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Uniqueness of two-convex closed ancient solutions to the mean curvature flow

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Abstract

In this paper we consider the classification of closed non-collapsed ancient solutions to the Mean Curvature Flow ($n \geq 2$) that are uniformly two-convex. We prove that they are either contracting spheres or they must coincide up to translations and scaling with the rotationally symmetric closed ancient non-collapsed solution first constructed by Brian White, and later by Robert Haslhofer and Or Hershkovits.

Contents

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1. Introduction

In this paper we consider closed non-collapsed ancient solutions $F(\cdot, t) : M^n \to \mathbb{R}^{n+1}$ to the mean curvature flow $(n \geq 2)$

$$\frac{\partial}{\partial t} F = -H\,\nu \tag{1.1}$$

for $t \in (-\infty, 0)$, where H is the inward mean curvature of $M_t := F(M^n, t)$ and ν is the outward unit normal vector. We know by Huisken's result [14] that compact convex surfaces M_t will contract to a point in finite time.

The main focus of the paper is the classification of two-convex *closed ancient solutions* to mean curvature flow, i.e., solutions that are defined for $t \in (-\infty, T)$ for some $T < +\infty$. Ancient solutions play an important role in understanding the singularity formation in geometric flows, as such solutions are usually obtained after performing a blow-up near points where the curvature is very large. In fact, Perelman's famous work on the Ricci flow [17] shows that the high curvature regions in 3D Ricci flow are modeled on ancient solutions that have non-negative curvature and are κ-non-collapsed. Similar results for mean curvature flow were obtained in [13], [20], [21] assuming mean convexity and embeddedness.

Daskalopoulos, Hamilton and Sesum previously established the complete classification of ancient compact convex solutions to the curve shortening flow in [8] and ancient compact solutions to the Ricci flow on S^2 in [9]. The higher dimensional cases have remained open for both the mean curvature flow and the Ricci flow.

In [18] W. Sheng and X. J. Wang introduced the following notion of non-collapsed solutions to the MCF, which is the analogue to the κ-non-collapsing condition for the Ricci flow discussed above. Furthermore, in [19] Xu-Jia Wang provided a number of results regarding the asymptotic behavior of ancient solutions, as $t \to -\infty$, and he also constructed new examples of ancient MCF solutions.

Definition 1.1. Let $K^{n+1} \subset \mathbb{R}^{n+1}$ be a domain whose boundary is a smooth mean convex hypersurface M^n. We say that M^n is α-non-collapsed if for every $p \in M^n$, there are balls B_1 and B_2 of radius at least $\frac{\alpha}{H(p)}$ such that $\bar{B}_1 \subset K^{n+1}$ and $\bar{B}_2 \subset \mathbb{R}^{n+1} \setminus \mathrm{Int}(K^{n+1})$, and such that B_1 and B_2 are tangent to M^n at the point p, from the interior and exterior of K^{n+1}, respectively. (In the limiting case $H(p) \equiv 0$, this means that K^{n+1} is a halfspace.) A smooth mean curvature flow $\{M_t\}$ is α-non-collapsed if M_t is α-non-collapsed for every t.

In [1] Andrews showed that the α-non-collapsedness property is preserved along mean curvature flow; namely, if the initial hypersurface is α-non-collapsed at time $t = t_0$, then evolving hypersurfaces M_t are α-non-collapsed for all later times for which the solution exists. Haslhofer and Kleiner [13] showed that every closed, ancient, and α-non-collapsed solution is necessarily convex.

In recent breakthrough works, Brendle and Choi [7], [6] gave the complete classification of non-compact ancient solutions to the mean curvature flow that are both strictly convex and uniformly two-convex. More precisely, they show that any non-compact and complete ancient solution to mean curvature flow (1.1) that is strictly convex, uniformly two-convex, and non-collapsed is the Bowl soliton, up to scaling and ambient isometries. Recall that the Bowl soliton is the unique rotationally-symmetric, strictly convex solution to mean curvature flow that translates with unit speed. It has the approximate shape of a paraboloid, and its mean curvature is largest at the tip. The uniqueness of the Bowl soliton among non-collapsed and uniformly two-convex translating solitons was established by Haslhofer in [11].

While the α-non-collapsedness property for mean curvature flow is preserved forward in time, it is not necessarily preserved going back in time. Indeed, Xu-Jia Wang [19] exhibited examples of ancient compact convex mean curvature flow solutions $\{M_t \mid t < 0\}$ that are not uniformly α-non-collapsed for any $\alpha > 0$. Such solutions lie in slab regions. The methods in [19] rely on the level set flow. Recently, Bourni, Langford and Tinaglia [5] provided a detailed construction of the Xu-Jia Wang solutions by different methods, showing also that the solution they construct is unique within the class of rotationally symmetric mean curvature flows that lie in a slab of a fixed width. In the present paper we will not consider these ancient collapsed solutions and will focus on the classification of ancient closed non-collapsed mean curvature flows.

Ancient self-similar shrinking solutions to MCF are of the form $M_t = \sqrt{T-t}\,\bar{M}$ for some fixed surface $\bar{M}$ and some "blow-up time" T. We rewrite a general ancient solution $\{M_t : t < T\}$ as

$$M_t = \sqrt{T-t}\,\bar{M}_\tau, \qquad \tau := -\log(T-t). \tag{1.2}$$

Haslhofer and Kleiner [13] proved that every closed ancient non-collapsed mean curvature flow with strictly positive mean curvature sweeps out the whole space. By Xu-Jia Wang's result [19], it follows that in this case the backward limit as $\tau \to -\infty$ of the type-I rescaling $\bar{M}_\tau$ of the original solution M_t, defined by (1.2), is either a sphere or a generalized cylinder $\mathbb{R}^k \times S^{n-k}$ of radius

$\sqrt{2(n-k)}$. In [3] we showed that if the backward limit is a sphere, then the ancient solution $\{M_t\}$ has to be a family of shrinking spheres itself. Note that in [13] it has been shown that if a closed ancient non-collapsed solution is self-similar, then it has to be the round sphere. Hence, we introduce the following definition.

Definition 1.2. We say an ancient mean curvature flow $\{M_t : -\infty < t < T\}$ is an *Ancient Oval* if it is compact, smooth, non-collapsed, and not self-similar.

Definition 1.3. We say that an ancient solution $\{M_t : -\infty < t < T\}$ is *uniformly two-convex* if there exists a uniform constant $\beta > 0$ so that

$$\lambda_1 + \lambda_2 \geq \beta H \qquad \text{for all } t < T. \tag{1.3}$$

Throughout the paper we will be using the following observation: *if an Ancient Oval M_t is uniformly two-convex, then by results in* [19], *the backward limit of its type-I parabolic blow-up must be a shrinking round cylinder $\mathbb{R}\times S^{n-1}$, with radius $\sqrt{2(n-1)}$.*

Based on formal matched asymptotics, Angenent [2] conjectured the existence of an Ancient Oval, that is, of an ancient solution that for $t \to 0$ collapses to a round point, but for $t \to -\infty$ becomes more and more oval in the sense that it looks like a round cylinder $\mathbb{R}\times S^{n-1}$ in the middle region, and like a rotationally symmetric translating soliton (the Bowl soliton) near the tips. A variant of this conjecture was proved already by White in [21]. By considering convex regions of increasing eccentricity and using limiting arguments, White proved the existence of ancient flows of compact, convex sets that are not self-similar. Haslhofer and Hershkovits [12] carried out White's construction in more detail, including, in particular, the study of the geometry at the tips. As a result they gave a rigorous and simple proof for the existence of an Ancient Oval.

Our main result in this paper is as follows.

THEOREM 1.4 (Uniqueness of Ancient Ovals). *Let $\{M_t, -\infty < t < T\}$ be a uniformly two-convex Ancient Oval. Then it is unique up to rotation, scaling and translation in time, and hence it must be the solution constructed by White in* [21] *and later by Haslhofer and Hershkovits in* [12].

An immediate consequence of our Theorem 1.4 and the definition of Ancient Ovals is the following classification result.

THEOREM 1.5. *Let $\{M_t, -\infty < t < T\}$ be an ancient mean curvature flow that is compact, uniformly two-convex and non-collapsed. Then, it is either the contracting spheres or the solution constructed by White in* [21] *and later by Haslhofer and Hershkovits in* [12].

The proof of Theorem 1.4 will follow from the two Theorems 1.6 and 1.7 stated below.

THEOREM 1.6 (Rotational symmetry of Ancient Ovals). *If* $\{M_t : -\infty < t < 0\}$ *is an Ancient Oval that is uniformly two-convex, then it is rotationally symmetric.*

Our proof of Theorem 1.6 closely follows the arguments by Brendle and Choi in [7], [6] on the uniqueness of strictly convex, non-compact, uniformly two-convex, and non-collapsed ancient mean curvature flow. It was shown in [7] that such solutions are rotationally symmetric. Then, by analyzing the rotationally symmetric solutions, Brendle and Choi showed that such solutions agree with the Bowl soliton.

Given Theorem 1.6, we may assume in our proof of Theorem 1.4 that any Ancient Oval M_t is rotationally symmetric. After applying a suitable Euclidean motion we may assume that its *axis of symmetry is the* x_1*-axis*. Then, M_t can be represented as

$$M_t = \big\{(x, x') \in \mathbb{R} \times \mathbb{R}^n : -d_1(t) < x < d_2(t), \|x'\| = U(x,t)\big\} \tag{1.4}$$

for some function $\|x'\| = U(x,t)$, and from now on we will set $x = x_1$ and $x' = (x_2, \ldots, x_{n+1})$. We call the points $(-d_1(t), 0)$ and $(d_2(t), 0)$ *the tips* of the surface. The function $U(x,t)$, which we call the *profile* of the hypersurface M_t, is only defined for $x \in [-d_1(t), d_2(t)]$. Any surface M_t defined by (1.4) is automatically invariant under $O(n)$ acting on $\mathbb{R} \times \mathbb{R}^n$. Convexity of the surface M_t is equivalent to concavity of the profile U; i.e., M_t is convex if and only if $U_{xx} \le 0$.

A family of surfaces M_t defined by $\|x'\| = U(x,t)$ evolves by mean curvature flow if and only if the profile $U(x,t)$ satisfies

$$\frac{\partial U}{\partial t} = \frac{U_{xx}}{1 + U_x^2} - \frac{n-1}{U}. \tag{1.5}$$

If M_t satisfies MCF, then its parabolic rescaling $\bar{M}_\tau$ defined by (1.2) evolves by the *rescaled MCF*

$$\nu \cdot \frac{\partial \bar{F}}{\partial \tau} = -H + \tfrac{1}{2}\bar{F} \cdot \nu,$$

where $\bar{F}(x,\tau) = e^{\tau/2} F(x, T - e^{-\tau})$ is the parametrization of $\bar{M}_\tau$, and $\nu = \nu(x,\tau)$ is the corresponding unit normal. Also,

$$\bar{M}_\tau = \{(y, y') \in \mathbb{R} \times \mathbb{R}^n \mid -\bar{d}_1(\tau) \le y \le \bar{d}_2(\tau), \ \ \|y'\| = u(y,\tau)\} \tag{1.6}$$

for a profile function u, which is related to U by the parabolic rescaling

$$U(x,t) = \sqrt{T-t}\, u(y,\tau), \qquad y = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t).$$

The points $(-\bar{d}_1(\tau), 0)$ and $(\bar{d}_2(\tau), 0)$ are referred to as the tips of rescaled surface $\bar{M}_\tau$. Equation (1.5) for $U(x,t)$ is equivalent to the following equation for $u(y,\tau)$:

$$\frac{\partial u}{\partial \tau} = \frac{u_{yy}}{1+u_y^2} - \frac{y}{2}\, u_y - \frac{n-1}{u} + \frac{u}{2}. \tag{1.7}$$

It follows from the discussion above that our most general result (1.4) reduces to the following classification under the presence of rotational symmetry.

THEOREM 1.7 (Uniqueness of $O(n)$-invariant Ancient Ovals). *Let $(M_1)_t$ and $(M_2)_t$, $-\infty < t < T$, be two $O(n)$-invariant Ancient Ovals with the same axis of symmetry. Then, they are the same up to translations along the axis of symmetry, translations in time and parabolic rescaling.*

Our proof of Theorem 1.7 relies on our previous result [3], which we state below for the reader's convenience.

THEOREM 1.8 (Angenent, Daskalopoulos, Sesum in [3]). *Let $\{M_t\}$ be any $O(1) \times O(n)$ invariant Ancient Oval. Then the solution $u(y,\tau)$ to (1.7) has the following asymptotic expansions*:

(i) *For every $M > 0$,*

$$u(y,\tau) = \sqrt{2(n-1)}\Big(1 - \frac{y^2-2}{4|\tau|}\Big) + o(|\tau|^{-1}), \qquad |y| \le M$$

as $\tau \to -\infty$.

(ii) *Define $z := y/\sqrt{|\tau|}$ and $\bar{u}(z,\tau) := u(z\sqrt{|\tau|}, \tau)$. Then,*

$$\lim_{\tau \to -\infty} \bar{u}(z,\tau) = \sqrt{(n-1)\,(2-z^2)}$$

uniformly on compact subsets in $|z| < \sqrt{2}$.

(iii) *Denote by p_t any of the two tips of $M_t \subset \mathbb{R}^{n+1}$, and define for any $t_* < 0$ the rescaled flow at the tip*

$$\tilde{M}_{t_*}(t) = \lambda(t_*)\big\{M_{t_*+t\lambda(t_*)^{-2}} - p_{t_*}\big\},$$

where

$$\lambda(t) := H(p_t, t) = H_{\max}(t) = \sqrt{\tfrac{1}{2}|t|\log|t|}\big(1+o(1)\big).$$

Then, as $t_* \to -\infty$, the family of mean curvature flows $\tilde{M}_{t_*}(\cdot)$ converges to the unique unit speed Bowl soliton, i.e., the unique convex rotationally symmetric translating soliton with velocity one.

In [3] we proved this theorem with the additional assumption that the solutions are reflection symmetric (i.e., they were $O(1) \times O(n)$ invariant). In Appendix 9 we show how to remove the assumption of reflection symmetry.

Finally, in Appendix 10 we recall that for non-collapsed convex hypersurfaces, the intrinsic and extrinsic distances are equivalent.

In previous classifications of ancient solutions to mean curvature flow and Ricci flow, [8], [9], [7], [6], an essential role in the proofs was played by the fact that all such solutions were either given in closed form or that they were solitons. The techniques in our current work overcome such a requirement and potentially can be used in many other parabolic equations and particularly in other geometric flows. To our knowledge, our work and the recent work by Bourni, Langford and Tinaglia [5] are the first classification results of geometric ancient solutions where the solutions are not given in closed form and they are not solitons. Let us also point out that our current techniques are reminiscent of the significant work by Merle and Zaag in [16] which has provided an inspiration for us.

The outline of the proof of Theorem 1.4 is as follows. In Section 2 we prove Theorem 1.6. Once we know our Ancient Oval is rotationally symmetric, we devote the rest of the paper to the proof of Theorem 1.7. In Section 3 we give a detailed outline of the proof of Theorem 1.7. We prove various a priori estimates for a solution to (1.7) in Section 5. In Sections 6 and 7, we consider the difference of two solutions and prove coercive estimates with respect to appropriately chosen weights in the cylindrical region and tip region respectively. Finally in Section 8 we discuss how to combine those two estimates together to conclude the proof of Theorem 1.4. In Section 9 we show that the asymptotics showed in Theorem 1.8 in [3] hold under the assumption on rotational symmetry, that is, $O(n)$-symmetry only rather than $O(1) \times O(n)$-symmetry.

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2. **Rotational symmetry**

The main goal in this section is to prove Theorem 1.6. Our proof of Theorem 1.6 follows closely the arguments of the recent work by Brendle and Choi [7], [6] on the uniqueness of strictly convex, uniformly two-convex, noncompact and non-collapsed ancient solutions of mean curvature flow in $\mathbb{R}^{n+1}$. It was shown in [7] and [6] that such solutions are rotationally symmetric. Then by analyzing the rotationally symmetric solutions, Brendle and Choi showed that such solutions agree with the Bowl soliton. For the reader's convenience we state their result next.

THEOREM 2.1 (Brendle and Choi [7], [6]). *Let $\{M_t : t \in (-\infty, 0)\}$ be a non-compact ancient mean curvature flow in $\mathbb{R}^{n+1}$ that is strictly convex, non-collapsed, and uniformly two-convex. Then M_t agrees with the Bowl soliton, up to scaling and ambient isometries.*

In the proof of Theorem 1.6 we will use both the key results that led to the proof of the main theorem in [6] (see Propositions 2.5 and 2.6 below) and the uniqueness result as stated in Theorem 2.1.

Before we proceed with the proof of Theorem 1.6, let us recall some standard notation. Our solution M_t is embedded in $\mathbb{R}^{n+1}$, for all $t \in (-\infty, T)$ and in the mean curvature flow, time scales like distance squared. We denote by $\mathcal{P}(\bar{x}, \bar{t}, r)$ the *parabolic cylinder* centered at $(\bar{x}, \bar{t}) \in \mathbb{R}^{n+1} \times \mathbb{R}$ of radius $r > 0$ and duration $T > 0$, namely, the set

$$\mathcal{P}(\bar{x}, \bar{t}, r, T) := \mathcal{B}(\bar{x}, r) \times [\bar{t} - T, \bar{t}],$$

where $\mathcal{B}(x, r) := \{x \in \mathbb{R}^{n+1} : |x - \bar{x}| \leq r\}$ denotes the *closed* Euclidean ball of radius r in $\mathbb{R}^{n+1}$. If we do not specify the duration T, then we choose the default value $T = r^2$ that corresponds to parabolic scaling.

Also, following the notation in [15] and [6], we denote by $\hat{\mathcal{P}}(\bar{x}, \bar{t}, r, T)$ the *rescaled by mean curvature* parabolic cylinder centered at $(\bar{x}, \bar{t}) \in \mathbb{R}^{n+1} \times \mathbb{R}$ of radius $r > 0$ and duration T, namely, the set

$$\hat{\mathcal{P}}(\bar{x}, \bar{t}, r, T) := \mathcal{P}(\bar{x}, \bar{t}, \hat{\rho}(\bar{x}, \bar{t})\, r, \hat{\rho}(\bar{x}, \bar{t})^2\, T), \qquad \hat{\rho}(\bar{x}, \bar{t}) := \frac{n}{H(\bar{x}, \bar{t})}.$$

The default value for the duration T is always assumed to be $T = r^2$; therefore $\hat{\mathcal{P}}(\bar{x}, \bar{t}, r) := \hat{\mathcal{P}}(\bar{x}, \bar{t}, r, r^2)$.

Note that in [15, §7] Huisken and Sinestrari consider parabolic cylinders with respect to the intrinsic metric $g(t)$ on the solution M_t, which in our case and in the case of [7] and [6] is equivalent to the extrinsic metric on space-time that we are considering here. See Appendix 10.

We recall Brendle and Choi's [6] definition of a mean curvature flow being ϵ-symmetric in terms of the normal components of rotation vector fields. In what follows we identify $\mathfrak{so}(n)$ with the subalgebra of $\mathfrak{so}(n+1)$ consisting of skew symmetric matrices of the form

$$J = \begin{bmatrix} 0 & 0 \\ 0 & J' \end{bmatrix}, \qquad \text{with} \qquad J' \in \mathfrak{so}(n).$$

Thus $\mathfrak{so}(n)$ acts on the second factor in the splitting $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$. Any $J \in \mathfrak{so}(n+1)$ generates a vector field on $\mathbb{R}^{n+1}$ by $\vec{v}(x) = Jx$. If $\varPhi(x) = Sx + p$ is a Euclidean motion, with $p \in \mathbb{R}^{n+1}$ and $S \in O(n+1)$, then the pushforward of the vector field $\vec{v}(x) = Jx$ under $\varPhi$ is given by

$$\varPhi_* \vec{v}(x) = d\varPhi_x \cdot \vec{v}(\varPhi^{-1}x) = SJS^{-1}(x - p).$$

Any vector field of this form is a *rotation vector field.*

Definition 2.2. A collection of vector fields $\mathcal{K} := \{K_\alpha \mid 1 \leq \alpha \leq \frac{1}{2}n(n-1)\}$ on $\mathbb{R}^{n+1}$ is a *normalized set of rotation vector fields* if there exist an orthonormal basis $\{J_\alpha \mid 1 \leq \alpha \leq \frac{1}{2}n(n-1)\}$ of $\mathfrak{so}(n) \subset \mathfrak{so}(n+1)$, a matrix $S \in O(n+1)$, and a point $q \in \mathbb{R}^{n+1}$ such that

$$K_\alpha(x) = SJ_\alpha S^{-1}(x-q).$$

Definition 2.3. Let M_t be a solution of mean curvature flow. We say that a point $(\bar{x},\bar{t})$ is ϵ-*symmetric* if there exists a normalized set of rotation vector fields $\mathcal{K}^{(\bar{x},\bar{t})} = \{K_\alpha^{(\bar{x},\bar{t})} \mid 1 \leq \alpha \leq \frac{1}{2}n(n-1)\}$, such that $\max_\alpha |K_\alpha| H \leq 10n$ at the point $(\bar{x},\bar{t})$ and $\max_\alpha |\langle K_\alpha, \nu\rangle| H \leq \epsilon$ in the parabolic neighborhood $\hat{\mathcal{P}}(\bar{x},\bar{t},10,100)$.

Lemma 4.2 in [6] allows us to control how the axis of rotation of a normalized set of rotation vector fields $\mathcal{K}^{(x,t)}$ varies as we vary the point (x,t).

The proof of Theorem 1.6 relies on the following two key propositions which were both shown in [7] and [6] for dimensions $n=2$ and $n\geq 3$ respectively. The first proposition is directly taken from [7], [6] (see Theorem 4.4 in [7], [6]). The second proposition required some adjustments of the arguments in [7], [6] and hence we present those parts requiring modifications of the proof below (see Proposition 2.6).

Definition 2.4. A point (x,t) of a mean curvature flow *lies on an* (ϵ, L)-*neck* if there are a Euclidean transformation $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and a scale $\lambda > 0$ such that

- Φ maps x to $(0, \sqrt{2(n-1)}, 0, \ldots, 0)$;
- for all $\tau \in [-L^2, 0]$, the hypersurface $\lambda^{-1}\Phi\big(M_{t+\lambda^2\tau}\big)$ is ϵ-close in C^{20} in the ball $\mathcal{B}(0,L)$ to the cylinder of length L, radius $\sqrt{2(n-1)(1-\tau)}$ and with the x_1-axis as symmetry axis.

Proposition 2.5 (Neck Improvement: Theorem 4.4 in [7], [6]). *There exist a large constant L_0 and a small constant ϵ_0 with the following property. Suppose that M_t is a mean curvature flow, and suppose that $(\bar{x},\bar{t})$ is a point in space-time with the property that every point in $\hat{\mathcal{P}}(\bar{x},\bar{t},L_0)$ is ϵ-symmetric and lies on an $(\epsilon_0, 10)$-neck, where $\epsilon \leq \epsilon_0$. Then $(\bar{x},\bar{t})$ is $\frac{\epsilon}{2}$-symmetric.*

Proof. The proof is given in Theorem 4.4 in [7], [6]. □

The next result will be shown by slight modification of arguments in the proof of Theorem 5.2 in [6]. The proof of Proposition 2.6 below follows closely the arguments in [7].

Proposition 2.6 (Cap Improvement [7]). *Let L_0 and ϵ_0 be chosen as in the Neck Improvement Proposition 2.5. Then there exist constants $L_1 \geq 2L_0$, $T_1 > 0$, and $\epsilon_1 \leq \frac{\epsilon_0}{2}$ with the following property. Suppose that $\{M_t\}$ is a mean* *curvature flow solution defined in $\hat{\mathcal{P}}(\bar{x}, \bar{t}, L_1, T_1)$. Moreover, assume that after scaling to make $H(\bar{x}, \bar{t}) = 1$, $M_t \cap \mathcal{B}(\bar{x}, L_1)$ is, for $t \in [\bar{t} - T_1, \bar{t}]$, ϵ_1-close in the C^{20}-norm to a piece of a Bowl soliton, and that the tip of the Bowl soliton lies in $\hat{\mathcal{P}}(\bar{x}, \bar{t}, \frac{1}{2}L_1, T_1)$. If for some $\epsilon \leq \epsilon_0$ every point in $M_t \cap \mathcal{B}(\bar{x}, L_1)$ with $t \in [\bar{t} - T_1, \bar{t}]$ is ϵ-symmetric, then $(\bar{x}, \bar{t})$ is $\frac{\epsilon}{2}$-symmetric.*

Proof. Without loss of generality we may assume $\bar{t} = -1$ and $H(\bar{x}, -1) = 1$.

The Hessian of the mean curvature around the maximum mean curvature point in a Bowl soliton is strictly negative definite. Thus, the assumptions in the proposition imply that if we take ϵ_1 sufficiently small, then we may assume that the maximum of $H(\cdot, t)$ in $\mathcal{B}(\bar{x}, L_1) \cap M_t$ is attained at a unique interior point $q_t \in B(\bar{x}, L_1) \cap M_t$ and also that the Hessian of the mean curvature at q_t is negative definite. Hence, q_t varies smoothly in t. We now conclude that if $(x_0, t_0) \in \hat{\mathcal{P}}(\bar{x}, -1, L_1, T_1)$, then

$$\frac{d}{dt}|x_0 - q_t| < 0 \text{ for } -1 - T_1 \leq t \leq -1. \tag{2.1}$$

The proof of (2.1) is the same as the proof of Lemma 5.2 in [7].

We claim that there exists a uniform constant s_* with the property that every point $(x, t) \in \hat{\mathcal{P}}(\bar{x}, -1, L_1, T_1)$ with $|x - q_t| \geq s_*$ lies on an $(\epsilon_0, 10)$-neck and satisfies $|x - q_t| H(x, t) \geq 1000 L_0$. Indeed, knowing the behavior of the Bowl soliton, it is a straightforward computation to check that these claims are true on the Bowl soliton, with a constant, for example, $2000 L_0$. By our assumption, the part of the solution within $\hat{\mathcal{P}}(\bar{x}, -1, L_1, T_1)$ is ϵ_1-close to the Bowl soliton and hence the claims are true for our solution as well.

If $|\bar{x} - q_{-1}| \geq s_*$, the proposition follows immediately from the Neck Improvement Proposition 2.5. Thus, we may assume that $|\bar{x} - q_{-1}| \leq s_*$. Then we have the following claim, in which we abbreviate

$$\theta := 2^{\frac{1}{100}}.$$

Claim 2.7. Suppose that M_t is an ancient solution of mean curvature flow. Given any positive integer j, there exist large constants L_j, T_j, and a small constant $\epsilon_j > 0$ with the following property: if the parabolic neighborhood $\hat{\mathcal{P}}(\bar{x}, -1, L_j, T_j)$ is ϵ_j-close in the C^{20}-norm to a piece of the Bowl soliton that includes the tip, and every point in $\hat{\mathcal{P}}(\bar{x}, -1, L_j, T_j)$ is ϵ-symmetric, where $\epsilon < \epsilon_j$, then every point $(x, t) \in \hat{\mathcal{P}}(\bar{x}, -1, L_j, T_j)$ with $t \in [-\theta^{3j}, -1]$ and $s_* \theta^j \leq |x - q_t| \leq s_* \theta^{j+1}$ is $2^{-j}\epsilon$-symmetric.

Proof of Claim 2.7. The proof is by induction on j. If $j = 0$, the statement is obviously satisfied. Let us prove that the statement holds for $j = 1$. For $j = 1$, we have $s_* \leq |x - q_t| \leq s_* \theta$ and $t \leq -1$, as long as $(x, t) \in \hat{\mathcal{P}}(\bar{x}, -1, L_1, T_1)$. By the above discussion we have that (x, t) lies on an $(\epsilon_0, 10)$-neck and $|x - q_t| \geq$

$\frac{1000L_0}{H(x,t)}$. This implies $\frac{L_0}{H(x,t)} \leq \frac{\theta}{1000} s_*$ and hence $\hat{\mathcal{P}}(x,t,L_0,L_0^2) \subset \hat{\mathcal{P}}(\bar{x},-1,L_1,L_1^2)$ if we choose L_1 sufficiently big compared to s_*. Therefore, every point in $\hat{\mathcal{P}}(x,t,L_0,L_0^2)$ is ϵ-symmetric and lies on an $(\epsilon_0,10)$-neck (where $\epsilon < \epsilon_0$). By Proposition 2.5 we conclude (x,t) is $\frac{\epsilon}{2}$-symmetric. The same argument works to show that actually all points (x,t), where $|x-q_t| \geq s_*$ and $-1-T_1/2 \leq t \leq -1$, as long as $|x-\bar{x}| \leq L_1/2$, are $\frac{\epsilon}{2}$-symmetric. Note that we have to choose L_1 and T_1 big relative to s_*.

We will now assume that the statement of the claim holds for $j-1$ and prove that it also holds for j. Due to a repeated application of Proposition 2.5 we may assume that $\hat{\mathcal{P}}(\bar{x},-1,L_j,T_j)$ is ϵ_j close to a Bowl soliton and that all points (x,t) such that $|x-q_t| \geq \theta^{j-1}s_*$, $-1-T_j/2 \leq t \leq -1$, as long as $|\bar{x}-x| \leq L_j/2$, are $2^{-j+1}\epsilon$-symmetric. Let (x,t) be such that $\theta^j s_* \leq |x-q_t| \leq \theta^{j+1}s_*$ and $-\theta^{3j} \leq t \leq -1$. First, the arguments from above show that every such point (x,t) lies on an $(\epsilon_0,10)$-neck. Moreover, we claim that for every such point (x,t), we have that

$$\begin{aligned}\hat{\mathcal{P}}(x,t,L_0,L_0^2) \subset \{(y,s) \mid \ |y-q_s| \geq \theta^{j-1}s_*, \\ -1-T_j/2 \leq s \leq -1,\ |y-\bar{x}| \leq L_j/2\}.\end{aligned} \tag{2.2}$$

Indeed, let $(y,s) \in \hat{\mathcal{P}}(x,t,L_0,L_0^2)$. Then by (2.1) and the triangle inequality, we have

$$|y-q_s| \geq |y-q_t| \geq |q_t-x| - |x-y| \geq \theta^j s_* - \frac{L_0}{H(x,t)}.$$

By the same discussion as at the beginning of the proof of the claim we have that

$$\theta^{j+1}s_* \geq |x-q_t| \geq \frac{1000L_0}{H(x,t)} \tag{2.3}$$

implying that $\frac{L_0}{H(x,t)} \leq \frac{\theta^{j+1}s_*}{1000}$, which shows that $|y-q_s| \geq \theta^{j-1}\, s_*$, as desired. We also have $s \geq -1-T_j/2$. Indeed, using (2.3) we have

$$t - \frac{\theta^{2(j+1)}\, s_*^2}{1000^2} \leq t - \frac{L_0^2}{H(x,t)^2} \leq s \leq t \leq -1.$$

We also have $t \geq -\theta^{3j}$, and hence $-1 \geq s \geq -T_j/2-1$, if we choose T_j sufficiently big. If L_j is sufficiently big, then $|y-\bar{x}| \leq L_j/2$ easily follows. Finally, (2.2) holds and every point in the set on the right-hand side of the inclusion in (2.2) is $2^{-j+1}\epsilon$ - symmetric. By Proposition 2.5 we conclude that every point (x,t) such that $\theta^j s_* \leq |x-q_t| \leq \theta^{j+1}s_*$ and $-\theta^{3j} \leq t \leq -1$ is $2^{-j}\epsilon$-symmetric. It is clear from the proof of (2.1) that can be found in [7] that if we take a bigger parabolic cylinder around $(\bar{x},-1)$ of size L_j, in order to still have (2.1) one needs to require that $\hat{\mathcal{P}}(\bar{x},-1,L_j,T_j)$ is ϵ_j close to a Bowl soliton, where ϵ_j needs to be taken very small, depending on L_j. $\square$

In the following, j will denote a large integer, which will be determined later. Moreover, assume that $L \geq L_j$ and $\epsilon \leq \epsilon_j$. Using the claim, we conclude that for every point (x,t) with $-\theta^{3j} \leq t \leq -1$ and $s_*\theta^j \leq |x - q_t| \leq s_*\theta^{j+1}$, there exists a normalized set of rotation vector fields $\mathcal{K}^{(x,t)} = \{K_\alpha^{(x,t)} \mid 1 \leq \alpha \leq \frac{1}{2}n(n-1)\}$, such that $\max_\alpha |\langle K_\alpha^{(x,t)}, \nu\rangle| H \leq 2^{-j}\epsilon$ on $\bar{\mathcal{P}}(x,t,10)$. Moreover, since $|x - q_t| \geq s_*$ implies $H(x,t)|x - q_t| \geq 1000 L_0$, we have

$$\max_\alpha |\langle K_\alpha^{(x,t)}, \nu\rangle| \leq \frac{2^{-j}\epsilon}{H} \leq \frac{\theta^{j+1} 2^{-j} s_*}{1000 L_0}\,\epsilon \leq C\, 2^{\frac{j}{100} - j}\,\epsilon, \qquad j \geq j_0$$

for a uniform constant C that is independent of j and ϵ. Lemma 4.3 in [7] allows us to control how the axis of rotation of $\mathcal{K}^{(x,t)}$ varies as we vary the point (x,t). More precisely, as in [7], if (x_1,t_1) and (x_2,t_2) both satisfy $-\theta^{3j} \leq t_i \leq -1$ and $s_*\theta^j \leq |x_i - q_t| \leq s_*\theta^{j+1}$, and if $(x_2,t_2) \in \hat{\mathcal{P}}(x_1,t_1,1,1)$, then

$$\inf_{w \in O(n)} \sup_{B_{10 r_2}(x_2)} \max_\alpha \left| K_\alpha^{(x_1,t_1)} - \sum_{\beta=1}^{\frac{1}{2}n(n-1)} w_{\alpha\beta} K_\beta^{(x_2,t_2)} \right| \leq C 2^{-j} r_2,$$

where $r_2 = H(x_2,t_2)^{-1}$. Hence we can find a normalized set of rotation vector fields $\mathcal{K}^{(j)} = \{K_\alpha^{(j)} \mid 1 \leq \alpha \leq \frac{1}{2}n(n-1)\}$ so that if $-\theta^{3j} \leq t \leq -1$ and $s_*\theta^j \leq |x - q_t| \leq s_*\theta^{j+1}$, then,

$$\inf_{w \in O(n)} \max_\alpha \left| K_\alpha^{(j)} - \sum_{\beta=1}^{\frac{1}{2}n(n-1)} w_{\alpha\beta} K_\beta^{(x,t)} \right| \leq C 2^{-j/2},$$

at the point (x,t). As in [7] we conclude that $\max_\alpha |\langle K_\alpha^{(j)}, \nu\rangle| \leq C 2^{-\frac{j}{2}}$, whenever $-\theta^{3j} \leq t \leq -1$ and $s_*\theta^j \leq |x - q_t| \leq s_*\theta^{j+1}$. Finally, note that $\max_\alpha |\langle K_\alpha^{(j)}, \nu\rangle| \leq C 2^{\frac{j}{100}}$, whenever $s_*\theta^j \leq |x - q_t| \leq s_*\theta^{j+1}$ and $t = -\theta^{3j}$.

As in [7], for each $\alpha \in \{1, \ldots, \frac{1}{2}n(n-1)\}$, we define a function $f_\alpha^{(j)}$: $\{(x,t) \mid t \in [-\theta^{3j}, -1], \quad |x - q_t| \leq s_*\theta^j\} \to \mathbb{R}$ by

$$f_\alpha^{(j)} := e^{\theta^{-2} t} \frac{\langle K_\alpha^{(j)}, \nu\rangle}{H - \theta^{-1}}.$$

The same computation as in [7] implies that by the maximum principle applied to the evolution of $f_\alpha^{(j)}$ we get

$$|f_\alpha^{(j)}(x,t)| \leq C\, 2^{-j/4}$$

in the region

$$-\theta^{3j} \leq t \leq -1, \quad s_*\theta^j \leq |x - q_t| \leq s_*\theta^{j+1}.$$

Standard interior estimates for parabolic equations give estimates for the higher order derivatives of $\langle K_\alpha^{(j)}, \nu\rangle$.

Hence, if we choose j sufficiently big, then the same reasoning as in [7] implies $(\bar{x}, -1)$ is $\frac{\epsilon}{2}$-symmetric. Having chosen j in this way, we finally define $L_1 = L_j$ and $\epsilon_1 := \epsilon_j$. Then L_1 and ϵ_1 have the desired properties as stated in Proposition 2.6. □

The goal of the remaining part of this section is to show how we can employ Propositions 2.5 and 2.6 to prove Theorem 1.6.

Observe that by the crucial work of Haslhofer and Kleiner in [13] we know that a strictly convex α-non-collapsed ancient solution to mean curvature flow sweeps out the whole space. Hence, the well-known important result of X. J. Wang in [19] shows that the rescaled flow, after a proper rotation of coordinates, converges, as time goes to $-\infty$, uniformly on compact sets, to a round cylinder of radius $\sqrt{2(n-1)}$.

This has as as a consequence that $M_t \cap B_{8(n-1)\sqrt{|t|}}$ is a neck with radius $\sqrt{2(n-1)|t|}$ for $t \ll -1$. The complement $M_t \backslash B_{8(n-1)\sqrt{|t|}}$ has two connected components, call them Ω_1^t and Ω_2^t, both compact. Thus, for every t, the maximum of H on Ω_1^t is attained at least at one point in Ω_1^t and similarly for Ω_2^t.

For every t, we define the *tip* points p_t^1 and p_t^2 as follows. Let p_t^k for $k = 1, 2$ be a point such that

$$|\langle F, \nu\rangle(p_t^k, t)| = |F|(p_t^k, t) \quad \text{and} \quad |F|(p_t^k, t) = \max_{\Omega_k^t} |F|(\cdot, t).$$

Write $d_k(t) := |F|(p_t^k, t)$ for $k \in \{1, 2\}$.

Throughout the rest of the section we will be using the next observation about possible limits of our solution around arbitrary sequence of points (x_j, t_j) with $x_j \in M_{t_j}$, $t_j \to -\infty$ when rescaled by $H(x_j, t_j)$.

Lemma 2.8. *Let M_t, $t \in (-\infty, 0)$ be an Ancient Oval satisfying the assumptions in Theorem 1.6. Fix a $k \in \{1, 2\}$. Then for every sequence of points $x_j \in M_{t_j}$ and any sequence of times $t_j \to -\infty$, the rescaled sequence of solutions $F_j(\cdot, t) := Q_j(F(\cdot, t_j + tQ_j^{-2}) - x_j)$, where $Q_j := H(x_j, t_j)$, subconverges to either a Bowl soliton or a shrinking round cylinder.*

Proof. By the global convergence theorem (Theorem 1.12) in [13] we have that after passing to a subsequence, the flow M_t^j converges, as $j \to \infty$, to an ancient solution M_t^∞, for $t \in (-\infty, 0]$, which is convex and uniformly two-convex. Note that $H(0, 0) = 1$ on the limiting manifold. By the strong maximum principle applied to H we have that $H > 0$ everywhere on M_t^∞, where $t \in (-\infty, 0]$. Assume the limit M_t^∞ is non-compact. Then, if M_t^∞ is strictly convex, by the classification result in [7] we have that it is a Bowl soliton. If the limit is not strictly convex, by the strong maximum principle it splits off a line and hence it is of the form $N_t^{n-1} \times \mathbb{R}$, where N_t^{n-1} is an $n-1$-dimensional ancient solution. On the other hand, the uniform two-convexity assumption on our solution implies the inequality $\lambda_{\min}(N_t^{n-1}) \geq \beta H(N_t^{n-1})$ for a uniform constant $\beta > 0$. Thus, Lemma 3.14 in [13] implies that the limiting flow M_t^∞ is a family of round shrinking cylinders $S^{n-1} \times \mathbb{R}$.

To complete the proof of the lemma we still need to show the limit M_t^∞ is non-compact. We argue by contradiction. Assume it is compact, implying that

$$\limsup_{j\to\infty} H(x_j, t_j)\operatorname{diam}(M_{t_j}) < \infty$$

and

$$\limsup_{j\to\infty} H(x_j, t_j)^{-1} \sup_M H(\cdot, t_j) < \infty.$$

Combining these two we obtain $\limsup_{j\to\infty}\operatorname{diam}(M_{t_j})\sup_M H(x, t_j) < \infty$. This, in particula,r implies M_{t_j} cannot contain arbitrarily long necks. On the other hand, since we know the rescaled flow converges uniformly on compact sets to a round cylinder, M_{t_j} must contain arbitrarily long necks if j is sufficiently big. Hence we get contradiction, and M_t^∞ must be non-compact as claimed. □

We will next show that points that are away from the tip points in both regions Ω_t^k, $k = 1, 2$ are cylindrical.

Lemma 2.9. *Let M_t, $t \in (-\infty, 0)$, be an Ancient Oval satisfying the assumptions of Theorem 1.6, and fix $k \in 1, 2$. Then, for every $\eta > 0$, there exist $\bar{L}$ and t_0, so that for all $x \in \Omega_t^k$, the following holds*:

$$|x - p_t^k| \geq \frac{\bar{L}}{H(x,t)} \implies \frac{\lambda_{\min}}{H}(x,t) < \eta. \tag{2.4}$$

We may chose $\bar{L}$ so that (2.4) holds for both $k = 1, 2$.

Proof. Without loss of generality we may assume that $k = 1$, and we will argue by contradiction. If the statement is not true, then there exist $L_j \to \infty$ and sequences of times $t_j \to -\infty$ and points $x_j \in M_{t_j}$ so that

$$|x_j - p_{t_j}^1| \geq \frac{L_j}{H(x_j, t_j)} \quad \text{and} \quad \frac{\lambda_{\min}}{H}(x_j, t_j) \geq \eta. \tag{2.5}$$

Rescale the flow around (x_j, t_j) by $Q_j := H(x_j, t_j)$ as in Lemma 2.8, and call the rescaled manifolds M_t^j. Then

$$|0 - \bar{p}_j^1| \geq L_j \to \infty, \qquad \text{as } j \to \infty, \tag{2.6}$$

where the origin and $\bar{p}_j^1$ correspond to x_j and tip points $p_{t_j}^1$ after rescaling, respectively. By Lemma 2.8 we have that passing to a subsequence M_t^j converges to either a Bowl soliton or a cylinder. Since $\frac{\lambda_{\min}}{H}$ is a scaling invariant quantity, (2.5) implies that on the limiting manifold we have $\frac{\lambda_{\min}}{H}(0,0) \geq \eta$ which immediately excludes the cylinder. Thus the limiting manifold must be the Bowl soliton.

Let us look next at the tip points $p^2_{t_j}$ of our solution that lie on the other side $\Omega^2_{t_j}$ and denote by $\bar{p}^2_j$ the corresponding points on our rescaled solution. Then we must have that $|0 - \bar{p}^2_j| \leq C_0$ for some constant C_0. Otherwise, if we had that $\limsup_{j\to+\infty} |0 - \bar{p}^2_j| \to +\infty$, this together with (2.6), the convexity of our surface, the fact that the furthest points $p^1_{t_j}$ and $p^2_{t_j}$ lie on the opposite side of a necklike piece and the splitting theorem would imply that the limit of M^j_t would split off a line. This and Lemma 2.8 would yield that the limit of M^j_t around (x_j, t_j) would have been the cylinder that we have already ruled out. Thus, $|0 - \bar{p}^2_j| \leq C_0$, which in terms of our unrescaled solution M_t means that $|x_j - p^2_{t_j}| \leq \frac{C_0}{H(x_j,t_j)}$.

Since $x_j \in \Omega^1_{t_j}$ and $p^2_{t_j} \in \Omega^2_{t_j}$, we then have that the whole neck-like region that divides the sets $\Omega^1_{t_j}$ and $\Omega^2_{t_j}$ lies at a distance less than equal to $\frac{C_0}{H(x_j,t_j)}$ from x_j. This implies that the whole neck-like region has to lie on a compact set of the Bowl soliton, implying that $\frac{\lambda_{\min}}{H}(\cdot, t_j) \geq c_0 > 0$ holds for some constant c_0, independent of j. This is a contradiction, since on the neck-like region of our solution the scaling invariant quantity $\lambda_{\min} H^{-1} \to 0$ as $t_j \to -\infty$. The above discussion shows that $|x - p^1_t| \geq \frac{L}{H(x,t)}$ implies that $\frac{\lambda_{\min}}{H}(x,t) < \eta$, thus finishing the proof of the lemma. □

In the following lemma we show that mean curvature of an ancient oval solution satisfying the assumptions of Theorem 1.6, around the tip points on Ω^k_t, for a fixed $k = 1, 2$, are uniformly equivalent in a quantitative way.

LEMMA 2.10. *Let M_t, $t \in (-\infty, 0)$, be an Ancient Oval satisfying the assumptions of Theorem 1.6, and fix $k = 1, 2$. For every $L > 0$, there exist uniform constants $c > 0$, $C < \infty$ and $t_0 \ll -1$ so that for all $t \leq t_0$, we have*

$$c\, H(p^k_t, t) \leq H(x,t) \leq C\, H(p^k_t, t) \quad \text{if } |x - p^k_t| < \frac{L}{H(x,t)},\ x \in \Omega^k_t. \tag{2.7}$$

We may chose c, C so that (2.7) holds for both $k = 1, 2$.

Proof. Let us take, without loss of generality, $k = 1$. First let us show the estimate from below. Assume the statement is false. This implies there exist a sequence of times $t_j \to -\infty$ and a sequence of constants $C_j \to \infty$ so that

$$H(p^1_{t_j}, t_j) \geq C_j\, H(x_j, t_j) \qquad \forall\, j \tag{2.8}$$

for some $x_j \in \Omega^1_{t_j}$ such that $|x_j - p^1_{t_j}| < \frac{L}{H(x_j,t_j)}$. Rescale the flow around (x_j, t_j) by $Q_j := H(x_j, t_j)$. By the global convergence Theorem 1.12 in [13], the sequence of rescaled flows subconverges uniformly on compact sets to an ancient non-collapsed solution. Points x_j get translated to the origin, and points $p^1_{t_j}$ get translated to points $\tilde{p}^1_{t_j}$ under rescaling. Since by our assumption we have

$$|0 - \tilde{p}^1_{t_j}| = H(x_j, t_j)\, |x_j - p^1_{t_j}| < L,$$

then due to uniform convergence of the rescaled flow on bounded sets we have

$$H_j(\tilde{p}^1_{t_j}, 0) \le C, \qquad j \ge j_0$$

for a uniform constant $C < \infty$, which depends on L, but is independent of j. This implies

$$H(p^1_{t_j}, t_j) \le C\, H(x_j, t_j), \qquad j \ge j_0,$$

which contradicts (2.8).

To prove the upper bound in (2.7) note that the lower bound in (2.7), which we have just proved, implies $|x - p^1_t| \le \frac{L}{H(x,t)} \le \frac{L}{c\, H(p^1_t, t)}$. Hence, we can switch the roles of x and p^1_t in the proof above. This ends the proof of the lemma. □

Remark 2.11. Note that we can choose uniform $c > 0$ and $t_0 \ll -1$ so that the conclusion of Lemma 2.10 holds for both $k = 1$ and $k = 2$.

Let $\epsilon > 0$ be a small number. By our assumption the flow is α-non-collapsed and uniformly two-convex, meaning that (1.3) holds. By the cylindrical estimate ([13], [15]) we can find an $\eta = \eta(\epsilon, \alpha, \beta) > 0$ so that if the flow is defined in the normalized parabolic cylinder $\hat{\mathcal{P}}(x, t, \eta^{-1})$ and if

$$\frac{\lambda_1}{H}(x, t) < \eta,$$

then the flow M_t is ϵ-close to a shrinking round cylinder $S^{n-1} \times \mathbb{R}$ near (x, t). Being ϵ-close to a shrinking round cylinder near (x, t) means that after parabolic rescaling by $H(x, t)$, shifting (x, t) to $(0, 0)$ and a rotation, the solution becomes ϵ-close in the $C^{[\frac{1}{\epsilon}]}$-norm on $\mathcal{P}(0, 0, 1/\epsilon)$ to the standard shrinking cylinder with $H(0, 0) = 1$. (See [13] for more details.)

PROPOSITION 2.12. *Fix a $k \in \{1, 2\}$, and let $L > 0$ be any fixed constant. Let M_t be an Ancient Oval that satisfies the assumptions of Theorem 1.6. Then for any sequence of times $t_j \to -\infty$, and any sequence of points $x_j \in \Omega^k_{t_j}$ such that $|x_j - p^k_{t_j}| \le \frac{L}{H(x_j, t_j)}$, the rescaled limit around (x_j, t_j) by factors $H(x_j, t_j)$ subconverges to a Bowl soliton.*

In the course of proving this proposition we need the following observation.

LEMMA 2.13. *For all $t \ll 0$, each of the two components Ω^j_t of $M_t \setminus B(0, \sqrt{8(n-1)|t|})$ contains at least one point at which $\lambda_{\min}$ is not a simple eigenvalue.*

Proof. Suppose $\lambda_{\min}$ is a simple eigenvalue at each point on Ω_t^1. Then the corresponding eigenspace defines a one dimensional subbundle of the tangent bundle TM_t. Since M_t is simply connected, any one dimensional bundle over M_t is trivial and thus has a section $v : M_t \to TM_t$ with $v(p) \neq p$ for all p. Within the region $\bar{B}(0, \sqrt{8(n-1)|t|})$ the hypersurfaces M_t are asymptotic in a C^2 sense to a cylinder with radius $\sqrt{2(n-1)|t|}$ (which simply follows from the fact that within the region $B(0, \sqrt{8(n-1)})$, the rescaled hypersurfaces $\tilde{M}_\tau = \frac{M_t}{\sqrt{|t|}}$, converge, as $\tau \to -\infty$ in a C^2 sense to a cylinder with radius $\sqrt{2(n-1)}$) so within this region $\lambda_{\min}$ is a simple eigenvalue, and the eigenvector $v(p)$ will be transverse to the boundary $\partial\Omega_t^1$. We may assume that it points outward relative to Ω_t^1.

The component Ω_t^1 is diffeomorphic with the unit ball $B^n \subset \mathbb{R}^n$, and under this diffeomorphism the vector field $v : \Omega_t^1 \to T\Omega_t^1$ is mapped to non-zero vector field $\tilde{v} : B^n \to \mathbb{R}^n$, which points outward on the boundary $S^{n-1} = \partial B^n$. The normalized map $\hat{v} = \tilde{v}/|\tilde{v}| : S^{n-1} \to S^{n-1}$ is therefore homotopic to the unit normal, i.e., the identity map $\mathrm{id} : S^{n-1} \to S^{n-1}$. Its degree must then equal $+1$, which is impossible because $\hat{v}$ can be extended continuously to $\hat{v} = \tilde{v}/|\tilde{v}| : B^n \to S^{n-1}$. □

Proof of Proposition 2.12. Without any loss of generality take $k = 1$, and let $\tilde{L} > 0$ be an arbitrary fixed constant. Let $t_j \to -\infty$ be an arbitrary sequence of times, and let $x_j \in \Omega_{t_j}^1$ be an arbitrary sequence of points such that $|x_j - p_{t_j}^1| \leq \frac{\tilde{L}}{H(x_j,t_j)}$. Rescale our solution around (x_j, t_j) by scaling factors $H(x_j, t_j)$. By Lemma 2.8 we know that the sequence of our rescaled solutions subconverges to either a Bowl soliton or a round shrinking cylinder. If the limit is a Bowl soliton, we are done. Hence, assume the limit is a shrinking round cylinder, which is a situation we want to rule out. By Lemma 2.10 we have that for j large enough, the curvatures $H(p_{t_j}^1, t_j)$ and $H(x_j, t_j)$ are uniformly equivalent. This together with $|x_j - p_{t_j}^1| \leq \frac{\tilde{L}}{H(x_j,t_j)}$ implies that if we rescale our solution around points $(p_{t_j}^1, t_j)$ by factors $H(p_{t_j}^1, t_j)$, after taking a limit we also get a shrinking round cylinder.

Since the limit around $(p_{t_j}^1, t_j)$ is a round shrinking cylinder, for every $\epsilon > 0$ there exists a j_0 so that for $j \geq j_0$, we have $\frac{\lambda_{\min}(p_{t_j}^1,t_j)}{H(p_{t_j}^1,t_j)} < \epsilon$. In the following two claims, $p_{t_j}^1 \in \Omega_{t_j}^1$ will be a sequence of the tip points as above, such that the limit of the sequence of rescaled solutions around $(p_{t_j}^1, t_j)$ by factors $H(p_{t_j}^1, t_j)$ is a shrinking round cylinder.

In the first claim we show that the ratio $\frac{\lambda_{\min}}{H}$ can be made arbitrarily small not only at points $p_{t_j}^1$, but also at all the points that are at bounded distances away from them.

For every $\epsilon > 0$ and every $C_0 > 0$, there exists a j_0 so that for $j \geq j_0$, we have

$$\frac{\lambda_{\min}(p, t_j)}{H(p, t_j)} < \epsilon, \quad \text{whenever} \quad |p - p^1_{t_j}| \leq \frac{C_0}{H(p, t_j)} \quad \text{and} \quad p \in \Omega^1_{t_j}. \tag{2.9}$$

Proof of the claim. Assume the claim is not true, meaning there exist constants $\epsilon > 0$, $C_0 > 0$, a subsequence we still denote by t_j, and points $p_j \in \Omega^1_{t_j}$ so that

$$|p_j - p^1_{t_j}| \leq \frac{C_0}{H(p_j, t_j)} \qquad \text{but} \qquad \frac{\lambda_{\min}(p_j, t_j)}{H(p_j, t_j)} \geq \epsilon. \tag{2.10}$$

Consider the sequence of rescaled flows around (p_j, t_j) by factors $H(p_j, t_j)$. Lemma 2.8 and the second inequality in (2.10) imply the above sequence subconverges to a Bowl soliton. On the other hand, since $|p_j - p^1_{t_j}| \leq \frac{C_0}{H(p_j, t_j)}$, by Lemma 2.10, the curvatures $H(p_j, t_j)$ and $H(p^1_{t_j}, t_j)$ are uniformly equivalent.

At the same time, this together with our assumption on $(p^1_{t_j}, t_j)$ and the first inequality in (2.10) imply that the sequence, after rescaling around (p_j, t_j) by factors $H(p_j, t_j)$, subconverges to a round shrinking cylinder. Hence, we get a contradiction. This proves the claim. □

Next we claim that for sufficiently big j, even far away from the tip points $p^1_{t_j}$ we see the cylindrical behavior. Assume $\bar{L}$ is big enough so that the conclusion of Lemma 2.9 holds. The immediate consequence of the Lemma 2.9 is that for every $\epsilon > 0$, there exists a j_0 so that for $j \geq j_0$, we have

$$\frac{\lambda_{\min}(p, t_j)}{H(p, t_j)} < \epsilon, \quad \text{whenever} \quad p \in \Omega^1_{t_j} \quad \text{and} \quad |p - p^1_{t_j}| \geq \frac{\bar{L}}{H(p, t_j)}. \tag{2.11}$$

We now continue proving Proposition 2.12. Estimates (2.9), after taking $C_0 = \bar{L}$ and (2.11), yield for every $\epsilon > 0$ that there exists a j_0 so that for $j \geq j_0$,

$$\frac{\lambda_{\min}}{H}(p, t_j) < \epsilon, \quad \text{on all of} \quad \Omega^1_{t_j}. \tag{2.12}$$

By the cylindrical estimate ([15], [13]) we have that for every $\epsilon > 0$ there exists a j_0 so that for $j \geq j_0$.

$$\frac{|\lambda_p - \lambda_q|}{H}(p, t_j) < \epsilon \quad \text{for all} \;\; n \geq p, q \geq 2, \quad \text{on } \Omega^1_{t_j}. \tag{2.13}$$

For small enough $\epsilon > 0$, the conditions (2.12) and (2.13) imply that $\lambda_{\min}$ is a simple eigenvalue, hence contradicting Lemma 2.13. This finishes the proof of Proposition 2.12. □

LEMMA 2.14. *Let M_t, $t \in (-\infty, 0)$, be an Ancient Oval satisfying the assumptions of Theorem 1.6, and fix $k = 1, 2$. Then for every $\epsilon > 0$, there exist uniform constants $\rho_0 < \infty$ and $t_0 \ll -1$ so that for every $t \le t_0$, we have that $\hat{\mathcal{P}}(p^k_t, t, \rho_0, \rho_0^2)$ is ϵ-close to a piece of a Bowl soliton that includes the tip.*

Proof. First of all observe that by Proposition 2.12 it is easy to argue that for every $\epsilon > 0$ and any $\rho_0 < \infty$, there exists a $t_0 \ll -1$ so that for $t \le t_0$, the parabolic cylinder $\hat{\mathcal{P}}(p^k_t, t, \rho_0, \rho_0^2)$ is ϵ-close to a piece of a Bowl soliton. The point of this lemma is to show that we can find ρ_0 big enough, but uniform in $t \le t_0 \ll -1$ so that the piece of the Bowl soliton above includes the tip.

To prove the statement we argue by contradiction. Assume the statement is not true, meaning there exist an $\epsilon > 0$, a sequence $\rho_j \to \infty$ and a sequence $t_j \to -\infty$ so that $\hat{\mathcal{P}}(p^k_{t_j}, t_j, \rho_j, \rho_j^2)$ is ϵ-close to a piece of Bowl soliton that does not include the tip. Rescale the solution around $(p^k_{t_j}, t_j)$ by factors $H(p^k_{t_j}, t_j)$. By Proposition 2.12 we know that the rescaled solution subconverges to a piece of a Bowl soliton. Hence there exists a uniform constant C_0 so that the origin that lies on the limiting Bowl soliton and corresponds after scaling, to the points $(p^k_{t_j}, t_j)$, is at distance C_0 from the tip of the soliton (which is the point of maximum curvature). This implies that there exist points $q_{t_j} \in \Omega^k_{t_j}$ so that $|q_{t_j} - p^k_{t_j}| \le \frac{2C_0}{H(p^k_{t_j}, t_j)}$ for $j \ge j_0$, with the property that the points q_{t_j} converge to the tip of the Bowl soliton. Furthermore, for sufficiently big $j \ge j_0$, parabolic cylinders $\hat{\mathcal{P}}(p^k_{t_j}, t_j, 3C_0, 9C_0^2)$ are ϵ-close to a piece of the Bowl soliton that includes the tip. This contradicts our assumption that for every j, $\hat{\mathcal{P}}(p^k_{t_j}, t_j, \rho_j, \rho_j^2)$ is ϵ-close to a piece of Bowl soliton that does not include its tip. ∎

Finally we show the crucial, for our purposes, proposition below, which says that every point on M_t has a parabolic neighborhood of uniform size, around which it is either close to a Bowl soliton or to a round shrinking cylinder.

PROPOSITION 2.15. *Let M_t be an Ancient Oval that is uniformly two-convex. Let ϵ_0, ϵ_1, L_0, L_1 be the constants from Propositions 2.5 and 2.6, and let $\epsilon \le \min\{\epsilon_0, \epsilon_1\}$. Then, there exists $t_0 \ll -1$, depending on these constants, with the following property: for every $(\bar{x}, \bar{t})$ with $\bar{x} \in M_{\bar{t}}$ and $\bar{t} \le t_0$, either $\hat{\mathcal{P}}(\bar{x}, \bar{t}, L_0, L_0^2)$ lies on an $(\epsilon, 10)$-neck or every point in $\hat{\mathcal{P}}(\bar{x}, \bar{t}, L_1, L_1^2)$ is, after scaling by $H(\bar{x}, \bar{t})$, ϵ-close in the C^{20}-norm to a piece of a Bowl soliton that includes the tip.*

Proof. Recall that as a consequence of Hamilton's Harnack estimate [10], our ancient solution satisfies $H_t \ge 0$. This implies there exists a uniform constant C_0 so that

$$\max_{M_t} H(\cdot, t) \le C_0, \qquad t \le t_0. \tag{2.14}$$

Let $\bar\epsilon \ll \min\{\epsilon_0, \epsilon_1, L_0^{-1}\}$. For this $\bar\epsilon > 0$, find a $\delta = \delta(\bar\epsilon)$ as in Theorem 1.19 in [13] (see also [15] for the similar estimate) so that if

$$\frac{\lambda_{\min}}{H}(p,t) < \delta \tag{2.15}$$

and the flow is defined in $\hat{\mathcal{P}}(p,t,\delta^{-1})$, then the solution M_t is $\bar\epsilon$-close to a round cylinder around (p,t), in the sense that a rescaled flow by $H(p,t)$ around (p,t) is $\bar\epsilon$-close on $\mathcal{P}(0,0,\bar\epsilon^{-1})$ to a round cylinder with $H(0,0) = 1$. Take $\delta > 0$ as in (2.15). For this δ, choose $\bar L$ sufficiently big and $t_0 \ll -1$ so that Lemma 2.9 holds (after we take η in the lemma to be equal to δ).

Let $(\bar x, \bar t)$ be such that $\bar x \in M_{\bar t}$ and $\bar t < t_0$. Then either $\bar x \in M_{\bar t} \cap B_{8(n-1)\sqrt{|t|}}$, or $\bar x \in \Omega^1_{\bar t}$, or $\bar x \in \Omega^2_{\bar t}$. In the first case that has already been discussed above, for $-\bar t$ sufficiently large, we know that $M_{\bar t} \cap B_{16(n-1)\sqrt{|\bar t|}}$ is neck-like and hence there exists $t_0 \ll -1$ so that for $t \le t_0$,

$$\max_{M_{\bar t} \cap B_{16(n-1)\sqrt{|\bar t|}}} \frac{\lambda_{\min}}{H} < \delta,$$

where δ is as in (2.15). Thus every point $\bar x \in M_{\bar t} \cap B_{8(n-1)\sqrt{|t|}}$ has the property that every point in $\hat{\mathcal{P}}(\bar x, \bar t, L_0, L_0^2)$ lies at the center of an $(\epsilon, 10)$-neck.

We may assume from now on, with no loss of generality, that $\bar x \in \Omega^1_{\bar t}$, since the discussion for $\bar x \in \Omega^2_{\bar t}$ is equivalent. We either have $|\bar x - p^1_{\bar t}| \ge \frac{\bar L}{H(\bar x, \bar t)}$, or we have $|\bar x - p^1_{\bar t}| \le \frac{\bar L}{H(\bar x, \bar t)}$. In the first case, Lemma 2.9 gives that $\frac{\lambda_{\min}}{H}(\bar x, \bar t) < \delta$. As discussed above, the cylindrical estimate then implies that the rescaled flow $H(\bar x, \bar t)(F_{\bar t + H(\bar x, \bar t)^{-2} t} - \bar x)$ is $\bar\epsilon$-close to the round cylinder with $H(0,0) = 1$, in a parabolic cylinder $\mathcal{P}(0,0,\bar\epsilon^{-1})$. It is straightforward then to conclude that every point in the normalized cylinder $\hat{\mathcal{P}}(\bar x, \bar t, L_0, L_0^2)$ lies on an $(\epsilon, 10)$-neck, where we use that $L_0 \ll \bar\epsilon^{-1}$ and $\bar\epsilon \ll \epsilon$.

Assume now that $\bar x \in \Omega^1_{\bar t}$ and $|\bar x - p^1_{\bar t}| \le \frac{\bar L}{H(\bar x, \bar t)}$. Combining this with Lemmas 2.10 and 2.14 yield we can find a sufficiently large but uniform constant L_1 and constant $t_0 \ll -1$ so that for $\bar t \le t_0$, we have that $\hat{\mathcal{P}}(\bar x, \bar t, L_1, L_1^2)$ is ϵ_1-close to a piece of a Bowl soliton that also includes its tip. □

We can now conclude the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $L_0, L_1, \epsilon_0, \epsilon_1$ be chosen so that Propositions 2.5 and 2.6 hold. Let $\bar\epsilon \ll \epsilon := \min(\epsilon_0, \epsilon_1)$. Let $t_0 \ll -1$ be as in Proposition 2.15 so that for every $(\bar x, \bar t)$ with $\bar x \in M_{\bar t}$ and $\bar t \le t_0$, either $\hat P(\bar x, \bar t, L_0)$ lies on an $(\bar\epsilon, 10)$-neck (and hence on an $(\epsilon_0, 10)$-neck, since $\bar\epsilon \le \epsilon_0$), or every point in $\hat P(\bar x, \bar t, L_1)$ is, after scaling, $\bar\epsilon$-close in the C^{20}-norm to a piece of the Bowl soliton that includes the tip (and hence is also ϵ_1 close, since $\bar\epsilon \le \epsilon_1$). Note that the axis of symmetry of this Bowl soliton may depend on the point $(\bar x, \bar t)$.

The above implies that every point $(\bar{x}, \bar{t})$, for $\bar{x} \in M_{\bar{t}}$ and $\bar{t} \le t_0$, lies in a parabolic neighborhood of uniform size (after scaling) $\bar{\epsilon}$ close to a rotationally symmetric surface (either a round cylinder or a Bowl soliton). Hence, it follows that if we choose $\bar{\epsilon}$ sufficiently small relative to ϵ, then $(\bar{x}, \bar{t})$ is ϵ-symmetric (defined as in Definition 2.3). After applying Propositions 2.5 and 2.6 we then conclude that $(\bar{x}, \bar{t})$ is $\frac{\epsilon}{2}$-symmetric for all $\bar{x} \in M_{\bar{t}}$ and all $\bar{t} \le T$. Iterative application of Propositions 2.5 and 2.6 yields that $(\bar{x}, \bar{t})$ is $\frac{\epsilon}{2^j}$-symmetric for all $\bar{x} \in M_{\bar{t}}$, $\bar{t} \le t_0$ and all $j \ge 1$. Letting $j \to +\infty$ we finally conclude that M_t is rotationally symmetric for all $t \le t_0$, which also implies that M_t is rotationally symmetric for all $t \in (-\infty, 0)$. □

3. Outline of the proof of Theorem 1.7

Since the proof of Theorem 1.7 is quite involved, in this preliminary section we will give an outline of the main steps in the proof of the classification result in the presence of rotational symmetry. Our method is based on a priori estimates for various distance functions between two given ancient solutions in appropriate coordinates and measured in weighted L^2 norms. We need to consider two different regions: the *cylindrical* region and the *tip* region. Note that the tip region will be divided in two sub-regions: the *collar* and the *soliton* region. These are pictured in Figure 1 below. In what follows, we will define these regions, review the equations in each region and define appropriate weighted L^2 norms with respect to which we will prove coercive type estimates in the subsequent sections. At the end of the section we will give an outline of the proof of Theorem 1.7.

Let $M_1(t), M_2(t)$ be two rotationally symmetric ancient oval solutions satisfying the assumptions of Theorem 1.7. Being surfaces of rotation, they are each determined by a function $U = U_i(x,t)$, $(i = 1, 2)$, which satisfies the equation

$$U_t = \frac{U_{xx}}{1 + U_x^2} - \frac{n-1}{U}. \tag{3.1}$$

In the statement of Theorem 1.7 we claim the uniqueness of any two Ancient Ovals up to dilations and translations. In fact since equation (3.1) is invariant under translation in time, translation in space and also under parabolic dilations in space-time, each solution $M_i(t)$ gives rise to a three parameter family of solutions

$$M_i^{\alpha\beta\gamma}(t) = e^{\gamma/2}\, \Phi_\alpha(M_i(e^{-\gamma}(t - \beta))), \tag{3.2}$$

where Φ_α is a rigid motion that is just the translation of the hypersurface along x axis by value α. The theorem claims the following: *given two ancient oval solutions we can find α, β, γ and $t_0 \in \mathbb{R}$ such that*

$$M_1(t) = M_2^{\alpha\beta\gamma}(t) \qquad \textit{for } t \le t_0.$$

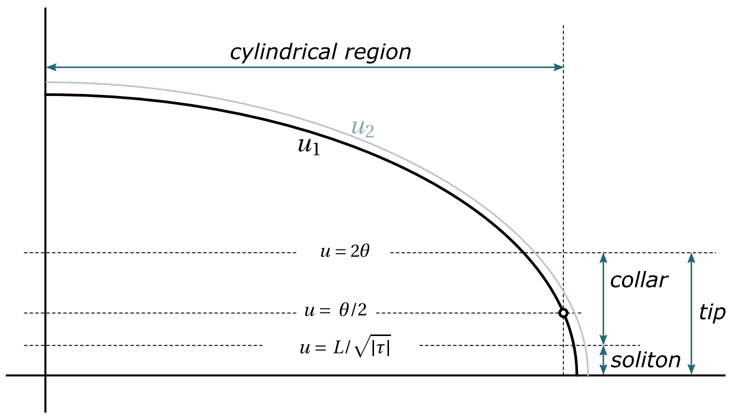


Figure 1. The three regions. The *cylindrical region* consists of all points with $u_1(y,\tau) \geq \theta/2$; the *tip region* contains all points with $u_1(y,\tau) \leq 2\theta$ and is subdivided into the *collar*, in which $u_1 \geq L/\sqrt{|\tau|}$, and the *soliton region*, where $u_1 \leq L/\sqrt{|\tau|}$.

The profile function $U_i^{\alpha\beta\gamma}$ corresponding to the modified solution $M_i^{\alpha\beta\gamma}(t)$ is given by

$$U_i^{\alpha\beta\gamma}(x,t) = e^{\gamma/2} U_i\Big(e^{-\gamma/2}(x-\alpha), e^{-\gamma}(t-\beta)\Big). \tag{3.3}$$

We rescale the solutions $M_i(t)$ by a factor $\sqrt{-t}$ and introduce a new time variable $\tau = -\log(-t)$; that is, we set

$$M_i(t) = \sqrt{-t}\,\bar{M}_i(\tau), \qquad \tau := -\log(-t). \tag{3.4}$$

These are again $O(n)$ symmetric with profile function u, which is related to U by

$$U(x,t) = \sqrt{-t}\,u(y,\tau), \qquad y = \frac{x}{\sqrt{-t}}, \quad \tau = -\log(-t). \tag{3.5}$$

If the U_i satisfy the MCF equation (3.1), then the rescaled profiles u_i satisfy (1.7), i.e.,

$$\frac{\partial u}{\partial \tau} = \frac{u_{yy}}{1+u_y^2} - \frac{y}{2}\,u_y - \frac{n-1}{u} + \frac{u}{2}.$$

Translating and dilating the original solution $M_i(t)$ to $M_i^{\alpha\beta\gamma}(t)$ has the following effect on $u_i(y,\tau)$:

$$u_i^{\alpha\beta\gamma}(y,\tau) = \sqrt{1+\beta e^{\tau}}\,u_i\left(\frac{y-\alpha e^{\tau/2}}{\sqrt{1+\beta e^{\tau}}}, \tau + \gamma - \log\big(1+\beta e^{\tau}\big)\right). \tag{3.6}$$

To prove the uniqueness theorem we will look at the difference $U_1 - U_2^{\alpha\beta\gamma}$, or equivalently at $u_1 - u_2^{\alpha\beta\gamma}$. The parameters α, β, γ will be chosen so that the projections of $u_1 - u_2^{\alpha\beta\gamma}$ onto positive eigenspace (which is spanned by two independent eigenvectors) and zero eigenspace of the linearized operator $\mathcal{L}$ at the cylinder are equal to zero at time τ_0, which will be chosen sufficiently close to $-\infty$. Correspondingly, we denote the difference $U_1 - U_2^{\alpha\beta\gamma}$ by $U_1 - U_2$ and $u_1 - u_2^{\alpha\beta\gamma}$ by $u_1 - u_2$. What we will actually observe is that the parameters α, β and γ can be chosen to lie in a certain range, which allows our main estimates to hold without having to keep track of these parameters during the proof. In fact, we will show in Section 8 that for a given small $\epsilon > 0$, there exists $\tau_0 \ll -1$ sufficiently negative for which we have

$$|\alpha| \le \epsilon \frac{e^{-\tau_0/2}}{|\tau_0|}, \qquad |\beta| \le \epsilon \, \frac{e^{-\tau_0}}{|\tau_0|}, \qquad |\gamma| \le \epsilon \, |\tau_0|, \tag{3.7}$$

and our estimates hold for $(u_1 - u_2^{\alpha\beta\gamma})(\cdot, \tau)$, $\tau \le \tau_0$. This inspires the following definition.

Definition 3.1 (Admissible triple of parameters (α, β, γ)). We say that the triple of parameters (α, β, γ) is admissible with respect to time τ_0 if they satisfy (3.7).

We will next define different regions and outline how we treat each region.

3.1. *The cylindrical region.* For a given $\tau \le \tau_0$ and constant θ positive and small, the cylindrical region is defined by

$$\mathcal{C}_\theta = \Big\{ (y, \tau) : u_1(y, \tau) \ge \frac{\theta}{2} \Big\}$$

(see Figure 1). We will consider in this region a *cut-off function* $\varphi_{\mathcal{C}}(y, \tau)$ with the following properties:

$$\text{(i) } \operatorname{supp} \varphi_{\mathcal{C}} \Subset \mathcal{C}_\theta, \quad \text{(ii) } 0 \le \varphi_{\mathcal{C}} \le 1, \quad \text{(iii) } \varphi_{\mathcal{C}} \equiv 1 \text{ on } \mathcal{C}_{2\theta}.$$

The solutions u_i, $i = 1, 2$, satisfy equation (1.7). Setting

$$w := u_1 - u_2^{\alpha\beta\gamma} \qquad \text{and} \qquad w_{\mathcal{C}} := w \, \varphi_{\mathcal{C}},$$

we see that $w_{\mathcal{C}}$ satisfies the equation

$$\frac{\partial}{\partial \tau} w_{\mathcal{C}} = \mathcal{L}[w_{\mathcal{C}}] + \mathcal{E}[w, \varphi_{\mathcal{C}}], \tag{3.8}$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L} = \partial_y^2 - \frac{y}{2} \partial_y + 1 \tag{3.9}$$

and where the error term $\mathcal{E}$ is described in detail in Section 6. We will see that

$$\mathcal{E}[w, \varphi_{\mathcal{C}}] = \mathcal{E}(w_{\mathcal{C}}) + \bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}],$$

where $\mathcal{E}(w_{\mathcal{C}})$ is the error introduced due to the non-linearity of our equation and is given by (6.3) and $\bar{\mathcal{E}}[w,\varphi_{\mathcal{C}}]$ is the error introduced due to the cut-off function $\varphi_{\mathcal{C}}$ and is given by (6.11). (To simplify the notation we have set $u_2 := u_2^{\alpha\beta\gamma}$.)

The differential operator $\mathcal{L}$ is a well-studied self-adjoint operator on the Hilbert space $\mathfrak{H} := L^2(\mathbb{R}, e^{-y^2/4}dy)$ with respect to the norm and inner product

$$\|f\|_{\mathfrak{H}}^2 = \int_{\mathbb{R}} f(y)^2 e^{-y^2/4}\, dy, \qquad \langle f, g\rangle = \int_{\mathbb{R}} f(y)g(y)e^{-y^2/4}\, dy. \tag{3.10}$$

We split $\mathfrak{H}$ into the unstable, neutral, and stable subspaces $\mathfrak{H}_+$, $\mathfrak{H}_0$, and $\mathfrak{H}_-$, respectively. The unstable subspace $\mathfrak{H}_+$ is spanned by all eigenfunctions with positive eigenvalues. (In this case $\mathfrak{H}_+$ is spanned by a constant function equal to $\psi_0 = 1$, which corresponds to eigenvalue 1, and by a linear function $\psi_1 = y$, which corresponds to eigenvalue $\frac{1}{2}$; that is, $\mathfrak{H}_+$ is two dimensional.) The neutral subspace $\mathfrak{H}_0$ is the kernel of $\mathcal{L}$ and is the one dimensional space spanned by $\psi_2 = y^2 - 2$. The stable subspace $\mathfrak{H}_-$ is spanned by all other eigenfunctions. Let $\mathcal{P}_\pm$ and $\mathcal{P}_0$ be the orthogonal projections on $\mathfrak{H}_\pm$ and $\mathfrak{H}_0$.

For any function $f : \mathbb{R}\times(-\infty,\tau_0] \to \mathbb{R}$, we define the cylindrical norm

$$\|f\|_{\mathfrak{H},\infty}(\tau) = \sup_{\sigma\le\tau}\Big(\int_{\sigma-1}^{\sigma} \|f(\cdot,s)\|_{\mathfrak{H}}^2\, ds\Big)^{\frac{1}{2}}, \qquad \tau \le \tau_0,$$

and we will often simply set

$$\|f\|_{\mathfrak{H},\infty} := \|f\|_{\mathfrak{H},\infty}(\tau_0). \tag{3.11}$$

In the course of proving necessary estimates in the cylindrical region we define yet another Hilbert space $\mathfrak{D}$ by

$$\mathfrak{D} = \{f \in \mathfrak{H} : f, f_y \in \mathfrak{H}\},$$

equipped with a norm

$$\|f\|_{\mathfrak{D}}^2 = \int_{\mathbb{R}} \{f(y)^2 + f'(y)^2\} e^{-y^2/4} dy.$$

We will write

$$(f,g)_{\mathfrak{D}} = \int_{\mathbb{R}} \{f'(y)g'(y) + f(y)g(y)\} e^{-y^2/4} dy$$

for the inner product in $\mathfrak{D}$. Similarly as above define the parabolic norm

$$\|f\|_{\mathfrak{D},\infty}(\tau) = \sup_{\sigma\le\tau}\Big(\int_{\sigma-1}^{\sigma} \|f(\cdot,s)\|_{\mathfrak{D}}^2\, ds\Big)^{\frac{1}{2}},$$

and also set $\|f\|_{\mathfrak{D},\infty} = \|f\|_{\mathfrak{D},\infty}(\tau)$.

Denote by $\mathfrak{D}^*$ the dual of $\mathfrak{D}$. Since we have a dense inclusion $\mathfrak{D} \subset \mathfrak{H}$, we also get a dense inclusion $\mathfrak{H} \subset \mathfrak{D}^*$ where every $f \in \mathfrak{H}$ is interpreted as a functional on $\mathfrak{D}$ via

$$g \in \mathfrak{D} \mapsto \langle f, g \rangle.$$

Because of this we will also denote the duality between $\mathfrak{D}$ and $\mathfrak{D}^*$ by

$$(f, g) \in \mathfrak{D} \times \mathfrak{D}^* \mapsto \langle f, g \rangle.$$

Since $\mathfrak{H} \subset \mathfrak{D}^*$, for every $f \in \mathfrak{H}$ we define the dual norm as usual by

$$\|f\|_{\mathfrak{D}^*} := \sup\{\langle f, g\rangle \,:\, \|g\|_{\mathfrak{D}} \le 1\,\},$$

the *dual parabolic norm* by

$$\|f\|_{\mathfrak{D}^*,\infty}(\tau) = \sup_{\sigma \le \tau} \Big(\int_{\sigma-1}^{\sigma} \|f(\cdot, s)\|^2_{\mathfrak{D}^*}\, ds \Big)^{\frac{1}{2}},$$

and for simplicity we set $\|f\|_{\mathfrak{D}^*,\infty} := \|f\|_{\mathfrak{D}^*,\infty}(\tau_0)$.

In Section 6 we will show a coercive estimate for $w_{\mathcal{C}}$ in terms of the error $E[w, \varphi_{\mathcal{C}}]$. However, as expected, this can only be achieved by removing the projection $\mathcal{P}_0 w_{\mathcal{C}}$ onto the kernel of $\mathcal{L}$, generated by ψ_2. More precisely, setting

$$\hat{w}_{\mathcal{C}} := \mathcal{P}_+ w_{\mathcal{C}} + \mathcal{P}_- w_{\mathcal{C}} = w_{\mathcal{C}} - \mathcal{P}_0 w_{\mathcal{C}},$$

we will prove that for any $\epsilon > 0$, there exist $\theta > 0$ and $\tau_0 \ll 0$ such that the following bound holds:

$$\|\hat{w}_{\mathcal{C}}\|_{\mathfrak{D},\infty} \le C\, \|E[w, \varphi_{\mathcal{C}}]\|_{\mathfrak{D}^*,\infty}, \tag{3.12}$$

provided $\mathcal{P}_+ w_{\mathcal{C}}(\tau_0) = 0$. This estimate is a consequence of the parabolic equation (3.8). In fact, we will show in Proposition 4.1 that the parameters α, β and γ can be adjusted so that for $w^{\alpha\beta\gamma} := u_1 - u_2^{\alpha\beta\gamma}$, we have

$$\mathcal{P}_+ w_{\mathcal{C}}(\tau_0) = 0 \qquad \text{and} \qquad \mathcal{P}_0 w_{\mathcal{C}}(\tau_0) = 0. \tag{3.13}$$

Thus (3.12) will hold for such a choice of α, β, γ and $\tau_0 \ll 0$. The condition $\mathcal{P}_0 w_{\mathcal{C}}(\tau_0) = 0$ is essential and will be used in Section 8 to give us that $w^{\alpha\beta\gamma} \equiv 0$. In addition, we will show in Proposition 4.1 that α, β and γ can be chosen to be admissible according to our Definition 3.1.

The norm of the error term $\|E[w, \varphi_{\mathcal{C}}]\|_{\mathfrak{D}^*,\infty}$ on the right-hand side of (3.12) will be estimated in Section 6, Lemmas 6.8 and 6.9. We will show that given $\epsilon > 0$ small, there exists a $\tau_0 \ll -1$ such that

$$\|E[w, \varphi_{\mathcal{C}}]\|_{\mathfrak{D}^*,\infty} \le \epsilon\, \big(\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty} + \|w\, \chi_{D_\theta}\|_{\mathfrak{D},\infty}\big), \tag{3.14}$$

where $D_\theta := \{(y, \tau) :\, \frac{\theta}{2} \le u_1(y, \tau) \le \theta\}$ contains the support of the derivative of $\varphi_{\mathcal{C}}$. Combining (3.12) and (3.14) yields the bound

$$\|\hat{w}_{\mathcal{C}}\|_{\mathfrak{D},\infty} \le \epsilon\, (\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty} + \|w\, \chi_{D_\theta}\|_{\mathfrak{H},\infty}), \tag{3.15}$$

holding for all $\epsilon > 0$ and $\tau_0 := \tau_0(\epsilon) \ll -1$.

To close the argument we need to estimate $\|w\chi_{D_\theta}\|_{\mathfrak{H},\infty}$ in terms of $\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty}$. This will be done by considering the tip region and establishing an appropriate a priori bound for the difference of our two solutions there.

3.2. *The tip region.* The tip region is defined by

$$\mathcal{T}_\theta = \{(u,\tau) : \; u_1 \le 2\theta, \, \tau \le \tau_0\}$$

(see Figure 1). In the tip region we switch the variables y and u in our two solutions, with u now becoming an independent variable. Hence, our solutions $u_1(y,\tau)$ and $u_2^{\alpha\beta\gamma}(y,\tau)$ become $Y_1(u,\tau)$ and $Y_2^{\alpha\beta\gamma}(u,\tau)$, where

$$u_j(Y_j(u,\tau),\tau) \equiv u.$$

Both functions $Y_1(u,\tau), Y_2^{\alpha\beta\gamma}$ satisfy the equation

$$Y_\tau = \frac{Y_{uu}}{1+Y_u^2} + \frac{n-1}{u} Y_u + \frac{1}{2}\big(Y - uY_u\big). \tag{3.16}$$

By inverting the definition (3.6) of $u^{\alpha\beta\gamma}$, we find that the transformed $Y_2^{\alpha\beta\gamma}$ and Y_2 are related by

$$Y_2^{\alpha\beta\gamma}(y,\tau) = \alpha e^{\tau/2} + \sqrt{1+\beta e^\tau}\, Y_2\Big(\frac{u}{\sqrt{1+\beta e^\tau}}, \tau + \gamma - \log(1+\beta e^\tau)\Big). \tag{3.17}$$

It follows from (3.16) that the difference $W := Y_1 - Y_2^{\alpha\beta\gamma}$ satisfies

$$W_\tau = \frac{W_{uu}}{1+Y_{1u}^2} + \Big(\frac{n-1}{u} - \frac{u}{2} + D\Big)\, W_u + \frac{1}{2} W, \tag{3.18}$$

where

$$D := -\frac{Y_{2,uu}^{\alpha\beta\gamma}\,(Y_{1,u} + Y_{2,u}^{\alpha\beta\gamma})}{(1+(Y_{1,u})^2)\,(1+(Y_{2,u}^{\alpha\beta\gamma})^2)}.$$

Our next goal is to define an appropriate weighted L^2 norm

$$\|W(\tau)\|^2 := \int_0^\theta W^2(u,\tau)\, e^{\mu(u,\tau)}\, du$$

in the tip region $\mathcal{T}_\theta$, by defining the weight $\mu(u,\tau)$. To this end we need to further distinguish between two regions in $\mathcal{T}_\theta$: for $L > 0$ sufficiently large to be determined in Section 7, we define the *collar* region to be the set

$$\mathcal{K}_{\theta,L} := \Big\{ y \mid \frac{L}{\sqrt{|\tau|}} \le u_1(y,\tau) \le 2\theta \Big\}$$

and the *soliton* region to be the set

$$\mathcal{S}_L := \Big\{ y \mid 0 \le u_1(y,\tau) \le \frac{L}{\sqrt{|\tau|}} \Big\}$$

(see Figure 1).

The soliton region is the set where our asymptotic result in Theorem 1.8 implies that the solutions Y_1 and Y_2 are very close to the Bowl soliton (after rescaling). To see this, we consider in the soliton region the change of variables

$$Y_i(u,\tau) = Y_i(0,\tau) + \frac{1}{\sqrt{|\tau|}}\, Z_i(\rho,\tau), \qquad \rho := u\,\sqrt{|\tau|} \tag{3.19}$$

for each of our two solutions Y_1 and Y_2. The asymptotic description in Theorem 1.8 (see also Corollary 5.1) implies that as $\tau \to -\infty$, both $Z_1(\rho,\tau)$ and $Z_2(\rho,\tau)$ converge to the unique rotationally symmetric translating Bowl solution $Z_0(\rho)$ with speed $\sqrt{2}/2$. The Bowl profile Z_0 is the unique solution of

$$\frac{Z_{0\rho\rho}}{1+Z_{0\rho}^2} + \frac{n-1}{\rho} Z_{0\rho} + \frac{1}{2}\sqrt{2} = 0, \qquad Z_0(0) = Z_0'(0) = 0. \tag{3.20}$$

For large and small ρ, the function $Z_0(\rho)$ satisfies

$$Z_0(\rho) = \begin{cases} -\sqrt{2}\rho^2/4(n-1) + \mathcal{O}(\log\rho) & \rho \to \infty, \\ -\sqrt{2}\rho^2/4n + \mathcal{O}(\rho^4) & \rho \to 0. \end{cases} \tag{3.21}$$

These expansions may be differentiated (see [4]).

The *collar region* is the *transition region* between the cylindrical and the tip regions. To deal with this region we need to refine our asymptotics in Theorem 1.8. A crucial part on this is played by the fact that for each profile $u(y,\tau)$, we have $(u^2)_{yy} \le 0$, namely, that $u^2(y,\tau)$ is a concave function in y. This will be shown in Proposition 5.2. A consequence of this fact is the estimate in Lemma 5.7 that implies one may regard the term D in (3.18) as an error term in $\mathcal{K}_{\theta,L}$ (since in this region D can be made arbitrarily small for $\tau_0 \ll -1$ and in addition by choosing L, θ appropriately).

Let us next define our *weight* $e^{\mu(u,\tau)}\,du$ *in the tip region* by choosing $\mu(u,\tau)$ so that it smoothly interpolates between

$$\mu(u,\tau) = -\frac{1}{4} Y_1^2(u,\tau)$$

for $u \ge \theta/2$ and

$$\mu(u,\tau) = -\frac{1}{4} Y_1^2(\theta,\tau) + \int_\theta^u \frac{n-1}{u}\left(1 + Y_{1,u}^2(u,\tau)\right) du$$

in the region $u \le \theta/4$. See (7.2) and (7.3) for the precise definitions. In the region $u \ge \theta/2$ this weight matches the Gaussian weight $e^{-y^2/4}dy$ that we use in the cylindrical region up to lower order factors.

For a function $W : [0, 2\theta] \times (-\infty, \tau_0] \to \mathbb{R}$, we define *the parabolic norms*

$$\|W\|_{2,\infty,\tau} = \sup_{\tau' \le \tau} \frac{1}{|\tau'|^{1/4}} \left[\int_{\tau'-1}^{\tau'} \int_0^{2\theta} W^2(u,\tau)\, e^{\mu(u,\tau)}\, du\, ds \right]^{1/2} \tag{3.22}$$

for any $\tau \le \tau_0$. We include the weight in time $|\tau|^{-1/4}$ to make the norms equivalent in the transition region between the cylindrical and the tip region, as will become apparent in Lemma 8.1. This is due to different rescalings in the two regions. We will also abbreviate

$$\|W\|_{2,\infty} := \|W\|_{2,\infty,\tau_0}. \tag{3.23}$$

Our main estimates in the tip region apply to a localized version of the difference W. To localize W to the tip region we choose a *cut-off function* $\varphi_T(u)$ with the following properties:

$$\text{(i) } \operatorname{supp}\varphi_T \Subset \mathcal{T}_\theta, \qquad \text{(ii) } 0 \le \varphi_T \le 1, \qquad \text{(iii) } \varphi_T \equiv 1, \text{ on } \mathcal{T}_{\theta/2} \tag{3.24}$$

and define

$$W = Y_1 - Y_2^{\alpha\beta\gamma} \qquad \text{and} \qquad W_T(u,\tau) := W(u,\tau)\,\varphi_T. \tag{3.25}$$

We will see in Section 7 that the following bound holds in the tip region:

$$\|W_T\|_{2,\infty} \le \frac{C}{|\tau_0|}\,\|W\chi_{[\theta,2\theta]}\|_{2,\infty}, \tag{3.26}$$

where $\chi_{[\theta,2\theta]}$ is the characteristic function of the interval $[\theta, 2\theta]$.

3.3. *The conclusion.* The statement of Theorem 1.7 is equivalent to showing that there exist parameters α, β and γ so that $u_1(y,\tau) = u_2^{\alpha\beta\gamma}(y,\tau)$, where $u_2^{\alpha\beta\gamma}(y,\tau)$ is defined by (3.6) and both functions, $u_1(y,\tau)$ and $u_2^{\alpha\beta\gamma}(y,\tau)$, satisfy equation (1.7). We set $w := u_1 - u_2^{\alpha\beta\gamma}$, where (α,β,γ) is an admissible triple of parameters with respect to τ_0, such that (3.13) holds for some $\tau_0 \ll -1$. Now for this τ_0, the main estimates in each of the regions, namely, (3.15) and (3.26) hold for w. Next, we want to combine (3.15) and (3.26). To this end we need to show that the norms of the difference of our two solutions, with respect to the weights defined in the cylindrical and the tip regions, are equivalent in the intersection between the cylindrical and the tip regions, the so-called *transition region*. More precisely, we will show in Section 8 that for every $\theta > 0$ small, there exist $\tau_0 \ll 0$ and uniform constants $c(\theta), C(\theta) > 0$, so that for $\tau \le \tau_0$, we have

$$c(\theta)\,\|W\chi_{[\theta,2\theta]}\|_{\mathfrak{H},\infty} \le \|w\,\chi_{\mathcal{D}_{2\theta}}\|_{\mathfrak{H},\infty} \le C(\theta)\,\|W\chi_{[\theta,2\theta]}\|_{\mathfrak{H},\infty}, \tag{3.27}$$

where $\mathcal{D}_{2\theta} := \{y \mid \theta \le u_1(y,\tau) \le 2\theta\}$ and $\chi_{[\theta,2\theta]}$ is the characteristic function of the interval $[\theta, 2\theta]$.

Combining (3.27) with (3.15) and (3.26) finally shows that in the norm $\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty}$, what actually dominates is $\|\mathcal{P}_0 w_{\mathcal{C}}\|_{\mathfrak{D},\infty}$. We will use this fact in Section 8 to conclude that $w(y,\tau) := w^{\alpha\beta\gamma}(y,\tau) \equiv 0$ for our choice of parameters α, β and γ. To do so we will look at the projection $a(\tau) := \mathcal{P}_0 w_{\mathcal{C}}$ and consider the norm

$$\|a\|_{\mathfrak{H},\infty}(\tau) = \sup_{\sigma\le\tau}\Big(\int_{\sigma-1}^{\sigma} \|a(s)\|^2\, ds\Big)^{\frac{1}{2}}, \qquad \tau \le \tau_0$$

with $\|a\|_{\mathfrak{H},\infty} := \|a\|_{\mathfrak{H},\infty}(\tau_0)$.

By projecting equation (3.8) onto the zero eigenspace spanned by ψ_2 and estimating error terms by $\|a\|_{\mathfrak{H},\infty}$ itself, we will conclude in Section 8 that $a(\tau)$ satisfies a certain differential inequality, which combined with our assumption that $a(\tau_0) = 0$ (which follows from the choice of parameters α, β, γ so that (3.13) hold) will yield that $a(\tau) = 0$ for all $\tau \le \tau_0$. On the other hand, since $\|a\|_{\mathfrak{H},\infty}$ dominates the $\|w_{\mathcal{C}}\|_{\mathfrak{H},\infty}$, this will imply that $w_{\mathcal{C}} \equiv 0$, thus yielding $w \equiv 0$, as stated in Theorem 1.7.

Remark 3.2. Note that our evolving hypersurface has $O(n)$ symmetry and can be represented as in (1.6). Due to asymptotics proved in Theorem 1.8, when considering the tip region, it is enough to consider our solutions and prove the estimates only around $y = \bar{d}_1(\tau)$, where after switching the variables as in (3.19), we have $\rho \ge 0$. There we have $Z(\rho,\tau) \le 0$ and $Z_\rho \le 0$. We also have $Z_{\rho\rho} \le 0$, due to our convexity assumption. The estimates around $y = -\bar{d}_2(\tau)$ are similar.

4. **Choice of parameters**

Recall that the zero eigenspace of $\mathcal{L}$ defined by (3.9) is spanned by the function $\psi_2(y) = y^2 - 2$ and the positive eigenspace is spanned by the eigenvectors $\psi_0(y) = 1$ (corresponding to eigenvalue 1) and $\psi_1(y) = y$ (corresponding to eigenvalue 1/2).

To prove Theorem 1.7 it turns out it is essential for our proof to have

$$\mathcal{P}_+ w_{\mathcal{C}}^{\alpha\beta\gamma}(\tau_0) = \mathcal{P}_0 w_{\mathcal{C}}^{\alpha\beta\gamma}(\tau_0) = 0. \tag{4.1}$$

We will next show that for every $\tau_0 \ll -1$, we can find parameters $\alpha = \alpha(\tau_0)$, $\beta = \beta(\tau_0)$ and $\gamma = \gamma(\tau_0)$ such that (4.1) holds, and we will also give their asymptotics relative to τ_0. Let us emphasize that we need to be able *for every* $\tau_0 \ll -1$ to find parameters α, β, γ so that (4.1) holds, since up to the final step of our proof we have to keep adjusting τ_0 by taking it even more negative so that our estimates hold. More precisely, we have the following result.

PROPOSITION 4.1. *There is a number $\tau_* \ll -1$ such that for all $\tau \le \tau_*$, there exist b, Γ and A such that the difference $w^{\alpha\beta\gamma} := u_1 - u_2^{\alpha\beta\gamma}$ satisfies*

$$\langle \psi_0, \varphi_{\mathcal{C}}\, w^{\alpha\beta\gamma}\rangle = \langle \psi_1, \varphi_{\mathcal{C}}\, w^{\alpha\beta\gamma}\rangle = \langle \psi_2, \varphi_{\mathcal{C}}\, w^{\alpha\beta\gamma}\rangle = 0.$$

In addition, the parameters α, β and γ can be chosen so that b, Γ and A defined in (4.5) satisfy

$$|b| = o\Big(|\tau|^{-1}\Big), \qquad |\Gamma| = o(1) \qquad \text{and} \qquad |A| = o(1), \qquad \text{as } \tau \to -\infty. \tag{4.2}$$

Equivalently, this means that the triple (α, β, γ) *is admissible with respect to* τ, *according to our Definition 3.1*.

For v_i related to u_i by $u_i = \sqrt{2(n-1)}(1+v_i)$, the corresponding dilations by (α, β, γ) are given by

$$v_i^{\alpha\beta\gamma}(y,\tau) = \sqrt{1+\beta\, e^{\tau}}\Big\{1 + v_i\Big(\frac{y - \alpha\, e^{\tau/2}}{\sqrt{1+\beta e^{\tau}}}, \tau + \gamma - \log\big(1+\beta e^{\tau}\big)\Big)\Big\} - 1.$$

Simply write v for v_1 and $\bar{v}$ for $v_2^{\alpha\beta\gamma}$.

Our asymptotics in Theorem 1.8 imply that each v_i satisfies the following estimates in the cylindrical region $\mathcal{C}_\theta$: for any $\epsilon_0 > 0$ and any number $M > 0$, there is a $\tau_{\epsilon_0,M} < 0$ such that

$$v_i(y,\tau) = -\frac{y^2-2}{4|\tau|} + \frac{\epsilon(y,\tau)}{|\tau|} \qquad \text{for } 0 \le y \le 2M,\ \tau \le \tau_{\epsilon_0,M}, \tag{4.3}$$

where $\epsilon(y,\tau)$ is a generic function whose definition may change from line to line, but that always satisfies

$$|\epsilon(y,\tau)| \le \epsilon_0 \qquad \text{for } 0 \le y \le 2M,\ \tau \le \tau_{\epsilon_0,M}. \tag{4.4}$$

We will next estimate the first three components of the truncated difference $\varphi_{\mathcal{C}}(\bar{v} - v)$,

$$\Big\langle \psi_j,\ \varphi_{\mathcal{C}}\,(\bar{v} - v)\Big\rangle \qquad (j = 0,1,2),$$

where $\varphi_{\mathcal{C}}$ is the cut-off function for the cylindrical region $\mathcal{C}_\theta$. We will show that the coefficients α, β and γ can be chosen so as to make these components vanish. Instead of working directly with α, β and γ it will be more convenient to use

$$b = \sqrt{1+\beta e^{\tau}} - 1, \qquad \Gamma = \frac{\gamma - \log(1+\beta e^{\tau})}{\tau}, \qquad A = \alpha\, e^{\tau/2}. \tag{4.5}$$

Then

$$\bar{v}(y,\tau) = b + (1+b)\, v_2\Big(\frac{y-A}{1+b}, (1+\Gamma)\tau\Big). \tag{4.6}$$

Our next goal is to show the following result.

The proof of the proposition will be based on the following estimate.

LEMMA 4.2. *For every* $\eta > 0$, *there exists* $\tau_\eta < 0$ *such that for all* $\tau \le \tau_\eta$, *and all* $b, \Gamma, A \in \mathbb{R}$ *with*

$$|b| \le \frac{1}{|\tau|}, \qquad |\Gamma| \le \frac{1}{2}, \qquad |A| \le 1,$$

one has

(4.7)

$$\Big|\langle\hat\psi_0, \varphi_{\mathcal C}(\bar v - v)\rangle - b + \frac{A^2}{4(\Gamma+1)|\tau|}\Big| + \Big|\langle\hat\psi_1, \varphi_{\mathcal C}(\bar v - v)\rangle - \frac{A}{2|\tau|(\Gamma+1)}\Big| + \Big|\langle\hat\psi_2, \varphi_{\mathcal C}(\bar v - v)\rangle - \frac{\Gamma}{4(\Gamma+1)|\tau|}\Big| \le \frac{\eta}{|\tau|},$$

where $\hat\psi_j = \psi_j/\langle\psi_j,\psi_j\rangle$.

The conditions on b, Γ and A are met if the original parameters α, β and γ satisfy

$$|\alpha\, e^{\tau/2}| \le 1, \qquad |\beta e^{\tau}| \le \frac{C}{|\tau|}, \qquad |\gamma| \le \tfrac13|\tau|.$$

Proof that Lemma 4.2 implies Proposition 4.1. Let $\eta_* > 0$ be given, and consider the disc

$$\mathcal B = \Big\{(b,\Gamma,A) \mid |\tau|^2 b^2 + \Gamma^2 + A^2 \le \eta_*^2\Big\}.$$

On this ball we define the map $\Phi : \mathcal B \to \mathbb R^3$ given by

$$\Phi(b,\Gamma) = \begin{pmatrix} |\tau|\langle\hat\psi_0, \varphi_{\mathcal C}(\bar v - v)\rangle \\ |\tau|\langle\hat\psi_1, \varphi_{\mathcal C}(\bar v - v)\rangle \\ |\tau|\langle\hat\psi_2, \varphi_{\mathcal C}(\bar v - v)\rangle \end{pmatrix}.$$

The map Φ is continuous because the solution $\bar v$ depends continuously on the parameters b, Γ, A.

It follows from (4.7) that if $\eta \ll \eta_*$ is chosen small enough, and if τ is restricted to $\tau < \tau_\eta$, with τ_η defined as in Lemma 4.2, then the map Φ restricted to the boundary of the ball $\mathcal B$ is homotopic to the injective map

$$(b,\Gamma,A) \mapsto \left(|\tau| b - \frac{A^2}{4(\Gamma+1)}, \frac{A}{2(\Gamma+1)}, \frac{\Gamma}{4(\Gamma+1)}\right),$$

through maps from $\partial\mathcal B$ to $\mathbb R^3 \setminus \{0\}$. The map Φ from the full ball to $\mathbb R^3$ therefore has degree one, and it follows that for some $(b',\Gamma',A') \in \mathcal B$, one has $\Phi(b',\Gamma',A') = 0$. From the definition of the disc $\mathcal B$ it follows that (b',Γ',A') satisfies (4.2). □

Proof of Lemma 4.2. By Lemma 5.14 in [3] it follows that our for any ancient solution u, we have

$$\left\|\left(u - \sqrt{2(n-1)} + \frac{\sqrt{2(n-1)}}{4|\tau|}\,\psi_2\right)\chi_{\mathrm{supp}(\varphi_{\mathcal C})}\right\| = o(|\tau|^{-1}),$$

where $\chi_{\mathrm{supp}(\varphi_{\mathcal{C}})}$ is the characteristic function of $\operatorname{supp}\varphi_{\mathcal{C}}$. Using this we have that on the support of $\varphi_{\mathcal{C}}$,

$$\begin{aligned}
&(\bar v(y,\tau)-v(y,\tau))\,\varphi_{\mathcal{C}}\\
&\quad= b+\frac{2b+b^2}{2(1+\Gamma)(1+b)}\frac{1}{|\tau|}\\
&\qquad+\Big\{1-\frac{1}{(1+b)}\frac{1}{(1+\Gamma)}\Big\}\frac{y^2-2}{4|\tau|}+\frac{2Ay-A^2}{4|\tau|(\Gamma+1)(b+1)}+\frac{5\epsilon(y,\tau)}{|\tau|}\\
&\quad= b+\frac{2b+b^2}{2(1+\Gamma)(1+b)}\frac{1}{|\tau|}\\
&\qquad+\frac{b+\Gamma+b\Gamma}{(1+\Gamma)(1+b)}\frac{y^2-2}{4|\tau|}+\frac{2Ay-A^2}{4|\tau|(\Gamma+1)(b+1)}+\frac{5\epsilon(y,\tau)}{|\tau|},
\end{aligned}$$

which holds in the L^2 sense, meaning that the $\int\epsilon(y,\tau)^2e^{-\frac{y^2}{4}}\,dy=o(1)$ as $\tau\to-\infty$. Given the assumptions on b, Γ and A in the statement of Lemma 4.2 we can rewrite this as

(4.8)

$$(\bar v(y,\tau)-v(y,\tau))\,\varphi_{\mathcal{C}}=b-\frac{A^2}{4(\Gamma+1)|\tau|}+\frac{\Gamma}{\Gamma+1}\frac{(y^2-2)}{4|\tau|}+\frac{Ay}{2|\tau|(\Gamma+1)}+R(y,\tau),$$

holding in the L^2-sense, where the remainder R satisfies

$$\Big(\int R(y,\tau)^2\,e^{-\frac{y^2}{4}}\,dy\Big)^{\frac12}=o(|\tau|^{-1}),\tag{4.9}$$

as $\tau\to-\infty$.

Components of the error. We estimate $\langle\psi_j,\varphi_{\mathcal{C}}(\bar v-v)\rangle$:

$$\begin{aligned}
\big\langle\psi_j,\varphi_{\mathcal{C}}(\bar v-v)\big\rangle&=\langle\psi_j,1\rangle\Big(b-\frac{A^2}{4(\Gamma+1)|\tau|}\Big)+\langle\psi_j,y\rangle\,\frac{A}{2(\Gamma+1)|\tau|}\\
&\qquad+\langle\psi_j,y^2-2\rangle\frac{\Gamma}{4(\Gamma+1)|\tau|}+\langle\psi_j,R\rangle.
\end{aligned}$$

In view of the fact that $\psi_0=1$, $\psi_1=y$ and $\psi_2=y^2-2$, we have

$$\frac{\langle\psi_0,\varphi_{\mathcal{C}}(\bar v-v)\rangle}{\langle\psi_0,\psi_0\rangle}=b-\frac{A^2}{4(\Gamma+1)|\tau|}+\frac{\langle\psi_0,R\rangle}{\langle\psi_0,\psi_0\rangle}$$

and

$$\begin{aligned}
\frac{\langle\psi_1,\varphi_{\mathcal{C}}(\bar v-v)\rangle}{\langle\psi_1,\psi_1\rangle}&=\frac{A}{2(\Gamma+1)|\tau|}+\frac{\langle\psi_1,R\rangle}{\langle\psi_1,\psi_1\rangle},\\
\frac{\langle\psi_2,\varphi_{\mathcal{C}}(\bar v-v)\rangle}{\langle\psi_2,\psi_2\rangle}&=\frac{\Gamma}{4(\Gamma+1)|\tau|}+\frac{\langle\psi_2,R\rangle}{\langle\psi_2,\psi_2\rangle}.
\end{aligned}$$

We claim that for every $\eta > 0$, there exist $\tau_\eta < 0$ such that for all $\tau \le \tau_\eta$ one has

$$\Big|\langle \psi_j, R\rangle\Big| \le \frac{\eta}{|\tau|}. \tag{4.10}$$

Indeed, this immediately follows by applying (4.9) and Hölder's inequality to

$$|\langle \psi_j R\rangle| \le \int |\psi_j||R| e^{-\frac{y^2}{4}}\, dy \le C\left(\int R^2 e^{-\frac{y^2}{4}}\, dy\right)^{\frac{1}{2}}.$$

This concludes the proof of the lemma. □

Remark 4.3 (The choice of parameters (α,β,γ)). We can choose $\tau_0 \ll -1$ to be any small number so that $\tau_0 \le \tau_*$, where τ_* is as in Proposition 4.1 and so that all our uniform estimates in previous sections hold for $\tau \le \tau_0$. Note also that having Proposition 4.1 we can decrease τ_0 if necessary and choose parameters α,β and γ again so that we still have $\mathcal{P}_+ w_{\mathcal{C}}(\tau_0) = \mathcal{P}_0 w_{\mathcal{C}}(\tau_0) = 0$, without affecting our estimates. Hence, from now on we will be assuming that we have fixed parameters α,β and γ at some time $\tau_0 \ll -1$, to have both projections zero at time τ_0. As a consequence of Proposition 4.1, which shows that the parameters (α,β,γ) are *admissible* with respect to τ_0, we can ensure all the estimates for $w = u_1 - u_2^{\alpha\beta\gamma}$ will then hold for all $\tau \le \tau_0$, *independently of our choice* of (α,β,γ).

5. A priori estimates

Let $u(y,\tau)$ be an ancient oval solution of (1.7) that satisfies the asymptotics in Theorem 1.8. In this section we will prove some further a priori estimates on $u(y,\tau)$ which hold for $\tau \ll -1$. These estimates will be used in the subsequent sections. Throughout this section we will use the notation introduced in the previous section and in particular the definition of $Y(u,\tau)$ as the inverse function of $u(y,\tau)$ in the tip region and $Z(\rho,\tau)$ given by (3.19).

Before we start discussing a priori estimates for our solution $u(y,\tau)$, we recall a corollary of Theorem 1.8 that will be used throughout the paper, especially in dealing with the tip region.

Corollary 5.1 (Corollary of Theorem 1.8). *Let M_t be any ancient oval satisfying the assumptions of Theorem* 1.8. *Consider the tip region of our solution as in part* (iii) *of Theorem* 1.8, *and switch the coordinates around the tip region as in formula* (3.19). *Then, $Z(\rho,\tau)$ converges, as $\tau \to -\infty$, uniformly smoothly to the unique rotationally symmetric translating Bowl solution $Z_0(\rho)$ with speed* $\sqrt{2}/2$.

Proof. According to the asymptotic description of the tip-region from [3] (see part (iii) of Theorem 1.8), the family of hypersurfaces that we get by translating the tip of M_t to the origin and then rescaling so that the maximal mean curvature becomes equal to one, converges to the translating Bowl soliton with velocity equal to one.

In defining $Z(\rho, \tau)$ by

$$Y(u, \tau) = Y(0, \tau) + \frac{1}{\sqrt{|\tau|}} Z(\rho, \tau) \tag{5.1}$$

we have in fact translated the tip to the origin and rescaled the surface M_t, first by a factor $1/\sqrt{|t|} = e^{\tau/2}$ (the cylindrical rescaling (3.4) that leads to $u(y, \tau)$ or equivalently $Y(u, \tau)$), and then by the factor $\sqrt{|\tau|}$ from (3.19). These two rescalings together shrink M_t by a factor $\sqrt{|t|/\log|t|}$. Since by Theorem 1.8 the maximal mean curvature at the tip satisfies

$$H_{\max}(t) = (1 + o(1))\sqrt{\frac{\log|t|}{2|t|}},$$

the hypersurface of rotation given by $z = Z(\rho, \tau)$ has maximal mean curvature $H_{\max}(t) \cdot \sqrt{|t|/\log|t|} = \sqrt{2}/2 + o(1)$. It therefore converges to the unique rotationally symmetric, translating Bowl solution $Z_0(\rho)$ with speed $\sqrt{2}/2$, which satisfies equation (3.20). □

Next we prove a proposition that will play an important role in obtaining the coercive type estimate (3.26) in the tip region.

PROPOSITION 5.2. *Let u be an ancient oval solution of (1.7) that satisfies the asymptotic estimates* (i)–(iii) *in Theorem 1.8. Then, there exists $\tau_0 \ll -1$ for which we have $(u^2)_{yy}(y, \tau) \le 0$ for all $\tau \le \tau_0$.*

The proof of this proposition will combine a contradiction argument based on scaling and the following maximum principle lemma.

LEMMA 5.3. *Under the assumptions of Proposition 5.2, there exists time $\tau_0 \ll -1$ such that*

$$\max_{\bar{M}_\tau}(u^2)_{yy}(\cdot, \tau) > 0 \quad \textit{and} \quad \tau \le \tau_0 \implies \frac{d}{d\tau}\max(u^2)_{yy}(\cdot, \tau) \le 0.$$

Proof. For the proof of this lemma, it is more convenient to work in the original scaling $(x, t, U(x, t))$ (see equation (1.5)) that is related to $(y, \tau, u(y, \tau))$ via the change of variables (3.5). Set

$$Q(x, t) := U^2(x, t), \quad q(y, \tau) = u^2(y, \tau).$$

The inequality we want to show is scaling invariant, namely, $(U^2)_{xx}(x, t) = (u^2)_{yy}(y, \tau)$. Hence, it is sufficient to show that there exists $t_0 \ll -1$ such that

$$\max_{M_t} Q_{xx}(\cdot, t) > 0 \quad \text{and} \quad t \le t_0 \implies \frac{d}{dt}\max_{M_t} Q_{xx}(\cdot, t) < 0.$$

To this end, we will apply the maximum principle to the evolution of Q_{xx}. Since U satisfies (1.5), a simple calculation shows that

$$Q_t = \frac{4QQ_{xx} - 2Q_x^2}{4Q + Q_x^2} - 2(n-1).$$

Differentiate this equation with respect to x to get

$$\begin{aligned} Q_{xt} &= \frac{4QQ_{xxx}}{4Q+Q_x^2} - (4QQ_{xx} - 2Q_x^2)\frac{4Q_x + 2Q_xQ_{xx}}{(4Q+Q_x^2)^2}, \\ &= \frac{4QQ_{xxx}}{4Q+Q_x^2} - (4QQ_{xx} - 2Q_x^2)(Q_{xx}+2)\frac{2Q_x}{(4Q+Q_x^2)^2}. \end{aligned} \tag{5.2}$$

We differentiate again, but this time we only consider points where Q_{xx} is either maximal or minimal, so that $Q_{xxx} = 0$. Note that

$$(4QQ_{xx} - 2Q_x^2)_x = 4QQ_{xxx} = 0 \qquad \text{and} \qquad (Q_{xx}+2)_x = Q_{xxx} = 0 \tag{5.3}$$

at those points. Also,

$$\begin{aligned} \Big(\frac{2Q_x}{(4Q+Q_x^2)^2}\Big)_x &= \frac{2Q_{xx}(4Q+Q_x^2) - 2(4Q_x + 2Q_xQ_{xx})(2Q_x)}{(4Q+Q_x^2)^3} \\ &= 2\,\frac{(4Q - 3Q_x^2)Q_{xx} - 8Q_x^2}{(4Q+Q_x^2)^3} \\ &= 2\frac{4Q-3Q_x^2}{(4Q+Q_x^2)^3}\Big(Q_{xx} - \frac{8Q_x^2}{4Q-3Q_x^2}\Big). \end{aligned}$$

Using these facts we now differentiate (5.2). This leads us to

$$\begin{aligned} &Q_{xxt} - \frac{4QQ_{xxxx}}{4Q+Q_x^2} \\ &\quad = -(Q_{xx}+2)(4QQ_{xx} - 2Q_x^2)\cdot 2\frac{4Q-3Q_x^2}{(4Q+Q_x^2)^3}\Big(Q_{xx} - \frac{8Q_x^2}{4Q-3Q_x^2}\Big), \end{aligned}$$

holding at the maximal or minimal points of Q_{xx}. Recall that since $Q = U^2$, we have $Q_x^2 = 4U^2U_x^2$. Thus the previous equation becomes

$$\big(Q_{xx}\big)_t - \frac{\big(Q_{xx}\big)_{xx}}{1+U_x^2} = -\frac{2}{4Q}(Q_{xx}+2)(Q_{xx} - 2U_x^2)\frac{(1-3U_x^2)Q_{xx} - 8U_x^2}{(1+U_x^2)^3}. \tag{5.4}$$

We will now use (5.4) to conclude that at a maximum point of Q_{xx}, such that $Q_{xx} > 0$, we have

$$\big(Q_{xx}\big)_t - \frac{\big(Q_{xx}\big)_{xx}}{1+U_x^2} < 0. \tag{5.5}$$

Since the equation becomes singular at the tip of the surface, we will first show that very near the tip we have $Q_{xx} < 0$. After going to the y variable and setting $q(y,\tau) := u^2(y,\tau)$, we have $Q_{xx} = q_{yy}$, where after switching coordinates,

$$q_{yy} = 2\,(u u_{yy} + u_y^2) = 2\Big(-u\,\frac{Y_{uu}}{Y_u^3} + \frac{1}{Y_u^2}\Big) = \frac{2}{Z_\rho^3}\,(Z_\rho - \rho Z_{\rho\rho}). \tag{5.6}$$

Since by Corollary 5.1 we have that $Z(\rho,\tau)$ converges uniformly smoothly, as $\tau \to -\infty$, on the set $\rho \le 1$, to the translating soliton $Z_0(\rho)$, it will be sufficient to show that $2Z_\rho^{-3}\,(Z_{0\rho} - \rho Z_{0\rho\rho}) < 0$ near $\rho = 0$. Since Z_0 is a smooth function, this can be easily seen using the Taylor expansion of Z_0 near the origin. Let $Z_0(\rho) = a\,\rho^2 + b\,\rho^4 + o(\rho^4)$, as $\rho \to 0$. A direct calculation using (3.20) shows that

$$a = -\frac{1}{2\sqrt{2}n} \qquad \text{and} \qquad b = -\frac{\sqrt{2}}{16n^3(2+n)},$$

implying that

$$\frac{2}{Z_\rho^3}\,(Z_{0\rho} - \rho Z_{0\rho\rho}) = \frac{1}{(2a\rho)^3}\,\frac{\sqrt{2}\,\rho^3}{2n^3(2+n)} + o(1) = -\frac{2}{2+n} + o(1) \tag{5.7}$$

as $\rho \to 0$. We conclude that for $\tau \le \tau_0 \ll -1$ and ρ sufficiently close to zero, we have

$$Q_{xx} = q_{yy} \le -\frac{1}{2+n} < 0. \tag{5.8}$$

We will now show that at a maximum point where $Q_{xx} > 0$, (5.5) holds. By (5.8) we know this point cannot be at the tip, and hence all derivatives are well defined at the maximum point of Q_{xx}. At such a point, $Q_{xx} + 2 > 0$. We also have $Q_{xx} = 2UU_{xx} + 2U_x^2$, so convexity of the surface implies $Q_{xx} - 2U_x^2 = 2UU_{xx} < 0$ on the entire solution. Thus we have

$$\forall x, t: \quad Q_{xx} < 2U_x^2, \tag{5.9}$$

so it suffices to show that when $Q_{xx} > 0$,

$$Q_{xx}(1 - 3U_x^2) - 8U_x^2 < 0 \tag{5.10}$$

holds.

We consider the two cases $3U_x^2 < 1$ and $3U_x^2 \ge 1$. If $3U_x^2 < 1$, then $Q_{xx}(1 - 3U_x^2) < Q_{xx} < 2U_x^2$ so that

$$Q_{xx}(1 - 3U_x^2) - 8U_x^2 < Q_{xx} - 8U_x^2 < -6U_x^2.$$

By (5.9) we have $U_x^2 > 0$ whenever $Q_{xx} > 0$, so that (5.10) holds at a positive maximum of Q_{xx}. If, on the other hand, $3U_x^2 \ge 1$, then in view of $Q_{xx} > 0$ we have $(1 - 3U_x^2)Q_{xx} - 8U_x^2 \le -8U_x^2 < 0$ so that (5.10) holds again. We conclude from both cases that at a maximum point where $Q_{xx} > 0$, (5.5) holds. □

Let Z_0 be the translating Bowl soliton that satisfies (3.20) and the asymptotics (3.21). Recall that we have $Z_0(0) = (Z_0)_\rho(0) = 0$ and the sign conventions $(Z_0)_\rho(\rho) < 0$ and $(Z_0)_{\rho\rho}(\rho) < 0$ for $\rho > 0$ (see Remark 3.2), which also imply that $Z_0(\rho) < 0$ for $\rho > 0$. By Corollary 5.1 we have $\lim_{\tau\to-\infty} Z(\rho,\tau) = Z_0(\rho)$, smoothly on compact sets in ρ. Thus (5.6) implies that

$$q_{yy} \sim \frac{2}{(Z_0)_\rho^3}\left((Z_0)_\rho - \rho(Z_0)_{\rho\rho}\right)$$

for $\tau \le \tau_0 \ll -1$. In the proof of the previous lemma we have shown that this quantity is negative near the origin $\rho = 0$. We will next show that it remains negative for all $\rho > 0$.

LEMMA 5.4. *On the translating Bowl soliton $Z_0(\rho)$ that satisfies equation (3.20), we have*

$$\frac{2}{(Z_0)_\rho^3}\big((Z_0)_\rho - \rho(Z_0)_{\rho\rho}\big) < 0$$

for any $\rho \ge 0$.

Proof. The proof simply follows from the maximum principle in a similar manner as the proof of Lemma 5.3. To use the calculations from before we need to flip the coordinates. Setting $x = Z_0(\rho)$, after we flip coordinates we have $\rho = U_0(x)$ for some function $U_0 > 0$. Since we have assumed above that $Z_0 \le 0$, we also have that $x \le 0$. Setting $Q := U_0^2$ we find that $Q_{xx} = \frac{2}{(Z_0)_\rho^3}\big((Z_0)_\rho - \rho(Z_0)_{\rho\rho}\big)$, hence it is sufficient to show that $Q_{xx} < 0$ for $x < 0$.

A direct calculation shows that U_0 satisfies the equation

$$\frac{(U_0)_{xx}}{1 + (U_0)_x^2} - \frac{n-1}{U} = \frac{\sqrt{2}}{2}\,(U_0)_x.$$

Note that in addition to $U > 0$ for $x < 0$, we have $(U_0)_x = 1/(Z_0)_\rho < 0$ and $(U_0)_{xx} = -(Z_0)_{\rho\rho}/(Z_0)_\rho^3 < 0$. Also since $(U_0)_x \to -\infty$ as $x \to 0$, the function U_0 fails to be a C^1 function near $x = 0$. However this is not a problem since we have shown in the proof of the previous lemma that (5.7) holds, implying that $Q_{xx} < 0$ for $|x| \le \eta$, if η chosen sufficiently small. In addition, a direct calculation where we use that $Z_0(\rho)$ satisfies the asymptotics

$$Z_0(\rho) = -\frac{\rho^2}{2\sqrt{2}(n-1)} + \log\rho + o(\log\rho), \qquad \text{as } \rho \to \infty,$$

as shown in by Proposition 2.1 in [4], leads to

$$Q_{xx} = \frac{2}{(Z_0)_\rho^3}\big((Z_0)_\rho - \rho(Z_0)_{\rho\rho}\big) < 0$$

for ρ sufficiently large, which is equivalent to $x < -\ell$ with $\ell > 0$ sufficiently large.

We will now use the maximum principle to conclude that $Q_{xx} < 0$ for $x \in [-\ell, -\eta]$. Similarly to the computation in the proof of the previous lemma, after setting $Q := U_0^2$, we find that

$$\frac{4QQ_{xx} - 2Q_x^2}{4Q + Q_x^2} - 2(n-1) = \frac{\sqrt{2}}{2}\, Q_x.$$

After we differentiate twice in x, following the same calculations as in the proof of Lemma 5.3, we find that Q_{xx} satisfies the equation

$$\begin{aligned} \frac{\sqrt{2}}{2}\Big(Q_{xx}\Big)_x - \frac{\big(Q_{xx}\big)_{xx}}{1+U_{0x}^2} \\ = -\frac{2}{4Q}(Q_{xx}+2)(Q_{xx}-2U_{0x}^2)\frac{Q_{xx}(1-3U_{0x}^2) - 8U_{0x}^2}{(1+U_{0x}^2)^3}. \end{aligned} \tag{5.11}$$

Assume that Q_{xx} assumes a *positive* maximum at some point $x_0 \in [-\ell, -\eta]$. Arguing exactly as in Lemma 5.3 we conclude that at a maximum point of Q_{xx} where $Q_{xx} > 0$, we have

$$-\frac{2}{4Q}(Q_{xx}+2)(Q_{xx}-2U_{0x}^2)\frac{Q_{xx}(1-3U_{0x}^2) - 8U_{0x}^2}{(1+U_{0x}^2)^3} < 0.$$

On the other hand, at this point we also have that $Q_{xxx} = 0$ and $Q_{xxxx} \le 0$, so we get a contradiction with (5.11). Hence, Q_{xx} cannot achieve a positive maximum on $[-\ell, -\eta]$ finishing the proof of our lemma. □

We will now proceed to the proof of Proposition 5.2.

Proof of Proposition 5.2. We will argue by contradiction. Assuming that our claim does not hold, we can find a decreasing sequence $\tau_j \to -\infty$ and points (y_j, τ_j) such that $q_{yy}(y_j, \tau_j) = \max_{\bar{M}_{\tau_j}} q_{yy}(\cdot, \tau_j) > 0$. We may assume without loss of generality that $y_j > 0$. It follows from Lemma 5.3 that the sequence $\{q_{yy}(y_j, \tau_j)\}$ is non-increasing, implying that

$$q_{yy}(y_j, \tau_j) := \max_{\bar{M}_{\tau_j}} q_{yy}(\cdot, \tau_j) \ge c > 0 \qquad \forall i. \tag{5.12}$$

This in particular implies that

$$u_y^2(y_j, \tau_j) \ge \frac{c}{2} > 0. \tag{5.13}$$

Set $\delta := \sqrt{c/2}$. After flipping the coordinates and using the change of variables (3.19) (or (5.1)) we find that for $u_j = u(y_j, \tau_j)$, $\rho_j = \sqrt{|\tau_j|}\, u_j$, we have

$$|u_y(y_j, \tau_j)| = \frac{1}{|Y_u(u_j, \tau_j)|} = \frac{1}{|Z_\rho(\rho_j, \tau_j)|} \ge \delta \implies |Z_\rho(\rho_j, \tau_j)| \le \frac{1}{\delta}.$$

The monotonicity of $Z_\rho(\rho,\tau)$ in ρ and the convergence $\lim_{\tau\to\infty} Z(\rho,\tau) = Z_0(\rho)$ smoothly on any compact set in ρ imply that $\rho_j \le \rho_\delta$, where ρ_δ is the point at which $|(Z_0)_\rho(\rho_\delta)| = 2/\delta$. We may assume without loss of generality that δ is small, which means that ρ_δ is large. The asymptotics (3.21) for $Z_0(\rho)$ as $\rho\to\infty$, give that $|(Z_0)_\rho(\rho)| \sim \rho/(\sqrt{2}(n-1))$, as $\rho\to+\infty$, implying that by choosing δ sufficiently small we have $2/\delta = |(Z_0)_\rho(\rho_\delta)| \sim \rho_\delta/(\sqrt{2}(n-1))$, or equivalently $\rho_\delta \sim 2\sqrt{2}(n-1)/\delta$. Since $\rho_j \le \rho_\delta$, we conclude that the points $(\rho_j,\tau_j,Z(\rho_j,\tau_j))$, or equivalently the points $(y_j,\tau_j,u(y_j,\tau_j))$, belong to the soliton region where we know that $q_{yy}<0$ by Lemma 5.4, contradicting our assumption (5.12). This implies (2.15) holds.

Since we have that (2.15) holds, this contradicts (5.13), hence finishing the proof of Proposition 5.2. □

In the rotationally symmetric case that we consider here, the principal curvatures of our hypersurface are given by

$$\lambda_1 = -\frac{u_{yy}}{(1+u_y^2)^{3/2}} \qquad \text{and} \qquad \lambda_2 = \cdots = \lambda_n = \frac{1}{u\,(1+u_y^2)^{1/2}}.$$

In [3] we showed that on our Ancient Ovals M_t we have

$$\lambda_1 \le \lambda_2.$$

We also showed $\lambda_1 = \lambda_2$ at the tip of the Ancient Ovals, at which the mean curvature is maximal as well. The quotient

$$R := \frac{\lambda_1}{\lambda_2} = -\frac{U\,U_{xx}}{1+U_x^2} = -\frac{u\,u_{yy}}{1+u_y^2}$$

is a scaling invariant quantity and in some sense measures how close we are to a cylinder, in a given region and at a given scale. It turns out that this quotient can be made arbitrarily small *outside* the *soliton* region $\mathcal{S}_L(\tau) := \big\{y \mid 0 \le u(y,\tau) \le \frac{L}{\sqrt{|\tau|}}\big\}$, by choosing $L \gg 1$ and $\tau \le \tau_0 \ll -1$. This is shown next.

PROPOSITION 5.5. *For every $\eta > 0$, there exist $L \gg 1$ and $\tau_0 \ll -1$ so that*

$$\frac{\lambda_1}{\lambda_2}(y,\tau) < \eta, \qquad \textit{if } u(y,\tau) > \frac{L}{\sqrt{|\tau|}} \ \textit{and} \ \ \tau \le \tau_0.$$

Proof. Having Lemma 2.9, to prove Proposition 5.5 it suffices to show the following claim.

Claim 5.6. For every $L > 0$ big, there exist $\bar{L} \gg 1$ and $\tau_0 \ll -1$ so that

$$u(y,\tau) \ge \frac{\bar{L}}{\sqrt{|\tau|}} \qquad \Longrightarrow \qquad |p_{y\tau} - p_\tau^k| \ge \frac{L}{H(p_{y\tau},\tau)}$$

for both $k \in \{1,2\}$ and where $p_{y\tau} \in \bar{M}_\tau$ is a point on a surface that is described by the profile function $u(y,\tau)$ at time τ.

Proof. Assume the claim is not true and that there exist a sequence (y_j, τ_j), with $\tau_j \to -\infty$ and $L_j \to \infty$ such that $u(y_j,\tau_j) \ge \frac{L_j}{|\tau_j|}$, but for example, for $k = 1$,

$$|\bar{p}_j - p^1_{t_j}| \le \frac{\bar{L}}{H(\bar{p}_j, \tau_j)}, \tag{5.14}$$

where we shortly denote $\bar{p}_j := p_{y_j\tau_j}$. By Proposition 2.12 we have that a rescaled limit around $(\bar{p}_j, \tau_j)$ by factors $H(\bar{p}_j,\tau_j)$ converges to a Bowl soliton. On the other hand, $u(y_j,\tau_j)\sqrt{|\tau_j|} \ge L_j$, or equivalently in the tip variables, $\rho_j \ge L_j$. In the switched variables around the tip we have

$$Y(0,\tau_j) - Y(u_j,\tau_j) = -\frac{1}{\sqrt{|\tau_j|}}\, Z(\rho_j,\tau_j),$$

which implies

$$|\bar{p}_j - p^1_{\tau_j}| = \frac{|Z(\rho_j,\tau_j)|}{\sqrt{|\tau_j|}} \ge \frac{|Z(L,\tau_j)|}{\sqrt{|\tau_j|}}, \tag{5.15}$$

since $|Z(\rho,\tau_j)|$ increases in $\rho > 0$ and $\rho_j \ge L_j \to \infty$. We can choose any L and the above inequality will hold for sufficiently big j; that is, the larger L we take, we may need to increase the j so that (5.15) holds. We know that $\lim_{j\to\infty} Z(L,\tau_j) = Z_0(L)$, where Z_0 is the Bowl soliton and $|Z_0(L)| \sim \frac{L^2}{2\sqrt{2}(n-1)}$ for L large enough. This together with (5.14) and (5.15) yield

$$\frac{\bar{L}}{H(\bar{p}_j,\tau_j)} \ge |\bar{p}_j - p^1_{\tau_j}| \ge \frac{L^2}{4\sqrt{2}(n-1)\sqrt{|\tau_j|}} \tag{5.16}$$

for $j \ge j_0$ sufficiently big.

On the other hand, since we have (5.14), by Lemmas 2.10 and 2.14 we have that $H(\bar{p}_j,\tau_j)$ and $H_{\max}(\tau_j)$ are uniformly equivalent, implying that $cH_{\max}(\tau_j) \le H(\bar{p}_j,\tau_j) \le H_{\max}(\tau_j)$ for a uniform constant $c > 0$ and for $j \ge j_0$. In [3] we proved that $H_{\max}(\tau_j) \sim \sqrt{\frac{|\tau_j|}{2}}$ for $j \gg 1$, and hence we have that $H(\bar{p}_j,\tau_j) \ge c\sqrt{|\tau_j|}$ for $j \gg 1$ for a uniform constant $c > 0$. Combining this and (5.16) yields contradiction for $j \ge j_0$ big enough, if we choose L so that $L^2 > \frac{8\sqrt{2}(n-1)\bar{L}}{c}$. □

Since we have Claim 5.6 and Lemma 2.9, the proof of Proposition 5.5 is now complete. □

We will finally use the convexity estimate shown in Proposition 5.2 to show the following crucial estimate that will be used in Section 7 and holds in the collar region $\mathcal{K}_{\theta,L} := \{u :\, L/\sqrt{|\tau|} \le u \le 2\theta\}$.

LEMMA 5.7. *Let u be an ancient oval solution of (1.7) that satisfies the asymptotics in Theorem 1.8. Then, for $0 < \theta \ll 1$ and $L \gg 1$ large, there exist $\epsilon(\theta, L)$ small and a $\tau_0 \ll -1$ for which we have*

$$\left|1 + \frac{u\,Y}{2(n-1)Y_u}\right| < \epsilon(\theta, L) \quad \text{in } \mathcal{K}_{\theta,L} \quad \text{for } \tau \le \tau_0.$$

Moreover, for $L \gg 1$ and $\theta \ll 1$, we can choose $\epsilon := \max\{4\theta^2, c(n)L^{-1}\}$.

Proof. By Proposition 5.2 we have that $(u^2)_{yy} \le 0$. We need to show that

$$1 - \epsilon \le -\frac{u\,Y}{2(n-1)Y_u} \le 1 + \epsilon$$

in the considered region, which is equivalent to

$$1 - \epsilon \le -\frac{1}{4(n-1)} y\,(u^2)_y \le 1 + \epsilon. \tag{5.17}$$

The intermediate region asymptotics in Theorem 1.8 imply that for $u = 2\theta$, we have

$$y = \sqrt{2|\tau|}\sqrt{1 - \frac{2\theta^2}{n-1} + o(1)}, \qquad \text{as } \tau \to -\infty. \tag{5.18}$$

It follows that at $u = 2\theta$ and for θ small, $y \ge \sqrt{2|\tau|}\,(1 - 4\theta^2)$. Hence, in the considered region $L/\sqrt{|\tau|} \le u \le 2\theta$, we have

$$\sqrt{2|\tau|}(1 - 4\theta^2) \le y \le \sqrt{2|\tau|}(1 + o(1)), \tag{5.19}$$

where $o(1) \to 0$, as $\tau \to -\infty$. Next, using the inequality $-(u^2)_{yy} \ge 0$, which was shown in Proposition 5.2, we can estimate

$$-(u^2)_y\big|_{u=2\theta} \le -(u^2)_y \le -(u^2)_y\big|_{u=L/\sqrt{|\tau|}}.$$

Furthermore, our intermediate region asymptotics from Theorem 1.8 imply that at $u = 2\theta$ and $\theta \ll 1$, we have

$$-(\bar{u}^2)_\zeta = 2(n-1)\,z + o(1),$$

which combined with (5.18) gives that

$$-(u^2)_y|_{u=2\theta} = 2(n-1)\frac{y}{|\tau|} + o\big(\frac{1}{\sqrt{|\tau|}}\big) = \frac{2\sqrt{2}(n-1)}{\sqrt{|\tau|}}\sqrt{1 - \frac{2\theta^2}{n-1} + o(1)}.$$

On the other hand, in the tip region the solutions are approximated by the Bowl soliton, so that at $u = L/\sqrt{|\tau|}$, we have

$$-(u^2)_y\Big|_{u=L/\sqrt{|\tau|}} = -\frac{2u}{Y_u}\Big|_{u=L/\sqrt{|\tau|}} = \frac{2L}{\sqrt{|\tau|}}\frac{1}{Z_\rho(L,\tau)}.$$

Combining the convergence $\lim_{\tau\to-\infty} Z(\rho,\tau) = Z_0(\rho)$ together with the asymptotics (3.21) implies that for $L \gg 1$, we have

$$Z_\rho(L,\tau) \geq \frac{L-c}{\sqrt{2}(n-1)}$$

for a fixed constant $c = c(n)$. Hence

$$-(u^2)_y\Big|_{u=L/\sqrt{|\tau|}} \leq \frac{2L}{\sqrt{|\tau|}}\,\frac{\sqrt{2}(n-1)}{L-c} = \frac{2\sqrt{2}(n-1)}{\sqrt{|\tau|}}\,(1+\epsilon),$$

for $\epsilon = c(n)L^{-1}$, for another fixed constant $c(n)$. We conclude that

$$\frac{2\sqrt{2}(n-1)}{\sqrt{|\tau|}}\sqrt{1-\frac{2\theta^2}{n-1}} \leq -(u^2)_y \leq \frac{2\sqrt{2}(n-1)}{\sqrt{|\tau|}}\,(1+\epsilon). \tag{5.20}$$

Combining (5.19) and (5.20) yields that for $\tau \ll -1$, we have the bounds

$$(1-4\theta^2)\sqrt{1-\frac{4\theta^2}{n-1}} \leq -\frac{1}{4(n-1)}y\,(u^2)_y \leq (1+\epsilon),$$

which yields (5.17) for $\epsilon := \max(4\theta^2, c(n)\,L^{-1})$ and $L \gg 1$, $\theta \ll 1$. ☐

6. The cylindrical region

Let $u_1(y,\tau)$ and $u_2(y,\tau)$ be the two solutions to equation (1.7) as in the statement of Theorem 1.7, and let $u_2^{\alpha\beta\gamma}$ be defined by (3.6). In this section we will estimate the difference $w := u_1 - u_2^{\alpha\beta\gamma}$ in the cylindrical region $\mathcal{C}_\theta = \{y \mid u_1(y,\tau) \geq \theta/2\}$ for a given number $\theta > 0$ small and any $\tau \leq \tau_0 \ll -1$. Recall all the definitions and notation introduced in Section 3.1.

Our goal in this section is to prove that the bound (3.15) holds as stated next.

Proposition 6.1. *For every $\epsilon > 0$ and $\theta > 0$ small, there exists a $\tau_0 \ll -1$ so that if $w(y,\tau)$ is a solution to (6.1) for which $\mathcal{P}_+ w_{\mathcal{C}}(\tau_0) = 0$, then we have*

$$\|\hat{w}_{\mathcal{C}}\|_{\mathfrak{D},\infty} \leq \epsilon\left(\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty} + \|w\,\chi_{D_\theta}\|_{\mathfrak{H},\infty}\right),$$

where $D_\theta := \{y \mid \theta/2 \leq u_1(y,\theta) \leq \theta\}$ and $\hat{w}_C = \mathcal{P}_- w_{\mathcal{C}} + \mathcal{P}_+ w_{\mathcal{C}}$.

The rest of this section will be devoted to the proof of Proposition 6.1. To simplify the notation for the rest of the section we will simply denote $u_2^{\alpha\beta\gamma}$ by u_2 and set $w := u_1 - u_2$. The difference w satisfies

$$w_\tau = \frac{w_{yy}}{1+u_{1y}^2} - \frac{(u_{1y}+u_{2y})u_{2yy}}{(1+u_{1y}^2)(1+u_{2y}^2)}w_y - \frac{y}{2}w_y + \frac{1}{2}w + \frac{n-1}{u_1 u_2}w, \tag{6.1}$$

which we can rewrite as

$$w_\tau = \mathcal{L}w + \mathcal{E}w \tag{6.2}$$

in which $\mathcal{L} = \partial_y^2 - \frac{y}{2}\partial_y + 1$ is as above, and where $\mathcal{E}$ is given by

$$\mathcal{E}[w] = -\frac{u_{1y}^2}{1+u_{1y}^2}\, w_{yy} - \frac{(u_{1y}+u_{2y})u_{2yy}}{(1+u_{1y}^2)(1+u_{2y}^2)}\, w_y + \frac{2(n-1)-u_1u_2}{2u_1u_2}\, w. \tag{6.3}$$

6.1. *The operator $\mathcal{L}$.* We recall the definition of the Hilbert spaces $\mathfrak{H}$, $\mathfrak{D}$ and $\mathfrak{D}^*$ are given in Section 3.1. The formal linear operator

$$\mathcal{L} = \partial_y^2 - \frac{y}{2}\partial_y + 1 = -\partial_y^*\partial_y + 1$$

defines a bounded operator $\mathcal{L} : \mathfrak{D} \to \mathfrak{D}^*$, meaning that for any $f \in \mathfrak{D}$, we have that $Lf \in \mathfrak{D}^*$ is the functional given by

$$\forall \phi \in \mathfrak{D} : \langle \mathcal{L}f, \phi\rangle = \int_{\mathbb{R}} (-f_y\phi_y + f\phi)\; e^{-y^2/4}\, dy.$$

By integrating by parts one verifies that if $f \in C_c^2$, one has

$$\langle f, \phi\rangle = \int_{\mathbb{R}} \Big(f_{yy} - \frac{y}{2} f_y + f\Big)\, \phi\, e^{-y^2/4} dy,$$

so that the weak definition of $\mathcal{L}f$ coincides with the classical definition.

6.2. *Operator bounds and Poincaré type inequalities.* The following inequality was shown in Lemma 4.12 in [3].

LEMMA 6.2. *For any $f \in \mathfrak{D}$, one has*

$$\int_{\mathbb{R}} y^2 f(y)^2 e^{-y^2/4} dy \le C \int_{\mathbb{R}} \big(f(y)^2 + f_y(y)^2\big)\, e^{-y^2/4} dy,$$

which implies the multiplication operator $f \mapsto yf$ is bounded from $\mathfrak{D}$ to $\mathfrak{H}$, i.e.,

$$\|yf\|_{\mathfrak{H}} \le C\|f\|_{\mathfrak{D}}$$

for all $f \in \mathfrak{D}$.

As a consequence we have the following two lemmas.

LEMMA 6.3. *The following operators are bounded both as operators from $\mathfrak{D}$ to $\mathfrak{H}$ and also as operators from $\mathfrak{H}$ to $\mathfrak{D}^*$:*

$$f \mapsto yf, \quad f \mapsto \partial_y f, \quad f \mapsto \partial_y^* f = \Big(-\partial_y + \frac{y}{2}\Big)\, f,$$

where ∂_y^* is the formal adjoint of the operator ∂_y, it satisfies $\langle f, \partial_y^* g\rangle = \langle \partial_y f, g\rangle$ for all $f, g \in \mathfrak{D}$.

LEMMA 6.4. *The following operators are bounded from $\mathfrak{D}$ to $\mathfrak{D}^*$:*

$$f \mapsto y^2 f, \quad f \mapsto y\partial_y f, \quad f \mapsto \partial_y^2 f.$$

Proof of Lemmas 6.3 and 6.4. By definition of the norms in $\mathfrak{D}$ and $\mathfrak{H}$, the operator ∂_y is bounded from $\mathfrak{D}$ to $\mathfrak{H}$, and by duality its adjoint $\partial_y^* = -\partial_y + \frac{y}{2}$ is bounded from $\mathfrak{H}$ to $\mathfrak{D}^*$.

The Poincaré inequality from Lemma 6.2 implies directly that $f \mapsto yf$ is bounded from $\mathfrak{D}$ to $\mathfrak{H}$. By duality the same multiplication operator is also bounded from $\mathfrak{H}$ to $\mathfrak{D}^*$; i.e., for every $f \in \mathfrak{H}$ the product yf defines a linear functional on $\mathfrak{D}$ by $\langle yf, \phi\rangle = \langle f, y\phi\rangle$ for every $\phi \in \mathfrak{D}$. We get

$$\|y\, f\|_{\mathfrak{D}^*} \le C\|f\|_{\mathfrak{H}}$$

for all $f \in \mathfrak{H}$.

Composing the multiplications $y : \mathfrak{D} \to \mathfrak{H}$ and $y : \mathfrak{H} \to \mathfrak{D}^*$ we see that multiplication with y^2 is bounded as operator from $\mathfrak{D}$ to $\mathfrak{D}^*$; i.e., for all $f \in \mathfrak{D}$, we have $y^2 f \in \mathfrak{D}^*$ and

$$\|y^2\, f\|_{\mathfrak{D}^*} \le C^2\|f\|_{\mathfrak{D}}.$$

Since $y : \mathfrak{D} \to \mathfrak{H}$ and $\partial_y : \mathfrak{D} \to \mathfrak{H}$ are both bounded operators, we find that $\partial_y^* = -\partial_y^* + \frac{y}{2}$ is also bounded from $\mathfrak{D}$ to $\mathfrak{D}$. By duality again, it follows that ∂_y is bounded from $\mathfrak{H}$ to $\mathfrak{D}^*$. This proves Lemma 6.3.

Each of the operators in Lemma 6.4 is the composition of two operators from Lemma 6.3, so they are also bounded. □

More generally, to estimate the operator norm of multiplication with some function $m : \mathbb{R} \to \mathbb{R}$, seen as operator from $\mathfrak{D}$ to $\mathfrak{H}$, we have

$$\|m\, f\|_{\mathfrak{H}} \le \sup_{y\in\mathbb{R}} \frac{|m(y)|}{1+|y|}\, \|f\|_{\mathfrak{D}}.$$

Indeed the following lemma can be easily shown.

LEMMA 6.5. *Let $m : \mathbb{R} \to \mathbb{R}$ be a measurable function, and consider the multiplication operator $\mathcal{M} : f \mapsto mf$. Then, the following hold*:

- *$\mathcal{M} : \mathfrak{H} \to \mathfrak{H}$ is bounded if $m \in L^\infty(\mathbb{R})$, and $\|\mathcal{M}\|_{\mathfrak{H}\to\mathfrak{H}} \le \|m\|_{L^\infty}$.*
- *$\mathcal{M} : \mathfrak{D} \to \mathfrak{H}$ is bounded if and only if $\mathcal{M} : \mathfrak{H} \to \mathfrak{D}^*$ is bounded. Both operators are bounded if $(1+|y|)^{-1}m(y)$ is bounded, and*

$$\|\mathcal{M}\|_{\mathfrak{H}\to\mathfrak{D}^*} = \|\mathcal{M}\|_{\mathfrak{D}\to\mathfrak{H}} \le C\,\text{ess sup}_{y\in\mathbb{R}} \frac{|m(y)|}{1+|y|}.$$

- *Finally, $\mathcal{M}$ is a bounded operator from $\mathfrak{D}$ to $\mathfrak{D}^*$ if $(1+|y|)^{-2}m(y)$ is bounded, and the operator norm is bounded by*

$$\|\mathcal{M}\|_{\mathfrak{D}\to\mathfrak{D}^*} \le \text{ess sup}_{y\in\mathbb{R}} \frac{|m(y)|}{(1+|y|)^2}.$$

6.3. *Eigenfunctions of* $\mathcal{L}$. There is a sequence of polynomials $\psi_n(y) = y^n + \cdots$ that are eigenfunctions of the operator $\mathcal{L}$ and agree with the Hermite polynomials, up to scaling. The n^{th} eigenfunction has eigenvalue $\lambda_n = 1 - \frac{n}{2}$. The first few eigenfunctions are given by

$$\psi_0(y) = 1, \quad \psi_1(y) = y, \quad \psi_2(y) = y^2 - 2$$

up to scaling.

The functions $\{\psi_n : n \in \mathrm{N}\}$ form an orthogonal basis in all three Hilbert spaces $\mathfrak{D}$, $\mathfrak{H}$ and $\mathfrak{D}^*$. The three projections $\mathcal{P}_\pm$ and $\mathcal{P}_0$ onto the subspaces spanned by the eigenfunctions with negative/positive, or zero eigenvalues are therefore the same on each of the three Hilbert spaces. Since ψ_2 is the eigenfunction with eigenvalue zero, they are given by

$$\mathcal{P}_+ f = \sum_{j=0}^{1} \frac{\langle \psi_j, f\rangle}{\langle \psi_j, \psi_j\rangle}\psi_j, \quad \mathcal{P}_- f = \sum_{j=3}^{\infty} \frac{\langle \psi_j, f\rangle}{\langle \psi_j, \psi_j\rangle}\psi_j, \quad \mathcal{P}_0 f = \frac{\langle \psi_2, f\rangle}{\langle \psi_2, \psi_2\rangle}\psi_2.$$

6.4. *Estimates for ancient solutions of the linear cylindrical equation.* In this section we will give energy type estimates for ancient solutions $f : (-\infty, \tau_0] \to \mathfrak{D}$ of the linear cylindrical equation

$$\frac{df}{d\tau} - \mathcal{L} f(\tau) = g(\tau). \tag{6.4}$$

LEMMA 6.6. *Let* $f : (-\infty, \tau_0] \to \mathfrak{D}$ *be a bounded solution of* (6.4). *Then there is a constant* $C < \infty$ *that does not depend on* f, *such that*

$$\sup_{\tau \le \tau_0} \|\hat{f}(\tau)\|_{\mathfrak{H}}^2 + \frac{1}{C}\int_{-\infty}^{\tau_0} \|\hat{f}(\tau)\|_{\mathfrak{D}}^2 \, d\tau \le \|f_+(\tau_0)\|_{\mathfrak{H}}^2 + C\int_{-\infty}^{\tau_0} \|\hat{g}(\tau)\|_{\mathfrak{D}^*}^2 \, d\tau, \tag{6.5}$$

where $f_+ = \mathcal{P}_+ f$ *and* $\hat{f} = \mathcal{P}_+ f + \mathcal{P}_- f$.

Proof. This is a standard cylindrical estimate applied to the infinite time domain $(-\infty, \tau_0]$. Since the operator $\mathcal{L}$ commutes with the projections $\mathcal{P}_\pm$ we can split $f(\tau)$ into its $\mathcal{P}_+$ and $\mathcal{P}_-$ components and estimate these separately.

Applying the projection $\mathcal{P}_-$ to both sides of the equation $f_\tau - \mathcal{L} f = g$ we get

$$f_-'(\tau) = \mathcal{L} f_-(\tau) + g_-(\tau),$$

where $g_-(\tau) = \mathcal{P}_- g(\tau)$. This implies

$$\frac{1}{2}\frac{d}{d\tau}\|f_-\|_{\mathfrak{H}}^2 = \langle f_-, \mathcal{L} f_-\rangle + \langle f_-, g_-\rangle.$$

Using the eigenfunction expansion of f_- we get

$$\langle f_-, \mathcal{L} f_-\rangle \le -\frac{1}{C}\,\|f_-\|_{\mathfrak{D}}^2.$$

We also have

$$\langle f_-, g_-\rangle \le \|f_-\|_{\mathfrak{D}}\,\|g_-\|_{\mathfrak{D}^*} \le \frac{1}{2C}\|f_-\|_{\mathfrak{D}}^2 + \frac{C}{2}\|g_-\|_{\mathfrak{D}^*}^2.$$

We therefore get

$$\frac{1}{2}\frac{d}{d\tau}\|f_-\|_{\mathfrak{H}}^2 \le -\frac{1}{2C}\|f_-\|_{\mathfrak{D}}^2 + \frac{C}{2}\|g_-\|_{\mathfrak{D}^*}^2.$$

Integrating in time over the interval $(-\infty, \tau]$ then leads to

$$\frac{1}{2}\|f_-(\tau)\|_{\mathfrak{H}}^2 + \frac{1}{2C}\int_{-\infty}^{\tau} \|f_-(\tau')\|_{\mathfrak{D}}^2\, d\tau' \le \frac{C}{2}\int_{-\infty}^{\tau} \|g_-(\tau')\|_{\mathfrak{D}^*}^2\, d\tau'.$$

Taking the supremum over $\tau \le \tau_0$ then gives us the $\mathcal{P}_-$ component of (6.6).

For the other component, $f_+(\tau) = \mathcal{P}_+ f$, we have

$$\langle f_+, \mathcal{L} f_+\rangle \ge \frac{1}{C}\,\|f_+\|_{\mathfrak{D}}^2.$$

A similar calculation then leads to

$$\frac{1}{2}\frac{d}{d\tau}\|f_+\|_{\mathfrak{H}}^2 \ge \frac{1}{2C}\|f_+\|_{\mathfrak{D}}^2 - \frac{C}{2}\,\|g_+\|_{\mathfrak{D}^*}^2.$$

Integrating this over the interval $[\tau, \tau_0]$ introduces the boundary term $\|f_+(\tau_0)\|_{\mathfrak{H}}^2$ and gives us the estimate

$$\frac{1}{2}\|f_+(\tau)\|_{\mathfrak{H}}^2 + \frac{1}{2C}\int_{\tau}^{\tau_0} \|f_+(\tau')\|_{\mathfrak{D}}^2\, d\tau' \le \frac{1}{2}\|f_+(\tau_0)\|_{\mathfrak{H}}^2 + \frac{C}{2}\int_{\tau}^{\tau_0} \|g_+(\tau')\|_{\mathfrak{D}^*}^2\, d\tau'.$$

Adding the estimates for $\mathcal{P}_+ f$ and $\mathcal{P}_- f$ yields (6.5). □

LEMMA 6.7. *Let $f : (-\infty, \tau_0] \to \mathfrak{D}$ be a bounded solution of equation (6.4). If $T > 0$ is sufficiently large, then there is a constant $C_\star$ such that*

$$(6.6)\qquad \sup_{\tau\le\tau_0} \|\hat f(\tau)\|_{\mathfrak{H}}^2 + \frac{1}{C_\star}\sup_{n\ge 0}\int_{I_n} \|\hat f(\tau)\|_{\mathfrak{D}}^2\, d\tau \le \|f_+(\tau_0)\|_{\mathfrak{H}}^2 + C_\star \sup_{n\ge 0}\int_{I_n} \|\hat g(\tau)\|_{\mathfrak{D}^*}^2\, d\tau,$$

where I_n is the interval $I_n = [\tau_0 - (n+1)T, \tau_0 - nT]$ and where $f_+ = \mathcal{P}_+ f$ and $\hat f = \mathcal{P}_+ f + \mathcal{P}_- f$.

Proof. To simplify notation we assume in this proof that $\mathcal{P}_0 f(\tau) = 0$, i.e., that $\hat f(\tau) = f(\tau)$ for all τ. Likewise we assume that $\hat g(\tau) = g(\tau)$ for all $\tau \le \tau_0$.

Choose a large number $T > 0$, and let $\eta \in C_c^\infty(\mathbb{R})$ be a smooth cut-off function with $\eta(t) = 1$ for $t \in [-T, 0]$, $\operatorname{supp}\eta \subset (-2T, +T)$. We may assume that

$$(6.7)\qquad |\eta'(\tau)| \le \frac{2}{T} \qquad \text{for all } \tau \in \mathbb{R}.$$

For any integer $n \ge 0$, we consider

$$f_n(\tau) = \eta_n(\tau) f(\tau), \quad \text{where} \quad \eta_n(\tau) = \eta(\tau - \tau_0 + nT).$$

The cut-off function η_n satisfies $\eta_n(\tau) = 1$ for $\tau \in I_n$, and $\operatorname{supp}\eta_n \subset J_n$ where,

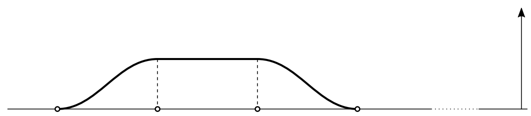


Figure 2. The cut-off function $\eta_n(\tau)$, and the intervals I_n and J_n.

by definition,

$$J_n = I_{n+1} \cup I_n \cup I_{n-1}.$$

The function f_n is a solution of

$$f_n'(\tau) - \mathcal{L} f_n(\tau) = \eta_n'(\tau) f(\tau) + \eta_n(\tau) g(\tau).$$

If $n \geq 1$, then we can apply Lemma 6.6 to f_n, with $f_n(\tau_0) = 0$. Since f_n and f coincide on I_n, we get

$$\begin{aligned}\sup_{\tau \in I_n} \|f(\tau)\|_{\mathfrak{H}}^2 + \frac{1}{C} \int_{I_n} \|f\|_{\mathfrak{D}}^2 d\tau &\leq \sup_{\tau \in J_n} \|f_n(\tau)\|_{\mathfrak{H}}^2 + \frac{1}{C} \int_{J_n} \|f_n\|_{\mathfrak{D}}^2 d\tau \\ &\leq C \int_{J_n} \|\eta_n' f + \eta_n g\|_{\mathfrak{D}^*}^2 d\tau.\end{aligned}$$

Here C is the constant from Lemma 6.6. Using $(a+b)^2 \leq 2(a^2+b^2)$ and also our bound (6.7) for $\eta_n'(\tau)$ we get

$$\sup_{\tau \in I_n} \|f(\tau)\|_{\mathfrak{H}}^2 + \frac{1}{C} \int_{I_n} \|f\|_{\mathfrak{D}}^2 d\tau \leq C \int_{J_n} \left\{ \frac{2}{T^2} \|f\|_{\mathfrak{D}^*}^2 + \|g\|_{\mathfrak{D}^*}^2 \right\} d\tau.$$

It follows that

$$\begin{aligned}&\sup_{\tau \in I_n} \|f(\tau)\|_{\mathfrak{H}}^2 + \frac{1}{C} \int_{I_n} \|f\|_{\mathfrak{D}}^2 d\tau \\ &\qquad \leq \frac{3C}{T^2} \sup_k \int_{I_k} \|f\|_{\mathfrak{D}^*}^2 d\tau + 3C \sup_k \int_{I_k} \|g\|_{\mathfrak{D}^*}^2 d\tau.\end{aligned} \tag{6.8}$$

For $n = 0$, the truncated function $f_n(\tau)$ is not defined for $\tau > \tau_0$ and we must use an estimate on $J_0 = I_1 \cup I_0$. We apply Lemma 6.6 to the function

$f_0(\tau)=\eta_0(\tau)f(\tau)$:

$$
\begin{aligned}
&\sup_{\tau\in I_0}\|f(\tau)\|_{\mathfrak{H}}^2+\frac{1}{C}\int_{I_0}\|f\|_{\mathfrak{D}}^2 d\tau \\
&\qquad\le \sup_{\tau\le\tau_0}\|f_0(\tau)\|_{\mathfrak{H}}^2+\frac{1}{C}\int_{-\infty}^{\tau_0}\|f_0\|_{\mathfrak{D}}^2 d\tau \\
&\qquad\le \|f_+(\tau_0)\|_{\mathfrak{H}}^2+C\int_{-\infty}^{\tau_0}\|\eta_0' f+\eta_0 g\|_{\mathfrak{D}^*}^2 d\tau \\
&\qquad\le \|f_+(\tau_0)\|_{\mathfrak{H}}^2+2C\int_{I_1}(\eta_0')^2\|f\|_{\mathfrak{D}^*}^2 d\tau+2C\int_{J_0}\|g\|_{\mathfrak{D}^*}^2 d\tau \\
&\qquad\le \|f_+(\tau_0)\|_{\mathfrak{H}}^2+\frac{2C}{T^2}\sup_k\int_{I_k}\|f\|_{\mathfrak{D}^*}^2 d\tau+2C\sup_k\int_{I_k}\|g\|_{\mathfrak{D}^*}^2 d\tau.
\end{aligned}
\tag{6.9}
$$

Combining (6.8) and (6.9) and taking the supremum over n yields

$$
\begin{aligned}
&\sup_{\tau\le\tau_0}\|f(\tau)\|_{\mathfrak{H}}^2+\frac{1}{C}\sup_n\int_{I_n}\|f\|_{\mathfrak{D}}^2 d\tau \\
&\qquad\le \|f_+(\tau_0)\|_{\mathfrak{H}}^2+\frac{3C}{T^2}\sup_k\int_{I_k}\|f\|_{\mathfrak{D}^*}^2 d\tau+3C\sup_k\int_{I_k}\|g\|_{\mathfrak{D}^*}^2 d\tau.
\end{aligned}
$$

Since $\|u\|_{\mathfrak{H}}\le\|u\|_{\mathfrak{D}}$ for all $u\in\mathfrak{D}$, it follows by duality that $\|u\|_{\mathfrak{D}^*}\le\|u\|_{\mathfrak{H}}$ for all $u\in\mathfrak{H}$, and thus we have $\|f(\tau)\|_{\mathfrak{D}^*}\le\|f(\tau)\|_{\mathfrak{D}}$. Therefore

$$
\begin{aligned}
&\sup_{\tau\le\tau_0}\|f(\tau)\|_{\mathfrak{H}}^2+\frac{1}{C}\sup_n\int_{I_n}\|f\|_{\mathfrak{D}}^2 d\tau \\
&\qquad\le \|f_+(\tau_0)\|_{\mathfrak{H}}^2+\frac{3C}{T^2}\sup_k\int_{I_k}\|f\|_{\mathfrak{D}}^2 d\tau+3C\sup_k\int_{I_k}\|g\|_{\mathfrak{D}^*}^2 d\tau.
\end{aligned}
$$

At this point we assume that T is so large that $3C/T^2\le 1/2C$, which lets us move the terms with f on the right to the left-hand side of the inequality:

$$
\sup_{\tau\le\tau_0}\|f(\tau)\|_{\mathfrak{H}}^2+\frac{1}{2C}\sup_n\int_{I_n}\|f\|_{\mathfrak{D}}^2 d\tau\le\|f_+(\tau_0)\|_{\mathfrak{H}}^2+3C\sup_k\int_{I_k}\|g\|_{\mathfrak{D}^*}^2 d\tau. \qquad \square
$$

6.5. *L^2-estimates for the error terms.* The two solutions u_1, u_2 of equation (1.7) that we are considering are only defined for $y^2\le(2+o(1))|\tau|$. This follows from the asymptotics in our previous work [3] (see also Theorems 1.8 and 9.1), where it was also shown that they satisfy the asymptotics

$$
u(y,\tau)=\sqrt{(n-1)\,(2-z^2)}+o(1), \qquad \text{as } \tau\to-\infty
$$

uniformly in z, where $z=\dfrac{y}{\sqrt{|\tau|}}$.

We have seen that $w := u_1-u_2$ satisfies (6.2) where the error term $\mathcal{E}$ is given by (6.3). We will now consider this equation only in the "cylindrical region," i.e., the region where

$$u > \frac{\theta}{2}, \qquad \text{i.e..} \qquad \frac{y}{\sqrt{|\tau|}} < \sqrt{2 - \frac{\theta^2}{4(n-1)}} + o(1).$$

To concentrate on this region, we choose a cut-off function $\Phi \in C^\infty(\mathbb{R})$ depending on the parameter θ that decreases smoothly from 1 to 0 in the interior of the interval

$$\sqrt{2 - \frac{\theta^2}{n-1}} < z < \sqrt{2 - \frac{\theta^2}{4(n-1)}}.$$

With this cut-off function we then define

$$\varphi_{\mathcal{C}}(y,\tau) = \Phi\Big(\frac{y}{|\tau|}\Big) \qquad \text{and} \qquad w_{\mathcal{C}}(y,\tau) = \varphi_{\mathcal{C}}(y,\tau) w(y,\tau).$$

The cut-off function $\varphi_{\mathcal{C}}$ satisfies the bounds

$$|(\varphi_{\mathcal{C}})_y|^2 + |(\varphi_{\mathcal{C}})_{yy}| + |(\varphi_{\mathcal{C}})_\tau| \le \frac{\bar{C}(\theta)}{|\tau|},$$

where $\bar{C}(\theta)$ is a constant that depends on θ and that may change from line to line in the text. The localized difference function $w_{\mathcal{C}}$ satisfies

$$w_{C,\tau} - \mathcal{L} w_{\mathcal{C}} = \mathcal{E}[w_{\mathcal{C}}] + \bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}], \tag{6.10}$$

where the operator $\mathcal{E}$ is again defined by (6.3) and where the new error term $\bar{\mathcal{E}}$ is given by the commutator

$$\bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}] = \big[\partial_\tau - (\mathcal{L} + \mathcal{E}), \varphi_{\mathcal{C}}\big] w,$$

i.e.,

$$\begin{aligned} \bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}] = \\ \Bigg\{\varphi_{\mathcal{C},\tau} - \varphi_{\mathcal{C},yy} + \frac{u_{1y}^2}{1+u_{1y}^2}\varphi_{\mathcal{C},yy} + \frac{(u_{1y}+u_{2y})u_{2yy}}{(1+u_{1y}^2)(1+u_{2y}^2)}(\varphi_{\mathcal{C}})_y + \frac{y}{2}(\varphi_{\mathcal{C}})_y\Bigg\} w \\ + \Big\{\frac{2u_{1y}^2}{1+u_{1y}^2}(\varphi_{\mathcal{C}})_y - 2(\varphi_{\mathcal{C}})_y\Big\} w_y. \end{aligned} \tag{6.11}$$

Equation (6.10) for $w_{\mathcal{C}}$ is not self contained because of the last term $\bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}]$, which involves w rather than $w_{\mathcal{C}}$. The extra non-local term is supported in the intersection of the cylindrical and tip regions because all the terms in it involve derivatives of $\varphi_{\mathcal{C}}$, but not $\varphi_{\mathcal{C}}$ itself.

Let us abbreviate the right-hand side in (6.10) to

$$g := \mathcal{E}[w_{\mathcal{C}}] + \bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}].$$

Apply Lemma 6.7 to $w_{\mathcal{C}}$ solving (6.10), to conclude that there exist $\tau_0 \ll -1$ and constant $C_* > 0$, so that if the parameters (α, β, γ) are chosen to ensure that $\mathcal{P}_+ w_{\mathcal{C}}(\tau_0) = 0$, then $\hat{w}_{\mathcal{C}} := \mathcal{P}_+ w_{\mathcal{C}} + \mathcal{P}_- w_{\mathcal{C}}$ satisfies the estimate

$$\|\hat{w}_{\mathcal{C}}\|_{\mathfrak{D},\infty} \leq C_* \, \|g\|_{\mathfrak{D}^*,\infty} \tag{6.12}$$

for all $\tau \leq \tau_0$.

In the next two lemmas we focus on estimating $\|g\|_{\mathfrak{D}^*}$.

LEMMA 6.8. *For every* $\epsilon > 0$, *there exist a* τ_0 *so that for* $\tau \leq \tau_0$, *we have*

$$\|\mathcal{E}[w_{\mathcal{C}}]\|_{\mathfrak{D}^*} \leq \epsilon \, \|w_{\mathcal{C}}\|_{\mathfrak{D}}.$$

Proof. Recall that

$$\mathcal{E}[w_{\mathcal{C}}] = -\frac{u_{1y}^2}{1+u_{1y}^2}(w_{\mathcal{C}})_{yy} - \frac{(u_{1y}+u_{2y})u_{2yy}}{(1+u_{1y}^2)(1+u_{2y}^2)}(w_{\mathcal{C}})_y + \frac{2(n-1)-u_1u_2}{2u_1u_2} w_{\mathcal{C}}.$$

In [3] we showed that for, $\tau \leq \tau_0 \ll -1$

$$|(u_i)_y| + |(u_i)_{yy}| + |(u_i)_{yyy}| \leq \frac{\bar{C}(\theta)}{\sqrt{|\tau|}} \qquad \text{for } (y,\tau) \in \mathcal{C}_\theta, \tag{6.13}$$

where $u_i, i = 1, 2$ is any of the two considered solutions. The constant $\bar{C}(\theta)$ depends on θ and may change from line to line, but it is independent of τ as long as $\tau \leq \tau_0 \ll -1$.

Using (6.13) and Lemma 6.4, we have

$$\Big\| \frac{u_{1y}^2}{1+u_{1y}^2}(w_{\mathcal{C}})_{yy} \Big\|_{\mathfrak{D}^*} \leq \frac{\bar{C}(\theta)}{|\tau|} \|(w_{\mathcal{C}})_{yy}\|_{\mathfrak{D}^*} \leq \frac{\bar{C}(\theta)}{|\tau|} \, \|w_{\mathcal{C}}\|_{\mathfrak{D}}, \tag{6.14}$$

while by (6.13) and Lemma 6.3, we have

$$\Big\| \frac{(u_{1y}+u_{2y})u_{2yy}}{(1+u_1y^2)(1+u_{2y}^2)} \, (w_{\mathcal{C}})_y \Big\|_{\mathfrak{D}^*} \leq \frac{\bar{C}(\theta)}{|\tau|} \|(w_{\mathcal{C}})_y\|_{\mathfrak{D}^*} \leq \frac{\bar{C}(\theta)}{|\tau|} \, \|w_{\mathcal{C}}\|_{\mathfrak{H}}. \tag{6.15}$$

Also,

$$\Big\| \frac{(2(n-1)-u_1u_2)}{2u_1u_2} \, w_{\mathcal{C}} \Big\|_{\mathfrak{D}^*} \leq \Big\| \frac{(2(n-1)-u_1^2)}{2u_1u_2} \, w_{\mathcal{C}} \Big\|_{\mathfrak{D}^*} + \Big\| \frac{(u_1-u_2)}{2u_2} \, w_{\mathcal{C}} \Big\|_{\mathfrak{D}^*}.$$

It is very similar to deal with either of the terms on the right-hand side, so we explain how to deal with the first one next: Lemma 6.3, the uniform boundedness of our solutions and the fact that $u_i \geq \theta/4$ in $\mathcal{C}$ for $i \in \{1, 2\}$, give

$$\begin{aligned} \Big\| \frac{(2(n-1)-u_1^2)}{2u_1u_2} \, w_{\mathcal{C}} \Big\|_{\mathfrak{D}^*} &\leq \frac{\bar{C}(\theta)}{\theta^2} \|(\sqrt{2(n-1)} - u_1) \, w_{\mathcal{C}}\|_{\mathfrak{D}^*} \\ &\leq \frac{\bar{C}(\theta)}{\theta^2} \Big\| \frac{(\sqrt{2(n-1)} - u_1)}{y+1} \, w_{\mathcal{C}} \Big\|_{\mathfrak{H}}. \end{aligned}$$

Then, for any $K > 0$, we have

$$\left\| \frac{(2(n-1) - u_1^2)}{2u_1 u_2} w_{\mathcal{C}} \right\|_{\mathfrak{D}^*} \leq \frac{\bar{C}(\theta)}{\theta^2} \left(\int_{0 \leq y \leq K} \frac{(\sqrt{2(n-1)} - u_1)^2}{(y+1)^2} w_{\mathcal{C}}^2 \, e^{-\frac{y^2}{4}} \, dy \right)^{\frac{1}{2}} + \frac{\bar{C}(\theta)}{\theta^2} \left(\int_{y \geq K} \frac{(\sqrt{2(n-1)} - u_1)^2}{(y+1)^2} w_{\mathcal{C}}^2 \, e^{-\frac{y^2}{4}} \, dy \right)^{\frac{1}{2}}.$$

Now for any given $\epsilon > 0$, we choose K large so that $\frac{\bar{C}(\theta)}{\theta^2 K^2} < \frac{\epsilon}{6}$, and then for that chosen K, we choose a $\tau_0 \ll -1$ so that $\frac{\bar{C}(\theta)}{\theta^2}(\sqrt{2(n-1)} - u_1) < \frac{\epsilon}{6}$ for all $\tau \leq \tau_0$ and $0 \leq y \leq K$. (Note that here we use that $u_i(y,\tau)$ converges uniformly on compact sets in y to $\sqrt{2(n-1)}$, as $\tau \to -\infty$.) We conclude that for $\tau \geq \tau_0$,

$$\left\| \frac{(2(n-1) - u_1^2)}{2u_1 u_2} w_{\mathcal{C}} \right\|_{\mathfrak{D}^*} \leq \frac{\epsilon}{3} \, \|w_{\mathcal{C}}\|_{\mathfrak{H}} \leq \frac{\epsilon}{3} \, \|w_{\mathcal{C}}\|_{\mathfrak{D}}. \tag{6.16}$$

Finally combining (6.14), (6.15) and (6.16) finishes the proof of the lemma. □

We will next estimate the error term $\bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}]$.

LEMMA 6.9. *There exists a* $\tau_0 \ll -1$ *and* $\bar{C}(\theta)$ *so that for all* $\tau \leq \tau_0$, *we have*

$$\|\bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}]\|_{\mathfrak{D}^*} \leq \frac{\bar{C}(\theta)}{\sqrt{|\tau_0|}} \, \|\chi_{D_\theta} \, w\|_{\mathfrak{H}},$$

where $\bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}]$ *is defined by* (6.11) *and* χ_{D_θ} *is the characteristic function of the set* $D_\theta := \{\theta/2 < u < \theta\}$.

Proof. Setting

$$a(y,\tau) := \varphi_{\mathcal{C},\tau} - \varphi_{\mathcal{C},yy} + \frac{u_{1y}^2}{1+u_{1y}^2} \, \varphi_{\mathcal{C},yy} + \frac{(u_{1y} + u_{2y}) u_{2yy}}{(1+u_{1y}^2)(1+u_{2y}^2)} \varphi_{\mathcal{C},y}$$

and

$$b(y,\tau) := (\varphi_{\mathcal{C}})_y \qquad \text{and} \qquad d(y,\tau) := \frac{2u_{1y}^2}{1+u_{1y}^2} (\varphi_{\mathcal{C}})_y - 2(\varphi_{\mathcal{C}})_y$$

we may write

$$\bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}] = a(y,\tau) w + \frac{y}{2} \, b(y,\tau) \, w + d(y,\tau) \, w_y. \tag{6.17}$$

Note that the support of all three functions, $a(y,\tau)$, $b(y,\tau)$ and $d(y,\tau)$ is contained in D_θ and

$$|a(y,\tau)| + |b(y,\tau)| + |d(y,\tau)| \leq \frac{\bar{C}(\theta)}{\sqrt{|\tau|}}.$$

Furthermore, by (6.13) and Lemma 6.3 we get

$$\|a(y,\tau)\, w\|_{\mathfrak{D}^*} \le \|a(y,\tau) w\|_{\mathfrak{H}} \le \frac{\bar C(\theta)}{\sqrt{|\tau|}}\, \|w \chi_{D_\theta}\|_{\mathfrak{H}},$$

$$\|\frac{y}{2}\, b(y,\tau) w\|_{\mathfrak{D}^*} \le \|b(y,\tau)\, w\|_{\mathfrak{H}} \le \frac{\bar C(\theta)}{\sqrt{|\tau|}}\, \|w\, \chi_{D_\theta}\|_{\mathfrak{H}}$$

and

$$\begin{aligned}\|d(y,\tau)\, w_y\|_{\mathfrak{D}^*} &\le \|(d(y,\tau) w)_y\|_{\mathfrak{D}^*} + \|w d_y(y,\tau)\|_{\mathfrak{D}^*}\\ &\le \|d(y,\tau)\, w\|_{\mathfrak{H}} + \frac{\bar C(\theta)}{\sqrt{|\tau|}} \|w\, \chi_{D_\theta}\|_{\mathfrak{H}}\\ &\le \frac{\bar C(\theta)}{\sqrt{|\tau|}} \|w\, \chi_{D_\theta}\|.\end{aligned}$$

The above estimates together with (6.17) readily imply the lemma. □

Finally, we now employ all the estimates shown above to conclude the proof of Proposition 6.1.

Proof of Proposition 6.1. By (6.12) with $g := \mathcal{E}[w_{\mathcal{C}}] + \bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}]$ and using also Lemmas 6.8, 6.9 and the assumption that $\mathcal{P}_+ w_{\mathcal{C}}(\tau_0) = 0$, we have that for every $\epsilon > 0$, there exists a $\tau_0 \ll -1$ so that

$$\|\hat w_C\|_{\mathfrak{D},\infty} \le \epsilon\, \|w_{\mathcal{C}}\|_{\mathfrak{D},\infty} + \frac{\bar C(\theta)}{\sqrt{|\tau_0|}}\, \|w \chi_{D_\theta}\|_{\mathfrak{H},\infty}.$$

This readily gives the proposition. □

7. The tip region

Let $u_1(y,\tau)$ and $u_2(y,\tau)$ be the two solutions to equation (1.7) as in the statement of Theorem 1.7, and let $u_2^{\alpha\beta\gamma}$ be defined by (3.6). We will now estimate the difference of these solutions in the tip region $\mathcal{T}_\theta = \{(y,\tau) \,|\, u_1 \le 2\theta\}$ for $\theta > 0$ sufficiently small, and $\tau \le \tau_0 \ll -1$, where τ_0 is to be chosen later (see Figure 1). In the tip region we invert the functions $y \mapsto u_1(y,\tau)$ and $u_2^{\alpha\beta\gamma}(y,\tau)$ to get $Y_1(u,\tau)$ and $Y_2^{\alpha\beta\gamma}(u,\tau)$. By the change of variables (3.19) and by the definition of $u_2(y,\tau) := u_2^{\alpha\beta\gamma}(y,\tau)$ as in (3.6), we have that

$$Z_2^{\alpha\beta\gamma}(\rho,\tau) = \sqrt{|\tau|}\, \Big\{ Y_2^{\alpha\beta\gamma}\big(\frac{\rho}{\sqrt{|\tau|}}, \tau\big) - Y_2^{\alpha\beta\gamma}(0,\tau) \Big\},$$

where

$$\begin{aligned} Y_2^{\alpha\beta\gamma}(u,\tau) &= \alpha e^{\tau/2} + \sqrt{1+\beta e^\tau}\, Y_2\Big(\frac{u}{\sqrt{1+\beta e^\tau}}, \sigma\Big),\\ \sigma &:= \tau + \gamma - \log(1+\beta e^\tau).\end{aligned}$$

Note that $Z_2^{\alpha\beta\gamma}$ actually does not depend on α.

LEMMA 7.1. *If (α,β,γ) are τ_0 admissible in the sense of Definition 3.1, then*

$$Z_2^{\alpha\beta\gamma}(\rho,\tau)\to Z_0(\rho)\quad \text{as } \tau\to-\infty,$$

where the convergence is in C^∞_{loc} for bounded ρ.

Proof. Combining the above two equations yields

$$Z_2^{\alpha\beta\gamma}(\rho,\tau)=\frac{\sqrt{|\tau|}\,\sqrt{1+\beta e^\tau}}{\sqrt{|\sigma|}}\,Z_2\Big(\rho\,\frac{\sqrt{|\sigma|}}{\sqrt{|\tau|}\sqrt{1+\beta e^\tau}},\sigma\Big).$$

Since (α,β,γ) is τ_0 admissible, Definition 3.1 guarantees that we have $|\beta e^\tau|\le \epsilon|\tau_0|^{-1}$, and $|\gamma|\le\epsilon|\tau_0|$ for all $\tau\le\tau_0$.

It follows that

$$|\sigma-\tau|\le|\gamma|+C|\beta e^\tau|\le\epsilon|\tau_0|+\frac{C\epsilon}{|\tau_0|},$$

and thus

$$\left|\frac{\sqrt{|\sigma|}}{\sqrt{|\tau|}\sqrt{1+\beta e^\tau}}-1\right|\le C\epsilon$$

for all $\tau\le\tau_0$, while

$$\frac{\sqrt{|\sigma|}}{\sqrt{|\tau|}\sqrt{1+\beta e^\tau}}\to 1\quad(\tau\to-\infty).$$

Since $Z_2(\rho,\tau)\to Z_0(\rho)$ in C^∞_{loc} for bounded ρ, as $\tau\to-\infty$, we conclude that the same must be true for $Z_2^{\alpha\beta\gamma}$. $\square$

Hence, it is easy to see that in all the estimates below we can find a uniform $\tau_0\ll-1$, independent of parameters α,β and γ (as long as they are admissible with respect to τ_0), so that all the estimates below hold for $Y_1(u,\tau)-Y_2^{\alpha\beta\gamma}(u,\tau)$ for all $\tau\le\tau_0$.

To measure the distance between the two solutions in the tip region we consider the difference $W=Y_1-Y_2^{\alpha\beta\gamma}$ and multiply it by the cut-off function defined in (3.24), namely, set $W_T:=\varphi_T\,W$. Recall the norm $\|\cdot\|_{2,\infty}$ as defined in (3.22)–(3.23). The goal in this section is to prove the following estimate.

PROPOSITION 7.2. *There exist θ with $0<\theta\ll1$, $\tau_0\ll-1$ and $C<+\infty$ such that*

$$\|W_T\|_{2,\infty}\le\frac{C}{|\tau_0|}\,\|W\,\chi_{[\theta,2\theta]}\|_{2,\infty}\tag{7.1}$$

holds.

To simplify the notation throughout this section we will drop the subscript on Y_1 and write $Y=Y_1$ instead. Also, we will denote $Y_2^{\alpha\beta\gamma}$ by Y_2. As already explained in Section 3.2, the proof of this proposition will be based on a Poincaré inequality for the function W_T that is supported in the tip region. These estimates will be shown to hold with respect to an appropriately chosen weight $e^{\mu(u,\tau)}\,du$, where $\mu(u,\tau)$ is given by (7.2) below. We will begin by establishing various properties of the weight $\mu(u,\tau)$. We will continue with the proof of the Poincaré inequality, and we will finish with the proof of Proposition 7.2. Recall that the definitions of the *collar region* $\mathcal{K}_{L,\theta}$ and the *soliton region* $\mathcal{S}_L$ are given in Section 3.2.

7.1. *Properties of* $\mu(u,\tau)$. Let us begin by recalling the definition of our weight $\mu(u,\tau)$ in the tip region. Let $\zeta(u)$ be a non-negative smooth decreasing function defined on $u \in (0,\infty)$ such that

$$\zeta(u) = 1 \quad \text{for } u \geq \theta/2 \qquad \text{and} \qquad \zeta(u) = 0 \quad \text{for } u \leq \theta/4.$$

Such a function can be chosen to satisfy the derivative estimate $0 \leq |\zeta'(u)| \leq 5\theta^{-1}$. We define our *weight* $\mu(u,\tau)$ in the *tip region* to be

$$\mu(u,\tau) = -\frac{Y^2(\theta,\tau)}{4} + \int_\theta^u \mu_u(u',\tau)\,du', \tag{7.2}$$

where

$$\mu_u := \zeta(u)\left(-\frac{Y^2}{4}\right)_u + (1-\zeta(u))\,\frac{n-1}{u}\,(1+Y_u^2). \tag{7.3}$$

Note that since $\zeta = 1$ for $u \geq \theta/2$, we have $\mu(u,\tau) = -\frac{1}{4}Y^2(u,\tau)$ in this region, hence $e^{\mu(u,\tau)}$ coincides with the Gaussian weight $e^{-y^2/4}$ under our coordinate change $y = Y(u,\tau)$. This is important as our norms in the intersection of the cylindrical and tip regions need to coincide.

In a few subsequent lemmas we show estimates for the weight $\mu(u,\tau)$. In our first two lemmas we summarize some bounds on quantities involving Y and its derivatives Y_u, Y_{uu} and Y_τ that will be used in the remainder of this section. The estimates in the next lemma hold on the collar region $\mathcal{K}_{\theta,L}$. They have been essentially shown in Section 5, but we state them here for the reader's convenience.

Lemma 7.3. *For any small $\eta > 0$, there exist $0 < \theta \ll 1$, $L \gg 1$, and $\tau_0 \ll -1$, all depending on η such that the bounds*

$$\frac{|Y_{uu}|}{1+Y_u^2} \leq \eta\,\frac{|Y_u|}{u} \qquad \text{and} \qquad \Big|1 + \frac{u\,Y}{2(n-1)Y_u}\Big| \leq \eta \tag{7.4}$$

hold on $\mathcal{K}_{\theta,L}$ for all $\tau \leq \tau_0$.

Proof. Fix $\eta > 0$ small. The first bound follows from Proposition 5.5 by observing that since $Y_{uu} = u_{yy}\,u_y^{-3}$ and $Y_u = u_y^{-1}$, we have

$$\frac{|uY_{uu}|}{|Y_u|(1+Y_u^2)} = \frac{|uu_{yy}|}{u_y^2(1+u_y^{-2})} = \frac{|u\,u_{yy}|}{1+u_y^2} = \frac{\lambda_1}{\lambda_2}.$$

Hence, Proposition 5.5 guarantees that

$$\frac{|uY_{uu}|}{|Y_u|(1+Y_u^2)} < \eta$$

for $L \gg 1$ and $\tau \le \tau_0 \ll -1$ (both L and τ_0 depending on η). This readily gives us the first bound.

The second bound simply follows from the estimate in Corollary 5.7 by choosing the parameters θ, L such that $\epsilon(\theta, L) := \max\{4\theta^2, c(n)L^{-1}\} < \eta$ and $\tau_0 \ll -1$. $\square$

The estimates in the next lemma hold on the whole tip region $\mathcal{T}_\theta$.

LEMMA 7.4. *For any small $\eta > 0$, there exist $0 < \theta \ll 1$ and $\tau_0 \ll -1$ depending on η, such that the bounds*

$$\frac{1}{2n}\sqrt{|\tau|} < \left|\frac{Y_u}{u}\right| < \sqrt{|\tau|} \qquad \text{and} \qquad |Y_\tau| \le \eta\,\frac{|Y_u|}{u} < \eta\sqrt{|\tau|} \tag{7.5}$$

hold on $\mathcal{T}_\theta$, for all $\tau \le \tau_0$.

Proof. Fix $\eta > 0$ small, and assume without loss of generality that we are in the region where $u_y < 0$, $Y_u < 0$. We begin by showing the first bounds from above and below. We use the crucial inequality $(u^2)_{yy} \le 0$, which holds everywhere on our solution for $\tau \le \tau_0 \ll -1$ and was shown in Proposition 5.2. Expanding the square gives $u\,u_{yy} + u_y^2 \le 0$ and can be expressed in terms of Y and its derivatives in u (under the assumption that $u_y < 0$) as $uY_{uu} - Y_u \le 0$. Hence, since $Y_u < 0$, we have

$$\left[\frac{|Y_u|}{u}\right]_u = -\frac{u\,Y_{uu} - Y_u}{u^2} \ge 0. \tag{7.6}$$

It follows that for all $(u, \tau) \in \mathcal{T}_\theta$ where $u \le 2\theta$, we have

$$\lim_{u\to 0}\frac{|Y_u(u,\tau)|}{u} \le \frac{|Y_u(u,\tau)|}{u} \le \frac{|Y_u(2\theta,\tau)|}{2\theta}.$$

To estimate $\lim_{u\to 0}\frac{|Y_u(u,\tau)|}{u}$ from below, we observe that for $\tau \ll -1$, we have

$$\lim_{u\to 0}\frac{|Y_u(u,\tau)|}{u} = Y_{uu}(0,\tau) = \sqrt{|\tau|}\,Z_{\rho\rho}(0,\tau) > \frac{1}{2n}\sqrt{|\tau|}$$

since $Z(\rho,\tau) \to Z_0(\rho)$ in C^∞_{loc} for $\rho \ge 0$ as $\tau \to -\infty$.

To estimate the ratio $\frac{|Y_u|}{u}$ at $u = 2\theta$, we use our intermediate region asymptotics from Theorem 1.8, which imply that

$$-(u^2)_y = 2(n-1)\frac{y}{|\tau|} + o\left(\frac{1}{\sqrt{|\tau|}}\right) \implies Y_u = \frac{1}{u_y} = -\frac{u\,|\tau|}{(n-1)Y} + o\left(\sqrt{|\tau|}\right)$$

for $u = 2\theta$ and $\tau \le \tau_0 \ll -1$ (τ_0 depending on θ). Using the intermediate region asymptotics from Theorem 1.8 again we find

$$Y(2\theta,\tau) = \sqrt{|\tau|}\sqrt{2 - \frac{4\theta^2}{2(n-1)}} = \sqrt{2|\tau|}\sqrt{1 - \frac{\theta^2}{(n-1)}} > 1.2\sqrt{|\tau|}$$

(the last bound holds for $0 < \theta \ll 1$ depending on n), and so we conclude that for θ sufficiently small, we have

$$\frac{|Y_u|}{u} = \frac{|\tau|}{(n-1)Y} + o(\sqrt{|\tau|}) < \sqrt{|\tau|}$$

provided that $\tau \le \tau_0 \ll -1$, which proves the desired bound.

We will next prove the bound on $|Y_\tau|$ and will first deal with the region $\mathcal{K}_{\theta,L}$ for $L \gg 1$. We rearrange the terms in equation (3.16) to get

$$Y_\tau = \frac{Y_{uu}}{1+Y_u^2} + \frac{(n-1)Y_u}{u}\Big(1 + \frac{uY}{2(n-1)\,Y_u}\Big) - \frac{u^2}{2}\frac{Y_u}{u}. \tag{7.7}$$

For our given $\eta > 0$, we use both inequalities in (7.4) with η replaced by $\frac{\eta}{4(n-1)}$ instead of η (these bounds hold on $\mathcal{K}_{\theta,L}$ and for $\tau \le \tau_0$) and the bound $u \le 2\theta$ (which holds on $\mathcal{T}_\theta$) to obtain the inequality

$$|Y_\tau| \le \frac{|Y_u|}{u}\Big(\frac{\eta}{4(n-1)} + \frac{\eta}{4} + 4\theta^2\Big) \le \eta\,\frac{|Y_u|}{u},$$

which holds if we choose θ with $4\theta^2 \le \eta/4$. Hence, the desired bound holds when $L/\sqrt{|\tau|} \le u \le 2\theta$ and $\tau \le \tau_0 \ll -1$.

Next, we show that the bound on $|Y_\tau|$ holds for $u \le L/\sqrt{|\tau|}$ by simply using the convergence of $Z(\rho,\tau) := \sqrt{|\tau|}\,\big(Y(u,\tau) - Y(0,\tau)\big)$, $\rho = \sqrt{|\tau|}\,u$ to the soliton $Z_0(\rho)$. We first express the right-hand side of equation (3.16) in terms of Z, which after substituting $Y = Y(0,\tau) + \sqrt{|\tau|}\,Z$ and factoring out $\sqrt{|\tau|}$ gives

$$Y_\tau = \sqrt{|\tau|}\Big(\frac{Z_{\rho\rho}}{1+Z_\rho^2} + \frac{(n-1)}{\rho}\,Z_\rho + \frac{1}{2\sqrt{|\tau|}}Y(0,\tau) + \frac{1}{2|\tau|}\,\big(Z - \rho\,Z_\rho\big)\Big). \tag{7.8}$$

To estimate $|Y_\tau|$ from (7.8), we use that $\frac{1}{2\sqrt{|\tau|}}Y(0,\tau) = \frac{\sqrt{2}}{2} + o(1)$, as $\tau \to -\infty$, that $\big|Z - \rho Z_\rho\big| < C(L)$, on $\rho \le L$, and the convergence $\lim_{\tau\to-\infty} Z(\rho,\tau) = Z_0(\rho)$ on $\rho \le L$, which implies that

$$\Big|\frac{Z_{\rho\rho}}{1+Z_\rho^2} + \frac{(n-1)}{\rho}\,Z_\rho + \frac{\sqrt{2}}{2}\Big| < \frac{\eta}{10}$$

for $\tau \le \tau_0 \ll -1$. Combining all these bounds readily gives that $|Y_\tau| < \eta\sqrt{|\tau|}$ holds on $0 \le u \le L/\sqrt{|\tau|}$, holds on $\rho := \sqrt{|\tau|}\,u \le L$ and for all $\tau \le \tau_0 \ll -1$, where τ_0 depends on η, L. This finishes the proof of the bound for Y_τ concluding the proof of the lemma. □

LEMMA 7.5. *For any small $\eta > 0$, there exists $\theta > 0$ small depending on η and $\tau_0 \ll -1$ depending on η, θ such that*

$$1 - \eta \le \frac{u\,\mu_u}{(n-1)\,(1+Y_u^2)} \le 1 + \eta \tag{7.9}$$

and

$$\mu_\tau \le \eta\,|\tau| \tag{7.10}$$

hold on $\mathcal{T}_\theta$ for all $\tau \le \tau_0$.

Proof. We begin with the proof of (7.9). By the definition of the weight $\mu(u,\tau)$, to satisfy (7.3) we have $\mu_u = \frac{n-1}{u}\,(1+Y_u^2)$ on $u \le \theta/4$ where $\zeta(u) = 0$. Hence, it is sufficient to show that

$$1 - \eta \le \frac{u}{(n-1)(1+Y_u^2)} \left(-\frac{Y^2}{4}\right)_u \le 1 + \eta \tag{7.11}$$

holds on the set where $\theta/4 \le u \le 2\theta$ (which is the intersection of $\mathcal{T}_\theta$ with $\{u \ge \theta/4\}$). This readily follows from the second bound in (7.4) since

$$\frac{u}{(n-1)(1+Y_u^2)} \left(-\frac{Y^2}{4}\right)_u = -\frac{u\,Y}{2(n-1)Y_u}\,\frac{Y_u^2}{1+Y_u^2}$$

and $Y_u^2 \gg 1$ in the considered region. Note that since we are interested in a bound that only holds on $\theta/4 \le u \le 2\theta$, the above bound holds if we choose θ sufficiently small depending on η and $\tau_0 \ll -1$ depending on η, θ.

We will now proceed with the proof of (7.10), which will follow from the definition of $\mu(u,\tau)$ in (7.2)–(7.3) and the bounds (7.5). Without loss of generality we will assume that we are in the region where $y > 0$, $u_y < 0$, or equivalently, $Y > 0$, $Y_u < 0$. We use the definition of $\mu(u,\tau)$ in (7.2)–(7.3) and that $\zeta \equiv 1$ for $u \ge \theta/2$. Integration by parts gives

(7.12)

$$\begin{aligned}
\mu_\tau &= \left(-\frac{Y^2(\theta,\tau)}{4}\right)_\tau + \int_\theta^u \left\{\zeta\left(-\frac{Y^2}{4}\right)_{u\tau} + (1-\zeta)\,\frac{(n-1)(1+Y_u^2)_\tau}{u}\right\} du \\
&= \left(-\frac{Y^2(\theta,\tau)}{4}\right)_\tau + \int_\theta^u \left\{\zeta\left(-\frac{Y^2}{4}\right)_{u\tau} + (1-\zeta)\,\frac{2(n-1)Y_u Y_{u\tau}}{u}\right\} du \\
&= \zeta\left(-\frac{Y^2(u,\tau)}{4}\right)_\tau - \int_u^\theta \zeta'\left(-\frac{Y^2}{4}\right)_\tau du + 2(n-1)\int_u^\theta \zeta'\,\frac{Y_u}{u}\,Y_\tau\,du \\
&\qquad + 2(n-1)(1-\zeta)\,\frac{Y_u\,Y_\tau}{u} + 2(n-1)\int_u^\theta (1-\zeta)\left[\frac{Y_u}{u}\right]_u Y_\tau\,du,
\end{aligned}$$

where, to simplify the notation, we will denote the variable of integration by u (instead of u') when there is no danger of confusion.

Fix $\eta > 0$ small. Observe first that the second bound in (7.5) and $Y \le 2\sqrt{2\,|\tau|}$ imply that for all $u \le 2\theta$, we have

$$\left|\left(-\frac{Y^2(u,\tau)}{4}\right)_\tau\right| = \left|\frac{Y\,Y_\tau}{2}\right| \le 2\sqrt{|\tau|}|Y_\tau| \le 2\eta\,|\tau| \tag{7.13}$$

for $\tau \ll -1$. This bound combined with (7.12) implies that in the region $u \ge \theta$, where $\zeta = 1, \zeta' = 0$, the desired bound (7.10) holds.

Assume now that $u \le \theta$. Using (7.13) to estimate the first two terms in the third line of (7.12), and using the bounds (7.5) to estimate the third term of (7.13), we get

$$\left|\zeta\left(-\frac{Y^2(u,\tau)}{4}\right)_\tau + \int_u^\theta \zeta'\left(-\frac{Y^2}{4}\right)_\tau du + 2(n-1)\int_u^\theta \zeta'\,\frac{Y_u}{u}\,Y_\tau du\right| < c(n)\eta\,|\tau| \tag{7.14}$$

holds on $\mathcal{T}_\theta$, for θ small and $\tau \le \tau_0 \ll -1$ (recall that ζ, ζ' are zero for $u \le \theta/4$) and $c(n)$ is a universal constant that depends only on a dimension.

Furthermore, the bounds (7.5) imply that

$$\left|2(n-1)(1-\zeta)\,\frac{Y_u\,Y_\tau}{u}\right| < 2(n-1)\eta\,|\tau|. \tag{7.15}$$

It remains to estimate the last integral in (7.12). To this end, recall (7.6), which gives (since $Y_u < 0$) the inequality

$$\left[\frac{Y_u}{u}\right]_u = \frac{u\,Y_{uu} - Y_u}{u^2} \le 0$$

and therefore

$$\int_u^\theta (1-\zeta)\left|\left[\frac{Y_u}{u}\right]_u\right| du \le \int_u^\theta \left[-\frac{Y_u}{u}\right]_u du \le \frac{|Y_u|(\theta,\tau)}{u} < \sqrt{\tau}.$$

This combined with our bound $|Y_\tau| \le \eta\,\sqrt{|\tau|}$ give us

$$2(n-1)\left|\int_u^\theta (1-\zeta)\,\left[\frac{Y_u}{u}\right]_u Y_\tau\,du\right| < 2(n-1)\eta\,|\tau|. \tag{7.16}$$

Finally, combining (7.12) with (7.14)–(7.16) shows that

$$\mu_\tau \le c(n)\,\eta\,|\tau|$$

from which the desired bound (7.10) follows if we start our estimates with $\eta/c(n)$ instead of η. □

7.2. *Poincaré inequality.* We will next show a *weighted Poincaré type estimate* with respect to weight $\mu(u,\tau)$ defined in (7.2)–(7.3). This inequality will play a crucial role in the proof of Proposition 7.2. For a fixed $\tau \le \tau_0$ where τ_0 is sufficiently negative, we recall that $\mathcal{T}_{\theta_0} := \{u : 0 \le u \le 2\theta_0\}$ and consider the solution $Y(u,\tau)$ and the weight profile $\mu(u,\tau)$ as functions of u for $u \in [0, 2\theta_0]$.

PROPOSITION 7.6 (Poincaré inequality). *There exist an absolute constant $C_0 > 0$ and a small absolute constant θ_0, such that*

$$|\tau| \int f^2(u)\, e^{\mu(u,\tau)}\, du \le C_0 \int \frac{f_u^2(u)}{1+Y_u^2}\, e^{\mu(u,\tau)}\, du \tag{7.17}$$

holds for any smooth compactly supported function $f(u)$ in $[0, 2\theta_0)$ with $f'(0) = 0$ and for all $\tau \le \tau_0 \ll -1$, where τ_0 depends on θ_0.

Proof. By the Peter-Paul inequality $2ab \le a^2 + b^2$, we have

$$-\frac{2\, f f_u}{u} \le \frac{4 f_u^2}{1+Y_u^2} + (1+Y_u^2)\, \frac{f^2}{4u^2}.$$

Multiply with $e^{\mu(u,\tau)}$ and integrate by parts over the interval $u_0 \le u \le 2\theta$ for some small $u_0 \in (0, 2\theta)$ to obtain

$$\begin{aligned}\int_{u_0}^{2\theta} \Big(\frac{4 f_u^2}{1+Y_u^2} + (1+Y_u^2)\, \frac{f^2}{4u^2}\Big)\, e^{\mu} du &\ge -\int_{u_0}^{2\theta} \frac{(f^2)_u}{u}\, e^{\mu} du \\ &= \frac{f(u_0)^2}{u_0} e^{\mu(u_0,\tau)} + \int_{u_0}^{2\theta} \frac{u\mu_u - 1}{u^2}\, f^2 e^{\mu} du.\end{aligned}$$

Rearranging terms leads to

$$\frac{f(u_0)^2}{u_0} e^{\mu(u_0,\tau)} + \int_{u_0}^{2\theta} \Big(u\mu_u - \tfrac14 (1+Y_u^2) - 1\Big)\, \frac{f^2}{u^2}\, e^{\mu} du \le 4 \int_{u_0}^{2\theta} \frac{f_u^2}{1+Y_u^2}\, e^{\mu} du. \tag{7.18}$$

We next apply (7.9) with $\eta = 1/4$, which shows that there exists $0 < \theta_0 \ll 1$ such that lower bound on $u\mu_u \ge \frac34 (n-1)(1+Y_u^2)$ holds on $\mathcal{T}_{\theta_0}$. Hence we find that

$$u\mu_u - \frac14 (1+Y_u^2) - 1 \ge \frac{3(n-1)-1}{4}\, (1+Y_u^2) - 1 \ge \frac12 (1+Y_u^2) - 1 \tag{7.19}$$

holds on $\mathcal{T}_{\theta_0}$ for $\tau \le \tau_0$. (Here τ_0 is an absolute constant, and we have used that $n \ge 2$.)

In (7.5) we found a lower bound for $|Y_u|/u$, which implies

$$1 + Y_u^2 \ge Y_u^2 \ge c_0(n)\, u^2 |\tau|.$$

If we choose u_0 depending on τ so that $c_0(n) u_0(\tau)^2 |\tau| \ge 4$, then

$$u\mu_u - \frac14 (1+Y_u^2) \ge \frac14 (1+Y_u^2),$$

and (7.19), (7.18) imply

$$\frac{f(u_0)^2}{u_0} e^{\mu(u_0,\tau)} + \frac{1}{4}\int_{u_0}^{2\theta} (1+Y_u^2)\, \frac{f^2}{u^2}\, e^{\mu} du \le 4\int_{u_0}^{2\theta} \frac{f_u^2}{1+Y_u^2}\, e^{\mu} du$$

holds for all $\tau \le \tau_0 \ll -1$ and with $u_0 = 2/\sqrt{-c_0(n)\tau}$.

Using $-Y_u/u \ge C\sqrt{-\tau}$, we can extract the following two estimates from this:

$$|\tau| \int_{u_0(\tau)}^{2\theta} f^2 e^{\mu}\, du \le C \int_{u_0(\tau)}^{2\theta} \frac{f_u^2}{1+Y_u^2} e^{\mu}\, du, \tag{7.20}$$

$$f(u_0)^2 e^{\mu(u_0,\tau)} \le 4u_0(\tau) \int_{u_0(\tau)}^{2\theta} \frac{f_u^2}{1+Y_u^2} e^{\mu}\, du\,. \tag{7.21}$$

To complete the proof we now apply the standard Poincaré inequality on the ball of radius u_0 in $\mathbb{R}^n$ to the function f. Recall that this inequality states that for all $f \in C^1([0,u_0))$ with $f(u_0)=0$, one has

$$\int_0^{u_0} f(u)^2 u^{n-1}\, du \le c(n) u_0^2 \int_0^{u_0} f_u^2 u^{n-1}\, du. \tag{7.22}$$

We may assume that $f(u_0) \ne 0$, in which case we use $f(u)^2 = \big(f(u) - f(u_0) + f(u_0)\big)^2 \le 2\big(f(u)-f(u_0)\big)^2 + 2f(u_0)^2$ and apply the above inequality to $f(u) - f(u_0)$ to get

$$\int_0^{u_0} f(u)^2 u^{n-1}\, du \le \frac{2}{n} u_0^n f(u_0)^2 + c(n) u_0^2 \int_0^{u_0} f_u^2 u^{n-1}\, du.$$

If $f(u_0)=0$, then we can directly use inequality (7.22). In the region $u \le u_0(\tau)$ one has $u \le \theta/4$ and thus $\mu_u = \frac{n-1}{u}(1+Y_u^2)$. Hence

$$\mu(u,\tau) - \mu(u_0,\tau) = (n-1)\log\frac{u}{u_0} + (n-1)\int_{u_0}^{u} \frac{1}{u} Y_u^2\, du.$$

Use $|Y_u/u| \le C\sqrt{-\tau}$ again to estimate

$$\left| \int_{u_0}^{u} \frac{1}{u} Y_u^2\, du \right| \le C|\tau| u_0(\tau)^2 \le C,$$

which implies that for some constant C,

$$\frac{1}{C} e^{\mu(u_0,\tau)} \left(\frac{u}{u_0}\right)^{n-1} \le e^{\mu(u,\tau)} \le C e^{\mu(u_0,\tau)} \left(\frac{u}{u_0}\right)^{n-1}$$

for all $u \in [0, u_0(\tau)]$ and $\tau \le \tau_0$ if $-\tau_0$ is sufficiently large. Using this we obtain

$$\begin{aligned}\int_0^{u_0(\tau)} f^2 e^{\mu(u,\tau)}\, du &\le C e^{\mu(u_0,\tau)} u_0^{-n+1} \int_0^{u_0(\tau)} f(u)^2 u^{n-1}\, du \\ &\le C e^{\mu(u_0,\tau)} u_0^{-n+1} \left\{ c(n) u_0^2 \int_0^{u_0(\tau)} f_u^2 u^{n-1}\, du + \frac{2}{n} u_0^n f(u_0)^2 \right\} \\ &\le c(n) C^2 u_0^2 \int_0^{u_0(\tau)} f_u^2 e^{\mu(u,\tau)}\, du + C u_0 e^{\mu(u_0,\tau)} f(u_0)^2.\end{aligned}$$

To continue, we use that $|Y_u(u,\tau)|$ is uniformly bounded in the region $0 \le u \le u_0(\tau)$, and we also use (7.21) to get

$$\begin{aligned}\int_0^{u_0(\tau)} f^2 e^{\mu(u,\tau)}\, du &\le C u_0^2 \int_0^{u_0(\tau)} \frac{f_u^2}{1+Y_u^2} e^{\mu(u,\tau)}\, du + C u_0^2 \int_{u_0}^{2\theta} \frac{f_u^2}{1+Y_u^2} e^{\mu(u,\tau)}\, du \\ &\le C u_0^2 \int_0^{2\theta} \frac{f_u^2}{1+Y_u^2} e^{\mu(u,\tau)}\, du.\end{aligned}$$

Finally recall that $u_0(\tau)^2 = 4/(c_0(n)|\tau|)$, and combine with the estimate (7.20) on the interval $[u_0(\tau), 2\theta]$ to arrive at

$$|\tau| \int_0^{u_0(\tau)} f^2 e^{\mu(u,\tau)}\, du \le C \int_0^{2\theta} \frac{f_u^2}{1+Y_u^2} e^{\mu(u,\tau)}\, du. \qquad \square$$

7.3. *Proof of Proposition* 7.2. In order to prove Proposition 7.2, we combine an energy estimate for the difference $W = Y_1 - Y_2$, which will be shown below, with our Poincaré inequality (7.17).

Proof of Proposition 7.2. Recall that $\varphi_T(u)$ denotes a standard smooth cut-off function supported on $0 \le u < 2\theta$, with $\varphi_T = 1$ on $0 \le u \le \theta$ and $\varphi_T = 0$ for $u \ge 2\theta$. To simplify the notation, in the proof below *we will drop the index T from φ_T and simply denote φ_T by φ and let $W_T := W\varphi_T = W\varphi$.*

We have seen in Section 3.2 that $W = Y - Y_2$ satisfies the equation

$$W_\tau = \frac{W_{uu}}{1+Y_u^2} + \left(\frac{n-1}{u} - \frac{u}{2} + D\right) W_u + \frac{1}{2} W, \tag{7.23}$$

where

$$D := -\frac{Y_{2uu}\,(Y_u + Y_{2u})}{(1+Y_u^2)\,(1+Y_{2u}^2)}. \tag{7.24}$$

As usual, multiplying (7.23) by $W\varphi^2 e^\mu$ and integrating by parts we obtain

$$\begin{aligned}\frac{d}{d\tau}\Big(\frac{1}{2}\int W_T^2\, e^\mu\, du\Big) &= -\int \frac{W_u^2}{1+Y_u^2}\,\varphi^2\, e^\mu\, du\\ &+\int\Big(\frac{n-1}{u}-\frac{u}{2}-\frac{\mu_u}{1+Y_u^2}+\frac{2Y_uY_{uu}}{(1+Y_u^2)^2}+D\Big)W_uW\varphi^2 e^\mu\, du\\ &-2\int\frac{1}{1+Y_u^2}\,W_uW\,\varphi\,\varphi_u\, e^\mu\, du+\int W_T^2\Big(\frac{1}{2}+\mu_\tau\Big)\, e^\mu\, du.\end{aligned}$$

Let us write

$$G := \frac{n-1}{u}-\frac{u}{2}-\frac{\mu_u}{1+Y_u^2}+\frac{2Y_uY_{uu}}{(1+Y_u^2)^2}+D. \tag{7.25}$$

Then, we have

$$\begin{aligned}\frac{d}{d\tau}\Big(\frac{1}{2}\int W_T^2\, e^\mu\, du\Big) &= -\int \frac{W_u^2\varphi^2}{1+Y_u^2}\, e^\mu\, du+\int G\,W_uW\,\varphi^2\, e^\mu\, du\\ &\quad+2\int\frac{1}{1+Y_u^2}\,W_uW\,\varphi\,\varphi_u\, e^\mu\, du+\int W_T^2\Big(\frac{1}{2}+\mu_\tau\Big)\, e^\mu\, du.\end{aligned}$$

Applying Cauchy-Schwarz to the term above that contains G, we have

$$\begin{aligned}\int G\,W_uW\,\varphi^2\, e^\mu\, du \leq &\frac{1}{2}\int\frac{W_u^2\varphi^2}{1+Y_u^2}\, e^\mu\, du\\ &+\frac{1}{2}\int G^2\,(1+Y_u^2)\,W_T^2\, e^\mu\, du,\end{aligned}$$

which inserting in the previous identity gives

$$\begin{aligned}\frac{d}{d\tau}\Big(\frac{1}{2}\int W_T^2 e^\mu\, du\Big) \leq &-\frac{1}{2}\int\frac{W_u^2\varphi^2}{1+Y_u^2}\, e^\mu\, du+\frac{1}{2}\int G^2\,(1+Y_u^2)\,W_T^2\, e^\mu\, du\\ &+2\int\frac{W\,W_u}{1+Y_u^2}\,\varphi\varphi_u\, e^\mu\, du+\int W_T^2\Big(\frac{1}{2}+\mu_\tau\Big)\, e^\mu\, du.\end{aligned}$$

Furthermore, using $(W_T)_u^2 = \big(W_u\varphi+W\varphi_u\big)^2$ to write

$$-\frac{1}{2}\,W_u^2\varphi^2 = -\frac{1}{2}\,(W_T)_u^2+\frac{1}{2}\,W^2\varphi_u^2+WW_u\,\varphi\varphi_u,$$

after combining and rearranging terms, we obtain the integral bound

$$\begin{aligned}\frac{d}{d\tau}\Big(\frac{1}{2}\int W_T^2\, e^\mu\, du\Big) \leq &-\frac{1}{2}\int\frac{(W_T)_u^2}{1+Y_u^2}\, e^\mu\, du+3\int\frac{WW_u}{1+Y_u^2}\,\varphi\varphi_u e^\mu\, du\\ &+\int\Big[\frac{1}{2}G^2\,(1+Y_u^2)+\frac{1}{2}+\mu_\tau\Big]\, e^\mu\, du+\frac{1}{2}\int\frac{W^2}{1+Y_u^2}\,\varphi_u^2 e^\mu\, du.\end{aligned}$$

Next, we use $W_u\varphi = (W_T)_u - W\,\varphi_u$ and the inequality $ab \leq \frac{a^2}{12}+3b^2$ to get

$$WW_u\varphi\varphi_u = (W_T)_uW\varphi_u - W^2\varphi_u^2 \leq \frac{1}{12}(W_T)_u^2+2W^2\varphi^2,$$

which implies the integral bound

$$3\int \frac{WW_u}{1+Y_u^2}\,\varphi\,\varphi_u\,e^\mu\,du \le \frac{1}{4}\int \frac{(W_T)_u^2}{1+Y_u^2}\,e^\mu\,du + 6\int \frac{W^2}{1+Y_u^2}\,\varphi_u^2\,e^\mu\,du.$$

Note that the support of φ_u is contained in the region $\{\theta \le u \le 2\theta\}$. Combining the above yields

$$\frac{d}{d\tau}\int W_T^2\,e^\mu\,du \le -\frac{1}{2}\int \frac{(W_T)_u^2}{1+Y_u^2}\,e^\mu\,du + \int \bar{G}\,W_T^2\,e^\mu\,du + C(\theta)\int_\theta^{2\theta} W^2 e^\mu\,du, \tag{7.26}$$

where

$$\bar{G} := G^2\,(1+Y_u^2) + 1 + 2\,\mu_\tau. \tag{7.27}$$

Claim 7.7. For $\eta > 0$ sufficiently small, there exist $0 < \theta \ll 1$ depending on η and $\tau_0 \ll -1$ depending on η, θ such that

$$G^2\,(1+Y_u^2)\;\le\;\frac{\eta}{3}\,|\tau| \tag{7.28}$$

on $\mathcal{T}_\theta$ for all $\tau \le \tau_0$.

Proof of Claim. Fix $\eta > 0$ sufficiently small. We will prove the bound

$$|G|\,\sqrt{1+Y_u^2} \le C(n)\,\eta\,\sqrt{|\tau|} \tag{7.29}$$

in which the constant $C(n)$ only depends on the dimension n. This will readily imply (7.28) if η is chosen sufficiently small. We begin by establishing that the desired bound holds on the collar region $\mathcal{K}_{\theta,L} := \{L/\sqrt{|\tau|} \le u \le 2\theta\}$ for $0 < \theta \ll 1, L \gg 1$ depending on η. First let us bound the first three terms in (7.25) multiplied by $\sqrt{1+Y_u^2}$ together. Using (7.9) and the bound $|Y_u|/u \le \sqrt{|\tau|}$ given in (7.5) to obtain that in the region where $Y_u^2 > 1$, we have

$$\begin{aligned}\Big|\frac{n-1}{u} - \frac{u}{2} - \frac{\mu_u}{1+Y_u^2}\Big|\,\sqrt{1+Y_u^2} &\le \frac{2(n-1)\,|Y_u|}{u}\left(\eta + \frac{u^2}{2}\right)\\ &\le 2(n-1)(\eta+2\theta^2)\,\sqrt{|\tau|} < \frac{\sqrt{\eta}}{10}\,\sqrt{|\tau|},\end{aligned} \tag{7.30}$$

where the last inequality holds for $\eta \ll 1$ (depending on n), $0 \le u \le 2\theta \ll 1$ (where θ depends on η) and $\tau \le \tau_0 \ll -1$.

To bound the fourth term in (7.25) multiplied by $\sqrt{1+Y_u^2}$, we use that $Y_u^2 > 1$ in this region, and we combine the first bound in (7.4) with the first bound in (7.5) to obtain that for $L/\sqrt{|\tau|} \le u \le 2\theta$ with $L \gg 1$ (depending on η) and $\tau \ll -1$, we have

$$\frac{2|Y_u Y_{uu}|\,\sqrt{1+Y_u^2}}{(1+Y_u^2)^2} \le 2\eta\,\frac{|Y_u|}{u}\;\frac{|Y_u|}{\sqrt{1+Y_u^2}} < 2\eta\,\sqrt{|\tau|}. \tag{7.31}$$

To bound the last term in (7.25) multiplied by $\sqrt{1+Y_u^2}$, we recall the definition of D in (7.24) to write

$$|D|\sqrt{1+Y_u^2} := \frac{|Y_{2uu}|\,|Y_u+Y_{2u}|}{\sqrt{1+Y_u^2}\,(1+Y_{2u}^2)} \le \frac{|Y_{2uu}|}{1+Y_{2u}^2}\,\frac{|Y_u|+|Y_{2u}|}{\sqrt{1+Y_u^2}}.$$

Using the first bound in (7.4) applied to Y_2 to estimate $\frac{|Y_{2uu}|}{1+Y_{2u}^2} \le \eta\,\frac{|Y_{2u}|}{u}$ and the bounds in (7.5) to estimate the ratio $\frac{|Y_{2u}|}{|Y_u|} < 2n$ and $\frac{|Y_{2u}|}{u} < \sqrt{|\tau|}$, we obtain the estimate

$$|D|\sqrt{1+Y_u^2} \le \eta\,\frac{|Y_{2u}|}{u}\,\frac{|Y_u|+|Y_{2u}|}{\sqrt{1+Y_u^2}} < (2n+1)\,\eta\,\sqrt{|\tau|}. \tag{7.32}$$

Combining (7.30), (7.31) and (7.32) yields that (7.29) holds on $L/\sqrt{|\tau|} \le u \le 2\theta$ for $\tau \le \tau_0 \ll -1$, provided that η is chosen sufficiently small depending on n and provided $\theta \ll 1$, $L \gg 1$ and $\tau \ll -1$ (all depending on η).

It remains to prove the inequality (7.29) in the soliton region $S_L := \{0 \le u \le L/\sqrt{|\tau|}\}$, where $L > 1$ is now fixed so that (7.29) holds in the collar region $\mathcal{K}_{\theta,L}$. Recall that in this region $\frac{u\mu_u}{(n-1)\,(1+Y_u^2)} = 1$, by the definition of our weight. Using also the change of variables

$$Y_i(u,\tau) = Y_i(0,\tau) + \frac{1}{\sqrt{|\tau|}}\,Z_i(\rho,\tau), \qquad \rho := u\sqrt{|\tau|}, \qquad i = 1,2,$$

we find that in S_L we have

$$G\sqrt{1+Y_u^2} = \Big(-\frac{\rho}{2|\tau|} + \frac{2Z_\rho Z_{\rho\rho}}{(1+Z_\rho^2)^2} - \frac{Z_{2\rho\rho}\,(Z_\rho+Z_{2\rho})}{(1+Z_\rho^2)\,(1+Z_{2\rho}^2)}\Big)\sqrt{1+Z_\rho^2}\,\sqrt{|\tau|}.$$

The C^∞ convergence of $Z_i(\rho,\tau) \to Z_0(\rho)$ on the soliton region S_L where $\rho \le L$ implies that

$$\Big|\frac{2Z_\rho Z_{\rho\rho}}{(1+Z_\rho^2)^2} - \frac{Z_{2\rho\rho}\,(Z_\rho+Z_{2\rho})}{(1+Z_\rho^2)\,(1+Z_{2\rho}^2)}\Big|\sqrt{1+Z_\rho^2} < \frac{\eta}{2}$$

on S_L if $\tau \ll -1$ depending on L and η. Hence,

$$|G|\sqrt{1+Y_u^2} \le \frac{L}{2|\tau|}\sqrt{1+Z_\rho^2} + \frac{\eta}{2} < \eta$$

provided $\tau \le \tau_0 \ll -1$, which readily implies that (7.29) also holds in the soliton region.

Finally squaring (7.29) and taking $\eta < C(n)^{-2}/3$ yields the bound (7.28). □

We now conclude the proof of Proposition 7.2. Let $\eta > 0$ be a sufficiently small number (depending on on dimension n) so that (7.28) holds on $\mathcal{T}_\theta$ for $\theta \ll 1$ and $\tau \le \tau_0 \ll -1$. Lemma 7.5 implies that by decreasing θ and τ_0, if necessary, the bound $\mu_\tau < \frac{\eta}{3}|\tau|$ holds on the whole tip region $\mathcal{T}_\theta$ for $\tau \le \tau_0$. The two bounds imply that

$$\bar{G} := G^2\,(1+Y_u^2) + 1 + 2\,\mu_\tau < \eta\,|\tau|$$

on $\mathcal{T}_\theta$ provided $\tau \le \tau_0$. Inserting this bound in (7.26) yields

$$\begin{aligned}\frac{d}{d\tau}\int W_T^2\,e^\mu\,du \le -\frac{1}{2}\int \frac{(W_T)_u^2}{1+Y_u^2}\,e^\mu\,du \\ + \eta|\tau|\int W_T^2\,e^\mu\,du + C(\theta)\int_\theta^{2\theta} W^2\,e^\mu\,du.\end{aligned}$$

On the other hand, our Poincaré inequality implies that

$$\int \frac{(W_T)_u^2}{1+Y_u^2}\,e^\mu\,du \ge c_0\,|\tau|\int W_T^2\,e^\mu\,du$$

for an absolute constant $c_0 > 0$ that is uniform in τ and independent of θ. This inequality holds if $\theta \le \theta_0$, where θ_0 is again an absolute constant. Finally choose $\eta := \dfrac{c_0}{4}$. Such an η is an absolute constant and determines θ and τ_0. Our Poincaré inequality then yields that

$$\begin{aligned}-\frac{1}{2}\int \frac{(W_T)_u^2}{1+Y_u^2}\,e^\mu\,du + \eta|\tau|\int W_T^2\,e^\mu\,du \\ \le -\frac{c_0}{2}\,|\tau|\int W_T^2\,e^\mu\,du + \eta\int W_T^2\,e^\mu\,du \\ \le -\frac{c_0}{4}\int |\tau|\,W_T^2\,e^\mu\,du\end{aligned}$$

holds provided $\tau \le \tau_0$, with τ_0. Combining this with our energy inequality we finally conclude that in the tip region $\mathcal{T}_\theta$ the following holds:

$$\frac{d}{d\tau}\int W_T^2\,e^\mu\,du \le -\frac{c_0}{4}\,|\tau|\int W_T^2\,e^\mu\,du + \frac{C(\theta)}{|\tau|}\int (W\chi_{[\theta,2\theta]})^2\,e^\mu\,du. \tag{7.33}$$

Define

$$f(\tau) := \int W_T^2\,e^\mu\,du, \qquad g(\tau) := \int (W\chi_{[\theta,2\theta]})^2\,e^\mu\,du.$$

Then equation (7.33) becomes

$$\frac{d}{d\tau}f(\tau) \le -\frac{c_0}{4}\,|\tau|\,f(\tau) + \frac{C(\theta)}{|\tau|}\,g(\tau).$$

Furthermore, setting $F(\tau) := \int_{\tau-1}^{\tau} f(s)\,ds$ and $G(\tau) := \int_{\tau-1}^{\tau} g(s)\,ds$, we have

$$\begin{aligned}\frac{d}{d\tau}F(\tau) = f(\tau) - f(\tau-1) = \int_{\tau-1}^{\tau}\frac{d}{ds}f(s)\,ds \\ \le \frac{c_0}{4}\int_{\tau-1}^{\tau} s f(s)\,ds + \int_{\tau-1}^{\tau}\frac{C(\theta)}{|s|}g(s)\,ds,\end{aligned}$$

implying

$$\frac{d}{d\tau}F(\tau) \le \frac{c_0}{8}\,\tau\, F(\tau) + \frac{C(\theta)}{|\tau|}G(\tau).$$

This is equivalent to

$$\frac{d}{d\tau}\big(e^{-c_0\tau^2/16}F(\tau)\big) \le \frac{C(\theta)}{|\tau|}e^{-c_0\tau^2/16}\,G(\tau).$$

Since W_T is uniformly bounded for $\tau \le \tau_0 \ll -1$, it follows that $f(\tau)$ and therefore also $F(\tau)$ are uniformly bounded functions for $\tau \le \tau_0$. Therefore, we have $\lim_{\tau\to-\infty} e^{-c_0\tau^2/16}F(\tau) = 0$, so for the last differential inequality we get

$$\begin{aligned} e^{-c_0|\tau|^2/16}\,F(\tau) &\le C\int_{-\infty}^{\tau}\frac{G(s)}{s^2}(|s|\,e^{-c_0 s^2/16})\,ds \\ &\le \frac{C}{|\tau|^{\frac{3}{2}}}\sup_{s\le\tau}\Big(|s|^{-\frac{1}{2}}G(s)\Big)\int_{-\infty}^{\tau}|s|\,e^{-c_0 s^2/16}\,ds \\ &\le \frac{C}{|\tau|^{\frac{3}{2}}}\sup_{s\le\tau}\Big(|s|^{-\frac{1}{2}}G(s)\Big)\,e^{-c_0\tau^2/16} \end{aligned}$$

with $C = C(\theta)$. This yields

$$\sup_{s\le\tau}\Big(|s|^{-\frac{1}{2}}F(s)\Big) \le \frac{C}{|\tau|^2}\sup_{s\le\tau}\Big(|s|^{-\frac{1}{2}}\,G(s)\Big),$$

or equivalently,

$$\|W_T\|_{2,\infty} \le \frac{C(\theta)}{|\tau_0|}\|W\chi_{[\theta,2\theta]}\|_{2,\infty}, \tag{7.34}$$

therefore concluding the proof of Proposition 7.2. □

8. Proofs of Theorems 1.4 and 1.7

We will now combine Propositions 6.1 and 7.2 to conclude the proof of our main result Theorem 1.7. Our most general result, Theorem 1.4, will then readily follow by combining Theorems 1.6 and 1.7. Recall that by Proposition 4.1 we found parameters (α,β,γ) so that we have the projections $\mathcal{P}_+ w_{\mathcal{C}}^{\alpha\beta\gamma} = \mathcal{P}_0 w_{\mathcal{C}}^{\alpha\beta\gamma} = 0$, where α, β and γ are admissible parameters (see also Remark 4.3). Our goal is to show that

$$w^{\alpha\beta\gamma} := u_1 - u_2^{\alpha\beta\gamma} \equiv 0.$$

Proposition 7.2 says that the weighted L^2-norm $\|W^{\alpha\beta\gamma}\|_{2,\infty}$ of the difference of our solutions $W^{\alpha\beta\gamma}(u,\tau) := Y_1(u,\tau) - Y_2^{\alpha\beta\gamma}(u,\tau)$ (after we switch the variables y and u) in the whole tip region $\mathcal{T}_\theta$ is controlled by $\|W^{\alpha\beta\gamma}\,\chi_{[\theta,2\theta]}\|_{2,\infty}$, where $\chi_{[\theta,2\theta]}(u)$ is supported in the transition region between the cylindrical and tip regions and is included in the cylindrical region $\mathcal{C}_\theta = \{(y,\tau) : u_1(y,\tau) \ge \theta/2\}$. Lemma 8.1 below says that the norms $\|W^{\alpha\beta\gamma}\,\chi_{D_{2\theta}}\|_{2,\infty}$ and

$\|w^{\alpha\beta\gamma}\,\chi_{D_{2\theta}}\|_{\mathfrak{H},\infty}$ are equivalent for every number $\theta>0$ sufficiently small. (Recall the definition of $\|\cdot\|_{\mathfrak{H},\infty}$ in (3.10)–(3.11).) Therefore combining Propositions 6.1 and 7.2 gives the crucial estimate (8.7), which will be shown in detail in Proposition 8.2 below. This estimate says that the norm of the difference $w_{\mathcal{C}}^{\alpha\beta\gamma}$ of our solutions when restricted in the cylindrical region is dominated by the norm of its projection onto the zero eigenspace of the operator $\mathcal{L}$ (the linearization of our equation on the limiting cylinder).

After having established that the projection onto the zero eigenspace $a(\tau):=\langle w_{\mathcal{C}}^{\alpha\beta\gamma},\psi_2\rangle$ dominates in $\|w_{\mathcal{C}}^{\alpha\beta\gamma}\|_{\mathfrak{H},\infty}$, the conclusion of Theorem 1.7 will follow by establishing an appropriate differential inequality for $a(\tau)$, for $\tau\le\tau_0\ll-1$, and also having that $a(\tau_0)=\mathcal{P}_0 w_{C}^{\alpha\beta\gamma}(\tau_0)=0$ at the same time.

As we pointed out above, we need to show next that the norms of the difference of our two solutions with respect to the weights defined in the cylindrical and the tip regions are equivalent in the intersection between the regions, the so-called *transition* region.

LEMMA 8.1 (Equivalence of the norms in the transition region). *Let w, W denote the difference of the two solutions $w:=u_1-u_2^{\alpha\beta\gamma}$ and $W:=Y_1-Y_2^{\alpha\beta\gamma}$ in the cylindrical and tip regions respectively. Then, for every $\theta>0$ small, there exist $\tau_0\ll-1$ and uniform constants $c(\theta), C(\theta)>0$, so that for $\tau\le\tau_0$, we have*

$$c(\theta)\,\|W\chi_{[\theta,2\theta]}\|_{2,\infty}\le\|w\,\chi_{D_{2\theta}}\|_{\mathfrak{H},\infty}\le C(\theta)\,\|W\chi_{[\theta,2\theta]}\|_{2,\infty}, \tag{8.1}$$

where $D_{2\theta}:=\{(y,\tau):\ \theta\le u_1(y,\tau)\le 2\theta\}$.

Proof. To simplify the notation we put $u_2:=u_2^{\alpha\beta\gamma}$ and $Y_2:=Y_2^{\alpha\beta\gamma}$ in this proof. Define $A_{2\theta}:=D_{2\theta}\cup\{(y,\tau):\ \theta\le u_2(y,\tau)\le 2\theta\}$. The convexity of both our solutions u_1 and u_2 imply that

$$\min_{A_{2\theta}}|(u_2)_y|\le\Big|\frac{u_1(y,\tau)-u_2(y,\tau)}{Y_1(u,\tau)-Y_2(u,\tau)}\Big|\le\max_{A_{2\theta}}|(u_2)_y|. \tag{8.2}$$

This easily follows from

$$\frac{|u_1(y,\tau)-u_2(y,\tau)|}{|Y_1(u,\tau)-Y_2(u,\tau)|}=\frac{|u_2(Y_1(u,\tau),\tau)-u_2(Y_2(u,\tau)|}{|Y_1(u,\tau)-Y_2(u,\tau)|}=|u_{2y}(\xi,\tau)|,$$

where ξ is a point in between $Y_1(u,\tau)$ and $Y_2(u,\tau)$.

The results in [3] (see also Theorem 1.8 in the current paper) show that by the asymptotics in the intermediate region for u_2, we have

$$\frac{c_1(\theta)}{\sqrt{|\tau|}}\le|u_{2y}(y,\tau)|\le\frac{C_1(\theta)}{\sqrt{|\tau|}}\qquad\text{for }\ \theta\le u_2(y,\tau)\le 2\theta \tag{8.3}$$

for uniform constants $c_1(\theta)>0$ and $C_1(\theta)>0$, independent of τ for $\tau\le\tau_0$. On the other hand, using that u_2 has the same asymptotics in the intermediate region as u_1, it is easy to see that for $\tau \le \tau_0 \ll -1$,

$$D_{2\theta} \subset \Big\{(y,\tau):\ \frac{\theta}{2} \le u_2(y,\tau) \le 3\theta\Big\}$$

and hence

$$\frac{c_1(\theta)}{\sqrt{|\tau|}} \le |u_{2y}| \le \frac{C_1(\theta)}{\sqrt{|\tau|}} \qquad \text{for} \ \ y \in D_{2\theta}.$$

Combining this, (8.3) and (8.2) yields

$$\frac{c_1(\theta)}{\sqrt{|\tau|}} \le \frac{|w(y,\tau)|}{|W(u,\tau)|} \le \frac{C_1(\theta)}{\sqrt{|\tau|}} \tag{8.4}$$

for all $y \in D_{2\theta}$, $u = u_1(y,\theta)$ and $\tau \le \tau_0 \ll -1$. See Figure 3.

By (7.2) and (7.3) we have $\mu(u,\tau) = -Y_1^2(u,\tau)/4$ for $u \in [\theta, 2\theta]$. Introducing the change of variables $y = Y_1(u,\tau)$ (or equivalently $u = u_1(y,\tau)$), the inequality (8.4) yields

$$\int_\theta^{2\theta} W^2\, e^{\mu(u,\tau)}\, du = \int_\theta^{2\theta} W^2\, e^{-\frac{Y_1^2(u,\tau)}{4}}\, du \le C(\theta)\sqrt{|\tau|} \int_{D_{2\theta}} w^2 e^{-\frac{y^2}{4}}\, dy,$$

where we used that $du = (u_1)_y\, dy$ and that due to our asymptotics from [3] in the intermediate region, we have

$$\frac{c_2(\theta)}{\sqrt{|\tau|}} \le |(u_1)_y| \le \frac{C_2(\theta)}{\sqrt{|\tau|}} \qquad \text{for } y \in D_{2\theta}. \tag{8.5}$$

In conclusion,

$$\|W \chi_{[\theta,2\theta]}\|_{2,\infty} \le C(\theta)\, \|w\, \chi_{D_{2\theta}}\|_{2,\infty},$$

which proves one of the inequalities in (8.1).

We will next show the other inequality in (8.1). To this end, we again use (8.4), the change of variables $u = u_1(y,\tau)$ (or equivalently $y = Y_1(u,\tau)$) and (8.5), to obtain

$$\int_{D_{2\theta}} w^2 e^{-\frac{y^2}{4}}\, dy \le \frac{C(\theta)}{\sqrt{|\tau|}} \int_\theta^{2\theta} W^2 e^{-\frac{Y_1^2(u,\tau)}{4}}\, du = \frac{C(\theta)}{\sqrt{|\tau|}} \int_\theta^{2\theta} W^2\, e^{\mu(u,\tau)}\, du \tag{8.6}$$

from which the bound

$$\|w\, \chi_{D_{2\theta}}\|_{2,\infty} \le C(\theta)\, \|W \chi_{[\theta,2\theta]}\|_{2,\infty}$$

readily follows. □

We will next combine the main results in the previous two sections, Propositions 6.1 and 7.2, with the estimate (8.1) above to establish our *crucial estimate*, which says that what actually dominates in the norm $\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty}$ is $\|\mathcal{P}_0 w_{\mathcal{C}}\|_{\mathfrak{D},\infty}$.

PROPOSITION 8.2. *For any $\epsilon > 0$, there exists a $\tau_0 \ll -1$ so that we have*

$$\|\hat{w}_{\mathcal{C}}\|_{\mathfrak{D},\infty} \le \epsilon\, \|\mathcal{P}_0 w_{\mathcal{C}}\|_{\mathfrak{D},\infty}. \tag{8.7}$$

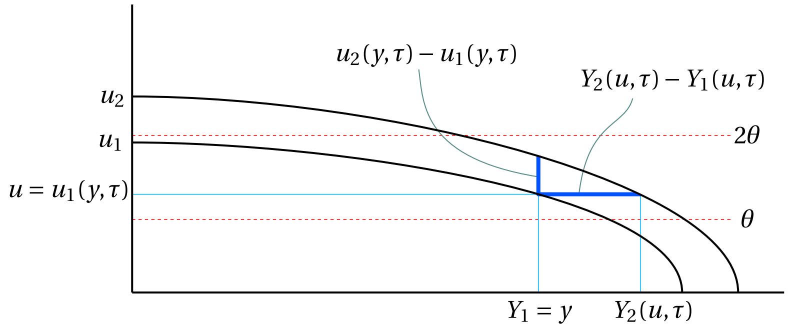


Figure 3. *Converting the vertical distance* $u_2(y,\tau) - u_1(y,\tau)$ *to the horizontal distance* $Y_2(u,\tau) - Y_1(u,\tau)$. Given a point (y,u) on the graph of $u_1(\cdot,\tau)$ we define $Y_1 = y$, $u = u_1(y,\tau)$, $Y_2 = Y_2(u,\tau)$. By the Mean Value Theorem the ratio $\frac{u_2-u_1}{Y_2-Y_1}$ must equal the derivative $-u_{2,y}(\tilde{y},\tau)$ at some $\tilde{y} \in (Y_1, Y_2)$.

Proof. Keeping in mind Remark 4.3, by Proposition 6.1, for every $\epsilon > 0$, there exists a $\tau_0 \ll -1$ so that

$$\|\hat{w}_C\|_{\mathfrak{D},\infty} < \frac{\epsilon}{3}(\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty} + \|w\chi_{D_\theta}\|_{\mathfrak{H},\infty}),$$

where $D_\theta = \{y \mid \theta/2 \le u_1(y,\tau) \le \theta\}$. Furthermore, by Lemma 8.1, by decreasing τ_0 if necessary we ensure that the following holds:

$$\begin{aligned} \|\hat{w}_C\|_{\mathfrak{D},\infty} &< \frac{\epsilon}{3}(\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty} + C(\theta)\|W\chi_{[\theta/2,\theta]}\|_{2,\infty}) \\ &< \frac{\epsilon}{3}\left(\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty} + C(\theta)\,\|W_T\|_{2,\infty}\right), \end{aligned} \tag{8.8}$$

where $\chi_{[\theta/2,\theta]}$ is the characteristic function of interval $u \in [\theta/2,\theta]$ and where we used the property of the cut-off function φ_T that $\varphi_T \equiv 1$ for $u \in [\theta/2,\theta]$. By Proposition 7.2, there exist $0 < \theta \ll 1$ and $\tau_0 \ll -1$ so that

$$\|W_T\|_{2,\infty} < \frac{C(\theta)}{\sqrt{|\tau_0|}}\,\|W\chi_{[\theta,2\theta]}\|_{2,\infty}.$$

By Lemma 8.1 we have

$$\|W_T\|_{2,\infty} \le \frac{C(\theta)}{\sqrt{|\tau_0|}}\,\|w\,\chi_{D_{2\theta}}\|_{\mathfrak{H},\infty} \le \frac{C(\theta)}{\sqrt{|\tau_0|}}\,\|w_{\mathcal{C}}\|_{\mathfrak{H},\infty},$$

where we also use that $\varphi_{\mathcal{C}} \equiv 1$ on $D_{2\theta}$. Combining this with (8.8) yields

$$\|\hat{w}_C\|_{\mathfrak{D},\infty} < \frac{\epsilon}{3}\left(\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty} + \frac{C(\theta)}{\sqrt{|\tau_0|}}\,\|w_{\mathcal{C}}\|_{\mathfrak{H},\infty}\right) < \frac{2\epsilon}{3}\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty}$$

by choosing $|\tau_0|$ sufficiently large relative to $C(\theta)$. The last estimate yields (8.7), finishing the proof of the proposition. $\square$

Proof of the Main Theorem 1.7. Recall that

$$w^{\alpha\beta\gamma}(y,\tau) = u_1(y,\tau) - u_2^{\alpha\beta\gamma}(y,\tau),$$

where α, β and γ are as in Remark 4.3. Denote this difference shortly by $w(y,\tau) = u_1(y,\tau) - u_2(y,\tau)$. Our goal is to show that for that choice of parameters, $w(y,\tau) \equiv 0$.

Following the notation from previous sections we have

$$\frac{\partial}{\partial\tau} w_{\mathcal{C}} = \mathcal{L}[w_{\mathcal{C}}] + \mathcal{E}[w_{\mathcal{C}}] + \bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}]$$

with $w_{\mathcal{C}} = \hat{w}_{\mathcal{C}} + a(\tau)\,\psi_2$, where $a(\tau) = \langle w_{\mathcal{C}}, \psi_2\rangle$. Projecting the above equation on the eigenspace generated by ψ_2 while using that $\langle \mathcal{L}[w_{\mathcal{C}}], \psi_2\rangle = 0$, we obtain

$$\frac{d}{d\tau} a(\tau) = \langle \mathcal{E}[w_{\mathcal{C}}] + \bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}], \psi_2\rangle.$$

Since $\frac{\langle \psi_2^2, \psi_2\rangle}{\|\psi_2\|^2} = 8$, we can write the above equation as

$$\frac{d}{d\tau} a(\tau) = \frac{2a(\tau)}{|\tau|} + F(\tau),$$

where

$$F(\tau) := \frac{\langle \mathcal{E}[w_{\mathcal{C}}] + \bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}] - \frac{a(\tau)}{4|\tau|}\psi_2^2, \psi_2\rangle}{\|\psi_2\|^2} \tag{8.9}$$
$$= \frac{\langle \bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}], \psi_2\rangle}{\|\psi_2\|^2} + \frac{\langle \mathcal{E}[w_{\mathcal{C}}] - \frac{a(\tau)}{4|\tau|}\psi_2^2, \psi_2\rangle}{\|\psi_2\|^2}.$$

Furthermore, solving the above ordinary differential equation for $a(\tau)$ yields

$$a(\tau) = \frac{C}{\tau^2} - \frac{\int_\tau^{\tau_0} F(s) s^2\,ds}{\tau^2}.$$

By Remark 4.3 we may assume $\alpha(\tau_0) = 0$ and hence $C = 0$, which implies

$$|a(\tau)| = \frac{|\int_\tau^{\tau_0} F(s) s^2\,ds|}{\tau^2}. \tag{8.10}$$

Define $\|a\|_{\mathfrak{H},\infty}(\tau) = \sup_{s\le\tau}\Big(\int_{s-1}^s |a(\zeta)|^2\,d\zeta\Big)^{\frac12}$. Since $\mathcal{P}_0 w_{\mathcal{C}}(\cdot,\tau) = a(\tau)\,\psi_2(\cdot)$, we have

$$\|\mathcal{P}_0 w_{\mathcal{C}}\|_{\mathfrak{D},\infty}(\tau) = \|a\|_{\mathfrak{H},\infty}(\tau)\,\|\psi_2\|_{\mathfrak{D}}.$$

Write $\|a\|_{\mathfrak{H},\infty} := \|a\|_{\mathfrak{H},\infty}(\tau_0)$. Note that

$$\Big|\int_\tau^{\tau_0} F(s)\,s^2\,ds\Big| \le \sum_{j=[\tau]-1}^{\tau_0} \Big|\int_j^{j+1} s^2 F(s)\,ds\Big| \le C \sum_{j=[\tau]-1}^{\tau_0} j^2 \int_j^{j+1} |F(s)|\,ds,$$

where with no loss of generality we may assume τ_0 is an integer. Next we need the following claim.

Claim 8.3. For every $\epsilon > 0$, there exists a τ_0 so that

$$\int_{\tau-1}^{\tau} |F(s)|\, ds \leq \frac{\epsilon}{|\tau|}\, \|a\|_{\mathfrak{H},\infty}$$

for $\tau \leq \tau_0$.

Assume for the moment that the claim holds. Then,

$$\begin{aligned}\Big|\int_{\tau}^{\tau_0} F(s)\, s^2\, ds\Big| &\leq \sum_{j=[\tau]}^{\tau_0} \int_{j-1}^{j} s^2 |F(s)|\, ds \leq \epsilon\, \|a\|_{\mathfrak{H},\infty} \sum_{j=[\tau]-1}^{\tau_0} |j| \\ &\leq \epsilon\, \|\alpha\|_{\mathfrak{H},\infty} \sum_{j=[\tau]-1}^{\tau_0} |j| \\ &\leq \epsilon\, |\tau|^2\, \|a\|_{\mathfrak{H},\infty}.\end{aligned}$$

Combining this with (8.10), where $\epsilon \leq 1/2$, yields

$$|a(\tau)| \leq \frac{1}{2}\|a\|_{\mathfrak{H},\infty}, \qquad \text{for all } \tau \leq \tau_0.$$

This implies

$$\|a\|_{\mathfrak{H},\infty} \leq \frac{1}{2}\, \|a\|_{2,\infty}$$

and hence $\|a\|_{\mathfrak{H},\infty} = 0$, which further gives

$$\|\mathcal{P}_0 w_{\mathcal{C}}\|_{\mathfrak{D},\infty} = 0.$$

Finally, (8.7) implies $\hat{w}_C \equiv 0$ and hence, $w_{\mathcal{C}} \equiv 0$ for $\tau \leq \tau_0$. By (8.1) and the fact that $\varphi_{\mathcal{C}} \equiv 1$ on $D_{2\theta}$, we have $W\chi_{[\theta,2\theta]} \equiv 0$ for $\tau \leq \tau_0$. Proposition 7.2 then yields that $W_T \equiv 0$ for $\tau \leq \tau_0$. All these imply $u_1(y,\tau) \equiv u_2^{\alpha\beta\gamma}(y,\tau)$ for $\tau \leq \tau_0$. By forward uniqueness of solutions to the mean curvature flow (or equivalently to cylindrical equation (1.7)), we have $u_1 \equiv u_2^{\alpha\beta\gamma}$, and hence $M_1 \equiv M_2^{\alpha\beta\gamma}$. Assuming Claim 8.3, this concludes the proof of Theorem 1.7. Hence, to complete the proof of Theorem 1.7 we now prove Claim 8.3.

Proof of Claim 8.3. Throughout the proof we will use the estimate

$$\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty} \leq C\, \|a\|_{\mathfrak{H},\infty} \qquad \text{for } \tau_0 \ll -1, \tag{8.11}$$

which follows from Proposition 8.2. By the proof of the same proposition we also have

$$\|w\, \chi_{D_\theta}\|_{\mathfrak{H},\infty} < \frac{C(\theta)}{\sqrt{|\tau_0|}}\, \|w_{\mathcal{C}}\|_{\mathfrak{H},\infty} \qquad \text{for } \tau_0 \ll -1.$$

Also throughout the proof we will use the a priori estimates on the solutions u_i shown in our previous work [3] that continue to hold here without the assumption of $O(1)$ symmetry, as we discuss in Theorem 9.1 below.

From the definition of $\bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}]$ given in (6.11) and the definition of the cut-off function $\varphi_{\mathcal{C}}$, we see that the support of $\mathcal{E}[w, \varphi_{\mathcal{C}}]$ is contained in

$$\left(\sqrt{2 - \frac{\theta^2}{n-1}} - \epsilon_1\right) \sqrt{|\tau|} \le |y| \le \left(\sqrt{2 - \frac{\theta^2}{4(n-1)}} + \epsilon_1\right) \sqrt{|\tau|},$$

where ϵ_1 is so tiny that $\sqrt{2 - \frac{\theta^2}{4(n-1)}} + \epsilon_1 < \sqrt{2}$. Also by the a priori estimates proved in [3, Lemma 4.1 and (5.28)], we have

$$|u_y| + |u_{yy}| \le \frac{C(\theta)}{\sqrt{|\tau|}} \qquad \text{for } |y| \le \left(\sqrt{2 - \frac{\theta^2}{4(n-1)}} + \epsilon_1\right) \sqrt{|\tau|}. \tag{8.12}$$

Furthermore, Lemma 5.14 in [3] shows that our ancient solutions u_i, $i \in \{1, 2\}$ satisfy

$$\begin{aligned} \left\| \left(u_i - \sqrt{2(n-1)} + \frac{\sqrt{2(n-1)}}{4|\tau|}\, \psi_2 \right) \chi_{\mathrm{supp}(\varphi_{\mathcal{C}})} \right\| &= o(|\tau|^{-1}), \\ \left\| \left(u_i + \frac{\sqrt{2(n-1)}}{4|\tau|}\, \psi_2 \right)_y \chi_{\mathrm{supp}(\varphi_{\mathcal{C}})} \right\| &= o(|\tau|^{-1}), \end{aligned} \tag{8.13}$$

where $\chi_{\mathrm{supp}(\varphi_{\mathcal{C}})}$ is the characteristic function of $\mathrm{supp}\, \varphi_{\mathcal{C}}$. In particular, this implies

$$\left\| u_i - \sqrt{2(n-1)} \right\| = O(|\tau|^{-1}) \qquad \text{and} \qquad \left\| (u_i)_y \right\| = O(|\tau|^{-1}). \tag{8.14}$$

We start by estimating the first term on the right-hand side in (8.9). Using Lemma 6.9, we conclude that

$$|\langle \bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}], \psi_2 \rangle| \le \|\bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}]\|_{\mathfrak{D}^*} \|\psi_2\, \bar{\chi}\|_{\mathfrak{D}} < \epsilon\, \|w_{\mathcal{C}}\|_{\mathfrak{D}}\, e^{-|\tau|/4}, \tag{8.15}$$

where $\bar{\chi}$ denotes a smooth function with a support in

$$|y| \ge (\sqrt{2 - \theta^2/(4(n-1))} - 2\epsilon_1)\, \sqrt{|\tau|}$$

being equal to one for $|y| \ge (\sqrt{2 - \theta^2/(4(n-1))} - \epsilon_1)\, \sqrt{|\tau|}$. This implies that for every $\epsilon > 0$ we can find a $\tau_0 \ll -1$ so that for $\tau \le \tau_0$, we have

$$\int_{\tau-1}^{\tau} |\langle \bar{\mathcal{E}}[w, \varphi_{\mathcal{C}}], \psi_2 \rangle|\, ds \le \frac{\epsilon \|a\|_{\mathfrak{H},\infty}}{|\tau|},$$

where we used (8.11).

We focus next on the second term on the right-hand side in (8.9). Let us write $w_{\mathcal{C}} = \hat{w}_{\mathcal{C}} + a(\tau)\psi_2$. Recall that

$$\mathcal{E}[w_{\mathcal{C}}] = \frac{2(n-1) - u_1 u_2}{2u_1 u_2} w_{\mathcal{C}} - \frac{u_{1y}^2}{1 + u_{1y}^2} (w_{\mathcal{C}})_{yy} - \frac{(u_{1y} + u_{2y}) u_{2yy}}{(1 + u_{1y}^2)(1 + u_{2y}^2)} (w_{\mathcal{C}})_y. \tag{8.16}$$

Then for the first term on the right-hand side of (8.16), we get

(8.17)
$$\begin{aligned}&\Big|\Big\langle\frac{2(n-1)-u_1u_2}{2u_1u_2}\,w_{\mathcal{C}}-\frac{a(\tau)}{4|\tau|}\,\psi_2^2,\psi_2\Big\rangle\Big|\\&\quad\le\Big|\Big\langle\frac{2(n-1)-u_1u_2}{2u_1u_2}\,\hat{w}_C,\psi_2\Big\rangle\Big|+|a(\tau)|\,\Big|\Big\langle\frac{2(n-1)-u_1u_2}{2u_1u_2}-\frac{1}{4|\tau|}\,\psi_2,\psi_2^2\Big\rangle\Big|.\end{aligned}$$

To estimate the first term on the right-hand side in (8.17), we write

(8.18)
$$\begin{aligned}&\Big|\Big\langle\frac{2(n-1)-u_1u_2}{2u_1u_2}\,\hat{w}_C,\psi_2\Big\rangle\Big|\le\Big|\Big\langle\frac{(\sqrt{2(n-1)}-u_1)(\sqrt{2(n-1)}+u_1)}{2u_1u_2}\hat{w}_C,\psi_2\Big\rangle\Big|\\&+\Big|\Big\langle\frac{u_1-\sqrt{2(n-1)}}{2u_2}\,\hat{w}_C,\psi_2\Big\rangle\Big|+\Big|\Big\langle\frac{\sqrt{2(n-1)}-u_2}{2u_2}\,\hat{w}_C,\psi_2\Big\rangle\Big|.\end{aligned}$$

Note that $u_i\ge\theta/2$ on the support of $\hat{w}_C$. Hence the arguments for estimating either of the terms on the right-hand side in (8.18) are analogous to estimating the second term in (8.18). Using Lemma 6.2, Proposition 8.2, (8.11) and (8.14), we get that for every $\epsilon>0$, there exists a $\tau_0\ll-1$ so that for $\tau\le\tau_0$, we have

$$\begin{aligned}&\Big|\Big\langle\frac{u_1-\sqrt{2(n-1)}}{2u_2}\,\hat{w}_C,\psi_2\Big\rangle\Big|\\&\quad\le C(\theta)\Big(\int\hat{w}_C^2|\psi_2|e^{-y^2/4}\,dy\Big)^{1/2}\Big(\int(\sqrt{2(n-1)}-u_1)^2|\psi_2|e^{-y^2/4}\,dy\Big)^{1/2}\\&\quad\le C(\theta)\|\hat{w}_C\|_{\mathfrak{D}}\,\|\sqrt{2(n-1)}-u_1\|_{\mathfrak{D}}\\&\quad<\frac{\epsilon}{|\tau|}\,\|\hat{w}_{\mathcal{C}}\|_{\mathfrak{D}}\end{aligned}$$

implying

$$\int_{\tau-1}^{\tau}\Big|\Big\langle\frac{2(n-1)-u_1u_2}{2u_1u_2}\,\hat{w}_C,\psi_2\Big\rangle\Big|ds<\frac{\epsilon}{|\tau|}\,\|a\|_{\mathfrak{H},\infty}.\tag{8.19}$$

Let us now estimate the second term on the right-hand side in (8.17). Writing $u_i=\sqrt{2(n-1)}(1+v_i)$, we get

$$\begin{aligned}&\Big\langle\frac{2(n-1)-u_1u_2}{2u_1u_2}-\frac{1}{4|\tau|}\psi_2,\psi_2^2\Big\rangle\\&\quad=-\Big\langle\frac{v_1+v_2+v_1v_2}{2(1+v_1)(1+v_2)}+\frac{1}{4|\tau|}\,\psi_2,\psi_2^2\Big\rangle\\&\quad=-\frac12\Big\langle\frac{v_1}{(1+v_1)(1+v_2)}+\frac{\psi_2}{4|\tau|},\psi_2^2\Big\rangle-\frac12\Big\langle\frac{v_2}{1+v_2}+\frac{\psi_2}{4|\tau|},\psi_2^2\Big\rangle.\end{aligned}\tag{8.20}$$

The two terms on the right-hand side in above equation can be estimated in the same way, so we will demonstrate how to estimate the second one. Using

(8.13), (8.14) and Hölder's inequality we get that for every $\epsilon > 0$, there exist K large enough and $\tau_0 \ll -1$ so that for $\tau \le \tau_0$, we have

$$\begin{aligned}
&\Big\langle \frac{v_2}{1+v_2} + \frac{\psi_2}{4|\tau|}, \psi_2^2 \Big\rangle \\
&\quad = \langle v_2 + \frac{\psi_2}{4|\tau|}, \psi_2^2\rangle - \langle \frac{v_2^2}{1+v_2}, \psi_2^2\rangle \\
&\quad \le C \Big\| v_2 + \frac{\psi_2}{4|\tau|} \Big\| + C \int_{\mathbb{R}} v_2^2\, y^4\, e^{-\frac{y^2}{4}}\, dy \\
&\quad \le \frac{o(1)}{|\tau|} + \Big(\int_{\mathbb{R}} v_2^2\, e^{-\frac{y^2}{4}}\, dy \Big)^{\frac12} \Big(\int_{\mathbb{R}} v_2^2\, y^8\, e^{-\frac{y^2}{4}}\, dy \Big)^{\frac12} \\
&\quad \le \frac{o(1)}{|\tau|} + \frac{C}{|\tau|} \Bigg(\Big(\int_{|y|\le K} v_2^2 y^8 e^{-\frac{y^2}{4}}\, dy \Big)^{\frac12} + \Big(\int_{|y|\ge K} v_2^2 y^8 e^{-\frac{y^2}{4}}\, dy \Big)^{\frac12} \Bigg) \\
&\quad < \frac{\epsilon}{4|\tau|}.
\end{aligned}$$

To justify the last inequality note that for a given $\epsilon > 0$ we can find K large enough so that $\Big(\int_{|y|\ge K} v_2^2 y^8 e^{-\frac{y^2}{4}}\, dy \Big)^{\frac12} < \frac{\epsilon}{6C}$. On the other hand, using our asymptotics result proven in [3], for a chosen K, we can find a $\tau_0 \ll -1$ so that for $\tau \le \tau_0$ we have $|v_i| < \frac{\epsilon}{6C\sqrt{K}}$. Finally, we conclude that for every $\epsilon > 0$, there exists a $\tau_0 \ll -1$, so that for all $\tau \le \tau_0$,

$$\Big| \Big\langle \frac{2(n-1) - u_1 u_2}{2 u_1 u_2} - \frac{\psi_2}{4|\tau|}, \psi_2^2 \Big\rangle \Big| < \frac{\epsilon}{2|\tau|}.$$

This implies

$$\int_{\tau-1}^{\tau} |a(s)| \Big| \Big\langle \frac{2(n-1) - u_1 u_2}{2 u_1 u_2} - \frac{\psi_2}{4|\tau|}, \psi_2^2 \Big\rangle \Big|\, ds \le \frac{\epsilon}{|\tau|}\, \|a\|_{\mathfrak{H},\infty}. \tag{8.21}$$

Since the first term on the right-hand side in (8.20) can be estimated in a similar manner, we conclude that this inequality holds.

It remains now to estimate the second and third terms in the error term (8.16), which involve first and second order derivative bounds for our solutions u_i. We claim that for every K there exist $\tau_0 \ll -1$ and a uniform constant C so that

$$|(u_i)_y| + |(u_i)_{yy}| \le \frac{C}{|\tau|} \qquad \text{for } \ |y| \le K,\ \tau \le \tau_0, \qquad i = 1, 2. \tag{8.22}$$

This follows by standard derivative estimates applied to the equation satisfied by each of the v_i, $i = 1, 2$ and the L^∞ bound $|v_i| \le \frac{C}{|\tau|}$, which holds on $|y| \le 2K$, $\tau \le \tau_0 \ll -1$.

Let us use (8.22) to estimate the projection involving the third term in (8.16): for every $\epsilon > 0$, there exists a $\tau_0 \ll -1$ so that for $\tau \le \tau_0$,

$$
\begin{aligned}
&\Big|\Big\langle \frac{(u_{1y}+u_{2y})u_{2yy}}{(1+u_{1y}^2)\,(1+u_{2y}^2)}\,(w_{\mathcal{C}})_y, \psi_2\Big\rangle\Big| \\
&\quad\le C\int_{|y|\le K}(|u_{1y}|+|u_{2y}|)\,|u_{2yy}|\,|(w_{\mathcal{C}})_y|\,(y^2+1)\,e^{-\frac{y^2}{4}}\,dy \\
&\qquad + C\int_{|y|\ge K}(|u_{1y}|+|u_{2y}|)\,|u_{2yy}|\,|(w_{\mathcal{C}})_y|\,y^2\,e^{-\frac{y^2}{4}}\,dy \\
&\quad\le \frac{C(K)}{|\tau|^2}\,\|w_{\mathcal{C}}\|_{\mathfrak{D}} + \frac{C}{|\tau|}\,\|w_{\mathcal{C}}\|_{\mathfrak{D}}\Big(\int_{|y|\ge K} y^4\,e^{-\frac{y^2}{4}}\,dy\Big)^{\frac12} \\
&\quad< \frac{\epsilon}{|\tau|}\,\|w_{\mathcal{C}}\|_{\mathfrak{D}},
\end{aligned}
$$

where we used Hölder's inequality, estimate (8.12) in the region $\{|y|\ge K\}\cap \operatorname{supp} w_{\mathcal{C}}\}$ and estimate (8.22) in the region $\{|y|\le K\}$. This implies that for every $\epsilon > 0$, there exists a $\tau_0 \ll -1$ so that

$$
\int_{\tau-1}^{\tau}\Big|\Big\langle \frac{(u_{1y}+u_{2y})u_{2yy}}{(1+u_{1y}^2)(1+u_{2y}^2)}\,(w_{\mathcal{C}})_y, \psi_2\Big\rangle\Big|\,ds < \frac{\epsilon}{|\tau|}\,\|w_{\mathcal{C}}\|_{\mathfrak{D},\infty} < \frac{\epsilon}{|\tau|}\,\|a\|_{\mathfrak{H},\infty}. \tag{8.23}
$$

Finally, to estimate the projection involving the second term in (8.16), we note that integration by parts yields

$$
\begin{aligned}
&\Big\langle \frac{u_{1y}^2}{1+u_{1y}^2}(w_{\mathcal{C}})_{yy}, \psi_2\Big\rangle \\
&\quad= -2\int_{\mathbb{R}} \frac{u_{1yy}u_{1y}}{1+u_{1y}^2}\,(w_{\mathcal{C}})_y\,\psi_2\,e^{-\frac{y^2}{4}}\,dy + 2\int_{\mathbb{R}} \frac{u_{1y}^3u_{1yy}}{(1+u_{1y}^2)^2}\,(w_{\mathcal{C}})_y\,\psi_2\,e^{-\frac{y^2}{4}}\,dy \\
&\qquad - \int_{\mathbb{R}} \frac{u_{1y}^2}{1+u_{1y}^2}\,(w_{\mathcal{C}})_y\,(\psi_2)_y\,e^{-\frac{y^2}{4}}\,dy + \frac12\int_{\mathbb{R}} \frac{u_{1y}^2}{1+u_{1y}^2}(w_{\mathcal{C}})_y\psi_2\,y\,e^{-\frac{y^2}{4}}\,dy.
\end{aligned}
\tag{8.24}
$$

It is easy to see that all terms on the right-hand side in (8.24) can be estimated very similarly as in (8.23). Hence, for every $\epsilon > 0$, there exists a τ_0 so that for all $\tau \le \tau_0$, we have

$$
\int_{\tau-1}^{\tau}\Big|\Big\langle \frac{u_{1y}^2}{1+u_{1y}^2}(w_{\mathcal{C}})_{yy}, \psi_2\Big\rangle\Big|\,ds < \frac{\epsilon}{|\tau|}\,\|a\|_{\mathfrak{H},\infty}. \tag{8.25}
$$

Combining (8.9), (8.15), (8.16), (8.19), (8.21), (8.23), (8.24) and (8.25) concludes Claim 8.3. ∎

The proof of our Theorem 1.7 is now also complete. ∎

9. **Reflection symmetry**

In this appendix we will justify why the conclusions of Theorem 1.8 proved in [3] under the assumption on $O(1) \times O(n)$ symmetry hold in the presence of $O(n)$-symmetry only. More precisely we will show the following result.

THEOREM 9.1. *If M_t is an Ancient Oval that is rotationally symmetric, then the conclusions of Theorem 1.8 hold.*

Proof. We will follow closely the arguments in Theorem 1.8 and point out below only steps in which the arguments slightly change because of the lack of reflection symmetry. All other estimates can be argued in exactly the same way.

Recall that we consider non-collapsed, ancient solutions (and hence convex due to [13]) that are $O(n)$-invariant hypersurfaces in $\mathbb{R}^{n+1}$. Such hypersurfaces can be represented as

$$\{(x, x') \ \in \mathbb{R} \times \mathbb{R}^n \ \mid \ -d_1(t) < x < d_2(t), \ \ \|x'\| = U(x, t)\}$$

for some function $\|x'\| = U(x, t)$. The points $(-d_1(t), 0)$ and $(d_2(t), 0)$ are called the tips of the surface. The profile function $U(x, t)$ is defined only for $x \in [-d_1(t), d_2(t)]$. After parabolic rescaling

$$U(x,t) = \sqrt{T-t}\, u(y,\tau), \qquad y = \frac{x}{\sqrt{T-t}}, \qquad \tau = -\log(T-t),$$

the profile function $u(y, \tau)$ is defined for $-\bar{d}_1(\tau) \leq y \leq \bar{d}_2(\tau)$. Theorem 1.11 in [13] and Corollary 6.3 in [19] imply that as $\tau \to -\infty$, surfaces M_τ converge in C^∞_{loc} to a cylinder of radius $\sqrt{2(n-1)}$, with axis passing through the origin.

Due to concavity, for every τ, there exists a $y(\tau)$ so that $u_y(\cdot, \tau) \leq 0$ for $y \geq y(\tau)$, $u_y(\cdot, \tau) \geq 0$ for $y \leq y(\tau)$ and $u_y(y(\tau), \tau) = 0$. To finish the proof of Theorem 9.1 we need the following lemma saying the maximum of H is attained at one of the tips.

LEMMA 9.2. *We have that $(\lambda_1)_y \geq 0$ for $y \in [y(\tau), \bar{d}_1(\tau))$ and $(\lambda_1)_y \leq 0$ for $y \in (-\bar{d}_2(\tau), y(\tau)]$. As a consequence, the mean curvature H on M_t attains its maximum at one of the tips $(-d_1(t), 0)$ or $(d_2(t), 0)$.*

Proof. We follow the proof of Corollary 3.8 in [3], where the result followed from the fact that the scaling invariant quantity

$$R := \frac{\lambda_n}{\lambda_1} = -\frac{u u_{yy}}{1 + u_y^2} \geq 0$$

satisfies

$$R \leq 1. \tag{9.1}$$

Let us then show that (9.1) still holds in our case. Note that at umbilic points one has $R = 1$. Both tips of the surface are umbilic points, and hence we have $R = 1$ at the tips for all τ. (Here we use that the surface is smooth and strictly convex and radially symmetric at the tips.) Hence, $R_{\max}(\tau)$ is achieved on the surface for all τ and is larger or equal than one. Thus it is sufficient to show that $R_{\max}(\tau) \le 1$. We first note that the quantity $Q := \dfrac{u_y^2}{u^2(1+u_y^2)}$ (considered also in [3]) satisfies $Q_y \ge 0$ for $y \ge y(\tau)$ and $Q_y \le 0$ for $y \le y(\tau)$.

To prove (9.1), we may assume $R_{\max}(\tau) = R(\bar{y}_\tau, \tau) > 1$ for all $\tau \le \tau_0$ and some $\bar{y}_\tau \in \bar{M}_\tau$, since otherwise the statement is true. The convergence to the cylinder in the middle implies that $|\bar{y}_\tau| \to +\infty$, as $\tau \to -\infty$. As in the proof of Lemma 3.5 in [3] it is enough to show that

$$\liminf_{\tau \to -\infty} Q(\bar{y}_\tau, \tau) \ge c > 0 \tag{9.2}$$

for a uniform constant $c > 0$ and all $\tau \le \tau_0$.

The same proof as in [3] implies there exists a uniform constant $c_1 > 0$ so that for all $\tau \le \tau_0 \ll -1$, we have

$$Q(y,\tau) \ge c_1, \qquad \text{whenever } \ R(y,\tau) = 1. \tag{9.3}$$

We claim that this implies (9.2). To prove this claim we argue by contradiction and hence assume that there exists a sequence $\tau_i \to -\infty$ for which $Q(\bar{y}_{\tau_i}, \tau_i) \to 0$ as $i \to \infty$. This implies that $\lim_{\tau \to -\infty} R(y,\tau) = 0$, uniformly for y bounded. We conclude that for all $\tau \le \tau_0$, there exists at least one point y_τ such that $R(y_\tau, \tau) = 1$. The convergence to the cylinder also implies that without loss of generality we may take a subsequence such that $y(\tau_i) < \bar{y}_{\tau_i}$. We consider two different cases.

Case 1. $R(y(\tau_i), \tau_i) \le 1$. Then, either $R(y(\tau_i), \tau_i) = 1$ (in which case set $\hat{y}_{\tau_i} := y(\tau_i)$), or $R(y(\tau_i), \tau_i) < 1$ (in which case we find $\hat{y}_{\tau_i} \in (y(\tau_i), \bar{y}_{\tau_i})$ so that $R(y_{\tau_i}, \tau_i) = 1$). In either case, since $R(\hat{y}_{\tau_i}, \tau_i) = 1$, (9.3) implies that $Q(\hat{y}_{\tau_i}, \tau_i) \ge c_1$ for $i \ge i_0$. Since $Q_y(\cdot, \tau) \ge 0$ for $y \ge y(\tau)$ and $\bar{y}_{\tau_i} \ge \hat{y}_{\tau_i} \ge y(\tau_i)$, we conclude that $Q(\bar{y}_{\tau_i}, \tau_i) \ge c_1 > 0$ for $i \ge i_0$, contradicting our assumption that the $\lim_{i\to\infty} Q(\bar{y}_{\tau_i}, \tau_i) = 0$.

Case 2. $R(y(\tau_i), \tau_i) > 1$. Recall that $u(y,\tau)$ satisfies the equation

$$\frac{\partial}{\partial \tau} u = \frac{u_{yy}}{1+u_y^2} - \frac{y}{2} u_y + \frac{u}{2} - \frac{n-1}{u} = -H\sqrt{1+u_y^2} - \frac{y}{2} u_y + \frac{u}{2}.$$

The maximum of $u(\cdot, \tau)$ is achieved at $y(\tau)$, and hence by (2.14) we have

$$\frac{d}{d\tau} u_{\max} \ge -C + \frac{u_{\max}}{2}$$

implying that

$$u(y(\tau), \tau) = u_{\max}(\tau) \le \max\{2C, u_{\max}(\tau_0)\} \qquad \text{for } \ \tau \le \tau_0.$$

On the other hand, due to the convergence to the cylinder of radius $\sqrt{2(n-1)}$ in the middle, we have that $u_{\max}(\tau) \geq u(0,\tau) \geq \frac{1}{2}\sqrt{2(n-1)}$ for $\tau \leq \tau_0 \ll -1$. All these imply that for $\tau \leq \tau_0 \ll -1$, we have

$$C_0 \geq H(y(\tau),\tau) \geq \frac{n-1}{u} \geq c_0 > 0.$$

Hence, we can take a limit around $(y(\tau_i), u(y(\tau_i),\tau_i)$ to conclude that the limit is a complete graph of a concave, non-negative function $\hat{u}(y,\tau)$ so that $\hat{u}_y(0,0) = 0$. All these yield $\hat{u} \equiv$ constant, that is, the limit is the round cylinder $\mathbb{R} \times S^{n-1}$, contradicting that $R(y(\tau_i),\tau_i) > 1$.

This finishes the proof of estimate (9.2). Next we can argue as in the proof of Lemma 3.5 in [3] to conclude the proof that $R \leq 1$ for $\tau \leq \tau_0 \ll -1$.

To finish the proof of Lemma 9.2, note that $R \leq 1$ on M_τ for $\tau \leq \tau_0$ implies that

$$(\lambda_1)_y \geq 0 \text{ for } y \in [y(\tau), \bar{d}_1(\tau)] \quad \text{and} \quad (\lambda_1)_y \leq 0 \text{ for } y \in [-\bar{d}_2(\tau), y(\tau)].$$

We now conclude as in the proof of Corollary 3.8 in [3] that

$$H(y,\tau) \leq \max\big(H(\bar{d}_1(\tau),\tau), H(\bar{d}_2(\tau),\tau)\big), \qquad y \in M_\tau$$

for all $\tau \leq \tau_0 \ll -1$, finishing the proof of Lemma 9.2. □

The a priori estimates from Section 4 in [3] hold as well in our case, one just has to use that $u_y \leq 0$ for $y \in [y(\tau), \bar{d}_1(\tau)]$ and $u_y \geq 0$ for $y \in [-\bar{d}_2(\tau), y(\tau)]$. By using the same barriers that we constructed in [3] one can easily see that we still have the inner-outer estimate we showed in Section 4.5 in [3]. Note that the same inner-outer estimates were proved and the same barriers were used in [7] without assuming any symmetry.

LEMMA 9.3. *There is an $L_n > 0$ such that for any rescaled Ancient Oval $u(y,\tau)$, there exist sequences $\tau_i, \tau_i' \to -\infty$ such that for all $i = 1,2,3,\dots,$ one has*

$$u(L_n,\tau_i) < \sqrt{2(n-1)} \quad \textit{and} \quad u(-L_n,\tau_i') < \sqrt{2(n-1)}.$$

Proof. Choose L_n so that the region $\{(y,u) : y \geq L_n,\ \ 0 \leq u \leq \sqrt{2(n-1)}\}$ is foliated by self-shinkers as in [3]; i.e., for each $a \in (0, \sqrt{2(n-1)})$, there is a unique solution $U_a : [L_n,\infty) \to \mathbb{R}$ of

$$\frac{U_{yy}}{1+U_y^2} - \frac{y}{2}U_y + \frac{1}{2}U - \frac{n-1}{U} = 0, \qquad U(L_n) = a. \tag{9.4}$$

To prove the lemma we argue by contradiction and assume that the sequence τ_i does not exist. This means that for some τ_*, one has $u(L_n,\tau) \geq \sqrt{2(n-1)}$ for all $\tau \leq \tau_*$. The same arguments as in [3, §4] then imply that $u(y,\tau) \geq U_a(y)$ for all $y \geq L_n$, any $\tau \leq \tau_*$ and any $a \in (0, \sqrt{2(n-1)})$. This implies that $u(y,\tau) \geq \sqrt{2(n-1)}$ for all $y \geq L_n$ and therefore contradicts the compactness of M_τ. □

For any of our rescaled rotationally symmetric Ancient Ovals $u(y,\tau)$, then we can consider the truncated difference

$$v(y,\tau) = \varphi\Big(\frac{y}{L}\Big)\Big(\frac{u(y,\tau)}{\sqrt{2(n-1)}} - 1\Big)$$

for some large L. This function satisfies

$$v_\tau = \mathcal{L}v + E(\tau), \tag{9.5}$$

where E contains the non-linear as well as the cut-off terms, and where $\mathcal{L}$ is the operator

$$\mathcal{L}\phi = \phi_{yy} - \frac{y}{2}\phi_y + \phi.$$

Using the fact that v comes from an ancient solution, and by comparing the Huisken functionals of M_τ with that of the cylinder, we can show as in [3] that for any $\epsilon > 0$, one can choose $L = L_\epsilon$ and $\tau_\epsilon < 0$ large enough so that

$$\|E(\tau)\|_{\mathfrak{H}} \le \epsilon \|v(\cdot,\tau)\|_{\mathfrak{H}} \tag{9.6}$$

holds for all $\tau \le \tau_\epsilon$.

As in [3] we can decompose v into eigenfunctions of the linearized equation, i.e.,

$$v(y,\tau) = v_-(y,\tau) + c_2(\tau)\psi_2(y) + v_+(y,\tau)$$

with the only difference that $v_\pm$ are no longer necessarily even functions of y. The component in the unstable directions now has two terms,

$$v_+(y) = c_0(\tau)\psi_0(y) + c_1(\tau)\psi_1(y) = c_0(\tau) + c_1(\tau)\, y.$$

The estimate (9.6) implies that the exponential growth rates of the various components v_-, c_2, c_1, c_0 are close to the growth rates predicted by the linearization; i.e., if we write $V_-(\tau) = \|v_-(\cdot,\tau)\|_{\mathfrak{H}}$, then we have

$$V_-'(\tau) \le -\tfrac{1}{2}V_-(\tau) + \epsilon\|v(\cdot,\tau)\|, \tag{9.7a}$$

$$|c_2'(\tau)| \le \epsilon\|v(\cdot,\tau)\|, \tag{9.7b}$$

$$|c_1'(\tau) - \tfrac{1}{2}c_1(\tau)| \le \epsilon\|v(\cdot,\tau)\|, \tag{9.7c}$$

$$|c_0'(\tau) - c_0(\tau)| \le \epsilon\|v(\cdot,\tau)\|. \tag{9.7d}$$

The total norm, which appears on the right in each of these inequalities, is given by Pythagoras:

$$\|v(\cdot,\tau)\|_{\mathfrak{H}}^2 = V_-(\tau)^2 + c_0(\tau)^2 + c_1(\tau)^2 + c_2(\tau)^2.$$

Using the ODE Lemma (see Lemma in [3]) we conclude that for $\tau \to -\infty$, exactly one of the four quantities $V_-(\tau)$, $c_0(\tau)$, $c_1(\tau)$, and $c_2(\tau)$ is much larger than the others. Similarly to [3], we will now argue that $c_2(\tau)$ is in fact the largest term:

Lemma 9.4. *For $\tau \to -\infty$, we have*

$$V_-(\tau) + |c_0(\tau)| + |c_1(\tau)| = o\big(|c_2(\tau)|\big).$$

Proof. We must rule out that any of the three components V_-, c_0, or c_1 dominates for $\tau \ll 0$.

The simplest is V_-, for if $\|v(\tau)\|_{\mathfrak{H}} = \mathcal{O}(V_-(\tau))$, then (9.7a) implies that $V_-(\tau)$ is exponentially decaying. Since $v(\cdot,\tau) \to 0$ as $\tau \to -\infty$, it would follow that $V_-(\tau) \equiv 0$, and thus $v(\cdot,\tau) \equiv 0$, which is impossible.

If $\|v(\cdot,\tau)\|_{\mathfrak{H}} = o\big(c_0(\tau)\big)$, then on any bounded interval $|y| \le L$ we have

$$v(y,\tau) = c_0(\tau)\big(1 + o(1)\big) \qquad (\tau \to -\infty).$$

In this case we derive a contradiction using the same arguments as in [3].

Finally, if $c_1(\tau)$ were the largest component, then we would have

$$v(y,\tau) = c_1(\tau)\big(y + o(1)\big) \qquad (\tau \to -\infty)$$

so that we would have either $v(L,\tau) > 0$, or $v(-L,\tau) > 0$ for all $\tau \ll 0$. This again contradicts Lemma 9.3. □

Once we have the result in Lemma 9.3, it follows as in [3] that

$$u(y,\tau) = \sqrt{2(n-1)}\left(1 - \frac{y^2-2}{4|\tau|}\right) + o(|\tau|^{-1}) \qquad |y| \le M$$

as $\tau \to -\infty$. This implies that $y(\tau)$, the maximum point of $u(y,\tau)$ (such that $u_y(y(\tau),\tau) = 0$) satisfies

$$|y(\tau)| = o(1), \qquad \text{as } \tau \to -\infty.$$

In particular, we have that $y(\tau) \le 1$ for $\tau \le \tau_0 \ll -1$. After we conclude this, the arguments in the intermediate and the tip region asymptotics in [3] go through in our current case where we lack the reflection symmetry. □

10. **Equivalence of intrinsic and extrinsic distance**

Let $\Omega \subset \mathbb{R}^{n+1}$ be a compact convex subset with smooth boundary. Recall that Ω is α-non-collapsed if at every point $Q \in \partial\Omega$ there is a $P \in \Omega$ with $Q \in \partial B_r(P) \subset \Omega$, where r satisfies $H(Q)r \ge \alpha$. Here $H(Q) > 0$ is the mean curvature of $\partial\Omega$ at Q. Since the sphere $\partial B_r(P)$ touches the hypersurface $\partial\Omega$ from one side, we have $H(Q) \le \frac{n}{r}$ so that a convex subset cannot be α-non-collapsed if $\alpha > n$.

For any pair of points $A, B \in \partial\Omega$, define $d(A,B)$ to be the intrinsic distance between A and B on the surface $\partial\Omega$. Then $d(A,B) \ge \|A - B\|$ always. Consider

$$L = \max_{A \ne B} \frac{d(A,B)}{\|A-B\|}.$$

Lemma 10.1. *There is a* $c_n \in \mathbb{R}$ *such that if* Ω *is* α*-non-collapsed for some* $\alpha \in (0, n]$*, then* $L \le c_n \alpha^{-1/2}$.

Except for the precise value of the constant c_n, this estimate is optimal, as shown by considering flat ellipsoids of rotation.

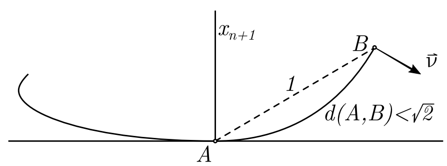
Proof. Throughout we will assume that $L \ge 2$.

Given Ω, choose $A, B \in \partial\Omega$ with $d(A, B) = L\|A - B\|$. The lemma is scaling invariant, so we may assume that $\|A - B\| = 1$, and after a Euclidean motion we may also assume that A is the origin, and that Ω is contained in the upper halfspace $x_{n+1} \ge 0$.

Let ν be the outward unit normal to Ω at B.

Claim 1. The outward normal ν at B points upwards, i.e., $\nu_{n+1} > 0$.

Indeed, assume $\nu_{n+1} \le 0$. Then B cannot lie on the x_{n+1} axis, for in this case we clearly would have $\nu_{n+1} > 0$. Thus B does not lie on the x_{n+1}-axis, and we can consider the two dimensional plane $\mathcal{P}$ containing the x_{n+1} axis and the point B. The intersection of this plane with $\partial\Omega$ is a convex plane curve



containing both A and B. One of the arcs in $\partial\Omega \cap \mathcal{P}$ connecting A and B is the graph of a convex increasing function whose length is at most $\sqrt{2}$. Thus $L = d(A, B) \le \sqrt{2}$, contradicting our assumption that $L \ge 2$.

After further rotation we may assume that $\nu_i \ge 0$ for $i = 1, 2, \ldots, n$.

Consider the point $C = \frac{1}{2}(A + B)$, and define

$$m_i = \sup\{m > 0 \mid C + me_i \in \Omega\}.$$

We also define $D_i = C + m_i e_i$.

Claim 2. $L \le \max\{2, 12m_i\}$ for each $i \in \{1, \ldots, n\}$.

To prove this consider the plane $\mathcal{P}$ through the points A, B, D_i. The curve $\partial\Omega \cap \mathcal{P}$ is convex and contains $\{A, B, D_i\}$.

The points $\{A, B\}$ split the curve in two arcs, each of which has length no less than $d(A, B) = L$. We consider the length of the arc that contains D_i. To this end consider two horizontal lines in the plane $\mathcal{P}$ through the points A and B, respectively. The line through BD_i intersects the horizontal line through A in the point F; the line through AD_i intersects the horizontal line through B at E. Since we had arranged our coordinate axes so that $\nu_i \ge 0$,

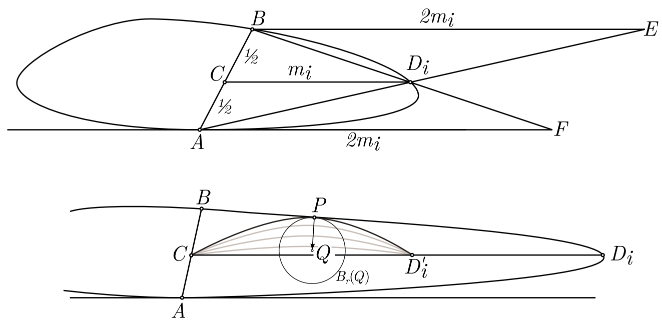


Figure 4. Proof that intrinsic and extrinsic distance on $\partial\Omega$ are equivalent.

the arc AD_iB lies below the horizontal line through B (i.e., $x_{n+1} \le 1$ on the arc AD_iB). (See Figure 4.)

Both line segments BE and AF have length $2m_i$. By convexity, the arc BD_i is shorter than $\|D_i - E\| + \|E - B\|$, and the arc AD_i is shorter than $\|A - F\| + \|F - D_i\|$.

The segments BD_i and D_iF have the same length, as do the segments AD_i and D_iF. By the triangle inequality we have $\|B - D_i\| \le \frac{1}{2} + m_i$ and $\|A - D_i\| \le \frac{1}{2} + m_i$.

Summing up, we find that the arc AD_iB is bounded by

$$\begin{aligned}\text{length}(AD_iB) &\le \|A - F\| + \|F - D_i\| + \|D_i - E\| + \|E - B\| \\ &\le 2m_i + (\tfrac{1}{2} + m_i) + (\tfrac{1}{2} + m_i) + 2m_i \\ &= 6m_i + 1.\end{aligned}$$

Hence $L \le 6m_i + 1$. If we assume that $L \ge 2$, then this implies $L \le 6m_i + \frac{1}{2}L$, and thus $L \le 12m_i$.

We now consider the hypersurface $\Phi : \Sigma \to \mathbb{R}^{n+1}$ parametrized by

$$\Phi_\lambda(t_1, \ldots, t_n) = C + \tfrac{1}{2}t_1 m_1 e_1 + \cdots + \tfrac{1}{2}t_n m_n e_n + \lambda f(t_1, \ldots, t_n) e_{n+1},$$

where Σ is the simplex $\Sigma = \{t \in \mathbb{R}^n \mid t_i \ge 0, \sum_i t_i \le 1\}$ and f is the function

$$f(t_1, \ldots, t_n) = (n+1)^{n+1} t_1 t_2 \cdots t_n (1 - t_1 - \cdots - t_n).$$

This function vanishes on $\partial\Sigma$, is positive in the interior of Σ, and attains its maximum when $t_1 = \cdots = t_n = 1/(n+1)$. The coefficient in f was chosen so that the maximal value of $f|_\Sigma$ is exactly $\max f(\Sigma) = 1$. By convexity of Ω we have $\Phi_\lambda(\partial\Sigma) \subset K$ for all $\lambda \ge 0$. If $\lambda > 1$, then $\Phi_\lambda(\frac{1}{n+1}, \ldots, \frac{1}{n+1}) \notin \Omega$, so there is a largest $\lambda_* \in [0, 1]$ for which the patch $\Phi_{\lambda_*}(\Sigma)$ is contained in Ω.

Since each of the m_i is bounded from below by $m_i \ge \frac{1}{12}L$, it follows that the angle between the unit normal anywhere on the patch $\Phi_{\lambda_*}(\Sigma)$ and the x_{n+1}-axis is bounded by C/L, while the curvature of the patch $\Phi_{\lambda_*}(\Sigma)$ is bounded by C/L^2 for some constant C. For large enough L, the unit normal anywhere on the patch $\Phi_{\lambda_*}(\Sigma)$ will be close to vertical. If $P \in \partial\Omega$ is a point of contact between $\Phi_{\lambda_*}(\Sigma)$ and $\partial\Omega$, then the inward pointing normal at P will also be almost vertical. If Ω is α-non-collapsed, then the sphere with radius α/H that touches $\partial\Omega$ from the inside at the point of contact must be contained in Ω. The mean curvature H at the contact point is bounded by C/L^2, so we find that the interior sphere with radius $\frac{\alpha}{C}L^2$ at the point of contact is contained in the region Ω. See Figure 4.

The point of contact has $x_{n+1} \leq 1$ while we have $x_{n+1} \geq 0$ throughout the convex region Ω. It follows that the radius of any interior sphere at the point of contact is at most $\sqrt{2}$. Therefore we have $\frac{\alpha}{C}L^2 \leq \sqrt{2}$, which implies that $L^2 \leq C/\alpha$, where the constant C only depends on the dimension n. □

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