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The energy of dilute Bose gases

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Abstract

For a dilute system of non-relativistic bosons interacting through a positive, compactly supported, L^1-potential v with scattering length a we prove that the ground state energy density satisfies the bound $e(\rho) \geq 4\pi a\rho^2(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho a^3}))$, thereby proving the Lee-Huang-Yang formula for the energy density.

Contents

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1. **Introduction**

Our goal in this paper is to solve the long standing conjecture in mathematical physics to rigorously establish the Lee-Huang-Yang (LHY) formula for the second correction to the thermodynamic (infinite volume) ground state energy per volume of a translation invariant Bose gas in the dilute limit. The formula (i.e., (1.3) below with an equality) is one of the most fundamental results in quantum many-body theory. It appeared for the first time as equation (25) in the seminal 1957 publication [15]. The striking feature of the formula is that the first two terms of the asymptotics of the ground state energy in the dilute limit depend on the interaction potential only through a single parameter, *the scattering length*. Fairly recently the LHY formula was tested experimentally as reported in [27]. Here the coefficient $\frac{128}{15\sqrt{\pi}} = 4.81$ was measured to be 4.4(5).

The derivation in [15] relies on the pseudo-potential method and offers deep insight into the problem, but nevertheless lacks in mathematical rigor. An alternative, but still non-rigorous, argument was proposed in [17]. We establish the LHY formula rigorously for a large family of two-body potentials (see Assumption 1.1 below) which, however, does not include the hard core potential.

The importance of the scattering length in understanding the energy and excitation spectrum for interacting many-body gases had already been observed in the celebrated 1947 paper of Bogolubov [5] where he introduced the Bogolubov approximation and laid the foundation for the theory of superfluidity. In this paper Bogolubov studies the excitation spectrum of a Bose gas and finds that it depends on the integral of the potential, not the scattering length. In a famous footnote Bogolubov thanks Landau for making the important remark that this must be wrong and that the correct answer must be to replace the integral of the potential by the scattering length. To establish this rigorously has been a major challenge ever since. The first major rigorous advance was achieved by Dyson in [10] where the leading order asymptotics for the ground state energy was established as an upper bound, but where the lower bound was off by a factor. The correct leading order asymptotics was finally established by Lieb and Yngvason in [23] for all positive interaction potentials with finite scattering length including the hard core potential. This result was extended to the Gross-Pitaevskii limit in the case of trapped gases in [19]. These leading order results are reviewed in the monograph [18], which also contains a non-rigorous derivation of the LHY formula using the Bogolubov approximation. To the best of our knowledge the first works to rigorously establish the validity of the Bogolubov approximation for a many-body problem were [21], [22], [31], which studied the one- and two-component charged Bose gases and established a conjecture of Dyson. Several ideas from [21] are important also in the present work.

The first work to show an upper bound to the LHY order was [11] by Erdős, Schlein, and Yau. This paper makes a very interesting observation about the Bogolubov approximation. The usual approach to the Bogolubov approximation is to approximate the Hamiltonian of the system by what is referred to as a quadratic Hamiltonian. As mentioned above this leads to a wrong approximation for the ground state energy where it will be expressed in terms of the integral of the potential rather than the scattering length. Quadratic Hamiltonians have ground states that are quasi-free (or Gaussian) states. In [11] it is observed that if we do not approximate the Hamiltonian by a quadratic Hamiltonian, but instead restrict the evaluation of the full Hamiltonian to quasi-free states, then miraculously the scattering length appears in the leading order term, but to LHY order the answer is still wrong. The work in [11] emphasizes that it may often be more fruitful to focus on classes of states rather than to approximate the Hamiltonian. This approach was further pursued in the papers [25], [26] where the positive temperature situation was analyzed for the Hamiltonian restricted to quasi-free states. The leading order correction to the positive temperature free energy for the full many-body problem in the dilute limit was established in [29], [33].

For gases confined to a region in the Gross-Pitaevskii regime, there is a formula for the second order correction to the ground state energy similar to the LHY formula. This has recently been established in an impressive series of papers by Boccato, Brennecke, Cenatiempo, and Schlein [2], [4], [3]. This, however, does not imply the formula in the original thermodynamic infinite volume setting discussed here. Our proof follows a very different strategy than the one applied in the confined case.

In the confined or trapped case it is also possible to analyze the excitation spectrum of the gas, which is particularly important for understanding superfluidity. The excitation spectrum is also studied in the papers by Boccato et. al. The first result in this direction is, however, due to Seiringer [30] and was also analyzed in [9], [14], [16], [24]. Getting the excitation spectrum in the thermodynamic case seems much more difficult.

The LHY formula in the translation invariant thermodynamic setting was finally rigorously established as an upper bound in the work [32] by Yau and Yin, where they consider smooth rapidly decaying interaction potentials. It is this work that we complement by establishing the lower bound in (1.3), in fact, for a much, larger class of interaction potentials. Thus the LHY formula has been proved for all compactly supported potentials satisfying the assumptions in [32]. We shall not discuss the upper bound further in this paper. In Bogolubov theory, the particles not in the condensate constitute pairs of opposite momentum. An important insight, confirmed by the contributions of [32] and the present work, is that in order to get the correct energy to LHY order, one has to go beyond these simple pairs and also consider "soft pairs." This means that not only pairs of particles of exactly opposite momentum contribute. Also pairs of particles with non-zero total momentum—although the individual momenta are much larger than the sum—are important for the energy to this precision.

The LHY formula had previously been established as a lower bound in the restricted case where the interaction potential is allowed to become softer as the gas becomes more dilute. This was first achieved in [12]. In this case, however, the potential still has a range much larger than the inter-particle spacing, which is why the paper has "high density" in the title. Allowing the potential to have range shorter than the inter-particle spacing, but still required to be soft, was recently achieved in [7]. The softness condition was removed in [6], but only to get the ground state energy to the correct LHY order, not with the correct asymptotics. Several of the methods developed in [7] and [6] are crucial to this work.

There has been a large literature also on the dynamics of interacting Bose gases, but we will not review that here.

We now turn to describing the problem in details. We consider N bosons in three dimensions described by the Hamiltonian

$$\mathcal{H}_N = \mathcal{H}_N(v) = \sum_{i=1}^{N} -\Delta_i + \sum_{1\leq i<j\leq N} v(x_i - x_j). \tag{1.1}$$

The first term above represents the kinetic energy, and the second term is the potential energy due to interactions.

We will allow interactions described by the following assumptions.

Assumption 1.1 (Potentials). *The potential $v \neq 0$ is non-negative and spherically symmetric, i.e., $v(x) = v(|x|) \geq 0$, and of class $L^1(\mathbb{R}^3)$ with compact support. We fix $R > 0$ such that* $\operatorname{supp} v \subset B(0,R)$.

We are interested in the thermodynamic limit of the ground state energy density as a function of the particle density ρ:

$$e(\rho, v) = \lim_{\substack{L\to\infty \\ N/L^3\to\rho}} L^{-3} \inf_{\Psi\in C_0^\infty([0,L]^N)\setminus\{0\}} \frac{\langle \Psi, \mathcal{H}_N(v)\Psi\rangle}{\|\Psi\|^2}. \tag{1.2}$$

We will omit the dependence on v from the notation and just write $e(\rho)$ when the potential is clear from the context. Here the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$ are in the Hilbert space $L^2(\Omega^N)$, where we have denoted $\Omega = [0,L]^3$. If we talk about bosons, the infimum above should be over all symmetric functions in $C_0^\infty(\Omega^N)$. It is, however, a well-known fact that the infimum over all functions is actually the same as if constrained to symmetric functions. When we restrict to functions with compact support in Ω we are effectively using Dirichlet boundary conditions, but it is not difficult to see that the thermodynamic energy is independent of the boundary condition used.[1]

The main result of this work is to establish the celebrated Lee-Huang-Yang formula that gives a two-term asymptotic formula for $e(\rho)$ in the dilute limit. We express the diluteness in terms of the scattering length a of the potential v. The definition of the scattering length and its basic properties will be given in Section 3.

THEOREM 1.2 (The Lee-Huang-Yang Formula). *If v satisfies Assumption 1.1, then in the limit $\rho a^3 \to 0$,*

$$e(\rho) \geq 4\pi\rho^2 a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} - \mathcal{C}(\rho a^3)^{1/2+\eta}\right), \tag{1.3}$$

where $\eta > 0$ and $\mathcal{C}$ depend on $\mathcal{R} = \int v/(8\pi a)$ and R/a as given explicitly in Theorem 6.8 below.

We have not attempted to optimize the dependence of the constant $\mathcal{C}$ on $\mathcal{R} = \int v/(8\pi a)$ and R/a. It follows from Theorem 6.8 that $\mathcal{R}$ and R/a only need to be bounded by an appropriate negative power of ρa^3. By an approximation argument, this would allow us to extend Theorem 1.2 to potentials that do not have compact support (but sufficiently rapid decay; see, e.g., [6, Th. 2.3]) and/or potentials that do not satisfy the L^1-assumption.

As reviewed above, an upper bound consistent with the Lee-Huang-Yang formula was given in [32] under more restrictive assumptions on the potential (see also [1]). Combined with Theorem 1.2 the second term of the energy asymptotics of the dilute Bose gas has therefore been established. It remains an interesting open problem to give upper bounds consistent with (1.3) under less restrictive assumptions on the potential than in [32], [1]. It remains, in particular, an open problem to obtain upper and lower bounds for the hard core potential.

[1] See also [28], where the condition on the interaction potential is slightly more restrictive than ours.

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2. Strategy of the proof and organization of the paper

It is an important first step in Bogolubov's approach that the ground state of the Bose gas is close, in an appropriate sense, to a condensate, i.e., the state corresponding to a product wave function where all particles are in the same one-body state. Establishing condensation in a thermodynamically extended Bose gas in the continuum is still one of the major open problems in the rigorous analysis of Bose gases. It turns out, however, that it is not necessary to establish condensation in order to prove the validity of the Bogolubov approximation or the LHY formula. It is only important that the state is close to a condensate on the relevant length scale. In fact, the relevant length is the distance at which the excited Bogolubov pairs correlate. This scale is often referred to as the "healing length," and it turns out to be of the order $(\rho a)^{-1/2}$.

An important ingredient in our rigorous proof is, therefore, to localize the problem to the healing length and to establish condensation there. Localization here means that we can achieve an appropriate lower bound on the thermodynamic energy density by considering a Hamiltonian restricted to a finite box. This is done in Section 6. It is, however, very delicate for several reasons. First of all, actually localizing to the healing length will interfere with the system and affect its ground state energy to the LHY order. Localization must necessarily be to a length scale somewhat longer than the healing length. At this longer scale we can unfortunately not directly control condensation. We therefore apply a double localization approach, where we first localize the Hamiltonian to a scale somewhat longer than the healing length and then further to a scale somewhat shorter than the healing length.

The key to achieve condensation is to have a localized kinetic energy with the property that constant functions in the box, representing the condensate, have zero kinetic energy and such that there is a gap in the kinetic energy spectrum above the zero energy. If this gap is large enough, it will allow us to control the number of excited particles, i.e., those not in the condensate. For the box much larger than the healing length, the gap is not sufficiently large to immediately control condensation. On the much smaller boxes the gap is, however, large enough to absorb many error terms and get an a priori lower bound on the energy that is almost of LHY order. This is done in Appendix B. The a priori bound will then allow us to get sufficient control on condensation in the larger box. Indeed, the gap in the large box and the a priori bound on the energy allow us in Section 7 to establish that the expected number of excited particles is sufficiently small. We, however, need to control higher powers of the number of excited particles, not just the expectation value. For this we apply in Section 8 the method of localization of large matrices introduced in [21].

The second reason localization is delicate is that it may affect the full kinetic energy spectrum. This would affect the effective scattering length and hence the leading term in the LHY energy asymptotics. Hence we must ensure that the localized kinetic energy is essentially unchanged for momenta much larger than $\sqrt{\rho a}$, i.e., the momentum corresponding to the healing length.

The delicate localizations of both the kinetic and potential energies are done in Section 6.2 using a sliding technique that appeared already in [7]. The sliding technique for localizing the potential energy is motivated by [21], [22], [8], [13].

After localization we no longer know the exact number of particles in the boxes. It is therefore convenient even before localization in Section 4 to reformulate the problem in a grand canonical setting where the total particle number is not fixed but a chemical potential is introduced and carried through in the localization.

An important step in controlling the energy in both the small and large boxes is to split the potential energy in terms of writing the identity in $L^2(\text{box})$ as $\mathbb{1} = P + Q$, where P is the projection onto constant functions, i.e., the condensate. The potential energy can then be written as a sum of 16 terms that contain 0–4 Q's. One of the main new ideas in the present paper is to identify in Section 6.4 an appropriate completion of the square containing the term with 4 Q's (see Lemma 6.9). After ignoring the positive square we will be left with *renormalized* terms with 0-3 Q's, where the potential has essentially been replaced by a renormalized potential whose integral is the scattering length. As already mentioned in the introduction, a naive approach to the Bogolubov approximation will give the integral of the potential and not the scattering length. The completion of the square that we introduce partly resolves this issue for a lower bound. It only resolves it partly because the integral of the potential appears in estimating errors when applying the method of localization of large matrices, which has to be done before the "completion of the square." This is the main reason why hard core potentials are not covered in our result.

The renormalized terms with 0–3 Q's must now be studied more carefully. In particular, the $3Q$ terms pose serious difficulties. They can be ignored in the small boxes, but not in the large box, as they include the effect of the soft pairs mentioned in the introduction. The part of the $3Q$ terms, which does not correspond to soft pairs may, however, be ignored. We therefore split the $3Q$ terms in a relevant part and an irrelevant part. Recall that Q projects onto the space of momenta above the gap. The relevant part of the $3Q$ terms corresponds to restricting to one of the three momenta being sufficiently low and the other two sufficiently high, corresponding to soft pairs. This is the second main new ingredient in the present proof. The splitting of the $3Q$ terms is done in two steps. First the error in restricting one momentum to be low is controlled using the gap and part of the positive completed square. This is the contents of Section 9. Restricting to the other two momenta being high is done in (10.13) after we have introduced second quantization in Section 10, which is the natural next step in the analysis.

We are finally ready for the detailed analysis of the renormalized terms with 0–2 Q's and the relevant renormalized $3Q$ terms. First we do c-number substitution in Section 10.2 using the approach introduced in [20]. This allows us to replace the annihilation operator for the constant function by a number and, in particular, consider the density of particles in the condensate as a number to be optimized. The optimization of this condensate density is done at the end after the careful calculation of the ground state energy. However, we need an initial a priori estimate on the condensate density to control errors. This is achieved from initial rough energy bounds given in Section 11.

Finally, we are then left with (see (12.7))

- terms with no Q's that can be explicitly calculated;
- a quadratic (in creation and annihilation operators) Hamiltonian $\mathcal{K}^{\text{Bog}}$ including also some linear terms (corresponding to $1Q$ terms);
- the $3Q$ terms that are left after the momentum cut-offs and additional quadratic and linear terms not included above.

The quadratic Hamiltonian is treated Section 12.1 using the simplified Bogolubov method in Appendix A. Our approach to localization of both the kinetic and potential energies is chosen to conveniently allow us to use the simplified Bogolubov method. This together with the no-Q terms will give the correct energy up to the LHY correction and a positive quadratic operator (the diagonalized Bogolubov excitation Hamiltonian); see (12.8). This positive quadratic operator together with the remaining $3Q$ and other terms not treated by Bogolubov's method is shown by a very detailed calculation in Section 12.2 to be bounded below by a term of lower order than LHY. We emphasize that the $3Q$ terms themselves do contribute to the LHY order, and our calculation shows that they exactly cancel the quadratic terms not included in the Bogolubov Hamiltonian $\mathcal{K}^{\text{Bog}}$. This calculation is the last new main ingredient in our proof. In Section 13 we put all the pieces together to arrive at the main result.

Before starting the whole analysis we review relevant facts about the scattering solution and the scattering length in Section 3. As the proof requires the consistent choices of many parameters, we have collected these choices in Section 5.

3. Facts about the scattering solution

In this short section we establish notation and recall results concerning the scattering length and associated quantities.

We suppose that v satisfies Assumption 1.1 and refer to Appendix C of [18] for details and a more general treatment. The scattering equation reads

$$\left(-\Delta + \frac{1}{2}v(x)\right)(1-\omega(x)) = 0, \qquad \text{with } \omega \to 0, \text{ as } |x| \to \infty. \tag{3.1}$$

The radial solution ω to this equation satisfies that there exists a constant $a > 0$ such that $\omega(x) = a/|x|$ for x outside supp v. This constant a is the *scattering length* of the potential v, and we will refer to ω as the *scattering solution*. Furthermore, ω is radially symmetric and non-increasing with

$$0 \leq \omega(x) \leq 1. \tag{3.2}$$

We introduce the function

$$g := v(1-\omega). \tag{3.3}$$

The scattering equation can be reformulated as

$$-\Delta\omega = \frac{1}{2}g. \tag{3.4}$$

From this we deduce, using the divergence theorem, that[2]

$$a = (8\pi)^{-1}\int g \tag{3.5}$$

and that the Fourier transform satisfies

$$\widehat{\omega}(k) = \frac{\hat{g}(k)}{2k^2}. \tag{3.6}$$

4. Grand canonical reformulation of the problem

To prove Theorem 1.2 we will reformulate the problem grand canonically on Fock space. Consider, for given $\rho_\mu > 0$, the following operator $\mathcal{H}_{\rho_\mu}$ on the symmetric Fock space $\mathcal{F}_{\mathrm{s}}(L^2(\Omega))$. The operator $\mathcal{H}_{\rho_\mu}$ commutes with particle number and satisfies, with $\mathcal{H}_{\rho_\mu,N}$ denoting the restriction of $\mathcal{H}_{\rho_\mu}$ to the

[2] We have here used the convention—which will be used throughout the paper—of writing $\int g$ instead of $\int g(x)\,dx$ when the integration variable is clear from the context.

N-particle subspace of $\mathcal{F}_{\rm s}(L^2(\Omega))$,

$$\begin{aligned}\mathcal{H}_{\rho_\mu,N} &= \mathcal{H}_N - 8\pi a\rho_\mu N = \sum_{i=1}^N -\Delta_i + \sum_{i<j} v(x_i-x_j) - 8\pi a \rho_\mu N \\ &= \sum_{i=1}^N \left(-\Delta_i - \rho_\mu \int_{\mathbb{R}^3} g(x_i-y)\,dy\right) + \sum_{i<j} v(x_i-x_j).\end{aligned} \tag{4.1}$$

The new term in $\mathcal{H}_{\rho_\mu,N}$ plays the role of a chemical potential justifying the notation.

Define the corresponding ground state energy density:

$$e_0(\rho_\mu) := \lim_{|\Omega|\to\infty} |\Omega|^{-1} \inf_{\Psi\in\mathcal{F}_{\rm s}\setminus\{0\}} \frac{\langle \Psi, \mathcal{H}_{\rho_\mu}\Psi\rangle}{\|\Psi\|^2}. \tag{4.2}$$

We formulate the following result, which will be a consequence of Theorems 6.7 and 6.8 below.

Theorem 4.1. *Suppose that v satisfies Assumption 1.1. Then the thermodynamic ground state energy density of $\mathcal{H}_{\rho_\mu}$ satisfies for $\rho_\mu a^3 \to 0$ that*

$$e_0(\rho_\mu) \geq -4\pi\rho_\mu^2 a \left(1 - \frac{128}{15\sqrt{\pi}}(\rho_\mu a^3)^{1/2} + \mathcal{C}(\rho_\mu a^3)^{1/2+\eta}\right), \tag{4.3}$$

where $\eta > 0$ and $\mathcal{C}$ depend on $\mathcal{R} = \int v/(8\pi a)$ and R/a as given explicitly in Theorem 6.8.

Proof of Theorem 1.2. It is easy to deduce Theorem 1.2 from Theorem 4.1. By inserting the ground state of $\mathcal{H}_N$ as a trial state in $\mathcal{H}_{\rho_\mu}$ one gets in the thermodynamic limit for all $\rho, \rho_\mu > 0$,

$$\begin{aligned} e(\rho) &\geq e_0(\rho_\mu) + 8\pi a\rho\rho_\mu \\ &\geq 8\pi a \rho\rho_\mu - 4\pi\rho_\mu^2 a\left(1 - \frac{128}{15\sqrt{\pi}}(\rho_\mu a^3)^{1/2} + \mathcal{C}(\rho_\mu a^3)^{1/2+\eta}\right),\end{aligned} \tag{4.4}$$

where we have used the lower bound from Theorem 4.1. If we therefore choose ρ_μ to be equal to ρ, we arrive at the LHY formula (1.3). □

5. The various parameters and their choices

As already mentioned in the introduction the important parameters given in the problem are

$$a, \int v, R.$$

All estimates will in the end depend on these. The most important combination is the diluteness parameter

$$\rho_\mu a^3.$$

The proof introduces a series of additional parameters. There is an integer

$$M \in \mathbb{N}$$

that determines the regularity of the localization function defined in Appendix C. It will be chosen explicitly below. We cannot choose $M = \infty$, which would correspond to χ being smooth, since it would complicate the double localization. (Technically, some of the estimates in Appendix C depend on the finiteness of M.) However, we need to choose M sufficiently large in the control of various error terms.

The remaining parameters will be chosen to depend on $\rho_\mu a^3$ and $\mathcal{R} = \int v/(8\pi a)$. There are dimensionless parameters $0 < s, d, \varepsilon_T$ that will be chosen small, and there are dimensionless parameters $1 < K_\ell, K_{\mathcal{M}}, \widetilde{K}_H, K_B$ that will be chosen large. The power in the error term will depend on the choice of these seven parameters in terms of $\rho_\mu a^3$ and $\mathcal{R} = \int v/(8\pi a)$.

Let us describe how these parameters enter into the proof and list all the conditions that they must satisfy. Finally we will make choices to show that these conditions can all be satisfied. The reader not interested in the description of all these conditions may just skip to (5.26) and the lines following it, where concrete choices of parameters are made. One can then verify all through the paper that these choices work at all stages of the proof.

As explained, the proof will use a double localization approach. First we localize into large boxes of length scale

$$\ell =: K_\ell (\rho_\mu a)^{-1/2}, \tag{5.1}$$

and then we localize further to small boxes of length scale

$$d\ell = dK_\ell (\rho_\mu a)^{-1/2} \ll (\rho_\mu a)^{-1/2}, \tag{5.2}$$

which gives us our first condition that $dK_\ell \ll 1$. Here and below, $f \ll g$ is used in the precise meaning that $(f/g) \leq (\rho_\mu a^3)^\varepsilon$ for some positive ε and likewise for $f \gg g$.

The parameters ε_T, d, s appear in the kinetic energy localization formulas of Section 6.2, and they must satisfy the conditions

$$d^{-5} s^{M+1} \ll 1, \tag{5.3}$$

$$(dK_\ell)^2 \ll \varepsilon_T K_\ell^{-2} \ll \varepsilon_T \ll sdK_\ell, \tag{5.4}$$

$$sK_\ell \gg 1, \tag{5.5}$$

$$sdK_\ell \gg K_B^{-1}. \tag{5.6}$$

Throughout the paper there will also be logarithmic factors. They are ignored here as they are always accommodated by the conditions given. Condition (5.3) is needed to prove the kinetic energy localization into the small boxes (see (B.13)). It relies on a result from [7]. The first condition in (5.4) is needed to have a sufficiently large gap in the small boxes, but in fact, this would only require $(dK_\ell)^2 \ll \varepsilon_T$. The need for the stronger condition will be explained below. The condition $dK_\ell \ll 1$ noted above is contained in (5.4). The last condition in (5.4) is required to finally get the correct LHY constant when the appropriate integral is estimated in Section 12. Condition (5.5) is also needed to control the same integral; in fact, this condition implies that the localized kinetic energy (see (6.20)) in the large boxes is essentially the original kinetic energy at the relevant Bogolubov scales. Finally, (5.6) introduces the parameter K_B to control that the small boxes are not too small. This is required in order to get a good lower bound on the the energy in the small boxes in Appendix B (see Theorem B.5) and hence for the a priori bound on the energy in the large boxes and consequently on the number of particles and excited particles in the large boxes (see Theorem 7.1). The parameter K_B has to satisfy the additional conditions that

$$K_B \ll (\rho_\mu a^3)^{-1/6}, \tag{5.7}$$

$$K_B^3 K_\ell^2 \ll (\rho_\mu a^3)^{-1/4}. \tag{5.8}$$

Here (5.7) is a very weak condition implying that the a priori lower bound on the energy in Theorem B.6 is at least better than the leading order term. Condition (5.8) ensures that the a priori bounds on the particle number and expected number of excited particles are both correct to leading order (see (7.2)).

The technique of localizing large matrices from [21] allows us to restrict the analysis to the subspace where the number of excited particles is bounded above by a number

$$\mathcal{M} =: K_{\mathcal{M}} (\rho_\mu a^3)^{-1/4}. \tag{5.9}$$

It must satisfy

$$K_{\mathcal{M}}^{-2} \int v/a \ll 1, \tag{5.10}$$

$$K_B^3 K_\ell^5 \ll \mathcal{M} = K_{\mathcal{M}} (\rho_\mu a^3)^{-1/4}, \tag{5.11}$$

$$K_{\mathcal{M}} K_\ell^{-3} \ll (\rho_\mu a^3)^{-1/4}. \tag{5.12}$$

Condition (5.10) is needed to control the error in the energy when restricting to the situation with a bounded number of excited particles. Condition (5.11) says that the upper bound $\mathcal{M}$ on the number of excited particles must be much bigger than the *expected* number of these particles, which in Theorem 7.1 is shown to be not much worse than $K_B^3 K_\ell^2 \rho_\mu \ell^3 (\rho_\mu a^3)^{1/2} \sim K_B^3 K_\ell^5$. Condition (5.12) is a very weak condition that ensures

$$\mathcal{M} \ll \rho_\mu \ell^3, \tag{5.13}$$

i.e., that the bound on the number of excited particles is much less than the total number of particles.

When splitting the $3Q$ terms in a relevant and an irrelevant part we introduce an upper cutoff for low momenta, which we choose to be $K_L(\rho_\mu a)^{1/2}$, and a lower cutoff for high momenta, which we choose to be (see Section 9)

$$\widetilde{K}_H^{-1}(\rho_\mu a^3)^{5/12}a^{-1}. \tag{5.14}$$

The relevance of the power $5/12$ is technical and will appear in the proof of Lemma 10.3. For convenience we also introduce the parameter $K_H = \widetilde{K}_H(\rho_\mu a^3)^{-5/12}$. We will not choose K_L as a new parameter, but take

$$K_L =: (K_\ell d^2)^{-1} \gg K_\ell, \tag{5.15}$$

where the estimate follows from (5.4).

We get the additional conditions

$$K_{\mathcal{M}} K_\ell^4 \ll \widetilde{K}_H^3, \tag{5.16}$$

$$(K_\ell K_L)^{1-M} = d^{2M-2} \ll (\rho_\mu a^3)^{1/2}, \tag{5.17}$$

$$K_L \widetilde{K}_H = (K_\ell d^2)^{-1}\widetilde{K}_H \ll (\rho_\mu a^3)^{-1/12}. \tag{5.18}$$

Condition (5.18) ensures that the high momenta are disjoint from the low momenta. Condition (5.17) will be ensured by choosing the integer M that appears in the explicit localization function large enough. The condition is needed to control errors that occur because of the localization function. This error will also appear in the final error on the lower bound on the energy (see (12.41)). Condition (5.16) is needed to control the error (see (10.13)) in cutting off the 3Q terms in momentum by absorbing it into the energy gap. It is here that the powers in the choice (5.14) become important.

After c-number substitution, we need to a priori control that the density in the c-number substituted condensate is sufficiently close to ρ_μ. This is done in Section 11 and requires the additional conditions

$$K_\ell^8(\rho_\mu \ell^3)^{-1}\mathcal{M} K_L^6 K_\ell^6 = K_{\mathcal{M}} K_\ell^5 d^{-12}(\rho_\mu a^3)^{1/4} \ll 1, \tag{5.19}$$

$$K_\ell^8(\rho_\mu \ell^3)^{-1}\mathcal{M}^3 K_L^6 \widetilde{K}_H^4(\rho_\mu a^3)^{4/3} = K_{\mathcal{M}}^3 \widetilde{K}_H^4 K_\ell^{-1} d^{-12}(\rho_\mu a^3)^{13/12} \ll 1. \tag{5.20}$$

These conditions ensure that δ_1 defined in Lemma 11.1 is small enough to satisfy (12.2). That (12.2) is, indeed, satisfied then follows from (5.6) and (5.8).

The treatment of the quadratic Bogolubov Hamiltonian $\mathcal{K}^{\mathrm{Bog}}$ given in Theorem 12.1 requires the condition

$$K_{\mathcal{M}} K_\ell^{-3/2}(K_\ell \widetilde{K}_H(\rho_\mu a^3)^{1/12})^{M-5} \ll (\rho_\mu a^3)^{1/2}. \tag{5.21}$$

Note that the term taken to the power M here is small by (5.18) and the estimate in (5.15).

The last detailed calculation estimating the $3Q$ terms, done in Section 12.2, requires the conditions (see Theorem 12.4)

$$K_L^3 K_{\mathcal{M}} = (K_\ell d^2)^{-3} K_{\mathcal{M}} \ll (\rho_\mu a^3)^{-1/4}, \tag{5.22}$$

$$K_\ell^2 \widetilde{K}_H^4 d^{-6} \ll (\rho_\mu a^3)^{-1/3}, \tag{5.23}$$

$$K_{\mathcal{M}} K_\ell^{-3} \widetilde{K}_H^{-2} \ll (\rho_\mu a^3)^{-1/12}, \tag{5.24}$$

$$K_{\mathcal{M}} K_\ell^{-3} d^{-12} (K_\ell^{-2} \widetilde{K}_H^2 (\rho_\mu a^3)^{1/6})^{M-1} \ll (\rho_\mu a^3)^{3/4}. \tag{5.25}$$

Conditions (5.24) and (5.25) are needed in order for the errors in Theorem 12.4 to be of lower order than LHY. There are two additional error terms in (12.41). One is, however, already controlled by condition (5.17), and the last term is small. Condition (5.4) above will also be needed in Section 12.2.

If we choose to let all the parameters depend on a small parameter $X \ll 1$ in the following way,

$$\begin{aligned} s = X,\ d = X^6,\ \varepsilon_T = X^{23/4},\ K_\ell = X^{-3/2}, \\ K_B = X^{-6},\ K_{\mathcal{M}} = X^{-1},\ \widetilde{K}_H = X^{-8/3}, \end{aligned} \tag{5.26}$$

then all conditions (5.4)–(5.6), (5.16) will be satisfied. In order to satisfy (5.7), (5.8), (5.11), (5.12), (5.18)–(5.20), (5.22)–(5.24) of which the most restrictive is (5.19), we can choose

$$X = (\rho_\mu a^3)^{1/323}. \tag{5.27}$$

We can choose the integer $M = 30$ to ensure that (5.3), (5.17),(5.21), and (5.25) hold. Finally, (5.10) holds if

$$\int v/a \ll (\rho_\mu a^3)^{-2/323}. \tag{5.28}$$

To get all the arguments to work we need the assumptions

$$R < d\ell,\ R \le K_B^{1/2} (\rho_\mu a^3)^{1/4} (\rho_\mu a)^{-1/2},\ R/\ell \ll (\rho_\mu a^3)^{1/4},\ R/a \ll (\rho_\mu a^3)^{-1/4}. \tag{5.29}$$

The fourth assumption (which could be improved slightly) is the most restrictive and is used in (12.8). The first and the second assumptions are used in Appendix B, and the third assumption says that the range of the potential should be sufficiently much smaller than the size of the large boxes.

6. Localization

6.1. *Setup and notation.* The main part of the analysis will be carried out on a box $\Lambda = [-\ell/2, \ell/2]^3$ of size ℓ given in (5.1). In this section we will carry out the localization to the box Λ. The main result is given at the end of the section as Theorem 6.7, which states that for a lower bound it suffices to consider a "box energy," i.e., the ground state energy of a Hamiltonian localized to a box of size ℓ. For convenience, in Theorem 6.8 we state the bound on the box energy that will suffice in order to prove Theorem 4.1.

It will be important to make an explicit choice of a localization function $\chi \in C_0^{M-1}(\mathbb{R}^3)$ for $M \in \mathbb{N}$ with support in $[-1/2, 1/2]^3$. It is given in Appendix C. The function will not be smooth, but it will be important in the analysis that we choose $M \in \mathbb{N}$ finite but sufficiently large. The explicit choice $M = 30$ was explained in the previous section. We choose χ to be even and such that

$$0 \leq \chi, \qquad \int \chi^2 = 1. \tag{6.1}$$

We will also use the notation

$$\chi_\Lambda(x) := \chi(x/\ell). \tag{6.2}$$

For given $u \in \mathbb{R}^3$, we define

$$\chi_u(x) = \chi(\frac{x}{\ell} - u) = \chi_\Lambda(x - u\ell). \tag{6.3}$$

Notice that χ_u localizes to the box $\Lambda(u) := \ell u + [-\ell/2, \ell/2]^3$.

We will also need the sharp localization function θ_u to the box $\Lambda(u)$, i.e.,

$$\theta_u := \mathbb{1}_{\Lambda(u)}. \tag{6.4}$$

Define P_u, Q_u to be the orthogonal projections in $L^2(\mathbb{R}^3)$ defined by

$$P_u\varphi := \ell^{-3}\langle\theta_u, \varphi\rangle\theta_u, \qquad Q_u\varphi := \theta_u\varphi - \ell^{-3}\langle\theta_u, \varphi\rangle\theta_u. \tag{6.5}$$

In the case $u = 0$ we will use the notation

$$P_{u=0} = P_\Lambda = P, \qquad Q_{u=0} = Q_\Lambda = Q. \tag{6.6}$$

Furthermore, define

$$W(x) := \frac{v(x)}{\chi * \chi(x/\ell)}. \tag{6.7}$$

That W is well-defined uses that $R < \ell$, which is a much weaker condition than (5.29). Manifestly W depends on ℓ and thus ρ_μ, but we will not reflect this in our notation.

Define the localized potentials

$$w_u(x, y) := \chi_u(x)W(x - y)\chi_u(y), \qquad w(x, y) := w_{u=0}(x, y). \tag{6.8}$$

Notice the translation invariance

$$w_{u+\tau}(x, y) = w_u(x - \ell\tau, y - \ell\tau). \tag{6.9}$$

For some estimates it is convenient to invoke the scattering solution, and thus we introduce the notation, which again is well-defined for $\rho_\mu a^3$ sufficiently small,

$$\begin{aligned} W_1(x) &:= W(x)(1-\omega(x)) = \frac{g(x)}{\chi * \chi(x/\ell)}, \\ w_1(x,y) &:= w(x,y)(1-\omega(x-y)), \\ W_2(x) &:= W(x)(1-\omega^2(x)) = \frac{g(x)+g\omega(x)}{\chi * \chi(x/\ell)}, \\ w_2(x,y) &:= w(x,y)(1-\omega^2(x-y)). \end{aligned} \tag{6.10}$$

If we add a subscript u, as above we mean the translated versions $w_{1,u}(x,y) = w_1(x-\ell u, y-\ell u)$. For $\rho_\mu a^3$ sufficiently small, a simple change of variables yields, for all $u \in \mathbb{R}^3$, the identities

$$\begin{aligned} \frac{1}{2}\ell^{-6}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \chi(\frac{x}{\ell})\chi(\frac{y}{\ell})W_1(x-y)\,dx\,dy &= \frac{1}{2}\ell^{-6}\iint_{\mathbb{R}^3\times\mathbb{R}^3} w_1(x,y)\,dx\,dy \\ &= \frac{1}{2}\ell^{-3}\int g = 4\pi a \ell^{-3} \end{aligned} \tag{6.11}$$

and likewise

$$\frac{1}{2}\ell^{-6}\iint_{\mathbb{R}^3\times\mathbb{R}^3} w_2(x,y)\,dx\,dy = \frac{1}{2}\ell^{-3}\int g(1+\omega). \tag{6.12}$$

The following simple lemma will often be useful.

LEMMA 6.1.

$$g(x) \leq W_1(x) \leq g(x)\left(1 + C\frac{R^2}{\ell^2}\right). \tag{6.13}$$

Proof. The proof is an easy estimate of the convolution, noting that its maximum is attained at the origin. We have used that $R < \ell$. □

LEMMA 6.2. *Suppose that* $f \in L^1(\mathbb{R}^3)$ *satisfies* $\operatorname{supp} f \subset B(0,R)$ *and* $f(-x) = f(x)$. *Then*

$$\left| f * \chi_\Lambda(x) - \chi_\Lambda(x)\int f \right| \leq \max_{i,j} \|\partial_i\partial_j\chi\|_\infty \left(\frac{R}{\ell}\right)^2 \int |f|. \tag{6.14}$$

Proof. The proof is an easy application of a Taylor expansion and the integral representation

$$f * \chi_\Lambda(x) - \chi_\Lambda(x)\int f = \int f(y)[\chi_\Lambda(x-y) - \chi_\Lambda(x)]\,dy. \qquad \square$$

LEMMA 6.3. *Suppose that $R/\ell \leq 1$. For some universal constant $C > 0$, we have*

$$\left|(2\pi)^{-3}\int \frac{\widehat{W}_1(k)^2}{2k^2}\,dk - \widehat{g\omega}(0)\right| \leq C(R/\ell)^2\widehat{g\omega}(0). \tag{6.15}$$

We also get

$$\int \frac{(\widehat{W}_1(k)-\widehat{g}(k))^2}{2k^2}\,dk \leq C\frac{R^4}{\ell^4}\widehat{g\omega}(0). \tag{6.16}$$

Proof. Recall that $\widehat{\omega}(k) = \frac{\widehat{g}(k)}{2k^2}$ by (3.6). Using the Fourier transformation and (6.13) we get

$$\begin{aligned}\left|(2\pi)^{-3}\int \frac{\widehat{W}_1^2(k)-\widehat{g}^2(k)}{2k^2}\,dk\right| &= C\iint \frac{(W_1-g)(x)(W_1+g)(y)}{|x-y|}\,dx\,dy\\ &\leq 3C\frac{R^2}{\ell^2}\iint\frac{g(x)g(y)}{|x-y|}\,dx\,dy\\ &= C'\frac{R^2}{\ell^2}\widehat{g\omega}(0).\end{aligned} \tag{6.17}$$

This finishes the proof of (6.15). The proof of (6.16) follows from a similar calculation and is omitted. □

6.2. *Localization of the kinetic and potential energies.* We will use a sliding localization technique developed in the paper [7], where we estimate the kinetic energy $-\Delta$ in $\mathbb{R}^3$ below by an integral over kinetic energy operators in the boxes $\Lambda(u)$. The following theorem is essentially Lemma 3.7 in [7].

LEMMA 6.4 (Kinetic energy localizaton). *Let $-\Delta_u^{\mathcal{N}}$ denote the Neumann Laplacian in $\Lambda(u)$. If the regularity of χ has $M \geq 5$ (e.g., for our choice 30) and the positive parameters ε_T, d, s, b are smaller than some universal constant, then for all $\ell > 0$, we have*

$$\int_{\mathbb{R}^3} \mathcal{T}_u du \leq -\Delta, \tag{6.18}$$

where

$$\begin{aligned}\mathcal{T}_u := {}&\frac{1}{2}\varepsilon_T(d\ell)^{-2}\frac{-\Delta_u^{\mathcal{N}}}{-\Delta_u^{\mathcal{N}}+(d\ell)^{-2}}\\ &+ b\ell^{-2}Q_u + b\varepsilon_T(d\ell)^{-2}Q_u\mathbb{1}_{(d^{-2}\ell^{-1},\infty)}(\sqrt{-\Delta})Q_u + \mathcal{T}'_u,\end{aligned} \tag{6.19}$$

with

(6.20)

$$\mathcal{T}'_u := Q_u\chi_u\Big\{(1-\varepsilon_T)\Big[\sqrt{-\Delta}-\frac{1}{2}(s\ell)^{-1}\Big]_+^2 + \varepsilon_T\Big[\sqrt{-\Delta}-\frac{1}{2}(ds\ell)^{-1}\Big]_+^2\Big\}\chi_u Q_u.$$

Proof. In Lemma 3.7 in [7] we have the same inequality except that the terms above

$$\frac{1}{2}\varepsilon_T(d\ell)^{-2}\frac{-\Delta_u^{\mathcal{N}}}{-\Delta_u^{\mathcal{N}}+(d\ell)^{-2}}+b\varepsilon_T(d\ell)^{-2}Q_u\mathbb{1}_{(d^{-2}\ell^{-1},\infty)}(\sqrt{-\Delta})Q_u$$

are replaced by the term $\varepsilon_T(d\ell)^{-2}\frac{-\Delta_u^{\mathcal{N}}}{-\Delta_u^{\mathcal{N}}+(d\ell)^{-2}}$.

Using scaling it is clear that we may assume $\ell=1$. The proof in Lemma 3.7 in [7] relies on the inequality (see (44) in [7])

$$d^{-2}\int_{\mathbb{R}^3}\frac{-\Delta_u^{\mathcal{N}}}{-\Delta_u^{\mathcal{N}}+d^{-2}}du\leq d^{-2}\frac{-\Delta}{-\Delta+d^{-2}}.$$

The lemma above will follow in the same way if we can also prove that

$$bd^{-2}\int_{\mathbb{R}^3}Q_u\mathbb{1}_{(d^{-2},\infty)}(\sqrt{-\Delta})Q_u du\leq\frac{1}{2}d^{-2}\frac{-\Delta}{-\Delta+d^{-2}}. \tag{6.21}$$

Using Lemma 3.3 in [7] (with $\chi_u=\theta_u=\mathbb{1}_{\Lambda(u)}$ and $\mathcal{K}(p)=bd^{-2}\mathbb{1}_{(d^{-2},\infty)}$) we can explicitly calculate the operator on the left in (6.21) to be $H(-i\nabla)$, where

(6.22)
$$\begin{aligned}H(p)=&(2\pi)^{-3}bd^{-2}\int_{|q|>d^{-2}}(\widehat{\theta}(p)\widehat{\theta}(q)-\widehat{\theta}(q-p))^2dq\\ \leq&(2\pi)^{-3}2bd^{-2}(\widehat{\theta}(p)-1)^2\\ &\times\int_{|q|>d^{-2}}\widehat{\theta}(q)^2dq+(2\pi)^{-3}2bd^{-2}\int_{|q|>d^{-2}}(\widehat{\theta}(q-p)-\widehat{\theta}(q))^2dq.\end{aligned}$$

We clearly have $H(0)=0$ and $0\leq H(p)\leq Cbd^{-2}$. We will improve this estimate if $|p|<d^{-1}$. We will use that $\widehat{\theta}(q)=\widehat{\theta}_1(q_1)\widehat{\theta}_1(q_2)\widehat{\theta}_1(q_3)$, where θ_1 is the characteristic function of $[-1/2,1/2]\subset\mathbb{R}$. We easily see that for all $s,t\in\mathbb{R}$, with $|s|<|t|/2$ we have

$$\widehat{\theta}_1(t)^2\leq C(1+t^2)^{-1},\qquad(\widehat{\theta}_1(t-s)-\widehat{\theta}_1(t))^2\leq C\frac{s^2}{(1+t^2)}.$$

As the set $|q|>d^{-2}$ is a subset of the union of the sets where $|q_i|>d^{-2}/\sqrt{3}$, $i=1,2,3$, we immediately see that the first term above in the estimate (6.22) on $H(p)$ is bounded by $Cb|p|^2$. For the second term we use that

$$\begin{aligned}\widehat{\theta}(q)-\widehat{\theta}(p-q)=&\widehat{\theta}_1(q_1)[\widehat{\theta}_1(q_2)\widehat{\theta}_1(q_3)-\widehat{\theta}_1(p_2-q_2)\widehat{\theta}_1(p_3-q_3)]\\&+[\widehat{\theta}_1(q_1)-\widehat{\theta}_1(p_1-q_1)]\widehat{\theta}_1(p_2-q_2)\widehat{\theta}_1(p_3-q_3)\end{aligned}$$

and that

$$\iint \left(\widehat{\theta}_1(q_2)\widehat{\theta}_1(q_3) - \widehat{\theta}_1(p_2 - q_2)\widehat{\theta}_1(p_3 - q_3)\right)^2 dq_2 dq_3$$
$$= C \iint_{[-1/2,1/2]^2} |1 - e^{i(x_1 p_1 + x_2 p_2}|^2 dx_2 dx_2$$
$$\leq C(p_2^2 + p_3^2)$$

to see that

$$bd^{-2} \int_{|q_1| > d^{-2}/\sqrt{3}} (\widehat{\theta}(q-p) - \widehat{\theta}(q))^2 dq \leq Cb|p|^2.$$

Here we have used that over the domain of integration, $|p_1| < |q_1|/2$ since $|p| < d^{-1}$ and d is chosen sufficiently small. The same holds for the integrals over $|q_2| > d^{-2}/\sqrt{3}$ and $|q_2| > d^{-2}/\sqrt{3}$. It then follows that $H(p) \leq Cb \min\{|p|^2, d^{-2}\}$. Hence (6.21) holds if b is smaller than a universal constant. □

Remark 6.5. The kinetic operator in (6.19) looks complicated. This is partly because we need to localize it even further into smaller boxes in order to get a priori estimates (see Appendix B). The first term in (6.19) will give us a Neumann gap in the small boxes. The second term in (6.19) is a Neumann gap in the large boxes. The third term in (6.19) will control errors coming from excited particles with very large momenta. (See Lemma 9.1 and the estimate (12.49) in Lemma 12.5.) Finally the term $\mathcal{T}'_u$ is the main kinetic energy term in the large boxes.

The localization of the potential energy is much simpler and relies on the identity in the following lemma, which is a straightforward computation similar to Proposition 3.1 in [7].

Lemma 6.6 (Potential energy localization). *For points $x_1, \ldots, x_N \in \mathbb{R}^3$, we have with the definitions of $w_{1,u}$ and w_u in (6.8) and (6.10) that*

$$-\rho_\mu \sum_{i=1}^N \int g(x_i - y)\, dy + \sum_{i<j} v(x_i - x_j)$$
$$= \int_{\mathbb{R}^3} \Big[-\rho_\mu \sum_{i=1}^N \int w_{1,u}(x_i, y)\, dy + \sum_{i<j} w_u(x_i, x_j) \Big] du. \tag{6.23}$$

6.3. *The localized Hamiltonian.* The localized Hamiltonian $\mathcal{H}_{\Lambda,u}$ will be an operator on the symmetric Fock space over $L^2(\mathbb{R}^3)$ preserving particle number. Its action on the N-particle sector is as

$$(\mathcal{H}_{\Lambda,u}(\rho_\mu))_N := \sum_{i=1}^N \mathcal{T}_u^{(i)} - \rho_\mu \sum_{i=1}^N \int w_{1,u}(x_i, y)\, dy + \sum_{i<j} w_u(x_i, x_j), \tag{6.24}$$

where the kinetic energy operator was given in (6.19) above. We abbreviate

$$\mathcal{T} := \mathcal{T}_{u=0}, \qquad \mathcal{H}_\Lambda(\rho_\mu) := \mathcal{H}_{\Lambda,u=0}(\rho_\mu). \tag{6.25}$$

We will also write

$$\chi_\Lambda := \chi_{u=0} = \chi(\,\cdot\,/\ell).$$

Define the ground state energy and energy density in the box by

$$E_\Lambda(\rho_\mu) := \inf \operatorname{Spec} \mathcal{H}_\Lambda(\rho_\mu), \tag{6.26}$$

$$e_\Lambda(\rho_\mu) := \ell^{-3} \inf \operatorname{Spec} \mathcal{H}_\Lambda(\rho_\mu) = \ell^{-3} E_\Lambda(\rho_\mu). \tag{6.27}$$

With these conventions, we find

Theorem 6.7. *We have*

$$e_0(\rho_\mu) \geq e_\Lambda(\rho_\mu). \tag{6.28}$$

Proof. The proof of this statement follows from the fact that $(\mathcal{H}_{\Lambda,u}(\rho_\mu))_N$ and $(\mathcal{H}_{\Lambda,u'}(\rho_\mu))_N$ are unitarily equivalent by (6.9). Thus, using Lemmas 6.4 and 6.6 we find that

$$\mathcal{H}_{\rho_\mu,N} \geq \int_{\ell^{-1}(\Omega+B(0,\ell/2))} (\mathcal{H}_{\Lambda,u}(\rho_\mu))_N \, du \geq \ell^{-3} |\Omega + B(0,\ell/2)| E_\Lambda(\rho_\mu). \tag{6.29}$$

Now the desired result follows upon using that $|\Omega + B(0,\ell/2)|/|\Omega| \to 1$ in the thermodynamic limit. □

It is clear, using Theorem 6.7, that Theorem 4.1 is a consequence of the following theorem on the box Hamiltonian. Therefore, the remainder of the paper will be dedicated to the proof of Theorem 6.8 below.

Theorem 6.8. *Suppose that v satisfies Assumption 1.1, (5.28) and (5.29). Then with $\mathcal{R} = (8\pi a)^{-1}\int v$ and the parameters chosen in (5.26) with X as in (5.27), and with M as chosen in Section 5, we have in the limit $\rho_\mu a^3 \to 0$,*

$$\begin{aligned} e_\Lambda(\rho_\mu) \geq{}& -4\pi\rho_\mu^2 a + 4\pi\rho_\mu^2 a \frac{128}{15\sqrt{\pi}}(\rho_\mu a^3)^{\frac12} \\ & - C\rho_\mu^2 a(\rho_\mu a^3)^{1/2}\left(X^2\mathcal{R} + \frac{R^2}{a^2}(\rho_\mu a^3)^{\frac12} + X^{\frac15}\right). \end{aligned} \tag{6.30}$$

6.4. *Potential energy splitting.* Using that $P+Q=\mathbb{1}_\Lambda$ we will in Lemma 6.9 below arrive at a very useful decomposition of the potential.

Define the (commuting) operators

$$n_0 = \sum_{i=1}^N P_i, \qquad n_+ = \sum_{i=1}^N Q_i, \qquad n = \sum_{i=1}^N \mathbb{1}_{\Lambda,i} = n_0 + n_+. \tag{6.31}$$

We furthermore define

$$\rho_+ := n_+\ell^{-3}, \qquad \rho_0 := n_0\ell^{-3}. \tag{6.32}$$

A crucial idea in this paper is to write the potential energy in the form given in the next lemma, where the important observation is to identify the positive term $\mathcal{Q}_4^{\rm ren}$, which we will ignore in our lower bound.

LEMMA 6.9 (Potential energy decomposition). *We have*

(6.33)
$$-\rho_\mu \sum_{i=1}^N \int w_1(x_i,y)\,dy + \frac{1}{2}\sum_{i\neq j} w(x_i,x_j) = \mathcal{Q}_0^{\rm ren} + \mathcal{Q}_1^{\rm ren} + \mathcal{Q}_2^{\rm ren} + \mathcal{Q}_3^{\rm ren} + \mathcal{Q}_4^{\rm ren},$$

where[3]

(6.34)
$$\mathcal{Q}_4^{\rm ren} := \frac{1}{2}\sum_{i\neq j}\Big[Q_iQ_j + (P_iP_j + P_iQ_j + Q_iP_j)\omega(x_i-x_j)\Big]w(x_i,x_j) \times \Big[Q_jQ_i + \omega(x_i-x_j)(P_jP_i + P_jQ_i + Q_jP_i)\Big],$$

(6.35)
$$\mathcal{Q}_3^{\rm ren} := \sum_{i\neq j} P_iQ_jw_1(x_i,x_j)Q_jQ_i + \text{h.c.}$$

(6.36)
$$\mathcal{Q}_2^{\rm ren} := \sum_{i\neq j} P_iQ_jw_2(x_i,x_j)P_jQ_i + \sum_{i\neq j} P_iQ_jw_2(x_i,x_j)Q_jP_i - \rho_\mu\sum_{i=1}^N Q_i\int w_1(x_i,y)\,dyQ_i + \frac{1}{2}\sum_{i\neq j}(P_iP_jw_1(x_i,x_j)Q_jQ_i + \text{h.c.}),$$

(6.37)
$$\mathcal{Q}_1^{\rm ren} := \sum_{i,j} P_jQ_iw_2(x_i,x_j)P_iP_j - \rho_\mu\sum_i Q_i\int w_1(x_i,y)\,dyP_i + \text{h.c.},$$

(6.38)
$$\mathcal{Q}_0^{\rm ren} := \frac{1}{2}\sum_{i\neq j} P_iP_jw_2(x_i,x_j)P_jP_i - \rho_\mu\sum_i P_i\int w_1(x_i,y)\,dyP_i.$$

Proof. The identity (6.33) follows using simple algebra and the identities (6.10). We simply write $P_i + Q_i = 1_{\Lambda,i}$ for all i. Inserting this identity in both i and j on both sides of $w(x_i,x_j)$ and expanding yields 16 terms, which we have organized in a positive $\mathcal{Q}_4$ term and terms depending on the number of Q's occurring. □

[3]Here and in the rest of the paper, we have used that standard abbreviation "h.c." for "hermitian conjugate." To be precise, for an operator A we have $A + \text{h.c.} = A + A^*$, where A^* is the adjoint of A.

It will be useful to rewrite and estimate these terms as in the following lemma.

LEMMA 6.10. *If v and hence W_1 are non-negative, then we have*

$$\begin{aligned}\mathcal{Q}_0^{\rm ren} &= \frac{n_0(n_0-1)}{2|\Lambda|^2}\iint w_2(x,y)\,dxdy - \rho_\mu \frac{n_0}{|\Lambda|}\iint w_1(x,y)\,dxdy \\ &= \frac{n_0(n_0-1)}{2|\Lambda|}\Big(\widehat{g}(0)+\widehat{g\omega}(0)\Big) - \rho_\mu n_0 \widehat{g}(0),\end{aligned} \tag{6.39}$$

$$\begin{aligned}\mathcal{Q}_1^{\rm ren} &= (n_0|\Lambda|^{-1}-\rho_\mu)\sum_i Q_i \chi_\Lambda(x_i) W_1 * \chi_\Lambda(x_i) P_i + \text{h.c.} \\ &\quad + n_0|\Lambda|^{-1}\sum_i Q_i\chi_\Lambda(x_i)(W_1\omega)*\chi_\Lambda(x_i)P_i + \text{h.c.}\end{aligned} \tag{6.40}$$

and

$$\begin{aligned}\mathcal{Q}_2^{\rm ren} \geq& \sum_{i\neq j} P_i Q_j w_2(x_i,x_j) P_j Q_i + \frac{1}{2}\sum_{i\neq j}(P_iP_j w_1(x_i,x_j)Q_jQ_i + \text{h.c.}) \\ &+ \Big((\rho_0-\rho_\mu)\widehat{W_1}(0) + \rho_0 \widehat{W_1\omega}(0)\Big) \\ &\times \sum_i Q_i \chi_\Lambda(x_i)^2 Q_i - C(\rho_\mu+\rho_0)a(R/\ell)^2 n_+.\end{aligned} \tag{6.41}$$

Proof. The rewriting of $\mathcal{Q}_0$ is straightforward using (6.11) and (6.12). The rewriting of $\mathcal{Q}_1^{\rm ren}$ follows from

$$\begin{aligned}\mathcal{Q}_1^{\rm ren} =& \Big((n_0|\Lambda|^{-1}-\rho_\mu)\sum_i Q_i \int w_1(x_i,y)\,dy P_i + \text{h.c.}\Big) \\ &+ \Big(n_0|\Lambda|^{-1}\sum_i Q_i \int w_1(x_i,y)\omega(x_i-y)\,dy P_i + \text{h.c.}\Big).\end{aligned}$$

We carry out the similar calculation on the part of the $2Q$-term where P acts in the same variable on both sides of the potential, i.e., the second term in (6.36), to get

$$\begin{aligned}\mathcal{Q}_2^{\rm ren} =& \sum_{i\neq j} P_i Q_j w_2(x_i,x_j)P_jQ_i + \frac{1}{2}\sum_{i\neq j}(P_iP_jw_1(x_i,x_j)Q_jQ_i + \text{h.c.}) \\ &+ (\rho_0-\rho_\mu)\sum_i Q_i\chi_\Lambda(x_i)W_1 * \chi_\Lambda(x_i)Q_i \\ &+ \rho_0 \sum_i Q_i \chi_\Lambda(x_i)(W_1\omega)*\chi_\Lambda(x_i)Q_i.\end{aligned}$$

At this point we invoke Lemma 6.2 to get, for example,

(6.42)
$$\sum_i Q_i\chi_\Lambda(x_i)W_1*\chi_\Lambda(x_i)Q_i \geq \Big(\int W_1\Big)\sum_i Q_i\chi_\Lambda(x_i)^2 Q_i - \max_{i,j}\|\partial_i\partial_j\chi\|_\infty (R/\ell)^2\Big(\int W_1\Big)\|\chi\|_\infty n_+. \quad \square$$

The decomposition in Lemma 6.9 easily implies a simple lower bound on the potential energy.

Lemma 6.11 (Simple bound on the potential energy). *If the 2-body potential $v \geq 0$, we have for all $x_1,\dots,x_N \in \mathbb{R}^3$ the following bound on the potential energy*:

(6.43)
$$-\rho_\mu\sum_{i=1}^N\int w_1(x_i,y)\,dy + \frac{1}{2}\sum_{i\neq j} w(x_i,x_j) \geq -C(n^2\ell^{-3}+\rho_\mu^2\ell^3)a + \frac{1}{2}\mathcal{Q}_4^{\rm ren}.$$

Moreover, we also have the bounds

$$\pm\mathcal{Q}_1^{\rm ren} \leq C(n^2\ell^{-3}+\rho_\mu^2\ell^3)a, \tag{6.44}$$

$$\pm\Big(\sum_{i\neq j}Q_jQ_iw_1(x_i,x_j)P_iP_j + \text{h.c.}\Big) \leq Cn^2\ell^{-3}a + \frac{1}{4}\mathcal{Q}_4^{\rm ren}, \tag{6.45}$$

$$\pm\Big(\sum_{i,j}P_jQ_i'w_1(x_i,x_j)Q_iQ_j + \text{h.c.}\Big) \leq Cn^2\ell^{-3}a + \frac{1}{4}\mathcal{Q}_4^{\rm ren} \tag{6.46}$$

for any (not necessarily self adjoint) operator Q' on $L^2(\mathbb{R}^3)$ with $QQ' = Q'$ and $\|Q'\| \leq 1$.

Proof. Since $0 \leq \int W_1 \leq Ca$, we have

$$0 \leq \rho_\mu\sum_{i=1}^N\int w_1(x_i,y)\,dy \leq Ca\|\chi_\Lambda\|_\infty^2\rho_\mu n \leq Ca\|\chi_\Lambda\|_\infty^2(\rho_\mu^2\ell^3+n^2\ell^{-3}). \tag{6.47}$$

The off-diagonal terms in the one-body potential can be estimated using a Cauchy-Schwarz inequality relying on the positivity of w_1

(6.48)
$$\pm\rho_\mu\Big(\sum_{i=1}^N P_i\int w_1(x_i,y)\,dyQ_i + \text{h.c.}\Big) \leq \rho_\mu\sum_{i=1}^N P_i\int w_1(x_i,y)\,dyP_i + \rho_\mu\sum_{i=1}^N Q_i\int w_1(x_i,y)\,dyQ_i \leq Ca(1+\|\chi_\Lambda\|_\infty^2)\rho_\mu n.$$

We also have

$$\begin{aligned}0 &\le \sum_{i,j} P_iQ_jw_1(x_i,x_j)P_iQ_j\\&= n_0|\Lambda|^{-1}\sum_j Q_j\chi_\Lambda(x_j)W_1*\chi_\Lambda(x_j)Q_j \le Cn_0n_+\ell^{-3}a\|\chi_\Lambda\|_\infty^2\end{aligned}$$

or more generally, using again Cauchy-Schwarz inequalities, we have for all $k=0,1,\dots,$

$$\begin{aligned}&\pm\Big(\sum_{i,j} P_iQ_j'(w_1\omega^k)(x_i,x_j)P_iQ_j + \text{h.c.}\Big)\\&\qquad\le Cn_0\ell^{-3}a\|\chi_\Lambda\|_\infty^2\Big(\varepsilon n_+ + \varepsilon^{-1}\sum_i Q_i'Q_i'^*\Big),\end{aligned}\tag{6.49}$$

$$\begin{aligned}&\pm\Big(\sum_{i,j} P_iQ_j'(w_1\omega^k)(x_i,x_j)P_jQ_i + \text{h.c.}\Big)\\&\qquad\le Cn_0\ell^{-3}a\|\chi_\Lambda\|_\infty^2\Big(\varepsilon n_+ + \varepsilon^{-1}\sum_i Q_i'Q_i'^*\Big),\end{aligned}\tag{6.50}$$

$$\begin{aligned}&\pm\Big(\sum_{i,j} P_jQ_i'(w_1\omega^k)(x_i,x_j)P_iP_j + \text{h.c.}\Big)\\&\qquad\le \sum_{i,j} P_jQ_i'(w_1\omega^k)(x_i,x_j)Q_i'^*P_j\\&\qquad\quad+\sum_{i,j} P_jP_i(w_1\omega^k)(x_i,x_j)P_iP_j\\&\qquad\le Cn_0a\ell^{-3}\Big(\|\chi_\Lambda\|_\infty^2\sum_i Q_i'Q_i'^* + n_0\Big)\end{aligned}\tag{6.51}$$

for all $\varepsilon>0$, where we have abbreviated

$$(w_1\omega^k)(x_1,x_2) = w_1(x_1,x_2)\omega(x_1-x_2)^k.$$

In this proof we will choose $\varepsilon=1$ and use $\sum_i Q_i'Q_i'^* \le n_+ \le n$. The freedom to choose $\varepsilon\neq 1$ will be used in the proof of Corollary 6.12 below. The estimates in (6.49)–(6.51) prove (6.44) if we recall that $w_2 = w_1(1+\omega)$ and choose $Q'=Q$.

To prove (6.46) we rewrite the terms in $\mathcal{Q}_3^{\text{ren}}$ as follows:

$$\begin{aligned}&\sum_{i,j} P_iQ_j'w_1(x_i,x_j)Q_jQ_i\\&= \sum_{i,j}\Big(P_iQ_j'w_1(x_i,x_j)\Big[Q_jQ_i + \omega(x_i-x_j)(P_jP_i+P_jQ_i+Q_jP_i)\Big]\Big)\\&\quad-\sum_{i,j}\Big(P_iQ_j'w_1(x_i,x_j)\omega(x_i-x_j)(P_jP_i+P_jQ_i+Q_jP_i)\Big)\end{aligned}\tag{6.52}$$

and likewise for the Hermitian conjugate terms. Thus applying a Cauchy-Schwarz inequality and the estimates (6.49)–(6.51) we arrive at

$$\pm\Big(\sum_{i,j} P_i Q'_j w_1(x_i,x_j) Q_j Q_i + \text{h.c.}\Big) \\ \leq \frac{1}{2}\mathcal{Q}_4^{\text{ren}} + C\sum_{i\neq j} P_i Q'_j w_1(x_i,x_j)(1-\omega(x_i-x_j))Q'^*_j P_i + Cn^2 a\ell^{-3},$$

which implies (6.46). The estimate (6.45) follows in the same way. Finally, the estimate (6.43) follows from (6.45), (6.46), and (6.47)–(6.51) with $Q' = Q$. □

In our more detailed analysis of the $\mathcal{Q}_3$ terms in Section 9 we will need the following more refined version of the estimate in (6.46).

Corollary 6.12. *With the same notation as in Lemma 6.11 we have for all* $0 < \varepsilon < 1$,

$$\sum_{i,j}\Big(P_j Q'_i w_1(x_i,x_j) Q_i Q_j + P_j Q'_i w_1(x_i,x_j)\omega(x_i-x_j) P_i P_j\Big) + \text{h.c.} \\ \geq -Cn_0\ell^{-3} a\Big(\varepsilon^{-1}\sum_i Q'_i Q'^*_i + \varepsilon n_+\Big) - \frac{1}{4}\mathcal{Q}_4^{\text{ren}}. \tag{6.53}$$

Proof. We again use the identity (6.52) and perform the same Cauchy-Schwarz as above, but the term with three P operators now appear on the left and we do not have to estimate it using (6.51). We, however, use (6.49) and (6.50) with $0 < \varepsilon < 1$. □

7. A priori bounds on particle number and excited particles

In the section we will give some important a priori bounds on the particle number n, the number of excited particles n_+ as well as on some of the potential energy terms. The bounds on n and n_+ essentially say that for states with sufficiently low energy, n is close to what one would expect; i.e., $\rho_\mu \ell^3$ and the expectation of n_+ is smaller with a factor that is not much worse than the relative LHY error. These bounds are difficult to prove and are given in (7.2) below. The proof is in Appendix B. They rely on a very detailed analysis of a further localization into smaller boxes.

Theorem 7.1 (A priori bounds). *Assume that conditions (5.3), (5.4), (5.6), (5.7), and (5.29) on K_B, R, ε_T, s, and d are satisfied and that $\rho_\mu a^3$ is small enough. Then there is a universal constant $C > 0$ such that if $\Psi \in \mathcal{F}_s(L^2(\Lambda))$ is an n-particle normalized state in the bosonic Fock space over $L^2(\Lambda)$ satisfying*

$$\langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi\rangle \leq -4\pi\rho_\mu^2 a\ell^3(1 - J(\rho_\mu a^3)^{\frac{1}{2}}) \tag{7.1}$$

for a $0 < J \leq K_B^3$ (*the freedom to take* $J < K_B^3$ *will be used in Lemma* 8.2), *then*

$$|n\ell^{-3} - \rho_\mu| \leq \rho_\mu C K_B^{3/2} K_\ell (\rho_\mu a^3)^{1/4} \quad \textit{and} \quad \langle \Psi, n_+ \Psi \rangle \leq C \rho_\mu \ell^3 K_B^3 K_\ell^2 (\rho_\mu a^3)^{1/2}. \tag{7.2}$$

Moreover, we also have

$$0 \leq \langle \Psi, \mathcal{Q}_4^{\mathrm{ren}} \Psi \rangle \leq C \rho_\mu^2 a \ell^3 \tag{7.3}$$

and

$$\begin{aligned} &\Big| \langle \Psi, \rho_\mu \sum_{i=1}^N \Big(P_i \int w_1(x_i, y) dy Q_i + \mathrm{h.c.} \Big) \Psi \rangle \Big| \\ &\qquad + \Big| \langle \Psi, \sum_{i \neq j} (Q_j P_i w(x_i, x_j) P_i P_j + \mathrm{h.c.}) \Psi \rangle \Big| \\ &\qquad + \Big| \langle \Psi, \sum_{i,j} (P_j Q_i w(x_i, x_j) Q_i Q_j + \mathrm{h.c.}) \Psi \rangle \Big| \\ &\qquad + \Big| \langle \Psi, \sum_{i \neq j} (Q_j Q_i w(x_i, x_j) P_i P_j + \mathrm{h.c.}) \Psi \rangle \Big| \\ &\leq C \rho_\mu^2 \ell^3 \int v. \end{aligned} \tag{7.4}$$

Remark 7.2. Note that the expressions on the left of (7.4) above contain w instead of w_1, which appeared in (6.44)–(6.46). We will need the estimates (7.4) in the next section, and this will be the only place where an estimate containing $\int v$ will be used.

Proof. As explained, the bounds (7.2) are proved in Theorem B.6. Due to our assumptions they, in particular, imply that $n \leq C \rho_\mu \ell^3$.

This a priori bound on n, the positivity of the kinetic energy $\mathcal{T}$, and the bound in (6.43) immediately imply

$$\langle \Psi, \mathcal{H}_\Lambda(\rho_\mu) \Psi \rangle \geq -C \rho_\mu^2 a \ell^3 + \frac{1}{2} \langle \Psi, \mathcal{Q}_4^{\mathrm{ren}} \Psi \rangle,$$

which by the assumption on Ψ gives the bound (7.3).

The bounds on the first two terms in (7.4) follow exactly as the proofs of (6.49)–(6.51) for $k = 0$ and with w_1 replaced by w such that a has to be replaced by $\int v \geq 8\pi a$ in the bounds. The bounds on the last two terms in (7.4) follow the same lines as the proof of (6.45) and (6.46). We sketch it for the last term in (7.4). We rewrite

$$\begin{aligned}(7.5)\qquad &\sum_{i\neq j} P_iP_jw(x_i,x_j)Q_iQ_j\\ &= \sum_{i\neq j} P_iP_jw(x_i,x_j)(Q_iQ_j+\omega(x_i-x_j)(P_iP_j+Q_iP_j+P_iQ_j))\\ &\quad -\sum_{i\neq j} P_iP_jw(x_i,x_j)\omega(x_i-x_j)(P_iP_j+Q_iP_j+P_iQ_j)\end{aligned}$$

and likewise for the Hermitian conjugate. If we recall that $0\leq\omega\leq 1$, the last sum is estimated as in the case of (6.49)–(6.51) again with a replaced by $\int v$. The first term above together with its complex conjugate is after a Cauchy-Schwarz controlled by a similar term and Q_4^{ren}; i.e., we get

$$\left\langle \Psi, \Big(\sum_{i\neq j} P_iP_jw(x_i,x_j)Q_iQ_j + \mathrm{h.c.}\Big)\Psi\right\rangle \leq C\rho_\mu^2\ell^3\int v + C\langle\Psi, Q_4^{\mathrm{ren}}\Psi\rangle,$$

which by the bound (7.3) implies what we want. □

8. **Localization of the number of excited particles** n_+

As in [7] we shall use the following theorem from [21] to restrict the number of excited particles.

THEOREM 8.1 (Localization of large matrices). *Suppose $\mathcal{A}$ is an $(N+1)\times(N+1)$ Hermitian matrix, and let $\mathcal{A}^{(k)}$, with $k=0,1,\ldots,N$, denote the matrix consisting of the k^{th} supra- and infra-diagonal of $\mathcal{A}$. Let $\psi\in\mathbb{C}^{N+1}$ be a normalized vector, and set $d_k=\langle\psi,\mathcal{A}^{(k)}\psi\rangle$ and $\lambda=\langle\psi,\mathcal{A}\psi\rangle=\sum_{k=0}^N d_k$. (Note that ψ need not be an eigenvector of $\mathcal{A}$.) Choose some positive integer $\mathcal{M}'\leq N+1$. Then, with $\mathcal{M}'$ fixed, there is some $n'\in[0,N+1-\mathcal{M}']$ and some normalized vector $\varphi\in\mathbb{C}^{N+1}$ with the property that $\varphi_j=0$ unless $n'+1\leq j\leq n'+\mathcal{M}'$ (i.e., φ has localization length $\mathcal{M}'$) and such that*

$$(8.1)\qquad \langle\varphi,\mathcal{A}\varphi\rangle \leq \lambda + \frac{C}{\mathcal{M}'^2}\sum_{k=1}^{\mathcal{M}'-1} k^2|d_k| + C\sum_{k=\mathcal{M}'}^{N}|d_k|,$$

where $C>0$ is a universal constant. (Note that the first sum starts at $k=1$.)

This will allow us to prove the following result using the estimate (7.4). We emphasize that this is the only place in the proof of our main result where an estimate depends explicitly on $\int v$ and not just on a.

LEMMA 8.2 (Restriction on n_+). *Let $\mathcal{M}$ be as defined in (5.9) and satisfying (5.10) and (5.11). Assume, moreover, that $\rho_\mu a^3$ is small enough. There is then a universal $C>0$ such that if there is a normalized n-particle*

$\Psi \in \mathcal{F}_s(L^2(\Lambda))$ satisfying (7.1) under the assumptions in Theorem 7.1 with $J = \frac{1}{2}K_B^3$, then there is also a normalized n-particle wave function $\widetilde{\Psi} \in \mathcal{F}_{\rm s}(L^2(\Lambda))$ with the property that

$$\widetilde{\Psi} = 1_{[0,\mathcal{M}]}(n_+)\widetilde{\Psi}, \tag{8.2}$$

i.e., only values of n_+ smaller than $\mathcal{M}$ appear in $\widetilde{\Psi}$, and such that

$$\Big\langle \widetilde{\Psi}, \mathcal{H}_\Lambda(\rho_\mu)\widetilde{\Psi}\Big\rangle \leq \langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi\rangle + CK_{\mathcal{M}}^{-2}\rho_\mu^2\ell^3(\rho_\mu a^3)^{1/2}\int v. \tag{8.3}$$

Proof. We may assume from (5.11) that $\mathcal{M} \geq 5$ and that $\mathcal{M} \leq n$ since otherwise there is nothing to prove.

We shall apply Theorem 8.1 on localization of large matrices to the $(n+1) \times (n+1)$-matrix with elements

$$\mathcal{A}_{i,j} = \|\mathbb{1}_{n_+=i}\Psi\|^{-1}\|\mathbb{1}_{n_+=j}\Psi\|^{-1}\langle \mathbb{1}_{n_+=i}\Psi, H_\Lambda(\rho_\mu)\mathbb{1}_{n_+=j}\Psi\rangle.$$

(If any of the norms are zero, we set the element to zero.) Then we get a normalized vector $\psi = (\|\mathbb{1}_{n_+=0}\Psi\|, \ldots, \|\mathbb{1}_{n_+=n}\Psi\|)$ in $\mathbb{C}^{n+1}$ and

$$\langle \psi, \mathcal{A}\psi\rangle = \langle \Psi, H_\Lambda(\rho_\mu)\Psi\rangle.$$

Moreover, for the matrix $\mathcal{A}$, using the notation of Theorem 8.1, only the $\mathcal{A}^{(k)}$ with $k = 0, 1, 2$ are non-vanishing. In fact, we have

$$\begin{aligned} d_1 &= \langle \psi, \mathcal{A}^{(1)}\psi\rangle \\ &= \Big\langle \Psi, \Big(-\rho_\mu \sum_{i=1}^N (P_i \int w_1(x_i, y)dyQ_i + \text{h.c.}) \\ &\quad + \sum_{i\neq j}(Q_jP_iw(x_i,x_j)P_iP_j + \text{h.c.}) + \sum_{i\neq j}(P_iQ_jw(x_i,x_j)Q_jQ_i + \text{h.c.})\Big)\Psi\Big\rangle \end{aligned}$$

and

$$d_2 = \langle \psi, \mathcal{A}^{(2)}\psi\rangle = \Big\langle \Psi, \Big(\sum_{i\neq j}(P_iP_jw(x_i,x_j)Q_jQ_i + \text{h.c.})\Big)\Psi\Big\rangle.$$

It thus follows from (7.4) that $|d_1|, |d_2| \leq C\rho_\mu^2\ell^3 \int v$.

The theorem on localization of large matrices tells us that if we choose $\mathcal{M}'$ equal to the integer part of $\mathcal{M}/2$, we can find a normalized $\varphi \in \mathbb{C}^{n+1}$ with localization length $\mathcal{M}'$ such that

$$\begin{aligned} \langle \varphi, \mathcal{A}\varphi\rangle &\leq \langle \psi, \mathcal{A}\psi\rangle + C\mathcal{M}'^{-2}(|d_1| + |d_2|)) \\ &\leq \langle \Psi, H_\Lambda(\rho_\mu)\Psi\rangle + CK_{\mathcal{M}}^{-2}\rho_\mu^2\ell^3(\rho_\mu a^3)^{1/2}\int v. \end{aligned} \tag{8.4}$$

Let $\widetilde{\varphi} \in \mathbb{C}^{n+1}$ be given by $\widetilde{\varphi}_i = \varphi_i$ if $\|\mathbb{1}_{n_+=i}\Psi\| \neq 0$ and $\widetilde{\varphi}_i = 0$ if $\|\mathbb{1}_{n_+=i}\Psi\| = 0$. Then $\|\widetilde{\varphi}\| \leq 1$. We then have

$$\langle \widetilde{\varphi}, \mathcal{A}\widetilde{\varphi}\rangle = \langle \varphi, \mathcal{A}\varphi\rangle \leq \langle \Psi, H_\Lambda(\rho_\mu)\Psi\rangle + CK_{\mathcal{M}}^{-2}\rho_\mu^2\ell^3(\rho_\mu a^3)^{1/2}\int v < 0, \tag{8.5}$$

where the negativity follows from $J = \frac{1}{2}K_B^3$, (5.7), and (5.10). In particular, $\widetilde{\varphi} \neq 0$. Define

$$\widetilde{\Psi} = \|\widetilde{\varphi}\|^{-1}\sum_{i=0}^{n}\widetilde{\varphi}_i\|\mathbb{1}_{n_+=i}\Psi\|^{-1}\mathbb{1}_{n_+=i}\Psi.$$

Then $\widetilde{\Psi}$ is normalized and satisfies

$$\left\langle\widetilde{\Psi}, H_\Lambda(\rho_\mu)\widetilde{\Psi}\right\rangle = \|\widetilde{\varphi}\|^{-2}\langle\widetilde{\varphi}, \mathcal{A}\widetilde{\varphi}\rangle \leq \langle\widetilde{\varphi}, \mathcal{A}\widetilde{\varphi}\rangle,$$

since the term on the right is negative and $\|\widetilde{\varphi}\|^{-2} \geq 1$. This proves that $\widetilde{\Psi}$ satisfies (8.3). It remains to prove that $\widetilde{\Psi}$ satisfies (8.2). We know from the construction that the possible values of n_+ that occur in $\widetilde{\Psi}$ lie in an interval of length $\mathcal{M}'$. We need to prove that this interval lies close to zero. This follows from the estimate (8.3), $J = \frac{1}{2}K_B^3$, and (5.10). which imply that we may use the a priori bound (7.2) on the expectation value of n_+ in $\widetilde{\Psi}$. The consequence is that the interval of n_+ values in $\widetilde{\Psi}$ must be contained in

$$[0, \mathcal{M}' + C\rho_\mu\ell^3K_B^3K_\ell^2(\rho_\mu a^3)^{1/2}] = [0, \mathcal{M}' + CK_B^3K_\ell^5] \subseteq [0, \mathcal{M}]$$

by (5.11). □

9. **Localization of the $3Q$-term**

In this section we will absorb an unimportant part of the $3Q$ term in the positive $4Q$ term.

We first define the "low" and "high" momentum regions as follows:

$$P_L := \{|p| \leq K_L\sqrt{\rho_\mu a}\}, \quad P_H := \{|p| \geq \widetilde{K}_H^{-1}(\rho_\mu a^3)^{\frac{5}{12}}a^{-1}\} = \{|p| \geq K_H^{-1}a^{-1}\}, \tag{9.1}$$

where $K_L, \widetilde{K}_H$ were defined in Section 5. The somewhat peculiar definition of P_H is convenient for later estimates. (See the proof of Lemma 10.3.) We will always assume that (5.18) is satisfied. This assures that P_L and P_H are disjoint.

We will define the low momentum localization operator Q_L as follows. Let $f \in C^\infty(\mathbb{R})$ be a monotone non-increasing function satisfying that $f(s) = 1$ for $s \leq 1$ and $f(s) = 0$ for $s \geq 2$. We further define

$$f_L(s) := f\Big(\frac{s}{K_L\sqrt{\rho_\mu a}}\Big); \tag{9.2}$$

i.e., f_L is a smooth localization to the low momenta P_L. With this notation, we define

$$Q_L := Qf_L(\sqrt{-\Delta}), \qquad \overline{Q_L} := Q(1 - f_L(\sqrt{-\Delta})). \tag{9.3}$$

Notice that Q_L is not self-adjoint.

We will choose K_L such that $K_L\sqrt{\rho_\mu a} = d^{-2}\ell^{-1}$—this is equivalent to (5.15)—where d is from the definition of the "small boxes" (see (5.2)).

We define

$$n_+^H := \sum_{j=1}^N Q_j \mathbb{1}_{(d^{-2}\ell^{-1},\infty)}(\sqrt{-\Delta_j})Q_j. \tag{9.4}$$

With this definition and the choice of K_L above, we have

$$\sum_{j=1}^N \overline{Q}_{L,j}(\overline{Q}_{L,j})^* \leq n_+^H. \tag{9.5}$$

Lemma 9.1. *Define*

$$\widetilde{Q}_3^{(1)} := \sum_{i\neq j}(P_i Q_{L,j} w_1(x_i,x_j)Q_j Q_i + \text{h.c.}). \tag{9.6}$$

We assume (5.4), (5.17) *and* (7.2). *With the notation from* (6.34), (6.35), *we get*

$$\begin{aligned} &\mathcal{Q}_3^{\text{ren}} + \frac{1}{4}\mathcal{Q}_4^{\text{ren}} + \frac{b}{100}\left(\ell^{-2}n_+ + \varepsilon_T(d\ell)^{-2}n_+^H\right) \\ &\qquad \geq \widetilde{Q}_3^{(1)} - C\rho_\mu^2 a\ell^3\left((K_\ell K_L)^{1-M} + \frac{R^2}{\ell^2}\right). \end{aligned} \tag{9.7}$$

Proof. Using Corollary 6.12, with $Q' = \overline{Q}_L$ and $\varepsilon = cK_\ell^{-2}$ for some sufficiently small constant c, as well as (9.5) we find

$$\begin{aligned} &\frac{1}{4}\mathcal{Q}_4^{\text{ren}} + \frac{b}{100\ell^2}n_+ + \mathcal{Q}_3^{\text{ren}} - \widetilde{Q}_3^{(1)} \\ &\qquad \geq \sum_{i,j}\Big(P_j\overline{Q}_{L,i}w_1(x_i,x_j)\omega(x_i-x_j)P_iP_j + \text{h.c.}\Big) - C\ell^{-2}K_\ell^4 n_+^H. \end{aligned} \tag{9.8}$$

Using (5.4) it is clear that the n_+^H term is dominated by half of the positive n_+^H term from (9.7).

To estimate the remaining terms in (9.8) we start by using the estimate on the convolution from Lemma 6.2 to get

$$\begin{aligned} &-\sum_{i\neq j}\left(P_i\overline{Q}_{L,j}w_1(x_i,x_j)\omega(x_i-x_j)P_jP_i + \text{h.c.}\right) \\ &\qquad \geq -I\ell^{-3}\Big(n_0\sum_j \overline{Q}_{L,j}\chi_\Lambda^2(x_j)P_j + \text{h.c.}\Big) - Can^2\ell^{-3}\frac{R^2}{\ell^2}, \end{aligned} \tag{9.9}$$

where $I := \int W_1(y)\omega(y) \leq Ca$.

To complete the proof we write, with $M-1 \le 2\widetilde{M} \le M$,

$$\overline{Q}_L \chi_\Lambda^2 P + \text{h.c.} = \overline{Q}_L (\ell^{-2}-\Delta)^{-\widetilde{M}} \left[(\ell^{-2}-\Delta)^{\widetilde{M}} \chi_\Lambda^2 \right] P + \text{h.c.} \tag{9.10}$$

and notice that

$$|(\ell^{-2}-\Delta)^{\widetilde{M}} \chi_\Lambda^2| \le C \ell^{-2\widetilde{M}}. \tag{9.11}$$

Therefore,

$$\begin{aligned} \overline{Q}_L \chi_\Lambda^2 P + \text{h.c.} &\le \varepsilon_2 P + \varepsilon_2^{-1} \ell^{2\widetilde{M}} \overline{Q}_L (\ell^{-2}-\Delta)^{-2\widetilde{M}} (\overline{Q}_L)^* \\ &\le \varepsilon_2 P + \varepsilon_2^{-1} (K_\ell K_L)^{-2\widetilde{M}} \overline{Q}_L (\overline{Q}_L)^*. \end{aligned} \tag{9.12}$$

Choosing $\varepsilon_2 = (K_\ell K_L)^{-2\widetilde{M}}$ and using again (5.4), we get (9.7) upon summing this estimate in the particle indices and absorbing the n_+^H term as before. □

10. **Second quantized operators**

10.1. *Creation/annihilation operators.* We will use $a, a^\dagger$ to denote the standard bosonic annihilation/creation operators on the bosonic Fock space $\mathcal{F}_s(L^2(\Lambda))$.

We define a_0 as the annihilation operator associated to the condensate function for the box Λ, i.e., $a_0 = \ell^{-3/2} a(\theta)$, where we recall that θ defined in (6.4) is the characteristic function of the box. In more detail, for $\Psi \in \otimes_s^N L^2(\Lambda)$, we have

$$(a_0 \Psi)(x_2, \ldots, x_N) := \frac{\sqrt{N}}{\ell^{3/2}} \int_\Lambda \Psi(y, x_2, \ldots, x_N)\, dy.$$

Therefore,

$$\langle \Psi, n_0 \Psi \rangle = \langle \Psi \,|\, a_0^\dagger a_0 \Psi \rangle = \frac{N}{\ell^3} \int \left| \int_\Lambda \Psi(y, x_2, \ldots, x_N)\, dy \right|^2 dx_2 \cdots dx_N. \tag{10.1}$$

Due to the localization function χ_Λ it is convenient to work with the localized annihilation/creation operators $a_k, a_k^\dagger$ defined in (10.3) below. However, we will also need the non-localized versions $\widetilde{a}_k, \widetilde{a}_k^\dagger$. Since these are more standard, we give their definition first.

For $k \in \mathbb{R}^3 \setminus \{0\}$, we let

$$\widetilde{a}_k := \ell^{-3/2} a(Q(e^{ikx}\theta)), \qquad \widetilde{a}_k^\dagger := \ell^{-3/2} a^\dagger(Q(e^{ikx}\theta)) \tag{10.2}$$

Clearly, for $k, k' \in \mathbb{R}^3 \setminus \{0\}$,

$$[\widetilde{a}_k, \widetilde{a}_{k'}] = 0, \qquad [\widetilde{a}_k, \widetilde{a}_{k'}^\dagger] = \ell^{-3} \langle e^{ikx}\theta, Q e^{ik'x}\theta \rangle. \tag{10.3}$$

We also define, for $k \in \mathbb{R}^3 \setminus \{0\}$,

$$a_k := \ell^{-3/2} a(Q(e^{ikx}\chi_\Lambda)) \qquad \text{and} \qquad a_k^\dagger := \ell^{-3/2} a(Q(e^{ikx}\chi_\Lambda))^*. \tag{10.4}$$

Then, for all $k, k' \in \mathbb{R}^3 \setminus \{0\}$,

$$[a_k, a_{k'}] = 0 \tag{10.5}$$

and

$$[a_k, a_{k'}^\dagger] = \ell^{-3}\langle Q(e^{ikx}\chi_\Lambda), Q(e^{ik'x}\chi_\Lambda)\rangle = \widehat{\chi^2}((k-k')\ell) - \widehat{\chi}(k\ell)\overline{\widehat{\chi}(k'\ell)}. \tag{10.6}$$

In particular,

$$[a_k, a_k^\dagger] \le 1. \tag{10.7}$$

Furthermore, we introduce the Fourier multiplier corresponding to the localized kinetic energy (after the separation of the constant term), i.e.,

$$\tau(k) := (1-\varepsilon_T)\Big[|k| - \frac{1}{2}(s\ell)^{-1}\Big]_+^2 + \varepsilon_T\Big[|k| - \frac{1}{2}(ds\ell)^{-1}\Big]_+^2. \tag{10.8}$$

We can express the different parts of the Hamiltonian $\mathcal{H}_\Lambda(\rho_\mu)$ in second quantized formalism. We give this as the following Lemma 10.1. The proof is a standard calculation and will be omitted.

Lemma 10.1. *We have the following expressions for the operators in second quantized formalism* (*with $\mathcal{T}'$ the part of the kinetic energy operator defined in* (6.20)) *acting on the N-particle sector of Fock space*:

(10.9)

$$\begin{aligned}
n_0 &= \Big(a_0^\dagger a_0\Big)_N, \\
n_0^2 &= \Big((a_0^\dagger a_0)^2\Big)_N = \Big((a_0^\dagger)^2 a_0^2 - a_0^\dagger a_0\Big)_N, \\
n_+ &= \Big((2\pi)^{-3}\ell^3 \int \widetilde{a}_k^\dagger \widetilde{a}_k \, dk\Big)_N \\
\sum_{j=1}^N \mathcal{T}_j' &= \Big((2\pi)^{-3}\ell^3 \int_{k\in\mathbb{R}^3} \tau(k) a_k^\dagger a_k\Big)_N, \\
\sum_{i\neq j}^N P_i P_j w_1(x_i,x_j) Q_j Q_i &= \Big((2\pi)^{-3} \int \widehat{W}_1(k) a_0^\dagger a_0^\dagger a_k a_{-k}\, dk\Big)_N, \\
\sum_{j\neq s}^N P_j Q_s w_2(x_i,x_j) P_s Q_j &= \Big((2\pi)^{-3} \int \widehat{W}_2(k) a_{-k}^\dagger a_0^\dagger a_0 a_{-k}\, dk\Big)_N, \\
\sum_{i=1}^N Q_i f(x_i)\chi_\Lambda(x_i) P_i &= \Big((2\pi)^{-3} \int \widehat{f}(k) a_k^\dagger a_0 \, dk\Big)_N, \\
\sum_{i\neq j}^N P_i Q_{L,j} w_1(x_i,x_j) Q_j Q_i &= \Big(\ell^3 (2\pi)^{-6} \iint f_L(s) \widehat{W}_1(k) a_0^\dagger \widetilde{a}_s^\dagger a_{s-k} a_k \, dk\, ds\Big)_N.
\end{aligned}$$

PROPOSITION 10.2. *Assume that $\widetilde{\Psi}$ satisfies (8.2) and (8.3) and that the parameters satisfy (5.16), (5.4), and (5.17). Then, the operator $\mathcal{H}_\Lambda(\rho_\mu)$ defined in (6.24) satisfies*

$$\langle\widetilde{\Psi}, \mathcal{H}_\Lambda(\rho_\mu)\widetilde{\Psi}\rangle \geq \langle\widetilde{\Psi}, \mathcal{H}_\Lambda^{\mathrm{2nd}}(\rho_\mu)\widetilde{\Psi}\rangle - C\rho_\mu^2 a\ell^3\left((K_\ell K_L)^{1-M} + \frac{R^2}{\ell^2}\right), \tag{10.10}$$

where

$$\begin{aligned}\mathcal{H}_\Lambda^{\mathrm{2nd}} &= (2\pi)^{-3}\ell^3\int \tau(k)a_k^\dagger a_k\,dk + \frac{b}{2\ell^2}n_+ + \varepsilon_T\frac{b}{2d^2\ell^2}n_+^H\\
&+ \frac{1}{2}\ell^{-3}a_0^\dagger a_0^\dagger a_0 a_0\Big(\widehat{g}(0) + \widehat{g\omega}(0)\Big) - \rho_\mu\widehat{g}(0)a_0^\dagger a_0\\
&+ \Big((\ell^{-3}a_0^\dagger a_0 - \rho_\mu)\widehat{W}_1(0)(2\pi)^{-3}\int \widehat{\chi}_\Lambda(k)a_k^\dagger a_0\,dk + \mathrm{h.c.}\Big)\\
&+ \Big(\ell^{-3}a_0^\dagger a_0\widehat{W\omega_1}(0)(2\pi)^{-3}\int \widehat{\chi}_\Lambda(k)a_k^\dagger a_0\,dk + \mathrm{h.c.}\Big)\\
&+ (2\pi)^{-3}\int \left(\widehat{W}_1(k) + \widehat{W_1\omega}(k)\right)a_0^\dagger a_k^\dagger a_k a_0\,dk\\
&+ \frac{1}{2}\widehat{W}_1(k)\left(a_0^\dagger a_0^\dagger a_k a_{-k} + a_k^\dagger a_{-k}^\dagger a_0 a_0\right)\,dk\\
&+ \Big((\ell^{-3}a_0^\dagger a_0 - \rho_\mu)\widehat{W_1}(0) + \ell^{-3}a_0^\dagger a_0\widehat{W_1\omega}(0)\Big)(2\pi)^{-3}\ell^{-3}\int a_k^\dagger a_k\,dk\\
&+ \widetilde{Q}_3,\end{aligned} \tag{10.11}$$

where

$$\widetilde{Q}_3 := \ell^3(2\pi)^{-6}\iint_{\{k\in P_H\}} f_L(s)\widehat{W}_1(k)(a_0^\dagger\widetilde{a}_s^\dagger a_{s-k}a_k + a_k^\dagger a_{s-k}^\dagger\widetilde{a}_s a_0)\,dk\,ds. \tag{10.12}$$

Proof. Notice that (7.2) holds, using (8.3) and Theorem 7.1.

We apply Lemma 6.9. For the operators $\mathcal{Q}_0^{\mathrm{ren}}$ and $\mathcal{Q}_1^{\mathrm{ren}}$, we use the simplifications of Lemma 6.10 before making the explicit calculation of their second quantifications. For $\mathcal{Q}_2^{\mathrm{ren}}$, we also use the simplifications of Lemma 6.10. The error term in (6.41) is absorbed in the gap in the kinetic energy. This uses that $R \ll (\rho_\mu a)^{-1/2}$ and the relation $n \approx \rho_\mu\ell^3$ from (7.2).

Finally we consider $\mathcal{Q}_3^{\mathrm{ren}}$ and $\mathcal{Q}_4^{\mathrm{ren}}$. By Lemma 9.1 and the positivity of v we have the lower bound (9.7). What remains of $\mathcal{Q}_4^{\mathrm{ren}}$ will be discarded for a lower bound. The application of (9.7) also costs a bit of the gap in the kinetic energy. What remains is to compare $\widetilde{Q}_3^{(1)}$ with $\widetilde{Q}_3$; this is the content of Lemma 10.3 below. Using (5.16) the error term from (10.13) can be absorbed in the gap in the kinetic energy. This finishes the proof of Proposition 10.2. □

In the above proof we used the following localization of the 3Q-term.

LEMMA 10.3. *Assume that* $\widetilde{\Psi}$ *satisfies* (8.2) *and* (8.3). *Let* $\widetilde{Q}_3^{(1)}$ *be as defined in Lemma* 9.1 *and* $\widetilde{Q}_3$ *from* (10.12).

Then,

$$\langle \widetilde{\Psi}, \widetilde{Q}_3^{(1)} \widetilde{\Psi} \rangle \geq \langle \widetilde{\Psi}, \widetilde{Q}_3 \widetilde{\Psi} \rangle - Can \frac{n_+}{\ell^3} \widetilde{K}_H^{-3/2} K_{\mathcal{M}}^{1/2}. \tag{10.13}$$

Proof. Again (7.2) holds, using (8.3) and Theorem 7.1.

In second quantization we have

$$\widetilde{Q}_3^{(1)} = \ell^3 (2\pi)^{-6} \iint f_L(s) \widehat{W}_1(k) (a_0^\dagger \widetilde{a}_s^\dagger a_{s-k} a_k + a_k^\dagger a_{s-k}^\dagger \widetilde{a}_s a_0)\, dk\, ds, \tag{10.14}$$

so we have to estimate the part of the integral where $k \notin P_H$. Let $\varepsilon > 0$. Then,

(10.15)

$$\begin{aligned} &\Big\langle \widetilde{\Psi}, \ell^3 (2\pi)^{-6} \iint_{\{|k| \leq K_H^{-1} a^{-1}\}} f_L(s) \widehat{W}_1(k) (a_0^\dagger \widetilde{a}_s^\dagger a_{s-k} a_k + a_k^\dagger a_{s-k}^\dagger \widetilde{a}_s a_0)\, dk ds\, \widetilde{\Psi} \Big\rangle \\ &\geq -Ca\ell^3 (2\pi)^{-6} \iint_{\{|k| \leq K_H^{-1} a^{-1}\}} f_L(s) \Big(\varepsilon \langle \widetilde{\Psi}, \widetilde{a}_s^\dagger a_0^\dagger a_0 \widetilde{a}_s \widetilde{\Psi} \rangle \\ &\qquad\qquad + \varepsilon^{-1} \langle \widetilde{\Psi}, a_k^\dagger a_{s-k}^\dagger a_{s-k} a_k \widetilde{\Psi} \rangle \Big)\, dk ds \\ &\geq -Can \frac{n_+}{\ell^3} \left(\varepsilon \ell^3 (K_H a)^{-3} + \varepsilon^{-1} \frac{\mathcal{M}}{n} \right). \end{aligned}$$

Here we used, in particular, that $\langle \widetilde{\Psi}, n_+^2 \widetilde{\Psi} \rangle \leq \mathcal{M} n_+$ since $\widetilde{\Psi}$ satisfies (7.2). Observe that we have not assumed that $\widehat{W}_1(k)$ has a sign and that the Cauchy-Schwarz inequality in (10.15) is valid for $\widehat{W}_1(k)$ of variable sign.

We choose $\varepsilon = \left(\frac{\mathcal{M} K_H^3 a^3}{n \ell^3} \right)^{1/2}$. Using the relation $n \approx \rho_\mu \ell^3$ from (7.2), the error term in parenthesis in (10.15) becomes of magnitude $\sqrt{\frac{\mathcal{M}}{\rho_\mu a^3 K_H^3}} = \widetilde{K}_H^{-3/2} K_{\mathcal{M}}^{1/2}$. □

It will also be useful to notice the following representation in terms of the operators $\widetilde{a}_k$.

LEMMA 10.4. *We have the identities*

$$\left((2\pi)^{-6} \ell^6 \iint \widetilde{a}_k^\dagger \widehat{\chi^2} \left((k - k')\ell \right) \widetilde{a}_{k'} \right)_N = \sum_{j=1}^N Q_j \chi_\Lambda^2(x_j) Q_j \tag{10.16}$$

and

(10.17)

$$\left((2\pi)^{-6} \ell^6 \iint f_L(k) f_L(k') \widetilde{a}_k^\dagger \widehat{\chi^2} \left((k - k')\ell \right) \widetilde{a}_{k'} \right)_N = \sum_{j=1}^N Q_{L,j} \chi_\Lambda^2(x_j) Q_{L,j}.$$

10.2. *c-number substitution*. It is convenient to apply the technique of c-number substitution as described in [20].

Let $\Psi \in \mathcal{F}(L^2(\Lambda))$. We can think of $L^2(\Lambda) = \mathrm{Ran}(P) \oplus \mathrm{Ran}(Q)$, with $\mathrm{Ran}(P)$ being spanned by the constant vector θ (defined in (6.4)). This leads to the splitting $\mathcal{F}(L^2(\Lambda)) = \mathcal{F}(\mathrm{Ran}(P)) \otimes \mathcal{F}(\mathrm{Ran}(Q))$. We let Ω denote the vacuum vector in $\mathcal{F}(\mathrm{Ran}(P))$.

For $z \in \mathbb{C}$, we define

$$|z\rangle := \exp\left(-\frac{|z|^2}{2} - z a_0^\dagger\right)\Omega. \tag{10.18}$$

Given z and Ψ, we can define

$$\Phi(z) := \langle z|\Psi\rangle \in \mathcal{F}(\mathrm{Ran}(Q)), \tag{10.19}$$

where the inner product is considered as a partial inner product induced by the representation $\mathcal{F}(L^2(\Lambda)) = \mathcal{F}(\mathrm{Ran}(P)) \otimes \mathcal{F}(\mathrm{Ran}(Q))$.

It is a simple calculation that

$$1 = \pi^{-1}\int_{\mathbb{C}} |z\rangle\langle z|\, d^2 z \qquad \text{and} \qquad a_0|z\rangle = z|z\rangle. \tag{10.20}$$

THEOREM 10.5. *Define*

$$\rho_z := |z|^2 \ell^{-3} \tag{10.21}$$

and

$$\begin{aligned}
\mathcal{K}(z) = {}& \frac{b}{2\ell^2} n_+ + \varepsilon_T \frac{b}{2d^2\ell^2} n_+^H + \frac{1}{2}\rho_z^2 \ell^3 \Big(\widehat{g}(0) + \widehat{g\omega}(0)\Big) - \rho_\mu \widehat{g}(0)\rho_z \ell^3 \\
&+ (2\pi)^{-3}\ell^3 \int \Big(\tau(k) + \rho_z \widehat{W}_1(k)\Big) a_k^\dagger a_k + \frac{1}{2}\rho_z \widehat{W}_1(k)\left(a_k a_{-k} + a_k^\dagger a_{-k}^\dagger\right) dk \\
&+ (\rho_z - \rho_\mu)\widehat{W_1}(0)(2\pi)^{-3}\ell^3 \int a_k^\dagger a_k \, dk \\
&+ \mathcal{Q}_1(z) + \mathcal{Q}_1^{\mathrm{ex}}(z) + \mathcal{Q}_2^{\mathrm{ex}}(z) + \mathcal{Q}_3(z),
\end{aligned} \tag{10.22}$$

with

$$\mathcal{Q}_1(z) := \Big((\rho_z - \rho_\mu)\widehat{W}_1(0)(2\pi)^{-3}\int \widehat{\chi}_\Lambda(k) a_k^\dagger z \, dk + \mathrm{h.c.}\Big), \tag{10.23}$$

$$\mathcal{Q}_1^{\mathrm{ex}}(z) := \Big(\rho_z \widehat{W_1\omega}(0)(2\pi)^{-3}\int \widehat{\chi}_\Lambda(k) a_k^\dagger z \, dk + \mathrm{h.c.}\Big), \tag{10.24}$$

$$\mathcal{Q}_3(z) := \ell^3 (2\pi)^{-6} z \iint_{\{k\in P_H\}} f_L(s)\widehat{W}_1(k)(\widetilde{a}_s^\dagger a_{s-k} a_k + a_k^\dagger a_{s-k}^\dagger \widetilde{a}_s), \tag{10.25}$$

and

$$\mathcal{Q}_2^{\mathrm{ex}} = \mathcal{Q}_2^{\mathrm{ex}}(z) := (2\pi)^{-3}\rho_z \ell^3 \int \big(\widehat{W_1\omega}(k) + \widehat{W_1\omega}(0)\big) a_k^\dagger a_k. \tag{10.26}$$

Assume that $\widetilde{\Psi}$ *satisfies* (8.2). *Then,*

$$\langle\widetilde{\Psi},\mathcal{H}^{\mathrm{2nd}}_{\Lambda}(\rho_\mu)\widetilde{\Psi}\rangle \geq \inf_{z\in\mathbb{R}_+}\inf_{\Phi}\langle\Phi,\mathcal{K}(z)\Phi\rangle - C\rho_\mu a, \tag{10.27}$$

where C *is some universal constant and the second infimum is over all normalized* $\Phi\in\mathcal{F}(\mathrm{Ran}(Q))$ *with*

$$\Phi = 1_{[0,\mathcal{M}]}(n_+)\Phi. \tag{10.28}$$

Proof. As before (7.2) holds, using (8.3) and Theorem 7.1.

We define $\widetilde{\mathcal{K}}(z)$ to be the operator $\mathcal{H}^{\mathrm{2nd}}_{\Lambda}$ defined in (10.11) above, but where the following substitutions have been performed:

$$\begin{aligned}
&a_0^\dagger a_0^\dagger a_0 a_0 \mapsto |z|^4-4|z|^2+2,\\
&a_0^\dagger a_0 a_0 \mapsto |z|^2 z - 2z, \qquad a_0 a_0^\dagger a_0^\dagger \mapsto |z|^2\overline{z},\\
&a_0a_0 \mapsto z^2, \qquad a_0^\dagger a_0^\dagger \mapsto \overline{z}^2, \qquad a_0^\dagger a_0 \mapsto |z|^2-1,\\
&a_0\mapsto z, \qquad a_0^\dagger \mapsto \overline{z}.
\end{aligned} \tag{10.29}$$

Then, we will prove that

$$\begin{aligned}
\langle\widetilde{\Psi},\mathcal{H}^{\mathrm{2nd}}_{\Lambda}\widetilde{\Psi}\rangle &= \pi^{-1}\Re\int\langle\Phi(z),\widetilde{\mathcal{K}}(z)\Phi(z)\rangle\,d^2z\\
&=\pi^{-1}\Re\int\langle\widetilde{\Phi}(z),\widetilde{\mathcal{K}}(z)\widetilde{\Phi}(z)\rangle n^2(z)\,d^2z,
\end{aligned} \tag{10.30}$$

where $n(z)=\|\Phi(z)\|_{\mathcal{F}(\mathrm{Ran}(Q))}$ and $\widetilde{\Phi}(z)=\Phi(z)/n(z)$ and $\Re$ denotes the real part.

To obtain (10.30) we write all polynomials in $a_0, a_0^\dagger$ in anti-Wick ordering, for example, $a_0^\dagger a_0 = a_0 a_0^\dagger - 1$. Therefore,

$$\begin{aligned}
\langle\Psi,a_0^\dagger a_0\Psi\rangle &= \pi^{-1}\int\langle a_0^\dagger\Psi|z\rangle\langle z|a_0^\dagger\Psi\rangle - \langle\Psi|z\rangle\langle z|\Psi\rangle\,d^2z\\
&=\pi^{-1}\int(|z|^2-1)\langle\Phi(z)|\Phi(z)\rangle\,d^2z.
\end{aligned} \tag{10.31}$$

Performing this type of calculation for each term in $\mathcal{H}^{\mathrm{2nd}}_{\Lambda}$ yields (10.30).

Suppose that $\widetilde{\Psi}\in\mathcal{F}_{\mathrm{s}}(L^2(B))$ is such that

$$\widetilde{\Psi} = 1_{[0,\mathcal{M}]}(n_+)\widetilde{\Psi}. \tag{10.32}$$

Then, for all $z\in\mathbb{C}$, we have with $\widetilde{\Phi}(z) := \langle z|\widetilde{\Psi}\rangle\in\mathcal{F}(\mathrm{Ran}(Q))$,

$$\widetilde{\Phi}(z) = 1_{[0,\mathcal{M}]}(n_+)\widetilde{\Phi}(z), \tag{10.33}$$

with $\widetilde{\Phi}(z)=\langle z|\widetilde{\Psi}\rangle$ as above.

The next step of the proof is to remove the lower order terms coming from the substitutions in (10.29) above.

We first consider the negative term $-4|z|^2$ in the substitution of $a_0^\dagger a_0^\dagger a_0 a_0$. By undoing the integrations leading to $\widetilde{\mathcal{K}}(z)$ for this term, we see that it contributes with

$$\int \langle \Phi(z), -4\frac{1}{2}|z|^2\ell^{-3}\Big(\widehat{g}(0)+\widehat{g\omega}(0)\Big)\Phi(z)\rangle \, d^2z \geq -Ca\ell^{-3}\langle\widetilde{\Psi}, a_0 a_0^\dagger \widetilde{\Psi}\rangle \geq -Ca\ell^{-3}(n+1), \tag{10.34}$$

in agreement with the error term in (10.27) (using that $n \approx \rho_\mu \ell^3 \gg 1$).

We also estimate the term linear in z coming from the substitution of $a_0^\dagger a_0 a_0$ in (10.29). This substitution occurs twice, but we will only explicitly treat one of them, namely, the term

$$\begin{aligned} &\Re\int \Big\langle \Phi(z), -2\ell^{-3}\widehat{W}_1(0)(2\pi)^{-3}\int \widehat{\chi}_\Lambda(k)a_k^\dagger z \, dk\Phi(z)\Big\rangle d^2z \\ &\quad = -2\ell^{-3}\widehat{W}_1(0)(2\pi)^{-3}\int \Big\langle \Phi(z), \int \widehat{\chi}_\Lambda(k)(a_k^\dagger z + a_k\overline{z}) \, dk\Phi(z)\Big\rangle d^2z \\ &\quad \geq -Ca\ell^{-3}\int \Big\langle \Phi(z), \int |\widehat{\chi}_\Lambda(k)|(\varepsilon a_k^\dagger a_k + \varepsilon^{-1}|z|^2) \, dk\Phi(z)\Big\rangle d^2z, \end{aligned} \tag{10.35}$$

where $\varepsilon > 0$ will be chosen in the end. Notice that $|\widehat{\chi}_\Lambda(k)| = \ell^3|\widehat{\chi}(k\ell)|$ and that $\widehat{\chi} \in L^1(\mathbb{R}^3)$ for $M \geq 4$. Redoing the calculation in (10.34) we therefore find with $\varepsilon = \sqrt{\langle\widetilde{\Psi}, n_+\widetilde{\Psi}\rangle}/\sqrt{n+1}$ that

$$\Re\int \Big\langle \Phi(z), -2\ell^{-3}\widehat{W}_1(0)(2\pi)^{-3}\int \widehat{\chi}_\Lambda(k)a_k^\dagger z \, dk\Phi(z)\Big\rangle d^2z \geq -Ca\ell^{-3}\sqrt{n+1}\sqrt{\langle\widetilde{\Psi}, n_+\widetilde{\Psi}\rangle}. \tag{10.36}$$

This is also easily absorbed in the error term in (10.27).

The other error terms from the substitutions are (10.29) estimated in a similar manner, and we will leave out the details.

Finally, we need to restrict to non-negative z. Suppose $z = |z|e^{i\varphi}$. In the operator $\mathcal{K}(z)$ we can replace $a_{\pm k}$ by $e^{i\varphi}a_{\pm k}$. This substitution will not affect the commutation relations, and in this way all occurrences of z will be replaced by $|z|$. This finishes the proof. □

11. **First energy bounds**

In this section we will make a rough estimate on the energy. This rough estimate will be used to eliminate the values of ρ_z that are far away from ρ_μ from the minimization problem in (10.27).

LEMMA 11.1. *For any state Φ satisfying (10.28) and assuming that $\mathcal{M} \le C^{-1}\rho_\mu \ell^3$ for some sufficiently large constant C, we have the bound*

$$\begin{aligned}\langle \Phi, \mathcal{K}(z)\Phi\rangle \ge& -\frac{\widehat{g}(0)}{2}\rho_\mu^2\ell^3 \\ &+ \frac{\widehat{g}(0)}{2}(\rho_\mu - \rho_z)^2\ell^3 - a(\rho_z+\rho_\mu)^{3/2}\rho_\mu^{1/2}\ell^3\delta_1 - \rho_z^2 a\ell^3\delta_2 \\ &- C\rho_\mu^2 a\ell^3 \frac{\rho_\mu a^3}{K_\ell^6 (ds)^6},\end{aligned} \tag{11.1}$$

with

$$\begin{aligned}\delta_1 &:= C\sqrt{\frac{\mathcal{M}}{\rho_\mu \ell^3}}\Big(K_L^3 \widetilde{K}_H^2 (\rho_\mu a^3)^{2/3}\mathcal{M} + K_L^3 K_\ell^3\Big), \\ \delta_2 &:= C\left(\frac{R^2}{\ell^2} + \frac{a}{ds\ell}\Big(1+\log\Big(\frac{ds\ell}{a}\Big)\Big)\right).\end{aligned} \tag{11.2}$$

Before we give the proof of Lemma 11.1 we will state its main consequence, Proposition 11.2 below. Our choices of parameters in Section 5 ensure that $\delta_1 + \delta_2 \ll 1$.

PROPOSITION 11.2. *Suppose that $\delta_1 + \delta_2 \le \frac{1}{2}$. Suppose furthermore that for some sufficiently large universal constant $C > 0$, we have*

$$|\rho_z - \rho_\mu| \ge C\rho_\mu \max\left(\Big(\delta_1 + \delta_2 + \frac{\rho_\mu a^3}{K_\ell^6 (ds)^6}\Big)^{\frac{1}{2}}, (\rho_\mu a^3)^{\frac{1}{4}}\right). \tag{11.3}$$

Then, for any state Φ satisfying (10.28), we have

$$\langle \Phi, \mathcal{K}(z)\Phi\rangle \ge -\frac{\widehat{g}(0)}{2}\rho_\mu^2\ell^3 + 2\rho_\mu^2 a\ell^3 \frac{128}{15\sqrt{\pi}}\sqrt{\rho_\mu a^3}. \tag{11.4}$$

Proof. Using the convexity of $t \mapsto t^\sigma$, for $\sigma \in \{3/2, 2\}$ and Jensen's inequality, (11.1) implies the bound

$$\begin{aligned}\langle \Phi, \mathcal{K}(z)\Phi\rangle \ge& -\frac{\widehat{g}(0)}{2}\rho_\mu^2\ell^3 + \frac{\widehat{g}(0)}{2}(1-\delta_1-\delta_2)(\rho_\mu-\rho_z)^2\ell^3 \\ &- C\rho_\mu^2 a\ell^3\Big(\delta_1+\delta_2+\frac{\rho_\mu a^3}{K_\ell^6 (ds)^6}\Big) \\ \ge& -\frac{\widehat{g}(0)}{2}\rho_\mu^2\ell^3 + \frac{\widehat{g}(0)}{4}(\rho_\mu-\rho_z)^2\ell^3 - C\rho_\mu^2 a\ell^3\Big(\delta_1+\delta_2+\frac{\rho_\mu a^3}{K_\ell^6 (ds)^6}\Big).\end{aligned} \tag{11.5}$$

If (11.3) is satisfied, then the term quadratic in $(\rho_\mu - \rho_z)$ dominates both the error term above and the LHY correction. This finishes the proof of Proposition 11.2. □

Proof of Lemma 11.1. Since $za_k^\dagger + \overline{z}a_k \le \delta'|z|^2 + (\delta')^{-1}a_k^\dagger a_k$ for any $\delta' > 0$, we find

$$\begin{aligned}\int \widehat{\chi_\Lambda}(k)(za_k^\dagger + \overline{z}a_k)\,dk &\le \delta'|z|^2 \int |\widehat{\chi_\Lambda}(k)|\,dk + |\widehat{\chi_\Lambda}(0)|(\delta')^{-1}\int a_k^\dagger a_k\,dk \\ &\le C(\delta'|z|^2 + (\delta')^{-1}n_+).\end{aligned} \tag{11.6}$$

Therefore, setting $\delta' = \sqrt{\mathcal{M}/(\rho_z \ell^3)}$ and using (10.28) and the definitions in (10.23) and (10.24), we easily get

$$\langle \Phi, \big(\mathcal{Q}_1(z) + \mathcal{Q}_1^{\rm ex}(z)\big)\Phi\rangle \ge -C\ell^3 a\sqrt{\frac{\mathcal{M}|z|^2}{\ell^3}}(|\rho_z - \rho_\mu| + \rho_z), \tag{11.7}$$

in agreement with (11.1) (where we used that $K_L, K_\ell \ge 1$).

Quadratic terms of the form $\ell^3\int \widehat{W}(k)a_k^\dagger a_k\,dk$ are easily estimated as

$$\pm\langle \Phi, \ell^3 \int \widehat{W}(k)a_k^\dagger a_k\,dk\Phi\rangle \le Ca\mathcal{M}. \tag{11.8}$$

This allows us to estimate all the quadratic terms in $\mathcal{K}(z)$ except the kinetic energy and the off-diagonal quadratic terms and to absorb the corresponding terms in the error in (11.1) (using, in particular, that $\mathcal{M} \le \rho_\mu \ell^3$).

Therefore, to establish (11.1) all that remains is to estimate the sum of the kinetic energy, $\mathcal{Q}_3(z)$ and the "off-diagonal" quadratic terms. This we will do by first adding and subtracting an n_+ term, which is easily estimated as above. We will prove the following three inequalities, where $\varepsilon < 1/2$ is a (small) parameter that we will optimize in the end (see (11.22)), and where Φ is a state satisfying (10.28):

$$-\left\langle \Phi, (2\pi)^{-3}\ell^3\rho_z\varepsilon^{-1/2}a\int a_k^\dagger a_k\,dk\,\Phi\right\rangle \ge -C\varepsilon^{-1/2}\ell^3\rho_z a\frac{\mathcal{M}}{\ell^3}, \tag{11.9}$$

$$\begin{aligned}&\left\langle \Phi, \left((2\pi)^{-3}\ell^3\int \varepsilon\tau(k)a_k^\dagger a_k\,dk + \mathcal{Q}_3(z)\right)\Phi\right\rangle \\ &\qquad\ge -\varepsilon^{-1}C\rho_z a\frac{\mathcal{M}}{\ell^3}\ell^3\Big(K_L^3K_H^2K_\ell^3\frac{\mathcal{M}a^3}{\ell^3} + K_L^3K_\ell^3\Big),\end{aligned} \tag{11.10}$$

and

$$\begin{aligned}&(2\pi)^{-3}\ell^3\int \Big(\mathcal{A}_1(k)a_k^\dagger a_k + \frac{1}{2}\mathcal{B}_1(k)\left(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}\right)\Big)dk \\ &\qquad\ge -\frac{1}{2}\rho_z^2\ell^3\widehat{g\omega}(0) - C\ell^3\rho_z^2 a\left(\varepsilon + \frac{R^2}{\ell^2} + \frac{a}{ds\ell}\big(1+\log\big(\frac{ds\ell}{a}\big)\big)\right) \\ &\qquad\quad - C\rho_\mu^2 a\ell^3\frac{\rho_\mu a^3}{K_\ell^6(ds)^6},\end{aligned} \tag{11.11}$$

where we have introduced

$$\mathcal{A}_1(k) := (1-\varepsilon)\tau(k) + \rho_z \varepsilon^{-1/2} a, \qquad \mathcal{B}_1(k) := \rho_z \widehat{W}_1(k). \tag{11.12}$$

The estimate (11.9) is easy given the discussion above.

We proceed to prove (11.11). We symmetrize the term in k as

$$\begin{aligned}
&(2\pi)^{-3}\ell^3 \int \Big(\mathcal{A}_1(k) a_k^\dagger a_k + \frac{1}{2}\mathcal{B}_1(k) a_k^\dagger a_{-k}^\dagger + \frac{1}{2}\mathcal{B}_1(k) a_k a_{-k}\Big)\, dk \\
=&\frac{1}{2}(2\pi)^{-3}\ell^3 \int \Big(\mathcal{A}_1(k) a_k^\dagger a_k + \mathcal{A}_1(k) a_{-k}^\dagger a_{-k} + \mathcal{B}_1(k) a_k^\dagger a_{-k}^\dagger + \mathcal{B}_1(k) a_k a_{-k}\Big)\, dk.
\end{aligned} \tag{11.13}$$

At this point we apply the "Bogolubov lemma," Lemma A.5, to get

$$\begin{aligned}
\mathcal{A}_1(k) a_k^\dagger a_k + \mathcal{A}_1(k) a_{-k}^\dagger a_{-k} + \mathcal{B}_1(k) a_k^\dagger a_{-k}^\dagger + \mathcal{B}_1(k) a_k a_{-k}& \\
\geq -\Big(\mathcal{A}_1(k) - \sqrt{\mathcal{A}_1(k)^2 - |\mathcal{B}_1(k)|^2}\Big)&,
\end{aligned} \tag{11.14}$$

where we have also used (10.7).

Using (6.13), we have $|\mathcal{B}_1(k)|/\mathcal{A}_1(k) \leq C\varepsilon^{1/2}$. Therefore, for ε sufficiently small, a Taylor expansion gives

$$-\Big(\mathcal{A}_1(k) - \sqrt{\mathcal{A}_1(k)^2 - |\mathcal{B}_1(k)|^2}\Big) \geq -\Big(\frac{1}{2} + C\varepsilon\Big)\frac{|\mathcal{B}_1|^2}{\mathcal{A}_1}. \tag{11.15}$$

Below we will need the following estimate of an integral:

$$\begin{aligned}
&\int_{\{|k|\geq (ds\ell)^{-1}\}} \frac{|\mathcal{B}_1|^2}{2\tau(k)} \\
&\quad \leq \rho_z^2 \Big(\int \frac{\widehat{W}_1(k)^2}{2k^2} + C\frac{a^2}{ds\ell}\int_{\{(ds\ell)^{-1}\leq |k| \leq a^{-1}\}} |k|^{-3} + C\frac{a}{ds\ell}\int \frac{\widehat{W}_1(k)^2}{2k^2}\Big) \\
&\quad \leq \rho_z^2\Big(1 + \frac{R^2}{\ell^2}\Big)\widehat{g\omega}(0) + C\rho_z^2 a^2 (ds\ell)^{-1}(1 + \log(ds\ell/a)),
\end{aligned} \tag{11.16}$$

where we used that $0 \leq k^2 - \tau(k) \leq 2|k|(ds\ell)^{-1}$ for $|k| \geq (ds\ell)^{-1}$ and we also used (6.15).

Inserting these considerations, we find

(11.17)

$$
\begin{aligned}
&(2\pi)^{-3}\ell^3 \int \mathcal{A}_1(k) a_k^\dagger a_k + \frac{1}{2}\mathcal{B}_1(k)\left(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}\right) dk \\
&\geq -\Big(\frac{1}{2} + C\varepsilon\Big)(2\pi)^{-3}\ell^3 \Big(\int_{|k|\leq (ds\ell)^{-1}} \varepsilon^{1/2} \frac{|\mathcal{B}_1|^2}{2\rho_z a} + \int_{|k|\geq (ds\ell)^{-1}} \frac{|\mathcal{B}_1|^2}{2(1-\varepsilon)\tau(k)} \Big) \\
&\geq -\frac{1}{2}\rho_z^2 \ell^3 \widehat{g\omega}(0) - C\ell^3 \rho_z a \Big(\varepsilon^{1/2}(ds\ell)^{-3} + \rho_z\Big(\varepsilon + \frac{R^2}{\ell^2}\Big) \\
&\qquad\qquad + C\rho_z \frac{a}{ds\ell}\Big(1 + \log\Big(\frac{ds\ell}{a}\Big)\Big)\Big) \\
&\geq -\frac{1}{2}\rho_z^2 \ell^3 \widehat{g\omega}(0) - C\ell^3 \rho_z^2 a \left(\varepsilon + \frac{R^2}{\ell^2} + \frac{a}{ds\ell}\Big(1 + \log\Big(\frac{ds\ell}{a}\Big)\Big)\right) - Ca(ds)^{-6}\ell^{-3}.
\end{aligned}
$$

This implies (11.11).

To prove (11.10) we use a similar approach. By definition (10.25), the k-integral in $\mathcal{Q}_3(z)$ is restricted to the high momentum region P_H. For these momenta, we have $\tau(k) \geq \frac{1}{2}k^2$. Therefore, dropping a part of the kinetic energy, it suffices to bound

(11.18)

$$
\ell^3(2\pi)^{-3}\int_{\{k\in P_H\}} \left(\frac{\varepsilon}{2}k^2 a_k^\dagger a_k + (2\pi)^{-3}\int f_L(s)\widehat{W}_1(k)(\overline{z}\widetilde{a}_s^\dagger a_{s-k}a_k + a_k^\dagger a_{s-k}^\dagger \widetilde{a}_s z)\, ds \right) dk.
$$

We estimate, with $\widetilde{b}_k := a_k + 2(2\pi)^{-3}\int f_L(s)\frac{\widehat{W}_1(k)}{\varepsilon k^2} z a_{s-k}^\dagger \widetilde{a}_s\, ds$,

(11.19)

$$
\begin{aligned}
&\ell^3(2\pi)^{-3}\int_{\{k\in P_H\}} \Bigg(\frac{\varepsilon}{2}k^2 a_k^\dagger a_k + (2\pi)^{-3} \\
&\qquad\qquad \cdot \int f_L(s)\widehat{W}_1(k)(\overline{z}\widetilde{a}_s^\dagger a_{s-k}a_k + a_k^\dagger a_{s-k}^\dagger \widetilde{a}_s z)\, ds \Bigg) dk \\
&= \ell^3(2\pi)^{-3}\int_{\{k\in P_H\}} \Bigg(\frac{\varepsilon}{2}k^2 \widetilde{b}_k^\dagger \widetilde{b}_k - 4(2\pi)^{-6} \\
&\qquad\qquad \cdot \iint f_L(s)f_L(s')\frac{\widehat{W}_1(k)^2}{\varepsilon k^2}|z|^2 \widetilde{a}_{s'}^\dagger a_{s'-k}a_{s-k}^\dagger \widetilde{a}_s \Bigg) dk \\
&\geq -4\varepsilon^{-1}\ell^3(2\pi)^{-9}\int_{\{k\in P_H\}} \frac{\widehat{W}_1(k)^2}{k^2}|z|^2 \\
&\qquad\qquad \cdot \iint f_L(s)f_L(s')\widetilde{a}_{s'}^\dagger (a_{s-k}^\dagger a_{s'-k} + [a_{s'-k}, a_{s-k}^\dagger])\widetilde{a}_s.
\end{aligned}
$$

On the term without a commutator, we estimate $\widetilde{a}_{s'}^{\dagger}a_{s-k}^{\dagger}a_{s'-k}\widetilde{a}_s$ by Cauchy-Schwarz and (since $k \in P_H$), $\frac{\widehat{W_1}(k)^2}{k^2} \leq CK_H^2 a^4$. Therefore, for a Φ satisfying (10.28), we find

$$\begin{aligned}
&\left\langle \Phi, \ell^3 \int_{\{k\in P_H\}} \frac{\widehat{W_1}(k)^2}{k^2}|z|^2 \iint f_L(s)f_L(s')\widetilde{a}_{s'}^{\dagger}a_{s-k}^{\dagger}a_{s'-k}\widetilde{a}_s\Phi \right\rangle \\
&\quad \leq C\rho_z \left(\int_{\{|s|\leq 2K_L\sqrt{\rho_\mu a}\}} ds\right) K_H^2 a^4 \mathcal{M}^2 \\
&\quad \leq C\rho_z a\ell^3 K_L^3 K_\ell^3 \frac{a^3 K_H^2 \mathcal{M}^2}{\ell^6}.
\end{aligned} \tag{11.20}$$

For the commutator term, we estimate (using (10.6) and the Cauchy-Schwarz inequality)

$$\widetilde{a}_{s'}^{\dagger}[a_{s'-k}, a_{s-k}^{\dagger}])\widetilde{a}_s \leq 2\widetilde{a}_{s'}^{\dagger}\widetilde{a}_{s'} + 2\widetilde{a}_s^{\dagger}\widetilde{a}_s$$

and $\int \frac{\widehat{W_1}(k)^2}{k^2} \leq Ca$. This leads to (for a Φ satisfying (10.28))

$$\begin{aligned}
&\left\langle \Phi, \ell^3 \int_{\{k\in P_H\}} \frac{\widehat{W_1}(k)^2}{k^2}|z|^2 \iint f_L(s)f_L(s')\widetilde{a}_{s'}^{\dagger}[a_{s'-k}, a_{s-k}^{\dagger}]\widetilde{a}_s\Phi \right\rangle \\
&\quad \leq C\mathcal{M}a|z|^2 \int_{\{|s|\leq 2K_L\sqrt{\rho_\mu a}\}} ds \\
&\quad \leq Ca\rho_z \frac{\mathcal{M}}{\ell^3}\ell^3 K_L^3 K_\ell^3.
\end{aligned} \tag{11.21}$$

Combining the estimates (11.19), (11.20) and (11.21) proves (11.10).

We choose

$$\varepsilon = \frac{\mathcal{M}^{1/2}}{\sqrt{(\rho_\mu + \rho_z)\ell^3}}. \tag{11.22}$$

We will add the estimates of (11.9), (11.10) and (11.11) with this choice of ε. Since $\mathcal{M} \leq \rho_\mu \ell^3$, the contribution from (11.9) will be smaller than the terms appearing in the other estimates. Therefore, we get,

(11.23)
$$\begin{aligned}
&\left\langle \Phi, \left(\frac{1}{2}\rho_z^2\ell^3\widehat{g\omega}(0) + (2\pi)^{-3}\ell^3 \int \tau(k)a_k^{\dagger}a_k\,dk + \mathcal{Q}_3(z)\right)\Phi \right\rangle \\
&\quad \geq -C\rho_z a\ell^3 \left(\frac{\sqrt{\mathcal{M}(\rho_\mu+\rho_z)\ell^3}}{\ell^3}\left(K_L^3 K_H^2(\rho_\mu a^3)^{3/2}\mathcal{M} + K_L^3 K_\ell^3\right) + \rho_z \frac{R^2}{\ell^2}\right).
\end{aligned}$$

This finishes the proof of (11.1). □

12. More precise energy estimates

From Proposition 11.2 above, we see that the energy is too high unless $\rho_z \approx \rho_\mu$. We now focus on this regime. More precisely, we will in this section always assume that

$$|\rho_z - \rho_\mu| \leq \rho_\mu C \max\left(\left(\delta_1 + \delta_2 + \frac{\rho_\mu a^3}{K_\ell^6 (ds)^6}\right)^{\frac{1}{2}}, (\rho_\mu a^3)^{\frac{1}{4}}\right), \tag{12.1}$$

with the notation from Proposition 11.2.

We will need the condition that

$$K_\ell^2 \max\left(\left(\delta_1 + \delta_2 + \frac{\rho_\mu a^3}{K_\ell^6 (ds)^6}\right)^{\frac{1}{2}}, (\rho_\mu a^3)^{\frac{1}{4}}\right) \leq C^{-1} \tag{12.2}$$

for some sufficiently large universal constant. This condition is satisfied by (5.19), (5.20), (5.6) and (5.8).

Using (12.1) and (12.2) we have

$$\frac{|\rho_z - \rho_\mu|}{\rho_\mu} \leq C^{-1} K_\ell^{-2}. \tag{12.3}$$

For convenience of notation, we define the parameter δ to be the square of the ratio between $\sqrt{\rho_\mu a}$ and the inner radius of P_H, i.e.,

$$\delta := \frac{\rho_\mu a}{K_H^{-2} a^{-2}} = (\rho_\mu a^3)^{\frac{1}{6}} \widetilde{K}_H^2. \tag{12.4}$$

Using (5.18), we see that $\delta \ll 1$.

We define the quadratic Bogolubov Hamiltonian as follows:

$$\begin{aligned}\mathcal{K}^{\mathrm{Bog}} = \frac{1}{2}(2\pi)^{-3}\ell^3 \int \Big(&\mathcal{A}(k)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \mathcal{B}(k)(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}) \\ &+ \mathcal{C}(k)(a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k})\Big)\, dk,\end{aligned} \tag{12.5}$$

with

$$\begin{aligned}\mathcal{A}(k) &:= \tau(k) + \rho_z \widehat{W_1}(k), \quad \mathcal{B}(k) := \rho_z \widehat{W_1}(k), \\ \mathcal{C}(k) &:= \ell^{-3}(\rho_z - \rho_\mu)\widehat{W_1}(0)\widehat{\chi_\Lambda}(k) z.\end{aligned} \tag{12.6}$$

With this notation, we can rewrite/estimate $\mathcal{K}(z)$ from (10.22) as follows:

$$\begin{aligned}\mathcal{K}(z) &= \mathcal{K}^{\mathrm{Bog}} + \frac{1}{2}\rho_z^2\ell^3\Big(\widehat{g}(0)+\widehat{g\omega}(0)\Big) - \rho_\mu\widehat{g}(0)\rho_z\ell^3 \\ &\quad + \frac{b}{2\ell^2}n_+ + \varepsilon_T\frac{b}{2d^2\ell^2}n_+^H + (\rho_z-\rho_\mu)\widehat{W}_1(0)(2\pi)^{-3}\ell^3\int a_k^\dagger a_k\,dk \\ &\quad + \mathcal{Q}_1^{\mathrm{ex}}(z) + \mathcal{Q}_2^{\mathrm{ex}}(z) + \mathcal{Q}_3(z) \\ &\geq -\frac{1}{2}\rho_\mu^2\ell^3\widehat{g}(0) + \frac{1}{2}\rho_z^2\ell^3\widehat{g\omega}(0) + \frac{1}{2}(\rho_z-\rho_\mu)^2\ell^3\widehat{g}(0) + \mathcal{K}^{\mathrm{Bog}} \\ &\quad + \frac{b}{4\ell^2}n_+ + \varepsilon_T\frac{b}{2d^2\ell^2}n_+^H + \mathcal{Q}_1^{\mathrm{ex}}(z) + \mathcal{Q}_2^{\mathrm{ex}}(z) + \mathcal{Q}_3(z).\end{aligned} \tag{12.7}$$

Here we used (12.2) to absorb a quadratic part in the gap.

12.1. *The Bogolubov Hamiltonian.*

THEOREM 12.1 (Analysis of Bogolubov Hamiltonian). *Assume Φ satisfies* (10.28) *and that $\frac{1}{2}\rho_\mu \leq \rho_z \leq 2\rho_\mu$. Let δ be the parameter defined in* (12.4). *Then,*

$$\begin{aligned}\langle\Phi,\mathcal{K}^{\mathrm{Bog}}\Phi\rangle &\geq (2\pi)^{-3}\ell^3\left\langle\Phi,\int\mathcal{D}_k b_k^\dagger b_k\,dk\,\Phi\right\rangle \\ &\quad - \frac{1}{2}(2\pi)^{-3}\ell^3\int\left(\mathcal{A}(k)-\sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2}\right)dk \\ &\quad - (\rho_z-\rho_\mu)^2\frac{\widehat{g}(0)}{2}\ell^3\left(1+C\frac{R^2}{a^2}(\rho_\mu a^3)\right) \\ &\quad - C\rho_\mu^2 a\ell^3 K_{\mathcal{M}}K_\ell^{-3/2}(K_\ell^2\delta)^{\frac{M-5}{2}}.\end{aligned} \tag{12.8}$$

Here

$$\mathcal{D}_k := \frac{1}{2}\left(\mathcal{A}(k)+\sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2}\right) \tag{12.9}$$

and

$$b_k := a_k + \alpha_k a_{-k}^\dagger + c_k, \tag{12.10}$$

with

$$\alpha_k := \mathcal{B}(k)^{-1}\left(\mathcal{A}(k)-\sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2}\right) \tag{12.11}$$

and

$$c_k := \begin{cases}\frac{2\mathcal{C}(k)}{\mathcal{A}(k)+\mathcal{B}(k)+\sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2}}, & |k|\leq\frac{1}{2}K_H^{-1}a^{-1},\\ 0, & |k|>\frac{1}{2}K_H^{-1}a^{-1}.\end{cases} \tag{12.12}$$

Proof. To simplify later calculations we start by removing $\mathcal{C}(k)$ for $|k| > \frac{1}{2}K_H^{-1}a^{-1}$ from $\mathcal{K}^{\text{Bog}}$, so we aim to prove

$$\begin{aligned}\frac{1}{2}(2\pi)^{-3}\ell^3 \int_{\{|k|>\frac{1}{2}K_H^{-1}a^{-1}\}} &\mathcal{C}(k)(a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k})\,dk \\ &\geq -C\rho_\mu^2 a\ell^3 K_{\mathcal{M}} K_\ell^{-3/2}(K_\ell^2\delta)^{\frac{M-5}{2}}.\end{aligned} \tag{12.13}$$

Obviously,

$$a_k + a_k^\dagger \leq a_k^\dagger a_k + 1.$$

Therefore,

$$\begin{aligned}&\frac{1}{2}(2\pi)^{-3}\ell^3 \int_{\{|k|>\frac{1}{2}K_H^{-1}a^{-1}\}} \mathcal{C}(k)(a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k})\,dk \\ &\geq -(2\pi)^{-3}|\rho_z - \rho_\mu|\widehat{W_1}(0)|z| \int_{\{|k|>\frac{1}{2}K_H^{-1}a^{-1}\}} |\widehat{\chi_\Lambda}(k)|(a_k^\dagger a_k + 1)\,dk \\ &\geq -C|\rho_z - \rho_\mu|\widehat{W_1}(0)|z|(n_+ + 1)\varepsilon(\chi),\end{aligned} \tag{12.14}$$

where

$$\varepsilon(\chi) := \ell^{-3} \sup_{\{|k|>\frac{1}{2}K_H^{-1}a^{-1}\}} (1+(k\ell)^2)^2|\widehat{\chi_\Lambda}(k)| \leq C(K_\ell^{-2}\delta)^{\widetilde{M}-2}, \tag{12.15}$$

where we used Lemma C.1 to get the last estimate. Estimating n_+ using (10.28) and using (12.1) to control $|z|$, it is elementary to conclude (12.13).

By the estimate above, it suffices to consider

$$\begin{aligned}\widetilde{\mathcal{K}}^{\text{Bog}} := \frac{1}{2}(2\pi)^{-3}\ell^3 \int \Big(&\mathcal{A}(k)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \mathcal{B}(k)(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}) \\ &+ \widetilde{\mathcal{C}}(k)(a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k})\Big)\,dk,\end{aligned} \tag{12.16}$$

with $\mathcal{A}, \mathcal{B}$ from (12.5) and

$$\widetilde{\mathcal{C}}(k) := \begin{cases} 0, & |k| \geq \frac{1}{2}K_H^{-1}a^{-1}, \\ \ell^{-3}(\rho_z - \rho_\mu)\widehat{W_1}(0)\widehat{\chi_\Lambda}(k)z, & \text{otherwise.}\end{cases} \tag{12.17}$$

With the notation from Theorem 12.1 and using Theorem A.1 combined with (10.7) we find

$$\begin{aligned}\widetilde{\mathcal{K}}^{\text{Bog}} &\geq (2\pi)^{-3}\ell^3 \int \mathcal{D}_k b_k^\dagger b_k\,dk \\ &\quad - \frac{1}{2}(2\pi)^{-3}\ell^3 \int \Big(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\Big)\,dk \\ &\quad - (\rho_z - \rho_\mu)^2\widehat{W_1}(0)^2 z^2 (2\pi)^{-3}\ell^{-3} \int_{\{|k|\leq\frac{1}{2}K_H^{-1}a^{-1}\}} \frac{|\widehat{\chi}_\Lambda(k)|^2}{\mathcal{A}(k)+\mathcal{B}(k)}.\end{aligned} \tag{12.18}$$

It is elementary, using that W_1 is even, that

$$\left|\widehat{W}_1(k) - \widehat{W}_1(0)\right| \le Ca(kR)^2. \tag{12.19}$$

Therefore, we easily get the lower bound

$$\mathcal{A}(k) + \mathcal{B}(k) \ge 2\rho_z \widehat{W_1(0)} \left(1 - C(\rho_\mu a^3)\frac{R^2}{a^2}\right), \tag{12.20}$$

using that the kinetic energy is dominating, unless $|k| \le C\sqrt{\rho_\mu a}$.

Therefore, the last term in (12.18) becomes controlled as

$$\begin{aligned}
(\rho_z - \rho_\mu)^2 \widehat{W_1}(0)^2 z^2 (2\pi)^{-3}\ell^{-3} &\int_{\{|k|\le \frac{1}{2}K_H^{-1}a^{-1}\}} \frac{|\widehat{\chi}_\Lambda(k)|^2}{\mathcal{A}(k)+\mathcal{B}(k)} \\
&\le (\rho_z-\rho_\mu)^2 \frac{\widehat{W_1}(0)}{2}\ell^3\left(1 + C(\rho_\mu a^3)\frac{R^2}{a^2}\right) \\
&\le (\rho_z-\rho_\mu)^2 \frac{\widehat{g}(0)}{2}\ell^3\left(1 + C(\rho_\mu a^3)\frac{R^2}{a^2}\right),
\end{aligned} \tag{12.21}$$

where we used that $\ell^{-2} \ll \rho_\mu a$ to get the last estimate.

This finishes the proof of Theorem 12.1. □

Remark 12.2. We notice that following commutation relations (using the ones for the a_k's (10.6) and that $\widehat{\chi}$ is even and real),

$$[b_k, b_{k'}] = (\alpha_k - \alpha_{k'})\Big(\widehat{\chi^2}((k+k')\ell) - \widehat{\chi}(k\ell)\widehat{\chi}(k'\ell)\Big). \tag{12.22}$$

Also,

$$[b_k, b_{k'}^\dagger] = (1 - \alpha_k\alpha_{k'})\Big(\widehat{\chi^2}((k-k')\ell) - \widehat{\chi}(k\ell)\widehat{\chi}(k'\ell)\Big). \tag{12.23}$$

LEMMA 12.3. *Assume that* (12.1) *holds and that* $\frac{9}{10}\rho_\mu \le \rho_z \le \frac{11}{10}\rho_\mu$. *We have the estimate*

$$\begin{aligned}
-\frac{1}{2}(2\pi)^{-3}\ell^3 \int \Big(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\Big)\,dk& \\
\ge -\frac{\widehat{g\omega}(0)}{2}\rho_z^2\ell^3 + 4\pi\frac{128}{15\sqrt{\pi}}\rho_z^2 a\sqrt{\rho_z a^3}\ell^3& \\
- C\varepsilon(\rho_\mu,\rho_z)\rho_z^2 a\sqrt{\rho_z a^3}\ell^3 - C\rho_z^2\ell^3\frac{R^2}{\ell^2}\widehat{g\omega}(0),&
\end{aligned} \tag{12.24}$$

with C *a universal constant and*

(12.25)

$$\begin{aligned}
\varepsilon(\rho_\mu,\rho_z) = (\rho_\mu a)^{\frac14}\sqrt{R} + \varepsilon_T + (K_\ell s)^{-1}\left(1 + \log(d^{-1}) + \log\left(\frac{K_\ell d s}{(\rho_\mu a^3)^{1/2}}\right)\right)& \\
+ \varepsilon_T (K_\ell d s)^{-1}\left(1 + \log\left(\frac{K_\ell d s}{(\rho_\mu a^3)^{1/2}}\right)\right).&
\end{aligned}$$

Proof. We regularize the integral as

(12.26)
$$\begin{aligned}&\int \mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\, dk \\ &\qquad = \int \mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} - \rho_z^2 \frac{\widehat{W_1}(k)^2}{2k^2}\, dk + \rho_z^2 \int \frac{\widehat{W_1}(k)^2}{2k^2}\, dk.\end{aligned}$$

The last integral is controlled by (6.15) and contributes with the first and the last term in (12.24).

In the regularized integral in (12.26) we perform the change of variables $\sqrt{\rho_z a}\, t = k$. In this way we get

$$\int \mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} - \rho_z^2 \frac{\widehat{W_1}(k)^2}{2k^2}\, dk = \rho_z^2 \sqrt{\rho_z a^3} a I_1, \tag{12.27}$$

with

$$\begin{aligned}I_1 &= \int \alpha(t) - \sqrt{\alpha(t)^2 - \beta(t)^2} - \frac{\widehat{W_1}(\sqrt{\rho_z a} t)^2}{2a^2 t^2}\, dt,\\ \alpha(t) &= \widetilde{\tau}(t) + a^{-1}\widehat{W_1}(\sqrt{\rho_z a} t),\\ \beta(t) &= a^{-1}\widehat{W_1}(\sqrt{\rho_z a} t),\\ \widetilde{\tau}(t) &= (1-\varepsilon_T)\Big[|t| - \frac{1}{2K_\ell s}(\frac{\rho_\mu}{\rho_z})^{1/2}\Big]_+^2 + \varepsilon_T \Big[|t| - \frac{1}{2K_\ell d s}(\frac{\rho_\mu}{\rho_z})^{1/2}\Big]_+^2.\end{aligned} \tag{12.28}$$

We will prove that $I_1 \approx -64\pi^4 \frac{128}{15\sqrt{\pi}}$ with an error estimated by $\varepsilon(\rho_\mu, \rho_z)$ from (12.25). For this we write I_1 as

$$\begin{aligned}I_1 &= \int \alpha(t) - \sqrt{\alpha(t)^2 - \beta(t)^2} - \frac{\beta^2}{2t^2}\\ &= \int \alpha(t) - \frac{\beta^2}{2\alpha} - \sqrt{\alpha(t)^2 - \beta(t)^2} + \Big(\frac{\beta^2}{2\alpha} - \frac{\beta^2}{2t^2}\Big)\\ &= \int \alpha(t) - \frac{\beta^2}{2\alpha} - \sqrt{\alpha(t)^2 - \beta(t)^2} + \frac{\beta^2}{2}\frac{t^2 - \widetilde{\tau} - \beta}{t^2\alpha}\\ &= I_1' + I_1'',\end{aligned} \tag{12.29}$$

with

$$\begin{aligned}I_1' &:= \int \alpha(t) - \frac{\beta^2}{2\alpha} - \sqrt{\alpha(t)^2 - \beta(t)^2} - \frac{\beta^3}{2t^2\alpha},\\ I_1'' &:= \int \frac{\beta^2}{2}\frac{t^2 - \widetilde{\tau}}{t^2\alpha}.\end{aligned} \tag{12.30}$$

It is not difficult to apply dominated convergence to the integral I_1' to get

$$I_1' \approx \int_{\mathbb{R}^3} t^2 + 8\pi - \frac{(8\pi)^2}{2t^2} - \sqrt{(t^2+8\pi)^2 - (8\pi)^2}\, dt = -64\pi^4 \frac{128}{15\sqrt{\pi}}. \tag{12.31}$$

More precisely, we will prove that

$$\left| I_1' - \int_{\mathbb{R}^3} t^2 + 8\pi - \frac{(8\pi)^2}{2t^2} - \sqrt{(t^2+8\pi)^2 - (8\pi)^2}\, dt \right| \leq C\left((\rho_\mu a)^{\frac14}\sqrt{R} + \varepsilon_T + (K_\ell s)^{-1}\right). \tag{12.32}$$

This estimate is included in the error defined in (12.25).

The part of both integrals where $|t| \leq 10(K_\ell s)^{-1}$ is bounded by

$$C(K_\ell s)^{-1}$$

for sufficiently small ρ_μ (using that $\rho_z \approx \rho_\mu$). This is in agreement with (12.32).

For $|t| \geq 10(K_\ell s)^{-1}$, we will use

$$|\beta(t) - 8\pi| \leq C\sqrt{\rho_\mu a} R|t|, \qquad 0 \leq t^2 - \widetilde{\tau}(t) \leq \varepsilon_T t^2 + \frac{1}{K_\ell s}\left(\frac{\rho_\mu}{\rho_z}\right)^{1/2}|t|.t \tag{12.33}$$

It follows by interpolation that $|\beta(t) - 8\pi| \leq C(\rho_\mu a)^{\frac14} R^{\frac12}|t|^{\frac12}$ and also that $\widetilde{\tau} \geq \frac12 t^2$ when ε_T is sufficiently small (since $\frac{\rho_\mu}{\rho_z}$ is close to 1).

For $|t| \geq 100$ we use Taylor's formula with remainder (applied to $\sqrt{1-x}$) to write

$$\begin{aligned}
&\int_{\{|t|\geq 10(K_\ell s)^{-1}\}} \alpha(t) - \frac{\beta^2}{2\alpha} - \sqrt{\alpha(t)^2 - \beta(t)^2} - \frac{\beta^3}{2t^2\alpha}\, dt \\
&\qquad - \int_{\{|t|\geq 10(K_\ell s)^{-1}\}} t^2 + 8\pi - \frac{(8\pi)^2}{2t^2} - \sqrt{(t^2+8\pi)^2 - (8\pi)^2}\, dt \\
&= \int_{\{10(K_\ell s)^{-1} \leq |t| \leq 100\}} \Big((\alpha - t^2 - 8\pi) - \Big(\frac{\beta^2}{2\alpha} - \frac{(8\pi)^2}{2(t^2+8\pi)}\Big) \\
&\qquad\qquad - \Big(\sqrt{\alpha(t)^2 - \beta(t)^2} - \sqrt{(t^2+8\pi)^2 - (8\pi)^2}\Big)\Big)\, dt \\
&\quad + \int_{\{|t|\geq 100\}} \int_0^1 f(\widetilde{\tau}, \beta, \sigma) - f(t^2, 8\pi, \sigma)\, d\sigma\, dt \\
&\quad - \int_{\{|t|\geq 10(K_\ell s)^{-1}\}} \frac{\beta^3}{2t^2\alpha} - \frac{(8\pi)^3}{2t^2(t^2+8\pi)}\, dt,
\end{aligned} \tag{12.34}$$

with

$$f(\tau, \beta, \sigma) := \frac{-\beta^4}{4}[\tau^2 + 2\beta\tau + (1-\sigma)\beta^2]^{-3/2}(1-\sigma). \tag{12.35}$$

The last integral in (12.34) is easily estimated, as

(12.36)

$$\left| \int_{\{|t|\geq 10(K_\ell s)^{-1}\}} \frac{\beta^3}{2t^2\alpha} - \frac{(8\pi)^3}{2t^2(t^2+8\pi)}\, dt \right| \leq C\left((\rho_\mu a)^{\frac14}\sqrt{R} + \varepsilon_T + (K_\ell s)^{-1}\right),$$

in agreement with (12.32).

For the Taylor expansion part in (12.34), we use that $\widetilde{\tau}^2+2\beta\widetilde{\tau}+(1-\sigma)\beta \geq \frac{1}{4}t^4$, when $|t| \geq 100$. Therefore,

$$\begin{aligned}
&\Big|f(\widetilde{\tau},\beta,\sigma) - f(t^2,8\pi,\sigma)\Big| \\
&\quad \leq C|\beta^4-(8\pi)^4|t^{-6} \\
&\qquad + Ct^2\Big|[\widetilde{\tau}^2+2\beta\widetilde{\tau}+(1-\sigma)\beta^2]^{-2} - [t^4+16\pi t^2+(1-\sigma)(8\pi)^2]^{-2}\Big| \\
&\qquad + Ct^{-8}\frac{(\widetilde{\tau}^2+2\beta\widetilde{\tau}+(1-\sigma)\beta^2)-(t^4+16\pi t^2+(1-\sigma)(8\pi)^2)}{\sqrt{\widetilde{\tau}^2+2\beta\widetilde{\tau}+(1-\sigma)\beta^2}+\sqrt{t^4+16\pi t^2+(1-\sigma)(8\pi)^2}}.
\end{aligned} \tag{12.37}$$

Now the integrals can easily be estimated to get an error consistent with (12.32).

Finally, we consider the integral over $\{10(K_\ell s)^{-1} \leq |t| \leq 100\}$ in (12.34). Here one may estimate term by term and use the finiteness of the domain of integration. Therefore, this part is also consistent with (12.32), which finishes the proof of (12.32).

The integral I_1'' from (12.30) is split in three parts. For $|t| \leq 10(K_\ell s)^{-1}$, we have $0 \leq t^2 - \widetilde{\tau}(t) \leq t^2$. Therefore,

$$\left|\int_{\{|t|\leq 10(K_\ell s)^{-1}\}} \frac{\beta^2}{2}\frac{t^2-\widetilde{\tau}}{t^2\alpha}\right| \leq C(K_\ell s)^{-1}, \tag{12.38}$$

which is again included in the error defined in (12.25).

For $10(K_\ell s)^{-1} \leq |t| \leq 10(K_\ell ds)^{-1}$, we have (12.33) above. Therefore,

$$\left|\int_{\{10(K_\ell s)^{-1}\leq|t|\leq 10(K_\ell ds)^{-1}\}} \frac{\beta^2}{2}\frac{t^2-\widetilde{\tau}}{t^2\alpha}\right| \leq C\varepsilon_T(K_\ell ds)^{-1} + C(K_\ell s)^{-1}\log(d^{-1}), \tag{12.39}$$

which may again be absorbed in (12.25).

Finally, we turn to the case $|t| \geq 10(K_\ell ds)^{-1}$. Here, $0 \leq t^2 - \widetilde{\tau}(t) \leq C|t|((K_\ell s)^{-1} + \varepsilon_T(K_\ell ds)^{-1})$ and $\alpha \geq \frac{1}{2}t^2$. Therefore,

$$\begin{aligned}
&\left|\int_{\{|t|\geq 10(K_\ell ds)^{-1}\}} \frac{\beta^2}{2}\frac{t^2-\widetilde{\tau}}{t^2\alpha}\right| \\
&\quad \leq C((K_\ell s)^{-1}+\varepsilon_T(K_\ell ds)^{-1})\int_{\{10(K_\ell ds)^{-1}\leq|t|\leq(\rho_z a^3)^{-1/2}\}} |t|^{-3} \\
&\qquad + C((K_\ell s)^{-1}+\varepsilon_T(K_\ell ds)^{-1})(\rho_z a^3)^{1/2}a^{-2}\int \frac{\widehat{W_1}(\sqrt{\rho_z}at)^2}{t^2} \\
&\quad \leq C((K_\ell s)^{-1}+\varepsilon_T(K_\ell ds)^{-1})\left(\log\left(\frac{K_\ell ds}{(\rho_\mu a^3)^{1/2}}\right)+1\right).
\end{aligned} \tag{12.40}$$

This may again be absorbed in (12.25). This finishes the proof of Lemma 12.3. □

12.2. *The control of* $\mathcal{Q}_3(z)$. The diagonalized quadratic Bogolubov Hamiltonian $(2\pi)^{-3}\ell^3\int \mathcal{D}_k b_k^\dagger b_k\,dk$ from (12.8) turns out to control the $3Q$-term $\mathcal{Q}_3(z)$ from (10.25). This we summarize as follows

THEOREM 12.4. *Assume that* Φ *satisfies* (10.28). *Assume furthermore that* (12.1) *and* (5.29) *are satisfied. Let* δ *be as defined in* (12.4). *We will furthermore assume* (5.4), (5.11), (5.18), (5.19), (5.22), *and* (5.23).

Then,

(12.41)
$$\Big\langle \Phi, \Big((2\pi)^{-3}\ell^3 \int \mathcal{D}_k b_k^\dagger b_k\,dk + \mathcal{Q}_3(z) + \mathcal{Q}_2^{\rm ex} + \rho_z z\widehat{W_1\omega}(0)(2\pi)^{-3}\int \widehat{\chi_\Lambda^2}(s)(\widetilde{a}_s^\dagger+\widetilde{a}_s)\,ds + \frac{b}{50}\Big(\frac{1}{\ell^2}n_+ + \frac{\varepsilon_T}{(d\ell)^2}n_+^H\Big)\Big)\Phi\Big\rangle \geq -C\rho_\mu^2 a\ell^3\Bigg[\sqrt{\frac{\mathcal{M}}{|z|^2}}\Big(\widetilde{K}_H^{-1}(\rho_\mu a^3)^{\frac{5}{12}} + (K_\ell K_L)^{-M} + K_\ell^3K_L^3(K_\ell^{-2}\delta)^{\frac{M-1}{2}}\Big) + \delta^4\frac{a}{\ell} + \sqrt{\rho_\mu a^3}\Big(K_\ell^{-3}d^{-12}\delta^2\big(K_\ell^{-2}\delta\big)^{M-1}\Big)\Bigg].$$

Proof of Theorem 12.4. Notice that

$$|\mathcal{B}(k)/\mathcal{A}(k)| \leq C\delta \qquad \forall\, |k| \geq \frac{1}{2}\widetilde{K}_H^{-1}(\rho_\mu a^3)^{5/12}a^{-1}. \tag{12.42}$$

In particular, $|\mathcal{B}(k)/\mathcal{A}(k)| \leq \frac{1}{2}$ for ρ_μ sufficiently small.

This implies, by expansion of the square root, that

$$|\alpha_k| = |\mathcal{B}(k)|^{-1}\Big(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2}\Big)| \leq C\delta \tag{12.43}$$

for all $|k| \geq \frac{1}{2}\widetilde{K}_H^{-1}(\rho_\mu a^3)^{5/12}a^{-1}$. In particular, (12.42) and (12.43) are valid for $k = k'-s$, when $s \in P_L$ and $k' \in P_H$.

For later convenience, we reformulate the first-order operator in (12.41) in terms of the $\widetilde{a}_s$. (Recall definitions (10.2) and (10.4).) We get

$$\begin{aligned} &-\rho_z z\widehat{W_1\omega}(0)(2\pi)^{-3}\int \widehat{\chi_\Lambda}(s)(a_s^\dagger + a_s)\,ds \\ &\quad= -\rho_z z\widehat{W_1\omega}(0)(2\pi)^{-3}\int \widehat{\chi_\Lambda^2}(s)(\widetilde{a}_s^\dagger+\widetilde{a}_s)\,ds \\ &\quad= -\rho_z z\widehat{W_1\omega}(0)(2\pi)^{-3}\ell^3\int \widehat{\chi^2}(s\ell)(\widetilde{a}_s^\dagger+\widetilde{a}_s)\,ds. \end{aligned} \tag{12.44}$$

We start by rewriting $\mathcal{Q}_3(z)$ in terms of the b_k's defined in (12.10). Notice that $c_k, c_{s-k} = 0$ if $k \in P_H$ and $s \in P_L$. We find the basic relation (we will freely use that all involved functions are symmetric, e.g., $\alpha_k = \alpha_{-k}$)

$$a_{s-k} = \frac{1}{1-\alpha_{s-k}^2}\Big(b_{s-k} - \alpha_{s-k} b_{k-s}^\dagger\Big), \quad a_k = \frac{1}{1-\alpha_k^2}\Big(b_k - \alpha_k b_{-k}^\dagger\Big). \tag{12.45}$$

Therefore,

$$\begin{aligned} a_{s-k}a_k = \frac{1}{1-\alpha_k^2}\frac{1}{1-\alpha_{s-k}^2}\Big(& b_{s-k}b_k - \alpha_k b_{-k}^\dagger b_{s-k} - \alpha_{s-k} b_{k-s}^\dagger b_k \\ &+ \alpha_k\alpha_{s-k} b_{k-s}^\dagger b_{-k}^\dagger - \alpha_k [b_{s-k}, b_{-k}^\dagger]\Big). \end{aligned} \tag{12.46}$$

We will decompose $\mathcal{Q}_3(z)$ according to the different terms in (12.46), i.e.,

$$\mathcal{Q}_3(z) = \mathcal{Q}_3^{(1)}(z) + \mathcal{Q}_3^{(2)}(z) + \mathcal{Q}_3^{(3)}(z) + \mathcal{Q}_3^{(4)}(z), \tag{12.47}$$

where

(12.48)

$$\begin{aligned} \mathcal{Q}_3^{(1)}(z) &:= z\ell^3(2\pi)^{-6}\iint_{\{k\in P_H\}} \frac{f_L(s)\widehat{W}_1(k)}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)} \\ &\qquad \times \Big(\widetilde{a}_s^\dagger b_{s-k}b_k + \alpha_k\alpha_{s-k}\widetilde{a}_s^\dagger b_{k-s}^\dagger b_{-k}^\dagger + \text{h.c.}\Big), \\ \mathcal{Q}_3^{(2)}(z) &:= -z\ell^3(2\pi)^{-6}\iint_{\{k\in P_H\}} \frac{f_L(s)\widehat{W}_1(k)\alpha_k}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)}\Big(\widetilde{a}_s^\dagger b_{-k}^\dagger b_{s-k} + b_{s-k}^\dagger b_{-k}\widetilde{a}_s\Big), \\ \mathcal{Q}_3^{(3)}(z) &:= -z\ell^3(2\pi)^{-6}\iint_{\{k\in P_H\}} \frac{f_L(s)\widehat{W}_1(k)\alpha_{s-k}}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)}\Big(\widetilde{a}_s^\dagger b_{k-s}^\dagger b_k + b_k^\dagger b_{k-s}\widetilde{a}_s\Big), \end{aligned}$$

and

$$\mathcal{Q}_3^{(4)}(z) := (2\pi)^{-6}z\ell^3\iint_{k\in P_H} f_L(s)\widehat{W}_1(k)\frac{-\alpha_k}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)}[b_{s-k}, b_{-k}^\dagger](\widetilde{a}_s^\dagger + \widetilde{a}_s).$$

The different $\mathcal{Q}_3^{(j)}(z)$'s will be estimated individually. The result of this is summarized in Lemma 12.5. Theorem 12.4 follows by adding the estimates of Lemma 12.5. We have used that the K's ($K_\ell, K_{\mathcal{M}}, \widetilde{K}_H$, and K_B) are larger than 1 and (5.11) to simplify the total remainder. This finishes the proof. □

LEMMA 12.5. *Let δ be as defined in* (12.4). *Assume that Φ satisfies* (10.28). *Assume furthermore that* (5.29), (12.1), (5.18), (5.19), (5.22), (5.4) *and* (5.23) *are satisfied. Then,*

(12.49)
$$\Big\langle \Phi, \Big(\mathcal{Q}_3^{(1)}(z) + (1-\delta^2)(2\pi)^{-3}\ell^3 \int_{\{|k|\geq \frac{1}{2}K_H^{-1}a^{-1}\}} \mathcal{D}_k b_k^\dagger b_k + \mathcal{Q}_2^{\text{ex}} + \frac{b}{100}\left(\frac{1}{\ell^2}n_+ + \frac{\varepsilon_T}{(d\ell)^2}n_+^H\right)\Big)\Phi\Big\rangle$$
$$\geq -C\rho_\mu^2 a\ell^3\delta K_\ell^{-3/2}(\rho_\mu a^3)^{\frac14}(\mathcal{M} + (K_\ell^3 K_L^3))\left(K_\ell^{-2}\delta\right)^{\frac{M-1}{2}} - C\rho_\mu^2 a\ell^3\sqrt{\rho_\mu a^3}\left(K_\ell^{-3}d^{-12}\delta^2\left(K_\ell^{-2}\widetilde{K}_H^2(\rho_\mu a^3)^{\frac16}\right)^{M-1}\right),$$

(12.50)
$$\Big\langle \Phi, \Big(\mathcal{Q}_3^{(2)}(z) + \mathcal{Q}_3^{(3)}(z) + \delta^2(2\pi)^{-3}\ell^3 \int_{\{|k|\geq \frac{1}{2}K_H^{-1}a^{-1}\}} \mathcal{D}_k b_k^\dagger b_k\Big)\Phi\Big\rangle \geq -C\rho_\mu^2 a\ell^3(K_\ell^{-2}\delta)^{\widetilde{M}}K_L^3K_\ell^{3/2}(\rho_\mu a^3)^{\frac14},$$

(12.51)
$$\Big\langle \Phi, \Big(\mathcal{Q}_3^{(4)}(z) + \rho_z z\widehat{W_1\omega}(0)(2\pi)^{-3}\int \widehat{\chi_\Lambda^2}(s)(\widetilde{a}_s^\dagger + \widetilde{a}_s)\,ds + \frac{1}{100}\frac{b}{\ell^2}n_+\Big)\Phi\Big\rangle$$
$$\geq -C\rho_z^2 a\ell^3\sqrt{\frac{\mathcal{M}}{|z|^2}}\Big(\widetilde{K}_H^{-1}(\rho_\mu a^3)^{\frac{5}{12}} + (K_\ell K_L)^{-M} + K_\ell^{3/2}K_L^{3/2}(K_\ell^{-2}\delta)^{\frac{M-1}{2}}\Big) - C\rho_\mu^2 a\ell^3\delta^4\frac{a}{\ell}.$$

Proof of Lemma 12.5. The proofs of (12.49), (12.50) and (12.51) are each rather lengthy and will be carried out individually.

Proof of (12.51). Using Lemma C.1 applied to χ^2 we have

$$\Big\|\widehat{\chi_\Lambda^2}(s)\Big(1-f_L(s)\Big)\Big\|_\infty \leq C_0\ell^3(1+(K_\ell K_L)^2)^{-M}, \tag{12.52}$$

with $C_0 = \int \left|(1-\Delta)^M\chi^2\right|$. Therefore, by a simple application of the Cauchy-Schwarz inequality, we get for any state Φ satisfying (10.28),

$$\Big|\Big\langle \Phi, \int \widehat{\chi_\Lambda^2}(s)(\widetilde{a}_s^\dagger + \widetilde{a}_s)\,ds\,\Phi\Big\rangle\Big| \leq C\sqrt{\mathcal{M}} \tag{12.53}$$

and

$$\Big|\Big\langle \Phi, \int \widehat{\chi_\Lambda^2}(s)\Big(1-f_L(s)\Big)(\widetilde{a}_s^\dagger + \widetilde{a}_s)\,ds\,\Phi\Big\rangle\Big| \leq C\sqrt{\mathcal{M}}(K_\ell K_L)^{-M}. \tag{12.54}$$

Therefore, using Lemma 12.6 below to estimate the k-integral, we find

$$\begin{aligned} &\Big|\Big\langle \Phi, z\Big(\rho_z \widehat{W_1\omega}(0)(2\pi)^{-3}\int \widehat{\chi^2_\Lambda}(s)(\widetilde{a}_s^\dagger+\widetilde{a}_s)\,ds \\ &\qquad -(2\pi)^{-6}\iint_{k\in P_H} \widehat{W}_1(k)\alpha_k\widehat{\chi^2_\Lambda}(s)f_L(s)(\widetilde{a}_s^\dagger+\widetilde{a}_s)\,ds\Big)\Phi\Big\rangle\Big| \\ &\leq C\rho_z^2 a\ell^3\sqrt{\frac{\mathcal{M}}{|z|^2}}\Big(\widetilde{K}_H^{-1}(\rho_\mu a^3)^{\frac{5}{12}}+(K_\ell K_L)^{-M}\Big). \end{aligned} \tag{12.55}$$

The estimate is in agreement with the error term in (12.51).

What remains in order to prove (12.51) is to estimate a difference of two integrals over the same domain. Writing out the commutator using (12.23) we have to estimate

$$z(2\pi)^{-6}\ell^3\iint_{k\in P_H}\widehat{W}_1(k)\alpha_k\widehat{\chi^2}(s\ell)f_L(s)\left(1-\frac{1-\alpha_{s-k}\alpha_{-k}}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)}\right)(\widetilde{a}_s^\dagger+\widetilde{a}_s) \tag{12.56}$$

and

$$z(2\pi)^{-6}\ell^3\iint_{k\in P_H}\widehat{W}_1(k)\alpha_k f_L(s)\frac{1-\alpha_{s-k}\alpha_{-k}}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)}\widehat{\chi}(k\ell)\widehat{\chi}\big((k-s)\ell\big)(\widetilde{a}_s^\dagger+\widetilde{a}_s). \tag{12.57}$$

To estimate (12.56) we use (12.43), (12.61) and Cauchy-Schwarz to get

$$(12.56)\leq C\rho_z a\delta^2\ell^3\int\widehat{\chi^2}(s\ell)(\varepsilon^{-1}+\varepsilon\widetilde{a}_s^\dagger\widetilde{a}_s)\leq C\rho_z a\delta^2(\varepsilon^{-1}+\varepsilon n_+). \tag{12.58}$$

We choose $\varepsilon^{-1}=D\rho_z a\ell^2\delta^2$ for some sufficiently large constant D to allow the n_+ term to be absorbed in the kinetic energy gap. Thereby, the magnitude of the error (the ε^{-1}-term) becomes (using (12.1))

$$C\rho_\mu^2 a\ell^3\delta^4\frac{a}{\ell}, \tag{12.59}$$

which can clearly be absorbed in the error term in (12.51).

In the second integral (12.57) the terms $\widehat{\chi}(k\ell)$ are very small due to regularity of χ and the fact that $k\in P_H$. Therefore this integral is much smaller.

We easily get, for arbitrary $\varepsilon > 0$,

$$
\begin{aligned}
&\Big\langle \Phi, z(2\pi)^{-6}\ell^3 \iint_{k\in P_H} \widehat{W}_1(k)\alpha_k f_L(s) \\
&\qquad\qquad \cdot \frac{1-\alpha_{s-k}\alpha_{-k}}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)}\widehat{\chi}(k\ell)\widehat{\chi}\big((k-s)\ell\big)(\widetilde{a}_s^\dagger + \widetilde{a}_s)\Phi \Big\rangle \\
&\geq -Cz\rho_\mu a \sup_{k\in P_H} |\widehat{\chi}(k\ell)|\ell^3 \left\langle \Phi, \int f_L(s)(\varepsilon \widetilde{a}_s^\dagger \widetilde{a}_s + \varepsilon^{-1})\Phi \right\rangle \\
&\geq -C\rho_\mu^2 a \ell^3 \sqrt{\frac{\mathcal{M}}{|z|^2}} K_\ell^{3/2} K_L^{3/2} (K_\ell^{-2}\delta)^{\widetilde{M}},
\end{aligned}
\tag{12.60}
$$

where we optimized in ε and used Lemma C.1 to get the last estimate. This error term is clearly in agreement with (12.51). This finishes the proof of (12.51). ☐

In the proof of (12.51) we used the following result.

LEMMA 12.6. *Assume* (5.4), (5.18) *and* (12.1).
Then for sufficiently small values of ρ_μ, *we have*

$$
\left| \rho_z \widehat{W_1\omega}(0) - (2\pi)^{-3}\int_{k\in P_H} \widehat{W}_1(k)\alpha_k \, dk \right| \leq C\rho_z a (\rho_\mu a^3)^{\frac{5}{12}} \widetilde{K}_H^{-1}. \tag{12.61}
$$

Furthermore,

$$
\left| \widehat{W_1\omega}(0) - (2\pi)^{-3}\int_{k\in P_H} \frac{\widehat{W}_1(k)^2}{2\mathcal{D}_k} \, dk \right| \leq C a (\rho_\mu a^3)^{\frac{5}{12}} \widetilde{K}_H^{-1}. \tag{12.62}
$$

Proof. We will use the following weaker version of (5.18),

$$
(\rho_\mu a^3)^{-\frac{1}{12}} \geq \frac{\widetilde{K}_H}{dsK_\ell}(1+\log(K_H)), \tag{12.63}
$$

which follows from (5.18) using (5.4).

Collecting the estimates below, we really get

$$
\begin{aligned}
&\left| \rho_z \widehat{W_1\omega}(0) - (2\pi)^{-3}\int_{k\in P_H} \widehat{W}_1(k)\alpha_k \, dk \right| \\
&\qquad \leq C\rho_z a\Big(K_H^{-1} + (\rho_z a^3)K_H + R^2/\ell^2 + (\rho_z a^3)^2 K_H^3 + \frac{a}{ds\ell}(1+\log K_H)\Big).
\end{aligned}
\tag{12.64}
$$

From this (12.61) follows upon using (5.18), (12.63) and (5.29) to compare the magnitudes of the different terms.

We calculate

(12.65)

$$\begin{aligned}&\rho_z \widehat{W_1\omega}(0) - (2\pi)^{-3} \int_{k\in P_H} \widehat{W_1}(k)\alpha_k \, dk \\ &= (2\pi)^{-3} \int_{k\in P_H} \widehat{W_1}(k)\Big(\rho_z \frac{\widehat{g}(k)}{2k^2} - \alpha_k\Big)\, dk + (2\pi)^{-3} \int_{k\notin P_H} \rho_z \widehat{W_1}(k) \frac{\widehat{g}(k)}{2k^2}\, dk.\end{aligned}$$

We first estimate the last integral,

$$\Big| \int_{k\notin P_H} \widehat{W_1}(k) \frac{\widehat{g}(k)}{2k^2}\, dk \Big| \leq Ca^2 \int_{\{|k|\leq K_H^{-1}a^{-1}\}} k^{-2}\, dk = CaK_H^{-1}. \tag{12.66}$$

This is consistent with the error term in (12.64).

To continue, we write

$$\widehat{W_1}(k)\alpha_k = \rho_z^{-1}\mathcal{A}(k)\Big(1 - \sqrt{1 - \mathcal{B}(k)^2/\mathcal{A}(k)^2}\Big). \tag{12.67}$$

Notice that $|\mathcal{B}(k)/\mathcal{A}(k)| \leq \frac{1}{2}$ for ρ_μ sufficiently small using (12.42) and (5.18). Therefore,

$$\Big|\widehat{W_1}(k)\alpha_k - \frac{\rho_z \widehat{W_1}(k)^2}{2\mathcal{A}(k)}\Big| \leq C\rho_z^3 \frac{\widehat{W_1}(k)^4}{\mathcal{A}(k)^3} \leq C\rho_z^3 a^4 k^{-6}, \tag{12.68}$$

where we used that $\mathcal{A}(k) \geq \frac{1}{2}k^2$ in P_H. Upon integrating over P_H we find a term of magnitude

$$\int_{P_H} \Big|\widehat{W_1}(k)\alpha_k - \frac{\rho_z \widehat{W_1}(k)^2}{2\mathcal{A}(k)}\Big| \leq C\rho_z a(\rho_z a^3)^2 K_H^3, \tag{12.69}$$

in agreement with (12.64).

Finally, we estimate, using $0 \leq k^2 - \tau(k) \leq 2|k|(ds\ell)^{-1}$ in P_H,

(12.70)

$$\begin{aligned}&\rho_z \Big| \int_{k\in P_H} \widehat{W_1}(k)\Big(\frac{\widehat{g}(k)}{2k^2} - \frac{\widehat{W_1}(k)}{2\mathcal{A}(k)}\Big)\Big| \\ &\quad\leq \rho_z \Big| \int_{k\in P_H} \widehat{W_1}(k) \frac{\widehat{g}(k) - \widehat{W_1}(k)}{2k^2}\Big| + \rho_z \Big| \int_{k\in P_H} \frac{\widehat{W_1}(k)^2}{2k^2}\Big(1 - \frac{k^2}{\mathcal{A}(k)}\Big)\Big| \\ &\quad\leq \rho_z \Big| \int_{k\in P_H} \widehat{W_1}(k) \frac{\widehat{g}(k) - \widehat{W_1}(k)}{2k^2}\Big| + C\rho_z^2 a^3 \int_{k\in P_H} k^{-4} \\ &\qquad + C\rho_z (ds\ell)^{-1}\Big(\int_{\{K_H^{-1}\leq a|k|\leq 1\}} a^2|k|^{-3} + a\int \frac{\widehat{W_1}(k)^2}{2k^2}\Big) \\ &\quad\leq \rho_z a \frac{R^2}{\ell^2} + \rho_z a(\rho_z a^3)K_H + C\rho_z a^2 (ds\ell)^{-1}(1 + \log(K_H)),\end{aligned}$$

where the estimate of the first term follows from Cauchy-Schwarz and (6.16). This finishes the proof of (12.61).

The proof of (12.62) is similar. One can for instance use (12.61) and (12.69) and the fact that $|1-\frac{\mathcal{A}(k)}{\mathcal{D}_k}| \leq C\frac{\mathcal{B}(k)^2}{\mathcal{A}(k)^2} \leq C\rho_\mu^2 a^2 k^{-4}$ in P_H. Then (12.62) follows. $\square$

Proof of (12.50). The two operators $\mathcal{Q}_3^{(2)}(z)$ and $\mathcal{Q}_3^{(3)}(z)$ are very similar and can be estimated in identical fashion, so we will only explicitly consider the first. We decompose

$$\mathcal{Q}_3^{(2)}(z) = I + II, \tag{12.71}$$

where

(12.72)

$$I := -z\ell^3(2\pi)^{-6}\iint_{\{k\in P_H\}} \frac{f_L(s)\widehat{W}_1(k)\alpha_k}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)}\left(b_{-k}^\dagger \widetilde{a}_s^\dagger b_{s-k} + b_{s-k}^\dagger \widetilde{a}_s b_{-k}\right),$$

$$II := -z\ell^3(2\pi)^{-6}\iint_{\{k\in P_H\}} \frac{f_L(s)\widehat{W}_1(k)\alpha_k}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)}\left([\widetilde{a}_s^\dagger, b_{-k}^\dagger] b_{s-k} + b_{s-k}^\dagger [b_{-k}, \widetilde{a}_s]\right).$$

The second term II will be very small, due to the smallness of the commutator. (Notice that s and k are "far apart" since $s \in P_L$ and $k \in P_H$.) So the main term is I, which we estimate using Cauchy-Schwarz and (12.43) as

$$I \geq -C\ell^3 za\delta \iint_{k\in P_H} f_L(s)\left(\varepsilon b_{-k}^\dagger \widetilde{a}_s^\dagger \widetilde{a}_s b_{-k} + \varepsilon^{-1} b_{s-k}^\dagger b_{s-k}\right). \tag{12.73}$$

We estimate $\int \widetilde{a}_s^\dagger \widetilde{a}_s \leq \ell^{-3}\mathcal{M}$. Upon choosing $\varepsilon = \sqrt{K_\ell^3 K_L^3/\mathcal{M}}$ and using an easy bound on $\mathcal{D}_k$, this leads to the estimate

$$\begin{aligned}\langle\Phi, I\Phi\rangle &\geq -Cza\delta K_\ell^{3/2} K_L^{3/2}\mathcal{M}^{1/2}\langle\Phi, \int_{\{|k|\geq\frac{1}{2}K_H^{-1}a^{-1}\}} b_k^\dagger b_k \Phi\rangle \\ &\geq -C\delta^2\ell^3\Big(\frac{K_\ell^3 K_L^3\mathcal{M}}{\rho_\mu\ell^3}\Big)^{1/2}\langle\Phi, \int_{\{|k|\geq\frac{1}{2}K_H^{-1}a^{-1}\}} \mathcal{D}_k b_k^\dagger b_k\Phi\rangle.\end{aligned} \tag{12.74}$$

Using (5.22),

$$\frac{K_\ell^3 K_L^3\mathcal{M}}{\rho_\mu\ell^3} = K_L^3 K_{\mathcal{M}}(\rho_\mu a^3)^{\frac{1}{4}} \ll 1. \tag{12.75}$$

Therefore, I can be absorbed in the $\delta^2(2\pi)^{-3}\ell^3\int_{\{|k|\geq\frac{1}{2}K_H^{-1}a^{-1}\}} \mathcal{D}_k b_k^\dagger b_k$ term in (12.50).

We now return to the term II from (12.72). This is easily estimated as

(12.76)

$$\begin{aligned}II &\geq -2z\ell^3(2\pi)^{-6}\sup|[\widetilde{a}_s^\dagger, b_{-k}^\dagger]| \iint_{\{k\in P_H\}} f_L(s)|\widehat{W}_1(k)\alpha_k|\left(b_{s-k}^\dagger b_{s-k}+1\right) \\ &\geq -Cz\Big(\sup_{|p|\geq\frac{1}{2}K_H^{-1}a^{-1}}\widehat{\chi}(p\ell)\Big)(K_L K_\ell)^3\left(\rho_\mu a + a\delta\int_{|k|\geq\frac{1}{2}K_H^{-1}a^{-1}} b_k^\dagger b_k\right).\end{aligned}$$

The $b_k^\dagger b_k$ is easily absorbed in the $\delta^2\ell^3 \int_{\{|k|\geq \frac{1}{2}K_H^{-1}a^{-1}\}} \mathcal{D}_k b_k^\dagger b_k$ term in (12.50). Therefore, using (12.1) and Lemma C.1, II contributes with an error term of order

$$\rho_\mu^2 a\ell^3 (K_\ell^{-2}\delta)^{\widetilde{M}} K_L^3 K_\ell^{3/2} (\rho_\mu a^3)^{\frac{1}{4}} \tag{12.77}$$

to (12.50).

This finishes the proof of (12.50). □

Proof of (12.49) . Finally, we estimate $\mathcal{Q}_3^{(1)}(z)$. We rewrite

$$\begin{aligned}\mathcal{Q}_3^{(1)}(z) = z\ell^3(2\pi)^{-6} \iint_{\{k\in P_H\}} &\frac{f_L(s)\widehat{W}_1(k)}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)} \\ &\times \left(\widetilde{a}_s^\dagger b_{s-k} b_k + \alpha_k\alpha_{s-k}\widetilde{a}_{-s}^\dagger b_{s-k}^\dagger b_k^\dagger + \text{h.c.}\right),\end{aligned} \tag{12.78}$$

where we performed a change of variables in the second term to get the equality.

We combine this term with the diagonalized Bogolubov Hamiltonian. We leave a δ^2-part of this operator in order to control error terms appearing below.

Therefore, we consider

(12.79)

$$\begin{aligned}&(2\pi)^{-3}\ell^3 \int_{\{k\in P_H\}} (1-2\delta^2)\mathcal{D}_k b_k^\dagger b_k\,dk + z\ell^3(2\pi)^{-6}\iint_{\{k\in P_H\}} \frac{f_L(s)\widehat{W}_1(k)}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)} \\ &\quad\times \left(\widetilde{a}_s^\dagger b_{s-k} b_k + \alpha_k\alpha_{s-k}\widetilde{a}_{-s}^\dagger b_{s-k}^\dagger b_k^\dagger + \text{h.c.}\right) \\ &= (2\pi)^{-3}\ell^3 \int_{\{k\in P_H\}} (1-2\delta^2)\mathcal{D}_k c_k^\dagger c_k + T_1(k) + T_2(k) \\ &\geq (2\pi)^{-3}\ell^3 \int_{\{k\in P_H\}} T_1(k) + T_2(k).\end{aligned}$$

Here we have introduced the operators,

$$\begin{aligned}c_k := b_k + z(2\pi)^{-3}\int &\frac{f_L(s)\widehat{W}_1(k)}{(1-2\delta^2)\mathcal{D}_k(1-\alpha_k^2)(1-\alpha_{s-k}^2)} \\ &\times \left(b_{s-k}^\dagger \widetilde{a}_s + \alpha_k\alpha_{s-k}\widetilde{a}_{-s} b_{s-k}^\dagger\right) ds,\end{aligned} \tag{12.80}$$

$$T_1(k) := -z(2\pi)^{-3}\int \frac{f_L(s)\widehat{W}_1(k)\alpha_k\alpha_{s-k}}{(1-\alpha_k^2)(1-\alpha_{s-k}^2)} \left([b_k^\dagger, \widetilde{a}_{-s} b_{s-k}^\dagger] + \text{h.c.}\right) ds, \tag{12.81}$$

and
(12.82)

$$\begin{aligned}T_2(k) &:= -\frac{|z|^2\widehat{W}_1(k)^2}{(1-2\delta^2)\mathcal{D}_k(1-\alpha_k^2)^2}(2\pi)^{-6}\iint \frac{f_L(s)f_L(s')}{(1-\alpha_{s-k}^2)(1-\alpha_{s'-k}^2)} \\ &\qquad \times \left(\widetilde{a}_{s'}^\dagger b_{s'-k} + \alpha_k\alpha_{s'-k}b_{s'-k}\widetilde{a}_{-s'}^\dagger\right)\left(b_{s-k}^\dagger\widetilde{a}_s + \alpha_k\alpha_{s-k}\widetilde{a}_{-s}b_{s-k}^\dagger\right)\,ds\,ds' \\ &\geq -\left(1+C\delta^2\right)|z|^2\frac{\widehat{W}_1(k)^2}{\mathcal{D}_k}(2\pi)^{-6}\iint f_L(s)f_L(s') \\ &\qquad \times \left(\widetilde{a}_{s'}^\dagger b_{s'-k} + \alpha_k\alpha_{s'-k}b_{s'-k}\widetilde{a}_{-s'}^\dagger\right)\left(b_{s-k}^\dagger\widetilde{a}_s + \alpha_k\alpha_{s-k}\widetilde{a}_{-s}b_{s-k}^\dagger\right)\,ds\,ds',\end{aligned}$$

where we used (12.43) to get the estimate on T_2. Notice that
(12.83)

$$\widetilde{a}_{s'}^\dagger b_{s'-k} + \alpha_k\alpha_{s'-k}b_{s'-k}\widetilde{a}_{-s'}^\dagger = \left(\widetilde{a}_{s'}^\dagger + \alpha_k\alpha_{s'-k}\widetilde{a}_{-s'}^\dagger\right)b_{s'-k} + \alpha_k\alpha_{s'-k}[b_{s'-k},\widetilde{a}_{-s'}^\dagger].$$

The contribution from the commutator term is very small, both due to the factors of α and to the commutator, since $k\in P_H$, $s'\in P_L$. Therefore, we estimate

$$T_2(k) \geq (1+\varepsilon)T_2'(k) + (1+\varepsilon^{-1})T_2''(k), \tag{12.84}$$

where

$$\begin{aligned}T_2'(k) &:= -\left(1+C\delta^2\right)|z|^2\frac{\widehat{W}_1(k)^2}{\mathcal{D}_k}(2\pi)^{-6}\iint f_L(s)f_L(s') \\ &\qquad \times \left(\widetilde{a}_{s'}^\dagger + \alpha_k\alpha_{s'-k}\widetilde{a}_{-s'}^\dagger\right)b_{s'-k}b_{s-k}^\dagger\left(\widetilde{a}_s + \alpha_k\alpha_{s-k}\widetilde{a}_{-s}\right)\,ds\,ds' \\ T_2''(k) &:= -\left(1+C\delta^2\right)|z|^2\frac{\widehat{W}_1(k)^2}{\mathcal{D}_k}(2\pi)^{-6} \\ &\qquad \times \iint f_L(s)f_L(s')|\alpha_k|^2\alpha_{s'-k}\alpha_{s-k}[b_{s'-k},\widetilde{a}_{-s'}^\dagger][\widetilde{a}_{-s},b_{s-k}^\dagger].\end{aligned} \tag{12.85}$$

With this choice, we estimate using (12.62), (5.18) and (12.43),

$$\begin{aligned}&(2\pi)^{-3}\ell^3\int_{k\in P_H}(1+\varepsilon^{-1})T_2''(k)\,dk \\ &\quad\geq -C\rho_z a(K_\ell K_L)^6\delta^2 \sup_{k\in P_H,s\in P_L}|[\widetilde{a}_{-s},b_{s-k}^\dagger]|^2 \\ &\quad\geq -C\rho_\mu^2 a\ell^3(\rho_\mu a^3)^{\frac12}K_\ell^3K_L^6\delta^2 \sup_{k\in P_H,s\in P_L}|[\widetilde{a}_{-s},b_{s-k}^\dagger]|^2.\end{aligned} \tag{12.86}$$

We continue to estimate the other part of $T_2(k)$:

$$\begin{aligned}T_2'(k) &:= -\left(1+C\delta^2\right)|z|^2\frac{\widehat{W}_1(k)^2}{\mathcal{D}_k}(2\pi)^{-6}\iint f_L(s)f_L(s') \\ &\qquad \times \left(\widetilde{a}_{s'}^\dagger + \alpha_k\alpha_{s'-k}\widetilde{a}_{-s'}^\dagger\right)b_{s'-k}b_{s-k}^\dagger\left(\widetilde{a}_s + \alpha_k\alpha_{s-k}\widetilde{a}_{-s}\right)\,ds\,ds' \\ &= T_{2,\mathrm{comm}}'(k) + T_{2,\mathrm{op}}'(k),\end{aligned} \tag{12.87}$$

with

(12.88)
$$T'_{2,\mathrm{comm}}(k) := -\left(1+C\delta^2\right)|z|^2\frac{\widehat{W}_1(k)^2}{\mathcal{D}_k}(2\pi)^{-6}\iint f_L(s)f_L(s') \times\left(\widetilde{a}^\dagger_{s'}+\alpha_k\alpha_{s'-k}\widetilde{a}^\dagger_{-s'}\right)[b_{s'-k},b^\dagger_{s-k}]\left(\widetilde{a}_s+\alpha_k\alpha_{s-k}\widetilde{a}_{-s}\right)ds\,ds',$$
$$T'_{2,\mathrm{op}}(k) := -\left(1+C\delta^2\right)|z|^2\frac{\widehat{W}_1(k)^2}{\mathcal{D}_k}(2\pi)^{-6}\iint f_L(s)f_L(s') \times\left(\widetilde{a}^\dagger_{s'}+\alpha_k\alpha_{s'-k}\widetilde{a}^\dagger_{-s'}\right)b^\dagger_{s-k}b_{s'-k}\left(\widetilde{a}_s+\alpha_k\alpha_{s-k}\widetilde{a}_{-s}\right)ds\,ds'.$$

We start by estimating the last term in (12.87). We introduce the notation

$$\mathcal{C} := \sup_{s,s'\in P_L,k\in P_H}\left|[\widetilde{a}^\dagger_{s'}+\alpha_k\alpha_{s'-k}\widetilde{a}^\dagger_{-s'},b^\dagger_{s-k}]\right|\le 1. \tag{12.89}$$

In fact, it follows from (10.6), (12.10), (12.43), and (C.4) that

$$\mathcal{C}\le C\delta\left(K_\ell^{-2}\widetilde{K}_H^2(\rho_\mu a^3)^{\frac{1}{6}}\right)^{\frac{M-1}{2}}. \tag{12.90}$$

To estimate the last term in (12.87) we first apply Cauchy-Schwarz, then commute the $\widetilde{a}$'s through the b's and apply Cauchy-Schwarz to the commutator terms. This yields

(12.91)
$$\begin{aligned}&\left\langle\Phi,\iint f_L(s)f_L(s')\left(\widetilde{a}^\dagger_{s'}+\alpha_k\alpha_{s'-k}\widetilde{a}^\dagger_{-s'}\right)b^\dagger_{s-k}b_{s'-k}\left(\widetilde{a}_s+\alpha_k\alpha_{s-k}\widetilde{a}_{-s}\right)\Phi\right\rangle\\&\le 2\iint f_L(s)f_L(s')\left\langle\Phi,b^\dagger_{s-k}\left(\widetilde{a}^\dagger_{s'}+\alpha_k\alpha_{s'-k}\widetilde{a}^\dagger_{-s'}\right)\left(\widetilde{a}_{s'}+\alpha_k\alpha_{s'-k}\widetilde{a}_{-s'}\right)b_{s-k}\Phi\right\rangle\\&\quad+C\mathcal{C}\iint f_L(s)f_L(s')\left\langle\Phi,\left(\varepsilon b^\dagger_{s-k}b_{s-k}+C\varepsilon^{-1}\widetilde{a}_{s'}\widetilde{a}^\dagger_{s'}+\mathcal{C}\right)\Phi\right\rangle\\&\le C(\ell^{-3}\mathcal{M}+\varepsilon|P_L|\mathcal{C})\int f_L(s)\langle\Phi,b^\dagger_{s-k}b_{s-k}\Phi\rangle\\&\quad+C\varepsilon^{-1}|P_L|\mathcal{C}(\ell^{-3}\mathcal{M}+|P_L|)+C|P_L|^2\mathcal{C}^2.\end{aligned}$$

For simplicity, we choose $\varepsilon=\frac{\mathcal{M}}{\ell^3|P_L|\mathcal{C}}$ and get

(12.92)
$$\begin{aligned}&\left\langle\Phi,\iint f_L(s)f_L(s')\left(\widetilde{a}^\dagger_{s'}+\alpha_k\alpha_{s'-k}\widetilde{a}^\dagger_{-s'}\right)b^\dagger_{s-k}b_{s'-k}\left(\widetilde{a}_s+\alpha_k\alpha_{s-k}\widetilde{a}_{-s}\right)\Phi\right\rangle\\&\quad\le C\ell^{-3}\mathcal{M}\left\langle\Phi,\int f_L(s)b^\dagger_{s-k}b_{s-k}\Phi\right\rangle+C|P_L|^2\mathcal{C}^2\Big(1+\frac{\ell^3|P_L|}{\mathcal{M}}\Big).\end{aligned}$$

Therefore, using (12.62),

$$\begin{aligned}&\left\langle \Phi, (2\pi)^{-3}\ell^3 \int_{k\in P_H} T'_{2,\mathrm{op}}(k)\Phi \right\rangle \\ (12.93)\qquad &\geq -C\rho_\mu \ell^3 \frac{a^2}{(\min_{k\in\frac12 P_H}\mathcal{D}_k)^2}\mathcal{M}|P_L| \left\langle \Phi \int_{\{q\in \frac12 P_H\}} \mathcal{D}_q b_q^\dagger b_q \Phi\right\rangle \\ &\quad - \rho_\mu \ell^6 a |P_L|^2 \mathcal{C}^2\left(1+\frac{\ell^3|P_L|}{\mathcal{M}}\right).\end{aligned}$$

Notice that $\mathcal{D}_k \geq C^{-1}\widetilde{K}_H^{-2}(\rho_\mu a^3)^{\frac56} a^{-2}$ for $k\in \frac12 P_H$. Therefore, using (12.4) and (5.22),

$$(12.94)\qquad \rho_\mu \frac{a^2}{(\min_{k\in\frac12 P_H}\mathcal{D}_k)^2}\mathcal{M}|P_L| \leq \delta^2 K_L^3 K_{\mathcal{M}}(\rho_\mu a^3)^{\frac14} \ll \delta^2.$$

Therefore, the negative $\mathcal{D}_q b_q^\dagger b_q$-term in (12.93) can be absorbed in a fraction of the similar (positive) term left out in (12.79) exactly for this purpose.

Using (5.15), we see that $\ell^3|P_L| \leq C(K_L K_\ell)^3 = d^{-6}$. Therefore, it follows from (5.19) that $\frac{\ell^3|P_L|}{\mathcal{M}} \ll 1$. So, using (12.90) we can estimate the error term in (12.93) as

$$\begin{aligned}&- \rho_\mu \ell^6 a|P_L|^2\mathcal{C}^2\left(1+\frac{\ell^3|P_L|}{\mathcal{M}}\right) \\ (12.95)\qquad &\geq -C\rho_\mu^2 a\ell^3\sqrt{\rho_\mu a^3}\left(K_\ell^{-3} d^{-12}\delta^2\left(K_\ell^{-2}\widetilde{K}_H^2(\rho_\mu a^3)^{\frac16}\right)^{M-1}\right).\end{aligned}$$

This is clearly seen to agree with (12.49).

We next consider the commutator term $T'_{2,\mathrm{comm}}(k)$ from (12.87).

From (12.23) and using Lemma C.1, we see that

$$(12.96)\qquad \left|[b_{s'-k}, b_{s-k}^\dagger] - \widehat{\chi^2}((s-s')\ell)\right| \leq C\delta^2|\widehat{\chi^2}((s-s')\ell)| + C(K_\ell^{-2}\delta)^{\frac{M-1}{2}}.$$

Therefore, using that $M\geq 5$,

(12.97)

$$\begin{aligned}T'_{2,\mathrm{comm}}(k) \geq \left(1+C\delta^2\right)|z|^2\frac{\widehat{W_1}(k)^2}{\mathcal{D}_k}(2\pi)^{-6}\iint f_L(s)f_L(s')\widetilde{a}_{s'}^\dagger\widehat{\chi^2}((s-s')\ell)\widetilde{a}_s \\ - C|z|^2\frac{\widehat{W_1}(k)^2}{\mathcal{D}_k}\delta^2|P_L|\ell^{-3}n_+.\end{aligned}$$

Using (10.17) and (12.62) we see that

(12.98)
$$-(2\pi)^{-3}\ell^3\int_{k\in P_H}\big(1+C\delta^2\big)|z|^2\frac{\widehat{W_1}(k)^2}{\mathcal{D}_k}(2\pi)^{-6}\iint f_L(s)f_L(s')\widetilde{a}^\dagger_{s'}\widehat{\chi^2}((s-s')\ell)\widetilde{a}_s$$
$$=-\rho_z\big(1+C\delta^2\big)\left((2\pi)^{-3}\int_{k\in P_H}\frac{\widehat{W_1}(k)^2}{\mathcal{D}_k}\right)\sum_j Q_{L,j}\chi^2 Q_{L,j}$$
$$\geq -2\rho_z\big(1+C\delta^2\big)\widehat{W_1\omega}(0)\sum_j Q_{L,j}\chi^2 Q_{L,j}.$$

Here we used (5.18) to control the error from (12.62).

We now notice that, for all $\varepsilon>0$,

$$\sum_j Q_{L,j}\chi^2 Q_{L,j}\leq(1+\varepsilon)\sum_j Q_{L,j}\chi^2 Q_{L,j}+C\varepsilon^{-1}n_+^H. \tag{12.99}$$

We notice that $\rho_\mu a=(dK_\ell)^2\frac{1}{d^2\ell^2}$. Therefore, choosing ε proportional to $\varepsilon_T^{-1}(dK_\ell)^2$, we find, using (12.62),

(12.100)
$$-(2\pi)^{-3}\ell^3\int_{k\in P_H}\big(1+C\delta^2\big)|z|^2\frac{\widehat{W_1}(k)^2}{\mathcal{D}_k}(2\pi)^{-6}\iint f_L(s)f_L(s')\widetilde{a}^\dagger_{s'}\widehat{\chi^2}((s-s')\ell)\widetilde{a}_s$$
$$\geq -2\rho_z\big(1+C\delta^2+C\varepsilon_T^{-1}(dK_\ell)^2\big)\widehat{W_1\omega}(0)\sum_j Q_j\chi^2 Q_j-\frac{1}{100}\frac{1}{(d\ell)^2}n_+^H.$$

Using (5.4), (5.23), (12.4) and (12.1),

$$\rho_z a[\delta^2+\varepsilon_T^{-1}(dK_\ell)^2]\ll\ell^{-2}. \tag{12.101}$$

Therefore, the above error terms can be absorbed in the energy gap.

To estimate the error term in (12.97) we integrate

$$-(2\pi)^{-3}\ell^3\int_{k\in P_H}C|z|^2\frac{\widehat{W_1}(k)^2}{\mathcal{D}_k}\delta^2|P_L|\ell^{-3}n_+\geq -C\rho_\mu a\delta^2(\ell^3|P_L|)n_+. \tag{12.102}$$

By (5.23) and (5.15), we see that $\rho_\mu a\delta^2(\ell^3|P_L|)\ll\ell^{-2}$, so this term can also be absorbed in the energy gap.

We now estimate the other commutator term, namely $T_1(k)$ from (12.80). We clearly have

(12.103)
$$T_1(k)\geq -Cz\delta\sup_{k\in P_H,s\in P_L}\Big(|[b_k^\dagger,b_{s-k}^\dagger]|\Big)|\alpha_k\widehat{W_1}(k)|\int f_L(s)\Big(\widetilde{a}^\dagger_{-s}\widetilde{a}_{-s}+1\Big)\,ds$$
$$-Cz\delta\sup_{k\in P_H,s\in P_L}\Big(|[b_k^\dagger,\widetilde{a}^\dagger_{-s}]|\Big)|\alpha_k\widehat{W_1}(k)|\int f_L(s)\Big(b^\dagger_{s-k}b_{s-k}+1\Big)\,ds.$$

Therefore,

(12.104)

$$\begin{aligned}&\ell^3(2\pi)^{-3}\int_{k\in P_H} T_1(k)\,dk\\&\geq -Cz\delta\Big(\sup_{k\in P_H, s\in P_L}|[b_k^\dagger, b_{s-k}^\dagger]|\Big)\rho a(n_+ + (K_\ell K_L)^3)\\&\quad - Cz\delta\Big(\sup_{k\in P_H, s\in P_L}|[b_k^\dagger, \widetilde{a}_{-s}^\dagger]|\Big)\rho a(K_\ell K_L)^3 - Cz\delta^2\Big(\sup_{k\in P_H, s\in P_L}|[b_k^\dagger, \widetilde{a}_{-s}^\dagger]|\Big)\\&\quad\times\Big(\sup_{k\in P_H, s\in P_L}|D_{s-k}|^{-1}\Big)a(K_\ell K_L)^3\int_{\{|k|\geq\frac{1}{2}K_H^{-1}a^{-1}\}} D_k b_k^\dagger b_k.\end{aligned}$$

The last term in this inequality is easily seen to be estimated as

(12.105)

$$\geq -\delta^2\left\{\delta\frac{1}{\sqrt{\rho_\mu\ell^3}}K_\ell^3K_L^3\Big(\sup_{k\in P_H, s\in P_L}|[b_k^\dagger, \widetilde{a}_{-s}^\dagger]|\Big)\right\}\ell^3\int_{\{|k|\geq\frac{1}{2}K_H^{-1}a^{-1}\}} D_k b_k^\dagger b_k,$$

and using the properties of the commutator and Lemma C.1, we see that this term can easily be absorbed in the extra $\delta^2\ell^3\int_{\{|k|\geq\frac{1}{2}K_H^{-1}a^{-1}\}} D_k b_k^\dagger b_k$ omitted in (12.79).

The two remaining terms in (12.104) can be estimated (in particular, using Lemma C.1 and (12.43)) as

$$\geq -C\rho_\mu^2 a\ell^3\delta K_\ell^{-3/2}(\rho_\mu a^3)^{\frac{1}{4}}(\mathcal{M}+(K_\ell^3K_L^3))\left(K_\ell^{-2}\delta\right)^{\frac{M-1}{2}}. \tag{12.106}$$

This finishes the proof of (12.49) □

Now we have established all three inequalities (12.49), (12.50) and (12.51). This finishes the proof of Lemma 12.5. □

13. Proof of the main theorem

In this section we will combine the results of the previous sections in order to prove Theorem 1.2.

Proof of Theorem 1.2. As noted in Section 4, Theorem 1.2 follows from Theorem 4.1, which again—as observed in Section 6.3—follows from Theorem 6.8. We will use the concrete choice of parameters set down in (5.26) and (5.27) in Section 5. Recall, in particular, the notation X defined in (5.27).

To prove Theorem 6.8, let $\Psi\in\mathcal{F}_s(L^2(\Lambda))$ be a normalized n-particle trial state satisfying (7.1). If such a state does not exist, there is nothing to prove. Using Lemma 8.2 there exists a normalized n-particle wave function $\widetilde{\Psi}\in\mathcal{F}_s(L^2(\Lambda))$ satisfying (8.2) and such that

$$\langle\Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi\rangle \geq \left\langle\widetilde{\Psi}, \mathcal{H}_\Lambda(\rho_\mu)\widetilde{\Psi}\right\rangle - CX^2\mathcal{R}\rho_\mu^2 a\ell^3(\rho_\mu a^3)^{1/2}. \tag{13.1}$$

The error term in (13.1) is consistent with the error term in Theorem 6.8.

Using Proposition 10.2 we find that our localized state $\widetilde{\Psi}$ satisfies

(13.2)
$$\langle\widetilde{\Psi},\mathcal{H}_\Lambda(\rho_\mu)\widetilde{\Psi}\rangle \geq \langle\widetilde{\Psi},\mathcal{H}_\Lambda^{\rm 2nd}(\rho_\mu)\widetilde{\Psi}\rangle - C\rho_\mu^2 a\ell^3(\rho_\mu a^3)^{1/2}\left((\rho_\mu a^3)^{\frac{348}{323}-\frac{1}{2}} + X^3(Ra^{-1})^2(\rho_\mu a^3)^{1/2}\right),$$

where the error is clearly consistent with the error term in Theorem 6.8.

At this point, we can apply Theorem 10.5 to get the lower bound

$$\langle\widetilde{\Psi},\mathcal{H}_\Lambda^{\rm 2nd}(\rho_\mu)\widetilde{\Psi}\rangle \geq \inf_{z\in\mathbb{R}_+}\inf_{\Phi}\langle\Phi,\mathcal{K}(z)\Phi\rangle - C\rho_\mu a, \tag{13.3}$$

where the second infimum is over all normalized $\Phi \in \mathcal{F}(\mathrm{Ran}(Q))$ satisfying (10.28).

Since

$$\rho_\mu a = \rho_\mu^2 a\ell^3\sqrt{\rho_\mu a^3}K_\ell^{-3} = \rho_\mu^2 a\ell^3\sqrt{\rho_\mu a^3}X^{\frac{9}{2}}, \tag{13.4}$$

which is in agreement with the error term in Theorem 6.8, this implies that we need to prove that

$$\inf_{\Phi}\langle\Phi,\mathcal{K}(z)\Phi\rangle \geq -4\pi\rho_\mu^2 a\ell^3 + 4\pi\rho_\mu^2 a\ell^3\frac{128}{15\sqrt{\pi}}(\rho_\mu a^3)^{\frac{1}{2}} - C\rho_\mu^2 a\ell^3(\rho_\mu a^3)^{\frac{1}{2}}\left(\frac{R^2}{a^2}(\rho_\mu a^3)^{\frac{1}{2}} + X^{\frac{1}{5}}\right) \tag{13.5}$$

for all normalized Φ satisfying (10.28).

We will use that with our choice of parameters (12.2) is satisfied.

If $\rho_z = |z|^2/\ell^3$ satisfies (11.3), i.e., is "far away" from ρ_μ, then Proposition 11.2 provides a lower bound on $\langle\Phi,\mathcal{K}(z)\Phi\rangle$ that is larger than needed for (13.5) by a factor of 2 on the LHY-term. Since (12.2) is satisfied, the assumptions of Proposition 11.2 are verified.

If ρ_z satisfies the complementary inequality (12.1) and Φ satisfies (10.28), then by (12.7) (using again that (12.2) is satisfied) and Theorem 12.1 combined with Lemma 12.3 we get

(13.6)
$$\langle\Phi,\mathcal{K}(z)\Phi\rangle \geq -\frac{1}{2}\rho_\mu^2\ell^3\widehat{g}(0) + 4\pi\frac{128}{15\sqrt{\pi}}\rho_z a\sqrt{\rho_z a^3}\ell^3 + \left\langle\Phi,\left(\frac{b}{4\ell^2}n_+ + \varepsilon_T\frac{b}{2d^2\ell^2}n_+^H + \mathcal{Q}_1^{\rm ex}(z) + \mathcal{Q}_2^{\rm ex}(z) + \mathcal{Q}_3(z)\right)\Phi\right\rangle + (2\pi)^{-3}\ell^3\langle\Phi,\int\mathcal{D}_k b_k^\dagger b_k\,dk\,\Phi\rangle - \mathcal{E}_1,$$

where the error term $\mathcal{E}_1$ satisfies

$$\mathcal{E}_1 \leq C\rho_\mu^2 a\ell^3(\rho_\mu a^3)^{\frac{1}{2}}\left(\frac{R^2}{a^2}(\rho_\mu a^3)^{\frac{1}{2}} + X^{\frac{1}{5}} + (\rho_\mu a^3)^{\frac{1}{4}}(Ra^{-1})^{\frac{1}{2}}\right). \tag{13.7}$$

Here the error term in $X^{\frac{1}{5}}$ comes from the $\varepsilon(\rho_\mu, \rho_z)$ in Lemma 12.3. This error is compatible with (13.5) using Young's inequality for products.

Now we can apply Theorem 12.4 to obtain the inequality

$$(13.8)\quad \begin{aligned} &(2\pi)^{-3}\ell^3 \left\langle \Phi, \int \mathcal{D}_k b_k^\dagger b_k \, dk \, \Phi \right\rangle \\ &+ \left\langle \Phi, \left(\frac{b}{4\ell^2} n_+ + \varepsilon_T \frac{b}{2d^2\ell^2} n_+^H + \mathcal{Q}_1^{\rm ex}(z) + \mathcal{Q}_2^{\rm ex}(z) + \mathcal{Q}_3(z) \right) \Phi \right\rangle \geq -\mathcal{E}_2, \end{aligned}$$

with error term

$$(13.9)\quad \mathcal{E}_2 \leq C\rho_\mu^2 a \ell^3 \sqrt{\rho_\mu a^3} (\rho_\mu a^3)^{\frac{1}{6}} X^{-\frac{11}{12}}.$$

Here the dominant contribution to the error (with our choice of parameters) comes from the $\widetilde{K}_H^{-1}(\rho_\mu a^3)^{\frac{5}{12}}$-term. This error is clearly consistent with (13.5).

Combining (13.6) and (13.8), we get

$$(13.10)\quad \begin{aligned} \langle \Phi, \mathcal{K}(z)\Phi \rangle \geq& -\frac{1}{2}\rho_\mu^2 \ell^3 \widehat{g}(0) + 4\pi \frac{128}{15\sqrt{\pi}} \rho_\mu a \sqrt{\rho_\mu a^3} \ell^3 \\ &- \left(\mathcal{E}_1 + \mathcal{E}_2 + C \left| \rho_\mu a \sqrt{\rho_\mu a^3} - \rho_z a \sqrt{\rho_z a^3} \right| \ell^3 \right). \end{aligned}$$

This establishes (13.5) for ρ_z satisfying (12.1), since by (12.1), (12.2) and (5.26), we have

$$(13.11)\quad \left| \rho_\mu a \sqrt{\rho_\mu a^3} - \rho_z a \sqrt{\rho_z a^3} \right| \ell^3 \leq C \rho_\mu a \sqrt{\rho_\mu a^3} \ell^3 K_\ell^{-2} = C\rho_\mu a \sqrt{\rho_\mu a^3} \ell^3 X^3.$$

This finishes the proof of (13.5) and therefore of Theorem 6.8, which in turn implies Theorems 4.1 and 1.2. □

Appendix A. Bogolubov method

In this section we recall a simple consequence of the Bogolubov method (see [21, Th. 6.3] and [7])

THEOREM A.1 (Simple case of Bogolubov's method). *Let $a_\pm$ be operators on a Hilbert space satisfying $[a_+, a_-] = 0$. For $\mathcal{A} > 0$, $\mathcal{B} \in \mathbb{R}$ satisfying either $|\mathcal{B}| < \mathcal{A}$ or $\mathcal{B} = \mathcal{A}$ and arbitrary $\kappa \in \mathbb{C}$, we have the operator identity*

$$(A.1)\quad \begin{aligned} &\mathcal{A}(a_+^* a_+ + a_-^* a_-) + \mathcal{B}(a_+^* a_-^* + a_+ a_-) + \kappa(a_+^* + a_-) + \overline{\kappa}(a_+ + a_-^*) \\ &= \mathcal{D}(b_+^* b_+ + b_-^* b_-) - \frac{1}{2}\left(\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2} \right) ([a_+, a_+^*] + [a_-, a_-^*]) - \frac{2|\kappa|^2}{\mathcal{A} + \mathcal{B}}, \end{aligned}$$

where

$$(A.2)\quad \mathcal{D} := \frac{1}{2}\left(\mathcal{A} + \sqrt{\mathcal{A}^2 - \mathcal{B}^2} \right)$$

and

$$b_+ := a_+ + \alpha a_-^* + \overline{c_0}, \qquad b_- := a_- + \alpha a_+^* + c_0, \tag{A.3}$$

with

$$\alpha := \mathcal{B}^{-1}\left(\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2}\right), \qquad c_0 := \frac{2\overline{\kappa}}{\mathcal{A} + \mathcal{B} + \sqrt{\mathcal{A}^2 - \mathcal{B}^2}}. \tag{A.4}$$

In particular,

$$\begin{aligned} &\mathcal{A}(a_+^* a_+ + a_-^* a_-) + \mathcal{B}(a_+^* a_-^* + a_+ a_-) + \kappa(a_+^* + a_-^*) + \overline{\kappa}(a_+ + a_-) \\ &\geq -\frac{1}{2}\left(\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2}\right)([a_+, a_+^*] + [a_-, a*_-]) - \frac{2|\kappa|^2}{\mathcal{A} + \mathcal{B}}. \end{aligned} \tag{A.5}$$

Proof. The identity (A.1) is elementary. From here the inequality (A.5) follows by dropping the positive operator term $\mathcal{D}(b_+^* b_+ + b_-^* b_-)$. □

Appendix B. Localization to small boxes

The Hamiltonian $\mathcal{H}_B(\rho_\mu)$ defined in (6.24) (with $u = 0$) is localized to the box $\Lambda := \Lambda(0) = [-\ell/2, \ell/2]^3$. In order to arrive at the a priori bounds in Theorem 7.1 we will localize again to boxes with a length scale $\ell d \ll (\rho a)^{-1/2}$. The reason for this second localization is that we need a larger Neumann gap in order to absorb errors. We therefore introduce a new family of boxes (some of which will have a rectangular shape) given by

$$B(u) = [-\ell/2, \ell/2]^3 \cap \left(\ell d u + [-\ell d/2, \ell d/2]^3\right), \quad u \in \mathbb{R}^3. \tag{B.1}$$

The functions that localize to these boxes are

$$\chi_{B(u)}(x) = \chi\left(\frac{x}{\ell}\right)\chi\left(\frac{x}{d\ell} - u\right), \quad u \in \mathbb{R}^3, \tag{B.2}$$

where χ is given in (C.1) in terms of the positive integer M. Observe that

$$\iint \chi_{B(u)}(x)^2 dx du = \ell^3. \tag{B.3}$$

We consider the projections

$$P_{B(u)}\varphi = |B(u)|^{-1}\langle \mathbb{1}_{B(u)}, \varphi\rangle \mathbb{1}_{B(u)}, \quad Q_{B(u)}\varphi = \mathbb{1}_{B(u)}\varphi - P_{B(u)}\varphi.$$

In these small boxes we consider the Hamiltonian

$$\mathcal{H}_{B(u)}(\rho_\mu) = \sum_{i=1}^N \left(\mathcal{T}_{B(u),i} - \rho_\mu \int w_{1,B(u)}(x_i, y) dy\right) + \frac{1}{2}\sum_{i \neq j} w_{B(u)}(x_i, x_j), \tag{B.4}$$

where (omitting the index u)

$$\mathcal{T}_B = \frac{1}{2}\varepsilon_T(1 + \pi^{-2})^{-1}(d\ell)^{-2} Q_B + Q_B \chi_B[\sqrt{-\Delta} - (ds\ell)^{-1}]_+^{-2}\chi_B Q_B \tag{B.5}$$

and

(B.6)
$$w_B(x,y) = \chi_B(x) W^{\rm s}(x-y)\chi_B(y), \quad w_{1,B}(x,y) = \chi_B(x) W_1^{\rm s}(x-y)\chi_B(y),$$

with (where the superscript s refers to small)

$$W^{\rm s}(x) = \frac{W(x)}{\chi * \chi(x/(d\ell))}, \quad W_1^{\rm s}(x) = \frac{W_1(x)}{\chi * \chi(x/(d\ell))}. \tag{B.7}$$

Here we use that $R < d\ell$ by (5.29). As in the large boxes we will also need

$$w_{2,B}(x,y) = \chi_B(x) W_2^{\rm s}(x-y)\chi_B(y), \quad W_2^{\rm s}(x) = \frac{W_2(x)}{\chi * \chi(x/(d\ell))}. \tag{B.8}$$

Since $\omega \leq 1$, we have

$$\int W_2^{\rm s} \leq 2\int W_1^{\rm s}(x) \leq Ca. \tag{B.9}$$

We have by a Schwarz inequality that

(B.10)
$$\iint w_{1,B}(x,y)dxdy \leq \iint \chi_B(x)^2 W_1^{\rm s}(x-y)dxdy \leq ({\rm cst.})a\int \chi_B^2 \leq Ca|B|.$$

Observe also that

$$\iiint w_{1,B(u)}(x,y)dxdydu = \ell^3 \int g = 8\pi a\ell^3. \tag{B.11}$$

It was proved in [7, Th. 3.10] that the operator $\mathcal{H}_\Lambda(\rho_\mu)$ defined in (6.24) and (6.25) can be bounded below by (we are for the lower bound ignoring the third term in $\mathcal{T}$ in (6.19))

$$\mathcal{H}_\Lambda(\rho_\mu) \geq \sum_{i=1}^N \frac{b}{2} Q_{\Lambda,i}\ell^{-2} + \int_{\mathbb{R}^3} \mathcal{H}_{B(u)}(\rho_\mu)du \tag{B.12}$$

if

$$\varepsilon_T, s, ds^{-1}, \text{ and } (s^{-2}+d^{-3})(sd)^{-2}s^M \tag{B.13}$$

are smaller than some universal constant. Note that, if $\rho_\mu a^3$ is small enough, this is satisfied for our choices in Section 5, in particular, due to (5.3).

In the integral above the operators $\mathcal{H}_{B(u)}(\rho_\mu)$ are, however, not unitarily equivalent. Depending on u the boxes $B(u)$ can be rather small and rectangular. We denote by $\lambda_1(u) \leq \lambda_2(u) \leq \lambda_3(u) \leq d\ell$ the side lengths of the boxes $B(u)$. To avoid boxes that are very small, i.e., where $\lambda_1(u) \leq \rho_\mu^{-1/3}$, we will restrict the integral above to u such that

$$\|\ell du\|_\infty \leq \frac{\ell}{2}(1+d) - \rho_\mu^{-1/3}.$$

Note that since the full integral would be over the set where $\|\ell du\|_\infty \leq \frac{\ell}{2}(1+d)$ we see that the restriction implies that all boxes will satisfy $\lambda_1(u) \geq \rho_\mu^{-1/3}$.

For the kinetic energy and the repulsive potential this restriction will only give a further lower bound. For the chemical potential term we will use the following result.

Lemma B.1. *For all $x \in \Lambda$, we have the estimate*

(B.14)
$$\begin{aligned}-\rho_\mu \iint & w_{1,B(u)}(x,y)dydu \\ & \geq -\rho_\mu \int_{\|u\|_\infty - \frac{1}{2}\left(\frac{1}{d}+1\right) \leq -(\ell d \rho_\mu^{1/3})^{-1}} \int w_{1,B(u)}(x,y)dydu \\ & \quad - 3\rho_\mu \int_{-2(\ell d \rho_\mu^{1/3})^{-1} \leq \|u\|_\infty - \frac{1}{2}\left(\frac{1}{d}+1\right) \leq -(\ell d \rho_\mu^{1/3})^{-1}} \int w_{1,B(u)}(x,y)dydu.\end{aligned}$$

Proof. The estimate above follows if we can show that for all $x, y \in \Lambda$ we have

(B.15)
$$\begin{aligned}\chi * \chi & \left(\frac{x-y}{\ell d}\right) \\ & \leq \int_{\|u\|_\infty - \frac{1}{2}\left(\frac{1}{d}+1\right) \leq -(\ell d \rho_\mu^{1/3})^{-1}} \chi\left(\frac{x}{\ell d} - u\right) \chi\left(\frac{y}{\ell d} - u\right) du \\ & \quad + 3 \int_{-2(\ell d \rho_\mu^{1/3})^{-1} \leq \|u\|_\infty - \frac{1}{2}\left(\frac{1}{d}+1\right) \leq -(\ell d \rho_\mu^{1/3})^{-1}} \chi\left(\frac{x}{\ell d} - u\right) \chi\left(\frac{y}{\ell d} - u\right) du.\end{aligned}$$

We have

$$\begin{aligned}\chi * \chi\left(\frac{x-y}{\ell d}\right) & - \int_{\|u\|_\infty - \frac{1}{2}\left(\frac{1}{d}+1\right) \leq -(\ell d \rho_\mu^{1/3})^{-1}} \chi\left(\frac{x}{\ell d} - u\right) \chi\left(\frac{y}{\ell d} - u\right) du \\ & = \int_{\|u\|_\infty - \frac{1}{2}\left(\frac{1}{d}+1\right) \geq -(\ell d \rho_\mu^{1/3})^{-1}} \chi\left(\frac{x}{\ell d} - u\right) \chi\left(\frac{y}{\ell d} - u\right) du.\end{aligned} \tag{B.16}$$

Since $x, y \in \Lambda$, the integral on the right is supported on $\|u\|_\infty - \frac{1}{2}\left(\frac{1}{d}+1\right) \leq 0$. Using the fact that $\rho_\mu^{-1/3} < \ell d/2$ and that χ is a product of symmetric decreasing functions of the coordinates u_1, u_2, u_3 respectively, we may observe that for fixed u_2, u_3, we have

$$\begin{aligned}& \max_{\frac{1}{2}\left(\frac{1}{d}+1\right) - (\ell d \rho_\mu^{1/3})^{-1} \leq |u_1| \leq \frac{1}{2}\left(\frac{1}{d}+1\right)} \chi\left(\frac{x}{\ell d} - u\right) \chi\left(\frac{y}{\ell d} - u\right) \\ & \leq \min_{\frac{1}{2}\left(\frac{1}{d}+1\right) - 2(\ell d \rho_\mu^{1/3})^{-1} \leq |u_1| \leq \frac{1}{2d} + \frac{1}{2} - (\ell d \rho_\mu^{1/3})^{-1}} \chi\left(\frac{x}{\ell d} - u\right) \chi\left(\frac{y}{\ell d} - u\right).\end{aligned} \tag{B.17}$$

Using this repeatedly (also with u_1, u_2 and u_1, u_3 fixed) gives the result in the lemma. □

As a consequence of the lemma we find from (B.12), if (B.13) is satisfied, that

$$\mathcal{H}_\Lambda(\rho_\mu) \geq \frac{b}{2}\ell^{-2}\sum_{i=1}^N Q_{\Lambda,i} + \int_{\|\ell du\|_\infty \leq \frac{1}{2}\ell(1+d)-\rho_\mu^{-1/3}} \mathcal{H}_{B(u)}(m(u)\rho_\mu)du, \tag{B.18}$$

where $m(u)=1$ if $\|\ell du\|_\infty \leq \frac{1}{2}\ell(1+d)-2\rho_\mu^{-1/3}$ and $m(u)=4$ otherwise, i.e., for u near the boundary.

The goal in the rest of this section is to give a lower bound on the ground state energy of the operators $\mathcal{H}_{B(u)}(m(u)\rho_\mu)$ to conclude an a priori lower bound on the ground state energy of $\mathcal{H}_\Lambda(\rho_\mu)$. We may now assume that the shortest side length of $B(u)$ satisfies $\lambda_1(u) \geq \rho_\mu^{-1/3}$, and we will make use of the fact that the range R of the potential satisfies $R \ll \rho_\mu^{-1/3}$. For simplicity, we will often omit the parameter u. A main ingredient in getting a lower bound is to get a priori bounds on the operators

$$n = \sum_{i=1}^N \mathbb{1}_{B,i}, \quad n_0 = \sum_{i=1}^N P_{B,i}, \quad n_+ = \sum_{i=1}^N Q_{B,i}. \tag{B.19}$$

Note that the operator n commutes with $\mathcal{H}_B$, so we may consider n a number.

Applying the decomposition of the potential energy in Section 6.4 to the small boxes we arrive at the following lemma.

Lemma B.2. *There is a constant $C>0$ such that on any small box B, we have*

$$-\rho_\mu \sum_{i=1}^N \int w_{1,B}(x,y)\,dy + \frac{1}{2}\sum_{i\neq j} w_B(x_i,x_j) \geq A_0 + A_2 - Ca(\rho_\mu + n_0|B|^{-1})n_+, \tag{B.20}$$

where

$$\begin{aligned} A_0 = {} & \frac{n_0(n_0-1)}{2|B|^2}\iint w_{2,B}(x,y)\,dxdy \\ & - \left(\rho_\mu \frac{n_0}{|B|} + \frac{1}{4}\left(\rho_\mu - \frac{n_0-1}{|B|}\right)^2\right)\iint w_{1,B}(x,y)\,dxdy \end{aligned} \tag{B.21}$$

and

$$A_2 = \frac{1}{2}\sum_{i\neq j} P_{B,i}P_{B,j} w_{1,B}(x_i,x_j) Q_{B,j}Q_{B,i} + \text{h.c.} \tag{B.22}$$

Proof. We use the identity (6.33), which also holds in the small boxes with P, Q and w, w_1, w_2 replaced by P_B, Q_B and $w_B, w_{1,B}, w_{2,B}$ respectively. Let us denote the corresponding terms $\mathcal{Q}^{\rm ren}_{i,B}$, $i = 0, \ldots, 4$. Then

$$\mathcal{Q}^{\rm ren}_{0,B} = \frac{n_0(n_0-1)}{2|B|^2} \iint w_{2,B}(x,y)\, dxdy - \rho_\mu \frac{n_0}{|B|} \iint w_{1,B}(x,y)\, dxdy.$$

As in the proof of Lemma 6.11 we apply a Cauchy-Schwarz inequality—using the positivity of w_B—to absorb $\mathcal{Q}^{\rm ren}_{3,B}$ in $\mathcal{Q}^{\rm ren}_{4,B}$. This results in the following inequality:

$$\begin{aligned}
&\mathcal{Q}^{\rm ren}_{3,B} + \mathcal{Q}^{\rm ren}_{4,B} \\
&\quad \geq -C \sum_{i\neq j} P_{B,i} Q_{B,j} w_{1,B}(x_i,x_j) Q_{B,j} P_{B,i} \\
&\qquad - \sum_{i\neq j} \Big(P_{B,i} Q_{B,j} w_{1,B}\omega(x_i,x_j) P_{B,j} P_{B,i} + \text{h.c.} \Big) \\
&\qquad - 2\sum_{i\neq j} \Big(P_{B,i} Q_{B,j} w_{1,B}\omega P_{B,j} Q_{B,i} + \text{h.c.} \Big) \\
&\quad \geq -C \sum_{i\neq j} P_{B,i} Q_{B,j} w_{1,B}(x_i,x_j) Q_{B,j} P_{B,i} \\
&\qquad\quad - \sum_{i\neq j} \Big(P_{B,i} Q_{B,j} w_{1,B}\omega(x_i,x_j) P_{B,j} P_{B,i} + \text{h.c.} \Big) \\
&\quad \geq - \sum_{i\neq j} \Big(P_{B,i} Q_{B,j} w_{1,B}\omega(x_i,x_j) P_{B,j} P_{B,i} + \text{h.c.} \Big) - C a n_0 |B|^{-1} n_+,
\end{aligned} \tag{B.23}$$

where we have used the pointwise inequality $0 \leq \omega \leq 1$, an additional Cauchy-Schwartz inequality in the second inequality, and

$$\begin{aligned}
&\sum_{i\neq j} P_{B,i} Q_{B,j} w_{1,B}(x_i,x_j) Q_{B,j} P_{B,i} \\
&\qquad \leq C \|\chi_B\|_\infty^2 n_0 |B|^{-1} n_+ \int W_1^{\rm s} \leq C a n_0 |B|^{-1} n_+,
\end{aligned} \tag{B.24}$$

which follows from

$$\int \chi_B(x) W_1^{\rm s}(x-y) \chi_B(y)\, dy \leq \|\chi_B\|_\infty^2 \int W_1^{\rm s}.$$

If we rewrite $\mathcal{Q}_{B,1}^{\rm ren}$ as in (6.40), the first term on the right side of (B.23) cancels the second line of (6.40). The remaining part of $\mathcal{Q}_{B,1}^{\rm ren}$ we estimate as follows:

(B.25)

$$\begin{aligned}
&|B|^{-1}(n_0-\rho_\mu|B|)\sum_i Q_{B,i}\chi_B(x_i)W_1^{\rm s}*\chi_B(x_i)P_{B,i}+\text{h.c.}\\
&\quad=|B|^{-1}({n_0}^{1/2}+(\rho_\mu|B|)^{1/2})\\
&\qquad\times\sum_i Q_{B,i}\chi_B(x_i)W_1^{\rm s}*\chi_B(x_i)P_{B,i}((n_0-1)^{1/2}-(\rho_\mu|B|)^{1/2})+\text{h.c.}\\
&\quad\geq -4|B|^{-1}\left(n_0^{1/2}+(\rho_\mu|B|)^{1/2}\right)^2\sum_i Q_{B,i}\chi_B(x_i)W_1^{\rm s}*\chi_B(x_i)Q_{B,i}\\
&\qquad-\frac{1}{4}|B|^{-1}\left((n_0-1)^{1/2}-(\rho_\mu|B|)^{1/2}\right)^2\sum_i P_{B,i}\chi_B(x_i)W_1^{\rm s}*\chi_B(x_i)P_{B,i}.
\end{aligned}$$

The first term above we estimate similarly to the estimate in (B.24). The last term above is equal to

$$\begin{aligned}
-\frac{1}{4}\frac{n_0}{|B|^2}&\left((n_0-1)^{1/2}-(\rho_\mu|B|)^{1/2}\right)^2\iint w_{1,B}(x,y)\,dxdy\\
&\geq-\frac{1}{4}\left(\frac{n_0-1}{|B|}-\rho_\mu\right)^2\iint w_{1,B}(x,y)\,dxdy,
\end{aligned}$$

where we used that $\rho_\mu|B|\geq 1$ to get the last inequality. This, together with $\mathcal{Q}_{0,B}^{\rm ren}$, give the A_0 term in the lemma.

The first three terms in $\mathcal{Q}_{2,B}^{\rm ren}$ are absorbed into the last term in (B.20) using again the same Cauchy-Schwartz as in the second inequality in (B.23). Finally, the last terms in $\mathcal{Q}_{2,B}^{\rm ren}$ are exactly the terms collected in A_2. □

We express the term A_2 from the lemma in second quantization. Introducing the operators

$$b_p^\dagger=|B|^{-1/2}a^\dagger(Q_B\chi_Be^{-ipx})a_0$$

we can write

$$A_2=\frac{1}{2}(2\pi)^{-3}\int\widehat{W_1^{\rm s}}(p)(b_p^\dagger b_{-p}^\dagger+b_{-p}b_p)dp.$$

We shall control A_2 using Bogolubov's method. In order to do this we will add and subtract a term

$$A_1=(2\pi)^{-3}K_{\rm s}a\int(b_p^\dagger b_p+b_{-p}^\dagger b_{-p})dp, \tag{B.26}$$

with the constant $K_{\rm s}>0$ chosen appropriately. Note that we have

$$A_1\leq K_{\rm s}a\frac{n_0+1}{|B|}n_+\|\chi_B\|_\infty^2\leq CK_{\rm s}a\frac{n_0+1}{|B|}n_+. \tag{B.27}$$

LEMMA B.3 (Bogolubov's method in small boxes). *There exists a constant* $C > 0$ *such that*

(B.28)
$$\sum_{i=1}^{N} Q_{B,i}\chi_{B,i}[\sqrt{-\Delta_i} - (ds\ell)^{-1}]_+^{-2}\chi_{B,i}Q_{B,i} + A_2$$
$$\geq -\frac{1}{2}(1 + C(R/(d\ell))^2)(1 + C(ds\ell)^{-1}a)\widehat{g\omega}(0)\frac{(n+1)n}{|B|^2}\int \chi_B^2$$
$$- C\left(a^2(ds\ell)^{-1}\log(ds\ell a^{-1})\frac{n+1}{|B|} + a^4(ds\ell)^3\left(\frac{n+1}{|B|}\right)^3 + a(ds\ell)^{-3}\right)\frac{n}{|B|}\int \chi_B^2$$
$$- Ca\frac{n+1}{|B|}n_+.$$

Moreover, for all $\varepsilon > 0$, *there is a* $C_\varepsilon > 0$ *such that if*

$$(R/d\ell)^2 < C_\varepsilon^{-1}, \quad a(ds\ell)^{-1}\log(ds\ell a^{-1}) < C_\varepsilon^{-1}, \tag{B.29}$$

then

$$\begin{aligned}\sum_{i=1}^{N} Q_{B,i}\chi_{B,i}[\sqrt{-\Delta_i} - (ds\ell)^{-1}]_+^{-2}\chi_{B,i}Q_{B,i} + A_2 \\ \geq -\frac{1}{2}\left((1+\varepsilon)\widehat{g\omega}(0) + \varepsilon a\right)\frac{(n+1)n}{|B|^2}\int \chi_B^2 \\ - C_\varepsilon a(ds\ell)^{-3}\frac{n}{|B|}\int \chi_B^2 - C_\varepsilon a\frac{n+1}{|B|}n_+.\end{aligned} \tag{B.30}$$

Proof. We add A_1 from (B.26) to the term we want to estimate. Using $n_0 \leq n$ we may write

$$\sum_{i=1}^{N} Q_{B,i}\chi_{B,i}[\sqrt{-\Delta_i} - (ds\ell)^{-1}]_+^{-2}\chi_{B,i}Q_{B,i} + A_1 + A_2 \geq (2\pi)^{-3}\frac{1}{2}\int h(p)dp,$$

where h is the operator

$$h(p) = \left(\frac{|B|}{n+1}[|p| - (ds\ell)^{-1}]_+^2 + 2K_{\mathrm{s}}a\right)(b_p^\dagger b_p + b_{-p}^\dagger b_{-p}) + \widehat{W_1^{\mathrm{s}}}(p)(b_p^\dagger b_{-p}^\dagger + b_{-p}b_p).$$

We observe that

$$[b_p, b_p^\dagger] \leq n_0|B|^{-1}\int \chi_B^2 \leq n|B|^{-1}\int \chi_B^2.$$

We will now apply the simple case of Bogolubov's method in Theorem A.1 with

$$\mathcal{A}(p) = \frac{|B|}{n+1}[|p| - (ds\ell)^{-1}]_+^2 + 2K_{\mathrm{s}}a, \quad \mathcal{B}(p) = \widehat{W_1^{\mathrm{s}}}(p).$$

We have by (B.9) that

$$|\mathcal{B}(p)| = |\widehat{W_1^{\mathrm{s}}}(p)| \le \int W_1^{\mathrm{s}} \le C_0 a.$$

If we therefore choose $K_{\mathrm{s}} \ge C_0$, we see that $|\mathcal{B}|/\mathcal{A} \le 1/2$, and we get the following lower bound from Theorem A.1:

$$h(p) \ge -\frac{1}{2}\left(\mathcal{A}(p) - \sqrt{\mathcal{A}(p)^2 - \mathcal{B}(p)^2}\right) n_0 |B|^{-1} \int \chi_B^2.$$

Using that $|\mathcal{B}|/\mathcal{A} \le 1/2$ we have

$$h(p) \ge -C\frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} n|B|^{-1} \int \chi_B^2.$$

We use this for $|p| < 2(ds\ell)^{-1}$, and for the integral over $|p| < 2(ds\ell)^{-1}$, we find

$$\text{(B.31)} \qquad \int_{|p|<2(ds\ell)^{-1}} \frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} dp \le \frac{C_0^2}{K_{\mathrm{s}}} a \int_{|p|<2(ds\ell)^{-1}} 1 dp \le \frac{C_0^2}{K_{\mathrm{s}}} (ds\ell)^{-3}.$$

For the simple bound (B.30), we may choose K_s large depending on ε to have

$$h(p) \ge -\frac{1}{2}(1+\varepsilon/2)\frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} n|B|^{-1} \int \chi_B^2$$

and use this in the range $|p| > 2(ds\ell)^{-1}$. For the more refined bound (B.28), in the range $|p| > 2(ds\ell)^{-1}$, we use

$$h(p) \ge -\left(\frac{1}{2}\frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} + C\frac{\mathcal{B}(p)^4}{\mathcal{A}(p)^3}\right) n|B|^{-1} \int \chi_B^2.$$

For $|p| > 2(ds\ell)^{-1}$, we have

$$\frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} \le \frac{n+1}{|B|} \frac{\widehat{W_1^{\mathrm{s}}}(p)^2}{(|p| - (ds\ell)^{-1})^2} \le \frac{\widehat{W_1^{\mathrm{s}}}(p)^2}{p^2}(1 + C(ds\ell)^{-1}|p|^{-1})\frac{n+1}{|B|},$$

and hence by splitting the integral over the error in $|p| < a^{-1}$ and $|p| > a^{-1}$ we obtain

$$\int_{|p|>2(ds\ell)^{-1}} \frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} dp \\ \le (1 + C(ds\ell)^{-1}a)\frac{n+1}{|B|} \int_{\mathbb{R}^3} \frac{\widehat{W_1^{\mathrm{s}}}(p)^2}{p^2} dp + Ca^2(ds\ell)^{-1}\frac{n+1}{|B|}\log(ds\ell a^{-1}).$$

Finally, we use that

(B.32)
$$\begin{aligned}\frac{1}{4}(2\pi)^{-3}\int_{\mathbb{R}^3}\frac{\widehat{W_1^{\rm s}}(p)^2}{p^2}dp =&\frac{1}{4}\iint\frac{W_1^{\rm s}(x)W_1^{\rm s}(y)}{4\pi|x-y|}dxdy\\ \leq&\frac{1}{4}(1+C(R/(d\ell))^2)\iint\frac{g(x)g(y)}{4\pi|x-y|}dxdy\\ =&\frac{1}{4}(1+C(R/(d\ell))^2)(2\pi)^{-3}\int_{\mathbb{R}^3}\frac{\widehat{g}(p)^2}{p^2}dp\\ =&\frac{1}{2}(1+C(R/(d\ell))^2)\widehat{g\omega}(0).\end{aligned}$$

Finally, to get (B.28) we estimate

(B.33)
$$\begin{aligned}&\int_{|p|>2(ds\ell)^{-1}}\frac{\mathcal{B}(p)^4}{\mathcal{A}(p)^3}\\ &\leq (\mathrm{cst.})a^4\left(\frac{n+1}{|B|}\right)^3\int_{|p|>2(ds\ell)^{-1}}|p|^{-6}dp = Ca^4\left(\frac{n+1}{|B|}\right)^3(ds\ell)^3.\end{aligned}$$

Using the estimate (B.27) on A_1 gives the last term in (B.28). □

In order to use this lemma we will control the negative term quadratic in n in (B.30) in terms of the positive term quadratic in n_0 in (B.21). The difference between n and n_0 will be absorbed in the Neumann gap of $\mathcal{T}_B$. It is, however, important to establish the result in the following lemma.

LEMMA B.4. *There is a constant $C>0$ such that if the shortest side length λ_1 of the box B satisfies $R\leq\frac{1}{2}C^{-1/2}\min\{\lambda_1,\ell d\}$, then*

(B.34)
$$\iint w_{1,B}(x,y)dxdy\geq 8\pi a\left(1-C\Big(\frac{R}{\lambda_1}\Big)^2\right)\left(1-C\Big(\frac{R}{\ell d}\Big)^2\right)\int\chi_B^2,$$

(B.35)
$$\begin{aligned}&\iint w_{2,B}(x,y)dxdy\\ &\quad\geq\iint w_{1,B}(x,y)dxdy+\left(1-C\Big(\frac{R}{\lambda_1}\Big)^2\right)\left(1-C\Big(\frac{R}{\ell d}\Big)^2\right)\widehat{g\omega}(0)\int\chi_B^2.\end{aligned}$$

Moreover, for any $0<\varepsilon<1/10$, we can find a $C'_\varepsilon>0$ such that if $R\leq (C'_\varepsilon)^{-1/2}\min\{\lambda_1,\ell d\}$, then

(B.36)
$$\iint w_{2,B}(x,y)dxdy\geq\frac{3}{4}\iint w_{1,B}(x,y)dxdy+((1+\varepsilon)\widehat{g\omega}(0)+\varepsilon a)\int\chi_B^2.$$

Proof. The estimate (B.35) follows from

$$\begin{aligned}\iint w_{2,B}(x,y)dxdy - \iint w_{1,B}(x,y)dxdy \\ &= \iint \omega(x-y)w_{1,B}(x,y)dxdy \\ &\geq \int \omega(x)W_1^{\mathrm{s}}(x)dx\left(\int \chi_B^2 - CR^2\|\nabla^2\chi_B\|_\infty \int \chi_B\right) \\ &\geq \left(1-C(R\lambda_1^{-1})^2\right)\int \omega(x)W_1^{\mathrm{s}}(x)dx \int \chi_B^2 \\ &\geq \left(1-C(R\lambda_1^{-1})^2\right)\left(1-C(R/(\ell d))^2\right)\left(\int g\omega\right)\int \chi_B^2,\end{aligned}$$

where we have used that ωW is spherically symmetric, that $|B|^{-1}\left(\int \chi_B\right)^2 \leq \int \chi_B^2$, and that

$$\|\partial_i\partial_j\chi_B\|_\infty \leq C_M\lambda_1^{-2}|B|^{-1}\int \chi_B, \tag{B.37}$$

which is a simple exercise (see Appendix C). The estimate (B.34) follows in the same way without ω and using $\int g = 8\pi a$. Finally, (B.36) follows from $\omega \leq 1$. □

We are now ready to give the bound on the energy in the small boxes.

Theorem B.5 (Lower bound on energy in small boxes). *Assume B is a box with shortest side length $\lambda_1 \geq \rho_\mu^{-1/3}$. There are universal constants $C, C' > 1$ and $0 < c < 1/2$ such that for all $1 \leq K_B \leq C'^{-1}(\rho_\mu a^3)^{-1/6}$, we have for the Hamiltonian defined in (B.4) that*

$$\begin{aligned}\mathcal{H}_B(\rho_\mu) \geq &\left(c\frac{(n|B|^{-1}-\rho_\mu)^2}{1+\frac{n}{|B|\rho_\mu}} - \frac{1}{2}\rho_\mu^2\right)\iint w_{1,B}(x,y)dxdy \\ &- C\rho_\mu^2 a\left((R\lambda_1^{-1})^2 + K_B^3(\rho_\mu a^3)^{1/2}\right)\int \chi_B^2 - C\rho_\mu a\end{aligned} \tag{B.38}$$

if

$$C'\varepsilon_T^{-1/2}(\rho_\mu a^3)^{1/2} \leq a(d\ell)^{-1} \leq a(ds\ell)^{-1}\ln(ds\ell a^{-1}) \leq K_B(\rho_\mu a^3)^{1/2} \tag{B.39}$$

and

$$R \leq K_B^{1/2}(\rho_\mu a^3)^{1/4}(\rho_\mu a)^{-1/2}. \tag{B.40}$$

We are assuming that $\varepsilon_T, s, d \leq 1$.

Note that all the assumptions on K_B, R, ε_T, s, and d are satisfied with our choices in Section 5 if $\rho_\mu a^3$ is small enough. Specifically, the assumption on K_B is a consequence of (5.7), formula (B.39) follows from (5.4), (5.6), and (5.7), and (B.40) was given in (5.29).

Proof. Note that (B.39) is equivalent to

$$(\rho_\mu a)^{-1/2} K_B^{-1} \le \frac{sd\ell}{\ln(ds\ell a^{-1})} \le d\ell \le \frac{\varepsilon_T^{1/2}}{C'}(\rho_\mu a)^{-1/2}.$$

This, in particular, implies that

$$\sum_{i=1}^{N} \frac{1}{2}\varepsilon_T (1+\pi^{-2})^{-1}(\ell d)^{-2} Q_{B,i} \ge C'^2(1+\pi^{-2})^{-1}\rho_\mu a n_+. \tag{B.41}$$

Moreover, we see from (B.40) that

$$R\rho_\mu^{1/3} \le K_B^{1/2}(\rho_\mu a^3)^{1/12}, \quad R/(\ell d) \le K_B^{3/2}(\rho_\mu a^3)^{1/4}.$$

We now first choose ε so small, e.g., to be 1/20, so that we can apply Lemma B.4. Hence if C' is large enough, we can, since $\lambda_1 > \rho_\mu^{-1/3}$, use (B.34), (B.35), and (B.36) from Lemma B.4. We choose the same ε in (B.30) and again, by assuming that C' large enough, we can ensure that (B.29) is satisfied.

We may of course assume that $n > 0$, since the inequality we want to prove is obviously satisfied if $n = 0$ since the operator is 0 whereas the lower bound is negative in this case. We choose a constant $\Xi > 2$ to be determined precisely below (see estimate (B.43)) to depend only on the constants C and C_ε in Lemmas B.3 and B.4. Our final choice of the constant C' in the theorem will also depend on the choice of Ξ. Observe that $\rho_\mu|B| \ge 1$. Hence we can choose an integer n' in the interval $[\Xi\rho_\mu|B|, (\Xi+1)\rho_\mu|B|)$, and we may write $n = mn' + n''$ with m, n', n'' non-negative integers and $n'' < n' < (\Xi+1)\rho_\mu|B|$. We will get a lower bound on the energy if in the Hamiltonian we think of dividing the particles in m groups of n' particles and one group of n'' particles ignoring the positive interaction between the groups. It is not important that the Hamiltonian is no longer symmetric between the particles since we are not considering it as an operator on the symmetric subspace, but only calculating its expectation value in a symmetric state. We arrive at the conclusion that if we denote by $e_B(n, \rho_\mu)$ the ground state energy of $\mathcal{H}_B(\rho_\mu)$ restricted to states with n particles in the box B, then

$$e_B(n, \rho_\mu) \ge m e_B(n', \rho_\mu) + e_B(n'', \rho_\mu). \tag{B.42}$$

We have that both n' and n'' are less than $(\Xi+1)\rho_\mu|B| \le 2\Xi\rho_\mu|B|$. This means that the last terms in (B.20), (B.28), and (B.30) in both cases can be absorbed in the positive term from (B.41) if we choose $C' \ge C\Xi^{1/2}$. Using (B.10) we see that the same is also true for the errors we get by replacing n_0' and n_0'' by n' and n'' respectively everywhere in A_0 in (B.21).

In the case of the m groups of n' particles we will use Lemma B.2 and (B.30) to arrive at

$$\begin{aligned} e_B(n',\rho_\mu) \geq \frac{n'^2}{2|B|^2} \iint w_{2,B}(x,y)\,dxdy \\ - \left(\rho_\mu \frac{n'}{|B|} + \frac{1}{4}\left(\rho_\mu - \frac{n'}{|B|}\right)^2\right) \iint w_{1,B}(x,y)\,dxdy \\ - \frac{1}{2}\left((1+\varepsilon)\widehat{g\omega}(0) + \varepsilon a\right)\frac{n'^2}{|B|^2}\int \chi_B^2 - C\rho_\mu a n'|B|^{-1}\int \chi_B^2, \end{aligned}$$

where we have used that (B.39) and the assumption on K_B imply that $a(ds\ell)^{-3} \leq \rho_\mu a$. We have also used that the error in replacing $n'-1$ by n' in several terms can also be absorbed in the last term. Thus applying (B.36) we arrive at

$$e_B(n',\rho_\mu) \geq \frac{1}{8}\left(\rho_\mu - \frac{n'}{|B|}\right)^2 \iint w_{1,B}(x,y)\,dxdy - C\rho_\mu a n'|B|^{-1}\int \chi_B^2.$$

It follows, using (B.34), that if we choose the constant Ξ large enough depending only on the constants in Lemmas B.3 and B.4, then

$$\text{(B.43)} \qquad \begin{aligned} e_B(n',\rho_\mu) &\geq \frac{1}{9}\left(\rho_\mu - \frac{n'}{|B|}\right)^2 \iint w_{1,B}(x,y)\,dxdy \\ &\geq \frac{1}{18}\rho_\mu \frac{n'}{|B|}\iint w_{1,B}(x,y)\,dxdy \geq 0. \end{aligned}$$

This is what fixes the choice of Ξ. Hence

$$\text{(B.44)} \qquad \begin{aligned} m e_B(n',\rho_\mu) &\geq \frac{1}{18}\rho_\mu \frac{mn'}{|B|}\iint w_{1,B}(x,y)\,dxdy \\ &= \frac{1}{18}\rho_\mu \frac{n-n''}{|B|}\iint w_{1,B}(x,y)\,dxdy. \end{aligned}$$

We turn to the group of n'' particles. We can again replace n_0'' by n'' by absorbing the resulting error terms in the positive gap. If we apply Lemma B.2 and (B.28), we see that since $n'' \leq 2\Xi\rho_\mu|B|$, we have

$$\begin{aligned} e_B(n'',\rho_\mu) \geq \frac{n''^2}{2|B|^2}\iint w_{2,B}(x,y)\,dxdy \\ - \left(\rho_\mu \frac{n''}{|B|} + \frac{1}{4}\left(\rho_\mu - \frac{n''}{|B|}\right)^2\right)\iint w_{1,B}(x,y)\,dxdy \\ - \frac{1}{2}\widehat{g\omega}(0)\frac{n''^2}{|B|^2}\int \chi_B^2 - C\Xi^4\rho_\mu^2 a K_B^3(\rho_\mu a^3)^{1/2}\int \chi_B^2 - C\Xi\rho_\mu a. \end{aligned}$$

The last term comes from repeatedly replacing $n''-1$ by n'' in the leading terms, which leads to an error $n'' a|B|^{-2}\int \chi_B^2 \leq Cn''|B|^{-1}a$. In the error terms we can, for the same replacement, alternatively use that $1 \leq \rho_\mu|B|$.

If we now apply the estimate (B.35) in Lemma B.4, we find that

(B.45)
$$e_B(n'',\rho_\mu) \geq \frac{1}{4}\left(\frac{n''}{|B|}-\rho_\mu\right)^2 \iint w_{1,B}(x,y)\,dxdy - \frac{1}{2}\rho_\mu^2 \iint w_{1,B}(x,y)\,dxdy - C\rho_\mu^2 a\left((R\lambda_1^{-1})^2 + K_B^3(\rho_\mu a^3)^{1/2}\right)\int \chi_B^2 - C\rho_\mu a,$$

where we have now ignored the explicit dependence on Ξ, which is after all now a chosen constant.

We have arrived at the bound that

$$e_B(n,\rho_\mu) \geq \left(\frac{1}{4}\left(\frac{n''}{|B|}-\rho_\mu\right)^2 + \frac{1}{18}\rho_\mu\frac{n-n''}{|B|}\right)\iint w_{1,B}(x,y)\,dxdy - \frac{1}{2}\rho_\mu^2 \iint w_{1,B}(x,y)\,dxdy - C\rho_\mu a^2\left((R\lambda_1^{-1})^2 + K_B^3(\rho_\mu a^3)^{1/2}\right)\int \chi_B^2 - C\rho_\mu a.$$

This easily implies the result in the theorem. □

We will now apply the small box estimate from the previous theorem to get an a priori bound on the energy and on the number of particles n and excited particles n_+ in the large box.

THEOREM B.6 (A priori estimates in large box). *Assume* (5.1), (B.39), (B.40). *Then there is a constant $C>0$ such that if $1 \leq K_B \leq C'^{-1}(\rho_\mu a^3)^{-1/6}$ and $\rho_\mu a^3$ is smaller than some universal constant, then we have*

$$\mathcal{H}_\Lambda(\rho_\mu) \geq -4\pi\rho_\mu^2 a\ell^3(1 + CK_B^3(\rho_\mu a^3)^{1/2}) + \frac{b}{2\ell^2}n_+. \tag{B.46}$$

Moreover, if there exists a normalized $\Psi \in \mathcal{F}_{\rm s}(L^2(\Lambda))$ with n particles in Λ, such that (7.1) *holds for a $0 < J \leq K_B^3$, then the a priori bounds* (7.2) *on n and n_+ hold.*

As explained just after Theorem B.5 the assumptions (B.39), (B.40), and the assumption on K_B are satisfied with our choices in Section 5.

Proof. We use (B.18) together with the estimate in Theorem B.5. We will denote by $n(u), n_0(u)$, and $n_+(u)$ the operators defined in (B.19). The corresponding operators in the large box Λ will be denoted n, n_0, and n_+. On the set

$$\mathcal{I} = \left\{u \in \left[-\frac{1}{2}(1+\frac{1}{d}), \frac{1}{2}(1+\frac{1}{d})\right]^3 \;\middle|\; \frac{1}{2}\ell(1+d) - 2\rho_\mu^{-1/3} \leq \|\ell du\|_\infty \leq \frac{1}{2}\ell(1+d) - \rho_\mu^{-1/3}\right\}$$

we have that ρ_μ is replaced by $4\rho_\mu$. On this set we have, according to (C.6), that $|\chi_{B(u)}(x)| \leq C(\rho_\mu^{-1/3}/\ell)^M \leq C(\rho_\mu a^3)^{M/6}$ with (C depending on M), and therefore

$$\int_{\mathcal{I}} \int \chi_{B(u)}(x)^2 dx\, du \leq C(\rho_\mu a^3)^{M/3}(\ell d)^3 d^{-3} \leq C(\rho_\mu a^3)^{M/3}\ell^3. \tag{B.47}$$

If we use Theorem B.5 and (B.10) to get the the rough estimate

$$\mathcal{H}_B(4\rho_\mu) \geq -C\rho_\mu^2 a \int \chi_B^2 - C\rho_\mu a,$$

we obtain

$$\int_{\mathcal{I}} \mathcal{H}_B(4\rho_\mu) \geq -C\rho_\mu^2 a(\rho_\mu a^3)^{M/3}\ell^3 - C\rho_\mu a d^{-3}. \tag{B.48}$$

In order to apply the estimate in Theorem B.5 over the remaining u, we need to control

$$\begin{aligned} &\int (R\lambda_1(u)^{-1})^2 \int \chi_{B(u)}^2(x) dx du \\ &\leq CR^2(d\ell)^{-2} \int (\lambda_1(u)/(d\ell))^{M-2} \int \chi_{B(u)}(x) dx du \\ &\leq C(R/(\ell d)^2)\ell^3 \leq CK_B^3 \ell^3, \end{aligned} \tag{B.49}$$

where we have used (C.5), i.e., $\|\chi_B\|_\infty \leq C(\lambda_1/(d\ell))^M$ and $\iint \chi_{B(u)}(x) dx du = C\ell^3$. If we combine this with (B.48) (with $M = 8$), (B.18), (B.11), (B.3), and the estimate in Theorem B.5, we arrive at the final a priori lower bound

$$\begin{aligned} \langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi\rangle \geq & \mathcal{R}_\Psi + \left\langle \Psi, \frac{b}{2\ell^2} n_+ \Psi \right\rangle - 4\pi\rho_\mu^2 a\ell^3 - CK_B^3(\rho_\mu a^3)^{1/2}\ell^3 - C\rho_\mu a d^{-3} \\ \geq & \mathcal{R}_\Psi + \left\langle \Psi, \frac{b}{2\ell^2} n_+ \Psi \right\rangle - 4\pi\rho_\mu^2 a\ell^3 \left(1 + CK_B^3(\rho_\mu a^3)^{1/2}\right), \end{aligned}$$

where

$$0 \leq \mathcal{R}_\Psi = \left\langle \Psi, \left(\int_{\mathcal{I}_-} F\left(\frac{n(u)}{|B(u)|}\right) \iint w_{1,B(u)}(x,y) dx dy du \right) \Psi \right\rangle$$

with $F(t) = c\frac{(t-\rho_\mu)^2}{1+t\rho_\mu^{-1}}$ and

$$\mathcal{I}_- = \left\{ u \in \mathbb{R}^3 \mid \|\ell d u\|_\infty \leq \frac{1}{2}\ell(1+d) - 2\rho_\mu^{-1/3} \right\}.$$

Since $\mathcal{R}_\Psi$ and n_+ are non-negative, this immediately gives (B.46) and

$$R_\Psi \leq C\rho_\mu^2 a\ell^3 K_B^3(\rho_\mu a^3)^{1/2} \quad \text{and} \quad \langle \Psi, n_+\Psi\rangle \leq C\rho_\mu \ell^3 K_B^3 K_\ell^2 (\rho_\mu a^3)^{1/2} \tag{B.50}$$

for a normalized n-particle Ψ satisfying (7.1). It remains to establish the a priori bound on n in (7.2).

Using that the function F is convex and denoting

$$\mathcal{C} = \int_{\mathcal{I}_-} \iint w_{1,B(u)}(x,y)dxdydu,$$

we obtain

$$R_\Psi \geq \mathcal{C} F\left(\mathcal{C}^{-1}\left\langle \Psi, \left(\int_{\mathcal{I}_-} \frac{n(u)}{|B(u)|} \iint w_{1,B(u)}(x,y)dxdydu\right)\Psi\right\rangle\right). \tag{B.51}$$

We have by (B.11) that

$$8\pi a\ell^3(1 - C(\rho_\mu a^3)^{M/3}) \leq \mathcal{C} \leq 8\pi a\ell^3,$$

where we used (B.10) and as in (B.47) that $|\chi_{B(u)}(x)| \leq C(\rho_\mu^{-1/3}/\ell)^M \leq C(\rho_\mu a^3)^{M/6}$ for u outside $\mathcal{I}_-$.

We may write

$$\mathcal{C}^{-1}\int_{\mathcal{I}_-} \frac{n(u)}{|B(u)|} \iint w_{1,B(u)}(x,y)dxdydu = \sum_{i=1}^N U(x_i),$$

where

$$U(z) = \mathcal{C}^{-1}\int_{\mathcal{I}_-} |B(u)|^{-1}\mathbb{1}_{B(u)}(z) \iint w_{1,B(u)}(x,y)dxdydu.$$

Using the form of F and the a priori bound on $\mathcal{R}_\Psi$ in (B.50), we see that

$$\left|\left\langle \Psi, \sum_i U(x_i)\Psi\right\rangle - \rho_\mu\right| \leq C\rho_\mu K_B^{3/2}(\rho_\mu a^3)^{1/4}. \tag{B.52}$$

Note that by (B.10) and $\int \mathbb{1}_{B(u)}du = \mathbb{1}_\Lambda$, we have that $U(z) \leq C\ell^{-3}$, and that

$$P_\Lambda U P_\Lambda = P_\Lambda |\Lambda|^{-1}\int_\Lambda U(z)dz = P_\Lambda \ell^{-3}.$$

Using that for all $\varepsilon > 0$,

$$\begin{aligned}(1-\varepsilon)\sum_{i=1}^N (P_\Lambda U P_\Lambda)_i - \varepsilon^{-1}\sum_{i=1}^N (Q_\Lambda U Q_\Lambda)_i \\ \leq \sum_{i=1}^N U(x_i) \leq (1+\varepsilon)\sum_{i=1}^N (P_\Lambda U P_\Lambda)_i + (1+\varepsilon^{-1})\sum_{i=1}^N (Q_\Lambda U Q_\Lambda)_i,\end{aligned}$$

we see that

$$(1-\varepsilon)n_0\ell^{-3} - C\varepsilon^{-1}n_+\ell^{-3} \leq \sum_{i=1}^N U(x_i) \leq (1+\varepsilon)n_0\ell^{-3} + (1+\varepsilon^{-1})Cn_+\ell^{-3}.$$

Choosing $\varepsilon = K_B^{3/2}K_\ell(\rho_\mu a^3)^{1/4}$ and using the a priori bounds on the expectation values of n_+ in (B.50) and U in (B.52), we conclude the result in the theorem. □

Appendix C. The explicit localization function

In this section we discuss the explicit choice of the localization function χ and its properties. Define

$$\zeta(y) = \begin{cases} \cos(\pi y), & |y| \leq 1/2, \\ 0, & |y| > 1/2 \end{cases}$$

and

$$\chi(x) = C_M \left(\zeta(x_1)\zeta(x_2)\zeta(x_3)\right)^M . \tag{C.1}$$

Here $M \in \mathbb{N}$ is to be chosen large enough; we explained the need to choose $M = 30$ in Section 5. The constant C_M is chosen such that the normalization $\int \chi^2 = 1$ from (6.1) holds. We have $0 \leq \chi \in C^{M-1}(\mathbb{R}^3)$.

LEMMA C.1. *Let χ be the localization function from (C.1). Let $\widetilde{M} = \max\{n \in \mathbb{Z} | 2n \leq M\}$. Then, for all $k \in \mathbb{R}^3$,*

$$|\widehat{\chi}(k)| \leq C_\chi (1 + |k|^2)^{-\widetilde{M}}, \tag{C.2}$$

where

$$C_\chi = \int \left| (1 - \Delta)^{\widetilde{M}} \chi \right| . \tag{C.3}$$

In particular, when $|k| \geq \frac{1}{2} \widetilde{K}_H^{-1} (\rho_\mu a^3)^{\frac{5}{12}} a^{-1}$, with the notation from (5.14), we have

$$|\widehat{\chi_\Lambda}(k)| = \ell^3 |\widehat{\chi}(k\ell)| \leq C \ell^3 (K_\ell^{-2} \widetilde{K}_H^2 (\rho_\mu a^3)^{\frac{1}{6}})^{\widetilde{M}}. \tag{C.4}$$

The proof of Lemma C.1 is elementary and will be omitted.

The explicit choice of χ is important when we analyze the behavior of the small box localization function. Recall that according to (B.2) and the explicit choice of χ, we may write $\chi_B(x) = C_M^2 F(x)^M$, where

$$F(x) = h_{u_1}(x_1) h_{u_2}(x_2) h_{u_3}(x_3)$$

and

$$h_v(t) = \zeta\left(\frac{t}{\ell}\right) \zeta\left(\frac{t}{\ell d} - v\right).$$

If we denote by λ_1 the shortest side length in the box B, we see by estimating one of the ζ factors of scale $d\ell$ and using that it must vanish at one of the sides that

$$\chi_B(x) \leq C C_M^2 (\lambda_1/(d\ell))^M. \tag{C.5}$$

If the shortest side length λ_1 of the box B satisfies that $\lambda_1 < d\ell$, we can improve this slightly to

$$\chi_B(x) \leq C C_M^2 (\lambda_1/\ell)^M. \tag{C.6}$$

This follows by estimating a ζ factor of scale ℓ and using that *it* vanishes at one of the sides.

In the rest of this short appendix we will briefly sketch how to get the estimate (B.37) on χ_B. Our first claim is that

$$\|\chi_B\|_\infty \leq C'_M |B|^{-1} \int \chi_B,$$

for some constant C'_M depending on M. It is enough to show this for the function $h_v(t)^M$. Since ζ is concave on its support we have that if h_v is supported on $[a, b]$ and takes its maximum in c then

$$h_v(t) \geq \|h_v\|_\infty \min\left\{\frac{(t-a)^2}{(c-a)^2}, \frac{(t-b)^2}{(c-b)^2}\right\}.$$

In particular, h_v is bigger than $\frac{1}{4}\|h_v\|_\infty$ on half the interval. The claim follows from this.

Our second claim is that

$$\max_i \|\partial_i \chi_B\|_\infty \leq C'_M \lambda_1^{-1} \|\chi_B\|_\infty, \quad \max_{i,j} \|\partial_i \partial_j \chi_B\|_\infty \leq C'_M \lambda_1^{-2} \|\chi_B\|_\infty.$$

It is easy to see that it is enough to show these properties for h_v, i.e., that

$$\|h'_v\|_\infty \leq C'(b-a)^{-1}\|h_v\|_\infty, \qquad \|h''_v\|_\infty \leq C'(b-a)^{-2}\|h_v\|_\infty.$$

In the case when $(b-a) < \ell d$, we have that one factor in h_v vanishes at one end point and the other factor vanishes at the other endpoint. It is then easy to see that $\|h'_v\|_\infty \leq C(b-a)/(d\ell^2)$, $\|h''_v\|_\infty \leq C(\ell^{-2} + (\ell d)^{-2})\|h_v\|_\infty + C(\ell^2 d)^{-1}$, and $\|h_v\|_\infty \geq c(b-a)^2(d\ell^2)^{-1}$. In case $b - a = \ell d$. Both endpoints occur when the second ζ factor in h_v vanish. Without loss of generality we may consider $v > 0$ and let $D = |\ell(1/2 - dv)|$ denote the distance from the middle of the support of h_v, i.e., ldv to the right endpoint of the support of the first ζ factor, i.e., $\ell/2$. Then $\ell d/2 \leq D \leq \ell/2$ and

$$\begin{aligned}
\|h'_v\| &\leq C'(\ell^2 d)^{-1} D, \\
\|h''_v\| &\leq C'(\ell^{-2} + (\ell d)^{-2})\|h_v\|_\infty + C'(\ell^2 d)^{-1}, \\
\|h_v\|_\infty &\geq cD/\ell.
\end{aligned}$$

References

[1] A. Aaen, The ground state energy of a dilute Bose gas in dimension $n > 3$, 2014, Ph.D. thesis, Department of Mathematics, Aarhus University.

[2] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein, Complete Bose-Einstein condensation in the Gross-Pitaevskii regime, *Comm. Math. Phys.* **359** no. 3 (2018), 975–1026. MR 3784538. Zbl 1391.82004. https://doi.org/10.1007/s00220-017-3016-5.

[3] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein, Bogoliubov theory in the Gross-Pitaevskii limit, *Acta Math.* **222** no. 2 (2019), 219–335. MR 3974476. Zbl 1419.82064. https://doi.org/10.4310/ACTA.2019.v222.n2.a1.

[4] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein, The excitation spectrum of Bose gases interacting through singular potentials, *J. Eur. Math. Soc.* (*JEMS*) **22** no. 7 (2020), 2331–2403. MR 4107508. Zbl 07227736. https://doi.org/10.4171/JEMS/966.

[5] N. Bogolubov, On the theory of superfluidity, *Acad. Sci. USSR. J. Phys.* **11** (1947), 23–32. MR 0022177.

[6] B. Brietzke, S. r. Fournais, and J. P. Solovej, A simple 2nd order lower bound to the energy of dilute Bose gases, *Comm. Math. Phys.* **376** no. 1 (2020), 323–351. MR 4093872. Zbl 07202521. https://doi.org/10.1007/s00220-020-03715-2.

[7] B. Brietzke and J. P. Solovej, The Second-Order Correction to the Ground State Energy of the Dilute Bose Gas, *Ann. Henri Poincaré* **21** no. 2 (2020), 571–626. MR 4056278. Zbl 1434.82042. https://doi.org/10.1007/s00023-019-00875-3.

[8] J. G. Conlon, E. H. Lieb, and H.-T. Yau, The $N^{7/5}$ law for charged bosons, *Comm. Math. Phys.* **116** no. 3 (1988), 417–448. MR 0937769. https://doi.org/10.1007/BF01229202.

[9] J. Dereziński and M. Napiórkowski, Excitation spectrum of interacting bosons in the mean-field infinite-volume limit, *Ann. Henri Poincaré* **15** no. 12 (2014), 2409–2439. MR 3272826. Zbl 1305.82042. https://doi.org/10.1007/s00023-013-0302-4.

[10] F. J. Dyson, Ground-state energy of a hard-sphere gas, *Phys. Rev.* **106** (1957), 20–26. Zbl 0077.23503.

[11] L. Erdős, B. Schlein, and H.-T. Yau, Ground-state energy of a low-density Bose gas: A second-order upper bound, *Phys. Rev. A* **78** no. 5 (2008), 053627. https://doi.org/10.1103/PhysRevA.78.053627.

[12] A. Giuliani and R. Seiringer, The ground state energy of the weakly interacting Bose gas at high density, *J. Stat. Phys.* **135** no. 5-6 (2009), 915–934. MR 2548599. Zbl 1172.82006. https://doi.org/10.1007/s10955-009-9718-0.

[13] G. M. Graf and D. Schenker, The free energy of systems with net charge, *Lett. Math. Phys.* **35** no. 1 (1995), 75–83. MR 1346046. Zbl 0833.58052. https://doi.org/10.1007/BF00739156.

[14] P. Grech and R. Seiringer, The excitation spectrum for weakly interacting bosons in a trap, *Comm. Math. Phys.* **322** no. 2 (2013), 559–591. MR 3077925. Zbl 1273.82007. https://doi.org/10.1007/s00220-013-1736-8.

[15] T. D. Lee, K. Huang, and C. N. Yang, Eigenvalues and eigenfunctions of a Bose system of hard spheres and its low-temperature properties, *Phys. Rev.* (2) **106** (1957), 1135–1145. MR 0088176. Zbl 0077.45003.

[16] M. Lewin, P. T. Nam, S. Serfaty, and J. P. Solovej, Bogoliubov spectrum of interacting Bose gases, *Comm. Pure Appl. Math.* **68** no. 3 (2015), 413–471. MR 3310520. Zbl 1318.82030. https://doi.org/10.1002/cpa.21519.

[17] E. H. Lieb, Simplified approach to the ground-state energy of an imperfect bose gas, *Phys. Rev.*, (1963), 2518–2528. https://doi.org/10.1103/PhysRev.130.2518.

[18] E. H. Lieb, R. Seiringer, J. P. Solovej, and J. Yngvason, *The Mathematics of the Bose Gas and its Condensation*, *Oberwolfach Semin.* **34**, Birkhäuser Verlag, Basel, 2005. MR 2143817. Zbl 1104.82012. https://doi.org/10.1007/b137508.

[19] E. H. Lieb, R. Seiringer, and J. Yngvason, Bosons in a trap: A rigorous derivation of the Gross-Pitaevskii energy functional, *Phys. Rev. A* **61** (2000). https://doi.org/10.1007/3-540-27056-6_47.

[20] E. H. Lieb, R. Seiringer, and J. Yngvason, Justification of c-number substitutions in Bosonic Hamiltonians, *Phys. Rev. Lett.* **94** (2005), 080401. https://doi.org/10.1103/PhysRevLett.94.080401.

[21] E. H. Lieb and J. P. Solovej, Ground state energy of the one-component charged Bose gas, *Comm. Math. Phys.* **217** no. 1 (2001), 127–163. MR 1815028. Zbl 1042.82004. https://doi.org/10.1007/s002200000353.

[22] E. H. Lieb and J. P. Solovej, Ground state energy of the two-component charged Bose gas, *Comm. Math. Phys.* **252** (2004), 485–534. MR 2104887. Zbl 1124.82303. https://doi.org/10.1007/s00220-004-1144-1.

[23] E. H. Lieb and J. Yngvason, Ground state energy of the low density Bose gas, *Phys. Rev. Lett.* **80** (1998), 2504–2507. https://doi.org/10.1103/PhysRevLett.80.2504.

[24] P. T. Nam and R. Seiringer, Collective excitations of Bose gases in the mean-field regime, *Arch. Ration. Mech. Anal.* **215** no. 2 (2015), 381–417. MR 3294406. Zbl 1317.35213. https://doi.org/10.1007/s00205-014-0781-6.

[25] M. Napiórkowski, R. Reuvers, and J. P. Solovej, The Bogoliubov free energy functional I: Existence of minimizers and phase diagram, *Arch. Ration. Mech. Anal.* **229** no. 3 (2018), 1037–1090. MR 3814596. Zbl 1404.82014. https://doi.org/10.1007/s00205-018-1232-6.

[26] M. Napiórkowski, R. Reuvers, and J. P. Solovej, The Bogoliubov free energy functional II: The dilute limit, *Comm. Math. Phys.* **360** no. 1 (2018), 347–403. MR 3795194. Zbl 1395.82077. https://doi.org/10.1007/s00220-017-3064-x.

[27] N. Navon, S. Nascimbene, F. Chevy, and C. Salomon, The equation of state of a low-temperature Fermi gas with tunable interactions, *Science* **328** (2010), 729–732. https://doi.org/10.1126/science.1187582.

[28] D. W. Robinson, *The Thermodynamic Pressure in Quantum Statistical Mechanics*, *Lecture Notes in Phys.* **9**, Springer-Verlag, New York, 1971. MR 0432122. https://doi.org/10.1007/3-540-05640-8.

[29] R. Seiringer, Free energy of a dilute Bose gas: lower bound, *Comm. Math. Phys.* **279** no. 3 (2008), 595–636. MR 2386722. Zbl 1148.82003. https://doi.org/10.1007/s00220-008-0428-2.

[30] R. Seiringer, The excitation spectrum for weakly interacting bosons, *Comm. Math. Phys.* **306** no. 2 (2011), 565–578. MR 2824481. Zbl 1226.82039. https://doi.org/10.1007/s00220-011-1261-6.

[31] J. P. Solovej, Upper bounds to the ground state energies of the one- and two-component charged Bose gases, *Comm. Math. Phys.* **266** no. 3 (2006), 797–818. MR 2238912. Zbl 1126.82006. https://doi.org/10.1007/s00220-006-0020-6.

[32] H.-T. Yau and J. Yin, The second order upper bound for the ground energy of a Bose gas, *J. Stat. Phys.* **136** no. 3 (2009), 453–503. MR 2529681. Zbl 1200.82002. https://doi.org/10.1007/s10955-009-9792-3.

[33] J. Yin, Free energies of dilute Bose gases: upper bound, *J. Stat. Phys.* **141** no. 4 (2010), 683–726. MR 2733402. Zbl 1208.82005. https://doi.org/10.1007/s10955-010-0066-x.

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