

Green functions and Glauberman degree-divisibility

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Abstract

The Glauberman correspondence is a fundamental bijection in the character theory of finite groups. In 1994, Hartley and Turull established a degree-divisibility property for characters related by that correspondence, subject to a congruence condition which should hold for the Green functions of finite groups of Lie type, as defined by Deligne and Lusztig. Here, we present a general argument for completing the proof of that congruence condition. Consequently, the degree-divisibility property holds in complete generality.

1. Introduction

This paper is mainly about representations of finite groups of Lie type, but the motivation comes from the general character theory of finite groups. Let Γ , S be finite groups of coprime order such that S is solvable and acts by automorphisms on Γ . Then the Glauberman correspondence [10] is a certain canonical bijection

$$\text{Irr}_S(\Gamma) \leftrightarrow \text{Irr}(C_\Gamma(S)), \quad \theta \leftrightarrow \theta^*,$$

where $\text{Irr}_S(\Gamma)$ is the set of S -invariant irreducible characters of Γ and $C_\Gamma(S)$ the subgroup of Γ fixed by all elements of S . It is of fundamental importance in various current trends of research; see, e.g., Navarro [26, §2.5]. Using the classification of finite simple groups, and subject to a certain congruence condition on Green functions of finite groups of Lie type, Hartley–Turull [11, Th. A] showed the following result, which gives a positive answer to a problem described as perhaps one of the deepest in character theory by Navarro [25, §1].

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GLAUBERMAN DEGREE-DIVISIBILITY. *Assume that $\theta \in \text{Irr}_S(\Gamma)$ and $\theta^* \in \text{Irr}(C_\Gamma(S))$ correspond to each other as above. Then $\theta^*(1)$ divides $\theta(1)$.*

That congruence condition on Green functions has been explicitly verified in [11, Prop. 7.5] for groups of type A_n and in [11, Prop. 7.7] for groups of type 3D_4 , 2B_2 , 2G_2 , 2F_4 ; by the comments on [11, p. 204] it is also known in a large number of further cases. In this paper, we present a general argument which completes the proof of that condition. Hence, the Glauberman degree-divisibility property will hold unconditionally and in complete generality.

We shall use the full power of the geometric representation theory of finite groups of Lie type, as developed by Lusztig [14], [16], [17], [18], [19], [20], [21], [23]; an essential role will also be played by the results of Shoji [28], [29] concerning the relation between irreducible representations and character sheaves.

Let us now explain that congruence condition on Green functions. We consider a connected reductive algebraic group G (over an algebraic closure of \mathbb{F}_p where p is a prime) and an endomorphism $F: G \rightarrow G$ such that some power of F is a Frobenius map. The Green function Q_T corresponding to an F -stable maximal torus $T \subseteq G$ is introduced by Deligne–Lusztig [4] (see also Carter [3]). It is a function defined on the set of unipotent elements of G^F , with values in \mathbb{Z} . The construction involves the theory of ℓ -adic cohomology applied to certain algebraic varieties on which the finite group G^F acts. In order to indicate the dependence on F , we shall write $Q_{T,F}$ instead of just Q_T . Now we can state

CONGRUENCE CONDITION (Hartley–Turull [11, Conditions 6.9, 6.10]). *Let $T \subseteq G$ be an F -stable maximal torus and $u \in G^F$ be unipotent. Let $r \in \mathbb{N}$ be a prime such that r does not divide the order of $|G^{F^r}|$. Then*

$$Q_{T,F}(u) \equiv Q_{T,F^r}(u) \pmod{r}.$$

A quick informal argument to establish this condition goes as follows. Since Suzuki and Ree groups have already been dealt with, we can assume that F is a Frobenius map defining an \mathbb{F}_q -rational structure on G , where q is a power of p . It is expected that $Q_{T,F}(u)$ is given by a well-defined polynomial in q with integer coefficients, such that $Q_{T,F^r}(u)$ is given by evaluating that same polynomial at q^r . Then it simply remains to use Fermat’s Little Theorem. For example, this works perfectly well if G is of type A_n , as already noted in [11, Prop. 7.5], and in many further cases; see Shoji [27, §6], [32, §5]. However, the required information is not yet available for all groups over fields of small characteristic. And even when it is known, then some additional care is needed since there are cases where the Green functions are only “PORC” (in the sense of Higman), that is, polynomial on residue classes of q ;

see Beynon–Spaltenstein [2]. Thus, it seems desirable to find a general argument, uniformly for all characteristics p and appropriately dealing with the “PORC” phenomenon — and this is what we will do in this paper.

It was first shown by Lusztig [21] (with some mild restrictions on q) and then by Shoji [28], [29] (in complete generality) that the original Green functions of [4] can be identified with another type of Green functions defined in terms of Lusztig’s character sheaves [17]. This provides new, extremely powerful tools.

In Section 2, we review the general plan for determining the Green functions, taking into account the above developments. The main result of Section 3 is Theorem 3.7, which is inspired by [7], [28] and implies a crucial “PORC” property. In Section 4 we recall the basic ingredients of the Lusztig–Shoji algorithm which reduces the computation of the Green functions to the determination of certain signs. Our Theorem 3.7 does not determine these signs, but it ensures that the signs behave well with respect to replacing F by F^r . In Section 5 we put all these pieces together to complete the proof of the Congruence Condition.

We only mention here that Theorem 3.7 is also useful in another direction, for computational purposes: in [8] it is the main theoretical tool to complete the computation of Green functions for several cases where these have not been previously known (e.g., type E_7 in characteristic 2, 3).

We assume some general familiarity with the theory of finite groups of Lie type and the character theory of these groups; see, e.g., [3], [13], [6].

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2. Green functions and character sheaves

Let p be a prime and $k = \overline{\mathbb{F}}_p$ be an algebraic closure of the field with p elements. Let G be a connected reductive algebraic group over k , and assume that G is defined over the finite subfield $\mathbb{F}_q \subseteq k$, where $q = p^m$ for some $m \geq 1$. Let $F: G \rightarrow G$ be the corresponding Frobenius map. Let $B_0 \subseteq G$ be an F -stable Borel subgroup and $T_0 \subseteq B_0$ be an F -stable maximal torus. Let $W = N_G(T_0)/T_0$ be the corresponding Weyl group. For each $w \in W$, let R_w be the virtual representation of the finite group G^F defined by Deligne–Lusztig [4, §1]. (In the setting of [3, §7.2], we have $\text{Tr}(g, R_w) = R_{T_w, 1}(g)$ for $g \in G^F$,

where $T_w \subseteq G$ is an F -stable maximal torus obtained from T_0 by twisting with w , and 1 stands for the trivial character of T_w^F .) This construction is carried out over $\overline{\mathbb{Q}}_\ell$, an algebraic closure of the ℓ -adic numbers where ℓ is a prime not equal to p . The corresponding Green function is defined by

$$Q_w: G_{\text{uni}}^F \rightarrow \overline{\mathbb{Q}}_\ell, \quad u \mapsto \text{Tr}(u, R_w),$$

where G_{uni} denotes the set of unipotent elements of G . It is known that $Q_w(u) \in \mathbb{Z}$ for all $u \in G_{\text{uni}}^F$; see [3, §7.6]. So the character formula [3, 7.2.8] shows that we also have $\text{Tr}(g, R_w) \in \mathbb{Z}$ for all $g \in G^F$.

The general plan for determining the values of Q_w is explained in Lusztig [20, Chap. 24] and Shoji [27, §5], [30, 1.1–1.3] (even for generalised Green functions, which we will not consider here). We will have to go through some of the steps of that plan.

2.1. *Almost characters.* The Frobenius map F induces an automorphism of W which we denote by $\gamma: W \rightarrow W$. Let $\text{Irr}(W)$ be the set of irreducible representations of W over $\overline{\mathbb{Q}}_\ell$ (up to isomorphism). Let $\text{Irr}(W)^\gamma$ be the set of all those $E \in \text{Irr}(W)$ which are γ -invariant; that is, there exists a bijective linear map $\sigma_E: E \rightarrow E$ such that $\sigma_E \circ w = \gamma(w) \circ \sigma_E: E \rightarrow E$ for all $w \in W$. Note that σ_E is only unique up to scalar multiples but, if γ has order $d \geq 1$, then one can always find some σ_E such that

$$\sigma_E^d = \text{id}_E \quad \text{and} \quad \text{Tr}(\sigma_E \circ w, E) \in \mathbb{Z} \quad \text{for all } w \in W;$$

see [14, 3.2]. In what follows, we assume that a fixed choice of σ_E satisfying the above conditions has been made for each $E \in \text{Irr}(W)^\gamma$. (For example, one could take the “preferred” choice for σ_E specified by Lusztig [19, 17.2].) For $E \in \text{Irr}(W)^\gamma$, the corresponding almost character is the class function $R_E: G^F \rightarrow \overline{\mathbb{Q}}_\ell$ defined by

$$R_E(g) := \frac{1}{|W|} \sum_{w \in W} \text{Tr}(\sigma_E \circ w, E) \text{Tr}(g, R_w) \quad \text{for all } g \in G^F.$$

We have $R_E(g) \in \mathbb{Q}$ for all $g \in G^F$. By [14, 3.9], the functions R_E are orthonormal with respect to the standard inner product on class functions of G^F . Furthermore, by [13, 3.19], we have

$$Q_w(u) = \sum_{E \in \text{Irr}(W)^\gamma} \text{Tr}(\sigma_E \circ w, E) R_E(u) \quad \text{for } w \in W, u \in G_{\text{uni}}^F.$$

Hence, knowing the values of all Green functions Q_w is equivalent to knowing the values of all R_E on G_{uni}^F .

2.2. *Constructible $\overline{\mathbb{Q}}_\ell$ -sheaves.* Let $\mathcal{D}G$ be the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$ -sheaves on G (in the sense of Beilinson, Bernstein, Deligne [1]). The “character sheaves” on G , defined by Lusztig [16], are certain simple

perverse sheaves in $\mathcal{D}G$, equivariant for the action of G on itself by conjugation. Consider such an object $A \in \mathcal{D}G$, and suppose that its inverse image F^*A under the Frobenius map is isomorphic to A in $\mathcal{D}G$. Let $\phi: F^*A \xrightarrow{\sim} A$ be an isomorphism. Then ϕ induces a linear map $\phi_{i,g}: \mathcal{H}_{F(g)}^i(A) \rightarrow \mathcal{H}_g^i(A)$ for each i and $g \in G$, where $\mathcal{H}^i(A)$ denotes the i -th cohomology sheaf of A and $\mathcal{H}_g^i(A)$ the stalk at $g \in G$. By [17, 8.4], this gives rise to a class function $\chi_{A,\phi}: G^F \rightarrow \overline{\mathbb{Q}}_\ell$, called a “characteristic function” of A , defined by

$$\chi_{A,\phi}(g) = \sum_i (-1)^i \text{Tr}(\phi_{i,g}, \mathcal{H}_g^i(A)) \quad \text{for } g \in G^F.$$

Note that, by a version of Schur’s Lemma, ϕ is unique up to a non-zero scalar; hence, $\chi_{A,\phi}$ is unique up to a non-zero scalar. If $F^*A \cong A$, then one can choose an isomorphism $\phi_A: F^*A \xrightarrow{\sim} A$ such that the values of χ_{A,ϕ_A} are cyclotomic integers and the standard inner product of χ_{A,ϕ_A} with itself is equal to 1. The precise conditions which guarantee these properties are formulated in [18, 13.8], [20, 25.1]; note that these conditions specify ϕ_A up to multiplication by a root of unity.

2.3. *The complexes A_E .* Lusztig [17, §8.1] describes a geometric induction process by which one obtains objects in $\mathcal{D}G$ from objects in $\mathcal{D}L$ where L is a Levi subgroup of some parabolic subgroup of G . Applying this to $L = T_0$ and the constant local system $\overline{\mathbb{Q}}_\ell$ on T_0 , we obtain a well-defined complex $K \in \mathcal{D}G$ together with a canonical isomorphism $\varphi: F^*K \xrightarrow{\sim} K$. The restriction of the corresponding characteristic function $\chi_{K,\varphi}: G^F \rightarrow \overline{\mathbb{Q}}_\ell$ to G_{uni}^F is an example of a “generalised Green function,” as defined in [17, §8.3]; see also [28, 1.7]. We have $\text{End}(K) \cong \mathbb{Q}_\ell[W]$ (see [17, 24.2]) and K has a canonical decomposition

$$K \cong \bigoplus_{E \in \text{Irr}(W)} V_E \otimes A_E,$$

where $A_E \in \mathcal{D}G$ is a character sheaf and $V_E = \text{Hom}(A_E, K)$ is an irreducible W -module isomorphic to $E \in \text{Irr}(W)$; see [30, 1.2]. Now let $E \in \text{Irr}(W)^\gamma$. Then we also have $F^*A_E \cong A_E$ and, using our choice of $\sigma_E: E \rightarrow E$ in Section 2.1, we can single out a particular isomorphism $\phi_{A_E}: F^*A_E \xrightarrow{\sim} A_E$ as in Section 2.2. Since this will be important later, let us briefly indicate how this is done, following [20, 24.2] or [30, 1.3]. We start with any isomorphism $\phi_{A_E}: F^*A_E \xrightarrow{\sim} A_E$. Then there is a unique linear map $\psi_E: V_E \rightarrow V_E$ such that $\phi_{A_E} \otimes \psi_E$ corresponds to $\varphi: F^*K \xrightarrow{\sim} K$ under the above direct sum decomposition; see [17, 10.4]. Furthermore, by [20, 24.2], ψ_E corresponds under a W -module isomorphism $V_E \cong E$ to a non-zero scalar $\zeta \in \overline{\mathbb{Q}}_\ell$ times the map $\sigma_E: E \rightarrow E$. Hence, replacing ϕ_{A_E} by a scalar multiple, we can achieve that $\zeta = 1$. Having fixed this choice of $\phi_{A_E}: F^*A_E \xrightarrow{\sim} A_E$, let $\chi_{A_E}: G^F \rightarrow \overline{\mathbb{Q}}_\ell$ be the corresponding characteristic function. Then, by the main result of Lusztig

[21] and by Shoji [29, Th. 5.5] (see also the argument in [28, 2.17, 2.18]), we have

$$R_E(g) = (-1)^{\dim T_0} \chi_{A_E}(g) \quad \text{for all } g \in G^F.$$

(In [28, 2.18], it is assumed that q is a sufficiently large power of p , but this condition is later removed thanks to [29, Th. 5.5].) The above identity is a special case of a more general conjecture about the relation between almost characters and characteristic functions of character sheaves; see [17, p. 226], [28], [29].

2.4. *The Springer correspondence.* Let \mathcal{N}_G be the set of all pairs (C, \mathcal{E}) , where C is a unipotent class in G and \mathcal{E} is a G -equivariant irreducible $\overline{\mathbb{Q}}_\ell$ -local system on C (up to isomorphism). The Springer correspondence defines an injective map

$$\iota_G: \text{Irr}(W) \hookrightarrow \mathcal{N}_G$$

such that, if $E \in \text{Irr}(W)$ and $\iota_G(E) = (C, \mathcal{E})$, then we have

$$\text{u-supp}(A_E) \subseteq \overline{C} \quad \text{and} \quad \mathcal{H}^i(A_E)|_C \cong \begin{cases} \mathcal{E} & \text{if } i = -\dim C - \dim T_0, \\ 0 & \text{otherwise.} \end{cases}$$

See Lusztig [15], [20, Chap 24], and the references there. Here, $\text{u-supp}(A)$ for any $A \in \mathcal{D}G$ is defined as the Zariski closure of

$$\{g \in G_{\text{uni}} \mid \mathcal{H}_g^i(A) \neq \{0\} \text{ for some } i\}.$$

Given $E \in \text{Irr}(W)$ and $\iota_G(E) = (C, \mathcal{E})$, we define

$$d_E := (\dim G - \dim C - \dim T_0)/2.$$

Note that $\dim C_G(g) \geq \dim T_0$ for $g \in G$. Furthermore, $d_E \in \mathbb{Z}_{\geq 0}$ since $\dim G - \dim T_0$ is always even and so is $\dim C$; see [3, §5.10] and the references there.

2.5. *The Y -functions.* Let $E \in \text{Irr}(W)^\gamma$ and $\iota_G(E) = (C, \mathcal{E})$. Then $F(C) = C$ and $F^*\mathcal{E} \cong \mathcal{E}$. Since $\mathcal{E} \cong \mathcal{H}^i(A_E)|_C$ for $i = -\dim C - \dim T_0$, the isomorphism $\phi_{A_E}: F^*A_E \xrightarrow{\sim} A_E$ induces a map $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ which we can write as $q^{d_E}\psi$ where $\psi: F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is an isomorphism. With this normalisation, ψ induces an automorphism of finite order $\psi_g: \mathcal{E}_g \rightarrow \mathcal{E}_g$ at each stalk \mathcal{E}_g where $g \in C^F$; see [20, 24.2.4]. For $g \in C^F$ and $i = -\dim C - \dim T_0$, we have

$$\chi_{A_E}(g) = (-1)^i \text{Tr}(\phi_{A_E, i, g}, \mathcal{H}_g^i(A_E)) = (-1)^i q^{d_E} Y_E(g),$$

where the class function $Y_E: G_{\text{uni}}^F \rightarrow \overline{\mathbb{Q}}_\ell$ is defined by

$$Y_E(g) = \begin{cases} \text{Tr}(\psi_g, \mathcal{E}_g) & \text{if } g \in C^F, \\ 0 & \text{otherwise;} \end{cases}$$

see [20, 24.2.3]. In particular, the values of Y_E are algebraic integers. Since $\dim C$ is an even number, we obtain that

$$R_E(g) = (-1)^{\dim T_0} \chi_{A_E}(g) = q^{d_E} Y_E(g) \quad \text{for all } g \in C^F.$$

Since the values of R_E are rational numbers (see Section 2.1), we conclude that

$$Y_E(g) \in \mathbb{Z} \quad \text{for all } E \in \text{Irr}(W)^\gamma \text{ and } g \in G_{\text{uni}}^F.$$

The Y -functions $\{Y_E \mid E \in \text{Irr}(W)^\gamma\}$ are linearly independent by [20, 24.2.7].

2.6. *The coefficients $p_{E',E}$.* Having established the above framework, in [20, Th. 24.4] Lusztig shows that we have unique equations

$$R_E|_{G_{\text{uni}}^F} = \sum_{E' \in \text{Irr}(W)^\gamma} q^{d_E} p_{E',E} Y_{E'} \quad \text{for all } E \in \text{Irr}(W)^\gamma,$$

where the coefficients $p_{E',E} \in \overline{\mathbb{Q}_\ell}$ are determined by a purely combinatorial algorithm which we will consider in more detail in Section 4. Note that the hypotheses of [20, Th. 24.4] (“cleanness”) are always satisfied by the main result of [23]. (Since we are only dealing with Green functions of G^F , and not with generalised Green functions, it would actually be sufficient to refer to [5, §3] instead of [23].) By [20, 24.5.2], we have

$$p_{E',E} \in \mathbb{Z} \quad \text{for all } E, E' \in \text{Irr}(W)^\gamma.$$

Furthermore, by [20, 24.2.10, 24.2.11], we have

$$p_{E,E} = 1 \quad \text{and} \quad p_{E',E} = 0 \quad \text{if } E' \neq E \text{ and } d_{E'} \geq d_E.$$

Consequently, for a suitable ordering of $\text{Irr}(W)^\gamma$, the matrix of coefficients $(p_{E',E})$ will be triangular with 1 along the diagonal (see also Section 4).

Thus, the whole problem of computing the Green functions Q_w of G^F is reduced to the determination of the functions $\{Y_E \mid E \in \text{Irr}(W)^\gamma\}$ (cf. Shoji [30, 1.3]).

As in [30, 1.3], the above discussion also applies, with additional technical complications, to the generalised Green functions defined in [17, §8.3], but in this article we restrict ourselves to the “ordinary” Green functions Q_w .

3. Evaluating the Y -functions

Combining and summarising the formulae in Section 2, we can state the following result about the values of the Green functions of G^F .

PROPOSITION 3.1. *Let $w \in W$ and $u \in G^F$ be unipotent. Then*

$$Q_w(u) = \sum_{E', E \in \text{Irr}(W)^\gamma} \text{Tr}(\sigma_E \circ w, E) q^{d_E} p_{E',E} Y_{E'}(u),$$

where $\sigma_E, d_E, Y_{E'}, p_{E',E}$ are defined in [Sections 2.1, 2.4, 2.5, and 2.6](#), respectively.

Proof. By [Section 2.1](#), we can express $Q_w(u)$ as a linear combination of $R_E(u)$ for various $E \in \text{Irr}(W)^\gamma$. By [Section 2.6](#), we can express each term $R_E(u)$ as a linear combination of $Y_{E'}(u)$ for various $E' \in \text{Irr}(W)^\gamma$. \square

In this section, we address the further evaluation of the terms $Y_{E'}(u)$. As we will use results from Shoji [\[28\]](#), [\[29\]](#) we will assume from now on that the Frobenius map $F: G \rightarrow G$ is given by

$$F = \tilde{\gamma} \circ F_p^m = F_p^m \circ \tilde{\gamma} \quad (m \geq 1),$$

where $\tilde{\gamma}: G \rightarrow G$ is an automorphism of finite order leaving T_0, B_0 invariant, and $F_p: G \rightarrow G$ is a Frobenius map corresponding to a split \mathbb{F}_p -rational structure, such that $F_p(t) = t^p$ for all $t \in T_0$. Note that F_p acts trivially on W and that $\tilde{\gamma}$ induces an automorphism of W which is just the automorphism $\gamma: W \rightarrow W$ induced by F considered earlier. Thus, if G is semisimple, then G^F is an untwisted or twisted Chevalley group, as in Steinberg [\[33, §11.6\]](#).

Remark 3.2. It is known that all unipotent classes of G are F_p -stable (since, in each case, representatives of the classes are known which lie in $G^{F_p} = G(\mathbb{F}_p)$; see, e.g., Liebeck–Seitz [\[12\]](#)). Let C be an F -stable unipotent class. We shall also make the following assumption:

- (♣) There exists an element $u_0 \in C^F$ such that F acts trivially on the finite group of components $A(u_0) := C_G(u_0)/C_G^\circ(u_0)$.

If (♣) holds, then there is a bijective correspondence between the conjugacy classes of $A(u_0)$ and the conjugacy classes of G^F which are contained in the set C^F (see, e.g., [\[12, Lemma 2.12\]](#)). For $a \in A(u_0)$, an element in the corresponding G^F -conjugacy class is given by $u_a = hu_0h^{-1}$, where $h \in G$ is such that $h^{-1}F(h) \in C_G(u_0)$ maps to a under the natural homomorphism $C_G(u_0) \rightarrow A(u_0)$. (The existence of h is guaranteed by Lang’s Theorem; note that h is not unique but $u_a = hu_0h^{-1}$ is well defined up to G^F -conjugacy.)

Let $E \in \text{Irr}(W)^\gamma$ and $\iota_G(E) = (C, \mathcal{E}) \in \mathcal{N}_G$. As in [Section 2.5](#), we have $F(C) = C$ and $F^*\mathcal{E} \cong \mathcal{E}$. Furthermore, there is a certain isomorphism $\psi: F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ which induces a map of finite order $\psi_g: \mathcal{E}_g \rightarrow \mathcal{E}_g$ for each $g \in C^F$. Now let us fix an element $u_0 \in C^F$ as in (♣), such that F acts trivially on $A(u_0)$.

LEMMA 3.3 (cf. Lusztig [\[22, 19.7\]](#)). *In the above setting, let $u_0 \in C^F$ be such that (♣) holds. There is a natural $A(u_0)$ -module structure on the stalk \mathcal{E}_{u_0} . We have $\mathcal{E}_{u_0} \in \text{Irr}(A(u_0))$, and the map $\psi_{u_0}: \mathcal{E}_{u_0} \rightarrow \mathcal{E}_{u_0}$ is given by scalar*

multiplication with a sign $\delta_E = \pm 1$. Furthermore, $Y_E(u_a) = \delta_E \operatorname{Tr}(a, \mathcal{E}_{u_0})$ for all $a \in A(u_0)$.

Note that $\operatorname{Tr}(a, \mathcal{E}_{u_0})$ is just an entry in the ordinary character table of $A(u_0)$. In particular, if $a = 1$, then u_1 is G^F -conjugate to u_0 and so $Y_E(u_0) = \delta_E \dim \mathcal{E}_{u_0}$.

Proof. The $A(u_0)$ -module structure on \mathcal{E}_{u_0} is explained in the proof of [22, 19.7]; this also shows that $\psi_{u_0} \circ a = a \circ \psi_{u_0} : \mathcal{E}_{u_0} \rightarrow \mathcal{E}_{u_0}$ for all $a \in A(u_0)$. (Recall from (♣) that F acts trivially on $A(u_0)$.) Since $\mathcal{E}_{u_0} \in \operatorname{Irr}(A(u_0))$, we conclude that ψ_{u_0} acts as a scalar times the identity on \mathcal{E}_{u_0} ; let us denote this scalar by $\delta_E \in \overline{\mathbb{Q}}_\ell$. Then, for any $g \in C^F$, we have $Y_E(g) = \delta_E \operatorname{Tr}(a, \mathcal{E}_{u_0})$, where $a \in A(u_0)$ is such that g is G^F -conjugate to u_a ; see [22, 19.7]. Since $\psi_{u_0} : \mathcal{E}_{u_0} \rightarrow \mathcal{E}_{u_0}$ is a map of finite order, the scalar δ_E is a root of unity. Since the values of the almost characters R_E are in \mathbb{Q} (see Section 2.1), and since $R_E(u_1) = q^{d_E} Y_E(u_1) = q^{d_E} \delta_E \operatorname{Tr}(1, \mathcal{E}_{u_0})$ (see Section 2.5), we conclude that we also have $\delta_E \in \mathbb{Q}$. Hence, we finally see that $\delta_E = \pm 1$. \square

Thus, the problem of computing the Green functions Q_w is further reduced to the determination of the signs $\delta_E = \pm 1$ for $E \in \operatorname{Irr}(W)^\gamma$ (cf. Shoji [30, 1.3, p. 161]).

Example 3.4. Let G be of type E_8 and $p > 5$. Let C be the unipotent class denoted by $D_8(a_3)$ in Mizuno [24], or by $E_8(b_6)$ in Carter [3, p. 407]. We have $\dim C = 28$ and, up to conjugation within G^F , there is a unique $u_0 \in C^F$ such that $|C_G(u_0)^F| = 6q^{28}$; see [24, p. 455]. We have $A(u_0) \cong \mathfrak{S}_3$ and F acts trivially on $A(u_0)$. Let $E \in \operatorname{Irr}(W)$ be the irreducible representation denoted by $\phi_{840,13}$ in [3, §13.2], or by 840_z in [14, 4.13.1]. Then $\iota_G(E) = (C, \mathcal{E})$, where the irreducible $A(u_0)$ -module \mathcal{E}_{u_0} corresponds to the sign representation of \mathfrak{S}_3 ; see [3, p. 432]. It is shown by Beynon–Spaltenstein [2, §3, Case 5] that

$$\delta_E = \begin{cases} 1 & \text{if } q \equiv 1 \pmod{3}, \\ -1 & \text{if } q \equiv -1 \pmod{3}. \end{cases}$$

In particular, there do exist cases in which $\delta_E = -1$.

Returning to the general setting, the following corollary interprets the signs $\delta_E = \pm 1$ somewhat more directly in terms of the character sheaves A_E in Section 2.3.

COROLLARY 3.5. *Let $E \in \operatorname{Irr}(W)^\gamma$ and $\iota_G(E) = (C, \mathcal{E}) \in \mathcal{N}_G$. Let $u_0 \in C^F$ be as in (♣). Then the isomorphism $\phi_{A_E} : F^* A_E \xrightarrow{\sim} A_E$ in Section 2.3 induces the scalar multiplication by $\delta_E q^{d_E}$ on the stalk $\mathcal{H}_{u_0}^i(A_E)$, where $i = -\dim C - \dim T_0$.*

Proof. This is just a reformulation of [Lemma 3.3](#), noting that $\mathcal{H}^i(A_E)|_C \cong \mathcal{E}$ and $\mathcal{H}^j(A_E)|_C = 0$ for $j \neq i$; see [Section 2.4](#). \square

Remark 3.6. Let $d \geq 1$ be the order of the automorphism $\gamma: W \rightarrow W$; we set

$$\mathcal{M} := \{r \in \mathbb{Z}_{\geq 1} \mid r \equiv 1 \pmod{d}\}.$$

Let $E \in \text{Irr}(W)^\gamma$ and $\iota_G(E) = (C, \mathcal{E})$, where C is F -stable and $F^* \mathcal{E} \cong \mathcal{E}$. Let $u_0 \in C^F$ be such that F acts trivially on $A(u_0)$; by [Lemma 3.3](#), we have a corresponding sign $\delta_E = \pm 1$ such that

$$R_E(u_0) = q^{d_E} \delta_E \dim \mathcal{E}_{u_0} \quad (\text{see } \a href="#">Section 2.5).$$

Now let $r \in \mathcal{M}$, and replace F by F^r . Thus, since $r \equiv 1 \pmod{d}$, the automorphism of W induced by F^r is again given by γ . Hence, we can use the chosen map $\sigma_E: E \rightarrow E$ (see [Section 2.1](#)) to define a corresponding almost character of G^{F^r} , which we denote by $R_E^{(r)}$. Finally, we still have $u_0 \in C^{F^r}$ and F^r acts trivially on $A(u_0)$. Hence, we also have a corresponding sign $\delta_E^{(r)} = \pm 1$ as in [Lemma 3.3](#), such that

$$R_E^{(r)}(u_0) = q^{d_{E^r}} \delta_E^{(r)} \dim \mathcal{E}_{u_0}.$$

The following result relates δ_E and $\delta_E^{(r)}$.

THEOREM 3.7. *With the above notation, we have $\delta_E^{(r)} = (\delta_E)^r$ for all $r \in \mathcal{M}$.*

Proof. We use the interpretation of δ_E in [Corollary 3.5](#). Starting with the isomorphism $\phi := \phi_{A_E}: F^* A_E \xrightarrow{\sim} A_E$ in [Section 2.3](#), we obtain natural isomorphisms

$$(F^*)^{j-1}(\phi): (F^*)^j A_E \xrightarrow{\sim} (F^*)^{j-1} A_E \quad \text{for } 1 \leq j \leq r.$$

These give rise to an isomorphism

$$\tilde{\phi}^{(r)} := \phi \circ F^*(\phi) \circ \dots \circ (F^*)^{r-1}(\phi): (F^*)^r A_E \xrightarrow{\sim} A_E.$$

We also have a canonical isomorphism $(F^*)^r A_E \cong (F^r)^* A_E$ which, finally, induces an isomorphism

$$\phi^{(r)}: (F^r)^* A_E \xrightarrow{\sim} A_E \quad (\text{see } [28, 1.1]).$$

We denote the corresponding characteristic function of A_E by $\chi^{(r)}: G^{F^r} \rightarrow \overline{\mathbb{Q}_\ell}$. Thus, we have

$$\chi^{(r)}(g) = \sum_i (-1)^i \text{Tr}(\phi_{i,g}^{(r)}, \mathcal{H}_g^i(A_E)) \quad \text{for } g \in G^{F^r}.$$

Now assume that g is an element in G^F , and not just in G^{F^r} . Then we have

$$\phi_{i,g}^{(r)} = (\phi_{i,g})^r: \mathcal{H}_g^i(A_E) \rightarrow \mathcal{H}_g^i(A_E) \quad \text{for all } i;$$

see [28, 1.1]. If we take $g = u_0$ (the chosen element in $C^F \subseteq C^{F^r}$) and let $i = -\dim C - \dim T_0$, then $\phi_{i,u_0}: \mathcal{H}_{u_0}^i(A_E) \rightarrow \mathcal{H}_{u_0}^i(A_E)$ is given by scalar multiplication with $\delta_E q^{d_E}$; see Corollary 3.5. So we conclude that

$$\phi_{i,u_0}^{(r)} = \text{scalar multiplication by } (\delta_E q^{d_E})^r = (\delta_E)^r q^{d_{E^r}} \text{ on } \mathcal{H}_{u_0}^i(A_E).$$

Hence, again in view of Corollary 3.5, it remains to show that the isomorphism $\phi^{(r)}: (F^r)^* A_E \xrightarrow{\sim} A_E$ constructed above is the particular isomorphism singled out in the discussion in Section 2.3. But this has already been checked essentially by Shoji in [28, 2.18.1]. For this purpose, we have to consider once more the decomposition

$$K \cong \bigoplus_{E \in \text{Irr}(W)} V_E \otimes A_E \quad (\text{see Section 2.3}).$$

As above, starting with the isomorphism $\varphi: F^* K \xrightarrow{\sim} K$, we obtain a natural isomorphism $\varphi^{(r)}: (F^r)^* K \xrightarrow{\sim} K$. Then we have a unique linear map $\psi_E^{(r)}: V_E \rightarrow V_E$ such that $\phi^{(r)} \otimes \psi_E^{(r)}$ corresponds to $\varphi^{(r)}$ under the above direct sum decomposition. By an argument completely analogous to that in [7, 3.7], we see that $\psi_E^{(r)} = (\psi_E)^r$, where $\psi_E: V_E \rightarrow V_E$ is determined by φ and ϕ_{A_E} as in Section 2.3. Since $r \equiv 1 \pmod d$ and ψ_E has order dividing d , we conclude that $\psi_E^r = \psi_E$. Thus, $\phi^{(r)}$ also satisfies the requirements in Section 2.3. \square

If $\tilde{\gamma} = \text{id}_G$ and $F = F_p^m$, then Theorem 3.7 shows that the determination of δ_E is reduced to the case where $m = 1$. This is exploited in [8] to compute the values of Green functions for groups of exceptional type in small characteristic.

Remark 3.8. Assume that $F = \tilde{\gamma} \circ F_0 = F_0 \circ \tilde{\gamma}$, where $F_0 := F_p^m$ as above and $\tilde{\gamma}: G \rightarrow G$ is non-trivial. Let $E \in \text{Irr}(W)^\gamma$ and $\iota_G(E) = (C, \mathcal{E}) \in \mathcal{N}_G$, where C is F -stable and $F^* \mathcal{E} \cong \mathcal{E}$. Assume that there exists an element $u_0 \in C$ such that

$$(\clubsuit') \quad F_p(u_0) = u_0 = \tilde{\gamma}(u_0) \quad \text{and} \quad F, F_0 \text{ act trivially on } A(u_0).$$

Then, by Lemma 3.3, we have signs $\delta_E = \pm 1$ (with respect to F) and $\delta_E^\circ = \pm 1$ (with respect to $F_0 = F_p^m$). We claim that

$$\delta_E \delta_E^\circ = \pm 1 \text{ does not depend on } m.$$

(This remark is used, for example, in the determination of Green functions for groups of type 2E_6 in [8, §7].) A proof is as follows. As noted in [28, 2.17], we have $\tilde{\gamma}^* A_E \cong A_E$. Let $\nu_{A_E}: \tilde{\gamma}^* A_E \xrightarrow{\sim} A_E$ be an isomorphism. Then ν_{A_E} induces linear maps $\nu_{A_E, i, u}: \mathcal{H}_{u_0}^i(A_E) \rightarrow \mathcal{H}_{u_0}^i(A_E)$ for all i . Let $\phi_{A_E}^\circ: F_0^* A_E \xrightarrow{\sim} A_E$ be an isomorphism as in Section 2.3. Since $F_0^*(\tilde{\gamma}^* A) \cong (F_0 \circ \tilde{\gamma})^* A = F^* A$, we obtain

$$\phi_{A_E} := \phi_{A_E}^\circ \circ F_0^*(\nu_{A_E}): F^* A_E \xrightarrow{\sim} A_E,$$

where $F_0^*(\nu_{A_E}) : F_0^*(\tilde{\gamma}^* A_E) \xrightarrow{\sim} F_0^* A_E$ is induced by $\tilde{\gamma}^* A_E \cong A_E$. Replacing ν_{A_E} by a scalar multiple if necessary, we can assume that the above isomorphism satisfies the requirements in [Section 2.3](#) (see also [28, 2.1, 2.17]). Now consider stalks at $u_0 \in C$. Since $F_p(u_0) = u_0 = \tilde{\gamma}(u_0)$, the above map ν_{A_E, i, u_0} agrees with the map induced by $(F_p^m)^*(\nu_{A_E})$ on $\mathcal{H}_{u_0}^i(A_E)$. Hence, we obtain that

$$\phi_{A_E, i, u_0} = \phi_{A_E, i, u_0}^\circ \circ \nu_{A_E, i, u_0} : \mathcal{H}_{u_0}^i(A_E) \rightarrow \mathcal{H}_{u_0}^i(A_E).$$

Now let $i = -\dim C - \dim T_0$. By [Corollary 3.5](#), ϕ_{A_E, i, u_0} is given by scalar multiplication with $\delta_E q^{d_E}$; similarly, ϕ_{A_E, i, u_0}° is given by scalar multiplication with $\delta_E^\circ q^{d_E}$. Hence, ν_{A_E, i, u_0} must also be given by scalar multiplication with a sign $\delta_E^{\tilde{\gamma}} = \pm 1$, such that $\delta_E = \delta_E^{\tilde{\gamma}} \delta_E^\circ$. Note that $\delta_E^{\tilde{\gamma}}$ only depends on $\tilde{\gamma}$.

4. Determining the coefficients $p_{E', E}$

We keep the notation from the previous section. Taking into account the information in [Section 2.6](#) and the orthogonality relations for Green functions, Lusztig [20, §24.4] has described a purely combinatorial algorithm for determining the coefficients $p_{E', E}$, which modifies and simplifies an earlier algorithm of Shoji [27, §5].

We say that $w, w' \in W$ are γ -conjugate, and we write $w \sim_\gamma w'$ if there exists some $x \in W$ such that $w' = x^{-1}w\gamma(x)$. This defines an equivalence relation on W ; the equivalence classes are called γ -conjugacy classes. For $w \in W$, we set

$$C_W^\gamma(w) := \{x \in W \mid w = x^{-1}w\gamma(x)\}.$$

Then the size of the γ -conjugacy class of w is given by the index of $C_W^\gamma(w)$ in W . By [3, Prop. 3.3.6] and [3, Prop. 7.6.2], the Green functions of G^F satisfy the following orthogonality relations, for any $w, w' \in W$:

$$\sum_{g \in G_{\text{uni}}^F} Q_w(g)Q_{w'}(g) = \begin{cases} [G^F : T_w^F] |C_W^\gamma(w)| & \text{if } w \sim_\gamma w', \\ 0 & \text{otherwise.} \end{cases}$$

We define the matrix $\tilde{\Omega} = (\tilde{\omega}_{E', E})_{E', E \in \text{Irr}(W)^\gamma}$, where

$$\tilde{\omega}_{E', E} := \frac{1}{|W|} \sum_{w \in W} [G^F : T_w^F] \text{Tr}(\sigma_{E' \circ w}, E') \text{Tr}(\sigma_{E \circ w}, E) \in \mathbb{Q};$$

here, $T_w \subseteq G$ denotes an F -stable maximal torus obtained from T_0 by twisting with w and the maps $\sigma_E : E \rightarrow E$, $\sigma_{E'} : E' \rightarrow E'$ are as in [Section 2.1](#). As in [13, 3.19], we have the formula

$$\sum_{E \in \text{Irr}(W)^\gamma} \text{Tr}(\sigma_{E \circ w}) \text{Tr}(\sigma_{E \circ w'}, E) = \begin{cases} |C_W^\gamma(w)| & \text{if } w \sim_\gamma w', \\ 0 & \text{otherwise.} \end{cases}$$

Note also that, for $E \in \text{Irr}(W)^\gamma$, the function $w \mapsto \text{Tr}(\sigma_E \circ w, E)$ is constant on γ -conjugacy classes (by the definition of σ_E). Combining this with the formulae in [Section 2.1](#), the above orthogonality relations can be restated as follows:

$$\sum_{g \in G_{\text{uni}}^F} R_{E'}(g) R_E(g) = \tilde{\omega}_{E',E} \quad \text{for all } E', E \in \text{Irr}(W)^\gamma.$$

LEMMA 4.1. *We have $\det(\tilde{\Omega}) = \prod_w [G^F : T_w^F]$, where w runs over a set of representatives of the γ -conjugacy classes of W .*

Proof. Let $w_1, \dots, w_n \in W$ be representatives of the γ -conjugacy classes of W . It is known that then $|\text{Irr}(W)^\gamma| = n$; see, e.g., [\[9, Lemma 7.3\]](#). Let us write $\text{Irr}(W)^\gamma = \{E_1, \dots, E_n\}$. We define a matrix

$$X = (x_{ii'})_{1 \leq i, i' \leq n}, \quad \text{where} \quad x_{ii'} := \text{Tr}(\sigma_{E_{i'}} \circ w_i, E_{i'}).$$

Then, by [Section 2.1](#), we have $Q_{w_i}(g) = \sum_{1 \leq i' \leq n} x_{ii'} R_{E_{i'}}(g)$ for all $g \in G_{\text{uni}}^F$ and so

$$\sum_{g \in G_{\text{uni}}^F} Q_{w_i}(g) Q_{w_j}(g) = \sum_{1 \leq i', j' \leq n} x_{ii'} x_{jj'} \tilde{\omega}_{E_{i'}, E_{j'}} = (X \cdot \tilde{\Omega} \cdot X^{\text{tr}})_{ij}.$$

On the other hand, the left-hand side equals $[G^F : T_{w_i}^F] |C_W^\gamma(w_i)|$ if $i = j$, and 0 otherwise. Hence, we find that

$$\det(X \cdot X^{\text{tr}}) \det(\tilde{\Omega}) = \det(X \cdot \tilde{\Omega} \cdot X^{\text{tr}}) = \prod_{1 \leq i \leq n} [G^F : T_{w_i}^F] |C_W^\gamma(w_i)|.$$

It remains to show that $\det(X \cdot X^{\text{tr}}) = \prod_{1 \leq i \leq n} |C_W^\gamma(w_i)|$. But this immediately follows from the identity

$$(X \cdot X^{\text{tr}})_{ij} = \sum_{1 \leq i' \leq n} \text{Tr}(\sigma_{E_{i'}} \circ w_i, E_{i'}) \text{Tr}(\sigma_{E_{i'}} \circ w_j, E_{i'}) = \begin{cases} |C_W^\gamma(w_i)| & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

which holds by the above-mentioned formula from [\[13, 3.19\]](#). \square

Following [\[20, 24.3.4\]](#), we define three matrices

$$P = (p_{E',E}), \quad \Omega = (\omega_{E',E}), \quad \Lambda = (\lambda_{E',E}),$$

where, in each case, the indices run over all $E', E \in \text{Irr}(W)^\gamma$. Here, $p_{E',E}$ are the coefficients in [Section 2.6](#); furthermore,

$$\omega_{E',E} := q^{-d_E - d_{E'}} \tilde{\omega}_{E',E} \quad \text{and} \quad \lambda_{E',E} := \sum_{g \in G_{\text{uni}}^F} Y_{E'}(g) Y_E(g).$$

(The integers d_E are defined in [Section 2.4](#).)

PROPOSITION 4.2 (Lusztig [\[20, 24.4\]](#)). *We have $P^{\text{tr}} \cdot \Lambda \cdot P = \Omega$. Furthermore, for all $E', E \in \text{Irr}(W)^\gamma$, we have $p_{E',E} \in \mathbb{Z}$, $\lambda_{E',E} \in \mathbb{Z}$, and $\omega_{E',E} \in \mathbb{Z}$.*

Proof. Recall from [Section 2.6](#) the following relations:

$$R_E|_{G_{\text{uni}}^F} = \sum_{E' \in \text{Irr}(W)^\gamma} q^{d_E} p_{E',E} Y_{E'} \quad \text{for all } E \in \text{Irr}(W)^\gamma.$$

This immediately implies the above matrix identity. The fact that $p_{E',E} \in \mathbb{Z}$ was already mentioned in [Section 2.6](#). The fact that $\lambda_{E',E} \in \mathbb{Z}$ follows from the fact that the Y -functions are integer-valued; see [Section 2.5](#). Finally, the above matrix identity implies that we also have $\omega_{E',E} \in \mathbb{Z}$. \square

We obtain further information about the matrices P and Λ by taking into account the additional information on $p_{E',E}$ in [Section 2.6](#). Let $E, E' \in \text{Irr}(W)^\gamma$ and $\iota_G(E) = (C, \mathcal{E}) \in \mathcal{N}_G$, $\iota_G(E') = (C', \mathcal{E}') \in \mathcal{N}_G$, where C, C' are F -stable and $F^* \mathcal{E} \cong \mathcal{E}$, $F^* \mathcal{E}' \cong \mathcal{E}'$. As in [\[20, 24.1\]](#), we write $E \sim E'$ if $C = C'$. This gives rise to a partition

$$\text{Irr}(W)^\sigma = \mathcal{I}_1 \sqcup \cdots \sqcup \mathcal{I}_h,$$

where $\mathcal{I}_1, \dots, \mathcal{I}_h$ are the equivalence classes for the relation \sim . Note that $d_E = d_{E'}$ if $E \sim E'$. Thus, we can define $d_i := d_E$, where $E \in \mathcal{I}_i$.

Remark 4.3. We fix a labelling of the equivalence classes $\mathcal{I}_1, \dots, \mathcal{I}_h$ such that

$$d_1 \geq d_2 \geq \cdots \geq d_h,$$

and we enumerate $\text{Irr}(W)^\sigma$ in a way which is compatible with the above partition of $\text{Irr}(W)^\sigma$. Then it is clear that Λ has a block diagonal shape, where the blocks correspond to the sets $\mathcal{I}_1, \dots, \mathcal{I}_h$ (see [\[20, 24.3.2\]](#)). Furthermore, P has an upper block triangular shape with identity matrices on the diagonal (see [\[20, 24.2.10, 24.2.11\]](#)). More precisely, we can write

$$P = \begin{bmatrix} I_{e_1} & P_{1,2} & \cdots & P_{1,h} \\ 0 & I_{e_2} & & \vdots \\ \vdots & & \ddots & P_{h-1,h} \\ 0 & \cdots & 0 & I_{e_h} \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \Lambda_h \end{bmatrix};$$

here, $e_j = |\mathcal{I}_j|$ and I_{e_j} denotes the identity matrix of size e_j . For $1 \leq i < j \leq h$, the block $P_{i,j}$ has size $e_i \times e_j$ and entries $p_{E',E}$ for $E' \in \mathcal{I}_i$ and $E \in \mathcal{I}_j$; similarly, the block Λ_j has size $e_j \times e_j$ and entries $\lambda_{E',E}$ for $E', E \in \mathcal{I}_j$.

With the additional requirement that P and Λ have block shapes as above, it easily follows that P, Λ are uniquely determined by Ω and the equation $P^{\text{tr}} \cdot \Lambda \cdot P = \Omega$; see [\[20, 24.4\]](#), [\[27, §5\]](#) (or the proof of [Lemma 4.4](#) below).

With these preparations, we can now establish a first step towards the proof of the Congruence Condition in [Section 1](#). Let $d \geq 1$ be the order of $\gamma: W \rightarrow W$ and

$$\mathcal{M} := \{r \in \mathbb{Z}_{\geq 1} \mid r \equiv 1 \pmod{d}\}$$

as in [Remark 3.6](#). Let $r \in \mathcal{M}$, and replace F by F^r . We obtain analogous matrices as above, which we now denote by

$$P^{(r)} = (p_{E',E}^{(r)}), \quad \Omega^{(r)} = (\omega_{E',E}^{(r)}), \quad \Lambda^{(r)} = (\lambda_{E',E}^{(r)})$$

and where the indices run again over all $E', E \in \text{Irr}(W)^\gamma$. (Note that γ is also the automorphism of W induced by F^r , since $r \in \mathcal{M}$.)

LEMMA 4.4. *Let $r \in \mathcal{M}$ be a prime such that*

$$r \nmid \det(\Omega) \quad \text{and} \quad \omega_{E',E}^{(r)} \equiv \omega_{E',E} \pmod r \quad \text{for all } E', E \in \text{Irr}(W)^\gamma.$$

Then $p_{E',E}^{(r)} \equiv p_{E',E} \pmod r$ and $\lambda_{E',E}^{(r)} \equiv \lambda_{E',E} \pmod r$ for all $E', E \in \text{Irr}(W)^\gamma$.

Proof. For $a \in \mathbb{Z}$, we denote by $\bar{a} \in \mathbb{F}_r$ the reduction modulo r . If $A = (a_{ij})$ is a matrix with entries in \mathbb{Z} , then we denote $\bar{A} = (\bar{a}_{ij})$. With this notation, we must show that $\bar{P} = \bar{P}^{(r)}$ and $\bar{\Lambda} = \bar{\Lambda}^{(r)}$ under the given assumptions, which mean that

$$\bar{\Omega}^{(r)} = \bar{\Omega} \quad \text{and} \quad \det(\bar{\Omega}^{(r)}) = \det(\bar{\Omega}) \neq 0.$$

Now note that the Springer correspondence $\iota_G: \text{Irr}(W) \rightarrow \mathcal{N}_G$ does not depend on any Frobenius map. Hence, the partition $\text{Irr}(W)^\gamma = \mathcal{S}_1 \sqcup \dots \sqcup \mathcal{S}_h$ and the block structure of P, Λ in [Remark 4.3](#) remain the same for $P^{(r)}, \Lambda^{(r)}$. Thus, Λ and $\Lambda^{(r)}$ are block diagonal matrices; furthermore, P and $P^{(r)}$ are upper block triangular matrices with identity blocks along the diagonal and so $\det(\bar{P}) = \det(\bar{P}^{(r)}) = 1$. Consequently, we have $\det(\bar{\Lambda}) = \det(\bar{\Omega}) \neq 0$ and $\det(\bar{\Lambda}^{(r)}) = \det(\bar{\Omega}^{(r)}) = \det(\bar{\Omega}) \neq 0$. So all of the above matrices are invertible. Hence, from the identities

$$(\bar{P}^{(r)})^{\text{tr}} \cdot \bar{\Lambda}^{(r)} \cdot \bar{P}^{(r)} = \bar{\Omega}^{(r)} = \bar{\Omega} = \bar{P}^{\text{tr}} \cdot \bar{\Lambda} \cdot \bar{P}$$

we can deduce the identity

$$(\bar{P}^{\text{tr}})^{-1} \cdot (\bar{P}^{(r)})^{\text{tr}} = \bar{\Lambda} \cdot \bar{P} \cdot (\bar{P}^{(r)})^{-1} \cdot (\bar{\Lambda}^{(r)})^{-1}.$$

Since the block shape remains the same when passing from F to F^r , the left-hand side of the above identity is a block lower triangular matrix with identity blocks along the diagonal, while the right-hand side is a block upper triangular matrix. Hence, we conclude that $\bar{P} = \bar{P}^{(r)}$ and $\bar{\Lambda} = \bar{\Lambda}^{(r)}$, as desired. \square

Remark 4.5. Arguing as in the first part of the proof of [\[20, Th. 24.8\]](#), one sees that there are well-defined polynomials $\pi_{E',E} \in \mathbb{Q}[\mathbf{q}]$ (where \mathbf{q} is an indeterminate) such that $p_{E',E}^{(r)} = \pi_{E',E}(q^r)$ for all $r \in \mathcal{M}$. With a little extra work, one can show that $\pi_{E',E} \in \mathbb{Z}[\mathbf{q}]$. This would, of course, also imply the conclusion of [Lemma 4.4](#), as far as the $p_{E',E}$ and $p_{E',E}^{(r)}$ are concerned.

5. Proof of the Congruence Condition

As remarked in [11, p. 202], it is sufficient to prove the Congruence Condition in Section 1 in the case where G is simple of adjoint type. We assume for the rest of this section that this is the case. Let us begin with an endomorphism $F' : G \rightarrow G$ such that some power of F' is a Frobenius map. Then $G^{F'}$ is one of the groups considered by Steinberg [33, §11.6]. It has been already shown in [11, Prop. 7.7] that the Congruence Condition holds for (G, F') of type ${}^3D_4, {}^2B_2, {}^2G_2, {}^2F_4$. So it remains to consider the case where $F' = F$ is a Frobenius map. Assume now that this is the case. Thus, we have

$$F = \tilde{\gamma} \circ F_p^m = F_p^m \circ \tilde{\gamma} \quad (m \geq 1)$$

as in Section 3, where $\tilde{\gamma} : G \rightarrow G$ is a graph automorphism of order $d \in \{1, 2, 3\}$ leaving T_0, B_0 invariant, and $F_p : G \rightarrow G$ is a Frobenius map corresponding to a split \mathbb{F}_p -rational structure, such that $F_p(t) = t^p$ for all $t \in T_0$. Then the map $\gamma : W \rightarrow W$ induced by $\tilde{\gamma}$ also has order d . Since groups of type 3D_4 have already been dealt with, we may actually assume that $d \in \{1, 2\}$. We set

$$\mathcal{M} := \{r \in \mathbb{Z}_{\geq 1} \mid r \equiv 1 \pmod{d}\} \quad (d \in \{1, 2\})$$

as in Remark 3.6. (Thus, \mathcal{M} consists of all positive integers if $d = 1$, and of all odd positive integers if $d = 2$.) Let $r \in \mathcal{M}$. Working with both F and F^r , we will have to consider two sets of matrices:

$$\begin{aligned} P &= (p_{E',E}), & \Omega &= (\omega_{E',E}), & \Lambda &= (\lambda_{E',E}), \\ P^{(r)} &= (p_{E',E}^{(r)}), & \Omega^{(r)} &= (\omega_{E',E}^{(r)}), & \Lambda^{(r)} &= (\lambda_{E',E}^{(r)}), \end{aligned}$$

defined as in the previous section. For each $E \in \text{Irr}(W)^\gamma$, we have an almost character R_E and a Y -function Y_E for G^F ; similarly, we have an almost character and a Y -function for G^{F^r} , which we denote by $R_E^{(r)}$ and $Y_E^{(r)}$, respectively. This notation will be used throughout this section.

LEMMA 5.1. *Let $r \in \mathcal{M}$ be a prime. Then we have $|G^{F^r}| \equiv |G^F| \pmod{r}$ and $|T_w^{F^r}| \equiv |T_w^F| \pmod{r}$ for any $w \in W$.*

Proof. Let $X^\vee(T_0) = \text{Hom}(k^\times, T_0)$ be the co-character group of T_0 (a free abelian group of rank equal to $\dim T_0$). Then $\tilde{\gamma}$ induces an automorphism $\tilde{\gamma}^* \in \text{Aut}(X^\vee)$ such that $\tilde{\gamma}^*(\nu)(x) = \tilde{\gamma}(\nu(x))$ for all $x \in k^\times$ and $\nu \in X^\vee$. We can also naturally regard W as a subgroup of $\text{Aut}(X^\vee)$; see [3, §1.9]. Let \mathbf{q} be an indeterminate over \mathbb{Z} , and define

$$f_w := \det(\mathbf{q} \text{id}_{X^\vee} - (\tilde{\gamma}^*)^{-1} \circ w) \in \mathbb{Z}[\mathbf{q}] \quad \text{for any } w \in W.$$

Then we have $|T_w^F| = f_w(q)$, where $q = p^m$; see [3, Prop. 3.3.5]. Furthermore, we have $|G^F| = f(q)$, where we define

$$f := \mathbf{q}^{|\Phi^+|} f_1 \sum_{w \in W, \gamma(w)=w} \mathbf{q}^{l(w)} \in \mathbb{Z}[\mathbf{q}];$$

see [3, §2.9]. Now let us replace F by F^r , where $r \in \mathcal{M}$. Then $F^r = \tilde{\gamma} \circ F_p^{rm} = F_p^{rm} \circ \tilde{\gamma}$. Consequently, we obtain that $|T_w^{F^r}| = f_w(q^r)$ and $|G^{F^r}| = f(q^r)$. If $r \in \mathcal{M}$ is a prime, then Fermat's Little Theorem yields the desired congruences. \square

LEMMA 5.2. *Let $r \in \mathbb{N}$ be a prime such that $r \nmid |G^{F^r}|$. Then $\omega_{E',E}^{(r)} \equiv \omega_{E',E} \pmod r$, $p_{E',E}^{(r)} \equiv p_{E',E} \pmod r$ and $\lambda_{E',E}^{(r)} \equiv \lambda_{E',E} \pmod r$ for all $E', E \in \text{Irr}(W)^\gamma$.*

Proof. Since G is non-trivial, the order of $|G^{F^r}|$ is even and so $r > 2$. Hence, we automatically have $r \in \mathcal{M}$. Let $w_1, \dots, w_n \in W$ be a set of representatives of the γ -conjugacy classes of W . First we consider the matrix $\tilde{\Omega}$. We have

$$\begin{aligned} \tilde{\omega}_{E',E} &= \sum_{1 \leq i \leq n} |C_W^\gamma(w_i)|^{-1} [G^F : T_{w_i}^F] \text{Tr}(\sigma_{E' \circ w_i}, E') \text{Tr}(\sigma_{E \circ w_i}, E), \\ \tilde{\omega}_{E',E}^{(r)} &= \sum_{1 \leq i \leq n} |C_W^\gamma(w_i)|^{-1} [G^{F^r} : T_{w_i}^{F^r}] \text{Tr}(\sigma_{E' \circ w_i}, E') \text{Tr}(\sigma_{E \circ w_i}, E). \end{aligned}$$

Since $G^F \subseteq G^{F^r}$, we also have $r \nmid |G^F|$. Hence, using Lemma 5.1, we obtain that

$$[G^{F^r} : T_w^{F^r}] \equiv [G^F : T_w^F] \pmod r \quad \text{for all } w \in W.$$

By [3, Prop. 3.3.6], we have $N_G(T_w^F)/T_w^F \cong C_W^\gamma(w)$. Since $N_G(T_w)^F \subseteq G^F$, we conclude that $r \nmid |C_W^\gamma(w)|$. Consequently, we have $\tilde{\omega}_{E',E}^{(r)} \equiv \tilde{\omega}_{E',E} \pmod r$. Now

$$\tilde{\omega}_{E',E} = q^{d_E + d_{E'}} \omega_{E',E} \quad \text{and} \quad \tilde{\omega}_{E',E}^{(r)} = q^{(d_E + d_{E'})r} \omega_{E',E}^{(r)}.$$

Using $r \nmid q$ and Fermat's Little Theorem, we conclude that we also have $\omega_{E',E}^{(r)} \equiv \omega_{E',E} \pmod r$. Finally, by Lemma 4.1, we have $r \nmid \det(\tilde{\Omega})$ and, hence, also $r \nmid \det(\Omega)$ (again, since $r \nmid q$). So we can apply Lemma 4.4. \square

LEMMA 5.3. *Let $u \in G^F$ be unipotent and $E \in \text{Irr}(W)^\gamma$. Let $r \in \mathbb{N}$ be a prime such that $r \nmid |G^{F^r}|$. Then $Y_E^{(r)}(u) = Y_E(u)$.*

Proof. As in the previous proof, we have $r > 2$ and $r \in \mathcal{M}$. Let $\iota_G(E) = (C, \mathcal{E})$, where C is F -stable and $F^* \mathcal{E} \cong \mathcal{E}$. If $u \notin C$, then $Y_E(u) = Y_E^{(r)}(u) = 0$. So now let $u \in C$. Since we are assuming that G is simple of adjoint type, it is known that there exists an element $u_0 \in C^F$ such that (\clubsuit) in Remark 3.2 holds. For G of classical type, such a representative $u_0 \in C$ is explicitly described by Shoji [31, §2]; see, e.g., [31, 2.10] for type 2D_n in arbitrary characteristic. If G

is of exceptional type, the existence of u_0 is guaranteed by [12, Lemma 20.16], the proof of which involves a certain amount of case-by-case considerations. For example, for type E_7, E_8 , one can simply look through the list of class representatives determined by Mizuno [24]. See also the discussion in [34, §2] (in good characteristic). In any case, let us now fix an element $u_0 \in C^F$ such that F acts trivially on $A(u_0)$. Then $u = gu_0g^{-1}$ for some $g \in G$, and we have $x := g^{-1}F(g) \in C_G(u_0)$; we denote by a the image of x in $A(u)$. Consequently, we have

$$Y_E(u) = \delta_E \text{Tr}(a, \mathcal{E}_{u_0}) \quad (\text{see Lemma 3.3}).$$

Note that, since the values of the Y -functions are integers, the same is true for $\text{Tr}(a, \mathcal{E}_{u_0})$. Let us now replace F by F^r . We still have $x' := g^{-1}F^r(g) \in C_G(u_0)$; consequently, if we denote by a' the image of x' in $A(u_0)$, then

$$Y_E^{(r)}(u) = \delta_E^{(r)} \text{Tr}(a', \mathcal{E}_{u_0})$$

(and, again, these values are integers). By Theorem 3.7, we have $\delta_E^{(r)} = (\delta_E)^r$. Since r is odd, we conclude that $\delta_E^{(r)} = \delta_E$. Hence, it remains to show that

$$\text{Tr}(a, \mathcal{E}_{u_0}) = \text{Tr}(a', \mathcal{E}_{u_0}).$$

Now, since F acts trivially on $A(u_0)$, we have $A(u_0) \cong C_G(u_0)^F / C_G^\circ(u_0)^F$ (see [3, p. 33]). Hence, since $r \nmid |G^F|$, we also have $r \nmid |A(u_0)|$. But then a well-known result in the character theory of finite groups implies that $\text{Tr}(a, \mathcal{E}_{u_0}) = \text{Tr}(a^r, \mathcal{E}_{u_0})$. So it will be enough to show that $a^r = a'$. This is seen as follows. We have $F(g) = gx$ and $F^r(g) = gx'$, with $x, x' \in C_G(u_0)$, which implies that

$$x' = xF(x)F(x)^2 \cdots F^{r-1}(x).$$

Since F acts trivially on $C_G(u_0)/C_G^\circ(u_0)$, we have $x^{-1}F^i(x) \in C_G^\circ(u_0)$ for all $i \geq 1$. This yields $x' = x^r c$ for some $c \in C_G^\circ(u_0)$ and, hence, $a' = a^r$, as desired. \square

We can now complete the proof of the Congruence Condition, as follows. Let $u \in G^F$ be unipotent and $T \subseteq G$ be an F -stable maximal torus. Let $w \in W$ be such that T is obtained from T_0 by twisting with w (relative to F). Thus, we have $Q_{T,F} = Q_w$. Let $r \in \mathbb{N}$ be a prime such that $r \nmid |G^{F^r}|$; then $r > 2$ and $r \in \mathcal{M}$ (see the above proofs). As pointed out in [11, p. 204], we also have that T is obtained by twisting with w relative to F^r . Thus, we have $Q_{T,F^r}(u) = Q_w^{(r)}(u)$ where, as usual, we indicate by the superscript “ (r) ” that we mean the Green function for G^{F^r} . Now, by Proposition 3.1, we have the

following formulae:

$$Q_w(u) = \sum_{E', E \in \text{Irr}(W)^\gamma} \text{Tr}(\sigma_{E \circ w}, E) q^{d_E} p_{E', E} Y_{E'}(u),$$

$$Q_w^{(r)}(u) = \sum_{E', E \in \text{Irr}(W)^\gamma} \text{Tr}(\sigma_{E \circ w}, E) q^{d_{E^r}} p_{E', E}^{(r)} Y_{E'}^{(r)}(u).$$

By Lemma 5.2, we have $p_{E', E}^{(r)} \equiv p_{E', E} \pmod r$; furthermore, by Lemma 5.3, we have $Y_E^{(r)}(u) = Y_E(u)$. Finally, by Fermat's Little Theorem, we have $q^{d_E} \equiv q^{d_{E^r}} \pmod r$. Hence, we conclude that $Q_w(u) \equiv Q_w^{(r)}(u) \pmod r$. Thus, the Congruence Condition in Section 1 is proved. \square

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