π be the (unique) local ψ -Howe lift of σ to $\widetilde{\mathrm{Sp}}_{2n}(k)$. By Theorem 2.2, π is an irreducible, supercuspidal, $\psi_{\widetilde{U},1}$ -generic representation of $\widetilde{\mathrm{Sp}}_{2n}(k)$. There is an element, which is a square in k^* , α^2 (and we may even take it to lie in $1 + \mathcal{P}_{\nu_0}^N$, where \mathcal{P}_{ν_0} is the maximal ideal in the ring of integers of k), such that there is a nontrivial character ψ_0 of \mathbb{A}/F (\mathbb{A} is the adèle ring of F), such that $\psi_{0,\nu_0} = \psi^{\alpha^2}$, i.e. $\psi_{0,\nu_0}(x) = \psi(\alpha^2 x)$, for all x in F_{ν_0} . Modifying ψ by ψ^{α^2} is not harmful, since $\omega_\psi \cong \omega_{\psi^{\alpha^2}}$ and π is $\psi_{\widetilde{U},1}$ -generic, if and only if it is $\psi_{\widetilde{U},1}^{\alpha^2}$ -generic. Note that π is generic with respect to the character

$$u \rightarrow \psi(u_{12+}, \dots, +u_{n-1,n} + u_{n,n+1}),$$

if and only if it is generic with respect to the character

$$u \rightarrow \psi(c_1 u_{12+}, \dots, +c_{n-1} u_{n-1,n} + c_n^2 u_{n,n+1}),$$

for any c_1, \dots, c_n in k^* (this is clear from the action by conjugation of the diagonal subgroup on \widetilde{U}). Thus, we may replace ψ by ψ^{α^2} , and hence we may just assume that $\alpha = 1$, so that there is a nontrivial character ψ_0 of \mathbb{A}/F , satisfying $\psi_{0,\nu_0} = \psi$.

Let S_0 be the (finite) set of finite places ν of F , which satisfy (at least) one of the following conditions.

- 1) The residual characteristic of F_ν is the same as that of F_{ν_0} .
- 2) The residual characteristic of F_ν is two.
- 3) $\psi_{0,\nu}$ is not normalized (i.e. its conductor is not the ring of integers at ν).
- 4) The residual characteristic at ν is equal to that of a place ν' , which satisfies the previous condition.

Choose for each place $\nu \in S_0$ an irreducible, supercuspidal, $(\psi_{0,\nu})_{\widetilde{U}_{\nu,1}}$ -generic representation π_ν of $\widetilde{\mathrm{Sp}}_{2n}(F_\nu)$, such that $\pi_{\nu_0} = \pi$. As in Theorem 2.2 of [V], there is an irreducible, automorphic, cuspidal, $(\psi_0)_{\widetilde{U}_{\mathbb{A},1}}$ -generic representation Π of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, such that $\Pi_\nu \cong \pi_\nu$, for all $\nu \in S_0$. (Here we abuse notation and denote by $\widetilde{U}_{\mathbb{A}}$ the adèle analogue of \widetilde{U} .)

Consider the global theta lift $\Theta(\Pi, \psi_0)$ of Π from $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ to $\mathrm{SO}_{2n+1}(\mathbb{A})$. To be consistent with our local set up, $\Theta(\Pi, \psi_0)$ is spanned by

$$g \mapsto \int_{\mathrm{Sp}_{2n}(F) \backslash \widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})} \theta_{\psi_0}^\phi(g, h) \overline{\xi(h)} dh.$$

Here $\theta_{\psi_0}^\phi(g, h)$ is the theta series for the dual pair $\mathrm{SO}_{2n+1}(\mathbb{A}) \times \widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ associated to ψ_0 and the Schwartz function ϕ , and ξ varies in the space of Π . By Proposition 3 in [F], $\Theta(\Pi, \psi_0)$ is nontrivial and generic (in the sense that the ψ_0 -Whittaker coefficient (and hence any other Whittaker coefficient) is nontrivial on $\Theta(\Pi, \psi_0)$). We claim that $\Theta(\Pi, \psi_0)$ is cuspidal. Otherwise, there is an integer $m < n$, such that $\Theta_m(\Pi, \psi_0)$, the theta lift of Π to $\mathrm{SO}_{2m+1}(\mathbb{A})$, $\Theta_m(\Pi, \psi_0)$, is nontrivial. Take the first such m . Then $\Theta_m(\Pi, \psi_0)$ is cuspidal. Let Σ be an irreducible summand of $\Theta_m(\Pi, \psi_0)$. We clearly have, at ν_0 ,

$$\mathrm{Hom}_{\mathrm{SO}_{2m+1}(k) \times \widetilde{\mathrm{Sp}}_{2n}(k)} \left(\omega_\psi^{(m,n)} \otimes \pi^\vee, \Sigma_{\nu_0} \right) \neq 0$$

where $\omega_\psi^{(m,n)}$ is the Weil representation for the dual pair $\mathrm{SO}_{2m+1}(k) \times \widetilde{\mathrm{Sp}}_{2n}(k)$. Thus π has a local ψ -Howe lift to $\mathrm{SO}_{2m+1}(k)$. This contradicts the tower principle of Theorem 2.1, since π already has a local ψ -Howe lift to the supercuspidal representation σ on $\mathrm{SO}_{2n+1}(k)$. Since $\Theta(\Pi, \psi_0)$ is cuspidal, we can decompose $\Theta(\Pi, \psi_0) = \bigoplus \Sigma_i$ into a direct sum of irreducible summands. Note that each summand is (irreducible and) cuspidal. Since $\Theta(\Pi, \psi_0)$ is generic, there is a summand, call it Σ , which is generic. Since Σ_{ν_0} is a local ψ -Howe lift of π , we find by Theorem 2.2, that $\Sigma_{\nu_0} \cong \sigma$. Similarly, for any other $\nu \in S_0$, Σ_ν is a local ψ -Howe lift of π_ν , and again, since π_ν is a supercuspidal, $(\psi_{0,\nu})_{\widetilde{U}_{\nu,1}}$ -generic representation π_ν of $\widetilde{\mathrm{Sp}}_{2n}(F_\nu)$, we get (by Theorem 2.2) that Σ_ν is a supercuspidal and generic. This proves:

PROPOSITION 2.5. *Let σ be an irreducible, supercuspidal, generic representation of $\mathrm{SO}_{2n+1}(k)$. Let π be the local ψ -Howe lift of σ to $\widetilde{\mathrm{Sp}}_{2n}(k)$. Let F be a number field, and let ν_0 be a place of F , such that $F_{\nu_0} \cong k$. Assume that there is ψ_0 as before. Then there exists an irreducible, automorphic, cuspidal, $(\psi_0)_{\widetilde{U}_{\mathbb{A},1}}$ -generic representation Π of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ such that*

- 1) $\Pi_{\nu_0} \cong \pi$.
- 2) *There exists an irreducible, automorphic, cuspidal, generic representation Σ of $\mathrm{SO}_{2n+1}(\mathbb{A})$, such that $\Sigma \subset \Theta(\Pi, \psi_0)$ and $\Sigma_{\nu_0} \cong \sigma$.*

Moreover, let, for each $\nu \in S_0, \nu \neq \nu_0$ (S_0 as before), π_ν be an irreducible, supercuspidal, $(\psi_{0,\nu})_{\widetilde{U}_{\nu,1}}$ -generic representation of $\widetilde{\mathrm{Sp}}_{2n}(F_\nu)$. Then we may take Π and Σ as above, such that $\Pi_\nu \cong \pi_\nu$ (and hence Σ_ν , being the local $\psi_{0,\nu}$ Howe lift of π_ν , is supercuspidal and generic), for all $\nu \in S_0, \nu \neq \nu_0$.

3. Local gamma factors

The basic theory of local gamma factors for $SO(2n + 1)$ and the twisted ones has been established by F. Shahidi ([Sh1] and [Sh2]) and D. Soudry ([S1], [S2], and [S3]), by different methods. We shall discuss the basic properties of local gamma factors related to various lifting problems of representations of p -adic groups. We will also discuss local gamma factors for metaplectic groups.

3.1. *Basic facts on local gamma factors.* Let k be a non-archimedean local field of characteristic zero with residual field consisting of q elements. We shall recall mainly from [S1] some basic facts on local gamma factors, for $SO(2n + 1)$ and later on we recall, mainly from [GRS2,3], local gamma factors for metaplectic groups.

Let σ be an irreducible admissible generic representation of $SO_{2n+1}(k)$ with $\mathcal{W}(\sigma, \psi)$ the associated standard Whittaker model with respect to the additive character ψ . Let ϱ be an irreducible admissible generic representation of $GL_l(k)$ with $\mathcal{W}(\varrho, \psi^{-1})$ the associated standard Whittaker model with respect to the additive character ψ^{-1} . Let $P_l = M_l N_l$ be the standard maximal parabolic subgroup of (split) $SO_{2l}(k)$ with the Levi subgroup $M_l \cong GL_l(k)$. Let $I(\varrho, s)$ be the unitarily induced representation of $SO_{2l}(k)$ from P_l , which is realized in the space of all smooth functions:

$$\Phi_{\varrho,s} : SO_{2l}(k) \rightarrow \mathcal{W}(\varrho, \psi^{-1})$$

satisfying the following condition:

$$\Phi_{\varrho,s}(m(a)ny)(x) = |\det a|^{s+\frac{l-1}{2}} \Phi_{\varrho,s}(y)(xa)$$

where $y \in SO_{2l}(k)$, $m(a) \in M_l(k)$, $n \in N_l(k)$, and $x \in GL_l(k)$. For the sake of convenience, we shall write $\Phi_{\varrho,s}$ as a \mathbb{C} -valued function with two variables:

$$\Phi_{\varrho,s}(y)(x) = \xi_{\varrho,s}(y; x),$$

where $y \in SO_{2l}(k)$ and $x \in GL_l(k)$.

The local Rankin-Selberg convolution $\mathcal{A}(W_\sigma; \xi_{\varrho,s})$ for $SO_{2n+1}(k) \times GL_l(k)$ is defined by formulas (1.2.3) and (1.3.1) in [S1] for $n \leq l$ and $n \geq l$, respectively. To illustrate the construction, we recall the definition of the integral $\mathcal{A}(W_\sigma; \xi_{\varrho,s})$ for the case $l \leq n$. For $d \in M_{(n-l) \times l}(k)$, we set

$$\bar{x}(d) := \begin{pmatrix} I_l & 0 & & & \\ d & I_{n-l} & & & \\ & & 1 & & \\ & & & I_{n-l} & \\ & & & d' & I_l \end{pmatrix} \in SO_{2n+1}(k).$$

Set $\bar{X}_{(l,n)} := \{\bar{x}(d) \mid d \in M_{(n-l) \times l}(k)\}$. For $W_\sigma \in \mathcal{W}(\sigma, \psi)$ and $\xi_{\varrho,s} \in I(\varrho, s)$, the local Rankin-Selberg convolution integral $\mathcal{A}(W_\sigma; \xi_{\varrho,s})$ is defined as formula

(1.3.1) in [S1] by

$$(3.1) \quad \mathcal{A}(W_\sigma; \xi_{\varrho,s}) := \int_{V_l \backslash \mathrm{SO}_{2l}(k)} \int_{\overline{X}_{(l,n)}} W_\sigma(\overline{x}(d)j_{l,n}(h))\xi_{\varrho,s}(h, I_l)d\overline{x}(d)dh,$$

where $j_{l,n}(h)$ is the embedding of SO_{2l} into SO_{2n+1} given by

$$h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & & B \\ & I_{2(n-l)+1} & \\ C & & D \end{pmatrix}.$$

V_l is the standard maximal unipotent subgroup of $\mathrm{SO}_{2l}(k)$. As in Chapter 9 of [S1], to obtain a functional equation for local Rankin-Selberg convolution integrals, one applies the intertwining operator

$$M(w_l, \cdot) : I(\varrho, s) \rightarrow I(\varrho^{w_l}, s),$$

which is defined by

$$M(w_l, \xi_{\varrho,s})(y; x) := \int_{N_l} \xi_{\varrho,s}(w_l n_l y; x) dn_l$$

where w_l is the Weyl element in SO_{2l} as defined in §9.1. Another local Rankin-Selberg convolution integral $\tilde{\mathcal{A}}(W_\sigma; \xi_{\varrho,s})$ for $\mathrm{SO}_{2n+1}(k) \times \mathrm{GL}_l(k)$ can be defined as in §9.2, §9.3, §9.4 and §9.5 of [S1] for various different cases. Now $\tilde{\mathcal{A}}(W_\sigma; \xi_{\varrho,s})$ is obtained with slight modification from $\mathcal{A}(W_\sigma; \xi_{\varrho,s})$ by substitution of $M(w_l, \xi_{\varrho,s})$ instead of $\xi_{\varrho,s}$.

By Theorem 10.1 of [S1], there exists a rational function in q^{-s} , $\Gamma(\sigma \times \varrho, s, \psi)$, such that

$$(3.2) \quad \Gamma(\sigma \times \varrho, s, \psi) \cdot \mathcal{A}(W_\sigma; \xi_{\varrho,s}) = \tilde{\mathcal{A}}(W_\sigma; \xi_{\varrho,s}).$$

Let $\gamma(\varrho, \Lambda^2, 2s - 1, \psi)$ be the local coefficient (local gamma factor) of Shahidi [Sh2], corresponding to the intertwining operator $M(w_l, \cdot)$. Then the local Rankin-Selberg convolution gamma factor $\gamma(\sigma \times \varrho, s, \psi)$ (or simply the local gamma factor of σ twisted by ϱ) is defined as in §10.1 in [S1] by the identity

$$(3.3) \quad \gamma(\sigma \times \varrho, s, \psi) \cdot \mathcal{A}(W_\sigma; \xi_{\varrho,s}) = \tilde{\mathcal{A}}^*(W_\sigma; \xi_{\varrho,s}),$$

where

$$\tilde{\mathcal{A}}^*(W_\sigma; \xi_{\varrho,s}) = \gamma(\varrho, \Lambda^2, 2s - 1, \psi) \cdot \tilde{\mathcal{A}}(W_\sigma; \xi_{\varrho,s}).$$

Hence we have the formula

$$(3.4) \quad \Gamma(\sigma \times \varrho, s, \psi) = \frac{\gamma(\sigma \times \varrho, s, \psi)}{\gamma(\varrho, \Lambda^2, 2s - 1, \psi)},$$

which implies that $\gamma(\sigma \times \varrho, s, \psi)$ is a rational function in q^{-s} .

We recall from Chapter 11 in [S1] and from [S3] the theorem on the multiplicativity of the local twisted gamma factor $\gamma(\sigma \times \varrho, s, \psi)$. See also Shahidi's work in [Sh1].

THEOREM 3.1 (Multiplicativity of gamma factors ([S1] and [S3])). (1) *Suppose that an irreducible admissible generic representation σ of $\mathrm{SO}_{2n+1}(k)$ is a subquotient of $\mathrm{Ind}_{P_r}^{\mathrm{SO}_{2n+1}(k)}(\tau_r \otimes \sigma_{n-r})$, the unitarily induced representation from a standard maximal parabolic subgroup P_r of $\mathrm{SO}_{2n+1}(k)$, where τ_r is an admissible generic representation of $\mathrm{GL}_r(k)$ and σ_{n-r} is an admissible generic representation of $\mathrm{SO}_{2(n-r)+1}(k)$. Then*

$$\gamma(\sigma \times \varrho, s, \psi) = \omega_\tau(-1)^n \gamma(\tau_r \times \varrho, s, \psi) \cdot \gamma(\sigma_{n-r} \times \varrho, s, \psi) \cdot \gamma(\tau_r^\vee \times \varrho, s, \psi),$$

for any irreducible admissible generic representations ϱ of $\mathrm{GL}_l(k)$ with l being any positive integer, where $\gamma(\tau_r \times \varrho, s, \psi)$ and $\gamma(\tau_r^\vee \times \varrho, s, \psi)$ are the local gamma factors defined as in [JP-SS] (τ^\vee is the contragredient representation of τ .)

(2) *Suppose that an irreducible admissible generic representation ϱ of $\mathrm{GL}_l(k)$ is a subquotient of $\mathrm{Ind}_{P_{r,l-r}(k)}^{\mathrm{GL}_l(k)}(\tau_r \otimes \tau_{l-r})$, the unitarily induced representation from a standard maximal parabolic subgroup $P_{r,l-r}$ of GL_l , where τ_r is an admissible generic representation of $\mathrm{GL}_r(k)$ and τ_{l-r} is an admissible generic representation of $\mathrm{GL}_{l-r}(k)$. Then*

$$\gamma(\sigma \times \varrho, s, \psi) = \gamma(\sigma \times \tau_r, s, \psi) \cdot \gamma(\sigma \times \tau_{l-r}, s, \psi),$$

for any irreducible admissible generic representations σ of $\mathrm{SO}_{2n+1}(k)$.

A similar theory of local gamma factors $\gamma(\pi \times \varrho, s, \psi)$ can be inferred from [GRS3], for a $\psi_{\tilde{U},1}$ -generic representation π of $\tilde{\mathrm{Sp}}_{2n}(k)$ and a generic representation ϱ of $\mathrm{GL}_l(k)$. In this paper we need the case $n < l$ only (more precisely $l = 2n$), and this is explained in [GRS3, §6.2] for k non-archimedean. (The case $n = l$ is covered in [GPS] and the case $n > l$ can be done similarly; it follows closely the analogous case of $\mathrm{SO}(2n+1) \times \mathrm{GL}(l)$, shown in [S1, §8.1, 8.2].) The case where k is archimedean can be done as well, similarly to [S2, §3]. However, in this paper we need less, namely we may assume that π and ϱ are components at one place of globally generic automorphic forms. In this case, the local functional equation at one place (giving rise to the corresponding local gamma factor) follows easily from the global functional equation satisfied by the global integrals. We review this in the appendix (§7.1). When we return to the non-archimedean local field k , π , ϱ , ψ , $n < l$, as above, the local gamma factor is the proportionality factor in a local functional equation of the form

$$\frac{\gamma(\pi \times \varrho, s, \psi)}{\gamma(\varrho, s - 1/2, \psi)\gamma(\varrho, \Lambda^2, 2s - 1, \psi)} \mathcal{B}(W, \varphi, \xi_{\varrho,s}) = \tilde{\mathcal{B}}(W, \varphi, M(w; \xi_{\varrho,s})).$$

Here W is a function in the ψ Whittaker model of π , φ is a Schwartz function on k^n , and $\xi_{\varrho,s}$ is a holomorphic section for $J(\varrho, s)$, the representation of $\mathrm{Sp}_{2l}(k)$ induced from the Siegel parabolic subgroup Q_l and the representation $\varrho|\det|^{s-\frac{1}{2}}$. The section $\xi_{\varrho,s}$ takes values in an appropriate Whittaker model of ϱ

and $M(w; \cdot)$ is the local intertwining operator attached to the Weyl element

$$w = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}.$$

In [GRS2], we defined

$$\mathcal{B}(W, \varphi, \xi_{\varrho, s}) = \int_{U(k) \backslash \mathrm{Sp}_{2n}(k)} W(g) J_{\psi, n}(\omega_{\psi}(g)\varphi, j(g)\xi_{\varrho, s}) dg$$

where U is the standard maximal unipotent subgroup of $\mathrm{Sp}(2n)$. The precise form of $J_{\psi, n}$ (a Fourier-Jacobi model) is given [GRS2, (1.11)], with ψ^{-1} replacing ψ . Here $j(g)$ is an appropriate embedding of g in $\mathrm{Sp}_{2l}(k)$ and $J_{\psi, n}(\varphi, \xi_{\varrho, s})$ is given by an integral which stabilizes on large compact open subgroups of a certain unipotent subgroup. In particular it is holomorphic in s , and actually, this is a polynomial in q^{-s} (q is the number of elements in the residue field of k). The right-hand side of the functional equation has a similar form, with $M(w; \xi_{\varrho, s})$ replacing $\xi_{\varrho, s}$.

3.2. Poles of local gamma factors. We study here the relation between the existence of possible poles of the twisted local gamma factors and the structure of the given irreducible admissible generic representations.

By the subquotient theorem of Jacquet [J] or the classification of irreducible admissible generic representation of $\mathrm{GL}_n(k)$ ([M], [BZ] and [Z]), an irreducible admissible generic representation τ of $\mathrm{GL}_n(k)$ is a subquotient of a certain induced representation. More precisely, the representation τ is a subquotient of the unitarily induced representation

$$\mathrm{Ind}_{P_{n_1, \dots, n_r}(k)}^{\mathrm{GL}_n(k)} (\tau_1 | \cdot |^{z_1} \otimes \tau_2 | \cdot |^{z_2} \otimes \dots \otimes \tau_r | \cdot |^{z_r}).$$

For the sake of convenience, we use the notation from [BZ] and denote it symbolically by

$$(3.5) \quad \tau \prec \tau_1 | \cdot |^{z_1} \times \tau_2 | \cdot |^{z_2} \times \dots \times \tau_r | \cdot |^{z_r}$$

where $n = \sum_{i=1}^r n_i$ and τ_i 's are irreducible unitary supercuspidal representations of $\mathrm{GL}_{n_i}(k)$. Without loss of generality, we may assume that z_i 's are real numbers satisfying the condition:

$$(3.6) \quad z_1 \geq z_2 \geq \dots \geq z_r,$$

because of the assumption of the unitarity of the τ_i 's. $(P_{n_1, \dots, n_r}, \tau_1 \otimes \dots \otimes \tau_r)$ is called the supercuspidal support of τ , which is determined by τ uniquely up to permutation, and (z_1, \dots, z_r) is called the exponent of τ .

PROPOSITION 3.1. *Assume that an irreducible admissible generic representation τ of $\mathrm{GL}_n(k)$ has the supercuspidal support $(P_{n_1, \dots, n_r}, \tau_1 \otimes \dots \otimes \tau_r)$ and the exponent (z_1, \dots, z_r) . Let ϱ be an irreducible unitary supercuspidal*

representation of $GL_l(k)$. Then the only possible real poles of the twisted local gamma factor $\gamma(\tau \times \varrho, s, \psi)$ occur at $s = 1 - z_i$ and the only possible real zeros occur at $s = -z_i$. In either case, we have $\varrho \cong \tau_i^\vee$ for some $i \in \{1, \dots, r\}$.

Proof. By the multiplicativity theorem for the twisted local gamma factors ([JP-SS]),

$$(3.7) \quad \gamma(\tau \times \varrho, s, \psi) = \prod_{i=1}^r \gamma(\tau_i \times \varrho, s + z_i, \psi).$$

For each factor $\gamma(\tau_i \times \varrho, s + z_i, \psi)$,

$$\gamma(\tau_i \times \varrho, s + z_i, \psi) = \epsilon(\tau_i \times \varrho, s + z_i, \psi) \cdot \frac{L(\tau_i^\vee \times \varrho^\vee, 1 - (s + z_i))}{L(\tau_i \times \varrho, s + z_i)}.$$

It follows that the local gamma factor $\gamma(\tau_i \times \varrho, s + z_i, \psi)$ may have a possible real pole at $s = 1 - z_i$ and a possible real zero at $s = -z_i$. If one of these two things occurs, then we must have

$$\varrho \cong \tau_i^\vee. \quad \square$$

This has the following useful consequence.

COROLLARY 3.1. *For any irreducible unitary supercuspidal representation ϱ of $GL_l(k)$, $s = 1 - z_r$ is the rightmost possible real pole of the product*

$$\prod_{i=1}^r \gamma(\tau_i \times \varrho, s + z_i, \psi).$$

If it occurs in one of the factors in the product, it cannot be cancelled by the possible zeros of other factors. In this case, one must have that the representation ϱ is equivalent to one of the τ_j^\vee 's with $z_j = z_r$.

We can prove the same result for $SO(2n + 1)$. First we prove the following lemma.

LEMMA 3.1. *Let σ be an irreducible generic supercuspidal representation of $SO_{2n+1}(k)$ and ϱ be an irreducible supercuspidal representation of $GL_l(k)$. Assume that the twisted local gamma factor $\gamma(\sigma \times \varrho, s, \psi)$ has a pole at $s = 1$. Then the pole must be simple and the representation ϱ must be self-dual. Moreover, $L(\varrho, \Lambda^2, s)$ has a pole at $s = 0$.*

Proof. By the definition of $\gamma(\sigma \times \varrho, s, \psi)$,

$$\begin{aligned} &\gamma(\sigma \times \varrho, s, \psi) \cdot \mathcal{A}(W_\sigma, \xi_{\varrho, s}) \\ &= \gamma(\varrho, \Lambda^2, 2s - 1, \psi) \cdot \tilde{\mathcal{A}}(W_\sigma, \xi_{\varrho, s}) \end{aligned}$$

$$\begin{aligned}
 &= \epsilon(\varrho, \Lambda^2, 2s - 1, \psi) \cdot \frac{L(\varrho^\vee, \Lambda^2, 2(1 - s))}{L(\varrho, \Lambda^2, 2s - 1)} \cdot \tilde{\mathcal{A}}(W_\sigma, \xi_{\varrho, s}) \\
 &= \epsilon(\varrho, \Lambda^2, 2s - 1, \psi) \cdot L(\varrho^\vee, \Lambda^2, 2(1 - s)) \cdot \frac{\tilde{\mathcal{A}}(W_\sigma, \xi_{\varrho, s})}{L(\varrho, \Lambda^2, 2s - 1)}.
 \end{aligned}$$

Since both σ and ϱ are irreducible, generic and supercuspidal, $\mathcal{A}(W_\sigma, \xi_{\varrho, s})$ is entire (as a function in s). By Theorem 5.1 in [CS], the normalized local intertwining operator

$$L(\varrho, \Lambda^2, 2s - 1)^{-1} \cdot M(w_l, \cdot)$$

is entire and hence

$$\frac{\tilde{\mathcal{A}}(W_\sigma, \xi_{\varrho, s})}{L(\varrho, \Lambda^2, 2s - 1)}$$

is entire. (Actually, at the point of our interest, $s = 1$, the intertwining integral converges.) Further, we can choose certain data such that $\mathcal{A}(W_\sigma, \xi_{\varrho, s})$ does not vanish. Hence, if $\gamma(\sigma \times \varrho, s, \psi)$ has a pole at $s = 1$, then we must have that $L(\varrho^\vee, \Lambda^2, s)$ has a pole at $s = 0$. It is known that if $L(\varrho, \Lambda^2, s)$ has a pole at $s = 0$, then

$$\varrho \cong \varrho^\vee$$

and the pole is simple. In other words, the gamma factor $\gamma(\sigma \times \varrho, s, \psi)$ has at most a simple pole at $s = 1$ and if the pole occurs, then ϱ is self-dual. \square

Here is an analogue needed on the metaplectic side.

LEMMA 3.2. *Let π be an irreducible, supercuspidal representation of $\widetilde{\mathrm{Sp}}_{2n}(k)$. Assume that it is $\psi_{\widetilde{U}, 1}$ generic. Assume that $l > n$ and let ϱ be a representation of $\mathrm{GL}_l(k)$ induced from $\varrho_1 \otimes \cdots \otimes \varrho_r$, where ϱ_i is an irreducible unitary supercuspidal representation of $\mathrm{GL}_{m_i}(k)$ with $m_1 + \cdots + m_r = l$. Assume also that ϱ_i is not isomorphic to ϱ_j^\vee for $i \neq j$. Then the order of the pole at $s = 1$ of $\gamma(\pi \times \varrho, s, \psi)$ is less than, or equal to that of $\prod_{i=1}^r L(\varrho_i, \Lambda^2, 2(1 - s))$. In particular, if $L(\varrho_i, \Lambda^2, z)$ is holomorphic at $z = 0$ (e.g., ϱ_i is not self-dual), for $1 \leq i \leq r$, then $\gamma(\pi \times \varrho, s, \psi)$ is holomorphic at $s = 1$.*

The proof follows (as in Lemma 3.1) from the local functional equation, and the fact that Whittaker functions for π are compactly supported modulo \widetilde{U}_n (we assume that π is supercuspidal.) The details are written in Proposition 1 and Corollary 1 of [GRS6]. Note also that

$$L(\varrho^\vee, \Lambda^2, z) = \prod_{1 \leq i < j \leq r} L(\varrho_i^\vee \times \varrho_j^\vee, z) \prod_{i=1}^r L(\varrho_i^\vee, \Lambda^2, z) = \prod_{i=1}^r L(\varrho_i^\vee, \Lambda^2, z).$$

We use this for $z = 2(1 - s)$.

Returning to the orthogonal group we consider more general cases. Let σ be an irreducible admissible generic representation of $\mathrm{SO}_{2n+1}(k)$. By Jacquet’s

subquotient theorem [J], there is a standard parabolic subgroup Q with its Levi part isomorphic to

$$\mathrm{GL}_{m_1} \times \dots \times \mathrm{GL}_{m_r} \times \mathrm{SO}_{2m_0+1},$$

($n = \sum_{i=0}^r m_i$) and there are irreducible unitary supercuspidal representations τ_i of $\mathrm{GL}_{m_i}(k)$ ($i = 1, 2, \dots, r$) and an irreducible generic supercuspidal representation σ_0 of $\mathrm{SO}_{2m_0+1}(k)$, such that the representation σ is a subquotient of the unitarily induced representation

$$(3.8) \quad \sigma \prec \mathrm{Ind}_Q^{\mathrm{SO}_{2n+1}}(\tau_1 |\det|^{z_1} \otimes \dots \otimes \tau_r |\det|^{z_r} \otimes \sigma_0).$$

Without loss of generality, we may assume that the parameters z_i are real and have the property that $z_1 \geq z_2 \geq \dots \geq z_r \geq 0$. With this assumption, we say that the representation σ has supercuspidal support $(Q; \tau_1, \tau_2, \dots, \tau_r; \sigma_0)$ and exponents (z_1, z_2, \dots, z_r) .

PROPOSITION 3.2. *Let σ be an irreducible admissible generic representation of $\mathrm{SO}_{2n+1}(k)$ with supercuspidal support $(Q; \tau_1, \tau_2, \dots, \tau_r; \sigma_0)$ and exponents (z_1, z_2, \dots, z_r) . Then $s = 1 + z_1$ is the rightmost real point at which the twisted local gamma factors $\gamma(\sigma \times \varrho, s, \psi)$ can possibly have a pole for any irreducible unitary supercuspidal representation ϱ of $\mathrm{GL}_l(k)$ where l is any positive integer. If the pole at $s = 1 + z_1$ occurs for some (l, ϱ) , then*

$$\varrho \cong \tau_{i_0}$$

where τ_{i_0} is a representation among the τ_i 's such that $z_{i_0} = z_1$.

Proof. By the multiplicativity of the local gamma factors (Theorem 3.1),

$$\begin{aligned} \gamma(\sigma \times \varrho, s, \psi) &= \omega_\varrho(-1)^{rn+m_0} \\ &\cdot \left[\prod_{i=1}^r \gamma(\tau_i \times \varrho, s + z_i, \psi) \gamma(\tau_i^\vee \times \varrho, s - z_i, \psi) \right] \gamma(\sigma_0 \times \varrho, s, \psi) \end{aligned}$$

where ϱ is any irreducible unitary supercuspidal representation of $\mathrm{GL}_l(k)$.

By the argument in the proof of Proposition 3.1, we get that

- (1) the factor $\gamma(\tau_i \times \varrho, s + z_i, \psi)$ may contribute a possible real pole at $s = 1 - z_i$ and a possible real zero at $s = -z_i$, and if one of these two things occurs, then $\varrho \cong \tau_i^\vee$,
- (2) the factor $\gamma(\tau_i^\vee \times \varrho, s - z_i, \psi)$ may contribute a possible real pole at $s = 1 + z_i$ and a possible real zero at $s = z_i$, and if one of these two things occurs, one has $\varrho \cong \tau_i$, and
- (3) the factor $\gamma(\sigma_0 \times \varrho, s, \psi)$ has no zero for $\mathrm{Re}(s) > 0$ ([Sh1, §5] and [Sh2, Prop. 7.2]) and may have a possible simple real pole at $s = 1$ (Lemma 3.1). If the pole occurs, we must have $\varrho \cong \varrho^\vee$ (and $L(\varrho, \Lambda^2, s)$ has a pole at $s = 0$).

Hence $s = 1 + z_1$ is the rightmost possible real pole of the product

$$\omega_\varrho(-1)^{rn+m_0} \left[\prod_{i=1}^r \gamma(\tau_i \times \varrho, s + z_i, \psi) \gamma(\tau_i^\vee \times \varrho, s - z_i, \psi) \right] \gamma(\sigma_0 \times \varrho, s, \psi).$$

If the pole occurs at $s = 1 + z_1$, it cannot be cancelled by any possible zero from other factors in the product and $\varrho \cong \tau_{i_0}$, where τ_{i_0} is a representation among the τ_i 's, such that $z_{i_0} = z_1$. □

COROLLARY 3.2. *Let σ and σ' be irreducible admissible generic representations of $\mathrm{SO}_{2n+1}(k)$ with supercuspidal supports $(Q; \tau_1, \tau_2, \dots, \tau_l; \sigma_0)$ and $(Q'; \tau'_1, \tau'_2, \dots, \tau'_l; \sigma'_0)$, and exponents (z_1, z_2, \dots, z_l) and $(z'_1, z'_2, \dots, z'_l)$, respectively. If the twisted gamma factors are the same, i.e.*

$$\gamma(\sigma \times \varrho, s, \psi) = \gamma(\sigma' \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ with $l = 1, 2, \dots, 2n - 1$, then $z_1 = z'_1$.

3.3. Gamma factors and functorial lift. The Langlands functorial lift (or transfer) conjecture describes the relation between automorphic representations of two different groups as long as their Langlands dual groups have an ‘admissible’ relation. One may find more details about the Langlands conjectures in [B]. In this paper, we need a special case, which we describe below in more detail.

Let F be a number field. The Langlands dual group of $\mathrm{SO}(2n + 1)$ is $\mathrm{Sp}_{2n}(\mathbb{C})$ and the Langlands dual group of $\mathrm{GL}(2n)$ is $\mathrm{GL}_{2n}(\mathbb{C})$. The natural embedding of $\mathrm{Sp}_{2n}(\mathbb{C})$ into $\mathrm{GL}_{2n}(\mathbb{C})$ is ‘admissible’, so that by Langlands functorial lift conjecture, any irreducible automorphic representation Σ of $\mathrm{SO}_{2n+1}(\mathbb{A})$ can be lifted to an irreducible automorphic representation \mathcal{T} of $\mathrm{GL}_{2n}(\mathbb{A})$, functorially. In other words, if we write

$$\Sigma = \otimes_v \Sigma_v, \quad \mathcal{T} = \otimes_v \mathcal{T}_v,$$

(as the restricted tensor product of the local components), then \mathcal{T} is a (global) Langlands functorial lift of Σ if and only if for each local place v , the local component \mathcal{T}_v is a (local) Langlands functorial lift of the local component Σ_v . Moreover, a lift from Σ to \mathcal{T} is called a weak Langlands functorial lift if the local component \mathcal{T}_v is a (local) Langlands functorial lift of the local component Σ_v for all archimedean places and for almost all places v of F , where \mathcal{T}_v and Σ_v are unramified.

In [CKP-SS] the weak Langlands functorial lift from $\mathrm{SO}_{2n+1}(\mathbb{A})$ to $\mathrm{GL}_{2n}(\mathbb{A})$ was proved to exist for irreducible generic cuspidal automorphic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$, by using the converse theorem for GL_{2n} .

Let Σ be an irreducible generic cuspidal automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$ and \mathcal{T} be a weak Langlands functorial lifting to $\mathrm{GL}_{2n}(\mathbb{A})$ of Σ .

The following theorem determines the explicit structure of the image of the weak Langlands functorial lifting without any assumption (which improves the results in [CKP-SS]).

THEOREM 3.2 ([GRS5]). *Let \mathcal{T} be an irreducible, automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ which is the weak Langlands functorial lifting of an irreducible generic cuspidal automorphic representation Σ of $\mathrm{SO}_{2n+1}(\mathbb{A})$, then \mathcal{T} is generic and self-dual. Moreover, \mathcal{T} is isomorphic to*

$$\mathrm{Ind}_{P_{2n_1, \dots, 2n_r}(\mathbb{A})}^{\mathrm{GL}_{2n}(\mathbb{A})}(\mathcal{T}_1 \otimes \dots \otimes \mathcal{T}_r)$$

where $P_{2n_1, \dots, 2n_r}$ is the standard parabolic subgroup of GL_{2n} corresponding to the partition $2n = \sum_{i=1}^r 2n_i$; \mathcal{T}_i for $i = 1, 2, \dots, r$, is an irreducible, unitary, self-dual, cuspidal, automorphic representation of $\mathrm{GL}_{2n_i}(\mathbb{A})$ such that the partial exterior square L -function $L^S(\mathcal{T}_i, \Lambda^2, s)$ has a pole at $s = 1$, and $\mathcal{T}_i \not\cong \mathcal{T}_j$ if $i \neq j$. In particular, \mathcal{T} is uniquely determined up to isomorphism by Σ .

Moreover, one has the following result on local gamma factors.

PROPOSITION 3.3. *Let Σ_v be the v -local component of an irreducible generic cuspidal automorphic representation Σ of $\mathrm{SO}_{2n+1}(\mathbb{A})$ and let \mathcal{T} be the weak Langlands functorial lifting to $\mathrm{GL}_{2n}(\mathbb{A})$ of Σ . Then for every supercuspidal representation τ_v of $\mathrm{GL}_l(F_v)$ where l is any positive integer, one has*

$$\gamma(\Sigma_v \times \tau_v, s, \psi_v) = \gamma(\mathcal{T}_v \times \tau_v, s, \psi_v).$$

It is the result of Corollary 5 in [CKP-SS] that the identity

$$\gamma(\Sigma_v \times \tau_v, s, \psi_v) = \gamma(\mathcal{T}_v \times \tau_v, s, \psi_v)$$

holds for every supercuspidal representation τ_v of $\mathrm{GL}_l(F_v)$ with $l = 1, 2, \dots, 2n - 1$. The argument in *loc. cit.* is valid for any l , with no restriction.

3.4. Structure of the image of local functorial lifting. Let σ be an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(F_{v_0})$ and let Σ be an irreducible generic cuspidal automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$ as constructed in Proposition 2.5. Then

$$\Sigma_{v_0} \cong \sigma.$$

Let \mathcal{T} be the image of Σ under the weak Langlands functorial lifting. Then, by Proposition 3.3,

$$(3.9) \quad \gamma(\Sigma_{v_0} \times \varrho, s, \psi_{v_0}) = \gamma(\mathcal{T}_{v_0} \times \varrho, s, \psi_{v_0}),$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(F_{v_0})$ (l any positive integer).

We shall determine the explicit structure of the v_0 -local component $\tau := \mathcal{T}_{v_0}$ in terms of the supercuspidal support by using the existence of poles of the local gamma factors, the global version of which was given in Theorem 3.2.

By the subquotient theorem or the classification of irreducible generic representations of $\mathrm{GL}_{2n}(k)$ ($k = F_{v_0}$) ([M], [BZ] and [Z]), the irreducible admissible generic self-dual representation τ of $\mathrm{GL}_{2n}(k)$ is a subquotient of an induced representation

$$(3.10) \quad \tau \prec \tau_1 |\cdot|^{z_1} \times \cdots \times \tau_r |\cdot|^{z_r} \times \eta_1 \times \cdots \times \eta_t \times \tau_r^\vee |\cdot|^{-z_r} \times \cdots \times \tau_1^\vee |\cdot|^{-z_1},$$

where τ_i 's are irreducible unitary supercuspidal representations of $\mathrm{GL}_{m_i}(k)$, and η_j 's are irreducible unitary supercuspidal self-dual representations of $\mathrm{GL}_{2n_j}(k)$ ($n = \sum_{i=1}^r m_i + \sum_{j=1}^t n_j$). From the given data, we may also assume that if $z_i = 0$ then τ_i is not self-dual. Without loss of generality, we may assume that z_i 's are real numbers satisfying the condition:

$$(3.11) \quad z_1 \geq z_2 \geq \cdots \geq z_r \geq 0.$$

THEOREM 3.3. *Let σ be an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(k)$. There exists a unique irreducible generic representation τ of $\mathrm{GL}_{2n}(k)$ such that*

$$(3.12) \quad \gamma(\sigma \times \varrho, s, \psi) = \gamma(\tau \times \varrho, s, \psi),$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ (l any positive integer). Moreover, in the notation of (3.10), τ must have the following properties:

- (1) $\tau = \eta_1 \times \cdots \times \eta_t$, where η_j are irreducible unitary supercuspidal self-dual representations of $\mathrm{GL}_{2n_j}(k)$ ($2n = \sum_{j=1}^t 2n_j$) and $\eta_i \neq \eta_j$ if $i \neq j$;
- (2) the local L -function $L(\eta_i, \Lambda^2, s)$ has a pole at $s = 0$ for $i = 1, 2, \dots, t$.

Proof. The existence of τ is given by the weak functorial lifting described above ($\tau = \mathcal{T}_{v_0}$ in (3.9)) and the uniqueness of τ follows from the local converse theorem for GL_n ([Hn2]). It remains to determine the structure of τ explicitly. Suppose that τ has supercuspidal support as in (3.10). Let ϱ be an irreducible, supercuspidal representation of $\mathrm{GL}_l(k)$. By Lemma 3.1, we know that the gamma factor $\gamma(\sigma \times \varrho, s, \psi)$ has at most a simple pole at $s = 1$ and if the pole occurs, then ϱ is self-dual. By the identity in (3.12), we know that the local gamma factor $\gamma(\tau \times \varrho, s, \psi)$ has at most a simple pole at $s = 1$ and if the pole occurs, then ϱ is self-dual. We are going to show that this information about the existence of a pole of $\gamma(\tau \times \varrho, s, \psi)$ at $s = 1$ and the order of the pole control the structure of τ .

By using the multiplicativity theorem for the twisted local gamma factors in this case (Theorem 3.1 in [JP-SS]), we have

$$\begin{aligned} &\gamma(\tau \times \varrho, s, \psi) \\ &= \left[\prod_{i=1}^r \gamma(\tau_i \times \varrho, s + z_i, \psi) \cdot \gamma(\tau_i^\vee \times \varrho, s - z_i, \psi) \right] \cdot \prod_{j=1}^t \gamma(\eta_j \times \varrho, s, \psi). \end{aligned}$$

By Corollary 3.1, we know that $1 + z_1$ is the rightmost possible real pole of the twisted local gamma factor $\gamma(\tau \times \varrho, s, \psi)$ and if it occurs, it cannot be cancelled by the zeros of other twisted local gamma factors in the product.

Now, we take $\varrho = \tau_1$. Then the local gamma factor $\gamma(\tau \times \tau_1, s, \psi)$ has a pole at $s = 1 + z_1$. By the identity in (3.12), we know that the local gamma factor $\gamma(\sigma \times \tau_1, s, \psi)$ has a pole at $s = 1 + z_1$. Repetition of the argument of Lemma 3.1 implies that the local exterior square L -function $L(\tau_1, \Lambda^2, s)$ has a pole at $s = -2z_1$. Since τ_1 is unitary (supercuspidal), we get that $\text{Re}(2z_1) = 0$, and hence $z_1 = 0$. We know by Lemma 3.1 that the pole is simple, and the representation τ_1 is self-dual. Now, this implies that

$$z_1 = \dots = z_r = 0$$

and the local gamma factor $\gamma(\tau \times \tau_1, s, \psi)$ can be expressed as

$$\gamma(\tau \times \tau_1, s, \psi) = [\gamma(\tau_1 \times \tau_1, s, \psi)]^2 \times [\dots].$$

Hence the local gamma factor $\gamma(\tau \times \tau_1, s, \psi)$ has a pole at $s = 1$ with order two or higher. This contradicts the simplicity of the pole of the local gamma factor $\gamma(\sigma \times \tau_1, s, \psi)$ because of (3.12). Therefore we conclude that the supercuspidal representation τ_1 should not occur in the supercuspidal support of τ .

By the same argument, we can conclude that all supercuspidal representations τ_i with $i = 1, 2, \dots, r$, do not occur in the supercuspidal support of τ . Namely we have that

$$\tau \prec \eta_1 \times \dots \times \eta_t.$$

Since the induced representation $\eta_1 \times \dots \times \eta_t$ is irreducible, we conclude that

$$\tau \cong \eta_1 \times \dots \times \eta_t.$$

Since the pole at $s = 1$ of the gamma factor $\gamma(\sigma \times \varrho, s, \psi)$ is at most simple, the representations η_j in the supercuspidal support of τ must be all distinct. Finally the existence of the pole of the local L -functions $L(\eta_i, \Lambda^2, s)$ at $s = 0$ for $i = 1, 2, \dots, t$ is now clear from the argument above (see Lemma 3.1 again). \square

COROLLARY 3.3. *Let Σ be an irreducible generic cuspidal automorphic representation of $\text{SO}_{2n+1}(\mathbb{A})$ and \mathcal{T} be the weak Langlands functorial lifting of Σ from $\text{SO}_{2n+1}(\mathbb{A})$ to $\text{GL}_{2n}(\mathbb{A})$. For a finite local place v of the number*

field F , if the v -local component Σ_v is supercuspidal, then the corresponding v -local component \mathcal{T}_v has form

$$\mathcal{T}_v = \eta_1 \times \cdots \times \widetilde{\eta}_{t_v}$$

where the η_j 's are irreducible unitary self-dual supercuspidal representations of $\mathrm{GL}_{2n_j}(F_v)$ such that

- (1) $\eta_i \neq \eta_j$ if $i \neq j$ ($2n = \sum_{j=1}^{t_v} 2n_j$);
- (2) the local L -function $L(\eta_i, \Lambda^2, s)$ has a pole at $s = 0$ for $i = 1, 2, \dots, t_v$.

3.5. *Gamma factors and Howe duality.* We discuss here the relation of local gamma factors under the local Howe duality. Of course, these gamma factors should remain invariant, but here we need much less. We need just the preservation of a pole at $s = 1$, as follows.

PROPOSITION 3.4. *Let σ be an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(k)$ and let π be the irreducible $\psi_{\widetilde{U},1}$ -generic supercuspidal representation of $\widetilde{\mathrm{Sp}}_{2n}(k)$, which is the local ψ -Howe lift of σ . Let τ be the local lift of σ to $\mathrm{GL}_{2n}(F)$, i.e. the representation given by Theorem 3.3. Write $\tau = \eta_1 \times \cdots \times \eta_t$ as in Theorem 3.3. Then $\gamma(\pi \times \tau, s, \psi)$ has a pole of order t at $s = 1$.*

Proof. Let Σ and Π be the representations constructed in Proposition 2.5. We keep the notation used prior to Proposition 2.5. (We may and do assume that $\psi_{0,\nu_0} = \psi$.) Recall the finite set of places S_0 , and the choices of $(\psi_{0,\nu})_{\widetilde{U}_\nu}$ generic supercuspidal representations π_ν of $\widetilde{\mathrm{Sp}}_{2n}(k_\nu)$, such that $\Pi_{\nu_0} \cong \pi$, and $\Pi_\nu \cong \pi_\nu$, for each place ν in S_0 . Recall also that Σ is the global theta lift of Π with respect to ψ_0 , and, in particular, Σ_ν is the local $\psi_{0,\nu}$ -Howe lift of π_ν , for each ν in S_0 (and hence $\Sigma_{\nu_0} \cong \sigma$). Recall that Σ is globally generic (and hence Σ_ν is generic, for each ν in S_0).

Let \mathcal{T} be an irreducible representation of $\mathrm{GL}_{2n}(\mathbb{A})$ of the form

$$\mathrm{Ind}_{P_{m_1, \dots, m_r}(\mathbb{A})}^{\mathrm{GL}_{2n}(\mathbb{A})}(\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_r)$$

where \mathcal{T}_i is an irreducible, automorphic, cuspidal, unitary representation of $\mathrm{GL}_{m_i}(\mathbb{A})$, $1 \leq i \leq r$, $m_1 + \cdots + m_r = 2n$. Note that \mathcal{T} is irreducible. We realize it automorphically by taking an Eisenstein series induced from $\mathcal{T}_1 |\det|^{s_1} \otimes \cdots \otimes \mathcal{T}_r |\det|^{s_r}$, and evaluating it at $(s_1, \dots, s_r) = (0, \dots, 0)$.

Let S be a finite set of finite places, containing S_0 , such that at finite places outside S , Π (hence Σ) and \mathcal{T} are unramified. For such places ν ,

$$\gamma(\Sigma_\nu \times \mathcal{T}_\nu, s, \psi_{0,\nu}) = \gamma(\Pi_\nu \times \mathcal{T}_\nu, s, \psi_{0,\nu}).$$

The functional equation satisfied by the global integrals for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_{2n}$ and for $\widetilde{\mathrm{Sp}}_{2n} \times \mathrm{GL}_{2n}$, means that the product over all places of the

corresponding gamma factors is one. From the last equalities outside S , we conclude

$$\gamma_\infty(\Sigma \times \mathcal{T}, s, \psi_0) \prod_{\nu \in S} \gamma(\Sigma_\nu \times \mathcal{T}_\nu, s, \psi_{0,\nu}) = \gamma_\infty(\Pi \times \mathcal{T}, s, \psi_0) \prod_{\nu \in S} \gamma(\Pi_\nu \times \mathcal{T}_\nu, s, \psi_{0,\nu}).$$

Here γ_∞ denotes the product of local gamma factors over all archimedean places. Write $S = \cup_{i=1}^N S(i)$, where $S(i)$ is the set of all places ν in S , such that the residual characteristic of k_ν is p_i , and p_1, \dots, p_N are different prime numbers. Put

$$A_i(s) = \prod_{\nu \in S(i)} \frac{\gamma(\Pi_\nu \times \mathcal{T}_\nu, s, \psi_{0,\nu})}{\gamma(\Sigma_\nu \times \mathcal{T}_\nu, s, \psi_{0,\nu})}.$$

Thus, $A_i(s)$ is in $\mathbb{C}(p_i^{-s})$. Write $A_i(s) = R_i(p_i^{-s})$, where $R_i(x) \in \mathbb{C}(x)$. We get

$$\prod_{i=1}^N R_i(p_i^{-s}) = \frac{\gamma_\infty(\Sigma \times \mathcal{T}, s, \psi_0)}{\gamma_\infty(\Pi \times \mathcal{T}, s, \psi_0)}.$$

We show in Section 7.2 that there is $M > 0$, such that for $|\text{Im}(s)| > M$, both $\gamma_\infty(\Sigma \times \mathcal{T}, s, \psi_0)$ and $\gamma_\infty(\Pi \times \mathcal{T}, s, \psi_0)$ are holomorphic with no zeroes. Thus, if s_0 is a pole (resp. a zero) of $R_j(p_j^s)$, then $s_0 + \frac{2\pi mi}{\log p_j}$ is a pole (resp. a zero) of $R_j(p_j^{-s})$, for all integers m . Since p_1, \dots, p_N are different, we see that $\prod_{i=1}^N R_i(p_i^{-s})$ has an unbounded sequence of poles (resp. zeroes) on the line $\text{Re}(s) = s_0$. This is impossible unless each $R_i(x)$ is an exponential. Thus there are a_i and b_i , such that

$$a_i e^{b_i s} \prod_{\nu \in S(i)} \gamma(\Sigma_\nu \times \mathcal{T}_\nu, s, \psi_{0,\nu}) = \prod_{\nu \in S(i)} \gamma(\Pi_\nu \times \mathcal{T}_\nu, s, \psi_{0,\nu}).$$

In particular, there is an exponential $\alpha(s)$, such that

$$\alpha(s) \prod_{\nu \in S'_0} \gamma(\Sigma_\nu \times \mathcal{T}_\nu, s, \psi_{0,\nu}) = \prod_{\nu \in S'_0} \gamma(\Pi_\nu \times \mathcal{T}_\nu, s, \psi_{0,\nu})$$

where S'_0 is the set of places ν in S , where the residual characteristic of k_ν is the same as that of k_{ν_0} (note that this is a subset of S_0).

Consider now the special case where $\mathcal{T}_{\nu_0} \cong \tau$, where τ is the local lift of σ to $\text{GL}_{2n}(k_{\nu_0})$, as in Theorem 3.3. More precisely, take $m_i = 2n_i$ and \mathcal{T}_i such that it has a trivial central character, and its ν_0 component is isomorphic to η_i . For each $i \leq t$, let χ_i be a unitary automorphic character of the adèles of k , such that $\chi_{i,\nu_0} = 1$ and for $\nu \neq \nu_0$ in S'_0 , $\mathcal{T}_{i,\nu} \chi_{i,\nu}$ is not isomorphic to the dual of $\mathcal{T}_{j,\nu} \chi_{j,\nu}$, for all $i, j \leq t$. (Such characters exist by Theorem 5, p.103 of [AT]. For example, it is enough to guarantee that $\chi_{i,\nu}^{2n_i} \chi_{j,\nu}^{2n_j} \neq 1$, for $\nu \neq \nu_0$ in S'_0 .) Let $\mathcal{T}(\chi)$ be the representation obtained from \mathcal{T} by twisting \mathcal{T}_i by χ_i . Repeating the last equality, we have an exponential $\alpha(s)$, such that

$$(3.13) \quad \alpha(s) \prod_{\nu \in S'_0} \gamma(\Sigma_\nu \times \mathcal{T}(\chi)_\nu, s, \psi_{0,\nu}) = \prod_{\nu \in S'_0} \gamma(\Pi_\nu \times \mathcal{T}(\chi)_\nu, s, \psi_{0,\nu}).$$

We have,

$$\gamma(\Sigma_{\nu_0} \times \mathcal{I}(\chi)_{\nu_0, s, \psi_{0, \nu_0}}) = \gamma(\sigma \times \tau, s, \psi) = \gamma(\tau \times \tau, s, \psi),$$

which, by the structure of τ , has a pole of order t at $s = 1$. The remaining factors of the left-hand side of (3.13) are holomorphic and nonzero at $s = 1$. This follows from Lemma 3.1, the multiplicativity property, and the choice of χ_i . We conclude that the left hand side of (3.13) has a pole of order t at $s = 1$. By Lemma 3.2, all factors on the right-hand side of (3.13), corresponding to $\nu \neq \nu_0$, are holomorphic at $s = 1$ (again, due to our choice of χ_i). We conclude from (3.13) that $\gamma(\pi \times \tau, s, \psi)$ has a pole of order t at $s = 1$. This completes the proof of the proposition. \square

4. The local converse theorem for $\mathrm{SO}(2n + 1)$: The supercuspidal case

In this section, we prove the local converse theorem for irreducible generic supercuspidal representations of $\mathrm{SO}(2n+1)$. The idea of the proof is to transfer the local converse theorem for $\mathrm{GL}_{2n}(k)$ to $\mathrm{SO}_{2n+1}(k)$ by combining various liftings.

THEOREM 4.1 (Local converse theorem for $\mathrm{SO}(2n + 1)$: The supercuspidal case). *Let σ and σ' be irreducible generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$. Assume that*

$$\gamma(\sigma \times \varrho, s, \psi) = \gamma(\sigma' \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$, $l = 1, 2, \dots, 2n-1$. Then the representations σ and σ' are equivalent.

Proof. Let σ and σ' be irreducible generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$, such that

$$(4.1) \quad \gamma(\sigma \times \varrho, s, \psi) = \gamma(\sigma' \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ with $l = 1, \dots, 2n - 1$. Let π and π' be the local ψ -Howe lifts of σ and σ' respectively to $\widetilde{\mathrm{Sp}}_{2n}(k)$ (Theorem 2.2). These are irreducible, $\psi_{\widetilde{U}, 1}$ -generic, supercuspidal representations of $\widetilde{\mathrm{Sp}}_{2n}(k)$.

By Theorem 3.3, there are irreducible generic representations τ and τ' of $\mathrm{GL}_{2n}(k)$ corresponding to σ and σ' , respectively, such that

$$(4.2) \quad \gamma(\sigma \times \varrho, s, \psi) = \gamma(\tau \times \varrho, s, \psi)$$

and

$$(4.3) \quad \gamma(\sigma' \times \varrho, s, \psi) = \gamma(\tau' \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ with any positive integer l . In particular,

$$\gamma(\tau \times \varrho, s, \psi) = \gamma(\tau' \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ with $l = 1, 2, \dots, 2n - 1$. By Theorem 1.1,

$$\tau \cong \tau'.$$

By Theorem 3.3 again,

$$(4.4) \quad \tau \cong \tau' \cong \eta_1 \times \dots \times \eta_t$$

such that each η_i is an irreducible unitary self-dual supercuspidal representation of $\mathrm{GL}_{2n_i}(k)$, $\eta_i \not\cong \eta_j$ if $i \neq j$, and the local exterior square L -function $L(\eta_i, \Lambda^2, s)$ has a pole at $s = 0$ for $i = 1, 2, \dots, t$. By Proposition 3.4, both gamma factors $\gamma(\pi \times \tau, s, \psi)$ and $\gamma(\pi' \times \tau, s, \psi)$ have a pole of order t at $s = 1$. By the theory of the local backward lifting from $\mathrm{GL}_{2n}(k)$ to $\widetilde{\mathrm{Sp}}_{2n}(k)$ in [GRS6], the image $\pi_\psi(\tau)$ of τ under the backward lifting is the unique irreducible $\psi_{\widetilde{U},1}$ -generic supercuspidal representation of $\widetilde{\mathrm{Sp}}_{2n}(k)$ with the property that the twisted local gamma factor $\gamma(\pi_\psi(\tau) \times \tau, s, \psi)$ has a pole of order t at $s = 1$. Since π and π' are supercuspidal and $\psi_{\widetilde{U},1}$ -generic, we conclude that

$$\pi_\psi(\tau) \cong \pi \cong \pi'.$$

Finally, it follows from Theorem 2.2 that the representation σ is equivalent to the representation σ' . □

5. The local converse theorem: general case

We shall use the information on poles of twisted gamma factors to determine explicitly the structure of the supercuspidal support of irreducible generic representations. Then the general case of the local converse theorem follows from the supercuspidal case.

THEOREM 5.1. *Let σ and σ' be irreducible generic representations of $\mathrm{SO}_{2n+1}(k)$ with supercuspidal supports $(Q; \tau_1, \tau_2, \dots, \tau_r; \sigma_0)$ and $(Q'; \tau'_1, \tau'_2, \dots, \tau'_{r'}; \sigma'_0)$, and exponents (z_1, z_2, \dots, z_r) and $(z'_1, z'_2, \dots, z'_{r'})$, respectively (see (3.8)). Assume that*

$$\gamma(\sigma \times \varrho, s, \psi) = \gamma(\sigma' \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ with $l = 1, 2, \dots, 2n - 1$. Then, after a possible rearrangement of $(\tau'_1, z'_1; \dots; \tau'_{r'}, z'_{r'})$, without affecting the decreasing order of $z'_1, \dots, z'_{r'}$,

- (1) $r = r'$ and $m_i = m'_i$ for $i = 0, 1, \dots, r$,

- (2) $z_i = z'_i$ and $\tau_i \cong \tau'_i$ for $i = 1, 2, \dots, r$,
- (3) for the representations σ_0 and σ'_0 of $\text{SO}_{2m_0+1}(k)$, the twisted gamma factors are the same, i.e.

$$\gamma(\sigma_0 \times \varrho, s, \psi) = \gamma(\sigma'_0 \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\text{GL}_l(k)$ with l as above.

Proof. By assumption,

$$(5.1) \quad \gamma(\sigma \times \varrho, s, \psi) = \gamma(\sigma' \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\text{GL}_l(k)$ with $l = 1, 2, \dots, 2n - 1$. By Proposition 3.2 and Corollary 3.2, we have $z_1 = z'_1$. Taking $\varrho = \tau_1$ we know that $\gamma(\sigma \times \tau_1, s, \psi)$ and hence $\gamma(\sigma' \times \tau_1, s, \psi)$ each has a pole at $s = 1 + z_1 = 1 + z'_1$. By the proof of Proposition 3.2 applied to the local gamma factor $\gamma(\sigma' \times \tau_1, s, \psi)$, we find that either (1) $z_1 = z'_i = 0$ and $\tau'_i \cong \tau_1^\vee$, for some i , or (2) $z_1 = z'_i$ and $\tau'_i \cong \tau_1$, for some i , or (3) $\tau_1 \cong \tau_1^\vee$ and $z_1 = 0$. We shall consider these three cases as follows.

If case (1) occurs, then $z'_1 = z_1 = 0$. This implies that

$$z_1 = \dots = z_r = 0 = z'_1 = \dots = z'_r.$$

In this case we change the order of the supercuspidal support $(Q'; \tau'_1, \tau'_2, \dots, \tau'_r; \sigma'_0)$ by making τ_i^\vee the first representation, so that now $\tau'_1 \cong \tau_1$. It follows that

$$\gamma(\tau_1 \times \varrho, s, \psi) = \gamma(\tau'_1 \times \varrho, s, \psi)$$

and

$$\gamma(\tau_1^\vee \times \varrho, s, \psi) = \gamma(\tau_1'^\vee \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\text{GL}_l(k)$. By Theorem 3.1 (multiplicativity of gamma factors),

$$\gamma(\sigma \times \varrho, s, \psi) = \omega_\varrho(-1)^{rn+m_0} \left[\prod_{i=1}^r \gamma(\tau_i \times \varrho, s, \psi) \gamma(\tau_i^\vee \times \varrho, s, \psi) \right] \gamma(\sigma_0 \times \varrho, s, \psi)$$

and

$$\gamma(\sigma' \times \varrho, s, \psi) = \omega_\varrho(-1)^{r'n+m'_0} \left[\prod_{i=1}^{r'} \gamma(\tau'_i \times \varrho, s, \psi) \gamma(\tau_i'^\vee \times \varrho, s, \psi) \right] \gamma(\sigma'_0 \times \varrho, s, \psi).$$

By cancelling the gamma factors

$$\gamma(\tau_1 \times \varrho, s, \psi) = \gamma(\tau'_1 \times \varrho, s, \psi)$$

and

$$\gamma(\tau_1^\vee \times \varrho, s, \psi) = \gamma(\tau_1'^\vee \times \varrho, s, \psi)$$

from equation (5.1), we obtain a new identity for products of gamma factors, i.e.

$$(5.2) \quad \omega_\varrho(-1)^{rn+m_0} \left[\prod_{i=2}^r \gamma(\tau_i \times \varrho, s, \psi) \gamma(\tau_i^\vee \times \varrho, s, \psi) \right] \gamma(\sigma_0 \times \varrho, s, \psi) \\ = \omega_\varrho(-1)^{r'n+m'_0} \left[\prod_{i=2}^{r'} \gamma(\tau'_i \times \varrho, s, \psi) \gamma(\tau'_i{}^\vee \times \varrho, s, \psi) \right] \gamma(\sigma'_0 \times \varrho, s, \psi).$$

It is clear that the arguments to prove Proposition 3.2 and Corollary 3.2 are applicable to the above identity. This reduces the proof to the case where the number of gamma factors in both sides of the identity is smaller by two.

If case (2) occurs, then we have $z_1 = z'_1$, $z_1 = z'_i$ and $\tau'_i \cong \tau_1$. Hence,

$$z'_1 = \cdots = z'_i.$$

If we change the order of the supercuspidal data $(Q'; \tau'_1, \tau'_2, \dots, \tau'_{r'}; \sigma'_0)$ by making τ'_i the first representation, so that $\tau'_1 \cong \tau_1$, then the same argument as in case (1) reduces the proof to (5.2).

Finally, we shall show that case (3) can reduce to cases (1) and (2), or yield a contradiction. If case (3) occurs, then τ_1 is self-dual, i.e. $\tau_1 \cong \tau_1^\vee$ and $z_1 = 0$. Hence,

$$z_1 = \cdots = z_r = 0 = z'_1 = \cdots = z'_{r'}.$$

By the conclusion of case (1) and case (2), we may assume that the representation τ_1 is not isomorphic to any one of the representations $\tau'_i, \tau'_i{}^\vee$, for $i = 1, 2, \dots, r'$. This implies that the pole at $s = 1$ ($z_1 = 0$) of the gamma factor $\gamma(\sigma' \times \tau_1, s, \psi)$ can only be achieved by the gamma factor $\gamma(\sigma'_0 \times \tau_1, s, \psi)$. Hence this gamma factor $\gamma(\sigma' \times \tau_1, s, \psi)$ has a simple pole at $s = 1$. On the other hand, since $\tau_1 \cong \tau_1^\vee$, the gamma factor $\gamma(\sigma \times \tau_1, s, \psi)$ can be written as

$$\gamma(\sigma \times \tau_1, s, \psi) = [\gamma(\tau_1 \times \tau_1, s, \psi)]^2 \cdot [\cdots].$$

It has at least a second order pole at $s = 1$. This is a contradiction.

Therefore, by repeating the above argument, we can finally conclude that (1) $r = r'$, (2) up to permutation, $z_i = z'_i$ and $\tau_i \cong \tau'_i$ for $i = 1, 2, \dots, r$, and

$$\gamma(\sigma_0 \times \varrho, s, \psi) = \gamma(\sigma'_0 \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $GL_l(k)$ with $l = 1, \dots, 2n - 1$. □

5.1. *The proof of the local converse theorem* (Theorem 1.2). Let σ and σ' be irreducible admissible generic representations of $SO_{2n+1}(k)$. Assume that σ and σ' have supercuspidal supports $(Q; \tau_1, \tau_2, \dots, \tau_r; \sigma_0)$ and $(Q'; \tau'_1, \tau'_2, \dots, \tau'_{r'}; \sigma'_0)$, and exponents (z_1, z_2, \dots, z_r) and $(z'_1, z'_2, \dots, z'_{r'})$, respectively. More precisely, the representation σ is a subquotient of the normalized induced representation

$$\sigma \prec \text{Ind}_Q^{\text{SO}_{2n+1}}(\tau_1 | \det |^{z_1} \otimes \cdots \otimes \tau_r | \det |^{z_r} \otimes \sigma_0)$$

and the representation σ' is a subquotient of the normalized induced representation

$$\sigma' \prec \text{Ind}_Q^{\text{SO}_{2n+1}}(\tau'_1 |\det|^{z'_1} \otimes \cdots \otimes \tau'_{r'} |\det|^{z'_{r'}} \otimes \sigma'_0).$$

Without loss of generality, we may assume that the exponents (z_1, z_2, \dots, z_r) and $(z'_1, z'_2, \dots, z'_{r'})$ satisfy the condition:

$$z_1 \geq z_2 \geq \cdots \geq z_r \geq 0; \quad z'_1 \geq z'_2 \geq \cdots \geq z'_{r'} \geq 0.$$

We assume now that the gamma factors attached to σ and σ' , twisted by any irreducible supercuspidal representation ϱ of $\text{GL}_l(k)$, are the same, i.e.

$$\gamma(\sigma \times \varrho, s, \psi) = \gamma(\sigma' \times \varrho, s, \psi)$$

with $l = 1, 2, \dots, 2n - 1$. By Theorem 5.1, we conclude that (1) $r = r'$ and $m_i = m'_i$ for $i = 0, 1, \dots, r$; (2) $z_i = z'_i$ and $\tau_i \cong \tau'_i$ for $i = 1, 2, \dots, r$ (after a possible reordering of the (τ'_i, z'_i) , which does not affect the decreasing order of the exponents; and (3) as representations of $\text{SO}_{2m_0+1}(k)$, the twisted gamma factors attached to σ_0 and σ'_0 are the same, i.e.

$$\gamma(\sigma_0 \times \varrho, s, \psi) = \gamma(\sigma'_0 \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\text{GL}_l(k)$ with $l = 1, 2, \dots, 2n - 1$. Since $n \geq m_0$, it follows from Theorem 4.1 that the representations σ_0 and σ'_0 are isomorphic. Hence both irreducible admissible generic representations σ and σ' have the same supercuspidal support $(Q; \tau_1, \tau_2, \dots, \tau_r; \sigma_0)$ and the same exponent (z_1, z_2, \dots, z_r) . In other words, both σ and σ' are irreducible generic constituents (up to equivalence) of the induced representation

$$\text{Ind}_Q^{\text{SO}_{2n+1}}(\tau_1 |\det|^{z_1} \otimes \cdots \otimes \tau_r |\det|^{z_r} \otimes \sigma_0).$$

By the uniqueness of the generic constituent in the induced representation

$$\text{Ind}_Q^{\text{SO}_{2n+1}}(\tau_1 |\det|^{z_1} \otimes \cdots \otimes \tau_r |\det|^{z_r} \otimes \sigma_0),$$

the irreducible admissible generic representations σ and σ' must be equivalent. This proves the local converse theorem (Theorem 1.2) in general. □

5.2. *Some direct applications.* We shall briefly discuss two global applications to the theory of automorphic representations of $\text{SO}_{2n+1}(\mathbb{A})$. First we obtain the injectivity of the weak Langlands functorial lifting established in [CKP-SS].

THEOREM 5.2. *Let $\mathcal{SO}_{2n+1}^{\text{igca}}(\mathbb{A})$ be the set of all equivalence classes of irreducible generic cuspidal automorphic representations of $\text{SO}_{2n+1}(\mathbb{A})$ and $\mathcal{GL}_{2n}^{\text{ia}}(\mathbb{A})$ be the set of all equivalence classes of irreducible automorphic representations of $\text{GL}_{2n}(\mathbb{A})$. Then the weak Langlands functorial lifting from $\mathcal{SO}_{2n+1}^{\text{igca}}(\mathbb{A})$ to $\mathcal{GL}_{2n}^{\text{ia}}(\mathbb{A})$, established in [CKP-SS], is an injective map.*

Proof. The proof is now a straightforward consequence of Proposition 3.3 and our local converse theorem (Theorem 1.2). (Recall again that the weak lift of [CKP-SS] is functorial at archimedean places.) \square

THEOREM 5.3 (rigidity theorem). *Let $\Sigma = \otimes_v \Sigma_v$ and $\Sigma' = \otimes_v \Sigma'_v$ belong to $\mathrm{SO}_{2n+1}^{\mathrm{igca}}(\mathbb{A})$ (as defined in Theorem 5.2). If Σ_v is equivalent to Σ'_v for almost all local places v , then Σ is equivalent to Σ' .*

Proof. Let \mathcal{T} and \mathcal{T}' be the weak Langlands functorial liftings of Σ and Σ' , respectively, as constructed in [CKP-SS]. This means that at all archimedean local places and unramified finite local places, the lifting is the local Langlands functorial lifting. By assumption we know that both representations \mathcal{T} and \mathcal{T}' are equivalent at almost all local places. By Theorem 3.2, both \mathcal{T} and \mathcal{T}' are irreducible generic self-dual unitary automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A})$. By the strong multiplicity-one property for $\mathrm{GL}(n)$ ([JS]), we conclude that

$$\mathcal{T} \cong \mathcal{T}'$$

as automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A})$. By Theorem 5.2, $\Sigma \cong \Sigma'$. \square

Remark 5.1. (1) The rigidity theorem for GL_n was proved by H. Jacquet and J. Shalika ([JS]) for generic automorphic representations of $\mathrm{GL}_n(\mathbb{A})$. By applying the multiplicity-one theorem for irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ ([Shl]), we see that the rigidity theorem implies the strong multiplicity-one theorem for $\mathrm{GL}(n)$. The rigidity theorem for some lower rank cases has also been studied in [B2], [R], and [S4].

(2) In [GRS5], the local converse theorem proved in this paper for generic representations of $\mathrm{SO}_{2n+1}(k)$ is used to prove the irreducibility of the backward lift to $\mathrm{SO}_{2n+1}(\mathbb{A})$ of a representation of $\mathrm{GL}_{2n}(\mathbb{A})$ of the form described in Theorem 3.2. This was conjectured in [GRS1].

6. On local Langlands conjectures

In this chapter, we establish the local Langlands functorial lift from irreducible generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$ to $\mathrm{GL}_{2n}(k)$ and the local Langlands reciprocity law for irreducible generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$. We shall consider the local Langlands conjectures for more general representations of $\mathrm{SO}_{2n+1}(k)$ and other relevant problems in a forthcoming work of ours [JngS].

6.1. *On local Langlands functorial lifting from $\mathrm{SO}(2n+1)$ to $\mathrm{GL}(2n)$.* Let $\mathcal{S}\mathcal{O}_{2n+1}^{\mathrm{igsc}}(k)$ be the set of all equivalence classes of irreducible generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$ and $\mathcal{G}\mathcal{L}_{2n}^{\mathrm{ifl}}(k)$ ('ifl' denotes the image of the

functorial lifting) be the set of all equivalence classes of irreducible admissible generic representations of $\mathrm{GL}_{2n}(k)$ of the form

$$\tau = \eta_1 \times \eta_2 \times \cdots \times \eta_t,$$

where η_i are irreducible unitary supercuspidal self-dual representations of $\mathrm{GL}_{2n_j}(k)$ ($j = 1, 2, \dots, t$) ($\sum_{j=1}^t n_j = n$) such that

- (1) $\eta_i \not\cong \eta_j$ if $i \neq j$, and
- (2) the local L -function $L(\eta_j, \Lambda^2, s)$ has a pole at $s = 0$ for $j = 1, 2, \dots, t$.

For any $\sigma \in \mathcal{SO}_{2n+1}^{\mathrm{igsc}}(k)$, the weak Langlands functorial lifting from SO_{2n+1} to GL_{2n} produces a map

$$\ell : \sigma \mapsto \tau = \ell(\sigma)$$

such that

$$\gamma(\sigma \times \varrho, s, \psi) = \gamma(\tau \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ where l is any positive integer (Proposition 3.3). Then by Theorem 3.3, we know that the image $\tau = \ell(\sigma)$ belongs to $\mathcal{GL}_{2n}^{\mathrm{ifl}}(k)$. So there is a map

$$(6.1) \quad \ell : \sigma \mapsto \tau = \ell(\sigma)$$

from $\mathcal{SO}_{2n+1}^{\mathrm{igsc}}(k)$ to $\mathcal{GL}_{2n}^{\mathrm{ifl}}(k)$, which preserves the twisted local gamma factors; i.e.,

$$(6.2) \quad \gamma(\sigma \times \varrho, s, \psi) = \gamma(\tau \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$, where l is any positive integer. In fact, we can prove that the map ℓ is a bijection.

PROPOSITION 6.1. *Let ℓ be the map defined above with property (6.2). Then the map ℓ is a bijection from $\mathcal{SO}_{2n+1}^{\mathrm{igsc}}(k)$ onto $\mathcal{GL}_{2n}^{\mathrm{ifl}}(k)$. Moreover, such a map is unique.*

Proof. The injectivity of the map ℓ follows from the local converse theorem for SO_{2n+1} (Theorem 1.2). The uniqueness of such a map ℓ follows from the local converse theorem for GL_n (Theorem 1.1). It remains to show that the map ℓ is surjective.

For any τ' in $\mathcal{GL}_{2n}^{\mathrm{ifl}}(k)$, and $\tau' = \eta'_1 \times \eta'_2 \times \cdots \times \eta'_{t'}$, we have to construct a representation σ in $\mathcal{SO}_{2n+1}^{\mathrm{igsc}}(k)$ such that

$$\gamma(\sigma \times \varrho, s, \psi) = \gamma(\tau' \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ where l is any positive integer. By the local backward lifting from $\mathrm{GL}_{2n}(k)$ to $\widetilde{\mathrm{Sp}}_{2n}(k)$ ([GRS6]),

there exists a unique (up to isomorphism) irreducible $\psi_{\tilde{U},1}$ -generic supercuspidal representation $\pi_\psi(\tau')$ of $\widetilde{\text{Sp}}_{2n}(k)$ such that the twisted local gamma factor $\gamma(\pi_\psi(\tau') \times \tau', s, \psi)$ has a pole of order t' at $s = 1$. By Part (2) of Theorem 2.2, there exists a unique nontrivial irreducible generic supercuspidal representation σ of $\text{SO}_{2n+1}(k)$, which is the local ψ -Howe lift of $\pi_\psi(\tau')$. By Theorem 3.3, there exists an irreducible admissible generic representation τ of $\text{GL}_{2n}(k)$ such that

$$\gamma(\tau \times \varrho, s, \psi) = \gamma(\sigma \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\text{GL}_l(k)$ where l is any positive integer, and the representation τ has the following properties:

- (1) $\tau = \eta_1 \times \cdots \times \eta_t$ with $\eta_i \neq \eta_j$ if $i \neq j$, where η_i is an irreducible unitary self-dual supercuspidal representation of $\text{GL}_{2n_i}(k)$;
- (2) the local L -function $L(\eta_i, \Lambda^2, s)$ has a pole at $s = 0$ for $i = 1, 2, \dots, t$.

In particular, $\gamma(\sigma \times \tau, s, \psi)$ has a pole of order t at $s = 1$. The proof of Proposition 3.4, with $\pi = \pi_\psi(\tau')$ can be repeated with simple modifications to conclude (from the fact that $\gamma(\pi \times \tau', s, \psi)$ has a pole of order t' at $s = 1$) that $\gamma(\sigma \times \tau', s, \psi)$ has a pole of order t' at $s = 1$. All we need to do is to take $\mathcal{T}_{\nu_0} \cong \tau'$, and then in the paragraph after (3.13), conclude first, using Lemma 3.2 that the right-hand side of (3.13) has a pole of order t' at $s = 1$, and hence conclude that the left-hand side of (3.13) has a pole of order t' at $s = 1$. Now, by Lemma 3.1, we see that $\gamma(\sigma \times \tau', s, \psi)$ has a pole of order t' at $s = 1$. Since

$$\gamma(\sigma \times \tau', s, \psi) = \gamma(\tau \times \tau', s, \psi)$$

we conclude, looking at the form of τ and τ' (both are in $\mathcal{GL}_{2n}^{\text{ifl}}(k)$), that $\tau \cong \tau'$. Therefore the representation σ just constructed has the property that

$$\ell(\sigma) = \tau$$

and we have proved the map ℓ is surjective. □

PROPOSITION 6.2. *The map*

$$\ell : \sigma \mapsto \tau = \ell(\sigma)$$

from $\mathcal{SO}_{2n+1}^{\text{igsc}}(k)$ to $\mathcal{GL}_{2n}^{\text{ifl}}(k)$ preserves the twisted local ϵ -factors and local L -factors; i.e.,

$$\epsilon(\sigma \times \varrho, s, \psi) = \epsilon(\tau \times \varrho, s, \psi)$$

and

$$L(\sigma \times \varrho, s) = L(\tau \times \varrho, s)$$

for all irreducible supercuspidal representations ϱ of $\text{GL}_l(k)$ where l is any positive integer.

Proof. Let $\tau = \ell(\sigma)$. Then we know that

$$\tau = \eta_1 \times \eta_2 \times \cdots \times \eta_t,$$

where η_i are irreducible unitary self-dual supercuspidal representations of $\mathrm{GL}_{2n_j}(k)$ ($j = 1, 2, \dots, t$) ($\sum_{j=1}^t n_j = n$) such that

- (1) $\eta_i \not\cong \eta_j$ if $i \neq j$, and
- (2) the local L -function $L(\eta_j, \Lambda^2, s)$ has a pole at $s = 0$ for $j = 1, 2, \dots, t$.

More importantly, we have

$$\gamma(\sigma \times \varrho, s, \psi) = \gamma(\tau \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\mathrm{GL}_l(k)$ where l is any positive integer (Theorem 3.3 and Proposition 6.1). It follows from [Sh1], [Sh2] and [JP-SS] that

$$(6.3) \quad \gamma(\sigma \times \varrho, s, \psi) = \epsilon(\sigma \times \varrho, s, \psi) \cdot \frac{L(\sigma \times \varrho^\vee, 1-s)}{L(\sigma \times \varrho, s)}$$

and

$$(6.4) \quad \gamma(\tau \times \varrho, s, \psi) = \epsilon(\tau \times \varrho, s, \psi) \cdot \frac{L(\tau \times \varrho^\vee, 1-s)}{L(\tau \times \varrho, s)}.$$

By assumption, we have

$$(6.5) \quad \epsilon(\sigma \times \varrho, s, \psi) \cdot \frac{L(\sigma \times \varrho^\vee, 1-s)}{L(\sigma \times \varrho, s)} = \epsilon(\tau \times \varrho, s, \psi) \cdot \frac{L(\tau \times \varrho^\vee, 1-s)}{L(\tau \times \varrho, s)}.$$

If the supercuspidal representation ϱ is not equivalent to any one of the η_i 's, up to unramified unitary twisting, then

$$L(\tau \times \varrho^\vee, 1-s) = 1 = L(\tau \times \varrho, s).$$

Hence, by equation (6.5), we have,

$$\frac{L(\sigma \times \varrho^\vee, 1-s)}{L(\sigma \times \varrho, s)} = \epsilon(\tau \times \varrho, s, \psi) \cdot \epsilon(\sigma \times \varrho, s, \psi)^{-1},$$

which is an exponential function in s .

We first assume that ϱ is unitary. Since both σ and ϱ are supercuspidal, by Proposition 7.2 in [Sh2], the possible poles of $L(\sigma \times \varrho^\vee, 1-s)$ lie in $\mathrm{Re}(s) \geq 1$, but the possible poles of $L(\sigma \times \varrho, s)$ lie in $\mathrm{Re}(s) \leq 0$. Hence,

$$L(\sigma \times \varrho^\vee, 1-s) = 1 = L(\sigma \times \varrho, s).$$

Therefore,

$$L(\sigma \times \varrho, s) = L(\tau \times \varrho, s)$$

and

$$\epsilon(\sigma \times \varrho, s, \psi) = \epsilon(\tau \times \varrho, s, \psi).$$

It is clear that the same argument works when ϱ is not necessarily unitary.

If the supercuspidal representation ϱ is isomorphic to one of the η_i 's, up to unramified unitary twisting, say

$$\varrho \cong \eta_i \cdot | \cdot |^y$$

where y is purely imaginary, then we know again that there are no cancellations between the poles of $L(\sigma \times \varrho^\vee, 1 - s)$ and the poles of $L(\sigma \times \varrho, s)$, and the same thing happens with $L(\tau \times \varrho^\vee, 1 - s)$ and $L(\tau \times \varrho, s)$. Hence, from equation (6.5), the set of the poles of $L(\sigma \times \varrho, s)$ is equal to the set of poles of $L(\tau \times \varrho, s)$. Thus, the polynomials $L(\sigma \times \varrho, s)^{-1}$ and $L(\tau \times \varrho, s)^{-1}$ are equal. Therefore

$$L(\sigma \times \varrho, s) = L(\tau \times \varrho, s).$$

It follows that the ϵ -factors are also equal, i.e.

$$\epsilon(\sigma \times \varrho, s, \psi) = \epsilon(\tau \times \varrho, s, \psi). \quad \square$$

The following theorem on local Langlands functoriality follows from Propositions 6.1 and 6.2.

THEOREM 6.1 (local Langlands functoriality). *There exists a unique bijective map*

$$\ell : \sigma \mapsto \tau = \ell(\sigma)$$

from $\mathcal{SO}_{2n+1}^{\text{igsc}}(k)$ to $\mathcal{GL}_{2n}^{\text{iff}}(k)$, which preserves the twisted local L -factors, ϵ -factors and gamma factors, i.e.

$$\begin{aligned} L(\sigma \times \varrho, s) &= L(\tau \times \varrho, s), \\ \epsilon(\sigma \times \varrho, s, \psi) &= \epsilon(\tau \times \varrho, s, \psi) \end{aligned}$$

and

$$\gamma(\sigma \times \varrho, s, \psi) = \gamma(\tau \times \varrho, s, \psi)$$

for all irreducible supercuspidal representations ϱ of $\text{GL}_l(k)$ where l is any positive integer.

6.2. On the local Langlands reciprocity law for $\text{SO}(2n + 1)$. We shall establish the local Langlands reciprocity law for $\mathcal{SO}_{2n+1}^{\text{igsc}}(k)$ by using the local Langlands reciprocity law for $\text{GL}(n)$ established by M. Harris and R. Taylor [HT] and by G. Henniart [Hn3].

Let W_k be the Weil group associated to the the local field k . We take

$$W_k \times \text{SL}_2(\mathbb{C})$$

as the Weil-Deligne group ([M] and [Kn]). Let $\mathcal{G}_n^{\text{ah}}(k)$ be the set of conjugacy classes of admissible homomorphisms ρ from $W_k \times \text{SL}_2(\mathbb{C})$ to $\text{GL}_n(\mathbb{C})$. If we write

$$\rho = \bigoplus_i \rho_i^0 \otimes \lambda_i^0,$$

then the admissibility of ρ means that ρ_i^0 's are continuous complex representations of W_k with $\rho_i^0(W_k)$ semi-simple and λ_i^0 's are algebraic complex representations of $\mathrm{SL}_2(\mathbb{C})$. Let $\mathcal{GL}_n^{\mathrm{is}}(k)$ be the set of equivalence classes of irreducible smooth representations of $\mathrm{GL}_n(k)$. Then the local Langlands conjecture (or local Langlands reciprocity law), now a theorem of Harris-Taylor [HT] and Henniart [Hn3], is the following.

THEOREM 6.2 (Harris-Taylor [HT] and Henniart [Hn3]). *There exists a (unique) bijection*

$$\mathfrak{r}_n : \rho \mapsto \tau = \mathfrak{r}_n(\rho)$$

from $\mathcal{G}_n^{\mathrm{ah}}(k)$ onto $\mathrm{GL}_n^{\mathrm{is}}(k)$ satisfying the following conditions.

- (1) For any $\rho \in \mathcal{G}_n^{\mathrm{ah}}(k)$, $\det(\rho)$ corresponds to $\omega_{\mathfrak{r}_n(\rho)}$, the central character;
- (2) For any $\rho \in \mathcal{G}_n^{\mathrm{ah}}(k)$ and any quasi-character χ of k^\times , one has $\mathfrak{r}_n(\chi \otimes \rho) = (\chi \circ \det) \otimes \mathfrak{r}_n(\rho)$;
- (3) For any $\rho \in \mathcal{G}_n^{\mathrm{ah}}(k)$, one has $\mathfrak{r}_n(\rho)^\vee = \mathfrak{r}_n(\rho^\vee)$;
- (4) For any $\rho \in \mathcal{G}_n^{\mathrm{ah}}(k)$ and $\rho' \in \mathcal{G}_{n'}^{\mathrm{ah}}(k)$, one has
 - (4L) $L(\rho \otimes \rho', s) = L(\mathfrak{r}_n(\rho) \times \mathfrak{r}_{n'}(\rho'), s)$,
 - (4 ϵ) $\epsilon(\rho \otimes \rho', s, \psi) = \epsilon(\mathfrak{r}_n(\rho) \times \mathfrak{r}_{n'}(\rho'), s, \psi)$,
 - (4 γ) $\gamma(\rho \otimes \rho', s, \psi) = \gamma(\mathfrak{r}_n(\rho) \times \mathfrak{r}_{n'}(\rho'), s, \psi)$;
- (5) If $\rho = (\rho^0, \delta)$, then $(\rho^0, 1)$ with ρ^0 irreducible corresponds to $\mathfrak{r}_n(\rho)$, which is irreducible and supercuspidal.

Remark 6.1. (1) The uniqueness of the reciprocity map in Theorem 6.2 follows from Henniart's local converse theorem (Theorem 1.1) and an induction argument on n .

(2) Theorem 6.2 has been proved for supercuspidal representations by Harris and Taylor ([HT]) and by Henniart ([Hn3]). The reduction of the general case to the supercuspidal case was given by A. Zelevinsky ([Z]). Various special cases of Theorem 6.2 were proved before by several authors. We refer to [H] and [K2] for detailed comments.

In order to establish the local Langlands reciprocity conjecture for $\mathcal{SO}_{2n+1}^{\mathrm{igsc}}(k)$, the set of equivalence classes of irreducible generic supercuspidal representations of $\mathrm{SO}_{2n+1}(k)$, it is sufficient to figure out the subset of the local Langlands parameters for $\mathcal{GL}_{2n}^{\mathrm{ifl}}(k)$ by using the explicit local Langlands functorial lift from $\mathrm{SO}_{2n+1}(k)$ to $\mathrm{GL}_{2n}(k)$ (Theorem 6.1) and the local Langlands reciprocity law for $\mathrm{GL}(n)$ (Theorem 6.2).

Recall that the set $\mathcal{GL}_{2n}^{\text{iff}}(k)$ consists of equivalence classes of representations of $\text{GL}_{2n}(k)$ of the form:

$$\tau = \eta_1 \times \eta_2 \times \cdots \times \eta_t,$$

where η_i are irreducible unitary supercuspidal self-dual representations of $\text{GL}_{2n_j}(k)$ ($j = 1, 2, \dots, t$) ($\sum_{j=1}^t n_j = n$) such that

- (1) $\eta_i \not\cong \eta_j$ if $i \neq j$, and
- (2) the local L -function $L(\eta_j, \Lambda^2, s)$ has a pole at $s = 0$ for $j = 1, 2, \dots, t$.

Each irreducible unitary supercuspidal self-dual representation η_j of $\text{GL}_{2n_j}(k)$ has the local Langlands parameter ρ_j^0 , which is an irreducible, $2n_j$ -dimensional, admissible, complex representation of W_k , by Theorem 6.2. Further, we have $\rho_i^0 \not\cong \rho_j^0$ if $i \neq j$. Hence the representation

$$\tau = \eta_1 \times \eta_2 \times \cdots \times \eta_t$$

has the local Langlands parameter

$$\rho^0 = \rho_1^0 \oplus \rho_2^0 \oplus \cdots \oplus \rho_t^0,$$

which is a $2n$ -dimensional, admissible, completely reducible, multiplicity-free, complex representation of W_k .

Recently, G. Henniart communicated to us ([Hn1]) that he can prove the following results among others satisfied by the local Langlands reciprocity map.

THEOREM 6.3 (Henniart [Hn1]). *The local Langlands reciprocity map has the following property: the gamma factor $\gamma(\rho, \Lambda^2, s, \psi)$ has the same poles as the local gamma factor $\gamma(\tau_n(\rho), \Lambda^2, s, \psi)$ for any irreducible ρ (i.e. for any $\tau_n(\rho)$ supercuspidal).*

By using Theorem 6.3, we can prove the following proposition.

PROPOSITION 6.3. (1) *Let ρ^0 be an irreducible, $2m$ -dimensional, admissible, complex representation of W_k and τ be an irreducible unitary supercuspidal representation of $\text{GL}_{2m}(k)$ with the properties that (i) τ has the local Langlands parameter ρ^0 and (ii) the local exterior square L -function $L(\tau, \Lambda^2, s)$ has a pole at $s = 0$. Then ρ^0 is symplectic, i.e.*

$$\rho^0(W_k) \subset \text{Sp}_{2m}(\mathbb{C}).$$

(2) *Let $\rho^0 = \rho_1^0 \oplus \rho_2^0$ be a $2m$ -dimensional, admissible, completely reducible, complex representation of W_k with the property that*

$$\text{Hom}_{W_k}(\rho_1^0 \otimes \rho_2^0, 1) = 0.$$

Then ρ^0 is symplectic if and only if ρ_1^0 and ρ_2^0 are both symplectic.

Proof. The proof of Part (1) follows from Theorem 6.3. More precisely it goes as follows. Since

$$\gamma(\tau^\vee, \Lambda^2, s, \psi) = \epsilon(\tau^\vee, \Lambda^2, s, \psi) \cdot \frac{L(\tau, \Lambda^2, 1 - s)}{L(\tau^\vee, \Lambda^2, s)},$$

and by the assumption, the local L -function $L(\tau, \Lambda^2, s)$ has a pole at $s = 0$, we obtain that the gamma factor $\gamma(\tau^\vee, \Lambda^2, s, \psi)$ has a pole at $s = 1$. By Theorem 6.3, the gamma factor $\gamma((\rho^0)^\vee, \Lambda^2, s, \psi)$ has a pole at $s = 1$. Because we also have

$$\gamma((\rho^0)^\vee, \Lambda^2, s, \psi) = \epsilon((\rho^0)^\vee, \Lambda^2, s, \psi) \cdot \frac{L(\rho^0, \Lambda^2, 1 - s)}{L((\rho^0)^\vee, \Lambda^2, s)},$$

we get that the L -function $L(\rho^0, \Lambda^2, s)$ has a pole at $s = 0$. Now it is an elementary fact that if $L(\rho^0, \Lambda^2, s)$ has a pole at $s = 0$, then the image $\rho^0(W_k)$ is included in $\mathrm{Sp}_{2m}(\mathbb{C})$, i.e. the parameter ρ^0 is symplectic. This proves Part (1).

Part (2) is basically proved by linear algebra. It is clear that if both ρ_1^0 and ρ_2^0 are symplectic, then ρ^0 is itself symplectic. Conversely, we use a basic fact from linear algebra that ρ^0 is symplectic if and only if $\Lambda^2(\rho^0)$ has W_k -invariant functionals ([GW, §5.1.7]). Since

$$\Lambda^2(\rho^0) = \Lambda^2(\rho_1^0) \oplus \Lambda^2(\rho_2^0) \oplus [\rho_1^0 \otimes \rho_2^0],$$

the W_k -invariant functionals will not vanish on at least one of $\Lambda^2(\rho_1^0)$, $\Lambda^2(\rho_2^0)$, since we assume that

$$\mathrm{Hom}_{W_k}(\rho_1^0 \otimes \rho_2^0, 1) = 0.$$

Without loss of generality, we assume that $\Lambda^2(\rho_1^0)$ supports a W_k -invariant functional. Hence ρ_1^0 is symplectic. Because ρ^0 is nondegenerate and ρ_2^0 is the complement of ρ_1^0 , we conclude that ρ_2^0 is also symplectic. \square

Let $\mathcal{G}_{2n}^0(k)$ be the set of conjugacy classes of all $2n$ -dimensional, admissible, completely reducible, multiplicity-free, symplectic complex representations ρ^0 of W_k . Then we have the following theorem.

THEOREM 6.4 (local Langlands reciprocity law). *There exists a unique bijection*

$$\mathfrak{A}_{2n} : \rho_{2n}^0 \mapsto \mathfrak{A}_{2n}(\rho_{2n}^0)$$

from the set $\mathcal{G}_{2n}^0(k)$ onto the set $\mathcal{SO}_{2n+1}^{\mathrm{igsc}}(k)$ such that

$$\begin{aligned} (L) \quad & L(\rho_{2n}^0 \otimes \rho_l^0, s) = L(\mathfrak{A}_{2n}(\rho_{2n}^0) \times \mathfrak{r}_l(\rho_l^0), s), \\ (\epsilon) \quad & \epsilon(\rho_{2n}^0 \otimes \rho_l^0, s, \psi) = \epsilon(\mathfrak{A}_{2n}(\rho_{2n}^0) \times \mathfrak{r}_l(\rho_l^0), s, \psi), \text{ and} \\ (\gamma) \quad & \gamma(\rho_{2n}^0 \otimes \rho_l^0, s, \psi) = \gamma(\mathfrak{A}_{2n}(\rho_{2n}^0) \times \mathfrak{r}_l(\rho_l^0), s, \psi) \end{aligned}$$

for all irreducible continuous representations of W_k of dimension l .

Proof. The theorem is a direct consequence of Theorem 6.1 and Proposition 6.3. □

Remark 6.2. The complete local Langlands reciprocity conjecture in this case states that the reciprocity map \mathfrak{R}_{2n} takes a local Langlands parameter ρ^0 in $\mathcal{G}_{2n}^0(k)$ to a finite set $\Pi(\rho^0)$ (local L -packet) of irreducible admissible representations of $\mathrm{SO}_{2n+1}(k)$. By our local converse theorem (Theorem 1.2), we know that in each local L -packet $\Pi(\rho^0)$, there is at most one generic member (i.e. with Whittaker model). It is a very interesting and difficult problem to give an explicit construction of the local L -packets.

7. Appendix: On gamma factors for $\widetilde{\mathrm{Sp}}_{2n} \times \mathrm{GL}_l$

7.1. *Review of the global theory* In [GRS3], L -functions for generic, automorphic, cuspidal representations on $\widetilde{\mathrm{Sp}}_{2n} \times \mathrm{GL}_l$, are represented via global integrals of Shimura type. We review this construction briefly. It yields local gamma factors at each place.

Let F be a number field. Denote by \mathbb{A} its adèle ring. Fix a nontrivial character ψ_0 of $F \backslash \mathbb{A}$. Let Π (resp. ρ) be an irreducible, automorphic, cuspidal representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ (resp. $\mathrm{GL}_l(\mathbb{A})$). Assume that Π is globally $(\psi_0)_{\widetilde{U}_{\mathbb{A},1}}$ -generic. In the sequel, the cuspidality of ρ is not important. What we need is that ρ is automorphic, realized in an irreducible subspace of automorphic forms on $\mathrm{GL}_l(\mathbb{A})$, and that ρ is globally generic (i.e. the Whittaker coefficient is nontrivial on the space of ρ). Although this is not pointed out in [GRS3], the proofs there do not use at all the cuspidality of ρ . Thus, we may take ρ to be an Eisenstein series induced from irreducible, automorphic, cuspidal representations at a point of holomorphy.

Let $\xi_{\rho,s}$ be a holomorphic section for $J_{\rho,s}$ – the representation of $\mathrm{Sp}_{2l}(\mathbb{A})$, induced from $\rho_s = \rho | \det \cdot |^{s-1/2}$ on the Siegel parabolic subgroup $Q_l(\mathbb{A})$, and denote by $E(\xi_{\rho,s})$ the corresponding Eisenstein series on $\mathrm{Sp}_{2l}(\mathbb{A})$. We distinguish two cases according to whether $n \geq l$ or $n < l$. In the first case, a Fourier-Jacobi coefficient of a cusp form in Π is paired against the Eisenstein series above, and in the second case, a cusp form in Π is paired against a Fourier-Jacobi coefficient of the Eisenstein series. As we need in this paper only the case $n < l$ (as a matter of fact, we need just the case $l = 2n$.) we assume from now on that $n < l$.

Let $w_{\psi_0^{-1}}$ be the Weil representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, corresponding to ψ_0^{-1} . Realize it in $\mathcal{S}(\mathbb{A}^n)$, and denote, for $\phi \in \mathcal{S}(\mathbb{A}^n)$, by $\theta_{\psi_0^{-1}}^\phi$ the corresponding theta series. Extend $w_{\psi_0^{-1}}$ and $\theta_{\psi_0^{-1}}^\phi$ to $\mathcal{H}_n(\mathbb{A})$ – the Heisenberg group in $2n + 1$ variables. Let $N_{l,n+1}$ be the unipotent radical of the standard parabolic

subgroup $Q_{l,n+1}$ of Sp_{2l} , whose Levi part is isomorphic to $\mathrm{GL}_1^{l-n-1} \times \mathrm{Sp}_{2(n+1)}$. Let $(\psi_0)_{N_{l,n+1}}$ be the restriction to $N_{l,n+1}(\mathbb{A})$ of the standard nondegenerate character defined by ψ_0 . Note that \mathcal{H}_n embeds naturally in $N_{l,n}$ so that $N_{l,n} = N_{l,n+1} \rtimes \mathcal{H}_n$. Extend $(\psi_0)_{N_{l,n+1}}$ to $N_{l,n}(\mathbb{A})$, so that it is trivial on $\mathcal{H}_n(\mathbb{A})$. Denote this extension by $\chi_{\psi_0;l,n}$. Denote by j the projection of $N_{l,n}$ to \mathcal{H}_n . Let j denote also the embedding of Sp_{2n} into the Levi part of $Q_{l,n}$. Note that j embeds $\mathrm{Sp}_{2n} \rtimes \mathcal{H}_n$ into the Levi part of $Q_{l,n+1}$.

Let f be an automorphic function on $\mathrm{Sp}_{2l}(\mathbb{A})$. A Fourier-Jacobi coefficient of f of type (ψ_0, n) , is a function on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ of the form

$$f_{\psi_0,n}^\phi(\tilde{g}) = \int_{N_{l,n}(F) \backslash N_{l,n}(\mathbb{A})} f(uj(g)) \theta_{\psi_0}^\phi(j(u)\tilde{g}) \chi_{\psi_0;l,n}^{-1}(u) du .$$

Here $\tilde{g} \in \widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ projects to $g \in \mathrm{Sp}_{2n}(\mathbb{A})$, and $\phi \in \mathcal{S}(\mathbb{A}^n)$. Let φ be a cusp form in (the space of) Π . Define

$$\mathcal{L}(\varphi, \phi, \xi_{\rho,s}) = \int_{\mathrm{Sp}_{2n}(F) \backslash \mathrm{Sp}_{2n}(\mathbb{A})} \varphi(g) E_{\psi_0,n}^\phi(\xi_{\rho,s}, g) dg .$$

(In case $n > l$, one takes $\varphi_{\psi_0,l}^\phi$ and pairs it with $E(\xi_{\tau,s}, \cdot)$ along $\mathrm{Sp}_{2l}(F) \backslash \mathrm{Sp}_{2l}(\mathbb{A})$, and in case $n = l$, one integrates $\varphi(g) \theta_{\psi_0}^\phi(g) E(\xi_{\rho,s}, g)$ along $\mathrm{Sp}_{2n}(F) \backslash \mathrm{Sp}_{2n}(\mathbb{A})$.) We have an Euler product decomposition, for decomposable data and $\mathrm{Re}(s) \gg 0$

$$\mathcal{L}(\varphi, \phi, \xi_{\rho,s}) = \prod_{\nu} \mathcal{B}(W_{\varphi,\nu}, \phi_{\nu}, \xi_{\rho_{\nu},s})$$

where $W = \prod_{\nu} W_{\varphi,\nu}$ is the Whittaker function of φ with respect to ψ_0 , and at each place ν ,

$$\begin{aligned} (7.1) \quad & \mathcal{B}(W_{\varphi,\nu}, \phi_{\nu}, \xi_{\rho_{\nu},s}) \\ &= \int_{U_n(F_{\nu}) \backslash \mathrm{Sp}_{2n}(F_{\nu})} \int_{N_{l,n}^{\gamma_{l,n}}(F_{\nu}) \backslash N_{l,n}(F_{\nu})} \\ & \quad W_{\varphi,\nu}(g) w_{\psi_0}^{-1}(j(u)g) \phi_{\nu}(e_0) f_{\xi_{\rho_{\nu},s}}(\gamma_{l,n} u j(g)) \chi_{\psi_0;l,n}^{-1}(u) du dg . \end{aligned}$$

Here U_n is the standard maximal unipotent subgroup of Sp_{2n} , $\gamma_{l,n}$ is a certain Weyl element, $N_{l,n}^{\gamma_{l,n}} = \gamma_{l,n}^{-1} Q_l \gamma_{l,n} \cap N_{l,n}$; and $e_0 = (0, \dots, 0, 1)$. We obtain $f_{\xi_{\rho,s}} = \Pi f_{\xi_{\rho_{\nu},s}}$ from $\xi_{\rho,s}$ after taking a certain Whittaker coefficient in the “ ρ -variable”. Thus, we consider sections $f_{\xi_{\rho_{\nu},s}}$ for $J_{\rho_{\nu},s}$, which take values in a certain Whittaker model ρ_{ν} .

For decomposable data $\varphi, \phi, \xi_{\rho,s}$, let S be a finite set of places, including those at infinity, those above 2, and such that outside S all data are unramified, and ψ_0 is normalized. Then (normalizing $W_{\varphi,\nu}(I) = 1$ outside S), we have

$$(7.2) \quad \mathcal{L}(\varphi, \phi, \xi_{\rho,s}) = \prod_{\nu \in S} \mathcal{B}(W_{\varphi,\nu}, \phi_{\nu}, \xi_{\rho_{\nu},s}) \frac{L_{\psi_0}^S(\Pi \times \rho, s)}{L^S(\rho, s + \frac{1}{2}) L^S(\rho, \Lambda^2, 2s)} .$$

This implies that $L^S_{\psi_0}(\Pi \times \rho, s)$ is meromorphic. Indeed $\mathcal{L}(\varphi, \phi, \xi_{\rho, s})$ is clearly meromorphic, and $L^S(\rho, s + \frac{1}{2}), L^S(\rho, \Lambda^2, 2s)$ are known to be meromorphic. For finite ν in S , we can choose data, such that $\mathcal{B}(W_{\varphi, \nu}, \phi_{\nu}, \xi_{\rho_{\nu, s}}) = 1$, for all s (see [GRS3, Prop. 6.6]) and given $s_0 \in \mathbb{C}$ and $\nu \in S$ which is archimedean, we can find a combination $\sum_{i=1}^N \mathcal{B}(W_{\nu}^{(i)}, \phi_{\nu}^{(i)}, \xi_{\rho_{\nu, s}}^{(i)})$ which is holomorphic and nonzero at s_0 (see [GRS3, Prop. 6.7]). From this we conclude, choosing data in the same way, at all places of S , but one place ν_0 , that $\mathcal{B}(W_{\varphi, \nu_0}, \phi_{\nu_0}, \xi_{\rho_{\nu_0, s}})$ is meromorphic. (This can be shown in general, without the assumption that the data are coming from global cusp forms. See [GRS2, §1.1] for the case $\nu < \infty$, where it follows that $\mathcal{B}(W_{\nu}, \phi_{\nu}, \xi_{\rho_{\nu, s}})$ is rational in q_{ν}^{-s} . The case where ν is infinite can be done exactly as in [S2].)

Applying in (7.1) the intertwining operator M , with respect to $\begin{pmatrix} & I_l \\ -I_l & \end{pmatrix}$ on $J_{\rho, s}$, we get

$$(7.3) \quad \mathcal{L}(\varphi, \phi, M(\xi_{\rho, s})) = \prod_{\nu \in S} \mathcal{B}(W_{\varphi, \nu}, \phi_{\nu}, M_{\nu}(\xi_{\rho_{\nu, s}})) \frac{L^S(\rho, s - \frac{1}{2})L^S(\rho, \Lambda^2, 2s - 1)}{L^S(\rho, s + \frac{1}{2})L^S(\rho, \Lambda^2, 2s)} \cdot \frac{L^S_{\psi_0}(\Pi \times \widehat{\rho}, 1 - s)}{L^S(\widehat{\rho}, \frac{3}{2} - s)L^S(\widehat{\rho}, \Lambda^2, 2 - 2s)}.$$

Using the functional equation satisfied by $E(\xi_{\rho, s, \cdot})$, we can equate (7.2) and (7.3) to get

$$(7.4) \quad \frac{L^S_{\psi_0}(\Pi \times \rho, s)L^S(\widehat{\rho}, \frac{3}{2} - s)L^S(\widehat{\rho}, \Lambda^2, 2 - 2s)}{L^S_{\psi_0}(\Pi \times \widehat{\rho}, 1 - s)L^S(\rho, s - \frac{1}{2})L^S(\rho, \Lambda^2, 2s - 1)} \prod_{\nu \in S} \mathcal{B}(W_{\varphi, \nu}, \phi_{\nu}, \xi_{\rho_{\nu, s}}) = \prod_{\nu \in S} \mathcal{B}(W_{\varphi, \nu}, \phi_{\nu}, M_{\nu}(\xi_{\rho_{\nu, s}})).$$

Fixing data at all places in S except one place ν_0 , we conclude from (7.4) that there is a meromorphic function $\Gamma(\Pi_{\nu_0} \times \rho_{\nu_0}, s, \psi_{0, \nu_0})$, which is rational in $q_{\nu_0}^{-s}$, in case ν_0 is finite, such that

$$\Gamma(\Pi_{\nu_0} \times \rho_{\nu_0}, s, \psi_{0, \nu_0})\mathcal{B}(W_{\varphi, \nu_0}, \phi_{\nu_0}, \xi_{\rho_{\nu_0, s}}) = \mathcal{B}(W_{\varphi, \nu_0}, \phi_{\nu_0}, M_{\nu_0}(\xi_{\rho_{\nu_0, s}}))$$

for all $W_{\varphi, \nu_0}, \phi_{\nu_0}, \xi_{\rho_{\nu_0, s}}$. We define the local gamma factor $\gamma(\Pi_{\nu_0} \times \rho_{\nu_0}, s, \psi_{0, \nu_0})$ by

$$\Gamma(\Pi_{\nu_0} \times \rho_{\nu_0}, s, \psi_{0, \nu_0}) = \frac{\gamma(\Pi_{\nu_0} \times \rho_{\nu_0}, s, \psi_{0, \nu_0})}{\gamma(\rho_{\nu_0}, s - \frac{1}{2}, \psi_{0, \nu_0})\gamma(\rho_{\nu_0}, \Lambda^2, 2s - 1, \psi_{0, \nu_0})}.$$

Thus

$$(7.5) \quad \gamma(\Pi_{\nu_0} \times \rho_{\nu_0}, s, \psi_{0, \nu_0})\mathcal{B}(W_{\varphi, \nu_0}, \phi_{\nu_0}, \xi_{\rho_{\nu_0, s}}) = \widetilde{\mathcal{B}}(W_{\varphi, \nu_0}, \phi_{\nu_0}, \xi_{\rho_{\nu_0, s}})$$

where $\tilde{\mathcal{B}}(W_{\varphi, \nu_0}, \phi_{\nu_0}, \xi_{\rho_{\nu_0, s}}) = \mathcal{B}(W_{\varphi, \nu_0}, \phi_{\nu_0}, M_{\nu_0}^*(\xi_{\rho_{\nu_0, s}}))$, and

$$M_{\nu_0}^*(\xi_{\rho_{\nu_0, s}}) = \gamma\left(\rho_{\nu_0}, s - \frac{1}{2}, \psi_{0, \nu_0}\right) \gamma(\rho_{\nu_0}, \Lambda^2, 2s - 1, \psi_{0, \nu_0}) M_{\nu_0}(\xi_{\rho_{\nu_0, s}}).$$

Note that at a finite place ν_0 where ψ_{0, ν_0} is normalized; and Π_{ν_0} and ρ_{ν_0} are unramified, we have

$$\gamma(\Pi_{\nu_0} \times \rho_{\nu_0}, s, \psi_{0, \nu_0}) = \frac{L_{\psi_{0, \nu_0}}(\Pi_{\nu_0} \times \hat{\rho}_{\nu_0}, 1 - s)}{L_{\psi_{0, \nu_0}}(\Pi_{\nu_0} \times \rho_{\nu_0}, s)}.$$

For such ν_0 , $L_{\psi_{0, \nu_0}}(\Pi_{\nu_0} \times \rho_{\nu_0}, s)$ is nothing but $L(\theta_{\psi_{0, \nu_0}}(\Pi_{\nu_0}) \times \rho_{\nu_0}, s)$, where $\theta_{\psi_{0, \nu_0}}(\Pi_{\nu_0})$ is the unramified representation of $\mathrm{SO}_{2n+1}(F_{\nu_0})$ corresponding to Π_{ν_0} by the local ψ_{0, ν_0} -Howe lift.

7.2. A result on gamma factors at archimedean places. Let ν_0 be an archimedean place of F . Put $\Pi_{\nu_0} = \pi$, $\rho_{\nu_0} = \tau$, $F_{\nu_0} = k$, $W_{\varphi, \nu_0} = W$, $\phi_{\nu_0} = \phi$, $\psi_{0, \nu_0} = \psi$. In this subsection, we denote V_{π} and V_{τ} to be the canonical models of the Harish-Chandra modules of π and τ , respectively. The canonical extension of a Harish-Chandra module has the C^{∞} -topology as given in [C]. From [C], such canonical extensions are unique, up to equivalence. Our goal in this subsection is to show that there is $A > 0$, such that for $|\mathrm{Im}(s)| > A$, $\gamma(\pi \times \tau, s, \psi)$ is holomorphic and nonzero. The global integral $\mathcal{L}(\varphi, \phi, \xi_{\rho, s})$ defined in subsection 7.1 is separately continuous in the C^{∞} -topology at the place ν_0 . Hence it remains continuous after the extension to the canonical models. Therefore, we may regard (the analytic continuation of) $\mathcal{B}(W, \phi, \xi_{\tau, s})$ as a continuous linear form T on $V_{\pi} \otimes \mathcal{S}(k^n) \otimes V_{J_{\tau, s}}$. Here the notions of separate continuity and of continuity coincide, and the two notions of tensor products (inductive $\overline{\otimes}$, projective $\widehat{\otimes}$) coincide. The proof of this is as follows.

First we note that V_{π} and $V_{J_{\tau, s}}$ are nuclear Fréchet spaces. Indeed, both representations are quotients of (differentiably) induced representations coming off Borel subgroups and quasi-characters, and since the spaces of such induced representations are images of surjective maps from spaces of the form $C^{\infty}(K)$, where K is compact, our spaces V_{π} and $V_{J_{\tau, s}}$ are quotients of such spaces. Since $C^{\infty}(K)$ is Fréchet and nuclear, so are V_{π} and $V_{J_{\tau, s}}$. (See [Tr, pp. 85, 94, 514, 530].) In particular, the two notions of tensor product for $V_{\pi} \otimes V_{J_{\tau, s}}$ coincide; i.e., $V_{\pi} \overline{\otimes} V_{J_{\tau, s}} \cong V_{\pi} \widehat{\otimes} V_{J_{\tau, s}}$, which (by [Tr, p. 514]) is nuclear. We conclude that the two notions of tensor product for $(V_{\pi} \otimes V_{J_{\tau, s}}) \otimes \mathcal{S}(k^n)$ coincide. Actually, $\mathcal{S}(k^n)$ is nuclear as well [Tr, p. 530].

We add some more related remarks. Note that the same proofs work for $V_{\pi} \otimes V_{\tau}$ and $(V_{\pi} \otimes V_{\tau}) \otimes \mathcal{S}(k^n)$ as well (i.e. \otimes can be replaced by either $\overline{\otimes}$ or $\widehat{\otimes}$). Note also that $V_{\pi} \otimes V_{\tau}$ is a Fréchet space as well, since it is a quotient of $C^{\infty}(K_1) \otimes C^{\infty}(K_2) \cong C^{\infty}(K_1 \times K_2)$, where K_1, K_2 are compact, such that V_{π} is a quotient of $C^{\infty}(K_1)$ and V_{τ} is a quotient of $C^{\infty}(K_2)$. We conclude that $V_{\pi} \otimes V_{\tau} \otimes \mathcal{S}(k^n)$ is a quotient of a Fréchet space, and hence is itself a

Fréchet space. Indeed, $V_\pi \otimes V_\tau \otimes \mathcal{S}(k^n)$ is a quotient of $C^\infty(K_1 \times K_2) \otimes \mathcal{S}(k^n)$ which is isomorphic to $\mathcal{S}(k^n; C^\infty(K_1 \times K_2))$ [Tr, p. 533], and the last space is isomorphic to $\mathcal{S}(k^n \times K_1 \times K_2)$, which is a Fréchet space [Tr, p. 92]. Note also that $V_\pi \otimes V_\tau \otimes \mathcal{S}(k^n)$ is nuclear as a tensor product of two nuclear spaces $V_\pi \otimes V_\tau$ and $\mathcal{S}(k^n)$. As a corollary, we obtain (see [Wr, p. 485]):

PROPOSITION 7.1. *Let M be a C^∞ -manifold, countable at infinity. Then*

$$(7.6) \quad C_c^\infty(M) \otimes (V_\pi \otimes V_\tau \otimes \mathcal{S}(k^n)) \cong C_c^\infty(M; V_\pi \otimes V_\tau \otimes \mathcal{S}(k^n)).$$

Again, in Proposition 7.1, each \otimes can be replaced by either $\overline{\otimes}$ or $\widehat{\otimes}$.

The linear form T (on $V_\pi \otimes \mathcal{S}(k^n) \otimes V_{J_{\tau,s}}$) is equivariant with respect to the subgroup $R = j(\mathrm{Sp}_{2n}(k))N_{l,n}(k)$.

$$(7.7) \quad T((\pi(\tilde{g}) \otimes w_{\psi^{-1}}(j(u)\tilde{g}) \otimes J_{\tau,s}(uj(g)))v) = \chi_{\psi;l,n}(u)T(v)$$

where \tilde{g} is (any) inverse image in $\widetilde{\mathrm{Sp}}_{2n}(k)$ of g in $\mathrm{Sp}_{2n}(k)$, and $u \in N_{l,n}(k)$. This follows easily from (7.1) (or even from the structure of the global integrals). We have a surjection

$$\begin{aligned} \tau_s^1 : C_c^\infty(\mathrm{Sp}_{2l}(k); V_\tau) &\longrightarrow V_{J_{\tau,s}} \\ \tau_s^1(\varphi)(h) &= \int_{Q_l(k)} \delta_{Q_l}^{-1/2}(p) \tau_s(p^{-1})(\varphi(ph)) d_r p, \end{aligned}$$

where $d_r p$ is a right invariant measure on $Q_l(k)$. Composing T with τ_s^1 yields a (continuous) linear form t on

$$\begin{aligned} C_c^\infty(\mathrm{Sp}_{2l}(k); V_\tau) \otimes \mathcal{S}(k^n) \otimes V_\pi &\cong (C_c^\infty(\mathrm{Sp}_{2l}(k)) \otimes V_\tau \otimes \mathcal{S}(k^n) \otimes V_\pi \\ &\cong C_c^\infty(\mathrm{Sp}_{2l}(k); V_\tau \otimes \mathcal{S}(k^n) \otimes V_\pi) \end{aligned}$$

(We used Proposition 7.1.) By (7.7) and

$$\tau_s^1(\lambda(p)\varphi) = \delta_{Q_l}^{1/2}(p) \tau_s^1(\tau_s(p^{-1} \circ \varphi)), \quad p \in Q_l(k)$$

($\lambda(p)$ denotes the left translation by p^{-1}), we conclude that t , when regarded as a $V_\tau \otimes \mathcal{S}(k^n) \otimes V_\pi$ -distribution on $\mathrm{Sp}_{2l}(k)$, satisfies

$$(7.8) \quad \begin{aligned} t(r(uj(g))f) &= \chi_{\psi;l,n}(u)t((1 \otimes w_{\psi^{-1}}(\tilde{g}^{-1}j(u^{-1})) \otimes \pi(\tilde{g}^{-1})) \circ f), \\ t(\lambda(p)f) &= \delta_{Q_l}^{1/2}(p)t((\tau_s(p^{-1}) \otimes 1 \otimes 1) \circ f). \end{aligned}$$

Here $u \in N_{l,n}(k)$, $g \in \mathrm{Sp}_{2n}(k)$, $p \in Q_l(k)$ and r denotes right translations. We are now at the situation of [Wr, 5.2.4]. (Note that in the notation [Wr, 5.2.4], $M = \mathrm{Sp}_{2l}(k)$ and $G = Q_l(k) \times R$ acts on M by $(p, r) \cdot h = phr^{-1}$. See also [Wr, p. 408].) We have a nice description of the set $Q_l(k) \backslash \mathrm{Sp}_{2l}(k) / R$. This is a finite set, and it has one open orbit $Q_l(k)\gamma_{l,n}R$. See [GRS3, Sec. 4]. The reference for the following is [Wr, 5.2.3, 5.2.4].

Let us show that the map

$$b : t \mapsto t \Big|_{C_c^\infty(Q_l(k)\gamma_{l,n}R; V_\tau \otimes \mathcal{S}(k^n) \otimes V_\pi)}$$

is injective on the space of $V_\tau \otimes \mathcal{S}(k^n) \otimes V_\pi$ -distributions on $\mathrm{Sp}_{2l}(k)$, which satisfy (7.8). Indeed, if $b(t) = 0$, then by Bruhat theory (see the proof of [Wr, Lemma 5.2.4.4]), t is supported in the complement of the open orbit $Q_l(k)\gamma_{l,n}R$. The dimension of the space of such distributions is majorized by

$$\sum_{\gamma_{l,n} \neq \gamma \in Q_l(k) \backslash \mathrm{Sp}_{2l}(k)/R} \sum_{m=1}^\infty \dim \left[\mathrm{Bil}_{Q_l(k) \cap \gamma R \gamma^{-1}} \left(\delta_{Q_l}^{-1/2} \tau_s \otimes (\chi_{\psi;l,n}^{-1} \otimes \pi \otimes w_{\psi^{-1}})^\gamma, \Lambda_{\gamma,m} \right) \right]$$

where $\Lambda_{\gamma,m}$ are certain algebraic finite-dimensional representations, coming from derivatives. Bil_H denotes H -equivariant bilinear forms. Let V_l denote the unipotent radical of Q_l . An element of

$$\mathrm{Bil}_{Q_l(k) \cap \gamma R \gamma^{-1}} \left(\delta_{Q_l}^{-1/2} \tau_s \otimes (\chi_{\psi;l,n}^{-1} \otimes \pi \otimes w_{\psi^{-1}})^\gamma, \Lambda_{\gamma,m} \right)$$

when regarded as a $V_l(k) \cap \gamma N_{l,n}(k)\gamma^{-1}$ -equivariant form embeds

$$\left(\chi_{\psi;l,n}^{-1} \cdot \psi^{-1} \Big|_{\mathrm{Center}(\mathcal{H}_n(k))} \right)^\gamma$$

on $E = V_l(k) \cap \gamma N_{l,n+1}(k) \cdot \mathrm{Center}(\mathcal{H}_n(k))\gamma^{-1}$ into the dual of $\Lambda_{\gamma,m}$. Since $\Lambda_{\gamma,m}$ is an algebraic finite-dimensional representation, it cannot have nontrivial eigenvalues on the unipotent subgroup E . Thus, we must have

$$\left(\chi_{\psi;l,n} \cdot \psi \Big|_{\mathrm{Center}(\mathcal{H}_n(k))} \right)^\gamma = 1.$$

By [GRS3, p. 212], this is impossible, unless $Q_l(F)\gamma R$ is the open orbit. This is a contradiction, and so the map b is injective.

Returning to $T(W \otimes \phi \otimes \xi_{\tau,s}) = \mathcal{B}(W, \phi, \xi_{\tau,s})$, let s_0 be a pole of order e of $\mathcal{B}(W, \phi, \xi_{\tau,s})$ in the sense that $(s - s_0)^e \mathcal{B}(W, \phi, \xi_{\tau,s})$ is holomorphic and not identically zero at s_0 . Let t be the linear form on $C_c^\infty(\mathrm{Sp}_{2l}(k); V_\tau) \otimes \mathcal{S}(k^n) \otimes V_\pi$ defined by

$$(7.9) \quad t(\varphi \otimes \phi \otimes W) = \lim_{s \rightarrow s_0} (s - s_0)^e \mathcal{B}(W, \phi, \tau_s^1(\varphi)) .$$

When viewed as a $V_\tau \otimes \mathcal{S}(k^n) \otimes V_\pi$ distribution on $\mathrm{Sp}_{2l}(F)$, t clearly satisfies (7.8). Then what we have just shown is that for φ supported in the open orbit $Q_l(k)\gamma_{l,n}R$, $t(\varphi \otimes \phi \otimes W)$ is not identically zero. This means that all poles of $\mathcal{B}(W, \phi, \xi_{\tau,s})$ are detected (with their orders) on the open orbit. Thus, in order to locate the poles, it is enough to take $\xi_{\tau,s}$ with compact support modulo

$Q_l(k)$ (independent of s) inside $Q_l(k)\gamma_{l,n}R$. We may take $\xi_{\tau,s} = \tau_s^1(\varphi)$, with φ supported in $Q_l(k)\gamma_{l,n}R$. For such $\xi_{\tau,s}$, the unipotent inner integration in (7.1) converges absolutely. Rewrite (7.1), for $\text{Re}(s) \gg 0$, following the Iwasawa decomposition $\text{Sp}_{2n}(k) = U_n(k)AK$

$$(7.10) \quad \mathcal{B}(W, \phi, \xi_{\tau,s}) = \int_{h \in K} \int_{V_0} \int_A \delta^{-1}(\hat{a})W(\hat{a}\tilde{h})w_{\psi^{-1}}(\hat{a}j(u)\tilde{h})\phi(e_0) \cdot \xi_{\tau,s}(\gamma_{l,n}uj(h); \begin{pmatrix} a & \\ & I_{l-n} \end{pmatrix})|\det a|^{s+s_{l,n}}\chi_{\psi;l,n}(u)dadudh.$$

Here V_0 is a compact subset of $N_{l,n}^{\gamma_{l,n}}(k)\backslash N_{l,n}(k)$ (the projection onto $N_{l,n}^{\gamma_{l,n}}(k)\backslash N_{l,n}(k)$ of the compact support modulo $Q_l(k)$ of $\xi_{\tau,s}$ inside $Q_l(k)\gamma_{l,n}R$), \hat{a} denotes an inverse image in $\widetilde{\text{Sp}}_{2n}(k)$ of $\begin{pmatrix} a & \\ & a^* \end{pmatrix}$, for a diagonal matrix a in $\text{GL}_n(k)$, and $s_{l,n}$ is a certain fixed translation of s . Denote the inner da integration, for fixed (h, u) , in (7.10) by $B(\pi(\tilde{h})W, w_{\psi^{-1}}(j(u)\tilde{h})\phi, J_{\tau,s}(uj(h))\xi_{\tau,s})$. For a fixed $(h, u) \in K \times V_0$, $W'_\pi = \pi(\tilde{h})W$, $\phi' = w_{\psi^{-1}}(j(u)\tilde{h})\phi$, $\xi'_{\tau,s} = J_{\tau,s}(uj(h))\xi_{\tau,s}$ and $W'_\tau(m) = \xi'_{\tau,s}(I; m)$, the above inner integration equals

$$(7.11) \quad \int_A \delta^{-1}(\hat{a})W'_\pi(\hat{a})w_{\psi^{-1}}(\hat{a})\phi(e_0)W'_\tau \begin{pmatrix} a & \\ & I_{l-n} \end{pmatrix} |\det a|^{s+s_{l,n}} da .$$

The analytic continuation of (7.11) is obtained when we replace $W'_\pi(\hat{a})$, $W'_\tau \begin{pmatrix} a & \\ & I_{l-n} \end{pmatrix}$ with their asymptotic expansions, obtained exactly as in [S1, §3.3]. (See also [S2, §4] which applies here in exactly the same way.) We conclude that (7.11) is a sum of elements of the form

$$(7.12) \quad \int f(a_1, \dots, a_n)\eta(a_1, \dots, a_n)\delta^{-1}(\hat{a})|\det a|^{s+s'_{l,n}} da$$

where $a = \text{diag}(a_1, \dots, a_n)$, $f \in \mathcal{S}(k^n)$ (f is independent of s) and η is a finite function varying in a finite set which depends on π and τ only ($s'_{l,n}$ is another fixed translation of s). Thus, there is a finite set of characters $X_{\pi,\tau}$ of k^* , and there is a polynomial $P_{\pi,\tau}(s)$, which depend on π and τ only, such that

$$(7.13) \quad \frac{\mathcal{B}(W'_\pi, \phi', \xi'_{\tau,s})}{P_{\pi,\tau}(s) \prod_{\mu \in X_{\pi,\tau}} L(\mu, s)} \text{ is holomorphic in the whole plane. We have } \frac{\mathcal{B}(W, \phi, \xi_{\tau,s})}{P_{\pi,\tau}(s) \prod_{\mu \in X_{\pi,\tau}} L(\mu, s)} = \int_{K \times V_0} \frac{\mathcal{B}(\pi(\tilde{h})W, w_{\psi^{-1}}(j(u)\tilde{h})\phi, J_{\tau,s}(uj(h))\xi_{\tau,s})}{P_{\pi,\tau}(s) \prod_{\mu \in X_{\pi,\tau}} L(\mu, s)} dudh.$$

The right-hand side of (7.13) is holomorphic since $K \times V_0$ is compact, the integrand is continuous in (u, h) and the convergence of the integral is uniform in s , as s varies in compact sets. Looking at the left-hand side of (7.13), we conclude:

PROPOSITION 7.2. *There is $A > 0$, such that $\mathcal{B}(W, \phi, \xi_{\tau,s})$ is holomorphic for $|\operatorname{Im}(s)| > A$, for all data.*

Remark 7.1. Since $\mathcal{B}(W, \phi, M^*(\xi_{\tau,s}))$ has a similar structure, we may take A in the last proposition so that $\tilde{\mathcal{B}}(W, \phi, \xi_{\tau,s})$ is holomorphic for $|\operatorname{Im}(s)| > A$, for all data, as well. Finally, we conclude:

PROPOSITION 7.3. *There is $A > 0$, such that for $|\operatorname{Im}(s)| > A$, $\gamma(\pi \times \tau, s, \psi)$ is holomorphic with no zeroes.*

Proof. We have

$$\gamma(\pi \times \tau, s, \psi)\mathcal{B}(W, \phi, \xi_{\tau,s}) = \tilde{\mathcal{B}}(W, \phi, \xi_{\tau,s}).$$

Let A be as in Proposition 7.2 and in the remark which follows. Given any $s_0 \in \mathbb{C}$, there is a combination $\sum_{i=1}^N \mathcal{B}(W_i, \phi_i, \xi_{\tau,s}^{(i)})$ which is holomorphic and nonzero at s_0 . (See [GRS3, Prop. 6.7]. Thus, if s_0 is a pole of $\gamma(\pi \times \tau, s, \psi)$, then s_0 is a pole of $\tilde{\mathcal{B}}(W, \phi, \xi_{\tau,s})$. This forces $|\operatorname{Im}(s_0)| \leq A$. Similarly, since

$$\gamma(\pi \times \tau, s, \psi)^{-1}\tilde{\mathcal{B}}(W, \phi, \xi_{\tau,s}) = \mathcal{B}(W, \phi, \xi_{\tau,s}),$$

if s_0 is a zero of $\gamma(\pi \times \tau, s, \psi)$, we may assume that M^* is an isomorphism between J_{τ,s_0} and $J_{\tau,1-s_0}$ (take $|\operatorname{Im}(s_0)|$ large enough), and then as before, we conclude, that s_0 is a pole of $\mathcal{B}(W, \phi, \xi_{\tau,s})$, which is impossible, if we assume that $|\operatorname{Im}(s_0)| > A$. \square

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