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Ax-Lindemann for $\mathcal{A}_g$

By Jonathan Pila and Jacob Tsimerman

Abstract

We prove the Ax-Lindemann theorem for the coarse moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension $g \geq 1$. We affirm the André-Oort conjecture unconditionally for $\mathcal{A}_g$ for $g \leq 6$, and under GRH for all g.

1. Introduction

In this paper we prove the "Ax-Lindemann" theorem for $\mathcal{A}_g = \mathcal{A}_{g,1}, g \geq 1$, the moduli space of principally polarized Abelian varieties of dimension g. The statement of the theorem is as follows. (For definitions and conventions see Section 2.) Let $\mathbb{H}_g$ be the Siegel upper-half space and $\pi_g : \mathbb{H}_g \to \mathcal{A}_g$ the $\mathrm{Sp}_{2g}(\mathbb{Z})$-invariant uniformisation.

Theorem 1.1. *Let $V \subset \mathcal{A}_g$ be a subvariety and $W \subset \pi_g^{-1}(V)$ a maximal algebraic subvariety. Then W is weakly special.*

As explained in [27], this theorem may be viewed as an analogue for the map π_g of part of Ax's theorem [4] establishing the differential field version of Schanuel's conjecture for the exponential function (see [22, p. 30]), namely the part that corresponds to the classical Lindemann (or Lindemann-Weierstrass) theorem (see [22, p. 77]); hence the neologism "Ax-Lindemann." The proof of Theorem 1.1 combines various arithmetic estimates with the Counting Theorem of Pila-Wilkie [29] and with the idea of Ullmo-Yafaev [40] to use hyperbolic volume at the boundary.

The André-Oort conjecture (AO) is a compositum of conjectures made by André [1] and Oort [25]. A full proof of AO under the assumption of GRH for CM fields has been announced by Klingler, Ullmo, and Yafaev [18], [42]. An appropriate Ax-Lindemann theorem is a key ingredient in proving cases of AO unconditionally using o-minimality and point-counting [7], [27], [28], [38], following the basic strategy originally proposed by Zannier for re-proving the

Manin-Mumford conjecture [30]. It provides a geometric characterization of the exceptional set in the Counting Theorem and in this role is analogous to functional transcendence statements in the context of other Zilber-Pink type problems (e.g., [23], [16]). The following theorem affirms AO unconditionally for $\mathcal{A}_g$ for $g \leq 6$, and for all g under the assumption of GRH.

THEOREM 1.2. *Let $V \subset \mathcal{A}_g, g \leq 6$ be a subvariety. Then V contains only finitely many maximal special subvarieties. Under the assumption of* GRH (*for* CM *fields*) *the same conclusion holds for all g.*

Both theorems rely on the definability in the o-minimal structure $\mathbb{R}_{\text{an, exp}}$ of the map $\pi_g : \mathbb{H}_g \to \mathcal{A}_g$ when restricted to a standard fundamental domain for the $\mathrm{Sp}_{2g}(\mathbb{Z})$ action. This result, stated in Section 2.7, is due to Peterzil-Starchenko [26]. The o-minimality of $\mathbb{R}_{\text{an, exp}}$ is due to van den Dries and Miller [11], building on the fundamental work of Wilkie [43]; for further references on o-minimality, see Section 2 .

The restriction to $g \leq 6$ for the unconditional statement in 1.2 is due to another ingredient that is crucial to the strategy: a suitable lower bound for the size of the Galois orbit of a special point. These have been established by the second author [37] unconditionally for $g \leq 6$, and for all g on GRH. (The former for $g \leq 3$ and the latter were shown independently by Ullmo-Yafaev [41].) However, we show that such bounds are the only remaining obstacle to proving AO for $\mathcal{A}_g$ in general. This gives a new proof of AO for $\mathcal{A}_g$ for all g assuming GRH, by different methods from the ones employed by Klingler-Ullmo-Yafaev [18], [42].

In the course of preparation of this manuscript, the preprint [38] by Ullmo appeared showing how to deduce AO from these ingredients (Ax-Lindemann, height upper bound for preimages of special points, Galois lower bounds, and uniformisation with suitable definability) for any Shimura variety. Ullmo thereby proves (in [38]) AO unconditionally for all projective Shimura subvarieties of $\mathcal{A}_6^n, n \geq 1$, using the Ax-Lindemann theorem for projective Shimura varieties established by Ullmo-Yafaev [40], the height upper bounds in [28], and the Galois lower bounds of [37], as the definability requirements are easily seen to be satisfied (in $\mathbb{R}_{\text{an}}$) in the case of a projective Shimura variety. The deductions of AO in [38] and here differ in detail, but both depend on the structure of weakly special subvarieties of $\mathcal{A}_g$ and make further crucial use of o-minimality (as in [27], [28]). We have retained our treatment in order to keep our paper self-contained.

We begin in Section 2 by reviewing $\mathbb{H}_g$ and our basic definitions and conventions. Then we prove some norm estimates (Section 3), estimates about volumes of curves in fundamental domains (Section 4) and near the boundary of $\mathbb{H}_g$ (Section 5). With these preparations we prove Theorem 1.1 in Section 6.

The deduction of AO from Theorem 1.1 and the various other "ingredients" is carried out in Section 7.

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2. **Basic conventions and notation**

2.1. $\mathrm{Sp}_{2g}(\mathbb{R})$ *and* $\mathbb{H}_g$. The *symplectic group* $\mathrm{Sp}_{2g}(R)$ with entries in a ring R is the group of matrices $T \in M_{2g}(R)$ satisfying

$$TJT^t = J,$$

where $J = \left(\begin{smallmatrix} 0 & 1_g \\ -1_g & 0 \end{smallmatrix}\right)$ is the standard alternating matrix of degree $2g$ and T^t is the transpose of T; see, e.g., [17]. If we write

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in M_g(R),$$

then the condition $TJT^t = J$ is equivalent (see [15, p. 183]) to

$$AB^t = BA^t, \quad CD^t = DC^t, \quad AD^t - BC^t = I_g.$$

We know (see [15, p. 184]) that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(R) \Rightarrow A^tC = C^tA, B^tD = D^tB,$$

so that $\mathrm{Sp}_{2g}(R)$ is closed under transposition.

The *Siegel upper half-space* $\mathbb{H}_g$ is defined to be

$$\mathbb{H}_g = \left\{ Z \in M_g(\mathbb{C}) : Z = Z^t, \mathrm{Im}(Z) > 0 \right\}.$$

Thus $\mathbb{H}_g$ is an open domain in the space $M_g(\mathbb{C})^{\mathrm{sym}}$ of symmetric complex $g \times g$ matrices, which we may identify with $\mathbb{C}^{g(g+1)/2}$. There is an action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathbb{H}_g$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}.$$

Denote by $\mathcal{A}_g$ the coarse moduli space of complex principally polarized abelian varieties of dimension g. The quotient $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$ is isomorphic to $\mathcal{A}_g$. We write $\pi_g : \mathbb{H}_g \to \mathcal{A}_g$ for the projection map, which may be given explicitly by scalar Siegel modular forms of suitable weight; see [15, §§10, 11].

2.2. *Varieties and subvarieties.* We identify varieties with their sets of complex-valued points. By a *subvariety* V of a quasiprojective variety W we mean a Zariski closed subset. An *irreducible algebraic subvariety* of $\mathbb{H}_g$, considered as a subset of $\mathbb{C}^{g(g+1)/2}$, means an irreducible component (as a complex analytic variety) of $\mathbb{H}_g \cap Y$ where $Y \subset \mathbb{C}^{g(g+1)/2}$ is a subvariety. Likewise, if $\mathbb{H}$ is any hermitian domain, then there is a natural algebraic variety $\mathbb{H}_0$ of which $\mathbb{H}$ is an open subset (see [39]), and we define an irreducible algebraic subvariety of $\mathbb{H}$ to be an irreducible component, as a complex analytic variety, of $\mathbb{H} \cap Y$, where Y is an algebraic subvariety of $\mathbb{H}_0$. For $A \subset \mathbb{H}_g$, a *maximal algebraic subvariety* of A is an irreducible algebraic subvariety $W \subset \mathbb{H}_g$ with $W \subset A$ such that if $W' \subset \mathbb{H}_g$ is an irreducible algebraic subvariety with $W \subset W' \subset A$, then $W = W'$.

2.3. *The metric.* There is an invariant metric on $\mathbb{H}_g$ for the action of $\mathrm{Sp}_{2g}(\mathbb{R})$, and this is given (see, e.g., [36, p. 17]) by

$$d\mu(Z) = \mathrm{Tr}(Y^{-1} dZ Y^{-1} d\bar{Z}).$$

2.4. *Shimura data and special subvarieties.* We gather here some basic facts about Shimura varieties; for a more thorough account with proofs see [8] or [9]. We work exclusively with connected Shimura varieties, of which an excellent account can be found in [31].

Define the Deligne torus $\mathbb{S}$ to be the real torus given by Weil restricting $\mathbb{G}_m$ from $\mathbb{C}$ to $\mathbb{R}$. Thus, the real points $\mathbb{S}(\mathbb{R})$ can be identified with $\mathbb{C}^\times$. A *connected Shimura datum* is defined to be a pair (G, X^+), where G is a reductive group over $\mathbb{Q}$ and X^+ is a connected component of a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the following conditions:

- for all $h \in X$, only the weights $(0,0), (1,-1), (-1,1)$ may occur in the adjoint action of $\mathbb{S}$ on the complexified Lie algebra of G;
- $\mathrm{ad} \circ h(i)$ is a Cartan involution on the adjoint group $G^{\mathrm{ad}}_{\mathbb{R}}$;
- G^{ad} has no simple $\mathbb{Q}$-factor H such that $H(\mathbb{R})$ is compact.

These axioms ensure that if we set K_∞ to be the stabilizer of some $h \in X$, then $X = G(\mathbb{R})/K_\infty$ is a finite union of hermitian symmetric domains on which $G(\mathbb{R})$ acts transitively and biholomorphically. We say that (H, X_H) is a *Shimura subdatum* of (G, X_G) if $H \subset G$ and $X_H \subset X_G$. A *connected Shimura variety* is defined to be $\Gamma \backslash X^+$, where $\Gamma \subset G(\mathbb{Q})$ is a congruence subgroup of $G(\mathbb{R})$ that stabilises X^+ ([31]). Thus, to give a connected Shimura variety is the same as to give a triple (G, X^+, Γ). A connected Shimura variety can be given the structure of an algebraic variety defined over a number field.

We denote by G^{ad} the adjoint form of a reductive group G. Given a Shimura datum (G, X_G) we get an associated Shimura datum $(G^{\mathrm{ad}}, X_{G^{\mathrm{ad}}})$ such that we can identify X^+_G with $X^+_{G^{\mathrm{ad}}}$.

A *morphism of connected Shimura varieties*

$$\phi : (H, X_H^+, \Gamma_H) \to (G, X_G^+, \Gamma_G)$$

is a morphism of $\mathbb{Q}$-groups $\phi : H \to G$ that carries X_H^+ to X_G^+ and Γ_H to Γ_G. The image $\phi(\Gamma_H \backslash X_H^+)$ is an algebraic subvariety of $\Gamma_G \backslash X_G^+$.

Definition. A subvariety $V \subset \Gamma_G \backslash X_G^+$ is called a *special subvariety* if there exists a morphism of connected Shimura varieties $\phi : (H, X_H^+, \Gamma_H) \to (G, X_G^+, \Gamma_G)$ such that $\phi(\Gamma_H \backslash X_H^+) = V$. By abuse of notation, we also refer to $\phi(X_H^+)$ as a special subvariety of X_G^+. Keep in mind that $\phi(X_H^+)$ is an irreducible algebraic subvariety of X_G^+ under its semi-algebraic structure as a hermitian symmetric space. A *special point* is a special subvariety of dimension zero (in $\Gamma_G \backslash X_G^+$ or in X_G^+).

For our purposes, we shall need a slightly more general notion (see [39]).

Definition. We say that $V \subset \Gamma_G \backslash X_G^+$ is a *weakly special subvariety* if there exist Shimura varieties (H_i, X_i^+, Γ_i) for $i = 1, 2$ and a Shimura subdatum $(H, X_H^+) \subset (G, X_G^+)$ such that

$$(H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}}^+) = (H_1, X_1^+) \times (H_2, X_2^+)$$

and $y \in \Gamma_2 \backslash X_2^+$ such that V is the image of $X_1^+ \times \{y\}$ in $\Gamma_G \backslash X_G^+$. By the same abuse of notation as above, we refer to the image of $X_1^+ \times \{y\}$ in X_G^+ as a weakly special subvariety of X_G^+.

A weakly special subvariety is special if and only if it contains a special point; see [24].

The following result of Ullmo-Yafaev is very useful in working with weakly special subvarieties:

LEMMA 2.1 ([39]). *An irreducible algebraic variety $S \subset \mathbb{H}_g$ is weakly special if and only if $\pi_g(S)$ is algebraic.*

COROLLARY 2.2. *An irreducible component of the intersection of weakly special subvarieties is weakly special.*

LEMMA 2.3. *Let (G, X_G^+) be a connected Shimura datum. Let V be a weakly special subvariety of X_G^+. Then there exists a semi-simple subgroup $G_0 \subset G$ defined over $\mathbb{Q}$, with no compact $\mathbb{Q}$-factors, and a point $Z_0 \in X_G^+$ such that the image of $G_{0,\mathbb{R}}$ in $G_{\mathbb{R}}^{\mathrm{ad}}$ is fixed by the Cartan involution corresponding to Z_0 and $V = G_0(\mathbb{R})^+ \cdot Z_0$.*

Proof. Since V is weakly special, there are connected Shimura data

$$(H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}}^+) = (H_1, X_1^+) \times (H_2, X_2^+),$$

where (H, X_H) is a Shimura subdatum of (G, X_G), and a point $y \in X_2^+$ such that V is the image of $\phi(X_1^+ \times \{y\})$ in $\Gamma_G \backslash X_G^+$.

Pick $h \in X_1^+$, and set $Z_0 = (h, y)$, where as always we identify $X_{H^{\mathrm{ad}}}^+$ with X_H^+. Finally, take $G_1 \subset H$ to be ϕ composed with the pullback under the map $H \to H^{\mathrm{ad}}$ of $H_1 \times \{1_{H_2}\}$. Then the Cartan involution corresponding to Z_0 is given by conjugation by a lift to $H(\mathbb{R})$ of $(h, y)(i)$. Thus, $G_{1,\mathbb{R}}$ is fixed under this involution. By definition, we have

$$V = X_1^+ \times \{y\} = G_1(\mathbb{R})^+ \cdot Z_0,$$

as desired. Finally, we let G_0 be the derived subgroup of G_1. The fact that G_0^{ad} has no compact $\mathbb{Q}$-factors follows from the fact that H_1^{ad} has none. $\square$

Now, given an irreducible variety $V \subset \mathcal{A}_g$, let $S(V)$ be the minimal weakly special subvariety containing V. Let V^{sm} be the smooth locus of V, and let U_0 be a connected component of $\pi_g^{-1}(V^{\mathrm{sm}})$. Define the *monodromy group* $\Gamma_V \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ to be the stabilizer of U_0. Let G_0 be the identity component of the Zariski closure of Γ_V.

LEMMA 2.4. *The group G_0 is semi-simple, and $\pi_g(G_0(\mathbb{R})^+ \cdot v) = S(V)$ for all $v \in U_0$.*

Proof. The semi-simplicity would follow from Theorem 1 of [2], except for the fact that $\mathrm{Sp}_{2g}(\mathbb{Z})$ does not act freely on $\mathbb{H}_g$ and thus $\mathcal{A}_g$ does not possess a family of polarized Hodge structures. However, consider the full congruence group $\Gamma(3) \subset \mathrm{Sp}_{2g}(\mathbb{Z})$. This subgroup does act freely on $\mathbb{H}_g$, so that $\mathcal{A}_{g,3} := \Gamma(3) \backslash \mathbb{H}_g$ does have a family of polarized Hodge structures. Now, we have the finite map $\phi_3 : \mathcal{A}_{g,3} \to \mathcal{A}_g$. Let $V_3 \subset \mathcal{A}_{g,3}$ be an irreducible component of $\phi_3^* V$. We can apply Theorem 1 of [2] to V_3 and get that the identity component of the Zariski closure of Γ_{V_3} is semi-simple. However, Γ_{V_3} is a finite index normal subgroup of Γ_V, and thus their Zariski closures have the same identity component, proving the semi-simplicity of G_0. The rest of the lemma follows from [24, Lemmas 3.7 and 3.9] $\square$

2.5. *Fundamental domains.* We set S_g to be the usual Siegel fundamental domain for the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathbb{H}_g$ (see [26], or [17, §3.3]).

LEMMA 2.5. *Suppose $X + iY \in S_g$. Then there exists a constant $c_g > 0$ such that*

(a) *all the coefficients of X are bounded in absolute value by $\frac{1}{2}$;*
(b) $|Y| \le \prod_{y=1}^{g} y_{ii} \le c_g |Y|$, *where $|Y|$ is the determinant of Y;*
(c) $\frac{\sqrt{3}}{2} \le y_{11} \le y_{22} \le y_{33} \cdots \le y_{gg}$;
(d) $\forall 1 \le i \ne j \le g, |y_{ij}| \le \frac{1}{2} \min(y_{ii}, y_{jj})$.

Proof. Parts (a) and (d) are true by definition, while (c) follows from [17, Lemma 15, p. 195]. Part (b) follows from Minkowski's second result on successive minima, together with [19, Th. 2]. □

2.6. *Norms.* We define the *norm* of a matrix $Z = (z_{ij}) = X + iY \in \mathbb{H}_g$, where $X, Y \in M_g(\mathbb{R})$, to be

$$h(Z) = \max(1, |z_{ij}|, |Y|^{-1}),$$

and we define the *norm* of a matrix $M = (m_{ij}) \in \mathrm{M}_n(\mathbb{R})$ to be

$$h(M) = \max(1, |m_{ij}|).$$

2.7. O-*minimality and definability.* In this paper *definable* will mean *definable in the* o-*minimal structure* $\mathbb{R}_{\text{an, exp}}$. See [27] for a brief introduction to o-minimality and [11], [10] for the properties of $\mathbb{R}_{\text{an, exp}}$. An essential input to enable the o-minimal machinery to be applied is the following result due to Peterzil-Starchenko [26]:

THEOREM 2.6. *The projection map* $\pi_g : S_g \to \mathcal{A}_g$ *is definable.*

3. **Some norm bounds**

In this section we prove some basic lemmas concerning the norms introduced in Section 2.6. We shall only care about asymptotic growth, and moreover we only wish to establish that certain quantities do not exhibit super-polynomial growth. Thus, we introduce some notation.

Definition. Let M be a set, and let F, G be functions mapping M to $\mathbb{R}_{>0}$. We say that F is *polynomially bounded* in G if there exist constants $a, b > 0$ with $F(m) \leq a \cdot G(m)^b$, and we write $F \prec G$. If $F \prec G$ and $G \prec F$, we write $F \asymp G$.

Clearly, $\prec$ is transitive, whereas $\asymp$ is an equivalence relation. It is also clear that $F \prec G \Leftrightarrow G^{-1} \prec F^{-1}$. We record some basic facts.

LEMMA 3.1. *If* $Z \in \mathbb{H}_g$ *and* $M_1, M_2 \in \mathrm{Sp}_{2g}(\mathbb{R})$, *we have*
(1) $h(M_1M_2) \prec h(M_1)h(M_2)$,
(2) $h(M_1) \asymp h(M_1^{-1})$,
(3) $h(M_1Z) \prec h(M_1)h(Z)$.

Proof. (1) Clear.

(2) By symmetry, it is enough to show $h(M_1^{-1}) \prec h(M_1)$. This is obvious because the minors of M_1 are polynomial in the entries of M_1, and $|M_1| = 1$.

(3) Write $M_1 = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Then $M_1Z = (AZ + B)(CZ + D)^{-1}$. We first bound $|CZ + D|$ from below. Note the identity ([15, p. 184])

$$(*) \qquad (C\overline{Z} + D)^t(AZ + B) - (A\overline{Z} + B)^t(CZ + D) = 2iY.$$

Let $f = |CZ + D|$. Then there exists a vector $\xi \in \mathbb{C}^g$ with $||\xi|| = 1$ and $||(CZ + D)\xi|| \prec f$, where $||\xi||$ is defined to be $\bar{\xi}^t \cdot \xi$.

Complete ξ to a unitary basis $\xi = \xi_1, \xi_2, \ldots, \xi_g$. Then we have

$$|Y| \leq \prod_{i=1}^{g} |\xi_i^t Y \bar{\xi}_i| \prec |\xi^t Y \bar{\xi}| h(Y)$$

so that

$$|\xi^t Y \bar{\xi}|^{-1} \prec h(Y) \leq h(Z).$$

Finally, $(*)$ gives

$$|\xi^t Y \bar{\xi}| \prec f h(M_1) h(Z),$$

so that $f^{-1} \prec h(M_1)h(Z)$. It is now clear that all the entries of $M_1 Z$ are polynomially bounded by $h(M_1)h(Z)$. It remains to show that if $M_1 Z = X' + iY'$, then $|Y'|^{-1} \prec h(M_1)h(Z)$.

Again using (*) and the fact that $M_1 Z$ is symmetric, we derive $(Y')^{-1} = (C\overline{Z} + D)Y^{-1}(CZ + D)^t$. Thus all the coefficients of $(Y')^{-1}$ are polynomially bounded by $h(M_1)h(Z)$, and thus so is $|Y'|^{-1}$. This completes the proof. $\square$

Here we restate Lemma 3.2 from [28]. For a point $Z \in \mathbb{H}_g$, there is a unique $\gamma_Z \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $\gamma_Z \cdot Z \in S_g$.

Lemma 3.2. $h(\gamma_Z) \prec h(Z)$.

4. Volumes of algebraic curves in $\mathbb{H}_g$

4.1. *Volumes of algebraic curves in* $\mathbb{H}_g$. Recall that there is a complex structure on $\mathbb{H}_g$ via the imbedding $\mathbb{H}_g \hookrightarrow M_g(\mathbb{C})^{\mathrm{sym}} = \mathbb{C}^{g(g+1)/2}$. Take $C \subset \mathbb{H}_g$ to be a curve in $\mathbb{H}_g$ (i.e., an irreducible algebraic subvariety of dimension 1). We define the degree of C to be the degree of the Zariski closure of C in $\mathbb{C}^{g(g+1)/2}$. The restriction of the metric $d\mu(Z)$ gives a Riemannian metric on C and thus an induced Volume form dC. Our goal in this section is to prove the following theorem:

Theorem 4.1. *For a curve $C \subset \mathbb{H}_g$ of degree k, we have the bound*

$$\int_{C \cap S_g} dC \ll k,$$

where the implied constant depends on g.

Proof. The main ingredient in the proof of the theorem is the following lemma:

Lemma 4.2. *Within S_g, we have*

$$d\mu(Z) = Tr(Y^{-1} dZ Y^{-1} d\overline{Z}) \leq O_g(1) \cdot \sum_{i,j} \frac{|dz_{ij}|^2}{y_{ii} y_{jj}}.$$

Proof. By Lemma 2.5 we have that $|Y| \gg \prod_{i=1}^{g} y_{ii}$, and

$$(*) \qquad \forall i \neq j, \quad |y_{ij}| \leq \min(y_{ii}, y_{jj})/2.$$

Now, let y'_{ij} denote the entries of Y^{-1}, and let M_{ij} be the (i,j)'th minor of Y. Then by $(*)$ and the expansion of $M_{i,j}$ along columns we see that $|M_{ij}| \ll \prod_{l\neq j} y_{ll}$, and thus $|y'_{ij}| = |Y|^{-1}|M_{ij}| \ll y_{jj}^{-1}$. Likewise $|y'_{ij}| \ll y_{ii}^{-1}$. Thus $y'_{ij} \leq |y_{ii} \cdot y_{jj}|^{-\frac{1}{2}}$. Thus we have

$$\begin{aligned} Tr(Y^{-1}dZY^{-1}d\overline{Z}) &= \sum_{i,j,m,l} y'_{ij} dz_{jm} y'_{ml} d\overline{z_{li}} \\ &\ll \sum_{i,j,m,l} |dz_{jm} dz_{li}| (y'_{ii} y'_{jj} y'_{mm} y'_{ll})^{-\frac{1}{2}} \\ &\leq \sum_{i,j,m,l} \frac{|dz_{jm}|^2}{y_{jj} y_{mm}} + \frac{|dz_{li}|^2}{y_{ii} y_{ll}}, \end{aligned}$$

which is what we wanted to show. $\square$

Since the volume form for the conformal metric $|dz_{ij}|^2$ is $dx_{ij}dy_{ij}$, by the lemma above, it is enough to show that for all (i,j), we have

$$\int_{C\cap S_g} \frac{dx_{ij}dy_{ij}}{y_{ii}y_{jj}} \ll k.$$

We first consider the case $i = j$. In this case, consider the projection map onto the z_{ii} coordinate, $\pi_{ii} : \mathbb{H}_g \to \mathbb{H}_1$. By Lemma 2.5 the image of S_g under π_{ii} is contained in the Siegel set

$$y_{ii} > \frac{\sqrt{3}}{2}, |x_{ii}| \leq \frac{1}{2}.$$

Moreover, when the map π_{ii} is restricted to C it is either constant, in which case the differential dz_{ii} vanishes along C, or it has finite fibers. In fact, since C has degree k, the map $\pi_{ii} : C \to \mathbb{H}_1$ is at most k to 1. Thus we have

$$\int_{C\cap S_g} \frac{dx_{ii}dy_{ii}}{y_{ii}^2} \leq k \int_{y_{ii}=\sqrt{3}/2}^{\infty} \int_{x_{ii}=-\frac{1}{2}}^{\frac{1}{2}} \frac{dx_{ii}dy_{ii}}{y_{ii}^2} = \frac{2k}{\sqrt{3}}.$$

We now consider the case of $i \neq j$. Here we use that since $y_{ii}, y_{jj} > \frac{\sqrt{3}}{2}$, we have $\frac{1}{y_{ii}y_{jj}} \leq \min(\frac{1}{y_{ij}^2}, \frac{4}{3})$. Thus projecting to the z_{ij} coordinate and reasoning as before, we get

$$\int_{C\cap S_g} \frac{dx_{ij}dy_{ij}}{y_{ii}y_{jj}} \leq k \int_{y_{ij}=-\infty}^{\infty} \int_{x_{ij}=-\frac{1}{2}}^{\frac{1}{2}} \min(\frac{4}{3}, \frac{1}{y_{ij}^2}) dx_{ij}dy_{ij} \ll k$$

as desired. $\square$

4.2. *Volumes of algebraic curves in weakly special subvarieties.* Let G be a connected semi-simple algebraic subgroup of Sp_{2g} defined over $\mathbb{Q}$, with no simple compact $\mathbb{Q}$-factors, and $Z_0 \in \mathbb{H}_g$ such that $\mathbb{H}_G := G(\mathbb{R})^+ \cdot Z_0$ is a weakly special subvariety with $G_0^{\mathrm{ad}}(\mathbb{R})$ fixed by the Cartan involution corresponding to Z_0. Our goal is to prove a similar statement to Theorem 4.1 with $\mathbb{H}_G$ replacing $\mathbb{H}_g$.

Let $V_G := \pi_g(\mathbb{H}_G)$ be the weakly special subvariety of $\mathcal{A}_g$, and let

$$\Gamma_G \subset \mathrm{Sp}_{2g}(\mathbb{Z})$$

be the lattice that stabilizes $\mathbb{H}_G$. Necessarily, Γ_G contains $G(\mathbb{Z})^+$, the intersection of $G(\mathbb{R})^+$ with $\mathrm{Sp}_{2g}(\mathbb{Z})$, as a subgroup of finite index. Note that $\pi_g^{-1}(V_G)$ is the (not disjoint) union over $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ of $\gamma\mathbb{H}_G$. Now, consider $\pi_g^{-1}(V_G) \cap S_g$. By Theorem 2.6 this intersection is definable and thus has only finitely many connected components. Thus, there are finitely many elements $\gamma_i \in \mathrm{Sp}_{2g}(\mathbb{Z}), 1 \leq i \leq m$ such that

$$\pi_g\left(\bigcup_{i=1}^m \gamma_i^{-1}\mathbb{H}_G \cap S_g\right) = V_G$$

and thus

$$\pi_g\left(\bigcup_{i=1}^m \mathbb{H}_G \cap \gamma_i S_g\right) = V_G.$$

Lemma 4.3. *With notation as above, $S_G^0 := \bigcup_{i=1}^m \gamma_i S_g \cap \mathbb{H}_G$ contains a fundamental domain for the action of Γ_G.*

Proof. Indeed, let $Z \in \mathbb{H}_G$. Then as S_g is a fundamental domain for $\mathrm{Sp}_{2g}(\mathbb{Z})$, there exists $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $\gamma \cdot Z \in S_g$. Now, as above this means that there is an i with $1 \leq i \leq m$ such that $\gamma \cdot Z \in \gamma_i^{-1} \cdot \mathbb{H}_G$ and $\gamma \cdot \mathbb{H}_G = \gamma_i^{-1} \cdot \mathbb{H}_G$. Thus there exists a $g \in \Gamma_G$ with $\gamma = \gamma_i^{-1} \cdot g$. Thus $\gamma_i^{-1} g Z \in S_g \cap \gamma_i^{-1}\mathbb{H}_G$, and so $gZ \in \gamma_i S_g \cap \mathbb{H}_G \subset S_G^0$ as desired. □

Now, by picking coset representatives for $G(\mathbb{Z})^+$ in Γ_G we can find a finite union of elements $\beta_j \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $S_G := \cup_j \beta_j S_G^0 \cap \mathbb{H}_G$ contains a fundamental domain for the action of $G(\mathbb{Z})^+$ on $\mathbb{H}_G$.

Summarizing the above discussion, we arrive at the following theorem:

Theorem 4.4. *There is a semi-algebraic set $S_G \subset \mathbb{H}_G$ such that*

(1) $G(\mathbb{Z})^+ \cdot S_G = \mathbb{H}_G$;
(2) *for an irreducible algebraic curve $C \subset \mathbb{H}_G$ of degree k, we have the bound*
$$\int_{C \cap S_G} dC \ll k,$$
where the implied constant depends on G;
(3) *the projection map $\pi_g : S_G \to \mathcal{A}_g$ is definable.*

Proof. Assertion (1) was already proven in the discussion above. For (2) we need only observe that if C is algebraic of degree k, then so is $\gamma \cdot C$. Hence

$$\int_{C\cap S_G} dC \leq \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{C\cap \beta_j\gamma_i S_g} dC = \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{(\beta_j\gamma_i)^{-1}C\cap S_g} d(\beta_j\gamma_i)^{-1}C \ll k,$$

where the last inequality follows from Theorem 4.1. Finally, (3) follows easily from the fact that $\pi_g : S_g \to \mathcal{A}_g$ is definable, as per Theorem 2.6. □

5. Volumes of algebraic curves near the boundary

Take $C \subset M_g(\mathbb{C})^{\mathrm{sym}}$ to be an irreducible algebraic curve of degree k in $\mathbb{H}_g$. Then its Zariski closure C^{zar} must intersect the boundary $\partial\mathbb{H}_g$ in some real algebraic curve C_0; to see that C^{zar} cannot be contained in $\mathbb{H}_g$, recall that there is a birational algebraic map taking $\mathbb{H}_g$ to a bounded set. Our goal in this section is to show that the volume of C near C_0 is large. This idea and its execution are due to Ullmo and Yafaev [40].

THEOREM 5.1. *For $M > 1$, set $C_M := \{Z \in C \mid h(Z) \leq M\}$. Then*

$$M \prec \int_{C_M} dC.$$

Proof. We proceed as in [40]. Pick a smooth compact piece $I \subset C_0$ and a point $p \in I$. For $0 < \alpha < \beta < 2\pi$, set

$$\Delta_{\alpha,\beta} := \{z = re^{i\theta} \mid 0 \leq r < 1, \alpha \leq \theta \leq \beta\}$$

and

$$C_{\alpha,\beta} := \{z = e^{i\theta} \mid \alpha \leq \theta \leq \beta\}.$$

We may find α, β and a real analytic map

$$\psi : \Delta_{\alpha,\beta} \to C$$

that extends to a real analytic function from a neighbourhood of $\Delta_{\alpha,\beta} \cup C_{\alpha,\beta}$ to C^{zar} such that $\psi(C_{\alpha,\beta}) \subset \partial\mathbb{H}_g$. Composing with $|Y|$ gives a real analytic function on a neighbourhood of $\Delta_{\alpha,\beta} \cup C_{\alpha,\beta}$ that is positive on $\Delta_{\alpha,\beta}$ and vanishes exactly when $1 - z\bar{z}$ vanishes. Thus, there exists $\lambda > 0$ such that $|Y|^2 = (1 - z\bar{z})^\lambda \cdot \psi_1(z)$, where $\psi_1(z)$ is a real analytic function that is positive on $\Delta_{\alpha,\beta}$ and that does not vanish identically on $C_{\alpha,\beta}$. Thus by changing α and β if necessary, we can ensure that $\psi_1(z)$ is nonvanishing on $C_{\alpha,\beta}$, so that

$$\log|Y| = \frac{\lambda}{2}\log(1 - z\bar{z}) + O(1). \tag{1}$$

Now, as in Ullmo-Yafaev we can also ensure that if ω denotes the Kähler form on $\mathbb{H}_g$ and $\omega_\Delta = idz \wedge d\bar{z}/(1 - |z|^2)^2$, then

$$\psi^*(\omega) = s\omega_\Delta + \eta, \tag{2}$$

where η is smooth in some neighbourhood of $C_{\alpha,\beta}$ and s is some positive integer. Finally, for $\delta < 1$ we set $I_\delta = \Delta_{\alpha,\beta} \cap \{|z| < 1-\delta\}$.

A computation gives that

$$\int_{I_\delta} \omega_\Delta \gg \frac{1}{\delta}.$$

Combining this with equations (1) and (2) gives the result. □

6. **Proof of Ax-Lindemann**

We can now prove the Ax-Lindemann (or Ax-Lindemann-Weierstrass) theorem for $\mathcal{A}_g$. We give a more general formulation applicable to semi-algebraic subsets $W \subset \pi_g^{-1}(V)$. Such a set W will be called *irreducible* if it is not the union of two nonempty relatively closed proper subsets in the topology induced on it by the Zariski topology of algebraic sets defined over $\mathbb{R}$ (see [14]).

THEOREM 6.1. *Let $V \subset \mathcal{A}_g$ be an irreducible algebraic variety, and suppose that $W \subset \pi_g^{-1}(V)$ is a connected irreducible semi-algebraic subset of $\mathbb{H}_g$. Then there exists a weakly special subvariety $S \subset V$ such that $W \subset \pi_g^{-1}(S)$.*

Proof. We proceed by induction on the dimension V, the case of $\dim(V)=0$ being obvious, as all points are weakly special; for the same reason we may assume W has positive dimension. We may assume without loss of generality that W is maximal; i.e., if W' is semi-algebraic with $W \subset W' \subset \pi_g^{-1}(V)$, then W' has W as a component. Likewise, we may assume by our induction hypothesis that V is minimal in the sense that there does not exist $V' \subset V$ with $\dim V' < \dim V$ and $W \subset \pi_g^{-1}(V')$. Note that since W is maximal, by Lemma 4.1 of [28], W must be a complex algebraic subvariety. To prove the theorem we must show that V is weakly special, so that W is an irreducible component of $\pi_g^{-1}(V)$, by Lemma 2.1.

Take now S_0 to be the minimal weakly special subvariety of $\mathcal{A}_g$ containing V, and write U_0 for the irreducible component of $\pi_g^{-1}(S_0)$ containing W (so we have to show that $S_0 = V$ and $W = U_0$).

Let $\Gamma_V \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ be the monodromy group of V, and let G_0 be the connected component of the Zariski closure of Γ_V. Then by Lemma 2.4, G_0 is a semi-simple group over $\mathbb{Q}$, and for any point $u_0 \in U_0$, we have $U_0 = G_0(\mathbb{R})^+ \cdot u_0$. Assume without loss of generality that $W \cap S_{G_0} \neq \emptyset$.

Write Y for a connected component of $\pi_g^{-1}(V)$ that intersects the open part of S_{G_0}, and set $Y^0 = Y \cap S_{G_0}$. Note that Y^0 is definable (in $\mathbb{R}_{\mathrm{an,exp}}$; [26]). Write

$$X = \{\gamma \in G_0(\mathbb{R}) \mid \dim\left(\gamma \cdot W \cap Y^0\right) = \dim W\},$$

which likewise is definable. Observe that $\gamma \cdot W \subset Y$ for all $\gamma \in X$ by the dimension assumptions and analytic continuation.

For a set $A \subset \mathrm{Sp}_{2g}(\mathbb{R})$ and a real number $T \geq 1$, define the counting function

$$N(A, T) = \#\{M \in A \cap \mathrm{Sp}_{2g}(\mathbb{Z}) : H(M) \leq T\}.$$

Here $H(M)$ is the multiplicative Weil height of M, i.e., the maximum size of its integer entries, so that $H(M) = h(M)$ as defined in Section 2.6 (as not all its entries can vanish).

LEMMA 6.2. *We have $T \prec N(X, T)$, implied constants depending on X.*

Proof. Let $C \subset W$ be an algebraic curve. For $T > 0$, define

$$C_T = \{Z \in C \mid h(Z) \leq T\}, \quad X_T = \{\gamma \in X \mid h(\gamma) \leq T\}.$$

Since S_{G_0} contains a fundamental domain, for each point $Z \in C$ there exists a $\gamma \in \Gamma_V$ such that $\gamma \cdot Z \in Y^0$. By Lemma 3.2 there exists $M > 0$ such that $H(\gamma) \leq h(Z)^M$ for sufficiently large $h(Z)$. Hence for $T \gg 1$, we must have

$$C_{T^{\frac{1}{M}}} \subset \bigcup_{\gamma \in X_T \cap \Gamma_V} \gamma^{-1} S_{G_0}.$$

In particular, we have that

$$\mathrm{Vol}(C_{T^{\frac{1}{M}}}) \leq \sum_{\gamma \in X_T \cap \Gamma_V} \mathrm{Vol}(\gamma \cdot C \cap S_{G_0}).$$

Combining Theorems 5.1 and 4.4 now gives the result. □

Since $N(X, T)$ grows faster than some positive power of T, the Counting Theorem (in the form [27, Th. 3.6]) implies that there are semi-algebraic varieties $W_1 \subset X$ of positive dimension containing arbitrarily many points $\gamma \in \Gamma_V$ such that $W_1 \cdot W \subset Y$. To see this, supposing $N(X, T) \gg T^\eta$ with $\eta > 0$, apply [27, Th. 3.6] with $\mu = 0, k = 1$, and $0 < \epsilon < \eta$. As the $\gg T^\eta$ rational points up to height T are contained in $\ll T^\epsilon$ blocks (as defined there) provided by the theorem, there must be a block containing $\gg T^{\eta-\epsilon}$ rational points. Note that the simpler version [27, Th. 3.2] (which is [29, Th. 1.9] in the case $k = 1$) does not give this conclusion. For such W_1 and $\gamma \in W_1 \cap \Gamma_V$, we have $\gamma^{-1} W_1 \cdot W$ contains W, and so by our maximality assumption we have

$$\gamma^{-1} W_1 \cdot W = W.$$

Now let Θ be the algebraic group over $\mathbb{R}$ that is the Zariski closure in $G_{0,\mathbb{R}}$ of the subset

$$\{\gamma \in G_0(\mathbb{R}) \mid \gamma \cdot W = W\}.$$

Let Θ^0 be its connected component. Since $\gamma^{-1} W_1 \subset \Theta$, it follows that Θ^0 has positive dimension. Let H be the maximal connected algebraic subgroup of G_0 defined over $\mathbb{Q}$ such that $H(\mathbb{R}) \subset \Theta^0(\mathbb{R})$. We know that Θ^0 has infinitely many rational (in fact integral) points, hence so does $H(\mathbb{R})$, and so $\dim(H(\mathbb{R})) > 0$.

Suppose that Y is *not* invariant under $H(\mathbb{R})^+$. Since $H(\mathbb{Q})$ is dense in $H(\mathbb{R})$ (see, e.g., [33, Cor. 3.5]), we can find an element $h \in H(\mathbb{Q})^+$ such that Y is not invariant under h. Let $Y' = Y \cap hY$. Then $\pi_g(Y')$ is a closed algebraic proper subvariety $V' \subset V$ with and $W \subset Y'$. (In fact V' is a component of the intersection of V with one of its Hecke translates.) This contradicts our minimality assumption on V, hence Y is invariant under $H(\mathbb{R})^+$.

Let H' be the smallest algebraic subgroup of G_0 containing the conjugates of H by all $\gamma \in \Gamma_V$. Since Y is invariant under $H(\mathbb{R})^+$ and under Γ_V, it is invariant under $H'(\mathbb{R})^+$. Now H' is an algebraic group that is invariant by conjugation under Γ_V, and hence also under G_0, the Zariski closure of Γ_V. Hence H' is normal in G_0. Note that $H'(\mathbb{R})^+ \cdot W \subset Y$, so by maximality of W we conclude that $H'(\mathbb{R})^+ \cdot W = W$, so that $H' = H$.

Next consider the map $\phi : G_0 \to G_0^{\mathrm{ad}}$, where G_0^{ad} is the adjoint form of G_0. We can therefore write $G_0^{\mathrm{ad}} = \prod_{i=1}^r G_i$, where the G_i are $\mathbb{Q}$-simple algebraic groups. Therefore there is some nonempty subset $I \subset \{1, \ldots, r\}$ such that

$$\phi(H) = \prod_{i \in I} G_i.$$

We write $U_0 \cong \prod_i U_i$, where the U_i are hermitian symmetric spaces associated to G_i. Thus W can be written as $\prod_{i \in I} U_i \times W'$, where W' is an irreducible algebraic subvariety of $\prod_{i \in I^c} U_i$. If W' is a point, then $\pi_g(W)$ is weakly special, and so we must have $W = U_0$ as desired. Hence, we assume from now on that $\dim(W') > 0$.

Now pick any point $u_I \in \prod_{i \in I} U_i$ and consider the variety

$$W^{\mathrm{new}} = u_I \times W'.$$

Moreover, let

$$V^{\mathrm{new}} = V \cap \pi_g(u_I \times \prod_{j \in I^c} U_j),$$

which is evidently an algebraic variety as $\prod_{j \in I^c} U_j$ is weakly special. Then

$$\pi_g(W^{\mathrm{new}}) \subset V^{\mathrm{new}},$$

and so by our induction hypothesis there is a weakly special variety

$$S^{\mathrm{new}} \subset u_I \times \prod_{j \in I^c} U_j$$

such that $\pi_g(S^{\mathrm{new}}) \subset V^{\mathrm{new}}$ and $W^{\mathrm{new}} \subset S^{\mathrm{new}}$. By projection, we can identify S^{new} with its image in $\prod_{j \in I^c} U_j$. Since weakly special varieties are closed under intersection (Corollary 2.2), we can consider $S_{\min} \subset \prod_{j \in I^c} U_j$ to be the minimal weakly special variety containing W'. By the discussion above, for each point $u_I \in \prod_{i \in I} U_i$, we know that $\pi_g(u_I \times S_{\min}) \subset V$. In other words, the variety

$$S = \prod_{i \in I} U_i \times S_{\min}$$

is weakly special and satisfies both $W \subset S$ and $\pi_g(S) \subset V$. By our maximality assumption, we conclude that $W = S = U_0$ and $V = S_0$ as desired. ☐

7. Application to Andre-Oort

In this section we give the application to the Andre-Oort conjecture. As already mentioned, the size of the Galois orbit of a special point plays a crucial role. For a point $x \in \mathcal{A}_g$, let A_x denote the corresponding g-dimensional principally polarized abelian variety, $R_x = Z(\mathrm{End}(A_x))$ the centre of the endomorphism ring of A_x, and $\mathrm{Disc}(R_x)$ its discriminant. (For further details see [37].) In general one expects to have the following lower bound suggested by Edixhoven in [12]:

Conjecture 7.1. *Let $g \geq 1$. Then, for a special point $x \in \mathcal{A}_g$,*

$$|\mathrm{Disc}(R_x)| \prec |\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x|$$

(*with the implied constants depending on g*).

For $g = 1$ this is known by the theory of complex multiplication of elliptic curves (see, e.g., [6]) and the Landau-Siegel lower bound for the class number of an imaginary quadratic field [21], [35]. As already mentioned, the second author [37] has affirmed this conjecture

(1) for $g \leq 6$, and
(2) for all g under GRH for CM fields (see also Ullmo-Yafaev [41]).

All the unconditional lower bounds are ineffective; hence Theorem 1.2, which we now establish by proving Theorem 7.1 below, is ineffective. (Other aspects of the proof are also ineffective, including the Counting Theorem, though this ineffectivity seems less serious.) The only nontrivial case of the Andre-Oort conjecture known unconditionally and effectively is that of the product of two modular curves, due independently to Kühne [20] and Bilu-Masser-Zannier [5]. (André's original proof [3] is ineffective.)

Theorem 7.1. *Suppose Conjecture 7.1 holds for g. Let V be an irreducible closed algebraic subvariety of $\mathcal{A}_g$. Then V contains only finitely many maximal special subvarieties.*

Before proving this theorem we need some lemmas, in which we take $V \subset \mathcal{A}_g$ to be an irreducible closed algebraic subvariety. We do not for the moment assume Conjecture 7.1.

We set $G \subset \mathrm{GSp}_{2g,\mathbb{Q}}$ to be the generic Mumford-Tate group of an irreducible component Z of $\pi_g^{-1}(V)$. Then, for any $z \in Z$, $\pi_g(G(\mathbb{R})^+ \cdot z)$ is the smallest special subvariety containing V. We may assume that V contains special points, so that this is also the smallest such weakly special subvariety. Let S_G be as defined above Theorem 4.4, and recall it contains a fundamental domain for $G(\mathbb{Z})^+$. Let $Y^0 = \pi_g^{-1}(V) \cap S_G$ which, as we have already observed in Theorem 4.4, is a definable set.

We consider orbits $H(\mathbb{R})^+ \cdot z, z \in \mathbb{H}_g$ that lie inside $\pi_g^{-1}(V)$, where $H \subset \mathrm{Sp}_{2g,\mathbb{R}}$ is a semi-simple group. As such orbits are real algebraic, we observe that if

$$\dim H(\mathbb{R})^+ \cdot z = \dim \left(H(\mathbb{R})^+ \cdot z \cap \left(\pi_g^{-1}(V) \cap S_G\right)\right)$$

then, by analytic continuation, we have $H(\mathbb{R})^+ \cdot z \subset \pi_g^{-1}(V)$. We call such an orbit *maximal* if it is not contained in an orbit $K(\mathbb{R})^+ \cdot z$, where $K \subset \mathrm{Sp}_{2g,\mathbb{R}}$ is semi-simple and $K(\mathbb{R})^+ \cdot z$ is contained in $\pi_g^{-1}(V)$ and has larger dimension than $H(\mathbb{R})^+ \cdot z$. By Ax-Lindemann (6.1), every maximal orbit is weakly-special, and Lemma 2.3 implies that there is a connected, semi-simple $\mathbb{Q}$-subgroup $F \subset G$ with no compact $\mathbb{Q}$-factors, such that $F(\mathbb{R})^+ \cdot z = H(\mathbb{R})^+ \cdot z$.

Definition. We set $\mathcal{C}$ to be the set of all connected semi-simple $\mathbb{Q}$-subgroups $F \subset G$ that have no compact $\mathbb{Q}$-factors and such that there exists $y \in \mathbb{H}_g$ for which $F(\mathbb{R})^+ \cdot y$ is weakly special.

For $F \in \mathcal{C}$ and for every weakly special subvariety $F(\mathbb{R})^+ \cdot y \subset \mathbb{H}_g$, there is a Shimura subdatum $(H, X_H) \subset (G, X_G)$ and a splitting

$$(H^{\mathrm{ad}}, X^+_{H^{\mathrm{ad}}}, \Gamma) = (H_1, X_1^+, \Gamma_1) \times (H_2, X_2^+, \Gamma_2),$$

and a point $h_2 \in X_2^+$ such that $X_1^+ \times \{h_2\} = F(\mathbb{R})^+ \cdot y$.

Consider the group

$$\Gamma' := \{\gamma \in G(\mathbb{Z})^+ \mid \gamma \cdot F(\mathbb{R})^+ \cdot y = F(\mathbb{R})^+ \cdot y\}.$$

This group has the property that $\Gamma' \backslash F(\mathbb{R})^+ \cdot y$ is finite volume. However, letting $\widetilde{H_1}$ denote the pullback of H_1 to H, the subgroups $F(\mathbb{Z})^+$ and $\widetilde{H_1}(\mathbb{Z})^+$ also have this property, and hence their images in G^{ad} are commensurable. Since neither F nor $\widetilde{H_1}$ have compact $\mathbb{Q}$-factors and both are connected, $F \cdot Z(\widetilde{H_1}) = \widetilde{H_1}$ by [32, Th. 4.10]. Letting $N(F)$ denote the connected component of the normalizer of F in G, it follows that $H \subset N(F)$.

Lemma 7.2. *Let $F \in \mathcal{C}$. Then $N(F)$ is a reductive $\mathbb{Q}$-group.*

Proof. Let $y \in \mathbb{H}_g$ be such that $F(\mathbb{R})^+ \cdot y$ is a weakly special subvariety. Recall that y gives a homomorphism $y : \mathbb{S} \to G_{\mathbb{R}}$, and conjugation by $y(i)$ gives a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$ that preserves the image of $F_{\mathbb{R}}$ in $G^{\mathrm{ad}}_{\mathbb{R}}$. Therefore, it must also preserve $N(F)_{\mathbb{R}}$. By [34, §I, 4.3], $N(F)$ is reductive. The fact that it is defined over $\mathbb{Q}$ follows easily from the fact that F is. ☐

Write $\mathrm{Com}(N(F))$ for the product of all compact $\mathbb{Q}$-factors of $N(F)^{\mathrm{ad}}$, and let $N(F)_{\mathrm{sh}}$ denote the identity component of the kernel of the projection $N(F) \to \mathrm{Com}(N(F))$. Then $y : \mathbb{S} \to G_{\mathbb{R}}$ factors through $\phi(H_1 \times H_2)$ and hence also through $N(F)_{\mathrm{sh}}$. Thus $N(F)_{\mathrm{sh}}(\mathbb{R})^+ \cdot y$ defines a special subvariety of $G(\mathbb{R})^+ \cdot y$.

Lemma 7.3. *Let $F \in \mathcal{C}$. Then the set of special subvarieties of $\mathbb{H}_g$ of the form $N(F)_{\mathrm{sh}}(\mathbb{R})^+ \cdot y$ such that $y(\mathbb{S}) \subset N(F)_{\mathrm{sh},\mathbb{R}}$ is finite as y varies over $\mathbb{H}_g$.*

Proof. The requirement is that $y : \mathbb{S} \to G_{\mathbb{R}}$ has image in $N(F)_{\mathrm{sh},\mathbb{R}}$, and we would like to show that the number of such images that arise is finite up to $N(F)_{\mathrm{sh}}(\mathbb{R})^+$ conjugacy. Since the homomorphisms we are interested in correspond to Shimura varieties, it means each such homomorphism also defines a hermitian symmetric domain, which has as its group of biholomorphisms the corresponding adjoint real group $N(F)^{\mathrm{ad}}_{\mathrm{sh},\mathbb{R}}$. It is well known that the number of such domains of a bounded dimension is finite, hence there are finitely many types $(N(F)^{\mathrm{ad}}_{\mathrm{sh},\mathbb{R}}, y : \mathbb{S} \to N(F)^{\mathrm{ad}}_{\mathrm{sh},\mathbb{R}})$ corresponding to a hermitian symmetric space up to an automorphism of $N(F)^{\mathrm{ad}}_{\mathrm{sh},\mathbb{R}}$. Since $N(F)^{\mathrm{ad}}_{\mathrm{sh}}$ is semi-simple, its outer automorphism group is finite. This means that, up to $N(F)_{\mathrm{sh}}(\mathbb{R})^+$ conjugacy, there are only finitely many homomorphisms $y : \mathbb{S} \to N(F)^{\mathrm{ad}}_{\mathrm{sh},\mathbb{R}}$ that arise from points $y \in \mathbb{H}_g$. But recall that we know that the weights of the action $\mathrm{ad} \circ y$ of $\mathbb{S}$ on the complexified Lie algebra of Sp_{2g} are restricted to a finite set. This implies that for each homomorphism to $N(F)^{\mathrm{ad}}_{\mathrm{sh},\mathbb{R}}$, only finitely many possible lifts to $N(F)_{\mathrm{sh},\mathbb{R}}$ actually occur in the $G(\mathbb{R})^+$-conjugacy class. This completes the proof of the assertion. □

For the proof of Theorem 7.1, it will be useful to make the following definition:

Definition. For each $F \in \mathcal{C}$, define $S(F)$ to be the union of $\pi_g(F(\mathbb{R})^+ \cdot z)$ over the maximal F-orbits $F(\mathbb{R})^+ \cdot z$, which are also special subvarieties and satisfy

$$\dim F(\mathbb{R})^+ \cdot z = \dim \left(F(\mathbb{R})^+ \cdot z \cap \left(\pi_g^{-1}(V) \cap S_G \right) \right).$$

We let $W(F)$ be the corresponding union over maximal F-orbits that are weakly special.

Finally, we need the following result, which is well known to the experts but for which we know of no easy reference in the literature:

Lemma 7.4. *For any $B > 0$, there are only finitely many special points $x \in \mathcal{A}_g$ such that $|\mathrm{Disc}(R_x)| < B$.*

Proof. Let x be such a point, and consider its lift Z to the fundamental domain S_g. Then by Theorem 3.1 in [28] we know that $H(Z) \prec |\mathrm{Disc}(R_x)|$, where $H(Z)$ is the height of the point Z. Since all the co-ordinates of Z are algebraic of degree at most $2g$, there are only finitely many such Z by Northcott's Theorem. This completes the proof. □

We can now prove Theorem 7.1.

Proof. We now assume that Conjecture 7.1 holds for g. The proof is by induction on $\dim V$, therefore we may assume that V contains a Zariski-dense set of special points, and we must then show that V is a special subvariety. Since special points are defined over $\overline{\mathbb{Q}}$, we must have that V is defined over a field K of finite degree over $\mathbb{Q}$.

Let x be a special point in V, and let Z be a preimage of x contained in the standard fundamental domain S_g of $\mathrm{Sp}_{2g}(\mathbb{Z})$ as in [28]. By Theorem 3.1 in [28], we know that the height $H(Z)$ of Z satisfies $H(Z) \prec |\mathrm{Disc}(R_x)|$. The conjugates x' of x over K again lie on V, are special points of the same discriminant, and have pre-images Z' in S_g. Thus by Conjecture 7.1, if $|\mathrm{Disc}(R_x)|$ is sufficiently large, the Counting Theorem (in the form [27, Th. 3.2]) implies that "many" of the x' (i.e., a positive power of $|\mathrm{Disc}(R_x)|$ of them) are contained in the algebraic part (as defined there) of Y^0, which thus contains a semi-algebraic set of positive dimension containing at least one (in fact many, if we use [29, 3.6], but one will suffice) preimage Z' of a Galois conjugate of x over K. By Ax-Lindemann (the semi-algebraic version: Theorem 6.1), Z' is contained in a weakly special subvariety $S \subset \pi^{-1}(V)$, which we may choose to be maximal, and is in fact special as x is special. This implies that $\pi_g(S)$ is algebraic, and hence the union of all Galois conjugates of $\pi_g(S)$ over K contains x and lies inside V. Hence, by Lemma 7.4, all but finitely many special points $x \in V$ are contained in a positive dimensional special subvariety of V.

It therefore suffices to establish that there are only finitely many positive dimensional maximal special subvarieties of V, for then the Zariski-density of special points will imply that V itself is special.

By Lemma 2.3, any such special subvariety is the image under π_g of an orbit of $F(\mathbb{R})^+$ for some $F \in \mathcal{C}$. We first show that, for each $F \in \mathcal{C}$, the set $S(F)$ is a finite union of special subvarieties.

We first consider the case where $N(F)_{\mathrm{sh}}$ is not equal to all of G. Then by Lemma 7.3, special subvarieties of V that are images under π_g of orbits of $F(\mathbb{R})^+$ lie in the union of finitely many special subvarieties whose generic Mumford-Tate group M satisfies $M^{\mathrm{der}} = N(F)_{\mathrm{sh}}^{\mathrm{der}}$. But the smallest special subvariety containing V has generic Mumford-Tate group G, and so the intersections of these special subvarieties with V must all be proper. Hence we are done in this case by induction on the dimension of V.

Now consider the case that $N(F)_{\mathrm{sh}}$ is equal to G. This means that F is normal in G, and so G splits as an almost direct product $G = F \cdot Z_G(F)$. Let y be a point in $\pi_g^{-1}(V) \cap S_G$, so that $G(\mathbb{R})^+ \cdot y$ is a hermitian symmetric space with $\pi_g(G(\mathbb{R})^+ \cdot y)$ the minimal special subvariety containing V. Then one has a splitting of the hermitian symmetric space $G(\mathbb{R})^+ \cdot y$ as $F(\mathbb{R})^+ \cdot y \times Z_G(F)(\mathbb{R})^+ \cdot y$.

Let $z_{CM} \in S_G$ be a special point. Then $F(\mathbb{R})^+ \cdot y \cap Z_G(F)(\mathbb{R})^+ \cdot z_{CM}$ consists of a single point, and moreover this point is special if and only if

$F(\mathbb{R})^+ \cdot y$ is a special subvariety, if and only if $F(\mathbb{R})^+ \cdot y$ contains special points (see [24, Th. 4.3]). Thus special subvarieties of the form $F(\mathbb{R})^+ \cdot y$ are in bijection with special points on $Z_G(F)(\mathbb{R})^+ \cdot z_{CM}$. Moreover, if we let X_F^+, X_G^+, and $X_{Z_G(F)}^+$ denote the hermitian symmetric spaces corresponding respectively to F, G and $Z_G(F)$, then we get a finite algebraic map of Shimura varieties

$$\psi : F(\mathbb{Z})^+\backslash X_F^+ \times Z_G(F)^+(\mathbb{Z})\backslash X_{Z_G(F)}^+ \to G(\mathbb{Z})^+\backslash X_G^+.$$

Put $W = \pi_g(Z_G(F)(\mathbb{R})^+ \cdot z_{CM})$ and let

$$V' = \{x \in W \mid x \in V \;\&\; F(\mathbb{Z})^+\backslash X_F^+ \times \mathrm{pr}_2\psi^{-1}\{x\} \subset \psi^{-1}(V)\}.$$

Then V' is a closed algebraic subvariety of $\mathcal{A}_g$ of dimension lower than V (as the orbits of $F(\mathbb{R})^+$ have positive dimension), and special points in V' correspond in a finite-to-one fashion to special subvarieties of the form $\pi_g(F(\mathbb{R})^+ \cdot y)$ in V.

By induction on dimension, the conclusion of Theorem 7.1 is true for V' and thus the set of special points in V' is contained in a finite union of special subvarieties of V'. Hence, the images of *maximal* orbits $\pi_g(F(\mathbb{R})^+ \cdot y)$ correspond to the finite set of isolated special points in V'. Thus we have finiteness also in this case.

Lemma 7.5. *The union of all positive dimensional weakly special subvarieties of V is a countable union of algebraic varieties.*

Proof. Since there are only countably many $\mathbb{Q}$-subgroups of G, we need only consider those weakly special subvarieties arising from a single Shimura subdatum $(H, X_H) \subset (G, X_G)$ and a splitting

$$(H^{\mathrm{ad}}, X_H^{\mathrm{ad},+}) = (H_1, X_1^+) \times (H_2, X_2^+).$$

We then get an algebraic morphism

$$\Gamma_1\backslash X_1^+ \times \Gamma_2\backslash X_2^+ \to G(\mathbb{Z})^+\backslash X_G^+,$$

where $\Gamma_1 \subset H_1$ and $\Gamma_2 \subset H_2$ are appropriate arithmetic subgroups. Then the set of points $y \in \Gamma_2\backslash X_2^+$ such that $X_1^+ \times \{y\}$ maps inside of V is clearly an algebraic variety. This completes the proof. □

From Lemma 7.5 it follows that $\bigcup_{F\in\mathcal{C}} W(F)$ is a countable union of algebraic varieties. We state two further lemmas (which do not require the assumption of Conjecture 7.1).

Lemma 7.6. *The union of all weakly special subvarieties of V of positive dimension is a definable subset of $\mathcal{A}_g(\mathbb{C})$.*

Proof. Let $T \subset V$ be weakly special. Since S_G contains a fundamental domain for the action of $G(\mathbb{Z})$, every point of T is contained in an image of the form $\pi_g(H(\mathbb{R})^+ \cdot z \cap S_G)$ for H a semi-simple subgroup of G and $z \in S_G$. There are only finitely many semi-simple real groups that embed into $\mathrm{Sp}_{2g,\mathbb{R}}$, and the embeddings come in finitely many families up to conjugacy by ([13, A.1]). We restrict for the moment to a particular semi-simple subgroup $H \subset G_{\mathbb{R}}$. For each $y \in G(\mathbb{R})$, let $H_y := yHy^{-1}$. Since the dimension of orbits is definable, it follows that the set

$$T_H := \{(y,z) \in \mathrm{Sp}_{2g}(\mathbb{R}) \times S_G : H_y(\mathbb{R})^+ \cdot z \text{ is maximal}\}$$

is definable (the maximality is definable as there are only finitely many conjugacy classes of groups to compare with), and the union of the

$$\pi_g(H_y(\mathbb{R})^+ \cdot z \cap S_G)$$

over $(y,z) \in T_H$ is thus also definable. Taking the union over the finitely many conjugacy classes of semi-simple groups H proves the result. □

Lemma 7.7. *Let A be a quasiprojective complex algebraic variety. Suppose V_i is a countable sequence of closed complex algebraic subvarieties of A whose union is definable. Then there is a natural number m such that*

$$\bigcup_{i\in\mathbb{N}} V_i = V_1 \cup V_2 \cup \cdots \cup V_m.$$

Proof. Let $X = \bigcup_{i\in\mathbb{N}} V_i$. Since X is definable (in $\mathbb{R}_{\text{an, exp}}$), it has an analytic cell decomposition ([11]), $X = \cup_j U_j$. For each j, there is an i such that $U_j \cap V_i$ has the same dimension as U_j. But since U_j is a connected real analytic variety, if a polynomial vanishes on an open set of U_j, it must vanish on U_j identically. Hence $U_j \subset V_i$. □

By applying the preceding lemmas we see that there is a finite set $\mathcal{V} \subset \mathcal{C}$ such that the union of all maximal weakly special subvarieties of V of positive dimension is $\bigcup_{F\in\mathcal{V}} W(F)$. Consider a positive dimensional special subvariety $T \subset V$. We have $T \subset \bigcup_{F\in\mathcal{V}} W(F)$. Now special points are Zariski-dense in T, but the special points in each $W(F)$ are contained in $S(F)$. Thus $T \subset \bigcup_{F\in\mathcal{V}} S(F)$. We conclude that all maximal special subvarieties of V of positive dimension are contained in the finite union $\bigcup_{F\in\mathcal{V}} S(F)$. Hence the theorem is true for V. □

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Mathematical Institute, University of Oxford, UK
E-mail: Jonathan.Pila@maths.ox.ac.uk

Harvard University, Cambridge, MA
E-mail: jacobt@math.harvard.edu