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On the Multiplicity One Conjecture in min-max theory

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Abstract

We prove that in a closed manifold of dimension between 3 and 7 with a bumpy metric, the min-max minimal hypersurfaces associated with the volume spectrum introduced by Gromov, Guth, Marques-Neves, are two-sided and have multiplicity one. This confirms a conjecture by Marques-Neves.

We prove that in a bumpy metric each volume spectrum is realized by the min-max value of certain relative homotopy class of sweepouts of boundaries of Caccioppoli sets. The main result follows by approximating such min-max value using the min-max theory for hypersurfaces with prescribed mean curvature established by the author with Zhu.

0. Introduction

Let (M^{n+1}, g) be a closed orientable Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. In [2], Almgren proved that the space of mod-2 cycles $\mathcal{Z}_n(M, \mathbb{Z}_2)$ is weakly homotopic the Eilenberg-MacLane space $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$. (See also [29] for a simpler proof.) Later, Gromov [15], [16], Guth [18], Marques-Neves [28] introduced the notion of volume spectrum as a nonlinear version of spectrum for the area functional in $\mathcal{Z}_n(M, \mathbb{Z}_2)$. In particular, the volume spectrum is a non-decreasing sequence of positive numbers

$$0 < \omega_1(M, g) \leq \cdots \leq \omega_k(M, g) \leq \cdots \to +\infty,$$

which is uniquely determined by the metric g in a given closed manifold M.

By adapting the celebrated min-max theory developed by Almgren [3], Pitts [31] (for $3 \leq (n+1) \leq 6$), and Schoen-Simon [33] (for $n+1 = 7$),

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Marques-Neves [28], [27] proved that each $\omega_k(M,g)$ is associated with an integral varifold V_k whose support is a disjoint collection of smooth, connected, closed, embedded, minimal hypersurfaces $\{\Sigma_1^k,\dots,\Sigma_{l_k}^k\}$, such that

$$\omega_k(M,g)=\sum_{i=1}^{l_k} m_i^k\cdot \mathrm{Area}(\Sigma_i^k), \tag{0.1}$$

where $\{m_1^k,\dots,m_{l_k}^k\}\subset\mathbb{N}$ is a set of positive integers, usually called *multiplicities*. We refer to [39], [6], [10], [17], [9], [23], [7], [47], [50], [32] for other variants of this theory.

Our main theorem states that all these integer multiplicities are identically equal to one for a bumpy metric. A metric g is called bumpy if every closed immersed minimal hypersurface is non-degenerate. White proved that the set of bumpy metrics is generic in Baire sense [42], [44].

THEOREM A. *Given a closed manifold M^{n+1} of dimension $3\le(n+1)\le 7$ with a bumpy metric g, the min-max minimal hypersurfaces $\{\Sigma_i^k : k\in\mathbb{N}, i=1,\dots,l_k\}$ associated with volume spectrum are all two-sided and have multiplicity one and index bounded by k. That is $m_i^k=1$ for all $k\in\mathbb{N}$, $1\le i\le l_k$,*

$$\omega_k(M,g)=\sum_{i=1}^{l_k}\mathrm{Area}(\Sigma_i^k)\ \textit{and}\ \sum_{i=1}^{l_k}\mathrm{index}(\Sigma_i^k)\le k.$$

Remark 0.1. This solves the *Multiplicity One Conjecture* of Marques-Neves [29, 1.2] (see also [27] for an earlier weaker version of this conjecture). We refer to Theorem 5.2 for a more detailed statement of this result. Note that by standard compactness analysis (see [35]), the same conclusion concerning two-sidedness and multiplicity one also holds true for a metric with positive Ricci curvature.

Remark 0.2. This conjecture was proved earlier for one-parameter min-max constructions under positive Ricci curvature assumption by Marques-Neves [25], the author [48, 49], and Ketover-Marques-Neves [22]. Later it was fully proved for one-parameter case by Marques-Neves [27]. Recently, Chodosh-Mantoulidis [5] proved this conjecture in dimension three $(n+1)=3$ for the Allen-Cahn setting (see [12] for earlier works along this direction); they also proved that the total index is exactly k for their k-min-max solutions when $(n+1)=3$. After our results were posted on arXiv, Marques-Neves finished their program and also proved the same optimal index estimates for $3\le(n+1)\le 7$ [29, Addendum].

One motivation of this conjecture is to prove the Yau's conjecture [46] on existence of infinitely many closed minimal surfaces in three manifolds.

Combining with the growth estimates of $\{\omega_k(M,g)\}$ by Marques-Neves [28, Ths. 5.1 and 8.1] and the Frankel Theorem [11], we have

THEOREM B. *Let M^{n+1} be a closed manifold of dimension $3\leq(n+1)\leq 7$.*

(a) *For each bumpy metric g, there exists infinitely many smooth, connected, closed, embedded, minimal hypersurfaces.*

(b) *If a metric g has positive Ricci curvature, then there exists a sequence of smooth, connected, closed, embedded, minimal hypersurfaces $\{\Sigma_k\}_{k\in\mathbb{N}}$, such that*

$$\mathrm{Area}(\Sigma_k)\sim k^{\frac{1}{n+1}} \text{ as } k\to\infty.$$

Remark 0.3. Result (a) was already known even without the bumpy assumption by combining Marques-Neves [28] and Song [40]. For a set of generic metrics, Irie-Marques-Neves [21] and Marques-Neves-Song [30] proved denseness and equi-distribution for the space of closed embedded minimal hypersurfaces, using the Weyl Law for volume spectrum by Liokumovich-Marques-Neves [24]. Their generic set in principle could be much smaller than the set of bumpy metrics.

Result (b) was also obtained by Chodosh-Mantoulidis [5] in dimension three $(n+1)=3$.

As a direct corollary of the compactness theory (see [35]), there is an equivalent formulation of Theorem A for general metrics. Note that a closed minimal hypersurface Σ is said to be *degenerately stable* if 0 is the lowest eigenvalue for the second variation of area of Σ, and hence $\mathrm{index}(\Sigma)=0$ (when Σ is one-sided, one has to pass to its two-sided double cover).

THEOREM C. *Let M^{n+1} be a closed manifold of dimension $3\leq(n+1)\leq 7$ with an arbitrary metric g. Then in (0.1), every connected component of $\{\Sigma_i^k : k\in\mathbb{N}, i=1,\dots,l_k\}$ which is not degenerately stable is two-sided and has multiplicity one. That is, if Σ_i^k is not degenerately stable, $k\in\mathbb{N}$, $1\leq i\leq l_k$, then Σ_i^k is two-sided, $m_i^k=1$, and*

$$\sum_{i=1}^{l_k}\mathrm{index}(\Sigma_i^k)\leq k.$$

Remark 0.4. This formulation was noticed by the author shortly after this article was first posted, and had formally appeared in the author's Oberwolfach Report for Partial Differential Equations, 2019.

This theorem can be viewed as a strengthened version of the *Weighted Morse Index Bound Conjecture* by Marques-Neves. As a significant milestone, Marques-Neves [27] proved the above index upper bound estimates. However, the estimates might not be sharp as the contribution of these multiple sheets were not counted. They made a conjecture that the sharp estimates should

contain the multiplicity as coefficients. Theorem C says that every component which contributes nonzero index is two-sided and has multiplicity equal to one, and hence proves sharpness of the estimates.

0.1. *Sketch of the proof.* The key idea of our proof is to approximate the Area-functional by the weighted $\mathcal{A}^h$-functional used in the prescribing mean curvature (PMC) min-max theory developed by the author with Zhu [52]. Note that the $\mathcal{A}^h$-functional is only defined for boundaries of Caccioppoli sets; see (1.1). A smooth critical point of $\mathcal{A}^h$ is a hypersurface whose mean curvature is prescribed by the restriction of h to itself. There are two crucial parts in the proof. In the first part, we consider min-max construction of minimal hypersurfaces using sweepouts of boundaries of Caccioppoli sets. We observe that in a bumpy metric if one approximates Area by a sequence $\{\mathcal{A}^{\varepsilon_k h}\}_{k\in\mathbb{N}}$ where $\{\varepsilon_k\}_{k\in\mathbb{N}} \to 0$, and if $h : M \to \mathbb{R}$ is carefully chosen, then the limit min-max minimal hypersurfaces (of min-max PMC hypersurfaces associated with $\mathcal{A}^{\varepsilon_k h}$) are all two-sided and have multiplicity one; see Theorem 4.1. In the second part, we show that in a bumpy metric the volume spectrum $\omega_k(M, g)$ can be realized by the area of some minimal hypersurfaces coming from min-max constructions using sweepouts of boundaries. We now elaborate the detailed ideas.

To implement the idea in the first part, we generalize the PMC min-max theory in [52] to multi-parameter families using continuous sweepouts. Since the space of Caccioppoli sets $\mathcal{C}(M)$ is contractible, there is no nontrivial free homotopy class to do min-max, so we have to consider relative homotopy classes. Heuristically, given a k-dimensional parameter space X, a subset $Z \subset X$, and a continuous map $\Phi_0 : X \to \mathcal{C}(M)$, we can consider its relative (X, Z)-homotopy class $\Pi = \Pi(\Phi_0)$ consisting of all maps $\Phi : X \to \mathcal{C}(M)$ that are homotopic to Φ_0 and such that $\Phi|_Z \equiv \Phi_0|_Z$. If the min-max value $\mathbf{L}^h = \inf\{\max_{x\in X} \mathcal{A}^h(\Phi(x)) : \Phi \in \Pi\}$ satisfies the nontriviality condition $\mathbf{L}^h > \max_{x\in Z} \mathcal{A}^h(\Phi_0(x))$ with respect to the $\mathcal{A}^h$-functional, and if h is chosen in a dense subset $\mathcal{S}(g) \subset C^\infty(M)$ (depending on the metric g, see [52, Prop. 0.2]), we prove the existence of a smooth closed hypersurface Σ^h of prescribed mean curvature h; moreover, it is represented as the boundary $\Sigma^h = \partial\Omega^h$ for some Caccioppoli set Ω^h and $\mathcal{A}^h(\Omega^h) = \mathbf{L}^h$; hence Σ^h is two-sided and have multiplicity one. Σ^h is usually called a min-max PMC hypersurface. We also established Morse index upper bounds following Marques-Neves [27]. That is, we prove that the Morse index of Σ^h is bounded from above by k (the dimension of parameter space).

Given a relative homotopy class Π as above, consider the min-max construction for the Area-functional and let $\mathbf{L} = \inf\{\max_{x\in X} \mathrm{Area}(\partial\Phi(x)) : \Phi \in \Pi\}$. If the nontriviality condition $\mathbf{L} > \max_{x\in Z} \mathrm{Area}(\partial\Phi_0(x))$ is satisfied, we can approximate $\mathbf{L}$ by $\mathbf{L}^{\varepsilon h}$ for a fixed $h \in \mathcal{S}(g)$ (to be chosen later) and small enough

$\varepsilon > 0$. We know that $\varepsilon \cdot h$ also belongs to the dense subset $\mathcal{S}(g)$. Denote Σ_ε as the min-max PMC hypersurface associated with $\mathbf{L}^{\varepsilon h}$. As the family $\{\Sigma_\varepsilon : \varepsilon > 0\}$ have uniformly bounded area and Morse index, we can pick a subsequence $\{\Sigma_k = \Sigma_{\varepsilon_k} : \varepsilon_k \to 0\}$ that converges as varifolds and also locally smooth and graphically away from finitely many points to some limit minimal hypersurface Σ_∞ with integer multiplicity such that $\mathrm{Area}(\Sigma_\infty) = \mathbf{L}$. The limit can be extended to a closed embedded minimal hypersurface Σ_∞ across the bad points, and Σ_∞ also has the same Morse index upper bound. Hence Σ_∞ is a min-max minimal hypersurface associated with $\mathbf{L}$. As a standard process, if the multiplicity is greater than one, or if a component is one-sided, one can obtain solutions of the Jacobi operator L_{Σ_∞} of Σ_∞ by taking the limit of the renormalizations of the heights between the top and bottom sheets of Σ_k. In particular, there are two possibilities for the limit depending on the orientations of the top and bottom sheets. For simplicity, let us assume that Σ_∞ is connected and two-sided. An easier case happens when the top and bottom sheets have the same orientation, and hence the limit is a nontrivial nonnegative solution φ of the Jacobi equation $L_{\Sigma_\infty}\varphi = 0$ which cannot happen in a bumpy metric. When the top and bottom sheets have opposite orientations, the limit is either a nontrivial nonnegative solution to the Jacobi equation, or is a solution φ of the following equation:

$$L_{\Sigma_\infty}\varphi = 2h|_{\Sigma_\infty}, \quad \text{such that } \varphi \text{ does not change sign.}$$

The key observation is that one can find a $h \in \mathcal{S}(g)$ so that the unique solution (as Σ_∞ is non-degenerate) of $L_{\Sigma_\infty}\varphi = 2h|_{\Sigma_\infty}$ must change sign, and hence Σ_∞ must have multiplicity one (see Lemma 4.2). Indeed, the set of minimal hypersurfaces with bounded area and Morse index in a bumpy metric is finite by the standard compactness results [35]. On each such Σ, we can construct a $h_\Sigma \in C^\infty(\Sigma)$ such that the unique solution f_Σ of $L_\Sigma f_\Sigma = 2h_\Sigma$ must change sign, and we can further make the support of all such h_Σ pairwise disjoint. Since $\mathcal{S}(g)$ is open and dense, we can pick a $h \in \mathcal{S}(g)$ that approximates all h_Σ on Σ as close as we want. Then the solution of $L_\Sigma\varphi = 2h|_\Sigma$ must also change sign. Up to here, we have elucidated how to construct two-sided min-max minimal hypersurfaces with multiplicity one for sweepouts of boundaries of Caccioppoli sets.

Lastly we apply the above multiplicity one result to the volume spectrum. Though the volume spectrum $\omega_k(M, g)$ is defined using cohomological relations, Marques-Neves proved in [27], using their Morse index estimates, that in a bumpy metric $\omega_k(M, g)$ is realized by the min-max value $\mathbf{L}(\Pi)$ for certain free homotopy class Π of maps $\Phi : X \to \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$, where X is some fixed k-dimensional parameter space and $\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ is the space of mod-2 cycles. It was observed by Marques-Neves [29] that the space of Caccioppoli sets $\mathcal{C}(M)$ forms a double cover of $\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ via the boundary

map $\partial : \mathcal{C}(M) \to \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$. Therefore, by lifting to the double cover, for each $\Phi \in \Pi$, we can produce a map $\tilde{\Phi} : \tilde{X} \to \mathcal{C}(M)$, where $\pi : \tilde{X} \to X$ is a double cover, such that $\partial\tilde{\Phi}(x) = \Phi(\pi(x))$. To produce a nontrivial relative homotopy class, we pick a map $\Phi_0 \in \Pi$ such that $\max_{x\in X} \mathrm{Area}(\Phi_0(x))$ is very close to $\mathbf{L}(\Pi) = \omega_k(M, g)$. Let $Z \subset X$ to be the subset where each $\Phi_0(x)$, $x \in Z$, is ε-distance away from the set of smooth closed embedded minimal hypersurface Σ with $\mathrm{Area}(\Sigma) \leq \mathbf{L}$ and $\mathrm{index}(\Sigma) \leq k$. Note that this set of minimal hypersurfaces is finite in a bumpy metric, hence for ε small enough the complement $Y = \overline{X \setminus Z} \subset X$ is topologically trivial in the sense that Y does not detect the generator of the cohomological ring of $\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$. Therefore, the pre-image $\tilde{Y} = \pi^{-1}(Y) \subset \tilde{X}$ is homeomorphic to two disjoint identical copies of Y, denoted as Y^+ and Y^-. On the other hand, since no element in $\Phi_0(Z)$ is regular, by Pitts's combinatorial argument, one can homotopically deform $\Phi_0|_Z$ so that $\max_{x\in Z} \mathrm{Area}(\Phi_0(x)) < \mathbf{L}$. Now consider the relative $(\tilde{X}, \tilde{Z})$-homotopy class $\tilde{\Pi}$ generated by the map $\tilde{\Phi}_0 : \tilde{X} \to \mathcal{C}(M)$. One key observation is that the min-max value $\mathbf{L}(\tilde{\Pi}) \geq \mathbf{L}(\Pi) > \max_{x\in Z} \mathrm{Area}(\Phi_0(x))$. To see this, given any homotopic deformation $\tilde{\Psi} : \tilde{X} \to \mathcal{C}(M)$ of $\tilde{\Phi}_0$ relative to $(\tilde{\Phi}_0)|_{\tilde{Z}}$, if $\max_{x\in Y^+} \mathrm{Area}(\partial\tilde{\Psi}(x)) < \mathbf{L}(\Pi)$, then we can pass it to quotient and obtain a continuous map $\Psi : X \to \mathcal{Z}_(M, \mathbb{Z}_2)$ as Y^+ and Y^- are disjoint and $\tilde{\Psi}|_{\tilde{Z}} \equiv (\tilde{\Phi}_0)|_{\tilde{Z}}$, so that $\max_{x\in X} \mathrm{Area}(\Psi(x)) < \mathbf{L}(\Pi)$, but this is a contradiction as Ψ is homotopic to Φ_0. Therefore, $\tilde{\Pi}$ is a nontrivial relative homotopy class in $\mathcal{C}(M)$, and its associated min-max minimal hypersurfaces are two-sided and have multiplicity one. Finally, as the metric is bumpy, the min-max value $\mathbf{L}(\tilde{\Pi})$ of $\tilde{\Pi}$ is equal to $\mathbf{L}(\Pi)$ when $\max_{x\in X} \mathrm{Area}(\Phi_0(x))$ is close enough to $\mathbf{L}(\Pi) = \omega_k(M, g)$. Hence we have explained how to construct two-sided min-max minimal hypersurfaces of multiplicity one whose areas realize the volume spectrum.

0.2. *Outline of the paper.* In Section 1, we establish the multi-parameter version of min-max theory for prescribing mean curvature hypersurfaces using continuous sweepouts. In Section 2, we prove several compactness results for prescribing mean curvature hypersurfaces with uniform area and Morse index upper bounds. In Section 3, we prove the Morse index upper bound for prescribing mean curvature hypersurfaces produced by our min-max theory. In Section 4, we prove that min-max minimal hypersurfaces associated with families of boundaries have multiplicity one in a bumpy metric. Finally, in Section 5, we prove the Multiplicity One Conjecture for volume spectrum.

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1. Multi-parameter min-max theory for prescribing mean curvature hypersurfaces

Here we present an adaption to multi-parameter families of the min-max theory for hypersurfaces with *prescribed mean curvature* (abbreviated as PMC) established by the author with Zhu [51], [52]. Let $\mathcal{S} = \mathcal{S}(g)$ (depending on the metric g) be the open and dense subset of $C^\infty(M)$ chosen as in [52, Prop. 0.2]. More precisely, $\mathcal{S}(g)$ consists of all Morse functions h such that the zero set $\Sigma_0 = \{h = 0\}$ is a smooth closed embedded hypersurface, and the mean curvature of Σ_0 vanishes to at most finite order. A hypersurface is *almost embedded* (sometime also called *strongly Alexandrov embedded*) if it locally decomposes into smooth embedded sheets that touch but do not cross. By [52, Th. 3.11], any almost embedded hypersurface of prescribed mean curvature $h \in \mathcal{S}$ has touching set $(n-1)$-rectifiable, and no component is minimal.

Notation. We collect some notions. We refer to [36] and [31, §2.1] for further materials in geometric measure theory.

Let (M^{n+1}, g) denote a closed, oriented, smooth Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. Assume that (M, g) is embedded in some $\mathbb{R}^L$, $L \in \mathbb{N}$. $B_r(p)$ denotes the geodesic ball of (M, g). We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure; $\mathbf{I}_k(M)$ (or $\mathbf{I}_k(M, \mathbb{Z}_2)$) the space of k-dimensional integral (or mod 2) currents in $\mathbb{R}^L$ with support in M; $\mathcal{Z}_k(M)$ (or $\mathcal{Z}_k(M, \mathbb{Z}_2)$) the space of integral (or mod 2) currents $T \in \mathbf{I}_k(M)$ with $\partial T = 0$; $\mathcal{V}_k(M)$ the closure, in the weak topology, of the space of k-dimensional rectifiable varifolds in $\mathbb{R}^L$ with support in M; $G_k(M)$ the Grassmannian bundle of un-oriented k-planes over M; $\mathcal{F}$ and $\mathbf{M}$ respectively the flat norm [36, §31] and mass norm [36, 26.4] on $\mathbf{I}_k(M)$; $\mathbf{F}$ the varifold $\mathbf{F}$-metric on $\mathcal{V}_k(M)$ and currents $\mathbf{F}$-metric on $\mathbf{I}_k(M)$ or $\mathbf{I}_k(M, \mathbb{Z}_2)$, [31, 2.1(19)(20)]; $\mathcal{C}(M)$ or $\mathcal{C}(U)$ the space of sets $\Omega \subset M$ or $\Omega \subset U \subset M$ with finite perimeter (Caccioppoli sets), [36, §14][14, §1.6]; and $\mathfrak{X}(M)$ or $\mathfrak{X}(U)$ the space of smooth vector fields in M or supported in U. $\partial\Omega$ denotes the (reduced)-boundary of $[[\Omega]]$ as an integral current, and $\nu_{\partial\Omega}$ denotes the outward pointing unit normal of $\partial\Omega$, [36, 14.2].

We also utilize the following definitions:

(a) Given $T \in \mathbf{I}_k(M)$, $|T|$ and $\|T\|$ denote respectively the integral varifold and Radon measure in M associated with T;

(b) Given $c > 0$, a varifold $V \in \mathcal{V}_k(M)$ is said to have *c-bounded first variation in an open subset $U \subset M$*, if

$$|\delta V(X)| \leq c \int_M |X| d\mu_V \text{ for any } X \in \mathfrak{X}(U);$$

here the first variation of V along X is $\delta V(X) = \int_{G_k(M)} div_S X(x) dV(x,S)$, [36, §39];

(c) Given a smooth immersed, closed, orientable hypersurface Σ in M, or a set $\Omega \in \mathcal{C}(M)$ with finite perimeter, $[[\Sigma]]$, $[[\Omega]]$ denote the corresponding integral currents with the natural orientation, and $[\Sigma]$ denotes the corresponding integer-multiplicity varifold.

As noted by Marques-Neves [29, §5], $\mathcal{C}(M)$ is identified with $\mathbf{I}_{n+1}(M, \mathbb{Z}_2)$. In particular, the flat $\mathcal{F}$-norm and the mass $\mathbf{M}$-norm are the same on $\mathcal{C}(M)$. Given $\Omega_1, \Omega_2 \in \mathcal{C}(M)$, the $\mathbf{F}$-distance between them is:

$$\mathbf{F}(\Omega_1, \Omega_2) = \mathcal{F}(\Omega_1 - \Omega_2) + \mathbf{F}(|\partial\Omega_1|, |\partial\Omega_2|).$$

Given $\Omega \in \mathcal{C}(M)$, we will denote $\overline{\mathbf{B}}^{\mathbf{F}}_{\varepsilon}(\Omega) = \{\Omega' \in \mathcal{C}(M) : \mathbf{F}(\Omega', \Omega) \leq \varepsilon\}$.

We are interested in the following weighted area functional defined on $\mathcal{C}(M)$. Given $h : M \to \mathbb{R}$, define the *$\mathcal{A}^h$-functional* on $\mathcal{C}(M)$ as

$$\mathcal{A}^h(\Omega) = \mathcal{H}^n(\partial\Omega) - \int_\Omega h \, d\mathcal{H}^{n+1}. \tag{1.1}$$

The *first variation formula* for $\mathcal{A}^h$ along $X \in \mathfrak{X}(M)$ is (see [36, 16.2])

$$\delta\mathcal{A}^h|_\Omega(X) = \int_{\partial\Omega} div_{\partial\Omega} X \, d\mu_{\partial\Omega} - \int_{\partial\Omega} h\langle X, \nu\rangle \, d\mu_{\partial\Omega}, \tag{1.2}$$

where $\nu = \nu_{\partial\Omega}$ is the outward unit normal on $\partial\Omega$.

When the boundary $\partial\Omega = \Sigma$ is a smooth immersed hypersurface, we have

$$div_\Sigma X = H\langle X, \nu\rangle,$$

where H is the mean curvature of Σ with respect to ν; if Ω is a critical point of $\mathcal{A}^h$, then (1.2) directly implies that $\Sigma = \partial\Omega$ must have mean curvature $H = h|_\Sigma$. In this case, we can calculate the *second variation formula* for $\mathcal{A}^h$ along normal vector fields $X \in \mathfrak{X}(M)$ such that $X = \varphi\nu$ along $\partial\Omega = \Sigma$ where $\varphi \in C^\infty(\Sigma)$, [4, Prop. 2.5],

$$\begin{aligned} \delta^2\mathcal{A}^h|_\Omega(X, X) &= II_\Sigma(\varphi, \varphi) \\ &= \int_\Sigma \left(|\nabla\varphi|^2 - \left(\mathrm{Ric}^M(\nu, \nu) + |A^\Sigma|^2 + \partial_\nu h \right) \varphi^2 \right) d\mu_\Sigma. \end{aligned} \tag{1.3}$$

In the above formula, $\nabla\varphi$ is the gradient of φ on Σ; Ric^M is the Ricci curvature of M; A^Σ is the second fundamental form of Σ.

1.1. *Min-max construction for (X, Z)-homotopy class.* In this part, we describe the setup for min-max theory for PMC hypersurfaces associated with multiple parameter families in $\mathcal{C}(M)$.

Let X^k be a cubical complex of dimension $k \in \mathbb{N}$ in some $I^m = [0,1]^m$ and $Z \subset X$ be a cubical subcomplex.

Let $\Phi_0 : X \to (\mathcal{C}(M), \mathbf{F})$ be a continuous map (with respect to the $\mathbf{F}$-topology on $\mathcal{C}(M)$). We let Π be the set of all sequences of continuous (in $\mathbf{F}$-topology) maps $\{\Phi_i : X \to \mathcal{C}(M)\}_{i\in\mathbb{N}}$ such that

(1) each Φ_i is homotopic to Φ_0 in the flat topology on $\mathcal{C}(M)$, and
(2) there exist homotopy maps $\{\Psi_i : [0,1] \times X \to \mathcal{C}(M)\}_{i\in\mathbb{N}}$ which are continuous in the flat topology, $\Psi_i(0,\cdot) = \Phi_i$, $\Psi_i(1,\cdot) = \Phi_0$, and satisfy

$$\limsup_{i\to\infty} \sup\{\mathbf{F}(\Psi_i(t,x), \Phi_0(x)) : t \in [0,1], x \in Z\} = 0. \tag{1.4}$$

Note that a sequence $\{\Phi_i\}_{i\in\mathbb{N}}$ with $\Phi_i = \Phi_0$ for all $i \in \mathbb{N}$ belongs to Π.

Definition 1.1. Given a pair (X, Z) and Φ_0 as above, $\{\Phi_i\}_{i\in\mathbb{N}}$ is called a *(X, Z)-homotopy sequence of mappings into $\mathcal{C}(M)$*, and Π is called the *(X, Z)-homotopy class of Φ_0*.

Remark 1.2. Π can be viewed as the relative homotopy class for Φ_0 in $(\mathcal{C}(M), \Phi_0|_Z)$. However, we cannot fix the values $\Phi_i|_Z$ to be exactly $\Phi_0|_Z$. In fact, in the later discretization/interpolation process, we will allow $\Phi_i|_Z$ to deviate slightly from $\Phi_0|_Z$; but the deviations will converge to zero as $i \to \infty$.

Definition 1.3. The *h-width* of Π is defined by:

$$\mathbf{L}^h = \mathbf{L}^h(\Pi) = \inf_{\{\Phi_i\}\in\Pi} \limsup_{i\to\infty} \sup_{x\in X} \{\mathcal{A}^h(\Phi_i(x))\}.$$

Definition 1.4. A sequence $\{\Phi_i\}_{i\in\mathbb{N}} \in \Pi$ is called a *min-max sequence* if

$$\mathbf{L}^h(\Phi_i) := \sup_{x\in X} \mathcal{A}^h(\Phi_i(x))$$

satisfies $\mathbf{L}^h(\{\Phi_i\}) := \limsup_{i\to\infty} \mathbf{L}^h(\Phi_i) = \mathbf{L}^h(\Pi)$.

Lemma 1.5. *Given Φ_0 and Π, there exists a min-max sequence.*

Proof. Take a sequence $\{\{\Phi_i^\alpha\}_{i\in\mathbb{N}}\}_{\alpha\in\mathbb{N}}$ in Π, such that

$$\lim_{\alpha\to\infty} \mathbf{L}^h(\{\Phi_i^\alpha\}_{i\in\mathbb{N}}) = \mathbf{L}^h(\Pi).$$

Now we pick up a new sequence by a diagonalization process. Take a sequence $\varepsilon_\alpha \to 0$. For each α, we pick $i_\alpha \in \mathbb{N}$, such that

$$\sup_{t\in[0,1],x\in Z} \mathbf{F}(\Psi^\alpha_{i_\alpha}(t,x),\Phi_0(x)) < \varepsilon_\alpha \text{ and}$$

$$\mathbf{L}^h(\{\Phi^\alpha_i\}) - \varepsilon_\alpha \le \sup_{x\in X} \mathcal{A}^h(\Phi^\alpha_{i_\alpha}(x)) \le \mathbf{L}^h(\{\Phi^\alpha_i\}) + \varepsilon_\alpha,$$

where $\Psi^\alpha_{i_\alpha}$ is the homotopy between $\Phi^\alpha_{i_\alpha}$ and Φ_0 in the flat topology. Hence the sequence $\{\Phi^\alpha_{i_\alpha}\}_{\alpha\in\mathbb{N}}$ belongs to Π and is a min-max sequence. □

Definition 1.6. The *image set* of $\{\Phi_i\}_{i\in\mathbb{N}}$ is defined by

$$\mathbf{K}(\{\Phi_i\}) = \{V = \lim_{j\to\infty} |\partial\Phi_{i_j}(x_j)| \text{ as varifolds} : x_j \in X\}.$$

If $\{\Phi_i\}_{i\in\mathbb{N}}$ is a min-max sequence in Π, the *critical set* of $\{\Phi_i\}$ is defined by

$$\begin{aligned}&\mathbf{C}(\{\Phi_i\})\\ &\quad = \{V = \lim_{j\to\infty} |\partial\Phi_{i_j}(x_j)| \text{ as varifolds} : \text{ with } \lim_{j\to\infty} \mathcal{A}^h(\Phi_{i_j}(x_j)) = \mathbf{L}^h(\Pi)\}.\end{aligned}$$

Now we are ready to state the continuous version of min-max theory for PMC hypersurfaces associated with a (X,Z)-homotopy class. It is a generalization of [52, Th. 4.8 and Proposition 7.3], and the proof is given in Section 1.4.

THEOREM 1.7 (Min-max theorem). *Let (M^{n+1},g) be a closed Riemannian manifold of dimension $3 \le (n+1) \le 7$, and $h \in \mathcal{S}(g)$ which satisfies $\int_M h \ge 0$. Given a map $\Phi_0 : X \to (\mathcal{C}(M),\mathbf{F})$ continuous in the $\mathbf{F}$-topology and the associated (X,Z)-homotopy class Π, suppose*

$$\mathbf{L}^h(\Pi) > \max\big\{\max_{x\in Z} \mathcal{A}^h(\Phi_0(x)), 0\big\}. \tag{1.5}$$

Let $\{\Phi_i\}_{i\in\mathbb{N}} \in \Pi$ be a min-max sequence for Π. Then there exists $V \in \mathbf{C}(\{\Phi_i\})$ induced by a nontrivial, smooth, closed, almost embedded hypersurface $\Sigma^n \subset M$ of prescribed mean curvature h with multiplicity one.

Moreover, $V = \lim_{j\to\infty} |\partial\Phi_{i_j}(x_j)|$ for some $\{i_j\} \subset \{i\}$, $\{x_j\} \subset X\backslash Z$, with $\lim_{j\to\infty} \mathcal{A}^h(\Phi_{i_j}(x_j)) = \mathbf{L}^h(\Pi)$, and $\Phi_{i_j}(x_j)$ converges in the $\mathbf{F}$-topology to some $\Omega \in \mathcal{C}(M)$ such that $\Sigma = \partial\Omega$ where its mean curvature with respect to the unit outer normal is h, and

$$\mathcal{A}^h(\Omega) = \mathbf{L}^h(\Pi).$$

1.2. *Pull-tight.* Now we describe the **pull-tight** process in [52, §5]. Let $c = \sup_M |h|$, and $L^c = 2\mathbf{L}^h + c\,\mathrm{Vol}(M)$. Denote

$$\begin{aligned}A^c_\infty = \{&V \in \mathcal{V}_n(M) : \|V\|(M)\\ &\le L^c, V \text{ has } c\text{-bounded first variation, or } V \in |\partial\Phi_0|(Z)\}.\end{aligned}$$

We can follow [51, §4] or [52, §5] to construct a continuous map:

$$H : [0,1] \times (\mathcal{C}(M), \mathbf{F}) \cap \{\mathbf{M}(\partial\Omega) \le L^c\} \to (\mathcal{C}(M), \mathbf{F}) \cap \{\mathbf{M}(\partial\Omega) \le L^c\}$$

such that

(i) $H(0,\Omega) = \Omega$ for all Ω;
(ii) $H(t,\Omega) = \Omega$ if $|\partial\Omega| \in A^c_\infty$;
(iii) if $|\partial\Omega| \notin A^c_\infty$,

$$\mathcal{A}^h(H(1,\Omega)) - \mathcal{A}^h(\Omega) \le -L(\mathbf{F}(|\partial\Omega|, A^c_\infty)) < 0;$$

here $L : [0,\infty) \to [0,\infty)$ is a continuous function with $L(0) = 0$, $L(t) > 0$ when $t > 0$;
(iv) for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in Z,\ \mathbf{F}(\Omega, \Phi_0(x)) < \delta \Longrightarrow \mathbf{F}(H(t,\Omega), \Phi_0(x)) < \varepsilon \text{ for all } t \in [0,1];$$

this is a direct consequence of (ii) since $|\partial\Phi_0|(Z) \subset A^c_\infty$.

Note that to construct H, the only modification of [52, §5.1] is to add $|\partial\Phi_0|(Z)$ into the definition of A^c_∞ as we want to fix the values assumed on Z in the tightening process; all other steps in [52, §5.1] carry out the same way. In particular, (using notions in [52, §5.1]), $H(t,\Omega) := \big(\Psi_{|\partial\Omega|}(t)\big)(\Omega)$.

LEMMA 1.8. *Given a min-max sequence $\{\Phi^*_i\}_{i\in\mathbb{N}} \in \Pi$, we define $\Phi_i(x) = H(1, \Phi^*_i(x))$ for every $x \in X$. Then $\{\Phi_i\}_{i\in\mathbb{N}}$ is also a min-max sequence in Π. Moreover, $\mathbf{C}(\{\Phi_i\}) \subset \mathbf{C}(\{\Phi^*_i\})$ and every element of $\mathbf{C}(\{\Phi_i\})$ either has c-bounded first variation, or belongs to $|\partial\Phi_0|(Z)$.*

Proof. By continuity of H, we know that Φ_i is homotopic to Φ^*_i in the flat topology. By (iv), $\{\Psi_i(t,x) = H(t, \Phi^*_i(x))\}$ satisfies (1.4), and hence $\{\Phi_i\} \in \Pi$. By (ii)(iii), $\mathcal{A}^h(\Phi_i(x)) \le \mathcal{A}^h(\Phi^*_i(x))$ for every $x \in X$, so $\{\Phi_i\}$ is also a min-max sequence. Finally, given any $V \in \mathbf{C}(\{\Phi_i\})$, then $V = \lim_{j\to\infty} |\partial\Phi_{i_j}(x_j)|$ where $\lim_{j\to\infty} \mathcal{A}^h(\Phi_{i_j}(x_j)) = \mathbf{L}^h$. Denote $V^* = \lim_{j\to\infty} |\partial\Phi^*_{i_j}(x_j)|$. By (iii), $\lim_{j\to\infty} \mathbf{F}(|\partial\Phi^*_{i_j}(x_j)|, A^c_\infty) = 0$ (as $\lim_{j\to\infty} \mathcal{A}^h(\Phi_{i_j}(x_j)) = \lim_{j\to\infty} \mathcal{A}^h(\Phi^*_{i_j}(x_j)) = \mathbf{L}^h$), so $V^* \in A^c_\infty$. On the other hand,

$$V = \lim_{j\to\infty} |\partial H(1, \Phi^*_{i_j}(x_j))| = H(1, \lim_{j\to\infty} |\partial\Phi^*_{i_j}(x_j)|) = H(1, V^*) = V^*.$$

(Note that H is also well defined as a continuous map $H : [0,1] \times \{V \in \mathcal{V}_n(M), \|V\|(M) \le L^c\} \to \{V \in \mathcal{V}_n(M), \|V\|(M) \le L^c\}$.) Hence $\mathbf{C}(\{\Phi_i\}) \subset \mathbf{C}(\{\Phi^*_i\})$ and the proof is finished. □

Definition 1.9. Let $c = \sup_M |h|$. Any min-max sequence $\{\Phi_i\}_{i\in\mathbb{N}} \in \Pi$ such that every element of $\mathbf{C}(\{\Phi_i\})$ has c-bounded first variation or belongs to $|\partial\Phi_0|(Z)$ is called *pulled-tight*.

1.3. *Discretization and interpolation results.* We record several discretization and interpolation results developed by Marques-Neves [26, 28]. Though these results were proven for sweepouts in $\mathcal{Z}_n(M,\mathbb{Z})$ or $\mathcal{Z}_n(M,\mathbb{Z}_2)$, they work well for sweepouts in $\mathcal{C}(M)$. We will point out necessary modifications.

We refer to Appendix A for the notion of cubic complex structure on X. We refer to [52, §4] for the notion of discrete sweepouts. Though all definitions therein were made when $X=[0,1]$, there is no change for discrete sweepouts on X.

Recall that given a map $\phi : X(k)_0 \to \mathcal{C}(M)$, the *fineness of* ϕ is defined as

$$\begin{aligned}\mathbf{f}(\phi) = \sup\{&\mathcal{F}(\phi(x)-\phi(y)) \\ &+ \mathbf{M}(\partial\phi(x)-\partial\phi(y)) : x,y \text{ are adjacent vertices in } X(k)_0\}.\end{aligned}$$

Definition 1.10 (cf. [28, §3.7]). Given a continuous (in the flat topology) map $\Phi : X \to \mathcal{C}(M)$, we say that Φ has *no concentration of mass* if

$$\lim_{r\to 0}\sup\{\|\partial\Phi(x)\|(B_r(p)), p\in M, x\in X\} = 0.$$

The purpose of the next theorem is to construct discrete maps out of a continuous map in flat topology.

Theorem 1.11. *Let $\Phi : X \to \mathcal{C}(M)$ be a continuous map in the flat topology that has no concentration of mass, and $\sup_{x\in X}\mathbf{M}(\partial\Phi(x)) < +\infty$. Assume that $\Phi|_Z$ is continuous under the $\mathbf{F}$-topology. Then there exist a sequence of maps*

$$\phi_i : X(k_i)_0 \to \mathcal{C}(M),$$

and a sequence of homotopy maps:

$$\psi_i : I(k_i)_0 \times X(k_i)_0 \to \mathcal{C}(M),$$

with $k_i < k_{i+1}$, $\psi_i(0,\cdot) = \phi_{i-1}\circ\mathbf{n}(k_i,k_{i-1})$, $\psi_i(1,\cdot)=\phi_i$, and a sequence of numbers $\{\delta_i\}_{i\in\mathbb{N}}\to 0$ such that

(i) *the fineness $\mathbf{f}(\psi_i) < \delta_i$;*

(ii)
$$\sup\{\mathcal{F}(\psi_i(t,x)-\Phi(x)) : t\in I(k_i)_0, x\in X(k_i)_0\} \le \delta_i;$$

(iii) *for some sequence $l_i\to\infty$, with $l_i < k_i$*

$$\mathbf{M}(\partial\psi_i(t,x)) \le \sup\{\mathbf{M}(\partial\Phi(y)) : x,y\in\alpha \text{ for some } \alpha\in X(l_i)\} + \delta_i;$$

and this directly implies that

$$\sup\{\mathbf{M}(\partial\phi_i(x)) : x\in X(k_0)_0\} \le \sup\{\mathbf{M}(\partial\Phi(x)) : x\in X\} + \delta_i.$$

As $\Phi|_Z$ is continuous in $\mathbf{F}$-topology, we have from (iii) that for all $t\in I(k_i)_0$ and $x\in Z(k_i)_0$

$$\mathbf{M}(\partial\psi_i(t,x)) \le \mathbf{M}(\partial\Phi(x)) + \eta_i$$

with $\eta_i \to 0$ *as* $i \to \infty$. *Applying* [26, Lemma 4.1] *with* $\mathcal{S} = \Phi(Z)$, *we get by* (ii) *that*

(iv)
$$\sup\{\mathbf{F}(\psi_i(t,x),\Phi(x)) : t \in I(k_i)_0, x \in Z(k_i)_0\} \to 0, \text{ as } i \to \infty.$$

Now given $h \in C^\infty(M)$, *denoting* $c = \sup_M |h|$, *then we have from* (*ii*)(*iii*) *that*

(v)
$$\mathcal{A}^h(\phi_i(x)) \le \sup\{\mathcal{A}^h(\Phi(y)) : \alpha \in X(l_i), x, y \in \alpha\} + (1+c)\delta_i;$$

and hence

$$\sup\{\mathcal{A}^h(\phi_i(x)) : x \in X(k_i)_0\} \le \sup\{\mathcal{A}^h(\Phi(x)) : x \in X\} + (1+c)\delta_i.$$

Proof. [26, Th. 13.1] and [28, Th. 3.9] proved this result when $\mathcal{C}(M)$ is replaced by $\mathcal{Z}_n(M)$ and $\mathcal{Z}_n(M,\mathbb{Z}_2)$ respectively. The adaption to $\mathcal{C}(M)$ was done in [49, Th. 5.1] when $X = [0,1]$ and it is the same for general X. □

The purpose of the next theorem is to construct a continuous map in the **F**-topology out of a discrete map with small fineness.

THEOREM 1.12. *There exist some positive constants* $C_0 = C_0(M,m)$ *and* $\delta_0 = \delta_0(M,m)$ *so that if* Y *is a cubical subcomplex of* $I(m,k)$ *and*

$$\phi : Y_0 \to \mathcal{C}(M)$$

has $\mathbf{f}(\phi) < \delta_0$, *then there exists a map*

$$\Phi : Y \to \mathcal{C}(M)$$

continuous in the **F***-topology and satisfying*

(i) $\Phi(x) = \phi(x)$ *for all* $x \in Y_0$;

(ii) *if* α *is some* j*-cell in* Y, *then* Φ *restricted to* α *depends only on the values of* ϕ *restricted on the vertices of* α;

(iii)
$$\sup\{\mathbf{F}(\Phi(x),\Phi(y)) : x, y \text{ lie in a common cell of } Y\} \le C_0\mathbf{f}(\phi).$$

Proof. This result was proved in [28, Th. 3.10] for when $\mathcal{C}(M)$ is replaced by $\mathcal{Z}_n(M,\mathbb{Z}_2)$. We can use the double cover $\partial : \mathcal{C}(M) \to \mathcal{Z}_n(M,\mathbb{Z}_2)$ (see [29, §5]) to lift the extension from $\mathcal{Z}_n(M,\mathbb{Z}_2)$ to $\mathcal{C}(M)$.

Let $C_0 = C_0(M,m)$ and $\delta_0 = \delta_0(M)$ be given in [28, Th. 3.10]. Denote $\tilde{\phi} = \partial \circ \phi : Y_0 \to \mathcal{Z}_n(M,\mathbb{Z}_2)$ as the projection of ϕ into $\mathcal{Z}_n(M,\mathbb{Z}_2)$. Then $\mathbf{f}(\tilde{\phi}) < \delta_0$, so by [28, Th. 3.10], there exists a map:

$$\tilde{\Phi} : Y \to \mathcal{Z}_n(M,\mathbf{M},\mathbb{Z}_2)$$

continuous in the **M**-topology and satisfying

(a) $\tilde{\Phi}(x) = \tilde{\phi}(x)$ for all $x \in Y_0$;
b) if α is some j-cell in Y, then $\tilde{\Phi}$ restricted to α depends only on the values of $\tilde{\phi}$ restricted on the vertices of α;
(c) $\sup\{\mathbf{M}(\tilde{\Phi}(x), \tilde{\Phi}(y)) : x, y \text{ lie in a common cell of } Y\} \leq C_0\mathbf{f}(\phi)$.

By [29, Claim 5.2], $\tilde{\Phi}$ can be uniquely lifted to a continuous map $\Phi : Y \to \mathcal{C}(M)$ such that $\partial \circ \Phi = \tilde{\Phi}$ and $\Phi(x) = \phi(x)$ for all $x \in Y_0$. In fact, given a j-cell α and a fixed vertex $x_0 \in \alpha_0$, there is a unique lift $\Phi : \alpha \to \mathcal{C}(M)$ such that $\Phi(x_0) = \phi(x_0)$. By the construction in [29, Claim 5.2], $\mathcal{F}(\Phi(x), \Phi(x_0)) = \mathcal{F}(\tilde{\Phi}(x), \tilde{\Phi}(x_0)) \leq C_0\mathbf{f}(\phi)$ for every $x \in \alpha$, so we know by the Constancy Theorem that $\Phi(x) = \phi(x)$ for each vertex $x \in \alpha_0$ when δ_0 is small enough. Thus Φ can be obtained by lifting $\tilde{\Phi}$ in each cell of Y.

Since $\partial\Phi(x)$ and $\tilde{\Phi}(x)$ represent the same varifold, Φ is continuous in the $\mathbf{F}$-topology. So we have proved (i) and (ii).

For (iii), we have

$$\mathbf{F}(\Phi(x), \Phi(y)) = \mathcal{F}(\Phi(x), \Phi(y)) + \mathbf{F}(|\partial\Phi(x)|, |\partial\Phi(y)|) \leq 2C_0\mathbf{f}(\phi). \qquad \square$$

Remark 1.13. Note that in general the mass of $\partial\Phi(x) - \partial\Phi(y)$ as element in $\mathcal{Z}_n(M)$ may not be equal to that of $\tilde{\Phi}(x) - \tilde{\Phi}(y)$, so we may not be able to prove the $\mathbf{M}$-continuity for Φ.

Following [28, 3.10], we call the map Φ given in Theorem 1.12 the *Almgren extension* of ϕ. We will record a few properties concerning the homotopy equivalence of Almgren's extensions.

Before stating the next result, we first recall the notion of homotopic equivalence between discrete sweepouts. Let Y be a cubical subcomplex of $I(m, k)$. Given two discrete maps $\phi_i : Y(l_i)_0 \to \mathcal{C}(M)$, we say ϕ_1 *is homotopic to* ϕ_2 *with fineness less than* η, if there exist $l \in \mathbb{N}$, $l > l_1, l_2$ and a map

$$\psi : I(1, k + l)_0 \times Y(l)_0 \to \mathcal{C}(M)$$

with fineness $\mathbf{f}(\psi) < \eta$ and such that

$$\psi([i - 1], y) = \phi_i(\mathbf{n}(k + l, k + l_i)(y)),\ i = 1, 2, y \in Y(l)_0.$$

The following result is analogous to [28, Prop. 3.11]. We provide a slightly different proof.

PROPOSITION 1.14. *With* ϕ_1, ϕ_2 *as above, if* $\eta < \delta_0(M, m)$ *in Theorem 1.12, then the Almgren extensions*

$$\Phi_1, \Phi_2 : Y \to \mathcal{C}(M)$$

of ϕ_1, ϕ_2, *respectively, are homotopic to each other in the* $\mathbf{F}$*-topology.*

Proof. By Theorem 1.12, the Almgren extension $\Psi : I \times Y \to \mathcal{C}(M)$ of ψ is continuous in $\mathbf{F}$-topology and is a homotopy between the Almgren extensions

Φ_1', Φ_2' of $\phi_1', \phi_2' : Y(l)_0 \to \mathcal{C}(M)$ (given by $\phi_i'(y) = \psi([i-1], y)$). Note that Φ_i' is just a reparametrization of the Almgren extension Φ_i of ϕ_i for $i = 1, 2$ respectively, so Φ_i is homotopic to Φ_i' in the $\mathbf{F}$-topology. Now let us describe the reparametrization map. Given an arbitrary cell α and $k \in \mathbb{N}$, we take α_c to be the center cell of $\alpha(k)$. We can define a map $\mathbf{n}_{\alpha,k} : \alpha \to \alpha$ such that it maps α_c to α linearly, and for each $x \in \alpha \backslash \alpha_c$, if we denote by x_c the nearest point projection of x to $\partial\alpha_c$ then $\mathbf{n}_{\alpha,k}$ maps x to $\mathbf{n}_{\alpha,k}(x_c)$. This map dilates α_c to α and compresses $\alpha \backslash \alpha_c$ to the boundary $\partial\alpha$, and it is homotopic to the identity map. With this notion $\Phi_i'|_\alpha = \Phi_i|_\alpha \circ \mathbf{n}_{\alpha, l-l_i}$ on each cell $\alpha \in Y(l_i)$. Hence we finish the proof. □

The following result is the counterpart of [28, Cor. 3.12].

PROPOSITION 1.15. *Let $\{\phi_i\}_{i\in\mathbb{N}}$ and $\{\psi_i\}_{i\in\mathbb{N}}$ be given by Theorem 1.11 applied to some Φ therein. Assume that Φ is continuous in the $\mathbf{F}$-topology on X. Then the Almgren extension Φ_i is homotopic to Φ in the $\mathbf{F}$-topology for sufficiently large i.*

In particular, for i large enough, there exist homotopy maps $\Psi_i : [0,1] \times X \to \mathcal{C}(M)$ continuous in the $\mathbf{F}$-topology, $\Psi_i(0,\cdot) = \Phi_i$, $\Psi_i(1,\cdot) = \Phi$, and

$$\limsup_{i\to\infty} \sup_{t\in[0,1], x\in X} \mathbf{F}(\Psi_i(t,x), \Phi(x)) \to 0.$$

Therefore, for given $h \in C^\infty(M)$, we have

$$\limsup_{i\to\infty} \sup_{x\in X} \mathcal{A}^h(\Phi_i(x)) \le \sup_{x\in X} \mathcal{A}^h(\Phi(x)).$$

Proof. For i large enough such that $\delta_i < \delta_0$ in Theorem 1.12, we let $\bar\Psi_i : I \times X \to \mathcal{C}(M)$ be the Almgren extensions of ψ_i. By Theorem 1.11(iv) (with $Z = X$) and Theorem 1.12(iii), we know that

$$\limsup_{i\to\infty} \sup_{t\in[0,1], x\in X} \mathbf{F}(\bar\Psi_i(t,x), \Phi(x)) \to 0. \tag{1.6}$$

As in the proof of the above Proposition, we can amend $\bar\Psi_i$ with the reparametrization maps associated with the two pairs $(\Phi_{i-1}', \Phi_{i-1})$ and (Φ_i', Φ_i), and abuse the notation and still denote them by $\bar\Psi_i$. Then $\bar\Psi_i$ is a continuous (in the $\mathbf{F}$-topology) homotopy between Φ_{i-1} and Φ_i. Note that the reparametrizations are done is small cells with sizes converging to zero, so (1.6) still holds true for the amended maps by Theorem 1.12(iii) again. For given i large enough, to construct the homotopy from Φ_i to Φ, we can just let $\Psi_i : [0,\infty] \times X \to \mathcal{C}(M)$ be the gluing of all $\{\bar\Psi_j\}_{j\ge i}$. Note that by (1.6), $\Psi_i(\infty,\cdot) = \Phi$ (we can identify $[0,\infty]$ with $[0,1]$ in the definition of Ψ_i), and (1.6) holds true with $\bar\Psi_i$ replaced by Ψ_i. Hence we finish the proof. □

1.4. *Proof of the min-max Theorem.* One key ingredient in the Almgren-Pitts theory to prove regularity of min-max varifold is to introduce the "almost minimizing" concept. Given $h \in \mathcal{S}(g)$, we refer to [52, §6] for the detailed notion of h-almost minimizing varifold and related properties. The existence of almost minimizing varifolds follows from a combinatorial argument of Pitts [31, p. 165-page 174] inspired by early work of Almgren [3]. Pitts's argument works well in the construction of min-max PMC hypersurfaces; see [52, Th. 6.4]. Marques-Neves has generalized Pitts's combinatorial argument to a more general form in [28, 2.12], and we can adapt their result to the PMC setting with no change. We now describe the adaption.

Consider a sequence of cubical subcomplexes Y_i of $I(m, k_i)$ with $k_i \to \infty$, and a sequence $S = \{\varphi_i\}_{i\in\mathbb{N}}$ of maps

$$\varphi_i : (Y_i)_0 \to \mathcal{C}(M)$$

with fineness $\mathbf{f}(\varphi_i) = \delta_i$ converging to zero. Define

$$\mathbf{L}^h(S) = \limsup_{i\to\infty} \sup\{\mathcal{A}^h(\varphi_i(y)) : y \in (Y_i)_0\},$$

$$\mathbf{K}(S) = \{V = \lim_{j\to\infty} |\partial\varphi_{i_j}(y_j)| \text{ as varifolds} : y_j \in (Y_{i_j})_0\},$$

and

$$\mathbf{C}(S) = \{V = \lim_{j\to\infty} |\partial\varphi_{i_j}(y_j)| \text{ as varifolds} : \text{ with } \lim_{j\to\infty} \mathcal{A}^h(\varphi_{i_j}(y_j)) = \mathbf{L}^h(S)\}.$$

We say that an element $V \in \mathbf{C}(S)$ is *h-almost minimizing in small annuli with respect to S* (cf. [52, Def. 6.3]), if for any $p \in M$ and any small enough annulus $A = A_{r_1,r_2}(p)$ centered at p with radii $0 < r_1 < r_2$, there exist sequences $\{i_j\}_{j\in\mathbb{N}} \subset \{i\}_{i\in\mathbb{N}}$ and $\{y_j : y_j \in (Y_{i_j})_0\}_{j\in\mathbb{N}}$, such that $V = \lim_{j\to\infty} |\partial\varphi_{i_j}(y_j)|$, $\lim_{j\to\infty} \mathcal{A}^h(\varphi_{i_j}(y_j)) = \mathbf{L}^h(S)$, and $\varphi_{i_j}(y_j) \in \mathscr{A}^h(A; \varepsilon_i, \delta_i; \mathbf{M})$ (see [52, Def. 6.1]) for some $\varepsilon_i, \delta_i \to 0$. The last condition is usually called *$(\varepsilon_i, \delta_i, h)$-almost minimizing*. Note that by [52, Prop. 6.5], V is also h-almost minimizing in small annuli in the sense of [52, Def. 6.3].

The following is a variant of [28, Th. 2.13] and [31, Th. 4.10].

THEOREM 1.16. *If no element $V \in \mathbf{C}(S)$ is h-almost minimizing in small annuli with respect to S, then there exists a sequence $\tilde{S} = \{\tilde{\varphi}_i\}$ of maps*

$$\tilde{\varphi}_i : Y_i(l_i)_0 \to \mathcal{C}(M),$$

for some $l_i \in \mathbb{N}$, such that

- *$\tilde{\varphi}_i$ is homotopic to φ_i with fineness converging to zero as $i \to \infty$;*
- *$\mathbf{L}^h(\tilde{S}) < \mathbf{L}^h(S)$.*

Proof. By the assumption of the theorem, for each $V \in \mathbf{C}(S)$, there exists a $p \in M$, such that for any $\tilde{r} > 0$, there exist $r, s > 0$, with $\tilde{r} > r+2s > r-2s > 0$ and $\varepsilon > 0$, such that, if $\mathcal{A}^h(\varphi_i(y)) > \mathbf{L}^h(S) - \varepsilon$ and $\mathbf{F}(|\partial\varphi_i(y)|, V) < \varepsilon$, then

$\varphi_i(y) \notin \mathscr{A}^h(A_{r-2s,r+2s}(p);\varepsilon,\delta;\mathbf{M})$ for any $\delta > 0$. As in the proof of [31, Th. 4.10], we denote $c = (3^m)^{3^m}$. By the compactness of $\mathbf{C}(S)$, we can find a uniform $\varepsilon > 0$ and $I \in \mathbb{N}$, and finitely many points $p_1, \dots, p_\nu \in M$, and for each p_j, we can find c concentric annuli $A_{j,1} \supset\supset \cdots \supset\supset A_{j,c}$ (centered at p_j), such that, if $\mathcal{A}^h(\varphi_i(y)) > \mathbf{L}^h(S) - \varepsilon$ and $i > I$, then there exists some $j \in \{1, \dots, \nu\}$, so that $\varphi_i(y) \notin \mathscr{A}^h(A_{j,a};\varepsilon,\delta;\mathbf{M})$ for all $a \in \{1, \dots, c\}$ and for any $\delta > 0$. From here the construction in [31, Page 165-174] can be applied to S so as to produce the desired $\tilde{S}$. □

Now we are ready to prove Theorem 1.7 following closely that of [27, Th. 3.8]. The only additional thing is to keep track of the volume term $\int_\Omega h d\mathcal{H}^{n+1}$ in $\mathcal{A}^h(\Omega)$ and the values of maps assumed on Z.

Proof of Theorem 1.7. Let $\{\Phi_i\}_{i\in\mathbb{N}}$ be a pulled-tight min-max sequence for Π. By the assumption $\int_M h \geq 0$, $\mathbf{L}^h(\Pi) > 0$, and [52, Lemma 4.11], every element in $\mathbf{C}(\{\Phi_i\})$ is nontrivial. Given $\Phi_i : X \to \mathcal{C}(M)$, it has no concentration of mass as it is continuous in the $\mathbf{F}$-topology, so applying Theorem 1.11 gives a sequence of maps:

$$\phi_i^j : X(k_i^j)_0 \to \mathcal{C}(M),$$

with $k_i^j < k_i^{j+1}$ and a sequence of positive $\{\delta_i^j\}_{j\in\mathbb{N}} \to 0$, satisfying (i)$\cdots$(v) in Theorem 1.11.

As Φ_i is continuous in the $\mathbf{F}$-topology, by the same reasoning as Theorem 1.11(iii)(iv), we further have that for every $x \in X(k_i^j)_0$,

$$\mathbf{M}(\partial\phi_i^j(x)) \leq \mathbf{M}(\partial\Phi_i(x)) + \eta_i^j$$

with $\eta_i^j \to 0$ as $j \to \infty$, and

$$\sup\{\mathbf{F}(\phi_i^j(x), \Phi_i(x)) : x \in X(k_i^j)_0\} \to 0, \text{ as } j \to \infty.$$

Now choose $j(i) \to \infty$ as $i \to \infty$, such that $\varphi_i = \phi_i^{j(i)} : X(k_i^{j(i)})_0 \to \mathcal{C}(M)$ satisfies

- $\sup\{\mathbf{F}(\varphi_i(x), \Phi_i(x)) : x \in X(k_i^{j(i)})_0\} \leq a_i$ with $a_i \to 0$ as $i \to \infty$;
- $\sup\{\mathbf{F}(\Phi_i(x), \Phi_i(y)) : x, y \in \alpha, \alpha \in X(k_i^{j(i)})\} \leq a_i$;
- the fineness $\mathbf{f}(\varphi_i) \to 0$ as $i \to \infty$;
- the Almgren extensions $\Phi_i^{j(i)} : X \to \mathcal{C}(M)$ is homotopic to Φ_i in the $\mathbf{F}$-topology with homotopy maps $\Psi_i^{j(i)}$, and

$$\limsup_{i\to\infty} \sup\{\mathbf{F}(\Psi_i^{j(i)}(t,x), \Phi_i(x)) : t \in [0,1], x \in X\} = 0,$$

and

$$\limsup_{i\to\infty} \sup_{x\in X} \mathcal{A}^h(\Phi_i^{j(i)}(x)) \leq \limsup_{i\to\infty} \sup_{x\in X} \mathcal{A}^h(\Phi_i(x)) = \mathbf{L}^h(\Pi),$$

by Proposition 1.15.

Therefore, if $S = \{\varphi_i\}$, then $\mathbf{L}^h(S) = \mathbf{L}^h(\{\Phi_i\})$ and $\mathbf{C}(S) = \mathbf{C}(\{\Phi_i\})$. By Theorem 1.16, if no element $V \in \mathbf{C}(S)$ is h-almost minimizing in small annuli with respect to S, we can find a sequence $\tilde{S} = \{\tilde{\varphi}_i\}$ of maps

$$\tilde{\varphi}_i : X(k_i^{j(i)} + l_i)_0 \to \mathcal{C}(M)$$

such that

- $\tilde{\varphi}_i$ is homotopic to φ_i with fineness converging to zero as $i \to \infty$;
- $\mathbf{L}^h(\tilde{S}) < \mathbf{L}^h(S)$.

By Proposition 1.14, the Almgren extensions of $\varphi_i, \tilde{\varphi}_i$:

$$\Phi_i^{j(i)}, \tilde{\Phi}_i : X \to \mathcal{C}(M),$$

respectively, are homotopic to each other in the $\mathbf{F}$-topology for i large enough, so $\tilde{\Phi}_i$ is homotopic to Φ_i in the $\mathbf{F}$-topology.

By assumptions (1.5) and (1.4), for i large enough, $\tilde{\varphi}_i$ is the identical to $\varphi_i \circ \mathbf{n}(k_i^{j(i)} + l_i, k_i^{j(i)})$ near $Z(k_i^{j(i)} + l_i)_0$; indeed, the deformation process in Theorem 1.16 was only made to those $\varphi_i(x)$ with $\mathcal{A}^h(\varphi_i(x))$ close to $\mathbf{L}^h(S)$. Therefore, the homotopy maps $\tilde{\Psi}_i$ between $\Phi_i^{j(i)}$ and $\tilde{\Phi}_i$ produced by Proposition 1.14 when restricted to Z are just the reparametrization maps described therein. Hence

$$\limsup_{i\to\infty} \sup\{\mathbf{F}(\tilde{\Psi}_i(t,x), \Phi_i^{j(i)}(x)) : t \in [0,1], x \in Z\} = 0.$$

Therefore, $\{\tilde{\Phi}_i\}_{i\in\mathbb{N}} \in \Pi$. However, by Theorem 1.12

$$\limsup_{i\to\infty} \sup\{\mathcal{A}^h(\tilde{\Phi}_i(x)) : x \in X\} \leq \mathbf{L}^h(\tilde{S}) < \mathbf{L}^h(S) = \mathbf{L}^h(\Pi).$$

This is a contradiction. So some $V \in \mathbf{C}(S) = \mathbf{C}(\{\Phi_i\})$ is h-almost minimizing in small annuli with respect to S, and hence is h-almost minimizing in small annuli in the sense of [52, Def. 6.3].

To finish the proof, we need to show that V has c-bounded first variation, and then [52, Th. 7.1 and Prop. 7.3] give the regularity of V and the existence of Ω. Indeed, by Definition 1.9, V either has c-bounded first variation or belongs to $|\partial\Phi_0|(Z)$. Being h-almost minimizing in small annuli implies that V has c-bounded first variation away from finitely many points by [52, Lemma 6.2]. If $V \in |\partial\Phi_0|(Z)$, then the proof of [19, Th. 4.1] implies that $\|V\|$ has at most $r^{n-\frac{1}{2}}$-volume growth near these bad points, so the first variation extends across these points, and hence V has c-bounded first variation in M. (Note that even if $V \in |\partial\Phi_0|(Z)$, the associated Ω does not belong to $\Phi_0(Z)$, as Ω may be equal to $M\backslash\Phi_0(z)$ for some $z \in Z$.) So we finish the proof. □

2. Compactness of PMC hypersurfaces with bounded Morse index

Now we present an adaption of Sharp's compactness theorem [35] (for minimal hypersurfaces) to the PMC setting and necessary modifications of the proof. Given a closed Riemannian manifold (M^{n+1}, g) and $h \in \mathcal{S}(g)$, denote by $\mathcal{P}^h$ the class of smooth, closed, almost embedded hypersurfaces $\Sigma \subset M$, such that Σ is represented as the boundary of some open subset $\Omega \subset M$ (in the sense of current), and the mean curvature of Σ with respect to the outer normal of Ω is prescribed by h, i.e.,

$$H_\Sigma = h|_\Sigma.$$

In the following we will sometime abuse the notation and identify Σ with Ω.

Note that when $h \in \mathcal{S}(g)$, the min-max PMC hypersurfaces produced in Theorem 1.7 satisfy the above requirements. Indeed, such $\Sigma = \partial\Omega$ is a critical point of the weighted $\mathcal{A}^h$ functional (1.1):

$$\mathcal{A}^h(\Omega) = \mathrm{Area}(\Sigma) - \int_\Omega h \, d\mathcal{H}^{n+1}.$$

The second variation formula for $\mathcal{A}^h$ along normal vector field $X = \varphi\nu \in \mathfrak{X}(M)$ is given by

$$\delta^2\mathcal{A}^h|_\Omega(X, X) = \int_\Sigma (|\nabla\varphi|^2 - (\mathrm{Ric}^M(\nu,\nu) + |A^\Sigma|^2 + \partial_\nu h)\varphi^2) d\mu_\Sigma.$$

The classical Morse index for Σ is defined as the number of negative eigenvalues of the the above quadratic form. However, since we will deal with hypersurfaces with self-touching, a weaker version of index is needed. We adopt a concept used by Marques-Neves [27, Def. 4.1]. As we will see, this weaker index works well for proving both compactness theory and Morse index upper bound.

Definition 2.1. Given $\Sigma \in \mathcal{P}^h$ with $\Sigma = \partial\Omega$, $k \in \mathbb{N}$ and $\varepsilon \geq 0$, we say that Σ is *k-unstable in an ε-neighborhood* if there exists $0 < c_0 < 1$ an a smooth family $\{F_v\}_{v\in\overline{B}^k} \subset \mathrm{Diff}(M)$ with $F_0 = \mathrm{Id}$, $F_{-v} = F_v^{-1}$ for all $v \in \overline{B}^k$ (the standard k-dimensional ball in $\mathbb{R}^k$) such that, for any $\Omega' \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Omega)$, the smooth function

$$\mathcal{A}^h_{\Omega'} : \overline{B}^k \to [0,\infty), \quad \mathcal{A}^h_{\Omega'}(v) = \mathcal{A}^h(F_v(\Omega'))$$

satisfies

- $\mathcal{A}^h_{\Omega'}$ has a unique maximum at $m(\Omega') \in B^k_{c_0/\sqrt{10}}(0)$;
- $-\frac{1}{c_0}\,\mathrm{Id} \leq D^2\mathcal{A}^h_{\Omega'}(u) \leq -c_0\,\mathrm{Id}$ for all $u \in \overline{B}^k$.

Since Σ is a critical point of $\mathcal{A}^h$, necessarily $m(\Omega) = 0$.

Remark 2.2. If a sequence Ω_i converges to Ω in the **F**-topology, then $\mathcal{A}^h_{\Omega_i}$ tends to $\mathcal{A}^h_\Omega$ in the smooth topology. Thus if a $\Sigma \in \mathcal{P}^h$ is k-unstable in a 0-neighborhood, then it is k-unstable in an ε-neighborhood for some $\varepsilon > 0$.

Definition 2.3. Given a $\Sigma \in \mathcal{P}^h$ and $k \in \mathbb{N}$, we say that *its Morse index is bounded (from above) by* k, denoted as

$$\operatorname{index}(\Sigma) \leq k,$$

if it is not j-unstable in 0-neighborhood for any $j \geq k+1$.

All the above concepts can be localized to an open subset $U \subset M$ by using $\operatorname{Diff}(U)$ in place of $\operatorname{Diff}(M)$. If Σ has index equal to 0 in U, we say Σ is *weakly stable* in U.

PROPOSITION 2.4. *If $\Sigma \in \mathcal{P}^h$ is smoothly embedded with no self-touching, then Σ is k-unstable (in 0-neighborhood) if and only if its classical Morse index is $\geq k$.*

Proof. The proof is the same as [27, Prop. 4.3]. □

We have the following curvature estimates as a variant of [52, Th. 3.6] (with relatively weaker stability assumptions).

THEOREM 2.5 (Curvature estimates for weakly stable PMC). *Let $3 \leq (n+1) \leq 7$, and let $U \subset M$ be an open subset. Let $\Sigma \in \mathcal{P}^h$ be weakly stable in U with $\operatorname{Area}(\Sigma) \leq C$. Then there exists C_1 depending only on $n, M, \|h\|_{C^3}, C$, such that*

$$|A^\Sigma|^2(x) \leq \frac{C_1}{\operatorname{dist}^2_M(x, \partial U)} \quad \text{for all } x \in \Sigma.$$

Proof. The curvature estimates follow from standard blowup arguments together with the Bernstein Theorem [34, Th. 2] and [33, Th. 3]. In particular, being weakly stable in U means that for any ambient vector field $X \in \mathfrak{X}(U)$ that generates the flow ϕ^X_t, we have

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}^h(\phi^X_t(\Omega)) \geq 0. \tag{2.1}$$

Assume the conclusion is false. Then there exists a sequence of weakly stable hypersurfaces $\{\Sigma_i\}_{i\in\mathbb{N}}$ with prescribing functions $\{h_i\}_{i\in\mathbb{N}}$ satisfying uniform bounds, but $\sup_U \operatorname{dist}^2_M(\cdot, \partial U)|A^{\Sigma_i}|^2(\cdot) \to \infty$. By the standard blowup process (cf. [41]), one can take a sequence of rescalings of Σ_i that converges locally in $C^{3,\alpha}$ and graphically to a non-flat minimal hypersurface Σ_∞ in $\mathbb{R}^{n+1}$. Note that the rescalings of $\{h_i\}$ converges to 0 locally uniformly in C^3. By the almost embedded assumption and the maximum principle for minimal hypersurfaces ([8]), Σ_∞ is embedded and hence is two-sided. By the classical monotonicity formula and area upper bound assumption on $\{\Sigma_i\}$, Σ_∞ has polynomial volume

growth. The key observation is that (2.1) is preserved under locally $C^{3,\alpha}$ convergence, and hence Σ_∞ is a stable minimal hypersurface. Therefore, it has to be flat by the Bernstein Theorem, but this is a contradiction. □

Given $h \in \mathcal{S}(g)$, $0 < \Lambda \in \mathbb{R}$ and $I \in \mathbb{N}$, let

$$\mathcal{P}^h(\Lambda, I) := \{\Sigma \in \mathcal{P}^h : \mathrm{Area}(\Sigma) \leq \Lambda, \mathrm{index}(\Sigma) \leq I\}.$$

THEOREM 2.6 (Compactness for PMCs with bounded index). *Let (M, g) be a closed Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. Assume that $\{h_k\}_{k\in\mathbb{N}}$ is a sequence of smooth functions in $\mathcal{S}(g)$ such that $\lim_{k\to\infty} h_k = h_\infty$ in smooth topology, where $h_\infty \in \mathcal{S}(g)$ or $h_\infty \equiv 0$. Let $\{\Sigma_k\}_{k\in\mathbb{N}}$ be a sequence of hypersurfaces such that $\Sigma_k \in \mathcal{P}^{h_k}(\Lambda, I)$ for some fixed $\Lambda > 0$ and $I \in \mathbb{N}$. Then,*

(i) *Up to a subsequence, there exists a smooth, closed, almost embedded hypersurface Σ_∞ with prescribed mean curvature h_∞, such that $\Sigma_k \to \Sigma_\infty$ (possibly with integer multiplicity) in the varifold sense, and hence also in the Hausdorff distance by monotonicity formula.*

(ii) *There exists a finite set of points $\mathcal{Y} \subset M$ with $\#\mathcal{Y} \leq I$, such that the convergence of $\Sigma_k \to \Sigma_\infty$ is locally smooth and graphical on $\Sigma_\infty \setminus \mathcal{Y}$.*

(iii) *If $h_\infty \in \mathcal{S}(g)$, then the multiplicity of Σ_∞ is 1, and $\Sigma_\infty \in \mathcal{P}^{h_\infty}(\Lambda, I)$.*

(iv) *Assuming $\Sigma_k \neq \Sigma_\infty$ eventually and $h_k = h_\infty = h \in \mathcal{S}(g)$ for all k such that every $\Sigma \in \mathcal{P}^h$ is properly embedded with no self-touching, then $\mathcal{Y} = \emptyset$, and the nullity of Σ_∞ with respect to $\delta^2\mathcal{A}^h$ is ≥ 1.*

(v) *If $h_\infty \equiv 0$, then the classical Morse index of Σ_∞ satisfies $\mathrm{index}(\Sigma_\infty) \leq I$ (without counting multiplicity).*

Remark 2.7. Here we choose to only present the proof when $h_\infty \in \mathcal{S}(g)$ or $h_\infty \equiv 0$. The results also hold true for arbitrary $h_\infty \in C^\infty(M)$, and details will appear elsewhere.

Proof. The proof follows in essentially the same way as [35, Th. 2.3] once we use Theorem 2.5 to replace [35, Th. 2.1]; we will provide necessary modifications.

Part 1. We first have the following variant of [35, Lemma 3.1]. Given any collection of $I+1$ pairwise disjoint open sets $\{U_i\}_{i=1}^{I+1}$, we have that Σ_k (we drop the sub-index k in this paragraph) is weakly stable in U_i for some $1 \leq i \leq I+1$. Indeed, suppose this is false. Then $\Sigma = \partial\Omega$ is at least 1-unstable in each U_i, and hence there exist $c_i \in (0,1)$ and $\{F_t^i\}_{t\in[-1,1]} \subset \mathrm{Diff}(U_i)$ with $F_{-t}^i = (F_t^i)^{-1}$, such that $-\frac{1}{c_i} \leq \frac{d^2}{dt^2}\mathcal{A}^h(F_t^i(\Omega)) \leq -c_i$. Now for $v = (v_1, \ldots, v_{I+1}) \in \overline{B}^{I+1}$, let $F_v(x) = F_{v_{I+1}} \circ \cdots \circ F_{v_1}(x)$. Since $\{U_i\}$ are pairwise disjoint, it is easy to see that $c_0 = \min\{c_i\}$ and $\{F_v\}$ give an $(I+1)$-unstable pair for Σ, and hence is a contradiction.

This fact, together with Theorem 2.5, implies that (up to a subsequence) Σ_k converges locally smoothly and graphically to an almost embedded hypersurface Σ_∞ of prescribed mean curvature h_∞ (possibly with integer multiplicity) away from at most I points, which we denote by $\mathcal{Y}$. Since as varifolds Σ_k have uniformly bounded first variation, by Allard's compactness theorem [1], Σ_k also converges as varifolds to an integral varifold represented by Σ_∞.

Now we prove that Σ_∞ extends smoothly as an almost embedded hypersurface across the singular points $\mathcal{Y}$; i.e., $\mathcal{Y}$ are removable. By the argument in [35, Claim 2, p. 326], for each $y_i \in \mathcal{Y}$, there exists some $r_i > 0$ such that Σ_∞ is weakly stable in $B_{r_i}(y_i)\backslash\{y_i\}$ in the following sense. Denote Ω_∞ as the weak limit of Ω_k as Caccioppoli sets where $\Sigma_k = \partial\Omega_k$. If $h_\infty \in \mathcal{S}(g)$, then by the first paragraph in Part 2 below, $\Sigma_\infty = \partial\Omega_\infty$ away from $\mathcal{Y}$, $H_{\Sigma_\infty} = h_\infty|_{\Sigma_\infty}$, and $\mathcal{A}^{h_\infty}(\Sigma_\infty) = \mathrm{Area}(\Sigma_\infty) - \int_{\Omega_\infty} h_\infty d\mathcal{H}^{n+1}$; if $h_\infty \equiv 0$, then Σ_∞ is a minimal hypersurface and $\mathcal{A}^{h_\infty}(\Sigma_\infty) = \mathrm{Area}(\Sigma_\infty)$. In both cases, we say that Σ_∞ is weakly stable in $B_{r_i}(y_i)\backslash\{y_i\}$ if for any $X \in \mathfrak{X}(B_{r_i}(y_i)\backslash\{y_i\})$ with the associated flow $\{\phi_t^X : t \in [-\varepsilon, \varepsilon]\}$, $\frac{d^2}{dt^2}\big|_{t=0}\mathcal{A}^{h_\infty}(\phi_t^X(\Sigma_\infty)) \geq 0$. Note that if this is not true for some $X \in \mathfrak{X}(B_{r_i}(y_i)\backslash\{y_i\})$, as $\mathcal{A}^{h_k}(\phi_t^X(\Sigma_k))$ converges to $\mathcal{A}^{h_\infty}(\phi_t^X(\Sigma_\infty))$ smoothly as functions of t, then Σ_k is not weakly stable in $B_{r_i}(y_i)\backslash\{y_i\}$ for k sufficiently large. Following [35, Claim 2, p. 326], we can deduce the required stability property for Σ_∞. Since Σ_∞ has bounded first variation, then by a classical removable singularity result (see Theorem B.1 when $h_\infty \in \mathcal{S}(g)$, and [33] when $h_\infty \equiv 0$), we get the smooth extension. Up to here, we have finished proving (i) and (ii).

Part 2. If $h_\infty \in \mathcal{S}(g)$, then [52, Th. 3.20] implies that Σ_∞ has multiplicity one and is the boundary of some open set Ω_∞ (note that when $h_\infty \in \mathcal{S}(g)$, only case (2) of [52, Th. 3.20] will happen). In fact, fix a point $p \in \Sigma_\infty$ where Σ_∞ is properly embedded. If the limit Σ_∞ has multiplicity ≥ 2, then for i sufficiently large and inside a neighborhood of p, Σ_i consists of several sheets with normal pointing to the same side of Σ_∞, but this cannot happen when Σ_i bounds a region Ω_i. We refer to the proof of [52, Th. 3.20] for more details.

If $\mathrm{index}(\Sigma_\infty) > I$, then there exist $c_0 \in (0,1)$ and $\{F_v : v \in \overline{B}^{I+1}\} \subset \mathrm{Diff}(M)$ such that $-\frac{1}{c_0}\,\mathrm{Id} \leq D^2\mathcal{A}^{h_\infty}(F_v(\Omega_\infty)) \leq -c_0\,\mathrm{Id}$ for all $v \in \overline{B}^{I+1}$. Since $\Sigma_k = \partial\Omega_k$ converges to Σ_∞ smoothly away from finitely many points, we know that Ω_k converges to Ω_∞ in the $\mathbf{F}$-topology as Caccioppoli sets, and then the sequence $v \mapsto \mathcal{A}^{h_k}(F_v(\Omega_k))$ converges to $v \mapsto \mathcal{A}^{h_\infty}(F_v(\Omega_\infty))$ smoothly as functions on $\overline{B}^{I+1}$. Therefore, for k large enough, $-\frac{2}{c_0}\,\mathrm{Id} \leq D^2\mathcal{A}^{h_k}(F_v(\Omega_k)) \leq -\frac{c_0}{2}\,\mathrm{Id}$, so Σ_k is $(I+1)$-unstable, which is a contradiction. This finishes the proof of (iii).

Part 3. Assuming $\Sigma_k \neq \Sigma_\infty$ eventually and $h_k = h_\infty = h \in \mathcal{S}(g)$ such that every element in $\mathcal{P}^h$ is properly embedded, we know $\mathcal{Y} = \emptyset$ by multiplicity

one convergence and the Allard regularity theorem [1]. Next we will produce a Jacobi field for the second variation $\delta^2\mathcal{A}^h$ along Σ_∞; this implies the nullity is ≥ 1.

By (1.3), the Jacobi operator associated with $\delta^2\mathcal{A}^h$ along a PMC $\Sigma \in \mathcal{P}^h$ is

$$L^h_\Sigma\varphi = -\triangle_\Sigma\varphi - \big(\operatorname{Ric}^M(\nu,\nu) + |A^\Sigma|^2 + \partial_\nu h\big)\varphi.$$

The smooth graphical convergence of $\Sigma_k \to \Sigma$ implies that for k sufficiently large, Σ_k can be written as a graph u_k in the normal bundle of Σ_∞, and $u_k \to 0$ uniformly in smooth topology. Subtracting the mean curvature operators between Σ_k and Σ_∞, we get

$$h(x,u_k) - h(x,0) = H_{\Sigma_k} - H_{\Sigma_\infty} = L_{\Sigma_\infty}u_k + o(u_k),$$

where $L_{\Sigma_\infty}u = -\triangle u - \big(\operatorname{Ric}^M(\nu,\nu) + |A^\Sigma|^2\big)u$ is the Jacobi operator for second variation of area, and the second equation follows from [37] and [35, p. 331]. (Note that though the calculation in [35, p. 331] is done assuming $h \equiv 0$, it does not depend on h.) The left-hand side equals to $\partial_\nu h(x, t(x)u_k)\cdot u_k$ by the mean value theorem. Let $\tilde{u}_k = u_k/\|u_k\|_{L^2(\Sigma_\infty)}$ be the renormalizations, then standard elliptic estimates imply that $\tilde{u}_k$ converges smoothly to a nontrivial $\varphi \in C^\infty(\Sigma_\infty)$ such that $\partial_\nu h \cdot \varphi = L_{\Sigma_\infty}\varphi$. This is the same as $L^h_{\Sigma_\infty}\varphi = 0$, so we finish proving (iv).

Part 4. Assuming $h_\infty \equiv 0$, then Σ_∞ is an embedded minimal hypersurface. Assume without loss of generality that Σ_∞ is connected with multiplicity $m \in \mathbb{N}$. Suppose the Morse index $\operatorname{index}(\Sigma_\infty) \geq I+1$. Then by similar argument as in (iii), we can deduce a contradiction. In particular, by [27, Prop. 4.3], there exist $c_0 \in (0,1)$ and $\{F_v : v \in \overline{B}^{I+1}\} \subset \operatorname{Diff}(M)$ such that $-\frac{1}{c_0}\operatorname{Id} \leq D^2\operatorname{Area}(F_v(\Sigma_\infty)) \leq -c_0\operatorname{Id}$ for all $v \in \overline{B}^{I+1}$. Since Σ_k converges to $m\cdot\Sigma_\infty$ as varifolds, and since $h_k \to 0$ uniformly, we know that $\mathcal{A}^{h_k}(F_v(\Omega_k))$ converges to $m\cdot\operatorname{Area}(F_v(\Sigma_\infty))$ smoothly as functions on $\overline{B}^{I+1}$. Therefore, for k large enough, Ω_k is $(I+1)$-unstable, which is a contradiction. So we finish proving (v). □

There is also a theorem analogous to the above one in the setting of changing ambient metrics on M; see [35, Th. A.6] for a similar result for minimal hypersurfaces. The proof proceeds the same way when one realizes that the constant C_1 in Theorem 2.5 depends only on the $\|g\|_{C^4}$ when g is allowed to change.

THEOREM 2.8. *Let M^{n+1} be a closed manifold of dimension $3 \leq (n+1) \leq 7$, and let $\{g_k\}_{k\in\mathbb{N}}$ be a sequence of metrics on M that converges smoothly to some limit metric g. Let $\{h_k\}_{k\in\mathbb{N}}$ be a sequence of smooth functions with $h_k \in \mathcal{S}(g_k)$ that converges smoothly to some limit $h_\infty \in \mathcal{S}(g)$. Let $\{\Sigma_k\}_{k\in\mathbb{N}}$ be a sequence of hypersurfaces with $\Sigma_k \in \mathcal{P}^{h_k}(\Lambda, I)$ for some fixed $\Lambda > 0$ and*

$I \in \mathbb{N}$. Then there exists a smooth, closed, almost embedded hypersurface Σ_∞ with prescribing mean curvature h_∞, such that all properties (i), (ii) and (iii) *in the above theorem are satisfied.*

3. Morse index upper bound

In this part, we will establish Morse index upper bound for min-max PMC hypersurfaces obtained in Theorem 1.7. We will follow closely the strategy of Marques-Neves [27, Th. 1.2], where they proved Morse index upper bound for min-max minimal hypersurfaces. Recall that the Morse index of an almost embedded PMC hypersurface Σ is given in Definition 2.3.

THEOREM 3.1. *Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \leq (n+1) \leq 7$, and let $h \in \mathcal{S}(g)$, which satisfies $\int_M h \geq 0$. Given a k-dimensional cubical complex X and a subcomplex $Z \subset X$, let $\Phi_0 : X \to \mathcal{C}(M)$ be a map continuous in the $\mathbf{F}$-topology, and let Π be the associated (X, Z)-homotopy class of Φ_0. Suppose*

$$\mathbf{L}^h(\Pi) > \max\big\{\max_{x\in Z} \mathcal{A}^h(\Phi_0(x)), 0\big\}. \tag{3.1}$$

Then there exists a nontrivial, smooth, closed, almost embedded hypersurface $\Sigma^n \subset M$, such that

- *Σ is the boundary of some $\Omega \in \mathcal{C}(M)$ where its mean curvature with respect to the unit outer normal of Ω is h, i.e.,*
$$H_\Sigma = h|_\Sigma;$$
- $\mathcal{A}^h(\Omega) = \mathbf{L}^h(\Pi)$;
- $\mathrm{index}(\Sigma) \leq k$.

3.1. *Preliminary lemmas.* Let $h \in \mathcal{S}(g)$. Assume that $\Sigma_0 = \partial\Omega_0 \in \mathcal{P}^h$ is k-unstable in an ε-neighborhood, $\varepsilon > 0$. Let $\{F_v\}_{v\in\overline{B}^k}$ be the associated smooth family given in Definition 2.1.

The first lemma is a counterpart of [27, Lemma 4.4].

LEMMA 3.2. *There exists $\bar\eta = \bar\eta(\varepsilon, \Sigma_0, \{F_v\}) > 0$, such that if $\Omega \in \mathcal{C}(M)$ with $\mathbf{F}(\Omega, \Omega_0) \geq \varepsilon$ satisfies*

$$\mathcal{A}^h(F_v(\Omega)) \leq \mathcal{A}^h(\Omega) + \bar\eta$$

for some $v \in \overline{B}^k$, then $\mathbf{F}(F_v(\Omega), \Omega_0) \geq 2\bar\eta$.

Proof. Assume by contradiction that there exist Ω_i, $\mathbf{F}(\Omega_i, \Omega_0) \geq \varepsilon$ satisfying

$$\mathcal{A}^h(F_{v_i}(\Omega_i)) \leq \mathcal{A}^h(\Omega_i) + \frac{1}{i}$$

for some $v_i \in \overline{B}^k$, but $\mathbf{F}(F_{v_i}(\Omega_i), \Omega_0) \leq \frac{2}{i}$.

Denote $v = \lim v_i$, and pass to the limit as $i \to \infty$. Then $\Omega_i \to F_{-v}(\Omega_0)$ in $\mathbf{F}$-metric, and $\mathcal{A}^h(\Omega_0) \leq \mathcal{A}^h(F_{-v}(\Omega_0))$, which implies that $v = 0$; hence $\Omega_i \to \Omega_0$ in the $\mathbf{F}$-metric, which is a contradiction. □

For each $\Omega \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Omega_0)$, consider the one-parameter flow $\{\phi^{\Omega}(\cdot, t) : t \geq 0\} \subset \mathrm{Diff}(\overline{B}^k)$ generated by the vector field

$$u \to -(1 - |u|^2)\nabla \mathcal{A}^h_{\Omega}(u), \quad u \in \overline{B}^k.$$

When $u \in \overline{B}^k$ is fixed, the function $t \to \mathcal{A}^h_{\Omega}(\phi^{\Omega}(u, t))$ is non-increasing.

The following lemma is a variant of [27, Lemma 4.5], and the proof is recorded in Appendix C.

LEMMA 3.3. *For any $\delta < 1/4$, there exists $T = T(\delta, \varepsilon, \Omega_0, \{F_v\}, c_0) \geq 0$ such that for any $\Omega \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Omega_0)$ and $v \in \overline{B}^k$ with $|v - m(\Omega)| \geq \delta$, we have*

$$\mathcal{A}^h_{\Omega}(\phi^{\Omega}(v, T)) < \mathcal{A}^h_{\Omega}(0) - \frac{c_0}{10} \text{ and } |\phi^{\Omega}(v, T)| > \frac{c_0}{4}.$$

3.2. *Deformation theorem.* Taking a min-max sequence $\{\Phi_i\}_{i\in\mathbb{N}}$, we will prove a deformation theorem as an adaption of [27, Th. 5.1] to our setting. Recall that $\mathcal{P}^h$ denotes the class of smooth, closed, almost embedded hypersurface $\Sigma \subset M$ represented as boundary $\Sigma = \partial\Omega$, and of prescribed mean curvature h.

Fix a $\sigma > 0$ such that $\mathbf{L}^h - \sup_{x\in Z} \mathcal{A}^h(\Phi_0(x)) > 2\sigma$. Denote

$$X_{i,\sigma} = \{x \in X, \text{ such that } \mathcal{A}^h(\Phi_i(x)) \geq \mathbf{L}^h - \sigma\}.$$

Note that when i is sufficiently large, $X_{i,\sigma} \subset X \backslash Z$.

Now we present the deformation theorem, and the proof follows closely that of [27, Th. 5.1]. Given two subsets $A, B \subset \mathcal{C}(M)$, we denote

$$\mathbf{F}(A, B) := \inf\{\mathbf{F}(\Omega_A, \Omega_B) : \Omega_A \in A, \Omega_B \in B\}.$$

THEOREM 3.4. *Suppose that*

(a) $\Sigma = \partial\Omega \in \mathcal{P}^h$ *is* $(k+1)$*-unstable*;
(b) $K \subset \mathcal{C}(M)$ *is a subset, so that* $\mathbf{F}(\{\Omega\}, K) > 0$ *and* $\mathbf{F}(\Phi_i(X_{i,\sigma}), K) > 0$ *for all* $i \geq i_0$;
(c) $\mathcal{A}^h(\Omega) = \mathbf{L}^h$.

Then there exist $\bar{\varepsilon} > 0$, $j_0 \in \mathbb{N}$, and another sequence $\{\Psi_i\}_{i\in\mathbb{N}}$, $\Psi_i : X \to (\mathcal{C}(M), \mathbf{F})$, so that

(i) Ψ_i *is homotopic to* Φ_i *in the* $\mathbf{F}$*-topology for all* $i \in \mathbb{N}$ *and* $\Psi_i|_Z = \Phi_i|_Z$ *for* $i \geq j_0$;
(ii) $\mathbf{L}^h(\{\Psi_i\}) \leq \mathbf{L}^h$;
(iii) $\mathbf{F}(\Psi_i(X_{i,\sigma}), \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}}(\Omega) \cup K) > 0$ *for all* $i \geq j_0$.

Proof. Denote $d = \mathbf{F}(\{\Omega\}, K) > 0$.

By (a), Σ is $(k+1)$-unstable in some ε-neighborhood. Let $\{F_v\}_{v\in\overline{B}^{k+1}}$, c_0 be the associated family and constant as in Definition 2.1. By possibly changing ε, $\{F_v\}$, c_0, we can assume that

$$\inf\{\mathbf{F}(F_v(\Omega'), K), v \in \overline{B}^{k+1}\} > \frac{d}{2} \text{ for all } \Omega' \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Omega). \tag{3.2}$$

Let $X(k_i)$ be a sufficiently fine subdivision of X so that

$$\mathbf{F}(\Phi_i(x), \Phi_i(y)) < \delta_i$$

for any x, y belonging to the same cell in $X(k_i)$ with $\delta_i = \min\{2^{-(i+k+2)}, \varepsilon/4\}$. We can also assume that

$$|m(\Phi_i(x)) - m(\Phi_i(y))| < \delta_i$$

for any x, y with $\mathbf{F}(\Phi_i(x), \Omega) \leq 2\varepsilon$, $\mathbf{F}(\Phi_i(y), \Omega) \leq 2\varepsilon$, and belonging to the same cell in $X(k_i)$.

For $\eta > 0$, let $U_{i,\eta}$ be the union of all cells $\sigma \in X(k_i)$ so that $\mathbf{F}(\Phi_i(x), \Omega) < \eta$ for all $x \in \sigma$. Then $U_{i,\eta}$ is a subcomplex of $X(k_i)$. If a cell $\beta \notin U_{i,\eta}$, then $\mathbf{F}(\Phi_i(x'), \Omega) \geq \eta$ for some $x' \in \beta$. Therefore, $\mathbf{F}(\Phi_i(x), \Omega) \geq \eta - \delta_i$ for all $x \in \beta$. By (c) (after possibly shrinking ε), we can assume

$$U_{i,2\varepsilon} \subset X_{i,\sigma}.$$

For each $i \in \mathbb{N}$ and $x \in U_{i,2\varepsilon}$, we simply denote $\mathcal{A}^h_{i,x} = \mathcal{A}^h_{\Phi_i(x)}$, $m_i(x) = m(\Phi_i(x))$ and $\phi_{i,x} = \phi^{\Phi_i(x)}$. The function $m_i : U_{i,2\varepsilon} \to \overline{B}^{k+1}$ is continuous, and the two families $\{\mathcal{A}^h_{i,x}\}_{x\in U_{i,2\varepsilon}}$, $\{\phi_{i,x}\}_{x\in U_{i,2\varepsilon}}$ are continuous in x. Following [27, 5.1] we can define a continuous map

$$\hat{H}_i : U_{i,2\varepsilon} \times [0,1] \to B^{k+1}_{1/2^i}(0) \text{ so that } \hat{H}_i(x, 0) = 0 \text{ for all } x \in U_{i,2\varepsilon}$$

and

$$\inf_{x\in U_{i,2\varepsilon}} |\hat{H}_i(x, 1) - m_i(x)| \geq \eta_i > 0 \text{ for some } \eta_i > 0. \tag{3.3}$$

The construction here is the same so we omit details. The crucial ingredient is the fact that $U_{i,2\varepsilon}$ has dimension less than or equal to k while the image set $\overline{B}^{k+1}$ has dimension $k+1$.

Let $c : [0, \infty) \to [0, 1]$ be a cutoff function that is non-increasing, is equal to 1 in a neighborhood of $[0, 3\varepsilon/2]$, and 0 in a neighborhood of $[7\varepsilon/4, +\infty)$. For $y \notin U_{i,2\varepsilon}$, $\mathbf{F}(\Phi_i(y), \Omega) \geq 2\varepsilon - \delta_i \geq 7\varepsilon/4$. Hence

$$c(\mathbf{F}(\Phi_i(y), \Omega)) = 0 \text{ for all } y \notin U_{i,2\varepsilon}.$$

Consider the map $H_i : X \times [0, 1] \to B^{k+1}_{2^{-i}}(0)$ defined as

$$H_i(x, t) = \hat{H}_i(x, c(\mathbf{F}(\Phi_i(x), \Omega))t) \text{ if } x \in U_{i,2\varepsilon}$$

and

$$H_i(x,t) = 0 \text{ if } x \in X \backslash U_{i,2\varepsilon}.$$

Then H_i is continuous.

With η_i as given in (3.3), let $T_i = T(\eta_i, \varepsilon, \Omega, \{F_v\}, c_0) \geq 0$ be given by Lemma 3.3. Now we set $D_i : X \to \overline{B}^{k+1}$ such that

$$D_i(x) = \phi_{i,x}(H_i(x,1), c(\mathbf{F}(\Phi_i(x), \Omega))T_i) \text{ if } x \in U_{i,2\varepsilon}$$

and

$$D_i(x) = 0 \text{ if } x \in X \backslash U_{i,2\varepsilon}.$$

Then D_i is continuous.

Define

$$\Psi_i : X \to \mathcal{C}(M), \quad \Psi_i(x) = F_{D_i(x)}(\Phi_i(x)).$$

In particular,

$$\Psi_i(x) = \Phi_i(x) \text{ if } x \in X \backslash U_{i,2\varepsilon}.$$

Hence $\Psi_i|_Z = \Phi_i|_Z$ for i sufficiently large.

Note that the map D_i is homotopic to the zero map in $\overline{B}^{k+1}$, so Ψ_i is homotopic to Φ_i in the $\mathbf{F}$-topology for all $i \in \mathbb{N}$. Thus far, we have proved (i).

Claim 1. $\mathbf{L}^h(\{\Psi\}_{i\in\mathbb{N}}) \leq \mathbf{L}^h$.

By the non-increasing property of $t \to \mathcal{A}^h_{i,x}(\phi_{i,x}(u,t))$, we have that for all $x \in X$,

$$\mathcal{A}^h(\Psi_i(x)) \leq \mathcal{A}^h(F_{H_i(x,1)}(\Phi_i(x))).$$

Using the fact that $H_i(x,1) \in B^{k+1}_{1/2^i}(0)$ for all $x \in X$ and that $\|F_v - \mathrm{Id}\,\|_{C^2} \to 0$ uniformly as $v \to 0$, we have that

$$\lim_{i\to\infty} \sup_{x\in X} \left|\mathcal{A}^h(\Phi_i(x)) - \mathcal{A}^h(F_{H_i(x,1)}(\Phi_i(x)))\right| = 0, \tag{3.4}$$

and this finishes proving Claim 1.

Claim 2. There exists $\bar{\varepsilon} > 0$, such that for all sufficiently large i,

$$\mathbf{F}(\Psi_i(X), \Omega) > \bar{\varepsilon}.$$

There are three cases. If $x \in X \backslash U_{i,2\varepsilon}$, then $\Psi_i(x) = \Phi_i(x)$ and so $\mathbf{F}(\Psi_i(x), \Omega) \geq \frac{7\varepsilon}{4}$.

If $x \in U_{i,2\varepsilon} \backslash U_{i,5\varepsilon/4}$, then $\mathbf{F}(\Phi_i(x), \Omega) \geq \varepsilon$. The non-increasing property of $t \to \mathcal{A}^h_{i,x}(\phi_{i,x}(u,t))$ implies

$$\mathcal{A}^h(\Psi_i(x)) = \mathcal{A}^h(F_{D_i(x)}(\Phi_i(x))) \leq \mathcal{A}^h(F_{H_i(x,1)}(\Phi_i(x))).$$

From (3.4), we have that for i large enough,

$$\mathcal{A}^h(F_{H_i(x,1)}(\Phi_i(x))) \leq \mathcal{A}^h(\Phi_i(x)) + \bar{\eta} \text{ for all } x \in X,$$

where $\bar{\eta}=\bar{\eta}(\varepsilon,\Omega,\{F_v\})>0$ is given by Lemma 3.2. Combining the two inequalities with Lemma 3.2 applied to $\Phi_i(x)$, $v=D_i(x)$, we get $\mathbf{F}(\Psi_i(x),\Omega)\geq 2\bar{\eta}$.

Finally when $x\in U_{i,5\varepsilon/4}$, $c(\mathbf{F}(\Phi_i(x),\Omega))=1$. Hence by Lemma 3.3 (with $\delta=\eta_i$, $\Omega=\Phi_i(x)$, $v=H_i(x,1)$) we have

$$\mathcal{A}^h(\Psi_i(x))=\mathcal{A}^h_{i,x}(\phi_{i,x}(H_i(x,1),T_i))<\mathcal{A}^h_{i,x}(0)-\frac{c_0}{10}=\mathcal{A}^h(\Phi_i(x))-\frac{c_0}{10}.$$

Note that there exists $\bar{\gamma}=\bar{\gamma}(\Omega,c_0)$ so that

$$\mathcal{A}^h(\Omega')\leq\mathcal{A}^h(\Omega)-\frac{c_0}{20}\Longrightarrow\mathbf{F}(\Omega',\Omega)\geq 2\bar{\gamma}.$$

By assumption (c), we can choose i sufficiently large so that

$$\sup_{x\in X}\mathcal{A}^h(\Phi_i(x))\leq\mathcal{A}^h(\Omega)+\frac{c_0}{20}.$$

So

$$\mathcal{A}^h(\Psi_i(x))\leq\mathcal{A}^h(\Omega)-\frac{c_0}{20}.$$

This implies that $\mathbf{F}(\Psi_i(x),\Omega)\geq 2\bar{\gamma}$, and hence ends the proof of Claim 2.

Claim 3. For all i, $\mathbf{F}(\Psi_i(X_{i,\sigma}),K)>0$.

If $x\in X_{i,\sigma}\backslash U_{i,2\varepsilon}$, then $\Psi_i(x)=\Phi_i(x)$ and so $\mathbf{F}(\Psi_i(X_{i,\sigma}\backslash U_{i,2\varepsilon}),K)>0$. If $x\in U_{i,2\varepsilon}$, then $\mathbf{F}(\Phi_i(x),\Omega)\leq 2\varepsilon$, and by (3.2) we have $\mathbf{F}(\Psi_i(x),K)\geq\frac{d}{2}$. So we finish proving Claim 3, and hence the theorem. □

3.3. *Proof of Morse index upper bound.* Let M^{n+1} be a closed manifold of dimension $3\leq(n+1)\leq 7$. A pair (g,h) consisting of a Riemannian metric g and a smooth function $h\in C^\infty(M)$ is called a *good pair* if

- $h\in\mathcal{S}(g)$, i.e., h is Morse and the zero set $\{h=0\}$ is a smooth embedded hypersurface in M with mean curvature H vanishing to at most finite order; and
- g is *bumpy* for $\mathcal{P}^h$, i.e., every $\Sigma\in\mathcal{P}^h$ is properly embedded (no self-touching), and is nondegenerate (nullity equal to zero).

Denote by $\mathcal{S}_0$ the class of smooth functions $h\in C^\infty(M)$ such that h is Morse and the zero set $\{h=0\}$ is a smooth embedded hypersurface. Note that $\mathcal{S}_0$ is open and dense in $C^\infty(M)$, and it is independent of the choice of a metric (see [52, Prop. 3.8]).

LEMMA 3.5. *Given $h\in\mathcal{S}_0$, the set of Riemannian metrics g on M with (g,h) as a good pair is generic in the Baire sense.*

Proof. By the proof of [52, Prop. 3.8], we know that the set of metrics g under which $\{h=0\}$ has mean curvature vanishing to at most finite order is an open and sense subset. In particular, openness follows as small smooth perturbations of g will bound the order of vanishing of $H_{\{h=0\}}$. To show

denseness,[1] note that it is proved in [52, Prop. 3.8] that for any $h \in \mathcal{S}_0$ and any metric g, one can first perturb g slightly so that $\{h = 0\}$ is not a minimal hypersurface. Then there exists a flow $\{F_t : t \in (-\varepsilon, \varepsilon)\} \subset \mathrm{Diff}(M)$ supported near $\{h = 0\}$, such that the zero set of $h \circ (F_t)^{-1}$ has mean curvature vanishing to at most finite order for $t > 0$. That is to say, the zero set $\{h = 0\}$ satisfies the requirement for the pull-back metrics $F_t^* g$.

In a series of celebrated papers [42], [44], [45], White proved that for a fixed $h \in \mathcal{S}_0$, the set of metrics under which all closed, simple immersed PMCs are non-degenerate and self-transverse is generic in the Baire sense. In fact, White proved in [42, §7] that the set of metrics under which all closed, simple immersed CMC hypersurfaces are non-degenerate is generic, and the proof is the same in a smooth neighborhood of an arbitrary pair (g, h) when $h \in \mathcal{S}(g)$, hence the result follows as the set of g where $h \in \mathcal{S}(g)$ is open and dense. In [45, Th. 33], White further proved self-transverse property for a generic set of metrics. Our almost embedded hypersurfaces are simple immersed. So for such generic metrics, almost embedded PMCs are properly embedded.

To finish the proof, we take the intersection of the two generic sets of metrics, which is still generic in the Baire sense. □

The following theorem is a counterpart of [27, Th. 6.1], and the proof follows closely. We remark that by Theorem 2.6(iv), if (g, h) is a good pair, then there are only finitely many elements in $\mathcal{P}^h(\Lambda, I)$.

THEOREM 3.6. *Assume that (g, h) is a good pair, and let $\{\Phi_i\}_{i\in\mathbb{N}}$ be a min-max sequence of Π such that $\mathbf{L}^h(\{\Phi_i\}_{i\in\mathbb{N}}) = \mathbf{L}^h(\Pi) = \mathbf{L}^h$ and (3.1) is satisfied.*

There exists a smooth, closed, properly embedded hypersurface $\Sigma = \partial\Omega \in \mathbf{C}(\{\Phi_i\}_{i\in\mathbb{N}})$ such that $\Sigma \in \mathcal{P}^h$ with

$$\mathbf{L}^h(\Pi) = \mathcal{A}^h(\Omega) \text{ and } \mathrm{index}(\Sigma) \leq k.$$

Proof. By the finiteness remark above, it suffices to show that, for every $r > 0$, there is a $\tilde{\Sigma} = \partial\tilde{\Omega} \in \mathcal{P}^h$ such that $\mathbf{F}([\tilde{\Sigma}], \mathbf{C}(\{\Phi_i\}_{i\in\mathbb{N}})) < r$,

$$\mathbf{L}^h(\Pi) = \mathcal{A}^h(\tilde{\Omega}) \text{ and } \mathrm{index}(\tilde{\Sigma}) \leq k.$$

Denote by $\mathcal{W}$ the set of all $\tilde{\Sigma} = \partial\tilde{\Omega} \in \mathcal{P}^h$ with $\mathcal{A}^h(\tilde{\Omega}) = \mathbf{L}^h$ and by $\mathcal{W}(r)$ the set

$$\{\Sigma \in \mathcal{W} : \mathbf{F}([\Sigma], \mathbf{C}(\{\Phi_i\}_{i\in\mathbb{N}})) \geq r\}.$$

[1] After this article was first posted, we noticed that denseness can also be proved by using conformal deformation of metrics; cf. [21, Prop. 2.3]. Indeed, by arbitrarily small conformal perturbation of the metric, we can make the mean curvature $H_{\{h=0\}}$ be a Morse function on $\{h = 0\}$.

LEMMA 3.7. *There exist* $i_0 \in \mathbb{N}$ *and* $\bar{\varepsilon}_0 > 0$ *such that* $\mathbf{F}(\Phi_i(X), \mathcal{W}(r)) > \bar{\varepsilon}_0$ *for all* $i \geq i_0$.

Proof. Suppose by contradiction for some subsequence $\{j\} \subset \{i\}$ that we have $x_j \in X$ and $\tilde{\Sigma}_j = \partial\tilde{\Omega}_j \in \mathcal{W}(r)$ so that

$$\lim_{j\to\infty} \mathbf{F}(\Phi_j(x_j), \tilde{\Omega}_j) = 0.$$

Since $\mathcal{A}^h(\tilde{\Omega}_j) \equiv \mathbf{L}^h$, we have $\lim_{j\to\infty} \mathcal{A}^h(\Phi_j(x_j)) = \mathbf{L}^h$. Hence a subsequence $|\partial\Phi_j(x_j)|$ will converge as varifolds to some $V \in \mathbf{C}(\{\Phi_i\}_{i\in\mathbb{N}})$, which is a contradiction to $\mathbf{F}(|\partial\tilde{\Omega}_i|, \mathbf{C}(\{\Phi_i\}_{i\in\mathbb{N}})) \geq r$. □

Denote $\mathcal{W}^{k+1}$ as the collection of elements in $\mathcal{W}$ with index greater than or equal to $(k+1)$. As (g,h) is a good pair, this set is countable by the remark above the theorem, and we can write $\mathcal{W}^{k+1}\backslash\overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_0}(\mathcal{W}(r)) = \{\Sigma_1, \Sigma_2, \cdots\}$, where $\Sigma_i = \partial\Omega_i$. Note that by possibly perturbing $\bar{\varepsilon}_0$, we can make sure $\mathcal{W}^{k+1} \cap \partial\overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_0}(\mathcal{W}(r)) = \emptyset$.

Using Theorem 3.4 (we can take $X_{i,\sigma}$ to be X) with $K = \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_0}(\mathcal{W}(r))$ and $\Sigma = \Sigma_1$, we find $\bar{\varepsilon}_1 > 0$, $i_1 \in \mathbb{N}$, and $\{\Phi^1_i\}_{i\in\mathbb{N}}$ so that

- Φ^1_i is homotopic to Φ_i in the $\mathbf{F}$-topology for all $i \in \mathbb{N}$ and $\Phi^1_i|_Z = \Phi_i|_Z$ for $i \geq i_1$;
- $\mathbf{L}^h(\{\Phi^1_i\}_{i\in\mathbb{N}}) \leq \mathbf{L}^h$;
- $\mathbf{F}(\Phi^1_i(X), \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_1}(\Omega_1) \cup \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_0}(\mathcal{W}(r))) > 0$ for $i \geq i_1$;
- no Ω_j belongs to $\partial\overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_1}(\Omega_1)$.

We consider Σ_2 now. If $\Omega_2 \notin \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_1}(\Omega_1)$, we apply Theorem 3.4 with $K = \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_1}(\Omega_1) \cup \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_0}(\mathcal{W}(r))$, $\Sigma = \Sigma_2$, and find $\bar{\varepsilon}_2 > 0$, $i_2 \in \mathbb{N}$, and $\{\Phi^2_i\}_{i\in\mathbb{N}}$ so that

- Φ^2_i is homotopic to Φ_i in the $\mathbf{F}$-topology for all $i \in \mathbb{N}$ and $\Phi^2_i|_Z = \Phi_i|_Z$ for $i \geq i_2$;
- $\mathbf{L}^h(\{\Phi^2_i\}_{i\in\mathbb{N}}) \leq \mathbf{L}^h$;
- $\mathbf{F}(\Phi^2_i(X), \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_1}(\Omega_1) \cup \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_2}(\Omega_2) \cup \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_0}(\mathcal{W}(r))) > 0$ for $i \geq i_2$;
- no Ω_j belongs to $\partial\overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_1}(\Omega_1) \cup \partial\overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_2}(\Omega_2)$.

If $\mathbf{F}(\Omega_2, \Omega_1) < \bar{\varepsilon}_1$, we skip it and repeat the construction with Σ_3.

By induction there are two possibilities. We can find for all $l \in \mathbb{N}$ a sequence $\{\Phi^l_i\}_{i\in\mathbb{N}}$, $\bar{\varepsilon}_l > 0$, $i_l \in \mathbb{N}$, and $\Sigma_{j_l} \in \mathcal{W}^{k+1}\backslash\overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_0}(\mathcal{W}(r))$ for some subsequences $\{j_l\} \subset \mathbb{N}$ so that

- Φ^l_i is homotopic to Φ_i in the $\mathbf{F}$-topology for all $i \in \mathbb{N}$ and $\Phi^l_i|_Z = \Phi_i|_Z$ for $i \geq i_l$;
- $\mathbf{L}^h(\{\Phi^l_i\}_{i\in\mathbb{N}}) \leq \mathbf{L}^h$;

- $\mathbf{F}(\Phi_i^l(X), \cup_{q=1}^l \overline{\mathbf{B}}^{\mathbf{F}}_{\bar\varepsilon_q}(\Omega_{j_q}) \cup \overline{\mathbf{B}}^{\mathbf{F}}_{\bar\varepsilon_0}(\mathcal{W}(r))) > 0$ for $i \geq i_l$;
- $\{\Omega_1, \dots, \Omega_l\} \subset \cup_{q=1}^l \overline{\mathbf{B}}^{\mathbf{F}}_{\bar\varepsilon_q}(\Omega_{j_q})$;
- no Ω_j belongs to $\partial\overline{\mathbf{B}}^{\mathbf{F}}_{\bar\varepsilon_q}(\Omega_{j_q})$ for all $q = 1, \dots, l$.

Or the process ends in finitely many steps. That means we can find some $m \in \mathbb{N}$, a sequence $\{\Phi_i^m\}_{i\in\mathbb{N}}$, $\bar\varepsilon_1, \dots, \bar\varepsilon_m > 0$, $i_m \in \mathbb{N}$ and $\Sigma_{j_1}, \dots, \Sigma_{j_m} \in \mathcal{W}^{k+1}\backslash\overline{\mathbf{B}}^{\mathbf{F}}_{\bar\varepsilon_0}(\mathcal{W}(r))$ so that

- Φ_i^m is homotopic to Φ_i in the $\mathbf{F}$-topology for all $i \in \mathbb{N}$ and $\Phi_i^m|_Z = \Phi_i|_Z$ for $i \geq i_m$;
- $\mathbf{L}^h(\{\Phi_i^m\}_{i\in\mathbb{N}}) \leq \mathbf{L}^h$;
- $\mathbf{F}(\Phi_i^m(X), \cup_{q=1}^m \overline{\mathbf{B}}^{\mathbf{F}}_{\bar\varepsilon_q}(\Omega_{j_q}) \cup \overline{\mathbf{B}}^{\mathbf{F}}_{\bar\varepsilon_0}(\mathcal{W}(r))) > 0$ for $i \geq i_m$;
- $\{\Omega_j : j \geq 1\} \subset \cup_{q=1}^m \overline{\mathbf{B}}^{\mathbf{F}}_{\bar\varepsilon_q}(\Omega_{j_q})$.

In the first case we choose an increasing sequence $p_l \geq i_l$ so that

$$\sup_{x\in X} \mathcal{A}^h(\Phi_{p_l}^l) \leq \mathbf{L}^h + \frac{1}{l},$$

and we set $\Psi_l = \Phi_{p_l}^l$. In the second case we set $p_l = l$ and $\Psi_l = \Phi_l^m$. The sequence $\{\Psi_l\}_{l\in\mathbb{N}}$ satisfies that

(i) Ψ_l is homotopic to Φ_{p_l} in the $\mathbf{F}$-topology, and $\Psi_l|_Z = \Phi_{p_l}|_Z$ for all l;

(ii) $\mathbf{L}^h(\{\Psi_l\}_{l\in\mathbb{N}}) \leq \mathbf{L}^h$;

(iii) given any subsequence $\{l_j\} \subset \{l\}$, $x_j \in X$, if $\lim_{j\to\infty} \mathcal{A}^h(\Psi_{l_j}(x_j)) = \mathbf{L}^h$, then $\{\Psi_{l_j}(x_j)\}_{j\in\mathbb{N}}$ does not converge in $\mathbf{F}$-topology to any element in $\mathcal{W}^{k+1} \cup \mathcal{W}(r)$.

The Min-max Theorem 1.7 applied to $\{\Psi_l\}_{i\in\mathbb{N}}$ implies that $\mathcal{W}\backslash(\mathcal{W}^{k+1} \cup \mathcal{W}(r))$ is not empty, and this proves the theorem. □

Now we can use the previous theorem and the Compactness Theorem 2.8 to prove Theorem 3.1.

Proof of Theorem 3.1. Given (g, h) as in the theorem, then $h \in \mathcal{S}(g) \subset \mathcal{S}_0$. By Lemma 3.5, there exists a sequence of metrics $\{g_j\}_{j\in\mathbb{N}}$ converging smoothly to g such that (g_j, h) is a good pair for all $j \in \mathbb{N}$. If $\mathbf{L}_j^h = \mathbf{L}_j^h(\Pi, g_j)$ is the h-width of Π with respect to g_j, then the sequence $\{\mathbf{L}_j^h\}_{j\in\mathbb{N}}$ tends to the h-width $\mathbf{L}^h(\Pi, g)$ with respect to g, and for j large enough (3.1) is satisfied with g_j in place of g. For each j large enough, the previous theorem gives a properly embedded closed hypersurface $\Sigma_j = \partial\Omega_j \in \mathcal{P}^h$ with $\mathcal{A}^{h_j}(\Omega_j) = \mathbf{L}_j^h$ and $\text{index}(\Sigma_j) \leq k$ (with respect to g_j). Let $\Sigma_\infty = \partial\Omega_\infty$ be the limit of $\{\Sigma_j\}_{j\in\mathbb{N}}$ given in Theorem 2.8. Then the locally smooth convergence implies that $\mathcal{A}^h(\Omega_\infty) = \mathbf{L}^h(\Pi, g)$ and $\text{index}(\Sigma_\infty) \leq k$. □

4. Min-max hypersurfaces associated with sweepouts of boundaries have multiplicity one in a bumpy metric

We present our first multiplicity one result. In particular, we will prove that the min-max minimal hypersurfaces associated with sweepouts of boundaries of Caccioppoli sets are two-sided and have multiplicity one in a bumpy metric. We will approximate the area functional by the weighted $\mathcal{A}^{\varepsilon h}$-functionals for some prescribing function h when $\varepsilon \to 0$. We know by Section 1 that the min-max PMC hypersurfaces are two-sided with multiplicity one, and we will prove that the limit minimal hypersurfaces (when $\varepsilon \to 0$) are also two-sided and have multiplicity one by choosing the right prescribing function h.

Recall that a Riemannian metric g is said to be *bumpy* if every smooth closed immersed minimal hypersurface is non-degenerate. White proved that the set of bumpy metrics is generic in the Baire sense [42], [44].

THEOREM 4.1 (Multiplicity one theorem for sweepouts of boundaries). *Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. Let X be a k-dimensional cubical complex and $Z \subset X$ be a subcomplex, and let $\Phi_0 : X \to \mathcal{C}(M)$ be a map continuous in the $\mathbf{F}$-topology. Let Π be the associated (X, Z)-homotopy class of Φ_0. Assume that*

$$\mathbf{L}(\Pi) > \max_{x \in Z} \mathbf{M}(\partial \Phi_0(x)), \tag{4.1}$$

where we let $h \equiv 0$ as in Section 1.1.

If g is a bumpy metric, then there exists a disjoint collection of smooth, connected, closed, embedded, two-sided, minimal hypersurfaces $\Sigma = \cup_{i=1}^N \Sigma_i$, such that

$$\mathbf{L}(\Pi) = \sum_{i=1}^N \mathrm{Area}(\Sigma_i) \text{ and } \mathrm{index}(\Sigma) = \sum_{i=1}^N \mathrm{index}(\Sigma_i) \leq k.$$

In particular, each component of Σ is two-sided and has exactly multiplicity one.

Proof. Pick a $h \in \mathcal{S}(g)$ with $\int_M h \geq 0$ (to be fixed at the end) and $\varepsilon > 0$ small enough so that

$$\mathbf{L}(\Pi) - \max_{x \in Z} \mathbf{M}(\partial \Phi_0(x)) > 2\varepsilon \sup_M |h| \cdot \mathrm{Vol}(M).$$

Note that for each $\Omega \in \mathcal{C}(M)$, we have

$$\mathbf{M}(\partial\Omega) - \varepsilon \sup_M |h| \cdot \mathrm{Vol}(M) \leq \mathcal{A}^{\varepsilon h}(\Omega) \leq \mathbf{M}(\partial\Omega) + \varepsilon \sup_M |h| \cdot \mathrm{Vol}(M). \tag{4.2}$$

The above two inequalities imply that if we consider the $\mathcal{A}^{\varepsilon h}$-functional in place of the mass $\mathbf{M}$-functional for the (X, Z)-homotopy class Π, we have

$$\mathbf{L}^{\varepsilon h}(\Pi) > \max\big\{ \max_{x \in Z} \mathcal{A}^{\varepsilon h}(\Phi_0(x)), 0 \big\}.$$

Note that when $h \in \mathcal{S}(g)$, εh also belongs to $\mathcal{S}(g)$. Therefore, Theorem 3.1 applies to Π and produces a nontrivial, smooth, closed, almost embedded hypersurface Σ_ε, such that

- Σ_ε is the boundary for some $\Omega_\varepsilon \in \mathcal{C}(M)$ where its mean curvature with respect to the unit outer normal ν (of Ω_ε) is $\varepsilon \cdot h$, i.e.,
$$H_{\Sigma_\varepsilon} = \varepsilon \cdot h|_{\Sigma_\varepsilon};$$
- $\mathcal{A}^{\varepsilon h}(\Omega_\varepsilon) = \mathbf{L}^{\varepsilon h}(\Pi)$;
- $\operatorname{index}(\Sigma_\varepsilon) \le k$.

We denote $\mathbf{L} = \mathbf{L}(\Pi)$ and $\mathbf{L}^\varepsilon = \mathbf{L}^{\varepsilon h}(\Pi)$. In the following, we proceed the proof by parts.

Part 1. $\mathbf{L}^\varepsilon \to \mathbf{L}$ when $\varepsilon \to 0$.

Proof. From (4.2), it is easy to see
$$\mathbf{L} - \varepsilon \sup_M |h| \operatorname{Vol}(M) \le \mathbf{L}^\varepsilon \le \mathbf{L} + \varepsilon \sup_M |h| \operatorname{Vol}(M).$$

Part 2. By Theorem 2.6, there exists a subsequence $\{\varepsilon_k\} \to 0$, such that $\Sigma_k = \Sigma_{\varepsilon_k}$ converges to some smooth, closed, embedded, minimal hypersurface Σ_∞ (with integer multiplicity) in the sense of Theorem 2.6(i) and (ii). We denote $\mathcal{Y}$ as the set of points where the convergence fails to be smooth. In particular, by (4.2) and Part 1 and Theorem 2.6(v), we have
$$\mathbf{M}(\Sigma_\infty) = \mathbf{L} \text{ and } \operatorname{index}(\Sigma_\infty) \le k.$$
That is to say, Σ_∞ is a min-max minimal hypersurface associated with Π.

Without loss of generality, we assume from Part 3 to Part 8 that Σ_∞ has only one connected component. If Σ_∞ is two-sided with the multiplicity equal to one, then we are done; otherwise we may assume that either the multiplicity $m > 1$ or Σ_∞ is one-sided.

Part 3. We first assume that Σ_∞ is two-sided. We will implicitly use exponential normal coordinates of Σ_∞ with respect to one fixed unit normal of Σ_∞. By the local, smooth graphical convergence $\Sigma_k \to \Sigma_\infty$ away from $\mathcal{Y}$, we know that there exists an exhaustion by compact domains $\{U_k \subset \Sigma_\infty \backslash \mathcal{Y}\}$ and some small $\delta > 0$, so that for k large enough, $\Sigma_k \cap (U_k \times (-\delta, \delta))$ can be written as a set of m-normal graphs $\{u_k^1, \dots, u_k^m : u_k^i \in C^\infty(U_k)\}$ over U_k, and such that
$$u_k^1 \le u_k^2 \le \cdots \le u_k^m \text{ and } u_k^i \to 0, \text{ in smooth topology as } k \to \infty.$$
Since Σ_k is the boundary of some set Ω_k, by the Constancy Theorem (applied to Ω_k in $U_k \times (-\delta, \delta)$), we know that the unit outer normal ν_k of Ω_k will alternate orientations along these graphs. In particular, if ν_k restricted to the

graph of u_k^i points upward (or downward), then ν_k restricted to the graph of u_k^{i+1} will point downward (or upward).

Part 4. We first deal with an easier case: m **is an odd number**. Hence $m \geq 3$. In this case ν_k restricted to the bottom (u_k^1) and top (u_k^m) sheets point to the same side of Σ_∞, and without loss of generality we may assume that ν_k points upward therein. This means that for $x \in U_k$,

$$H|_{\mathrm{Graph}(u_k^m)}(x) = \varepsilon_k h(x, u_k^m(x)) \text{ and } H|_{\mathrm{Graph}(u_k^1)}(x) = \varepsilon_k h(x, u_k^1(x)).$$

Here and in the following, the sign convention is made so that $H|_{\mathrm{Graph}(u)}$ is defined with respect to the upward pointing normal of $\mathrm{Graph}(u)$, and hence the linearized operator is positively definite.

Note that since $\varepsilon h \in \mathcal{S}(g)$, by the Strong Maximum Principle [52, Lemma 3.12] (applied to two sheets of the same orientation), we know

$$u_k^m(x) - u_k^1(x) > 0 \text{ for all } x \in U_k.$$

Now by subtracting the above two equations and using the fact that

$$H|_{\mathrm{Graph}(u_k^m)} - H|_{\mathrm{Graph}(u_k^1)} = L_{\Sigma_\infty}(u_k^m - u_k^1) + o(u_k^m - u_k^1)$$

(see [35, p. 331] and Part 3 in the proof of Theorem 2.6), we have

$$L_{\Sigma_\infty}(u_k^m - u_k^1) + o(u_k^m - u_k^1) = \varepsilon_k \cdot \partial_\nu h(x, v_k(x)) \cdot (u_k^m(x) - u_k^1(x)), \tag{4.3}$$

where $v_k(x) = t(x)u_k^m(x) + (1 - t(x))u_k^1(x)$ for some $t(x) \in [0, 1]$.

Now it is a standard argument to produce a nontrivial positive Jacobi field on $\Sigma_\infty \backslash \mathcal{Y}$. Let us present the details for completeness. Write $\mathbf{h}_k = u_k^m - u_k^1$, and pick a fixed point $p \in U_1$. Let $\tilde{\mathbf{h}}_k = \mathbf{h}_k / \mathbf{h}_k(p)$, then $\tilde{\mathbf{h}}_k(p) = 1$. By standard Harnack and elliptic estimates, $\tilde{\mathbf{h}}_k$ will converge locally smoothly to a positive function φ on any fixed $U \subset U_k$, and by a diagonalization process, we can extend φ to $\Sigma_\infty \backslash \mathcal{Y}$, and such that

$$L_{\Sigma_\infty}\varphi = 0 \text{ outside } \mathcal{Y}.$$

Part 5. Next we use White's local foliation argument [41] to prove that φ extends smoothly across $\mathcal{Y}$, and this will contradict the bumpy assumption of g.

Fix $y \in \mathcal{Y}$. We use the exponential normal coordinates $(x, z) \in \Sigma_\infty \times [-\delta, \delta]$. Let $\varepsilon > 0$ be as given in Proposition D.1. Fix a small radius $0 < \eta < \varepsilon$, and choose k large enough such that $\|u_k^1\|_{2,\alpha}, \|u_k^m\|_{2,\alpha} \ll \varepsilon\eta$ near $\partial B_\eta^n(y)$ so that some extensions of them to the whole $B_\eta^n(y)$ have $C^{2,\alpha}$-norms bounded by $\varepsilon\eta$. Let $v_{k,t}^1, v_{k,t}^m : B_\eta^n(y) \to \mathbb{R}$, $t \in [-\eta, \eta]$, be the PMC local foliations associated with $\varepsilon_k h$,

$$H_{\mathrm{Graph}(v_{k,t}^i)}(x) = \varepsilon_k h(x, v_{k,t}^i(x)),\ i = 1, m,\ x \in B_\eta^n(y)$$

and

$$v_{k,t}^i(x) = u_k^i(x) + t,\ i = 1, m,\ x \in \partial B_\eta^n(y).$$

By the Hausdorff convergence of $\Sigma_k \to \Sigma_\infty$ and the Strong Maximum Principle [52, Lemma 3.12] (applied to $\mathrm{Graph}(u_k^1)$ and $\{\mathrm{Graph}(v_{k,t}^1)\}$, $\mathrm{Graph}(u_k^m)$ and $\{\mathrm{Graph}(v_{k,t}^m)\}$), we have

$$u_k^m(x) - u_k^1(x) \leq v_{k,0}^m(x) - v_{k,0}^1(x) \text{ when } x \in U_k \cap B_\eta^n(y).$$

By subtracting the mean curvature equations for $\mathrm{Graph}(v_{k,0}^i)$, $i = 1, m$, we get an equation similar to (4.3),

$$L_{\Sigma_\infty}(v_{k,0}^m - v_{k,0}^1) + o(v_{k,0}^m - v_{k,0}^1) = \varepsilon_k \cdot \partial_\nu h(x, v_k(x)) \cdot (v_{k,0}^m(x) - v_{k,0}^1(x)).$$

Note that the two graphs $\mathrm{Graph}(v_{k,0}^i), i = 1, m$ must be disjoint by the Strong Maximum Principle. By elliptic estimates via the weak maximum principle [13, Th. 3.7], for η small enough and k sufficiently large and a uniform $C > 0$, we have

$$\max_{B_\eta^n}(v_{k,0}^m - v_{k,0}^1) \leq C \max_{\partial B_\eta^n}(v_{k,0}^m - v_{k,0}^1).$$

This implies

$$\max_{U_k \cap B_\eta^n}(u_k^m(x) - u_k^1(x)) \leq C \max_{\partial B_\eta^n}(u_k^m(x) - u_k^1(x)).$$

Hence $\max_{U_k \cap B_\eta^n} \tilde{\mathbf{h}}_k \leq C \max_{\partial B_\eta^n} \tilde{\mathbf{h}}_k$, so φ is uniformly bounded and hence extends smoothly across y.

Part 6. We now take care of the more interesting case: m **is an even number**. Hence $m \geq 2$. In this case ν_k restricted to the bottom (u_k^1) and top (u_k^m) sheets point to different side of Σ_∞, and without loss of generality we may assume that ν_k points downward on top sheet, and upward on bottom sheet. This means that for $x \in U_k$,

$$H|_{\mathrm{Graph}(u_k^m)}(x) = -\varepsilon_k h(x, u_k^m(x)) \text{ and } H|_{\mathrm{Graph}(u_k^1)}(x) = \varepsilon_k h(x, u_k^1(x)).$$

Note that

$$u_k^m(x) - u_k^1(x) \geq 0 \text{ for all } x \in U_k,$$

but it may take zeros in a co-dimension 1 subset by [52, Prop. 3.17].

Again by subtracting the above two equations, and using the fact that $H|_{\mathrm{Graph}(u_k^m)} - H|_{\mathrm{Graph}(u_k^1)} = L_{\Sigma_\infty}(u_k^m - u_k^1) + o(u_k^m - u_k^1)$, we have

$$L_{\Sigma_\infty}(u_k^m - u_k^1) + o(u_k^m - u_k^1) = -\varepsilon_k \cdot \big(h(x, u_k^1(x)) + h(x, u_k^m(x))\big). \tag{4.4}$$

Fix a point $p \in U_1$. We discuss the renormalization in two cases. Again write $\mathbf{h}_k = u_k^m - u_k^1$.

Case 1: $\limsup_{k\to\infty} \frac{\mathbf{h}_k(p)}{\varepsilon_k} = +\infty$. Consider renormalizations $\tilde{\mathbf{h}}_k(x) = \mathbf{h}_k(x)/\mathbf{h}_k(p)$. Then by the same reasoning as in Part 4, $\tilde{\mathbf{h}}_k$ converges locally smoothly to a nontrivial function $\varphi \geq 0$ on $\Sigma_\infty \backslash \mathcal{Y}$, and such that

$$L_{\Sigma_\infty}\varphi = 0 \text{ outside } \mathcal{Y}.$$

Case 2: $\limsup_{k\to\infty} \frac{\mathbf{h}_k(p)}{\varepsilon_k} < +\infty$. Consider renormalizations $\tilde{\mathbf{h}}_k(x) = \mathbf{h}_k(x)/\varepsilon_k$. Then again by the same reasoning, $\tilde{\mathbf{h}}_k$ converges locally smoothly to a nonnegative $\varphi \geq 0$ on $\Sigma_\infty \backslash \mathcal{Y}$, and such that

$$L_{\Sigma_\infty}\varphi = -2h|_{\Sigma_\infty} \text{ outside } \mathcal{Y}.$$

Part 7. We will follow a slightly different local foliation argument to prove removable singularity for φ. We inherit all the notation from Part 5. Without loss of generality, we may assume $\sup_M |h| = 1$. Let $v^1_{k,t}, v^m_{k,t} : B^n_\eta \to \mathbb{R}$, $t \in [-\eta, \eta]$, be the CMC local foliations associated with $-\varepsilon_k$ and ε_k respectively,

$$H_{\mathrm{Graph}(v^m_{k,t})}(x) = \varepsilon_k \text{ and } H_{\mathrm{Graph}(v^1_{k,t})}(x) = -\varepsilon_k,\ x \in B^n_\eta(y)$$

and

$$v^i_{k,t}(x) = u^i_k(x) + t,\ i = 1, m,\ x \in \partial B^n_\eta(y).$$

By the same reasoning as in Part 5, using the Strong Maximum Principle for varifolds by White [43], we get

$$\max_{U_k \cap B^n_\eta} (u^m_k(x) - u^1_k(x)) \leq \max_{B^n_\eta} (v^m_{k,0}(x) - v^1_{k,0}(x)).$$

Note that slightly differently from Part 5, we have

$$L_{\Sigma_\infty}(v^m_{k,0} - v^1_{k,0}) + o(v^m_{k,0} - v^1_{k,0}) = 2\varepsilon_k.$$

By [13, Th. 3.7], for η small enough, k large enough and for some uniform $C > 0$, we have

$$\max_{U_k \cap B^n_\eta} (u^m_k(x) - u^1_k(x)) \leq C\big(\max_{\partial B^n_\eta} (u^m_k(x) - u^1_k(x)) + \varepsilon_k\big).$$

Then for both Cases 1 and 2, this implies that φ is uniformly bounded and hence extends smoothly across $\mathcal{Y}$.

Note that if we flip the orientations of the top and bottom sheets, then in Case 2 the limit of renormalizations of heights will converge to a solution of $L_{\Sigma_\infty}\varphi = 2h|_{\Sigma_\infty}$, where $\varphi \geq 0$. Note that in the previous case, we can just flip the sign of φ and obtain

$$L_{\Sigma_\infty}\varphi = 2h|_{\Sigma_\infty} \text{ where } \varphi \leq 0.$$

Part 8. Now we briefly record the case when Σ_∞ is only one-sided. Then the convergence of Σ_k must have multiplicity at least 2; otherwise the convergence will be smooth by the Allard regularity theorem [1], and hence all Σ_k will be one-sided for k sufficiently large, which is a contradiction. Denote $\pi : \tilde{\Sigma}_\infty \to \Sigma_\infty$ as the two-sided double cover of Σ_∞, and denote $\tau : \tilde{\Sigma}_\infty \to \tilde{\Sigma}_\infty$ as the deck transformation map. By the same argument for the two-sided case applied to the double cover $\tilde{\Sigma}_\infty$, we can either construct a non-trivial Jacobi field φ on $\tilde{\Sigma}_\infty$ with $\varphi \circ \tau = \varphi$ and

$$L_{\tilde{\Sigma}_\infty}\varphi = 0;$$

or a smooth function φ on $\tilde{\Sigma}_\infty$ with $\varphi \circ \tau = \varphi$, such that φ does not change sign, and

$$L_{\tilde{\Sigma}_\infty}\varphi = 2h|_{\Sigma_\infty} \circ \pi.$$

By [44], the first case cannot happen in a bumpy metric.

Summarizing the discussion, we proved that if g is bumpy, then each connected two-sided component Σ_o of Σ_∞ with multiplicity bigger than one must carry a smooth solution φ to the equation

$$L_{\Sigma_o}\varphi = 2h|_{\Sigma_o}; \tag{4.5}$$

and the double cover $\tilde{\Sigma}_u$ of each one-sided component Σ_u of Σ_∞ must carry a smooth solution φ

$$L_{\tilde{\Sigma}_u}\varphi = 2h|_{\Sigma_u} \circ \pi \text{ and } \varphi \circ \tau = \varphi. \tag{4.6}$$

Moreover, in both cases φ does not change sign.

Part 9. We will show that for a nicely chosen $h \in \mathcal{S}(g)$, the (unique) solutions to (4.5) and (4.6) must change sign. Thus there is no one-sided component, and the multiplicity for two-sided component must be one.

LEMMA 4.2 (Key Lemma). *Assume that g is bumpy. Given $\mathbf{L} > 0$ and $k \in \mathbb{N}$, there exists $h \in \mathcal{S}(g)$, such that if Σ is a smooth, connected, closed, embedded minimal hypersurface with*

$$\text{Area}(\Sigma) \leq \mathbf{L} \text{ and } \text{index}(\Sigma) \leq k,$$

then the solution of (4.5) (*when Σ is two-sided*) *or* (4.6) (*when Σ is one-sided*) *must change sign.*

Proof. As g is bumpy, by the compactness analysis of Sharp [35], there are only finitely many such Σ with $\text{Area}(\Sigma) \leq \mathbf{L}$ and $\text{index}(\Sigma) \leq k$, and we can denote them as $\{\Sigma_1, \ldots, \Sigma_L\}$. If Σ_i is one-sided, we use $\pi_i : \tilde{\Sigma}_i \to \Sigma_i$ to denote the two-sided double cover, and $\tau_i : \tilde{\Sigma}_i \to \tilde{\Sigma}_i$ to denote the deck transformation map.

On each Σ_i, we can choose two disjoint open subsets U_i^+ and $U_i^- \subset \Sigma_i$, so that the collection of subsets $\{U_i^\pm\}_{i=1,\ldots,L}$ are pairwise disjoint. Moreover, by possibly changing $U_i^\pm$, we can make sure that the pre-images $\pi_i^{-1}(U_i^+)$, $\pi_i^{-1}(U_i^-)$ are diffeomorphic to two disjoint copies of U_i^+, U_i^- respectively. In that case, we will denote the two copies as $\tilde{U}_{i,1}^+, \tilde{U}_{i,2}^+$, and $\tilde{U}_{i,1}^-, \tilde{U}_{i,2}^-$. That is

$$\pi_i^{-1}(U_i^+) = \tilde{U}_{i,1}^+ \cup \tilde{U}_{i,2}^+ \text{ and } \pi_i^{-1}(U_i^-) = \tilde{U}_{i,1}^- \cup \tilde{U}_{i,2}^-.$$

For each $i \in \{1, \ldots, L\}$ such that Σ_i is two-sided, we can choose an arbitrary pair of nontrivial smooth functions $f_i^+ \in C_c^\infty(U_i^+)$, $f^- \in C_c^\infty(U_i^-)$ such

that

$$f_i^+ \geq 0 \text{ and } f_i^+(p_i^+) > 0 \text{ at some } p_i^+ \in U_i^+$$

and

$$f_i^- \leq 0 \text{ and } f_i^-(p_i^-) < 0 \text{ at some } p_i^- \in U_i^1.$$

Let $h_i^+ \in C_c^\infty(U_i^+)$ and $h_i^- \in C_c^\infty(U_i^-)$ be defined by

$$h_i^+ = L_{\Sigma_i} f_i^+, \quad h_i^- = L_{\Sigma_i} f_i^-.$$

If Σ_i is one-sided, we choose $\tilde{f}_{i,1}^\pm \in C_c^\infty(\tilde{U}_{i,1}^\pm), \tilde{f}_{i,2}^\pm \in C_c^\infty(\tilde{U}_{i,2}^\pm)$ in the same way, and we can make sure they are the same under deck transformation: $\tilde{f}_{i,1}^\pm \circ \tau = \tilde{f}_{i,2}^\pm$. In particular,

$$\tilde{f}_{i,1}^+ \geq 0 \text{ and } \tilde{f}_{i,1}^+ > 0 \text{ somewhere in } \tilde{U}_{i,1}^+$$

and

$$\tilde{f}_{i,1}^- \leq 0 \text{ and } \tilde{f}_{i,1}^- < 0 \text{ somewhere in } \tilde{U}_{i,1}^-.$$

Then we define $h_{i,1}^\pm, h_{i,2}^\pm$ in the same manner, so obviously $h_{i,1}^\pm \circ \tau = h_{i,2}^\pm$, and they pass to two functions

$$h_i^+ \in C_c^\infty(U_i^+) \text{ and } h_i^- \in C_c^\infty(U_i^-).$$

We can extend each $h_i^\pm$ to a function defined on Σ_i by letting it be zero outside $U_i^\pm$. Using the fact that the set of smooth functions $\mathcal{S}(g)$ is open and dense in $C^\infty(M)$, we can choose a $h \in \mathcal{S}(g)$ so that

h is as close to $h_i^\pm$ as we want in any $C^{k,\alpha}$-norm when restricted to Σ_i.

We may need to flip the sign of h to make $\int_M h \geq 0$, but the following argument proceeds the same way. Since all $\{\Sigma_i : i = 1, \ldots, L\}$ are non-degenerate (the Jacobi operator is an isomorphism), we know that if

$$L_{\Sigma_i}\varphi = 2h|_{\Sigma_i} \text{ when } \Sigma_i \text{ is two-sided, or}$$

$$L_{\tilde{\Sigma}_i}\varphi = 2h|_{\Sigma_i} \circ \pi_i \text{ when } \Sigma \text{ is one-sided,}$$

then

φ is as close to $f_i^\pm$ or $\tilde{f}_{i,j}^\pm$ $(j = 1, 2)$
as we want in $C^{k+2,\alpha}$-norm when restricted to Σ_i or $\tilde{\Sigma}_i$.

Then φ must change sign, and this is what we want to prove. □

Note that by Part 2, all connected components of a min-max minimal hypersurface must satisfy the area and index bound in Lemma 4.2. So we finish the proof of the theorem. □

Remark 4.3. Indeed, we can obtain more information. Since Σ_∞ has multiplicity one, the Allard regularity theorem [1] implies that the convergence $\Sigma_k \to \Sigma_\infty$ is smooth everywhere, and hence Σ_k is properly embedded for k large.

Remark 4.4. Without assuming that g is bumpy, our proof says that if the multiplicity of a two-sided component is greater than two, or if the multiplicity for a one-sided component is greater than one, then there exists a nontrivial, nonnegative Jacobi field. Let us point out the necessary details for two-sided case; the one-sided case follows the same way. Indeed, we only need to focus on the case when the multiplicity m is even and $m \geq 4$; and moreover, we can focus on Case 2 in Part 6. Using the notation from Parts 6 and 7, we consider the height difference between the two pairs (u_k^1, u_k^{m-1}) and (u_k^2, u_k^m):

$$\mathbf{h}_k^a = u_k^{m-1} - u_k^1, \quad \mathbf{h}_k^b = u_k^m - u_k^2.$$

Then both $\mathbf{h}_k^a$ and $\mathbf{h}_k^b$ are positive and satisfy equations of type (4.3) since the graphs of the two pairs have outer normals pointing to the same side. Consider the renormalizations $\tilde{\mathbf{h}}_k^a = \mathbf{h}_k^a/\varepsilon_k$ and $\tilde{\mathbf{h}}_k^b = \mathbf{h}_k^b/\varepsilon_k$. Then

$$\tilde{\mathbf{h}}_k^a,\ \tilde{\mathbf{h}}_k^b \leq \tilde{\mathbf{h}}_k \text{ and } \tilde{\mathbf{h}}_k^a + \tilde{\mathbf{h}}_k^b \geq \tilde{\mathbf{h}}_k.$$

Note that the limit of $\tilde{\mathbf{h}}_k$ cannot be identically zero, as then $h|_{\Sigma_\infty} \equiv 0$, violating the assumption $h \in \mathcal{S}(g)$. Then the above two inequalities and standard elliptic estimates imply that at least one limit of the two sequences $\{\tilde{\mathbf{h}}_k^a\}_{k\in\mathbb{N}}$ and $\{\tilde{\mathbf{h}}_k^b\}_{k\in\mathbb{N}}$ must be a smooth, nontrivial, nonnegative Jacobi field.

Part of the proof of the theorem can be summarized as the following multiplicity one convergence result, which we believe has its independent interests.

THEOREM 4.5 (Multiplicity one convergence). *Let (M^{n+1}, g) be a closed manifold of dimension $3 \leq (n+1) \leq 7$ with a bumpy metric g. Given $\mathbf{L} > 0$, $I \in \mathbb{N}$, then there exists a smooth function $h : M \to \mathbb{R}$, $h \in \mathcal{S}(g)$, such that the following holds*:

Let $\{\Sigma_k\}_{k\in\mathbb{N}}$ be a sequence of smooth, closed, almost embedded hypersurfaces, and let $\{\varepsilon_k\}_{k\in\mathbb{N}} \to 0$, such that

- *Σ_k is the boundary of some open set Ω_k, and the mean curvature of Σ_k with respect to the outer normal of Ω_k is prescribed by $\varepsilon_k h$;*
- *$\mathrm{Area}(\Sigma_k) \leq \mathbf{L}$ and $\mathrm{index}(\Sigma_k) \leq I$.*

Then up to a subsequence, $\{\Sigma_k\}_{k\in\mathbb{N}}$ converges smoothly to a smooth, closed, embedded, two-sided, minimal hypersurface Σ_∞ with multiplicity one.

5. Application to volume spectrum

In this part, we will show how to apply the result in Section 4 to study volume spectrum introduced by Gromov, Guth, and Marques-Neves. In particular, we will prove that in a bumpy metric, the volume spectrum can be realized by the area of min-max minimal hypersurfaces produced by Theorem 4.1. To do this, we will carefully pick a sequence of sweepouts of mod 2 cycles and open the parameter space so as to produce sweepouts of boundaries of Caccioppoli sets, whose relative homotopy classes satisfy (4.1). As the space of Caccioppoli sets forms a double cover of the space of mod 2 cycles, the parameter-space-opening process is achieved by lifting to the double cover.

We first recall the definition of volume spectrum following [28, §4]. Let (M^{n+1}, g) be a closed Riemannian manifold. Let X be a cubical subcomplex of $I^m = [0,1]^m$ for some $m \in \mathbb{N}$. Given $k \in \mathbb{N}$, a continuous map $\Phi : X \to \mathcal{Z}_n(M, \mathbb{Z}_2)$ is a *k-sweepout* if

$$\Phi^*(\bar{\lambda}^k) \neq 0 \in H^k(X, \mathbb{Z}_2),$$

where $\bar{\lambda} \in H^1(\mathcal{Z}_n(M, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2$ is the generator. Note that Φ is said to be *admissible* if it has no concentration of mass. Denote by $\mathcal{P}_k$ the set of all admissible k-sweepouts. Then

Definition 5.1. The *k-width* of (M, g) is

$$\omega_k(M, g) = \inf_{\Phi \in \mathcal{P}_k} \sup\{\mathbf{M}(\Phi(x)) : x \in \mathrm{dmn}(\Phi)\},$$

where $\mathrm{dmn}(\Phi)$ is the domain of Φ.

It was proved in [28, Ths. 5.1 and 8.1] that there exists some constant $C = C(M, g)$, such that

$$\frac{1}{C} k^{\frac{1}{n+1}} \leq \omega_k(M, g) \leq C k^{\frac{1}{n+1}}.$$

Assume from now on that the dimension satisfies $3 \leq (n+1) \leq 7$. It was later observed by Marques-Neves in [27] that one can restrict to a subclass of $\mathcal{P}_k$ in the definition of $\omega_k(M, g)$. In particular, let $\tilde{\mathcal{P}}_k$ denote those elements $\Phi \in \mathcal{P}_k$ that are continuous under the $\mathbf{F}$-topology and whose domain $X = \mathrm{dmn}(\Phi)$ has dimension k (and is identical to its k-skeleton). Then

$$\omega_k(M, g) = \inf_{\Phi \in \tilde{\mathcal{P}}_k} \sup\{\mathbf{M}(\Phi(x)) : x \in \mathrm{dmn}(\Phi)\}.$$

They also proved in [27] that for each $k \in \mathbb{N}$ there exists a disjoint collection of smooth, connected, closed, embedded minimal hypersurfaces $\{\Sigma_i^k : i = 1, \ldots, l_k\}$

with integer multiplicities $\{m_i^k : i = 1, \dots, l_k\} \subset \mathbb{N}$, such that

$$\omega_k(M, g) = \sum_{i=1}^{l_k} m_i^k \cdot \mathrm{Area}(\Sigma_i^k) \quad \text{and} \quad \sum_{i=1}^{l_k} \mathrm{index}(\Sigma_i^k) \leq k.$$

Now we are going to state and prove our main theorem.

THEOREM 5.2 (Theorem A). *If g is a bumpy metric and $3 \leq (n+1) \leq 7$, then for each $k \in \mathbb{N}$, there exists a disjoint collection of smooth, connected, closed, embedded, two-sided minimal hypersurfaces $\{\Sigma_i^k : i = 1, \dots, l_k\}$, such that*

$$\omega_k(M, g) = \sum_{i=1}^{l_k} \mathrm{Area}(\Sigma_i^k) \quad \textit{and} \quad \sum_{i=1}^{l_k} \mathrm{index}(\Sigma_i^k) \leq k.$$

That is to say, the min-max minimal hypersurfaces are all two-sided and have multiplicity one.

Proof. If g is bumpy, then there are only finitely many closed, embedded, minimal hypersurfaces with Area $\leq \Lambda$ and index $\leq I$ for given $\Lambda > 0, I \in \mathbb{N}$ by Sharp's result [35]. Using the Morse index upper bound estimates for min-max theory by Marques-Neves [27], we have

LEMMA 5.3. *Suppose g is bumpy. Then for each $k \in \mathbb{N}$, there exist a k-dimensional cubical complex X_k and a map $\Phi_{0,k} : X_k \to \mathcal{Z}_n(M, \mathbf{F}, \mathbb{Z}_2)$ continuous in the $\mathbf{F}$-topology with $\Phi_{0,k} \in \tilde{\mathcal{P}}_k$, such that*

$$\mathbf{L}(\Pi_k) = \omega_k(M, g),$$

where $\Pi_k = \Pi(\Phi_{0,k})$ is the class of all maps $\Phi : X_k \to \mathcal{Z}_n(M, \mathbf{F}, \mathbb{Z}_2)$ continuous in the $\mathbf{F}$-topology that are homotopic to $\Phi_{0,k}$ in flat topology.

Proof. From definition we know that

$$\omega_k(M, g) = \inf\{\mathbf{L}(\Pi(\Phi)),\ \Phi \in \tilde{\mathcal{P}}_k\}.$$

By area and index upper bounds and the finiteness result, the infimum is achieved. □

Now we fix $k \in \mathbb{N}$ and omit the sub-index k in the following. Take $\Pi = [\Phi_0 : X \to \mathcal{Z}_n(M, \mathbf{F}, \mathbb{Z}_2)]$ with $\mathbf{L}(\Pi) = \omega_k$. The following result is an outcome of the proof of [27, Th. 6.1].

LEMMA 5.4. *Suppose g is bumpy. Then there exists a pull-tight (see [27, 3.7]) min-max sequence $\{\Phi_i\}_{i\in\mathbb{N}}$ of Π such that if $\Sigma \in \mathbf{C}(\{\Phi_i\}_{i\in\mathbb{N}})$ has support a smooth, closed, embedded minimal hypersurface, then*

$$\|\Sigma\|(M) = \omega_k(M, g) \textit{ and } \mathrm{index}(\textit{support of } \Sigma) \leq k.$$

We proceed the proof by the following four steps.

Step 1. *In this and the next step, we show how to find another min-max sequence, still denoted as* $\{\Phi_i\}_{i\in\mathbb{N}}$, *such that for* i *sufficiently large, either* $|\Phi_i(x)|$ *is close to a regular min-max minimal hypersurface, or the mass* $\mathbf{M}(\Phi_i(x))$ *is strictly less than* $\omega_k(M,g)$.

We recall the following observation by [28, Claim 6.2]. Let $\mathcal{S}$ be the set of all stationary integral varifolds with Area $\leq \omega_k$ whose support is a smooth closed embedded minimal hypersurface with index(support) $\leq k$. Consider the set $\mathcal{T}$ of all mod 2 flat cycles $T \in \mathcal{Z}_n(M,\mathbb{Z}_2)$ with $\mathbf{M}(T) \leq \omega_k$ and such that either $T=0$ or the support of T is a smooth closed embedded minimal hypersurface with index $\leq k$. By the bumpy assumption, both sets $\mathcal{S}$ and $\mathcal{T}$ are finite. Moreover,

LEMMA 5.5 (Claim 6.2 in [28]). *For every* $\bar{\varepsilon}>0$, *there exists* $\varepsilon>0$ *such that*

$$T \in \mathcal{Z}_n(M,\mathbb{Z}_2) \text{ with } \mathbf{F}(|T|,\mathcal{S}) \leq 2\varepsilon \Longrightarrow \mathcal{F}(T,\mathcal{T}) < \bar{\varepsilon}.$$

We also need another observation by [28, Cor. 3.6]. Denote S^1 by the unit circle.

LEMMA 5.6 (Corollary 3.6 in [28]). *If* $\bar{\varepsilon}$ *is sufficiently small, depending on* $\mathcal{T}$, *then every map* $\Phi : S^1 \to \mathcal{Z}_n(M,\mathbb{Z}_2)$ *with*

$$\Phi(S^1) \subset B^{\mathcal{F}}_{\bar{\varepsilon}}(\mathcal{T}) = \{T \in \mathcal{Z}_n(M,\mathbb{Z}_2) : \mathcal{F}(T,\mathcal{T}) < \bar{\varepsilon}\}$$

is homotopically trivial.

Let $\{\Phi_i\}_{i\in\mathbb{N}}$ be chosen as in Lemma 5.4. We choose $\bar{\varepsilon}$ as in Lemma 5.6 and ε by Lemma 5.5. Take a sequence $\{k_i\}_{i\in\mathbb{N}} \to \infty$, such that

$$\sup\{\mathbf{F}(\Phi_i(x),\Phi_i(y)) : \alpha \in X(k_i), x,y \in \alpha\} \leq \varepsilon/2.$$

Consider Z_i to be the cubical subcomplex of $X(k_i)$ consisting of all cells $\alpha \in X(k_i)$ so that

$$\mathbf{F}(|\Phi_i(x)|,\mathcal{S}) \geq \varepsilon \text{ for every vertex } x \text{ in } \alpha.$$

Hence $\mathbf{F}(|\Phi_i(x)|,\mathcal{S}) \geq \varepsilon/2$ for all $x \in Z_i$.

Consider this sub-coordinating sequence $\{\Phi_i|_{Z_i}\}_{i\in\mathbb{N}}$. Then $\mathbf{L}(\{\Phi_i|_{Z_i}\})$ and $\mathbf{C}(\{\Phi_i|_{Z_i}\})$ are defined in the same way as in Section 1.1 with $\mathcal{A}^h$ replaced by $\mathbf{M}$.

LEMMA 5.7. *We have the following dichotomy*:

- *no element* $V \in \mathbf{C}(\{\Phi_i|_{Z_i}\}_{i\in\mathbb{N}})$ *is* $\mathbb{Z}_2$*-almost minimizing in small annuli* (*see* [28, 2.10]),
- *or*

$$\mathbf{L}(\{\Phi_i|_{Z_i}\}_{i\in\mathbb{N}}) < \mathbf{L}(\Pi) = \omega_k. \tag{5.1}$$

Proof. Suppose that (5.1) does not hold; then $\mathbf{L}(\{\Phi_i|_{Z_i}\}_{i\in\mathbb{N}}) = \mathbf{L}(\Pi)$. As $\{\Phi_i\}_{i\in\mathbb{N}}$ is pull-tight, we know that every $V \in \mathbf{C}(\{\Phi_i|_{Z_i}\}_{i\in\mathbb{N}})$ is stationary. If V is also $\mathbb{Z}_2$-almost minimizing in small annuli, then V is regular by the regularity of Pitts [31, Th. 7.11] and Schoen-Simon [33, Th. 4]. (See also [28, Th. 2.11] for the adaption to $\mathbb{Z}_2$-coefficients.) By Lemma 5.4, $V \in \mathcal{S}$, which is a contradiction. □

Let $Y_i = \overline{X\backslash Z_i}$. It then follows that

$$\mathbf{F}(|\Phi_i(x)|, \mathcal{S}) \leq \frac{3}{2}\varepsilon \text{ for all } x \in Y_i. \tag{5.2}$$

We also write $B_i = Y_i \cap Z_i$. In fact, B_i is the topological boundary of Y_i and Z_i. For later purpose, we consider the set

$$\mathbf{B}_i = \text{ the union of all cells } \alpha \in Z_i \text{ such that } \alpha \cap B_i \neq \emptyset.$$

Here $\mathbf{B}_i$ can be thought of as the "thickening" of B_i inside Z_i.

Let $\lambda = \Phi_i^*(\bar{\lambda}) \in H^1(X, \mathbb{Z}_2)$. Consider the inclusion maps $i_1 : Y_i \to X$ and $i_2 : Z_i \to X$. It then follows from (5.2) and Lemmas 5.5 and 5.6 that

$$i_1^*(\lambda) = 0 \in H^1(Y_i, \mathbb{Z}_2).$$

Then by [28, Claim 6.3], $(\Phi_i)|_{Z_i}$ is a $(k-1)$-sweepout, i.e.,

$$i_2^*(\lambda^{k-1}) \neq 0 \in H^{k-1}(Z_i, \mathbb{Z}_2).$$

Now we let $Y_i' = Y_i \cup \mathbf{B}_i$ and $Z_i' = \overline{Z_i\backslash\mathbf{B}_i}$, and we let $i_i' : Y_i' \to X$ and $i_2' : Z_i' \to X$ be the inclusion maps. Note that (5.2) is satisfied with $Y_i, \frac{3}{2}\varepsilon$ replaced by $Y_i', 2\varepsilon$ respectively, so by similar reasoning we have

$$(i_1')^*(\lambda) = 0 \in H^1(Y_i', \mathbb{Z}_2) \text{ and } (i_2')^*(\lambda^{k-1}) \neq 0 \in H^{k-1}(Z_i', \mathbb{Z}_2).$$

Step 2. The strategy is to follow the idea in the proof of Theorem 1.7 and apply [28, Th. 2.13] *(see also Theorem 1.16) to deform* $\{\Phi_i\}_{i\in\mathbb{N}}$ *so as to decrease* $\mathbf{L}(\{(\Phi_i)|_{Z_i}\}_{i\in\mathbb{N}})$ *and make* (5.1) *be satisfied.*

If (5.1) holds true, then we are done for this step. So let us assume that

$$\mathbf{L}(\{\Phi_i|_{Z_i}\}_{i\in\mathbb{N}}) = \mathbf{L}(\Pi) = \omega_k. \tag{5.3}$$

By Lemma 5.7 and our assumption (5.3), we know that no element $V \in \mathbf{C}(\{\Phi_i|_{Z_i}\}_{i\in\mathbb{N}})$ is $\mathbb{Z}_2$-almost minimizing in small annuli.

Since $\Phi_i : X \to \mathcal{Z}_n(M, \mathbf{F}, \mathbb{Z}_2)$ has no concentration of mass as it is continuous in $\mathbf{F}$-topology, we can apply [28, Th. 3.9] (the counterpart of Theorem 1.11 for maps to $\mathcal{Z}_n(M, \mathbb{Z}_2)$) to produce a sequence of maps

$$\phi_i^j : X(k_i + k_i^j)_0 \to \mathcal{Z}_n(M, \mathbb{Z}_2),$$

with $k_i^j \in \mathbb{N}$ and $k_i^j < k_i^{j+1}$ for all $j \in \mathbb{N}$ and a sequence of positive $\{\delta_i^j\}_{j\in\mathbb{N}} \to 0$, such that

(i) the fineness $\mathbf{f}(\phi_i^j) \leq \delta_i^j$;
(ii) $\sup\{\mathcal{F}(\phi_i^j(x) - \Phi_i(x)) : x \in X(k_i + k_i^j)_0\} \leq \delta_i^j$;
(iii) for some sequence $l_i^j \to \infty$ with $l_i^j < k_i^j$,

$$\mathbf{M}(\phi_i^j(x)) \leq \sup\{\mathbf{M}(\Phi_i(y)) : x, y \in \alpha \text{ for some } \alpha \in X(k_i + l_i^j)\} + \delta_i^j.$$

As Φ_i is continuous in $\mathbf{F}$-topology, we get from property (iii) that for all $x \in X(k_i + k_i^j)_0$,

$$\mathbf{M}(\phi_i^j(x)) \leq \mathbf{M}(\Phi_i(x)) + \eta_i^j,$$

with $\eta_i^j \to 0$ as $j \to \infty$. Applying [26, Lemma 4.1] with $\mathcal{S} = \Phi_i(X)$, we get by (ii) that

(iv) $\sup\{\mathbf{F}(\phi_i^j(x), \Phi_i(x)) : x \in X(k_i + k_i^j)_0\} \to 0$ as $j \to \infty$.

We can choose $j(i) \to \infty$ as $i \to \infty$ (then $k_i^{j(i)} \to \infty$) such that $\varphi_i = \phi_i^{j(i)} : X(k_i + k_i^{j(i)})_0 \to \mathcal{Z}_n(M, \mathbb{Z}_2)$ satisfies

- $\sup\{\mathbf{F}(\varphi_i(x), \Phi_i(x)) : x \in X(k_i + k_i^{j(i)})_0\} \leq a_i$ with $a_i \to 0$ as $i \to \infty$;
- $\sup\{\mathbf{F}(\Phi_i(x), \Phi_i(y)) : x, y \in \alpha, \alpha \in X(k_i + k_i^{j(i)})\} \leq a_i$;
- the fineness $\mathbf{f}(\varphi_i) \to 0$ as $i \to \infty$;
- the Almgren extension $\Phi_i^{j(i)} : X \to \mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2)$ (see [28, 3.10] for the definition; it is continuous in the $\mathbf{M}$-topology) is homotopic to Φ_i in the flat topology (by [28, Cor. 3.12]), and $\sup\{\mathbf{F}(\Phi_i^{j(i)}(x), \Phi_i(x)) : x \in X\} \to 0$ as $i \to \infty$ (by [28, 3.10]).

If we let $S = \{\varphi_i\}_{i\in\mathbb{N}}$ be a discrete sweepout, then we have $\mathbf{L}(S) = \mathbf{L}(\{\Phi_i\}_{i\in\mathbb{N}})$ and $\mathbf{C}(S) = \mathbf{C}(\{\Phi_i\}_{i\in\mathbb{N}})$. Moreover, consider the restrictions of φ_i to $Z_i(k_i^{j(i)})_0$:

$$S_Z = \{\varphi_i : Z_i(k_i^{j(i)})_0 \to \mathcal{Z}_n(M, \mathbb{Z}_2)\}.$$

Similarly, we have

$$\mathbf{L}(S_Z) = \mathbf{L}(\{\Phi_i|_{Z_i}\}_{i\in\mathbb{N}}) = \mathbf{L}(\Pi) \text{ and } \mathbf{C}(S_Z) = \mathbf{C}(\{\Phi_i|_{Z_i}\}_{i\in\mathbb{N}}).$$

As no $V \in \mathbf{C}(S_Z)$ is $\mathbb{Z}_2$-almost minimizing in small annuli, by [28, Th. 2.13] (which is a reformulation of Almgren-Pitts combinatorial argument [31, Th. 4.10]), we can find a sequence $\tilde{S}_Z = \{\tilde{\varphi}_i\}$ of maps

$$\tilde{\varphi}_i : Z_i(k_i^{j(i)} + l_i)_0 \to \mathcal{Z}_n(M, \mathbb{Z}_2)$$

and a sequence of homotopies

$$\psi_i : I(l_i)_0 \times Z_i(k_i^{j(i)} + l_i)_0 \to \mathcal{Z}_n(M, \mathbb{Z}_2),$$

such that

- $\psi_i([0], x) = \varphi_i \circ \mathbf{n}(k_i^{j(i)} + l_i, k_i^{j(i)})(x)$ and $\psi_i([1], x) = \tilde{\varphi}_i(x)$;
- the fineness of ψ_i tends to zero as $i \to \infty$;

- $\limsup_{i\to\infty} \sup\{\mathbf{M}(\psi_i(t,x)) : (t,x) \in I(l_i)_0 \times Z_i(k_i^{j(i)} + l_i)_0\} = \mathbf{L}(S_Z)$ — this property was not explicitly listed in [28, Th. 2.13], but it follows from the construction in [31, Th. 4.10];
- $\mathbf{L}(\tilde{S}_Z) < \mathbf{L}(S_Z)$.

Now we construct a new sequence $S^* = \{\varphi_i^*\}_{i\in\mathbb{N}}$ with

$$\varphi_i^* : X(k_i + k_i^{j(i)} + l_i)_0 \to \mathcal{Z}_n(M, \mathbb{Z}_2),$$

defined as

- $\varphi_i^*(x) = \varphi_i \circ \mathbf{n}(k_i^{j(i)} + l_i, k_i^{j(i)})(x)$, when $x \in Y_i(k_i^{j(i)} + l_i)_0$;
- $\varphi_i^*(x) = \psi_i(t(x), x)$, where $x \in \mathbf{B}_i(l_i)_0$ and $t(x) = \min\{3^{-l_i} \cdot \mathbf{d}(x, \mathbf{B}_i \cap Y_i), 1\} \in I(l_i)_0$ (here $\mathbf{d}$ is the distance function restricted to $\mathbf{B}_i(l_i)_0$ — see Appendix A;
- $\varphi_i^*(x) = \tilde{\varphi}_i(x)$, when $x \in Z_i'(k_i^{j(i)} + l_i)_0$ — note that $t(x) \geq 1$ when $x \in Z_i' \cap \mathbf{B}_i$.

By the construction, we see that

- φ_i^* is homotopic to φ_i with fineness tending to zero as $i \to \infty$;
- $\mathbf{L}(S^*) = \mathbf{L}(\Pi)$;
- $\limsup_{i\to\infty} \sup\{\mathbf{M}(\varphi_i^*(x)) : x \in Z_i'(k_i^{j(i)} + l_i)_0\} \leq \mathbf{L}(\tilde{S}_Z) < \mathbf{L}(\Pi)$.

Consider the Almgren's extension of φ_i^*:

$$\Phi_i^* : X \to \mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2).$$

Then

(a) Φ_i^* is homotopic to $\Phi_i^{j(i)}$ and hence to Φ_i in the flat topology by [28, 3.11] and by [28, 3.10];
(b) $\sup\{\mathbf{F}(\Phi_i^*(x), \Phi_i(x)) : x \in Y_i\} \to 0$;
(c) $\mathbf{L}(\{\Phi_i^*\}) = \mathbf{L}(S^*) = \mathbf{L}(\Pi)$;
(d) $\limsup_{i\to\infty} \sup\{\mathbf{M}(\Phi_i^*(x)) : x \in Z_i'\} \leq \mathbf{L}(\tilde{S}_Z) < \mathbf{L}(\Pi)$.

By summarizing what we have done (and abusing the notation $Y_i = Y_i'$ and $Z_i = Z_i'$), we produced another min-max sequence $\{\Phi_i^*\}_{i\in\mathbb{N}} \subset \Pi$ such that

(1) X can be decomposed to Y_i and Z_i with $Z_i = \overline{X \backslash Y_i}$, and for i large enough,

$$i_1^*(\lambda) = 0 \in H^1(Y_i, \mathbb{Z}_2) \text{ and } i_2^*(\lambda^{k-1}) \neq 0 \in H^{k-1}(Z_i, \mathbb{Z}_2);$$

(2) $\mathbf{L}(\{\Phi_i^*\}) = \mathbf{L}(\{\Phi_i\}) = \mathbf{L}(\Pi)$;
(3) $\limsup_{i\to\infty} \sup\{\mathbf{M}(\Phi_i^*(x)) : x \in Z_i\} < \mathbf{L}(\Pi)$.

Note that both Y_i and Z_i are nonempty for i large enough by (1)(3).

Step 3. *Now we want to produce sweepouts in* $\mathcal{C}(M)$ *by lifting to the double cover* $\partial : \mathcal{C}(M) \to \mathcal{Z}_n(M, \mathbb{Z}_2)$ *so as to produce sweepouts satisfying the assumption of Theorem* 4.1.

We abuse notation and still write $\{\Phi_i^*\}$ as $\{\Phi_i\}$. Since $(\Phi_i)^*(\bar{\lambda}) \neq 0 \in H^1(X, \mathbb{Z}_2) = \mathbb{Z}_2$, there exist a double cover $\pi : \tilde{X} \to X$ with deck transformation map $\tau : \tilde{X} \to \tilde{X}$ and the lifting maps

$$\tilde{\Phi}_i : \tilde{X} \to (\mathcal{C}(M), \mathbf{F}),$$

satisfying $\partial\tilde{\Phi}_i = \Phi_i \circ \pi$. Indeed, the cohomological condition implies that the induced maps $(\Phi_i)_* : \pi_1(X) \to \pi_1(\mathcal{Z}_n(M, \mathbb{Z}_2)) = \mathbb{Z}_2$ are surjective; see [28, Def. 4.1 (i)]. So the kernel of $(\Phi_i)_*$ is a subgroup of $\pi_1(X)$ with index 2. Then the existence of such liftings follows from [20, Props. 1.36 and 1.33].

Note that $i_1^*\lambda = 0 \in H^1(Y_i, \mathbb{Z}_2)$, so the pre-image of Y_i is disconnected and is a disjoint union of two copies of Y_i:

$$\tilde{Y}_i = \pi^{-1}(Y_i) = Y_i^+ \cup Y_i^-,$$

where both Y_i^+ and Y_i^- are homeomorphic to Y_i. In fact, the cohomological condition implies that every closed curve $\gamma : S^1 \to Y_i$ lies in the kernel of $(\Phi_i)_*$, so the lift $\tilde{\gamma}$ of γ to $\tilde{X}$ is still a closed curve. This means that $\tilde{Y}_i$ is disconnected as we want.

Denote by $\tilde{Z}_i$, $\tilde{B}_i$ and $\tilde{\mathbf{B}}_i$ the pre-images of $Z_i, B_i, \mathbf{B}_i$ under π respectively. Then $\tilde{\mathbf{B}}_i = \mathbf{B}_i^+ \cup \mathbf{B}_i^-$ is also a disjoint union of two copies of $\mathbf{B}_i$.

LEMMA 5.8. *For* i *large enough, if* $\tilde{\Pi}_i$ *is the* $(\tilde{X}, \tilde{Z}_i)$*-homotopy class associated with* $\tilde{\Phi}_i$*, then we have*

$$\mathbf{L}(\tilde{\Pi}_i) \geq \mathbf{L}(\Pi) > \max_{x \in \tilde{Z}_i} \mathbf{M}(\partial\tilde{\Phi}_i(x)).$$

Proof. Fix i large, so that

$$\sup_{x \in Z_i} \mathbf{M}(\Phi_i(x)) < \mathbf{L}(\Pi);$$

we will omit the sub-index in the following proof.

If the conclusion is not true, then we can find a sequence of maps $\{\tilde{\Psi}_j : \tilde{X} \to (\mathcal{C}(M), \mathbf{F})\}_{j \in \mathbb{N}} \subset \tilde{\Pi}$, such that

$$\limsup_{j \to \infty} \sup\{\mathbf{M}(\partial\tilde{\Psi}_j(x)) : x \in X\} < \mathbf{L}(\Pi),$$

and homotopy maps $\{H_j : [0,1] \times \tilde{X} \to \mathcal{C}(M)\}$ that are continuous in the flat topology, $H_j(0, \cdot) = \tilde{\Psi}_j$, $H_j(1, \cdot) = \tilde{\Phi}$, and

$$\limsup_{j \to \infty} \sup\{\mathbf{F}(H_j(t, x), \tilde{\Phi}(x)) : t \in [0,1], x \in \tilde{Z}\} = 0.$$

We construct a new sequence of maps $\{\tilde{\Psi}_j^*\}_{j \in \mathbb{N}}$ defined as

- $\tilde{\Psi}_j^*(x) = \tilde{\Psi}_j(x)$, if $x \in Y^+$, and $\tilde{\Psi}_j^*(x) = \tilde{\Psi}_j \circ \tau(x)$, if $x \in Y^-$;
- $\tilde{\Psi}_j^*(x) = H_j(t(x), x)$, where $t(x) = \min\{\mathrm{dist}(x, \mathbf{B}^+ \cap Y^+), 1\}$ if $x \in \mathbf{B}^+$, and $\tilde{\Psi}_j^*(x) = H_j \circ \tau(x)$, if $x \in \mathbf{B}^-$ — here dist is the distance function by viewing $\mathbf{B}$ as a cube complex in some $I(m, l)$;
- $\tilde{\Psi}_j^*(x) = \tilde{\Phi}(x)$, if $x \in \tilde{Z}'$ — note that $t(x) \geq 1$ for $x \in \tilde{Z}' \cap (\mathbf{B}^+ \cap \mathbf{B}^-)$.

Note that though $\tilde{\Psi}_j^*$ themselves may not be continuous as maps to $\mathcal{C}(M)$, Ψ_j^* can be passed to quotient as continuous maps from X to $\mathcal{Z}_n(M, \mathbb{Z}_2)$. This is essentially where we used the structures of $\tilde{Y}$ and $\tilde{\mathbf{B}}$, that is, (Y^+, Y^-) and $(\mathbf{B}^+, \mathbf{B}^-)$ are pairwise disjoint.

Denote the quotient maps of $\{\tilde{\Psi}_j^*\}_{j\in\mathbb{N}}$ by

$$\{\Psi_j^* = \partial \circ \tilde{\Psi}_j^* : X \to \mathcal{Z}_n(M, \mathbb{Z}_2)\}_{j\in\mathbb{N}}.$$

We have

- Ψ_j^* is homotopic to Φ in the flat topology;
- $\limsup_{j\to\infty} \sup\{\mathbf{M}(\Psi_j^*(x)) : x \in X\} < \mathbf{L}(\Pi) = \omega_k(M, g)$ (by the three above inequalities).

This will lead to a contradiction with the definition of k-width once we prove that Ψ_j^* is an admissible k-sweepout when j is sufficiently large. Indeed, the only thing left is to show that Ψ_j^* has no concentration of mass. This follows from the third inequality above. So we finish the proof. □

Step 4. *We are ready to finish the proof of Theorem* 5.2.

For i large enough as in Lemma 5.8, Theorem 4.1 applied to $\tilde{\Pi}_i$ gives a disjoint collection of smooth, connected, closed, embedded, two-sided, minimal hypersurfaces $\Sigma_i = \cup_{j=1}^{N_i} \Sigma_{i,j}$, such that

$$\mathbf{L}(\tilde{\Pi}_i) = \sum_{j=1}^{N_i} \mathrm{Area}(\Sigma_{i,j}) \text{ and } \mathrm{index}(\Sigma_i) \leq k.$$

Note also that $\mathbf{L}(\tilde{\Pi}_i) \leq \mathbf{L}(\Phi_i) \to \mathbf{L}(\Pi) = \omega_k$. Counting the fact that there are only finitely many smooth, closed, embedded minimal hypersurfaces with Area $\leq \omega_k + 1$ and index $\leq k$, for i sufficiently large we have

$$\mathbf{L}(\tilde{\Pi}_i) = \mathbf{L}(\tilde{\Pi}_{i+1}) = \cdots = \omega_k.$$

Hence we finish the proof of Theorem 5.2. □

Remark 5.9. By the course of the above proof, in a bumpy metric, the min-max minimal hypersurfaces associated with any homotopically nontrivial sweepouts of mod-2 cycles are always two-sided and have multiplicity one. In fact, if $\Phi : X \to \mathcal{Z}_n(M, \mathbb{Z}_2)$ is homotopically nontrivial, then the induced map $\Phi_* : \pi_1(X) \to \pi_1(\mathcal{Z}_n(M, \mathbb{Z}_2)) = \mathbb{Z}_2$ must be surjective. Otherwise by [20, Prop. 1.33], Φ can be lifted to a map $\tilde{\Phi} : X \to \mathcal{C}(M)$ that is then homotopically

trivial as $\mathcal{C}(M)$ is contractible. With this topological information, the above proof works the same way and implies the two-sidedness and multiplicity one for min-max minimal hypersurfaces associated with $\Pi(\Phi)$.

Remark 5.10. After this article was first posted on arXiv, we realized that the main consequences obtained in Steps 1 and 2 in the above proof are also contained in [29, Th. 4.9].

Appendix A. Cubical complex structures

Here we recall several cubical complex structures in [28, 2.1].

For each $k \in \mathbb{N}$, $I(1,k)$ denotes the cubical complex on the unit interval $I = [0,1]$ whose 1-cells and 0-cells (which are also called vertices) are, respectively,

$$[0,3^{-k}],[3^{-k},2\cdot 3^{-k}],\dots,[1-3^{-k},1] \text{ and } [0],[3^{-k}],\dots,[1-3^{-k}],[1].$$

We then denote by $I(m,k)$ the cell complex on I^m:

$$I(m,k) = I(1,k)\otimes\cdots\otimes I(1,k) \quad m \text{ times}.$$

Then $\alpha = \alpha_1\otimes\cdots\otimes\alpha_m$ is a q-cell of $I(m,k)$ if and only if α_i is a cell of $I(1,k)$ for each i, and $\sum_{i=1}^m \dim(\alpha_i) = q$. We often identify a q-cell α with its support $\alpha_1\times\cdots\times\alpha_m \subset I^m$. The distance function $\mathbf{d}$ on $I(m,k)_0$ is defined as $\mathbf{d}(x,y) = 3^k\sum_{i=1}^k |x_i - y_i|$, $x,y \in I(m,k)_0$, [31, 4.1(1)(e)].

Let $X \subset I^m$ be a cubical subcomplex. The cubical complex $X(k)$ is the union of all cells of $I(m,k)$ whose support is contained in some cell of X. We use the notation $X(k)_q$ to denote the set of all q-cells in $X(k)$, and particularly $X(k)_0$ to denote the set of vertices in $X(k)$. Two vertices $x,y \in X(k)_0$ are adjacent if they belong to a common cell in $X(k)_1$.

Let $Y \subset I(m,k)$ be a cubical subcomplex. Similarly, the cubical complex $Y(l)$ is the union of all cells of $I(m,k+l)$ whose support is contained in some cell of Y. Here $Y(k)_q$ is defined in the same way.

Given $k,l \in \mathbb{N}$, we define $\mathbf{n}(k,l) : X(k)_0 \to X(l)_0$ so that $\mathbf{n}(i,j)(x)$ is the element in $X(l)_0$ that is closest to x (see [31, p. 141]).

Appendix B. Removing singularity for weakly stable PMC

We record the following standard removable singularity result.

Theorem B.1. *Let (M^{n+1},g) be a closed Riemannian manifold of dimension $3 \le (n+1) \le 7$. Given $h \in \mathcal{S}(g)$ and $\Sigma \subset B_\varepsilon(p)\backslash\{p\}$ an almost embedded hypersurface with $\partial\Sigma \cap B_\varepsilon(p)\backslash\{p\} = \emptyset$, assume that Σ has prescribing mean curvature h, and Σ is weakly stable. If Σ represents a varifold of bounded first variation in $B_\varepsilon(p)$, then Σ extends smoothly across p as an almost embedded hypersurface in $B_\varepsilon(p)$.*

Proof. Given any sequence of positive $\lambda_i \to 0$, consider the blowups $\{\boldsymbol{\mu}_{p,\lambda_i}(\Sigma) \subset \boldsymbol{\mu}_{p,\lambda_i}(M)\}$, where $\boldsymbol{\mu}_{p,\lambda_i}(x) = \frac{x-p}{\lambda_i}$. Since Σ has bounded first variation, $\boldsymbol{\mu}_{p,\lambda_i}(\Sigma)$ converges (up to a subsequence) to a stationary integral rectifiable cone C in $\mathbb{R}^{n+1} = T_pM$. By weakly stability and Theorem 2.5, the convergence is locally smooth and graphical away from the origin, so C is an integer multiple of some embedded minimal hypercone; moreover, C is weakly stable, and hence is stable as an embedded minimal hypersurface away from 0. Therefore, C is an integer multiple of some n-plane P by Simons's classification [38], i.e., $C = m \cdot P$, where $m = \Theta^n(\Sigma, p)$. Note that a priori C may not be unique.

By the locally smooth and graphical convergence, there exists $\sigma_0 > 0$ small enough, such that for any $0 < \sigma \leq \sigma_0$, Σ has an m-sheeted, ordered (in the sense of [52, Def. 3.2]), graphical decomposition in the annulus $A_{\sigma/2,\sigma}(p) = B_\sigma(p)\backslash\overline{B}_{\sigma/2}(p)$:

$$\Sigma \cap A_{\sigma/2,\sigma}(p) = \cup_{i=1}^m \Sigma_i(\sigma).$$

Here each $\Sigma_i(\sigma)$ is a graph over $A_{\sigma/2,\sigma}(p) \cap P$ for some n-plane $P \subset T_pM$.

We can continue each $\Sigma_i(\sigma)$ all the way to $B_{\sigma_0}(p)\backslash\{p\}$, and we denote the continuation by Σ_i. Each Σ_i can be extended as a varifold across p with uniformly bounded first variation. (Since $\Sigma_i \subset \Sigma$ satisfies the area decay estimates, $\text{area}(\Sigma_i \cap B_\sigma(p)) \leq C\sigma^n$.) We claim that the density satisfies $\Theta^n(\Sigma_i, p) = 1$ for each i. In fact, $\Theta^n(\Sigma_i, p) \geq 1$ as any blowups of Σ_i converges to an n-plane, but $m = \Theta^n(\Sigma, p) = \sum_{i=1}^m \Theta^n(\Sigma_i, p)$. Now applying the Allard regularity theorem [1] to each Σ_i, we get that Σ_i extends as a $C^{1,\alpha}$ hypersurface across p. Higher regularity of Σ_i follows from the prescribing mean curvature equation and elliptic regularity. □

Appendix C. Proof of Lemma 3.3

[27, Lemma 4.5] is purely a result in finite dimensional multi-variable calculus. Let us translate the problem as follows: let $\mathbf{B}$ be some compact topological space with $0 \in \mathbf{B}$, and let $\{f^\omega \in C^\infty(\overline{B}^k) : \omega \in \mathbf{B}\}$ be a family of smooth functions defined on $\overline{B}^k$, such that $\omega \to f^\omega$ is a continuous map in the smooth topology on $C^\infty(\overline{B}^k)$. Moreover, we assume

- f^ω has a unique maximum $m(\omega) \in B^k_{c_0/\sqrt{10}}$, and $m(0) = 0$;
- $-\frac{1}{c_0}\,\text{Id} \leq D^2 f^\omega(u) \leq -c_0\,\text{Id}$ for all $u \in \overline{B}^k$ and for some $c_0 \in (0, 1)$.

So for each $\omega \in \mathbf{B}$, we have

$$f^\omega(m(\omega)) - \frac{1}{2c_0}|u - m(\omega)|^2 \leq f^\omega(u) \leq f^\omega(m(\omega)) - \frac{c_0}{2}|u - m(\omega)|^2 \tag{C.1}$$

for all $u \in \overline{B}^k$.

For each f^ω, consider the one-parameter flow $\{\phi^\omega(\cdot,t) : t \geq 0\} \subset \mathrm{Diff}(\overline{B}^k)$ generated by the vector field

$$u \to -(1-|u|^2)\nabla f^\omega(u), \quad u \in \overline{B}^k.$$

For fixed $u \in \overline{B}^k$, the function $t \to f^\omega(\phi^\omega(u,t))$ is non-increasing.

The prototype of [27, Lemma 4.5] is the following lemma, and the proof is essentially the same as therein so we omit it.

LEMMA C.1. *For any $\delta < \frac{1}{4}$, there exists $T = T(\delta, \mathbf{B}, \{f^\omega\}, c_0) \geq 0$ such that for any $\omega \in \mathbf{B}$ and $v \in \overline{B}^k$ with $|v - m(\omega)| \geq \delta$, we have*

$$f^\omega(\phi^\omega(v,T)) < f^\omega(0) - \frac{c_0}{10} \text{ and } |\phi^\omega(v,T)| > \frac{c_0}{4}.$$

Now we are ready to prove Lemma 3.3. Note that the ball $\overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Omega_0)$ is not compact under the $\mathbf{F}$-topology, so to apply Lemma C.1, we need to introduce a compactification of $\overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Omega_0)$.

Proof of Lemma 3.3. Given a $\mathbf{F}$-Cauchy sequence $\{\Omega_i\} \subset \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Omega_0)$, we denote $(V_\infty, \Omega_\infty) \in \mathcal{V}_n(M) \times \mathcal{C}(M)$ as the limit such that $V_\infty = \lim_{i\to\infty} |\partial\Omega_i|$ as varifolds and $\Omega_\infty = \lim_{i\to\infty} \Omega_i$ as Caccioppoli sets. If we define

$$\mathcal{A}^h_\infty(v) = \|(F_v)_\# V_\infty\|(M) - \int_{F_v(\Omega_\infty)} h \, d\mathcal{H}^{n+1} \text{ for each } v \in \overline{B}^k,$$

then $\mathcal{A}^h_{\Omega_i}$ converges smoothly to $\mathcal{A}^h_\infty$ as functions in $C^\infty(\overline{B}^k)$.

Now take $\mathbf{B}$ as the union of $\overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Omega_0)$ with the limits of the form $(V_\infty, \Omega_\infty)$, $f^\Omega = \mathcal{A}^h_\Omega$ and $f^{(V_\infty,\Omega_\infty)} = \mathcal{A}^h_\infty$. Then Lemma 3.3 follows from Lemma C.1. □

Appendix D. Existence of local PMC foliations

We recall the following classical result of White [41, Appendix and Rem. 2]. Note that the $\mathcal{A}^h$-functional can be locally expressed as the integration of an elliptic integrand.

PROPOSITION D.1. *Given a Riemannian metric g in a neighborhood U of $0 \in \mathbb{R}^{n+1}$, there exists an $\varepsilon > 0$, such that if $h : U \to \mathbb{R}$ is a smooth function with $\|h\|_{4,\alpha} < \varepsilon$, $r < \varepsilon$, and if*

$$w : B^n_r \subset \mathbb{R}^n \to \mathbb{R} \text{ satisfies } \|w\|_{2,\alpha} < \varepsilon r,$$

then for each $t \in [-r, r]$, there exists a $C^{2,\alpha}$-function $v_t : B^n_r \to \mathbb{R}$ whose graph G_t satisfies

$$H_{G_t} = h|_{G_t}$$

(*where* H_{G_t} *is evaluated with respect to the upward pointing normal of* G_t), *and*

$$v_t(x) = w(x) + t \text{ if } x \in \partial B_r^n.$$

Furthermore, v_t *depends on* r, t, h, w *in* C^1 *and the graphs* $\{G_t : t \in [-r, r]\}$ *form a foliation.*

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