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A minimizing valuation is quasi-monomial

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Abstract

We prove a version of Jonsson-Mustaţă's Conjecture, which says for any graded sequence of ideals, there exists a quasi-monomial valuation computing its log canonical threshold. As a corollary, we confirm Chi Li's conjecture that a minimizer of the normalized volume function is always quasi-monomial.

Applying our techniques to a family of klt singularities, we show that the volume of klt singularities is a constructible function. As a corollary, we prove that in a family of klt log Fano pairs, the K-semistable ones form a Zariski open set. Together with previous works by many people, we conclude that all K-semistable klt Fano varieties with a fixed dimension and volume are parametrized by an Artin stack of finite type, which then admits a separated good moduli space, whose geometric points parametrize K-polystable klt Fano varieties.

Contents

1. Introduction

Throughout this paper, we work over an algebraically closed field k of characteristic 0. In this note, we use recent developments in birational geometry, especially results from the minimal model program, to study invariants of singularities that are of an asymptotic nature. We aim to get some uniform results that cannot be obtained by previous methods.

Keywords: quasi-monomial, complement, local volume of klt singularities, K-moduli of Fano varieties

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1.1. *The valuation computing the log canonical threshold.* Our first theorem is to prove that for any graded sequence of ideals, there always exists a quasi-monomial valuation that computes the log canonical threshold.

THEOREM 1.1 ([JM12, Conj. B]). *If (X,Δ) is a Kawamata log terminal (klt) pair, and $\mathfrak{a}_\bullet := \{\mathfrak{a}_k\}_{k\in\mathbb{N}}$ is a graded sequence of ideals such that the log canonical threshold $\operatorname{lct}(\mathfrak{a}_\bullet) < \infty$, then there exists a quasi-monomial valuation v that calculates the log canonical threshold of $\mathfrak{a}_\bullet$, i.e.,*

$$\frac{A_{X,\Delta}(v)}{v(\mathfrak{a}_\bullet)} = \inf_w \frac{A_{X,\Delta}(w)}{w(\mathfrak{a}_\bullet)},$$

where w runs through all valuations whose center is on X.

This confirms the weak version of [JM12, Conj. B]. However, our techniques do not directly give the strong version, which predicts *any* valuation w computing the log canonical threshold of $\mathfrak{a}_\bullet$ is quasi-monomial. More precisely, for any such w, our approach produces a quasi-monomial valuation v with $A_{X,\Delta}(v) = A_{X,\Delta}(w)$, $v \geq w$, and v also computes the log canonical threshold of $\mathfrak{a}_\bullet$.

One consequence of the above theorem is the following statement.

THEOREM 1.2 ([Li18, Conj. 7.1.3]). *Let $x \in (X,\Delta)$ be a klt singularity. Any minimizer $v^{\rm m}$ of the normalized volume function*

$$\widehat{\rm vol}_{(X,\Delta),x}\colon {\rm Val}_{X,x} \to \mathbb{R}_{>0}\bigcup\{+\infty\}$$

is quasi-monomial.

This is one piece of a circle of conjectures about the minimizer of the normalized volume function $\widehat{\rm vol}_{(X,\Delta),x}$, which are all together packed into the Stable Degeneration Conjecture (see [Li18, Conj. 7.1], [LX18, Conj. 1.2]), and predict some deep information about an arbitrary klt singularity. We note that the Stable Degeneration Conjecture has been intensively studied (see, e.g., [Li17], [Blu18], [LL19], [LX16], [LX18]). Combining Theorem 1.2 with the previously known results, the main remaining part is to show that for the quasi-monomial minimizer v, its associated graded ring is finitely generated. While this is known when the rational rank is one ([LX16, Blu18]), this is still open when the rational rank of v is larger than one, except when $\dim(X) = 2$ (see [Cut18, Prop. 1.4]).

Another application of Theorem 1.1 is that it finishes the algebraic approach, originated from [JM14, Th. D], of solving the Demailly-Kollár's Openness Conjecture (see [DK01]). Recall that the Openness Conjecture says that for any germ of a pluri-subharmonic function ϕ at a point x on a complex manifold, denoted by $c_x(\phi)$, the complex singularity exponent of ϕ at x (see [DK01, Def. 0.1]), we have

if $c_x(\phi) < \infty$, then the function $\exp(-2c_x(\phi)\phi)$ is not locally integrable at x.

We note that the Openness Conjecture has been proved in [Ber15], [GZ15] by completely different methods with more analytic nature. On the other hand, it seems that our approach *cannot* yield the stronger version [JM14, Conj. C″], which would imply the Strong Openness Conjecture. Nevertheless, the latter is also proved in [GZ15] (see also [Hie14], [Lem17]).

The proof of Theorem 1.1 depends on a combination of two sets of recently established techniques. The first one is approximating a valuation computing the log canonical threshold by a sequence of valuations that can be better understood using birational geometry. This idea is developed in [LX16]. In particular, the proof of [LX16, Th. 1.3] essentially implies that in Theorem 1.1, we can find a valuation v computing the log canonical threshold that can be always approximated by a sequence of rescalings of Kollár components S_i (see Definition 2.6). Roughly speaking, Kollár components are divisorial valuations over $x \in (X, \Delta)$, which admits a log Fano structure.

The second main ingredient is the boundedness of complements which was recently established in [Bir19]. This difficult result together with an estimate established in [Li18], imply that all S_i can be obtained as log canonical places of *a bounded family of* $\mathbb{Q}$*-Cartier divisors* on (X, Δ). From this boundedness, we then could conclude that the limit is quasi-monomial.

1.2. *The volume function of klt singularities is constructible.* Applying our techniques to a family of klt singularities also leads to a proof of the following result.

THEOREM 1.3. *For a* $\mathbb{Q}$*-Gorenstein family of klt singularities* $(B \subset (X, \Delta)) \to B$ *over a smooth base, the volume function*

$$\widehat{\mathrm{vol}}_B \colon B \to \mathbb{R}_{>0} \qquad (s \to B) \to \widehat{\mathrm{vol}}(s, X_s, \Delta_s),$$

which sends each geometric point s *to the volume of singularity over* s, *is constructible in Zariski topology.*

In [BL18a], it is shown that $\widehat{\mathrm{vol}}_B$ is lower semi-continuous. Combining with the cone construction, we obtain the following theorem.

THEOREM 1.4. *For a* $\mathbb{Q}$*-Gorenstein family of log Fano pairs* $(X, \Delta) \to B$ *over a smooth base* B, *the locus* $B^\circ \subset B$ *that parametrizes K-semistable geometric fibers forms an open set.*

The openness of uniform K-stability in a $\mathbb{Q}$-Gorenstein family of log Fano pairs was previously proved in [BL18b]. Together with [Jia17], [BX19], and [ABHLX19], we conclude the following theorem.

THEOREM 1.5. *Fix* n *and* V. *The functor* $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ *of families of K-semistable* $\mathbb{Q}$*-Fano varieties of dimension* n *and volume* V *is an Artin stack of finite type.*

Moreover, it admits a good moduli space $X^{\mathrm{kps}}_{n,V}$, whose geometric points parametrize K-polystable $\mathbb{Q}$-Fano varieties.

We expect that $X^{\mathrm{kps}}_{n,V}$ is proper or even projective.

Remark 1.6. In the simultaneous work [BLX19], in a more global setting, i.e., for log Fano pairs (X,Δ), a similar strategy is applied to study the stability thresholds $\delta(X,\Delta)$. As a result, Theorem 1.4 as well as the consequence Theorem 1.5 are also proved there.

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2. **Preliminaries**

Notation and conventions. We use the standard notation as in [Laz04], [KM98] and [Kol13]. In particular, for X a normal variety, with a $\mathbb{Q}$-divisor Δ on X, we define (X,Δ) to be *klt*, *lc* as in [KM98, Def. 2.34]. For a lc (log canonical) pair (X,Δ), and a divisor E over (X,Δ), its log discrepancy $A_{X,\Delta}(E)$ is $1+a(E,X,\Delta)$, where $a(E,X,\Delta)$ is its discrepancy (see [KM98, Def. 2.25]). We say a divisor E over X is a *log canonical place* if the log discrepancy $A_{X,\Delta}(E)$ equals 0. We call $x\in X=\operatorname{Spec} R$ *a germ* if R is a local ring essentially of finite type over k and x is its closed point. When $x\in X$ is a germ of normal point, we can define (X,Δ) to be klt (resp. lc) for a $\mathbb{Q}$-divisor in the obvious way. By abuse of notation, in this case, we say $x\in(X,\Delta)$ is a klt (resp. lc) singularity.

A projective pair (X,Δ) is called a *log Fano pair* if (X,Δ) is klt and $-K_X-\Delta$ is ample.

For a morphism $X\to B$ and a point $s\in B$, we will use X_s to mean its fiber.

Given a ring R, an $\mathbb{N}$-graded sequence $\mathfrak{a}_\bullet=\{\mathfrak{a}_k\}_{k\in\mathbb{N}}$ of ideals is a set of ideals $\mathfrak{a}_k\subset R$ $(k\in\mathbb{N})$ satisfying that $\mathfrak{a}_k\cdot\mathfrak{a}_{k'}\subset\mathfrak{a}_{k+k'}$. We will sometimes also include $\mathfrak{a}_0=R$ in a graded sequence of ideals.

For two divisors, if $D=\sum_i d_iD_i$ and $D'=\sum_i d'_iD_i$, we define $D\wedge D'=\sum_i\min\{d_i,d'_i\}D_i$.

2.1. *The space of valuations.*

2.1.1. *Valuations.* Let X be a reduced, irreducible (separated) variety defined over k. A *real valuation* of its function field $K(X)$ is a non-constant map $v\colon K(X)^\times\to\mathbb{R}$, satisfying the following:

- $v(fg) = v(f) + v(g)$;
- $v(f+g) \geq \min\{v(f), v(g)\}$;
- $v(k^*) = 0$.

We set $v(0) = +\infty$. A valuation v gives rise to a valuation ring

$$\mathcal{O}_v := \{f \in K(X) \mid v(f) \geq 0\}.$$

We say a real valuation v is *centered at* a scheme-theoretic point $x = c_X(v) \in X$ if we have a local inclusion $\mathcal{O}_{x,X} \hookrightarrow \mathcal{O}_v$ of local rings. Notice that the center of a valuation, if it exists, is unique since X is separated. Denote by Val_X the set of real valuations of $K(X)$ that admits a center on X. For a closed point $x \in X$, we denote by $\mathrm{Val}_{X,x}$ the set of real valuations of $K(X)$ centered at $x \in X$. A valuation $v \in \mathrm{Val}_X$ is centered at $x \in X$ if $v(f) > 0$ for any $f \in \mathfrak{m}_x$.

For each valuation $v \in \mathrm{Val}_{X,x}$ and any positive integer k, we define the valuation ideal

$$\mathfrak{a}_k^v := \{f \in \mathcal{O}_{x,X} \mid v(f) \geq k\}.$$

Then it is clear that $\mathfrak{a}_k^v$ is an $\mathfrak{m}_x$-primary ideal for each k and $x = c_X(v)$.

Given a valuation $v \in \mathrm{Val}_X$ and a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_X$, we may evaluate $\mathfrak{a}$ along v by setting

$$v(\mathfrak{a}) := \min\{v(f) \mid f \in \mathfrak{a} \cdot \mathcal{O}_{c_X(v),X}\}.$$

It follows from the above definition that if $\mathfrak{a} \subset \mathfrak{b} \subset \mathcal{O}_X$ are nonzero ideals, then $v(\mathfrak{a}) \geq v(\mathfrak{b})$. Additionally, $v(\mathfrak{a}) > 0$ if and only if $c_X(v) \in \mathrm{Cosupp}(\mathfrak{a})$. We endow Val_X with the weakest topology such that, for every ideal $\mathfrak{a}$ on X, the map $\mathrm{Val}_X \to \mathbb{R} \cup \{+\infty\}$ defined by $v \mapsto v(\mathfrak{a})$ is continuous. The subset $\mathrm{Val}_{X,x} \subset \mathrm{Val}_X$ is endowed with the subspace topology. In some literature, the space $\mathrm{Val}_{X,x}$ is called the *non-archimedean link* of $x \in X$. For more background on valuation spaces, see [JM12, §4].

We say two valuations $v \geq w$ if $v(\mathfrak{a}) \geq w(\mathfrak{a})$ for any $\mathfrak{a} \subset \mathcal{O}_X$ (see [JM12, Def. 4.3]). Let D be a Cartier divisor on X and v a valuation. We can also define $v(D)$ to be $v(f)$, where f is a defining equation of D at $c_X(v)$. And similarly if D is $\mathbb{Q}$-Cartier, we can define $v(D) = \frac{1}{m}v(mD)$ for a sufficiently divisible positive integer m. For $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k \in \mathbb{N}}$, we define

$$v(\mathfrak{a}_\bullet) = \inf_k \frac{v(\mathfrak{a}_k)}{k} = \lim_{k \to \infty, \mathfrak{a}_k \neq 0} \frac{v(\mathfrak{a}_k)}{k}.$$

Let $Y \xrightarrow{\mu} X$ be a proper birational morphism with Y a normal variety. For a prime divisor E on Y, we define a valuation $\mathrm{ord}_E \in \mathrm{Val}_X$ that sends each rational function in $K(X)^\times = K(Y)^\times$ to its order of vanishing along E. Note that the center $c_X(\mathrm{ord}_E)$ is the generic point of $\mu(E)$. We say that $v \in \mathrm{Val}_X$ is a *divisorial valuation* if there exists E as above and $\lambda \in \mathbb{R}_{>0}$ such that $v = \lambda \cdot \mathrm{ord}_E$.

Next, we will introduce another important class of valuations, which are called *quasi-monomial valuations*. Let $\mu : Y \to X$ be a proper birational morphism and $\eta \in Y$ a point such that Y is regular at η. Given a system of parameters $y_1, \dots, y_r \in \mathcal{O}_{Y,\eta}$ at η and $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r_{\geq 0} \setminus \{0\}$, we define a valuation v_α as follows. For $f \in \mathcal{O}_{Y,\eta}$, we can write it as $f = \sum_{\beta \in \mathbb{Z}^r_{\geq 0}} c_\beta y^\beta$, with $c_\beta \in \widehat{\mathcal{O}_{Y,\eta}}$ either zero or a unit. We set

$$v_\alpha(f) = \min\{\langle \alpha, \beta \rangle \mid c_\beta \neq 0\}. \tag{2.1}$$

A *quasi-monomial valuation* is a valuation that can be written in the above form.

Let $(Y, E = \sum_{k=1}^N E_k)$ be a log smooth model of X; i.e., Y is smooth, E is simple normal crossing and $\mu : Y \to X$ is an isomorphism outside the support of E. We denote by $\mathrm{QM}_\eta(Y, E)$ the set of all quasi-monomial valuations v that can be described at the point $\eta \in Y$ with respect to coordinates $(y_1, \dots, y_r)$ such that each y_i defines at η an irreducible component of E. (Hence η is the generic point of a connected component of the intersection of some of the divisors E_i.) We put $\mathrm{QM}(Y, E) := \bigcup_\eta \mathrm{QM}_\eta(Y, E) \subset \mathrm{Val}_{X,x}$ where η runs over generic points of all irreducible components of intersections of some of the divisors E_i. Such a subspace $\mathrm{QM}(Y, E)$ can be naturally identified as a cone over the dual complex $\mathcal{D}(E)$ (see Definition 2.2).

Given a valuation $v \in \mathrm{Val}_{X,x}$, its *rational rank* $\mathrm{rat.rk}(v)$ is the rank of its value group. The *transcendental degree* $\mathrm{trans.deg}(v)$ of v is the transcendental degree of the field extension $k \hookrightarrow \mathcal{O}_v/\mathfrak{m}_v$. Let K be a field with transcendental degree n over k, $k \subset K_0 \subset K$ an intermediate field extension, v a valuation on K and v_0 its restriction to K_0. Then the Zariski-Abhyankar inequality states that

$$\mathrm{tr.deg}(v) + \mathrm{rat.rk}(v) \leq \mathrm{tr.deg}(v_0) + \mathrm{rat.rk}(v_0) + \mathrm{tr.deg}(K/K_0). \tag{2.2}$$

Taking $K_0 = k$, we have

$$\mathrm{trans.deg}(v) + \mathrm{rat.rk}(v) \leq n,$$

and a valuation satisfying the equality is called an *Abhyankar valuation*. By [ELS03, Prop. 2.8], we know that a valuation $v \in \mathrm{Val}_X$ is Abhyankar if and only if it is quasi-monomial.

2.1.2. *Log discrepancy*. Next, we give the definition of the log discrepancy $A_{X,\Delta}(v)$ (see Definition 2.1).

Definition 2.1 (Log discrepancy). Let (X, Δ) be a log canonical pair. We define the (*non-negative*) *log discrepancy function of valuations* $A_{X,\Delta} : \mathrm{Val}_X \to (0, +\infty]$ in successive generality.

(1) Let $\mu : Y \to X$ be a proper birational morphism from a normal variety Y. Let E be a prime divisor on Y. Then we define $A_{X,\Delta}(\mathrm{ord}_E) = A_{X,\Delta}(E)$, i.e.,

$$A_{X,\Delta}(\mathrm{ord}_E) := 1 + \mathrm{ord}_E(K_Y - \mu^*(K_X + \Delta)).$$

(2) Let $(Y, E = \sum_{k=1}^N E_k) \to X$ be a log smooth model of (X, Δ), i.e., Y is smooth and $E = \mathrm{Supp}(\mu_*^{-1}(\Delta) + \mathrm{Ex}(\mu))$ is simple normal crossing. Let η be the generic point of a connected component of $E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}$ of codimension r. Let $(y_1, \ldots, y_r)$ be a system of parameters of $\mathcal{O}_{Y,\eta}$ at η such that $E_{i_j} = (y_j = 0)$. Then for any $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r_{\geq 0} \setminus \{0\}$, we define $A_{X,\Delta}(v_\alpha)$ as

$$A_{X,\Delta}(v_\alpha) := \sum_{j=1}^r \alpha_j A_{X,\Delta}(\mathrm{ord}_{E_{i_j}}). \tag{2.3}$$

(3) In [JM12], it was shown that there exists a retraction map

$$\rho_Y : \mathrm{Val}_X \to \mathrm{QM}(Y, E)$$

for any log smooth model (Y, E) over X, such that it induces a homeomorphism $\mathrm{Val}_X \to \varprojlim_{(Y,E)} \mathrm{QM}(Y, E)$. For any real valuation $v \in \mathrm{Val}_X$, we define

$$A_{X,\Delta}(v) := \sup_{(Y,E)} A_{X,\Delta}(\rho(v)), \tag{2.4}$$

where (Y, E) ranges over all log smooth models over (X, Δ). For details, see [JM12] and [BdFFU15, Th. 3.1]. It is possible that $A_{X,\Delta}(v) = +\infty$ for some $v \in \mathrm{Val}_X$; see, e.g., [JM12, Rem. 5.12].

(4) For an lc pair (X, Δ) with an ideal $\mathfrak{a} \neq 0$ on X, for any $c \in \mathbb{Q}_{>0}$, we define

$$A_{X,\Delta + c\cdot\mathfrak{a}}(v) = A_{X,\Delta}(v) - c \cdot v(\mathfrak{a}).$$

In the above definition, if (X, Δ) is klt, then $A_{X,\Delta}$ is strictly positive on Val_X. For a klt singularity $x \in (X, \Delta)$, we denote by $\mathrm{Val}^{=1}_{X,x} \subset \mathrm{Val}_{X,x}$ the subspace consisting of all valuations with $A_{X,\Delta}(v) = 1$.

Definition 2.2 (Dual Complex). For a simple normal crossing pair (Y, E), we can form the dual complex $\mathcal{D}(E)$ (see [dFKX17, Def. 8]). So every E_i corresponds to a vertex v_i and any component Z of the intersection of r-components E_{i_j} $(j = 1, \ldots, r)$ corresponds to an $(r-1)$-cell

$$W_Z := \left\{ x = (a_1, \ldots, a_r) \in \mathbb{R}^r_{\geq 0} \,\middle|\, \sum_{i=1}^r a_i = 1 \right\}$$

glued on $v_{j_1}, \ldots, v_{j_r}$.

As a special case, for a log resolution $Y \to (X, \Delta)$ with the set of exceptional divisors $E = \sum_{i=1}^{r} E_i$ over x, if (X, Δ) is klt, then there is a natural embedding of

$$i_{X,\Delta}\colon \mathcal{D}(E) \to \mathrm{Val}^{=1}_{X,x}$$

by sending $v_i \to \frac{1}{A_{X,\Delta}(E_i)}\mathrm{ord}_{E_i}$ and the point on W_Z with coordinates

$$(a_1, \dots, a_r)\left(\sum_{i=1}^{r} a_i = 1\right)$$

to the quasi-monomial valuation $v_\alpha \in \mathrm{Val}_{X,x}$ defined in (2.1) where $\alpha = (\frac{a_1}{A_{X,\Delta}(E_1)}, \dots, \frac{a_r}{A_{X,\Delta}(E_r)})$.

Then the cone $i_{X,\Delta}\big(\mathcal{D}(E)\big) \times \mathbb{R}_{>0} \subset \mathrm{Val}_{X,x}$ consists of all valuations v such that $\lambda \cdot v \in i_{X,\Delta}(\mathcal{D}(E))$ for some rescaling $\lambda \in \mathbb{R}_{>0}$.

2.2. *Log canonical thresholds and Kollár components.*

2.2.1. *Log canonical thresholds.*

Definition 2.3. Given a non-zero ideal $\mathfrak{a}$ on a log canonical pair (X, Δ), we call c the *log canonical threshold* $c := \mathrm{lct}(X, \Delta; \mathfrak{a})$ if

$$c = \max_{t>0}\{\ t \mid (X, \Delta + t \cdot \mathfrak{a}) \text{ is log canonical; i.e., } A_{X,\Delta+t\cdot\mathfrak{a}}(v) \geq 0 \text{ for any } v\}.$$

We call any valuation v satisfying $A_{X,\Delta}(v) = c \cdot v(\mathfrak{a})$ *a valuation that computes the log canonical threshold of* (X, Δ) *with respect to* $\mathfrak{a}$.

We have the following well-known lemma.

Lemma 2.4. *Let* (X, Δ) *be a log canonical pair, and let* $c = \mathrm{lct}(X, \Delta; \mathfrak{a})$. *Let* $\mu\colon Y \to (X, \mathrm{Supp}(\Delta) \cup Z(\mathfrak{a}))$ *be a log resolution, where* $Z(\mathfrak{a})$ *is the subscheme defined by* $\mathfrak{a}$. *Let* $\mu^*(\mathfrak{a}) = \mathcal{O}_Y(-E)$. *Define* Δ_Y *by* $K_Y + \Delta_Y := \mu^*(K_X+\Delta) + cE$.

Then valuations v *that compute the log canonical threshold of* (X, Δ) *are precisely given by the points on the space*

$$i_{X,\Delta}\big(\mathcal{D}(\Gamma)\big) \times \mathbb{R}_{>0} \subset \mathrm{Val}_X,$$

where $\Gamma \subset \mathrm{Supp}(\Delta_Y)$ *consists of all the components in* Δ_Y *with coefficient* 1.

Proof. The case when v is a divisorial valuation follows from [KM98, Cor. 2.31].

When v is quasi-monomial, we can assume the model Y_v in Definition 2.1(2) is a log resolution of $(X, \mathrm{Supp}(\Delta) \cup Z(\mathfrak{a}))$. Let the center of v be a generic point of the intersection of $\bigcap_{j=1}^{r} E_j$, and assume $v = v_\alpha$ where $\alpha = (\alpha_1, \dots, \alpha_r)$ with

$\alpha_j > 0$ for all $1 \le j \le r$. By (2.4) and $v_\alpha(\mathfrak{a}) = \sum_{j=1}^r \alpha_j \mathrm{ord}_{E_j}(\mathfrak{a})$, we know that

$$A_{X,\Delta}(v) = \sum_{j=1}^r \alpha_j A_{X,\Delta}(E_j) \ge \sum_{j=1}^r c \cdot \alpha_j \mathrm{ord}_{E_j}(\mathfrak{a}) = c \cdot v(\mathfrak{a}),$$

and the equality holds if and only if $A_{X,\Delta}(E_j) = c \cdot \mathrm{ord}_{E_j}(\mathfrak{a})$ for all j.

Finally, for a general valuation v, we consider the quasi-monomial valuation $\rho_Y(v)$. Then we know that $v(\mathfrak{a}) = v(E) = \rho_Y(v)(E)$, and $A_{X,\Delta}(\rho_Y(v)) \le A_{X,\Delta}(v)$. Thus we know $A_{X,\Delta}(\rho_Y(v)) = A_{X,\Delta}(v)$, which implies $\rho_Y(v) = v$ (see [JM12, Cor. 5.4]). □

For a graded sequence $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k\in\mathbb{N}}$ of non-zero ideals on a klt pair, we can also define its log canonical threshold

$$\mathrm{lct}(X,\Delta;\mathfrak{a}_\bullet) := \limsup_k \mathrm{lct}\left(X,\Delta;\frac{1}{k}\mathfrak{a}_k\right) \in [0,+\infty].$$

It was shown in [JM12, Cor. 6.9] that

$$\mathrm{lct}(X,\Delta;\mathfrak{a}_\bullet) = \inf_{w\in \mathrm{Val}_X} \frac{A_{X,\Delta}(w)}{w(\mathfrak{a}_\bullet)}.$$

Moreover, by [JM12, Ths. A and 7.3], if $c := \mathrm{lct}(X,\Delta;\mathfrak{a}_\bullet) < +\infty$, then there always exists a valuation v satisfying $A_{X,\Delta}(v) = c\cdot v(\mathfrak{a}_\bullet)$, i.e.,

$$\frac{A_{X,\Delta}(v)}{v(\mathfrak{a}_\bullet)} = \inf_{w\in \mathrm{Val}_X} \frac{A_{X,\Delta}(w)}{w(\mathfrak{a}_\bullet)}.$$

However, if $\mathfrak{a}_\bullet$ is not finitely generated, we usually cannot expect that the log canonical threshold of $\mathfrak{a}_\bullet$ is computed by a divisorial valuation v (see, e.g., [JM12, Ex. 8.5]).

LEMMA 2.5 ([JM12, Prop. 7.10]). *Let (X,Δ) be a klt pair, and let $\mathfrak{a}_\bullet$ be a graded sequence of ideals with $\mathrm{lct}(X,\Delta;\mathfrak{a}_\bullet) < \infty$. Assume a valuation v computes its log canonical threshold with $v(\mathfrak{a}_\bullet) = 1$. Then if we let $\mathfrak{a}^v_\bullet = \{\mathfrak{a}^v_k\}_{k\in\mathbb{N}}$ be the graded sequence of valuation ideals associated to v, any valuation that computes the log canonical threshold of $\mathfrak{a}^v_\bullet$ must also compute the log canonical threshold of $\mathfrak{a}_\bullet$.*

Proof. Since $1 = v(\mathfrak{a}_\bullet) = \inf \frac{1}{k} v(\mathfrak{a}_k)$, we know that $\mathfrak{a}_k \subset \mathfrak{a}^v_k$. Thus, for any $w \in \mathrm{Val}_{X,x}$, since $w(\mathfrak{a}_\bullet) \ge w(\mathfrak{a}^v_\bullet)$, we have

$$A_{X,\Delta}(v) \le \frac{A_{X,\Delta}(w)}{w(\mathfrak{a}_\bullet)} \le \frac{A_{X,\Delta}(w)}{w(\mathfrak{a}^v_\bullet)}.$$

Since $v(\mathfrak{a}^v_\bullet) = 1$, this implies that v also computes the log canonical threshold of $\mathfrak{a}^v_\bullet$.

Moreover, for any v' that computes the log canonical threshold of $\mathfrak{a}^v_\bullet$, we may assume $v'(\mathfrak{a}^v_\bullet) = 1$, thus $A_{X,\Delta}(v) = A_{X,\Delta}(v')$. Since $v'(\mathfrak{a}^v_\bullet) \le v(\mathfrak{a}_\bullet)$, we conclude that $v'(\mathfrak{a}_\bullet) = 1$, and v' also computes the log canonical threshold of $\mathfrak{a}_\bullet$. □

2.2.2. *Kollár components.* The following special type of valuations will play a central role in our work.

Definition 2.6 (Kollár components). Let $x \in (X, \Delta)$ be a klt singularity. A prime divisor S over (X, Δ) is a *Kollár component* if there is a birational morphism $\mu\colon Y \to X$ of (X, Δ) such that μ is an isomorphism over $X \setminus \{x\}$, $\mu^{-1}(x) = S$ is $\mathbb{Q}$-Cartier, and if we write

$$(K_Y + \mu_*^{-1}\Delta + S)|_S = K_S + \Delta_S$$

(see [Kol13, Def. 4.2] for the meaning), then (S, Δ_S) is a log Fano pair.

By inversion of adjunction, this is equivalent to saying that $(Y, \mu_*^{-1}\Delta + S)$ is plt (purely log terminal) and $K_Y + \mu_*^{-1}\Delta + S \sim_{X,\mathbb{Q}} A_{X,\Delta}(S) \cdot S$ is anti-ample over X.

Such a morphism μ is called *a plt blow up* (see [Pro00]), as $(Y, \mu_*^{-1}\Delta + S)$ is a plt pair. Inspired by the construction in [Xu14] and the special test configuration construction in [LX14] in the global setting, Kollár components were systematically used to study various functions on the space of valuations in the local setting in [LX16].

Lemma 2.7. *Let $x \in (X, \Delta)$ be a klt singularity, and let $c = \mathrm{lct}(X, \Delta; \mathfrak{a})$ from some $\mathfrak{m}_x$-primary ideal $\mathfrak{a}$. Then there exists a Kollár component over $x \in (X, \Delta)$ that computes the log canonical thresholds of (X, Δ) with respect to $\mathfrak{a}$.*

Proof. See [LX16, Prop. 2.10]. □

For a graded sequence $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k \in \mathbb{N}}$ of ideals with a finite log canonical threshold, we have an approximation type result by Kollár components; see Proposition 3.1.

2.3. *Family of pairs.* Let $X \to B$ be a flat family with geometric integral fibers, and let D be a family of Weil divisors on X; i.e., $\mathrm{Supp}(D)$ does not contain any X_s. Then we call $(Y, E)/B \to (X, D)/B$ *a fiberwise log resolution* of $(X, D)/B$, where E is the sum of the birational transform of D and the exceptional divisor $\mathrm{Ex}(Y/X)$, if for each $s \in B$, $(Y_s, E_s) \to (X_s, D_s)$ is a log resolution and any strata of (Y, E), i.e., a component of the intersection $\cap E_i$ for components E_i of E, has geometric irreducible fibers over B.

Definition-Lemma 2.8. Let X be a variety over a finite type base B with geometrically integral fibers, and let $D \subset X$ be a codimension one subvariety that does not contain any fiber of $X \to B$. Then we can stratify B into a union of finitely many constructible subsets B'_α, such that over each B'_α, $(X,D)\times_B B'_\alpha$ admits a log resolution $\mu_\alpha\colon Y_\alpha \to (X,D)\times_B B'_\alpha$; i.e., $(Y_\alpha, E_\alpha := \mathrm{Ex}(\mu_\alpha)+\mu_{\alpha*}^{-1}D)$ is a simple normal crossing, and each stratum is log smooth over B'_α. In particular, for each $s\in B'_\alpha$, $(Y_s,E_s)\to(X_s,D_s)$ is a log resolution.

Moreover, we can replace B'_α by a finite étale cover B_α such that for each irreducible stratum Z of (Y_α,E_α), the fibers of $Z\to B_\alpha$ are also irreducible. So $(Y_\alpha,E_\alpha)/B_\alpha$ is *a fiberwise log resolution* of $(X,D)\times_B B_\alpha$.

Definition 2.9 ($\mathbb{Q}$-Gorenstein family of klt pairs). We call $(X,\Delta)\to B$ *a $\mathbb{Q}$-Gorenstein family of klt pairs* over a smooth base B if

(1) X is flat over B and $K_{X/B}+\Delta$ is $\mathbb{Q}$-Cartier;
(2) for any $s\in B$, X_s is normal and $\mathrm{Supp}(\Delta)$ does not contain X_s; and
(3) for any $s\in B$, the pair (X_s,Δ_s) is klt, where Δ_s is the cycle theoretic restriction over $s\in B$.

We call $B\subset (X,\Delta)\to B$ *a $\mathbb{Q}$-Gorenstein family of klt singularities* over a smooth base B, if $(X,\Delta)\to B$ is a family of klt pairs over B and there is a section $B\subset X$. We call $(X,\Delta)\to B$ *a $\mathbb{Q}$-Gorenstein family of log Fano pairs* over a smooth base B, if $(X,\Delta)\to B$ is a projective $\mathbb{Q}$-Gorenstein family of klt pairs over B, and the fiber (X_s,Δ_s) is log Fano for any $s\in B$.

Remark 2.10. Over a general (possibly singular) base, a correct definition of a family of klt pairs is subtle. See [Kol20] for a systematic study on this topic. For simplicity, in this note, we mostly only work over the smooth base. We could allow a more general base B. In fact, what is needed is that for any "*admissible*" morphism $B'\to B$, we have a compatible definition of the pullback of the family. Such a theory is worked out whenever B and B' are reduced in [Kol20, Ch. 4]. Using it, our results in this note can be extended to the case where B is reduced.

When $\Delta=0$, we can even work over non-reduced base as below.

Definition 2.11 (Locally stable family of klt varieties). We call $X\to B$ *a locally stable family of klt varieties* over a finite type base scheme B if

(1) X is flat over B and for any m, $\omega^{[m]}_{X/B}$ is flat over B and commutes with any base change $B'\to B$;
(2) for any point $s\in B$, X_s is klt.

See [Kol20, Ch. 3] for more background.

Lemma 2.12. *Let $(X,\Delta)\to B$ be a $\mathbb{Q}$-Gorenstein family of klt pairs over a smooth base B, and let $D\subset X\times_B U\to U$ be a family of effective Cartier divisors on X over a finite type variety $\pi\colon U\to B$. Fix a constant $c>0$.*

There is a constructible set $V \subset U$, such that if we denote by D_u the divisor corresponding to a point $u \to U$ and $(X_u, \Delta_u) = (X, \Delta) \times_B \{u\}$, then $\mathrm{lct}(X_u, \Delta_u; D_u) = c$ *if and only if u factors through V.*

Proof. See [Laz04, §9.5.D] or 2.13. □

2.13. If we apply Definition-Lemma 2.8 to the setting of Lemma 2.12, we can stratify V into a union of finitely many constructible subsets and take finite étale coverings to get finitely many varieties $\{V_\alpha\}$, such that $\bigsqcup_\alpha V_\alpha \to V$ is surjective and for any α

$$(X \times_B V_\alpha, \Delta \times_B V_\alpha + c \cdot D \times_U V_\alpha)$$

admits a fiberwise log resolution $(Y_\alpha, E_\alpha)/V_\alpha$. Let $(E_\alpha)_j$ $(j = 1, \dots, r)$ on Y_α be the log canonical places over $(X \times_B V_\alpha, \Delta \times_B V_\alpha + c \cdot D \times_U V_\alpha)$. Their reductions $(E_\alpha)_{u,j}$ over any point $u \in V_\alpha$ give precisely all divisors on Y_u that are log canonical places over $(X_s, \Delta_s + cD_u)$ where $s = \pi(u)$. So for any α, we can identify the dual complexes

$$\mathcal{D}\left(\Gamma_\alpha := \sum_{j=1}^{r} (E_\alpha)_j\right) \text{ and } \mathcal{D}\left(\Gamma_{\alpha,u} := \sum_{j=1}^{r} (E_\alpha)_{u,j}\right) \tag{2.5}$$

for any $u \in V_\alpha$.

2.4. *Boundedness of complements.* The concept of *complement* was an idea first introduced in [Sho92] to understand morphisms with a relative anti-ample canonical bundle. At the first sight, it seems to be technical. However, the boundedness of complement proved in [Bir19] is a major step forward to study birational geometry of Fano varieties. For this note, we need the following local result.

THEOREM 2.14 ([Bir19, Th. 1.8]). *Fix a positive integer n and a finite rational set $I \subset [0,1] \cap \mathbb{Q}$. Then there exists a positive integer $N_0 = N_0(n, I)$ depending only on n and I, such that for any klt singularity $x \in (X, \Delta)$ with* $\dim(X) = n$ *and the coefficients of Δ contained in I, if there is a Kollár component S given by the exceptional divisor of the plt blow up $\mu\colon Y \to (X, \Delta)$, then there is a divisor $\Delta^+ \geq \Delta$ that satisfies that (X, Δ^+) is log canonical, $N_0(K_X + \Delta^+) \sim 0$ and S is a log canonical place of (X, Δ^+).*

Proof. Denote by $\Delta_Y := \mu_*^{-1}\Delta$. By [Bir19, Th. 1.8], there is a constant $N_0 = N_0(n, I)$ that only depends on n and I, and a $\mathbb{Q}$-divisor $\Theta \geq 0$ such that $(Y, \Theta + \Delta_Y + S)$ is log canonical, and

$$N_0(K_Y + \Theta + \Delta_Y + S) \sim_X 0.$$

Push forward to X, and let $\Psi := \mu_*(\Theta)$ and $\Delta^+ := \Delta + \Psi$. Since

$$\mu^*(K_X + \Delta^+) = K_Y + \Delta_Y + S + \Theta,$$

we know that (X, Δ^+) is log canonical with S being a log canonical place. Moreover, $N_0(K_X + \Delta^+)$ is Cartier. □

2.5. *Local volumes.*

2.5.1. *Definitions.* For a valuation v centered on a klt singularity $x \in (X, \Delta)$, we give the definitions of two volume functions defined on $\mathrm{Val}_{X,x}$, namely, the volume $\mathrm{vol}_{X,x}(v)$ (or $\mathrm{vol}(v)$) as well as the normalized volume $\widehat{\mathrm{vol}}_{(X,\Delta),x}(v)$ (abbreviated as $\widehat{\mathrm{vol}}_{X,\Delta}(v)$ or simply $\widehat{\mathrm{vol}}(v)$ if there is no confusion).

Definition 2.15. Let X be an n-dimensional normal variety, and let $x \in X$ be a closed point. We define the *volume of a valuation* $v \in \mathrm{Val}_{X,x}$ following [ELS03] as

$$\mathrm{vol}_{X,x}(v) = \limsup_{k\to\infty} \frac{\ell(\mathcal{O}_{x,X}/\mathfrak{a}_k^v)}{k^n/n!},$$

where ℓ denotes the length of the artinian module.

Thanks to the works of [ELS03], [LM09], [Cut13], the above limsup is actually a limit.

The following invariant, which was defined first in [Li18], plays a key role for our study in the local stability.

Definition 2.16 ([Li18]). Let (X, Δ) be an n-dimensional klt log pair. Let $x \in X$ be a closed point. Then the *normalized volume function of valuations* $\widehat{\mathrm{vol}}_{(X,\Delta),x} : \mathrm{Val}_{X,x} \to (0, +\infty)$ is defined as

$$\widehat{\mathrm{vol}}_{(X,\Delta),x}(v) = \begin{cases} A_{X,\Delta}(v)^n \cdot \mathrm{vol}_{X,x}(v) & \text{if } A_{X,\Delta}(v) < +\infty, \\ +\infty & \text{if } A_{X,\Delta}(v) = +\infty. \end{cases}$$

The *volume of the klt singularity* $(x \in (X, \Delta))$ is defined as

$$\widehat{\mathrm{vol}}(x, X, \Delta) := \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}_{(X,\Delta),x}(v).$$

For a divisorial valuation ord_E, we will also use $\widehat{\mathrm{vol}}(E)$ for $\widehat{\mathrm{vol}}(\mathrm{ord}_E)$. It is known the minimum indeed exists by [Blu18] (see also Remark 3.8).

The minimizing problem for $\widehat{\mathrm{vol}}_{X,\Delta}$ is closely related to K-stability. The guiding question is called the *Stable Degeneration Conjecture*, which was formulated in [Li18, Conj. 7.1] and [LX18, Conj. 1.2]. See [LLX17] for more background. Theorem 1.2 settles one part of the conjecture.

We need the following result, which is a special case of the Stable Degeneration Conjecture.

Proposition 2.17. *Let (V,Δ_V) be a log Fano pair. Let r be a positive integer such that $H := -r(K_V+\Delta_V)$ is Cartier. Consider the cone $x\in(X,\Delta)=C(V,\Delta_V;H)$, with x being the vertex. Let $v^*\in \mathrm{Val}_{X,x}$ be the canonical divisorial valuation obtained by blowing up the vertex. Then v^* is a minimizer of $\widehat{\mathrm{vol}}_{(X,\Delta),x}$ if and only if (V,Δ_V) is K-semistable.*

Proof. This was proved in [LX16, Th. 4.5], after the works in [Li17] and [LL19]. □

2.5.2. *Invariance of local volumes.* The following theorem is a local version of [HMX13, Th. 4.2], and the proof is similar to the one there.

Theorem 2.18. *Let $B\subset(X,\Delta)\to B$ be a $\mathbb{Q}$-Gorenstein family of klt pairs over a smooth base B. Assume there is a fiberwise log resolutions $\mu\colon Y\to(X,\Delta)$ over B (see Definition-Lemma 2.8) with the exceptional divisor $E=\sum_{i=1}^k E_i$. If F is a prime toroidal divisor with respect to $(Y,\mathrm{Supp}(\mu_*^{-1}(\Delta))+E)$, with $A_{X,\Delta}(F)<1$, then the volume $\mathrm{vol}_{X_s,\Delta_s}(\mathrm{ord}_{F_s})$ is locally constant on $s\in B$.*

Proof. By restricting over a curve $C\to B$, we can assume B is a smooth curve. By taking a toroidal resolution, we can assume F is a divisor on Y. Write

$$\mu^*(K_X+\Delta)=K_Y+F_1-F_2,$$

where F_1 and F_2 are effective $\mathbb{Q}$-divisors without any common components. By our assumption, each stratum of $(Y,\mathrm{Supp}(F_1+F_2))$ is smooth over B and $F\subset\mathrm{Supp}(F_1)$. After possibly a further toroidal blow up, we may assume (Y,F_1) is terminal.

Fix s for a sufficiently small $\epsilon\in\mathbb{Q}_{>0}$ satisfying $\epsilon<\mathrm{mult}_F F_1$. The divisor $N_\sigma(Y_s/X_s;K_{Y_s}+(F_1)_s-\epsilon F_s)$ defined as in [Nak04, III.4] is a $\mathbb{Q}$-divisor, as $(Y_s,(F_1)_s-\epsilon F_s)$ has a relative good minimal model over X_s. In particular,

$$\Gamma_s := \Big((F_1)_s-\epsilon F_s\Big)-\Big(((F_1)_s-\epsilon F_s)\wedge N_\sigma\big(Y_s/X_s;K_{Y_s}+(F_1)_s-\epsilon F_s\big)\Big)$$

is also a $\mathbb{Q}$-divisor. Therefore, we can choose a divisor Γ supported on $\mathrm{Supp}(F_1)$ such that $\Gamma|_{Y_s}=\Gamma_s$.

We run a relative MMP program with scaling by [BCHM10] for $K_Y+\Gamma$ over X. Denote by $g^k\colon Y^k\dashrightarrow Y^{k+1}$ the k-th MMP step and by Γ^k the push-forward of Γ to Y^k. We will inductively prove that

(a) g^k is isomorphic at the generic point of every component of $\Gamma|_s$; and
(b) $g^k_s\colon Y^k_s\dashrightarrow Y^{k+1}_s$ is a birational contraction.

Assume this is true after $(k-1)$-steps. Then $(b)_{k-1}$ implies that no component of Γ^k_s is a component of the stable base locus of $K_{Y^k_s}+\Gamma^k_s$. Then if g^k is not an isomorphism at the generic point of a divisor D contained in Y^k_s, D

is covered by curves C such that

$$0 > C \cdot (K_{Y^k} + \Gamma^k) = C \cdot (K_{Y^k_s} + \Gamma^k_s).$$

It follows that D is a component of the stable base locus of $K_{Y^k_s} + \Gamma^k_s$, and thus D is not a component of Γ^k_s. This is $(a)_k$.

To see $(b)_k$, since the MMP is also a $(K_{Y^k_s} + \Gamma^k_s)$-negative morphism, if $g^k_s \colon Y^k_s \dashrightarrow Y^{k+1}_s$ is not a birational contraction, a component Θ in $\mathrm{Ex}((g^k_s)^{-1})$ will have non-positive discrepancy for $(Y^{k+1}_s, \Gamma^{k+1}_s)$, thus it has a negative discrepancy with respect to (Y^k_s, Γ^k_s). But as (Y^k_s, Γ^k_s) is terminal, and each step is $(K_{Y^k_s} + \Gamma^k_s)$-negative, we know Θ is component of $\mathrm{Supp}(\Gamma^k_s)$, which is a contradiction to $(a)_k$.

By [BCHM10], we obtain a relative minimal model $\phi \colon Y \dashrightarrow Z$ and $\psi \colon Z \to X$. For m sufficiently divisible,

$$\begin{aligned}
(\mu_s)_*\mathcal{O}_{Y_s}(-m\epsilon F_s) &= (\mu_s)_*\mathcal{O}_{Y_s}\big(-m(\epsilon F_s - (F_2)_s)\big)\\
&\cong (\mu_s)_*\mathcal{O}_{Y_s}\big(m(K_{Y_s} + (F_1)_s - \epsilon F_s)\big)\\
&= (\mu_s)_*\mathcal{O}_{Y_s}\big(m(K_{Y_s} + \Gamma_s)\big)\\
&= (\psi_s)_*\mathcal{O}_{Z_s}\big(m(K_{Z_s} + (\phi_s)_*\Gamma_s)\big)\\
&\cong \psi_*\mathcal{O}_Z\big(m(K_Z + \phi_*\Gamma)\big) \cdot \mathcal{O}_{X_s}\\
&\subset \mu_*\mathcal{O}_Y\big(m(K_Y + (F_1) - \epsilon F)\big) \cdot \mathcal{O}_{X_s}\\
&\cong \mu_*\mathcal{O}_Y\big(m(F_2 - \epsilon F)\big) \cdot \mathcal{O}_{X_s}\\
&= \mu_*\mathcal{O}_Y(-m\epsilon F) \cdot \mathcal{O}_{X_s},
\end{aligned}$$

where in the fourth line, we use (b) and that $(Y_s, \Gamma_s) \dashrightarrow (Z_s, (\phi_s)_*\Gamma_s)$ is $(K_{Y_s} + \Gamma_s)$-negative; in the fifth line, we use $\phi \colon Z \to X$ is a relative minimal model of $(Z, \phi_*\Gamma)$ and $\phi_*(\Gamma)|_{Z_s} = (\phi_s)_*(\Gamma_s)$ by (a) and (b).

So $\mu_*\mathcal{O}_Y(-mF) \cdot \mathcal{O}_{X_s} \cong (\mu_s)_*\mathcal{O}_{Y_s}(-mF_s)$ for any s and sufficiently divisible m, which implies

$$\frac{\mathcal{O}_X}{\mu_*\mathcal{O}_Y(-mF)} \cdot \mathcal{O}_{X_s} \cong \frac{\mathcal{O}_{X_s}}{(\mu_s)_*\mathcal{O}_{Y_s}(-mF_s)}.$$

Thus we conclude that the finite rank $\mathcal{O}_B$-module $\mathcal{O}_X/\mu_*\mathcal{O}_Y(-mF)$ is locally free, whose restriction over $\mathrm{Spec}\ k(s)$ is isomorphic to $\mathcal{O}_{X_s}/(\mu_s)_*\mathcal{O}_{Y_s}(-mF_s)$. As a result, we know that

$$\begin{aligned}
\mathrm{vol}(\mathrm{ord}_{F_\eta}) &= \lim_{m\to\infty} \frac{\mathrm{rk}\big(\mathcal{O}_X/\mu_*\mathcal{O}_Y(-mF)\big)}{m^n/n!}\\
&= \lim_{m\to\infty} \frac{\dim\big(\mathcal{O}_{X_s}/\mu_*\mathcal{O}_{Y_s}(-mF_s)\big)}{m^n/n!} = \mathrm{vol}(\mathrm{ord}_{F_s})
\end{aligned}$$

is a constant. ☐

The following corollary may be of independent interest. We will not need it in the rest of the paper.

COROLLARY 2.19. *We use the notation of Theorem 2.18. Let $\chi\colon Z^c \to X$ be the birational model that precisely extracts the birational transform of F, denoted by F', such that $-F'$ is ample over X. Then it satisfies that restricting over each $s \in B$, $\chi_s\colon Z^c_s \to X_s$ precisely extracts F'_s, which is the birational transform of F_s.*

Proof. In the proof of Theorem 2.18, we have shown under the assumption there that

$$\mu_*\mathcal{O}_Y(-mF)\cdot k_s \cong (\mu_s)_*\mathcal{O}_{Y_s}(-mF_s).$$

For any sufficiently divisible m, $\chi\colon Z^c \to X$ (resp. $\chi_s\colon Z^c_s \to X_s$) is indeed the blow up of the ideal sheaf $\mu_*\mathcal{O}_Y(-mF) \subset \mathcal{O}_X$ (resp. $(\mu_s)_*\mathcal{O}_{Y_s}(-mF_s) \subset \mathcal{O}_{X_s}$). Thus by the above isomorphism, Z^c_s is the birational transform of X_s under the morphism χ. Since $\mu_*\mathcal{O}_Y(-mF)$ is flat over B, we know Z^c is flat over B, and therefore $Z^c \times_B \{s\}$ coincides with Z^c_s. □

3. Quasi-monomial limit

3.1. *Approximation.* On a klt singularity $x \in (X,\Delta)$, for a graded sequence $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k\in\mathbb{N}}$ of $\mathfrak{m}_x$-primary ideals, unlike Lemma 2.7, usually we cannot find a divisorial valuation computing its log canonical threshold. However, we have the following result, whose proof slightly simplifies the one in [LX16, Th. 1.3].

PROPOSITION 3.1. *Let $x \in (X,\Delta)$ be a klt singularity. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k\in\mathbb{N}}$ be a graded sequence of $\mathfrak{m}_x$-primary ideals with $\mathrm{lct}(X,\Delta,\mathfrak{a}_\bullet) < +\infty$. Then we can find a valuation $v \in \mathrm{Val}^{=1}_{X,x}$ that is the limit of $\frac{1}{A_{X,\Delta}(S_j)}\cdot \mathrm{ord}_{S_j}$ for a sequence of Kollár components $\{S_j\}$, such that v calculates the log canonical threshold of $\mathfrak{a}_\bullet$.*

Later in Theorem 3.3, we will show such v is always quasi-monomial.

Proof. Let $c = \mathrm{lct}(X,\Delta,\mathfrak{a}_\bullet)$, and let $w \in \mathrm{Val}^{=1}_X$ be a valuation that calculates the log canonical threshold of $\mathfrak{a}_\bullet$; then $w(\mathfrak{a}_\bullet) = \frac{1}{c}$. Let $\mathfrak{a}_k$ ($k \in \mathbb{N}$) be the k-th element in the graded sequence of ideals. Let $c_k := \mathrm{lct}(X,\Delta;\frac{1}{k}\mathfrak{a}_k)$. In particular, $\lim_k c_k = c$.

By Lemma 2.7, there exists a Kollár component S_k with $c_k\cdot \mathrm{ord}_{S_k}(\mathfrak{a}_k) = k\cdot A_{X,\Delta}(S_k)$. We consider the valuation

$$v_k := \frac{c_k}{c\cdot A_{X,\Delta}(S_k)}\mathrm{ord}_{S_k} = \frac{k}{c\cdot \mathrm{ord}_{S_k}(\mathfrak{a}_k)}\mathrm{ord}_{S_k}.$$

Note that $A_{X,\Delta}(v_k) = \frac{c_k}{c} \le 1$.

Assume $\mathfrak{m}_x^p \subset \mathfrak{a}_1$ for some $p \gg 0$, as $\mathfrak{a}_1$ is $\mathfrak{m}_x$-primary. Then $\mathfrak{m}_x^{pk} \subset \mathfrak{a}_1^k \subset \mathfrak{a}_k$. Thus for any k,

$$v_k(\mathfrak{m}_x) \geq v_k(\mathfrak{a}_k) \cdot \frac{1}{pk} = \frac{1}{cp},$$

which is bounded from below. In particular, by the compactness result [JM12, Prop. 5.9] and [LX16, Prop. 3.9], we know that there is an infinite sequence $\{v_j\}$ that has a limit in $\mathrm{Val}_{X,x}$, denoted by $v = \lim_{j\to\infty} v_j$.

We have

$$A_{X,\Delta}(v) \leq \liminf_{j\to\infty} A_{X,\Delta}(v_j) \leq 1,$$

as $A_{X,\Delta}$ is lower semicontinuous (see [JM12, Lemma 5.7]). By definition, $v_k(\mathfrak{a}_k) = \frac{k}{c}$ for any k and $\mathfrak{a}_k^m \subset \mathfrak{a}_{mk}$ for any $m \in \mathbb{N}$. This implies

$$v_{mk}(\mathfrak{a}_k) \geq \frac{v_{mk}(\mathfrak{a}_{mk})}{m} = \frac{k}{c}.$$

Thus

$$v(\mathfrak{a}_\bullet) = \lim_{k\to\infty} \frac{v(\mathfrak{a}_k)}{k} = \lim_{k\to\infty} \left(\lim_{m\to\infty} \frac{v_{mk}(\mathfrak{a}_k)}{k} \right) \geq \frac{1}{c}.$$

Since

$$\frac{A_{X,\Delta}(v)}{v(\mathfrak{a}_\bullet)} \leq c = \inf_{w'} \frac{A_{X,\Delta}(w')}{w'(\mathfrak{a}_\bullet)},$$

this implies that $A_{X,\Delta}(v) = 1$ and $v(\mathfrak{a}_\bullet) = \frac{1}{c}$. Since $\lim_j c_j = c$, we have

$$v = \lim_j v_j = \lim_j \frac{c}{c_j} v_j = \lim_j \frac{1}{A_{X,\Delta}(S_j)} \mathrm{ord}_{S_j}. \qquad \square$$

We can slightly improve the result by showing the following.

LEMMA 3.2. *In the notation of Proposition 3.1. Assume $w \in \mathrm{Val}^{=1}_{X,x}$, which calculates the log canonical threshold of $\mathfrak{a}_\bullet$. Then we can choose v as in Proposition 3.1 such that $v \geq w$.*

Proof. Let $c = \mathrm{lct}(X, \Delta, \mathfrak{a}_\bullet)$ and then $w(\mathfrak{a}_\bullet) = \frac{1}{c}$. Let $w' = c \cdot w$. By Lemma 2.5, we can assume that $\mathfrak{a}_\bullet = \mathfrak{a}_\bullet^{w'}$. Then we can apply the construction in the proof of Proposition 3.1 to get v. It remains to show $v \geq w = \frac{1}{c} w'$. To verify it, we pick any $f \in R$ and write $w'(f) = p$ for some $p \in \mathbb{R}_{>0}$. For a fixed j, choose l such that

$$(l-1)p < j \leq lp.$$

Let $k = j$ in the previous construction. Then we have

$$\begin{aligned} w'(f) = p &\Longrightarrow w'(f^l) = pl, \\ &\Longrightarrow f^l \in \mathfrak{a}_{pl}, \\ &\Longrightarrow f^l \in \mathfrak{a}_j, \\ &\Longrightarrow v_j(f) \geq \frac{j}{cl} > \frac{p}{c} - \frac{p}{cl}. \end{aligned}$$

The fourth arrow is because $v_j(\mathfrak{a}_j) = \frac{j}{c}$. Thus

$$v(f) = \lim_j v_j(f) \geq \frac{p}{c} = w(f). \qquad \square$$

3.2. *Quasi-monomial limit.* Let $x \in (X = \mathrm{Spec}(R), \Delta)$ be a klt singularity. The main aim of this section is to prove the following theorem.

THEOREM 3.3. *Let $x \in (X, \Delta)$ be a klt singularity. Let $v_i = \frac{1}{A_{X,\Delta}(S_i)}(\mathrm{ord}_{S_i})$ be an infinite sequence of valuations, where S_i are Kollár components over $x \in (X, \Delta)$, and assume there is a uniform C and $\delta > 0$ such that either $\widehat{\mathrm{vol}}(v_i) < C$ or $v_i(\mathfrak{m}_x) > \delta$. Then there is an infinite subsequence that has a quasi-monomial limit $v \in \mathrm{Val}^{=1}_{X,x}$.*

LEMMA 3.4. *Fix constants C and $\epsilon > 0$. If there is a sequence of valuations $\{v_i\}_{i\in\mathbb{N}}$, such that either*

$$\big(\lim_{i\to\infty} v_i \to v \in \mathrm{Val}_{X,x}\big) \quad \textit{or} \quad \big(\widehat{\mathrm{vol}}(v_i) < C \textit{ and } A_{X,\Delta}(v_i) > \epsilon \textit{ for all } i\big),$$

then there is a positive $\delta > 0$, such that for any i, $v_i(\mathfrak{m}_x) \geq \delta$.

Proof. Let us first assume $\lim_{i\to\infty} v_i \to v \in \mathrm{Val}_{X,x}$. Let $f_1, \ldots, f_k$ be a set of generators of $\mathfrak{m}_x$. Then for any $1 \leq j \leq k$, $\lim_{i\to\infty} v_i(f_j) \to v(f_j) > 0$. In particular, we know that for all $1 \leq j \leq k$ and all i, $v_i(f_j)$ has a positive lower bound, which we denote by δ. Then $v_i(\mathfrak{m}_x) \geq \delta$ for any i.

If we assume $\widehat{\mathrm{vol}}(v_i) < C$ and $A_{X,\Delta}(v_i) > \epsilon$, then this follows from [Li18, Th. 1.1]. $\square$

PROPOSITION 3.5. *Let (X, Δ) be a klt pair. Let $\{S_i\}_{i\in\mathbb{N}}$ be a sequence of Kollár components such that*

$$\lim_{i\to\infty} \frac{1}{A_{X,\Delta}(S_i)} \mathrm{ord}_{S_i} = v.$$

Then there exist a constant N and a family of Cartier divisors $D \subset X \times V$ parametrized by a variety V of finite type, such that for any $u \in V$, $(X, \Delta + \frac{1}{N} D_u)$ is lc but not klt; and for any i, S_i computes the log canonical threshold of a pair $(X, \Delta + \frac{1}{N} D_{u_i})$ for some $u_i \in V$.

For a stronger statement, which will be needed later, see Proposition 4.2.

Proof. Let $v_i := \frac{1}{A_{X,\Delta}(S_i)} \cdot \mathrm{ord}_{S_i}$. Let $\mu_i \colon Y_i \to X$ be the plt blow up that extracts S_i.

By Theorem 2.14, we know that there is a uniform N_0 such that for each i, we can find an effective $\mathbb{Q}$-divisor Ψ_i with the property that $(X, \Delta + \Psi_i)$ is log canonical with S_i being a log canonical place. Define $\Delta^+_{S_i}$ by

$$K_{S_i} + \Delta^+_{S_i} = \mu_i^*(K_X + \Delta + \Psi_i)|_{S_i};$$

then $(S_i, \Delta^+_{S_i})$ is log canonical. Moreover, $N_0(K_X+\Delta+\Psi_i)$ is Cartier. Set $N = rN_0$ where r is a positive integer such that $r(K_X+\Delta)$ is Cartier, then both $N(K_X+\Delta)$ and $N\Psi_i$ are Cartier for all i. Thus we can assume $N\Psi_i$ is given by $\mathrm{div}(\psi_i)$ for some regular function ψ_i.

Fix $M \in \mathbb{N}$ such that $\delta \cdot M > N$ where δ is the positive constant obtained in Lemma 3.4 for the sequence $\{v_i\}$. Then let $g_1, \dots, g_m$ be m-elements in R, such that their reductions

$$[g_1], \dots, [g_m] \in \mathcal{O}_{x,X}/\mathfrak{m}_x^M$$

yield a k-basis. So for any i, there exists a k-linear combination of h_i of $g_1, \dots, g_m$ such that the image of ψ_i and h_i are the same in $\mathcal{O}_{x,X}/\mathfrak{m}_x^M$.

Claim 3.6. Let $\Phi_i := \mathrm{div}(h_i)$. Then $(X, \Delta + \frac{1}{N}\Phi_i)$ is log canonical and has S_i as its log canonical place.

Proof. Since $s_i = h_i - \psi_i \in \mathfrak{m}_x^M$, we have

$$v_i(s_i) \geq M \cdot v_i(\mathfrak{m}_x) > N.$$

On the other hand, since v_i computes the log canonical threshold of $(X, \Delta+\Psi_i)$, we know

$$N = N \cdot A_{X,\Delta}(v_i) = v_i(\psi_i) = v_i(h_i),$$

hence $A_{X,\Delta+\Psi_i}(v_i) = 0$. This implies that

$$\begin{aligned}\big(K_{Y_i} + S_i + \mu_{i*}^{-1}(\frac{1}{N}\Phi_i + \Delta_i)\big)|_{S_i} &= \mu_i^*(K_X + \Delta + \frac{1}{N}\Phi_i)|_{S_i} \\ &= \mu_i^*(K_X+\Delta+\Psi_i)|_{S_i} \\ &= K_{S_i} + \Delta^+_{S_i};\end{aligned}$$

see [Kol08, 35] for the second equality. Since $(S_i, \Delta^+_{S_i})$ is log canonical, by inversion of adjunction (see [Kaw07]), we know that $(Y_i, S_i + \mu_{i*}^{-1}(\Delta + \frac{1}{N}\Phi_i))$ is log canonical along S_i, which implies that $(X, \Delta + \frac{1}{N}\Phi_i)$ is log canonical, and S_i computes its log canonical threshold. □

Then applying Lemma 2.12 to the family of Cartier divisors $D_U \subset X \times U$, where

$$U = \{(x_1, \dots, x_m) \in \mathbb{A}^m_k \mid (x_1, \dots, x_m) \neq (0, \dots, 0)\}$$

and $D_U = (\sum_{i=1}^m x_i g_i = 0)$, we can find a bounded family of divisors $D \subset X \times V \to V$ for some $V \subset U$, such that $(X, \Delta + \frac{1}{N}D_u)$ is log canonical but not klt if and only if $u \in V$. From our argument, we know $D \subset X \times V$ is the desired family of Cartier divisors. □

The proof of of Claim 3.6 says that the log canonical thresholds of two functions are the same if they are sufficiently close in $\mathfrak{m}_x$-adic topology. As far as we know, this kind of argument first appeared in [Kol08] and [dFEM10].

Proof of Theorem 3.3. By Lemma 3.4, we can assume $v_i(\mathfrak{m}_x) > \delta$. Since $A_{X,\Delta}(v_i) = 1$, it follows from [LX16, Prop 3.9] that there is an infinite subsequence, which we still denote by v_i such that $v := \lim_i v_i$ exists. It remains to prove v is quasi-monomial.

Applying Proposition 3.5, we get a bounded family of Cartier divisors $(D \subset X \times V) \to V$ such that for any u, $(X, \Delta + \frac{1}{N} D_u)$ is log canonical but not klt, and any S_i is the lc place of $(X, \Delta + \frac{1}{N} D_{u_i})$ for some $u_i \in V$. Replacing V by an irreducible closed subset, we can further assume the set $\{u_i\}$ forms a dense set of points on V. We may further resolve V to be smooth.

Applying (2.13) to $(X \times V, \Delta \times V + \frac{1}{N} D)$ over V, after shrinking V to an open set and replacing it by a finite étale covering, we can assume $(X \times V, \Delta \times V + \frac{1}{N} D) \to V$ admits a fiberwise log resolution $\mu_V \colon Y \to (X \times V, \Delta \times V + \frac{1}{N} D)$ over V with the exceptional divisor $E = \sum_{i=1}^k E_i$ being simple normal crossing (see Definition-Lemma 2.8). We choose $\Gamma \subset E$ to be the subdivisor given by the components with log discrepancy 0 with respect to $(X \times_B V, \Delta \times_B V + \frac{1}{N} D)$. We also denote by $K := K(V)$ the function field of V and by η the point $x \times \mathrm{Spec}(K) \in X \times V$.

For any point u_i, since S_i computes the log canonical threshold of $(X, \Delta + \frac{1}{N} D_{u_i} (= \Phi_i))$, using the identification in (2.5), $S_i \in i_{X_s, \Delta_s}(\mathcal{D}(\Gamma|_{Y_{u_i}})) \times \mathbb{R}_{>0}$ can be regarded as a restriction of a divisor corresponding to a point on $i_{X,\Delta}(\mathcal{D}(\Gamma)) \times \mathbb{R}_{>0}$, whose restriction over the generic point $\mathrm{Spec}(K)$ of V will yield a divisor, denoted by T_i. Thus, the valuations $\frac{1}{A_{X_K, \Delta_K}(T_i)} \mathrm{ord}_{T_i}$ are contained in the image of the fixed dual complex $i_{X_K, \Delta_K}(\mathcal{D}(\Gamma|_{Y_K}))$ for all i.

Since $\mathcal{D}(\Gamma)$ is compact, after passing to an infinite subsequence of i, the valuations $\frac{1}{A_{X_K, \Delta_K}(T_i)} \mathrm{ord}_{T_i}$ converge to a quasi-monomial valuation w over X_K.

We claim that the restriction of w to $K(X) \subset K(X_K)$ is v. In fact, for any $f \in R$, we denote by f_K its image under the injection $K(X) \subset K(X_K)$. Then Lemma 3.7 implies that after passing to an infinite subsequence,

$$w(f_K) = \lim_{i \to \infty} \frac{1}{A_{X_K, \Delta_K}(T_i)} \mathrm{ord}_{T_i}(f_K) = \lim_{i \to \infty} \frac{1}{A_{X,\Delta}(S_i)} \mathrm{ord}_{S_i}(f) = v(f).$$

By the Zariski-Abhyankar inequality (see (2.2)),

$$\mathrm{rat.rk}(w) + \mathrm{tr.deg}(w) \le \mathrm{rat.rk}(v) + \mathrm{tr.deg}(v) + \mathrm{tr.deg}(K(X_K)/K(X)).$$

Since w is Abhyankar, the left-hand side is equal to $\dim(X) + \dim(V)$. Therefore,

$$\mathrm{rat.rk}(v) + \mathrm{tr.deg}(v) = \dim(X),$$

and thus v is an Abhyankar valuation on $K(X)$, which is the same as saying that it is a quasi-monomial valuation. ☐

LEMMA 3.7. *With the notation as above, for any* $f \in R$, $\mathrm{ord}_{T_i}(f_K) \le \mathrm{ord}_{S_i}(f)$, *and the equality holds for infinitely many* i.

Proof. The first inequality is straightforward. To see the equality, we can take a log resolution W of $\big(Y, E+\mu_{V*}^{-1}(p_1^*(\mathrm{div}(f)+\Delta))\big)$ where $p_1\colon X\times V\to X$. There is an open set $V^\circ\subset V$, such that

$$W\times_V V^\circ\to\big(Y,E+\mu_{V*}^{-1}(p_1^*(\mathrm{div}(f)+\Delta))\big)\times_V V^\circ\to V^\circ$$

could yield a fiberwise log resolution after a finite étale base change. Now it follows from [JM12, Proof of Lemma 4.6] that for any $u_i\in V^\circ$, we have $\mathrm{ord}_{T_i}(f_K)=\mathrm{ord}_{S_i}(f)$. □

Proof of Theorem 1.1. Let w compute the log canonical threshold of $\mathfrak{a}_\bullet$ on (X,Δ) (see [JM12, Th. 7.3]) with $c_X(w)=\eta$. We can replace $\mathfrak{a}_\bullet$ by $\mathfrak{a}_\bullet^w$ (see Lemma 2.5) and localize at η. Thus we reduce to the case that $\mathfrak{a}_\bullet$ is a graded sequence of $\mathfrak{m}_\eta$-primary ideals, where $\mathfrak{m}_\eta$ is the maximal ideal on a local ring of an essentially finite type.

Then we can apply Proposition 3.1, which says that there exists a valuation v that can be written as the limit of a sequence of valuations with the form $c_j\cdot\mathrm{ord}_{S_j}$ for Kollár components $\{S_j\}$ such that $A_{X,\Delta}(v)=A_{X,\Delta}(w)$, $v\geq w$, and v also calculates the log canonical thresholds of $\mathfrak{a}_\bullet$. By Lemma 3.4 , there exists $\delta>0$ such that $c_j\cdot\mathrm{ord}_{S_j}(\mathfrak{m}_x)>\delta$. Thus by Theorem 3.3, v has to be quasi-monomial. □

Proof of Theorem 1.2. By Theorem 1.1, we know that for a minimizer w, there exists a quasi-monomial valuation v that computes the log canonical threshold of $\mathfrak{a}_\bullet^w$. Since w is a minimizer of $\widehat{\mathrm{vol}}_{(X,\Delta),x}$, we conclude $v=\lambda w$ by [Blu18, Lemma 4.7] for some $\lambda>0$. □

Remark 3.8. In [Blu18], to show the existence of the minimizer, there is a technical assumption that the ground field k has to be uncountable. Our approach can indeed remove this assumption.

More precisely, we can always find a sequence of Kollár component S_i such that $\lim_i\widehat{\mathrm{vol}}(S_i)=\inf\widehat{\mathrm{vol}}(v)$ (see [LX16, Lemma3.8]). Therefore, after passing to a further infinite subsequence, as in the proof of Theorem 3.3, we can assume there is a family $(X\times V,\Delta\times V+\frac{1}{N}D)\to V$ that admits a fiberwise log resolution, and S_i is an lc place of $(X,\Delta+\frac{1}{N}D_{u_i})$ for some $u_i\in V$. Fix a closed point $u\in V$; then as before, S_i yields a divisor T_i that is an lc place of $(X,\Delta+\frac{1}{N}D_u)$ and yields the same point as S_i under the correspondence (2.5). By Theorem 2.18, $\widehat{\mathrm{vol}}(\mathrm{ord}_{S_i})=\widehat{\mathrm{vol}}(\mathrm{ord}_{T_i})$. Thus $w_i:=\frac{1}{A_{X,\Delta}(T_i)}\mathrm{ord}_{T_i}$ has a limit w. Since $\mathrm{vol}(\cdot)$ is continuous on a dual complex by [BFJ14, Cor. D], which implies that $\widehat{\mathrm{vol}}(\cdot)$ is also continuous, we have

$$\widehat{\mathrm{vol}}(w)=\lim_i\widehat{\mathrm{vol}}(\mathrm{ord}_{T_i})=\lim_i\widehat{\mathrm{vol}}(\mathrm{ord}_{S_i})=\inf\widehat{\mathrm{vol}}(v);$$

i.e., w is a minimizer of $\widehat{\mathrm{vol}}_{(X,\Delta),x}$.

4. **Family version**

In this section, we will use the techniques developed in the previous section to study a $\mathbb{Q}$-Gorenstein family of klt singularities, and we will prove the normalized volume function is a constructible function. As a consequence, we obtain the openness of the K-semistable locus among a $\mathbb{Q}$-Gorenstein family of log Fano pairs.

Let $(X,\Delta)\to B$ be family of klt singularities over a smooth base B with a section $\sigma\colon B\to X$.

LEMMA 4.1. *There is a uniform positive constant $\delta>0$ depending only on $B\subset(X,\Delta)$, such that for any geometric point $s\to B$, and a valuation $v_s\in \mathrm{Val}_{X_s,x_s}$ with $A_{X_s,\Delta_s}(v_s)<+\infty$, then*

$$\widehat{\mathrm{vol}}(v_s)\cdot v_s(\mathfrak{m}_s)>\delta\cdot A_{X_s,\Delta_s}(v_s).$$

Proof. This is a family version of the proper estimate in [Li18, Th. 1.1]. See [BL18a, Th. 21] for a proof. □

The following statement is a generalization of Proposition 3.5.

PROPOSITION 4.2. *Let $B\subset(X,\Delta)\to B$ be a $\mathbb{Q}$-Gorenstein family of klt singularities over a smooth base B. Fix a positive number C. There is a family of Cartier divisors $D\subset X\times_B V$ over a finite type variety $\pi\colon V\to B$ and a positive number N, such that for a geometric point $s\to B$ and a Kollár components S_s over (X_s,Δ_s) with $\widehat{\mathrm{vol}}(\mathrm{ord}_{S_s})\le C$, then there exists a point $u\in V\times_B\{s\}$ such that if we base change (X_s,Δ_s) and S_s to u, S_u is a log canonical place of the log canonical pair $(X_u,\Delta_u+\frac{1}{N}D_u)$, where $D_u:=D\times_V\{u\}$.*

Proof. Let δ be the constant as in Lemma 4.1, and let $\delta_0=\frac{\delta}{C}>0$. Then it follows from our assumption $\widehat{\mathrm{vol}}(\mathrm{ord}_{S_s})\le C$ that

$$\frac{1}{A_{X_s,\Delta_s}(S_s)}\mathrm{ord}_{S_s}(\mathfrak{m}_s)\ge\delta_0.$$

Fix M such that $M\delta_0>N:=rN_0$, where N_0 is the constant given by Lemma 2.14 that only depends on the dimension of X_s and the coefficients of Δ and r is a positive integer such that $r(K_X+\Delta)$ is Cartier. By shrinking B, we can assume $B=\mathrm{Spec}(T)$, $\mathcal{O}_X/(\mathfrak{m}_{\sigma(B)})^M$ is a free T-module with a basis $[g_1],\dots,[g_m]$ for $g_i\in\mathcal{O}_{B,X}$.

Let $E\to U:=(\mathbb{A}^m\setminus\{0\})_T$ be the space such that over point $t=(t_1,\dots,t_m)\in U$, the fiber E_t parametrizes the divisor of $(\sum_{j=1}^m t_jg_j=0)$. For the family

$$(X\times_B U,\Delta\times_B U+E)\to U\xrightarrow{\pi}B,$$

by Lemma 2.12, there is a constructible set $V \subset U$ such that for a point, $u \to U$ factors through V if and only if

$$\mathrm{lct}(X_u, \Delta_u; E_u) = \frac{1}{N}.$$

Let $D := E \times_U V$. If a Kollár component S_s over (X_s, Δ_s) for a geometric point $s \to B$ satisfies

$$\widehat{\mathrm{vol}}_{X_s,\Delta_s}(\mathrm{ord}_{S_s}) < C,$$

then the same argument for Proposition 3.4 shows that S_s is a log canonical place of the pair $(X_s, \Delta_s + \frac{1}{N}D')$ for some $D' = \mathrm{div}(g)$ with $g = \sum_{i=1}^m \lambda_i (g_i)_s$ for some $\lambda_i \in k(s)$, where $(g_i)_s$ is the reduction of g_i under the morphism $\mathcal{O}_{B,X} \to \mathcal{O}_{s,X_s}$. Thus $D' = D_u$ for $u = (\lambda_1, \dots, \lambda_m)$ over $s \to B$. □

Proof of Theorem 1.3. Let $C = n^n + 1$. Then we know that for any geometric point $s \in B$, $\widehat{\mathrm{vol}}(s, X_s, \Delta_s) < C$ by [LX19, Th. 1.6]. Apply Proposition 4.2 to such C, and let $(D \subset X \times_B V) \to V$ be the family of divisors given by it. Then after stratifying the base V into a disjoint union of finitely many constructible subsets and taking finite étale coverings, we can assume there exists a decomposition $V = \bigsqcup_\alpha V_\alpha$ into irreducible smooth strata V_α such that for each α, $\big(X \times_B V_\alpha, \mathrm{Supp}(\Delta \times_B V_\alpha + \frac{1}{N}D)\big)$ admits a fiberwise log resolution $\mu\colon Y_\alpha \to X \times_B V_\alpha$ over V_α with a simple normal crossing exceptional divisor $E_\alpha = \sum_{i=1}^k E_i$ (see Definition-Lemma 2.8). We choose $\Gamma_\alpha \subset E_\alpha$ to be the subdivisor given by the components with log discrepancy 0 with respect to $(X \times_B V, \Delta \times_B V + \frac{1}{N}D)$. Moreover, by the Noether induction, we can shrink B and assume each $V_\alpha \to B$ is surjective.

Then for any geometric point $s \to B$ and a Kollár component S_i over (X_s, Δ_s) with $\widehat{\mathrm{vol}}(\mathrm{ord}_{S_i}) \le C$, Proposition 4.2 implies that there is a point $u_i \in V_\alpha \times_B \{s\}$ such that $(X_{u_i}, \Delta_{u_i} + \frac{1}{N}D_{u_i})$ is log canonical and S_{u_i} is a log canonical place of the pair, where (X_{u_i}, Δ_{u_i}) and S_{u_i} are the base changes of (X_s, Δ_s) and S_i over u_i. Define $(Y_{u_i}, E_{u_i}) := (Y_\alpha, E_\alpha) \times_{V_\alpha} \{u_i\}$. Since $\mu_{u_i}\colon Y_{u_i} \to (X_{u_i}, \Delta_{u_i} + \frac{1}{N}D_{u_i})$ is a log resolution, we know that S_i will be a toroidal divisor over (Y_{u_i}, E_{u_i}), which then yields a toroidal divisor T_i over (Y_α, E_α) whose corresponding valuation is contained in $i_{X,\Delta}(\mathcal{D}(\Gamma_\alpha)) \times \mathbb{R}_{>0}$, such that S_i is given by the restriction of T_i over u_i.

For any toroidal divisor T with $\mathrm{ord}_T \in i_{X,\Delta}(\mathcal{D}(\Gamma_\alpha)) \times \mathbb{R}_{>0}$, since

$$\big(X \times_B V_\alpha, \Delta \times_B V_\alpha + (\frac{1}{N} - \epsilon)D\big)$$

is klt for any $\frac{1}{N} \ge \epsilon > 0$ and since we can choose ϵ sufficiently small such that

$$A_{X \times_B V_\alpha, \Delta \times_B V_\alpha + (\frac{1}{N}-\epsilon)D}(T) < 1,$$

then by Theorem 2.18 we can conclude that for any $u \in V_\alpha$, the function

$$u \in V_\alpha \to \widehat{\mathrm{vol}}_{X_u,\Delta_u}(T_u)$$

is a constant function on V_α. Thus, for each fixed α, the function

$$v_\alpha\colon u\in V_\alpha \to \inf_{T_u}\{\widehat{\mathrm{vol}}_{X_u,\Delta_u}(\mathrm{ord}_{T_u}) \mid \mathrm{ord}_{T_u}\in i_{X_u,\Delta_u}(\mathcal{D}(\Gamma|_{Y_u}))\times\mathbb{R}_{>0}\}$$

is a constant function.

For any geometric point $s\colon \mathrm{Spec}\ k\to B$, since the point can be lifted to $\mathrm{Spec}\ k\to V_\alpha$ for any α, then

$$\begin{aligned}&\widehat{\mathrm{vol}}(s,X_s,\Delta_s)\\ =&\inf_{S_i}\{\ \widehat{\mathrm{vol}}_{X_s,\Delta_s}(\mathrm{ord}_{S_i}) \mid \text{Kollár components } S_i \text{ with } \widehat{\mathrm{vol}}(S_i)\le n^n+1\}\\ \ge&\min_\alpha\{v_\alpha\}\\ \ge&\widehat{\mathrm{vol}}(s,X_s,\Delta_s),\end{aligned}$$

where the second relation holds since any Kollár component over (X_s,Δ_s) with normalized volume at most n^n+1 is isomorphic to T_u for some $u\in V_\alpha$ mapping to s; the last relation follows from the fact that there is a lift $\mathrm{Spec}\ k\to V_\alpha$ of $s\to B$ for every α. Thus $s\to\widehat{\mathrm{vol}}(s,X_s,\Delta_s)$ is a constant on all geometric points after we shrink B to a nonempty open set, which implies it is constructible. □

It is known that $\widehat{\mathrm{vol}}(s,X_s,\Delta_s)$ is also lower-semicontinuous by [BL18a, Th. 1]. Indeed, with Theorem 1.3, to see this we only need the weaker result that the normalized volume does not increase under a specialization of singularities.

It is well known that Theorem 1.4 follows from Theorem 1.3 via the cone construction.

Proof of Theorem 1.4. We take the relative cone over

$$Y:=C(X/B,-r(K_X+\Delta))$$

for sufficiently divisible r, i.e., $Y=\mathrm{Spec}_{\mathcal{O}_B}\bigoplus_{m=0}^{\infty}\big(f_*(-rm(K_X+\Delta))\big)$, and we can pull back the base Δ to get a boundary Δ_Y. It has a section $B\to(Y,\Delta_Y)$ given by the cone vertices, which makes it a $\mathbb{Q}$-Gorenstein family of klt singularities.

For any $s\in B$,

$$\widehat{\mathrm{vol}}(s,Y_s,\Delta_{Y_s})\le\widehat{\mathrm{vol}}_{(Y_s,\Delta_{Y_s}),s}(\mathrm{ord}_{V_s})=\frac{1}{r}(-K_{X_s}-\Delta_s)^n,$$

where V_s is the divisor obtained by blowing up the vertex. By Proposition 2.17, for any geometric point $s\in B$, (X_s,Δ_s) is K-semistable if and only if the equality holds.

By Theorem 1.3 and [BL18a, Th. 1], $\widehat{\mathrm{vol}}(s,Y_s,\Delta_{Y_s})$ is constructible and lower semi-continuous. Therefore, there is an open set B° of B, such that

$\widehat{\mathrm{vol}}(s, Y_s, \Delta_{Y_s})$ takes the possibly maximal value $\frac{1}{r}(-K_{X_s} - \Delta_s)^n$, which is precisely the locus where the geometric fibers are K-semistable. □

Remark 4.3. When $\Delta = 0$, the proof of Theorem 1.4 clearly can also be applied to any locally stable family of klt Fano varieties (see Definition 2.11).

It is known that openness of K-semistability was the last missing ingredient to prove Theorem 1.5 (see [ABHLX19, Cor. 1.2]). An outline of the construction is given in [BX19] for locally stable families of uniformly K-stable Fano varieties. To get Theorem 1.5, we only need to replace the ingredients, and then the same argument applies.

Proof of Theorem 1.5. We follow the proof of [BX19, Cor. 1.4]. Replacing the uniform K-stability by the K-semistability, and [BL18b] by Theorem 1.4 (see Remark 4.3), we conclude that $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ is parametrized by an Artin stack of finite type over k. Then by [BX19, Th. 1.1] and [ABHLX19, Cor. 1.2], we know the good moduli space $X^{\mathrm{kps}}_{n,V}$ exists and is separated. □

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