Arthur parameters and cuspidal automorphic modules of classical groups

By Dihua Jiang and Lei Zhang

Abstract

The endoscopic classification via the stable trace formula comparison provides certain character relations between irreducible cuspidal automorphic representations of classical groups and their global Arthur parameters, which are certain automorphic representations of general linear groups. It is a question of J. Arthur and W. Schmid that asks *how to construct concrete modules for irreducible cuspidal automorphic representations of classical groups in term of their global Arthur parameters*? In this paper, we formulate a general construction of concrete modules, using Bessel periods, for cuspidal automorphic representations of classical groups with generic global Arthur parameters. Then we establish the theory for orthogonal and unitary groups, based on certain well expected conjectures. Among the consequences of the theory in this paper is that the global Gan-Gross-Prasad conjecture for those classical groups is proved in full generality in one direction and with a global assumption in the other direction.

Contents

1. Introduction

Let F be a number field and $\mathbb{A}$ be the ring of the adeles of F. Let G be a classical group defined over F. The theory of endoscopic classification gives a parametrization of the irreducible automorphic representations of $G(\mathbb{A})$ occurring in the discrete spectrum of all square-integrable automorphic functions on $G(\mathbb{A})$, up to global Arthur packets, by means of global Arthur parameters. These parameters are formal sums of certain irreducible square-integrable automorphic representations of general linear groups. This fundamental theory has been established by J. Arthur in [3] for G to be either symplectic groups or F-quasisplit special orthogonal groups, with an outline on general orthogonal groups in [3, Ch. 9]. Following the fundamental work of Arthur ([3]),

several authors made progress for more general classical groups. C.-P. Mok established the theory for F-quasisplit unitary groups ([70]). More recently, Kaletha, Minguez, Shin, and White in [52] made progress on more general unitary groups. We refer to the work of B. Xu ([83]) for progress on the cases of similitude classical groups GSp_{2n} and GO_{2n}. We remark that all those works depend on the stabilization of the twisted trace formula, which has been achieved through a series of works of C. Mœglin and J.-L. Waldspurger that are now given in their books ([68] and [69]).

In Problem No. 5 in the *Open Problems in honor of W. Schmid* ([4]), Arthur explains that the trace formula method establishes certain *character relation* between irreducible cuspidal automorphic representations of classical groups and their global Arthur parameters. It was Schmid who asks *"What about modules...?"*. This means how to construct a concrete module for any irreducible cuspidal automorphic representation in terms of its global Arthur parameter. In [4], Arthur posed this question and pointed out that the work of the first named author ([36]) has the potential to give an answer to this question.

Our objective is to formulate, in the spirit of the constructive theory described in [36] and also [35], a general construction (Principle 1.1) of concrete modules for cuspidal automorphic representations of general classical groups, which provides an answer to the question of Arthur-Schmid.

In this paper we establish the theory of concrete modules (Conjecture 6.7), under certain well-expected conjectures (Conjecture 2.3, for instance) for cuspidal automorphic representations with generic global Arthur parameters (Theorem 7.1). The key idea in the theory is to introduce the method of *twisted automorphic descents*, which extends the method of automorphic descents of Ginzburg-Rallis-Soudry ([23]) from F-quasisplit classical groups to general classical groups, and from generic cuspidal automorphic representations to general cuspidal automorphic representations with generic global Arthur parameters.

One of the main technical issues in the method is to establish the global non-vanishing of the twisted automorphic descents that are constructed from the given data. This is treated by establishing the *reciprocal non-vanishing for Bessel periods* (Theorem 5.3), which depends heavily on the extension of the global and local theory of the global zeta integrals that represent the tensor product L-functions to the generality considered in this paper from the work of Ginzburg, Piatetski-Shapiro and Rallis ([19]), the work of the current authors ([45]), and the recent work of Soudry ([75] and [76]). Those previously done works mainly treat the F-quasisplit classical groups. The extension of the global theory is discussed in Section 4 of this paper, and that of the local theory is given in our joint work with Soudry in [43]. Another technical issue is to

prove the irreducibility of the concrete modules constructed via the twisted automorphic descents, which is carried out by using the local Gan-Gross-Prasad conjecture (Conjecture 3.1) as input. As a consequence, we are able to establish one direction of the global Gan-Gross-Prasad conjecture in full generality (Theorem 5.7), while we establish the other direction of the conjecture with a global assumption (Theorem 6.10), except some special cases (Corollary 6.11, and also [40]), where such a global assumption can be established.

The global Gan-Gross-Prasad conjecture that we refer to is Conjecture 24.1 (and Conjecture 26.1 for a different formulation) in [16]. It was first made by B. Gross and D. Prasad in [24] and [25] for orthogonal groups and was reformulated in full generality for all classical groups, including the metaplectic groups, by Gan, Gross and Prasad in [16]. The progress towards the proof of the global Gan-Gross-Prasad conjecture can be traced back to the pioneering work of Harder-Langlands-Rapoport on the Tate conjecture for Hilbert-Blumenthal modular surfaces ([26]), and it has been explained well in [16], [15], and also in [14].

It is important to point out that the work of W. Zhang ([88] and [87]) established the global Gan-Gross-Prasad conjecture for a special family of unitary groups with certain global and local constraints. The approach taken up in [88] and [87] is to use the relative trace formula developed by H. Jacquet and S. Rallis in [34] for unitary groups. However, such a relative trace formula that can be used to attack the global Gan-Gross-Prasad conjecture for orthogonal groups is so far not known to be available. The global Gan-Gross-Prasad conjecture for generic cuspidal automorphic representations with simple global Arthur parameters was considered in [20], [21], and [22] for symplectic and metaplectic groups, orthogonal groups, and unitary groups, respectively. The method is a combination of the Bessel or Fourier-Jacobi periods of certain residual representations with the Arthur truncation method. It was recently discovered that there is a technical gap in the argument towards the end of the proof, which needs to be filled up. A similar approach with the Arthur truncation replaced by the Jacquet-Lapid-Rogawski truncation is applied to the case of $U_{n+1} \times U_n$ by A. Ichino and S. Yamana in [33]. We refer to Section 5.5 for a more detailed account.

The approach taken up in this paper treats the global Gan-Gross-Prasad conjecture uniformly for unitary groups and orthogonal groups, and it can be used to take care of the symplectic group and metaplectic group situation by using the Fourier-Jacobi periods ([46]). It avoids the technical difficulties that occur in the literature ([20], [21], and [22]), which seem hopeless to be smoothly handled when one considers general cuspidal automorphic representations with generic global Arthur parameters and general classical groups. More importantly, the approach in this paper is much naturally related to the theory of

twisted automorphic descents and the general Rankin-Selberg method, so that one may regard the global Gan-Gross-Prasad conjecture as part of the theory developed in our work. Finally, the results on the global Gan-Gross-Prasad conjecture in this paper (Theorems 5.7 and 6.10) do not assume that the cuspidal multiplicity should be one, while this cuspidal multiplicity one assumption was taken for the global Gan-Gross-Prasad conjecture in [16]. This cuspidal multiplicity one issue was also discussed by H. Xue in [85, §6].

We also refer to [88] and [87] for a beautiful explanation of the relation between the Gan-Gross-Prasad conjecture and certain important problems in arithmetic and geometry, and for a more complete account of the progress on lower rank examples and other special cases towards the global conjecture and its refinement.

It is worthwhile to mention that the basic theoretic framework and technical results developed in this paper have been used in some recent work ([39] and [48]) to study the *automorphic branching problem and its reciprocal problem*, and to establish certain cases of the global Gan-Gross-Prasad conjecture for *non-tempered* global Arthur parameters, which has been recently formulated as [17, Conj. 9.1].

1.1. *Main ideas and arguments in the theory.* In order to illustrate the main ideas and arguments of the theory in this introduction, we take G to be an odd special orthogonal group. The general case will be discussed in the main body of this paper.

We denote by $G_n^* = \mathrm{SO}(V^*, q^*)$ the F-split odd special orthogonal group of $2n + 1$ variables. Let $G_n = \mathrm{SO}(V, q)$ be the odd special orthogonal group defined by a $2n + 1$ dimensional non-degenerate quadratic space (V, q) over F. Then G_n is a pure inner form of G_n^* over F, in the sense of Vogan (in [80] and also in [16], [50] and [51]). Following the work of Arthur ([3, Ch. 9, in particular]), the discrete spectrum of G_n are parametrized by the G_n-relevant, global Arthur parameters of G_n^*, the set of which is denoted by $\widetilde{\Psi}_2(G_n^*)_{G_n}$. The global Arthur parameters of G_n^* are multiplicity-free formal sums of the type

$$(1.1) \qquad \psi = (\tau_1, b_1) \boxplus \cdots \boxplus (\tau_r, b_r) \in \widetilde{\Psi}_2(G_n^*),$$

where τ_i is an irreducible unitary self-dual cuspidal automorphic representation of $\mathrm{GL}_{a_i}(\mathbb{A})$ for $i = 1, 2, \ldots, r$, having the property that when τ_i is of orthogonal type, the integer b_i must be even, and when τ_i is of symplectic type, the integer b_i must be odd.

Following [3], a global Arthur parameter ψ is called *generic* if $b_i = 1$ for $i = 1, 2, \ldots, r$. The subset of the generic parameters is denoted by $\widetilde{\Phi}_2(G_n^*)$ and that of the G_n-relevant ones is denoted by $\widetilde{\Phi}_2(G_n^*)_{G_n}$. Hence the generic global Arthur parameters are of the form

$$(1.2) \qquad \phi = (\tau_1, 1) \boxplus \cdots \boxplus (\tau_r, 1).$$

It follows that for a generic global Arthur parameter ϕ in (1.2), the cuspidal automorphic representations $\tau_1, \ldots, \tau_r$ are all of symplectic type and τ_i is not equivalent to τ_j if $i \neq j$.

By [3], in particular, [3, Ch. 9], for any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, the set of equivalence classes of irreducible automorphic representations of G_n that occur in the cuspidal spectrum, there is a G_n-relevant, global Arthur parameter $\psi \in \widetilde{\Psi}_2(G_n^*)$ such that $\pi \in \widetilde{\Pi}_\psi(G_n)$, the global Arthur packet of G_n associated to ψ. One has the following diagram,

$$\widetilde{\Psi}_2(G_n^*)$$

(1.3)
$$\psi$$
$$\swarrow \qquad \searrow$$
$$\mathcal{A}_{\mathrm{disc}}(G_n) \cap \widetilde{\Pi}_\psi(G_n) \qquad \Longleftrightarrow \qquad \mathcal{A}_{\mathrm{disc}}(G_n^*) \cap \widetilde{\Pi}_\psi(G_n^*),$$

where $\mathcal{A}_{\mathrm{disc}}(G_n)$ is the set of equivalence classes of irreducible automorphic representations of $G_n(\mathbb{A})$ that occur in the discrete spectrum.

When a parameter $\psi \in \widetilde{\Psi}_2(G_n^*)$ is generic, i.e., $\psi = \phi$ as given in (1.2), the global packet $\widetilde{\Pi}_\phi(G_n^*)$ contains an irreducible generic cuspidal automorphic representation π_0 of $G_n^*(\mathbb{A})$. This π_0 can be constructed by the automorphic descent of Ginzburg, Rallis and Soudry in [23] and in [42]. This construction produces a concrete module for π_0 by using only the generic global Arthur parameter ϕ. However, it remains a *big problem* to construct other cuspidal members in the global packet $\widetilde{\Pi}_\phi(G_n^*)$, and even more generally, to construct all cuspidal members in $\widetilde{\Pi}_\psi(G_n)$ for all pure inner forms G_n of G_n^*.

It seems clear from diagram (1.3) that one has to take more invariants of π into consideration in order to develop a reasonable theory that constructs concrete modules of all cuspidal members in $\widetilde{\Pi}_\psi(G)$ for general classical groups G. One of the natural choices is to utilize the structure of Fourier coefficients of cuspidal automorphic representations π, in addition to the global Arthur parameters ψ. We use $\mathcal{F}(\pi, G)$ to denote a certain piece of information about the structure of Fourier coefficients of π. Here is the principle of the theory.

PRINCIPLE 1.1 (Concrete Modules). *Let G^* be an F-quasisplit classical group and G be a pure inner form of G^*. For an irreducible cuspidal automorphic representation π of $G(\mathbb{A})$, assuming that π has a G-relevant global Arthur parameter $\psi \in \widetilde{\Psi}_2(G^*)$, there exists a datum $\mathcal{F}(\pi, G)$ such that one is able to construct a concrete irreducible module $\mathcal{M}(\psi, \mathcal{F}(\pi, G))$, depending on the data $(\psi, \mathcal{F}(\pi, G))$, with the property that*

$$\pi \cong \mathcal{M}(\psi, \mathcal{F}(\pi, G)).$$

Moreover, if π occurs in the cuspidal spectrum of G with multiplicity one, then $\pi = \mathcal{M}(\psi, \mathcal{F}(\pi, G))$.

We remark that if $G = G^*$ is F-quasisplit and π is generic, the concrete module expected in Principle 1.1 should coincide with the module constructed from the automorphic descents of Ginzburg-Rallis-Soudry in [23]. This will be explained in Corollary 7.2.

We still take G_n to be an odd special orthogonal group. For the case when the global Arthur parameter ψ is generic, we propose the *Main Conjecture* (Conjecture 6.7) of the theory developed in this paper that specifies the *Principle of Concrete Modules* (Principle 1.1) with the datum $\mathcal{F}(\pi, G_n)$ explicitly given in Conjecture 2.3. The nature of Conjecture 2.3 will be briefly discussed in Section 2.3 and will be considered in our future work. With $\mathcal{F}(\pi, G_n)$ as described in Conjecture 2.3, and with the generic global Arthur parameter ϕ for π, the construction of the concrete module $\mathcal{M}(\phi, \mathcal{F}(\pi, G_n))$ for the given π is carried out by the *twisted automorphic descent* as illustrated in diagram (6.7).

One of the key results in this paper is Theorem 5.3, which gives a *reciprocal non-vanishing for Bessel periods*. Such a non-vanishing property is proved using a refined theory of the global zeta integrals for the tensor product L-functions for G_n and a general linear group. The global theory of the global zeta integrals goes back to the pioneering work of Ginzburg, Piatetski-Shapiro and Rallis for orthogonal groups ([19]), which has been extended to a more general setting, including unitary groups by the authors of this paper in [45]. We established the global results of the global zeta integrals for the most general situation in Section 4. In order to obtain Theorem 5.3, we need the explicit unramified computation of the local zeta integrals. This is done in [43], which extends the work of Soudry ([75] and [76]) for split orthogonal groups to the generality considered in this paper.

By using the *reciprocal non-vanishing for Bessel periods*, we are able to show that certain Fourier coefficient of a residual representation, which is denoted by $\mathcal{F}^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau \otimes \sigma})$, is non-zero. The notation is referred to in Theorem 5.3. This is the candidate for the concrete module of π, as explained in the main conjecture of the theory (Conjecture 6.7). Conjecture 6.7 for G_n asserts that $\mathcal{F}^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau \otimes \sigma})$ is an irreducible cuspidal automorphic representation of $G_n(\mathbb{A})$ that is isomorphic to the given π. We note that when G_n is an even special orthogonal group, this assertion has to be modified due to the extra outer involution. We refer to Conjecture 6.7 for detail.

One of the main results of this paper (Theorem 7.1) is to prove that Conjecture 6.7 holds under the assumption of Conjectures 2.3 and 3.1, In two special cases, the results are stronger as given in Corollaries 7.2 and 7.4. It is worthwhile to mention that by a different argument, this theory recovered the classical Jacquet-Langlands correspondence for PGL(2) in [40].

We remark that the irreducibility of $\mathcal{F}^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau \otimes \sigma})$ is deduced from Conjecture 3.1, which is the local Gan-Gross-Prasad conjecture for local Vogan packets. The recent progress towards this conjecture is recorded as Theorem 3.2, according to the work of Mœglin-Waldspurger ([67]), the work of R. Beuzart-Plessis ([9] and [8]), the work of Gan-Ichino ([18]), the work of H. He ([28]), and the Ph.D. thesis of Zhilin Luo ([59]).

Based on our theory, Theorem 5.7 proves one direction of the global Gan-Gross-Prasad conjecture in full generality for the classical groups considered in this paper, while Theorem 6.10 proves the other direction of the conjecture with a global assumption (Conjecture 6.8), which is about a certain structure of Fourier coefficients of the relevant residual representations. We refer to [36, §4] and [37] for discussion of the general issue related to the conjecture.

1.2. *Structure of this paper.* A more detailed description of the content in each section is in order. In Section 2.1, we discuss the family of classical groups considered in this paper and recall their basic structures. The global Arthur parameters and the discrete spectrum for those classical groups are discussed in Section 2.2. We recall from [36] and [37] the general notion of Fourier coefficients of automorphic forms associated to the partitions or nilpotent orbits in Section 2.3 and give a more detailed account for the special type of Fourier coefficients, which is often called the *Bessel-Fourier coefficients.* Based on the *tower property* for Bessel-Fourier coefficients of cuspidal automorphic forms (Proposition 2.2), we state Conjecture 2.3. This is our starting point in the theory of construction of concrete modules for irreducible cuspidal automorphic representations for general classical groups via the twisted automorphic descents. In Section 2.4, we show (Proposition 2.6) that the construction illustrated by diagram (6.7) covers all the classical groups considered in this paper as described in Section 2.1.

The local Gan-Gross-Prasad conjecture (as in Conjecture 3.1) is one of the key inputs in the proof of the irreducibility of the constructed modules. We recall from [16] the cases considered in this paper in Section 3, and we state Conjecture 3.1, which is needed for one of the main results in the paper (Theorem 7.1). The known cases of Conjecture 3.1 are stated in Theorem 3.2. In Section 4, we consider a family of global zeta integrals, which represent the tensor product L-functions for the classical groups defined in Section 2.1 and the general linear groups. We show that they can be written as an Euler product of local zeta integrals (Theorems 4.5 and 4.7). With the explicit results on the unramified calculation of the local zeta integrals in terms of the local L-factors (Theorem 4.8), the global zeta integral can be written in a formula in (4.49). Based on what was discussed in Section 4, we establish in Section 5 the necessary analytic properties of the local zeta integrals

in Section 5.3, which are needed to establish the *reciprocal non-vanishing for Bessel periods* (Theorem 5.3). While some of the properties of the local zeta integrals can be deduced from the global argument based on the formula in (4.49) for the global zeta integrals, one of the most technical local results is Proposition 5.5, which asserts a general non-vanishing of the local zeta integrals for the data with certain global constraints, and which will be proved in Appendix A. It is also important to mention that Theorem 5.1 on the analytic properties of the normalized local intertwining operators is another key input in this theory. We will prove Theorem 5.1 in Appendix B. As a consequence, we obtain in Theorem 5.7 one direction of the global Gan-Gross-Prasad conjecture in full generality. With Conjecture 2.3 and Theorem 5.3, in addition to Theorem 5.1, we are able to obtain the non-vanishing of the Fourier coefficient of the particular residual representation $\mathcal{F}^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau \otimes \sigma})$, which is one of the key points in the theory. The basic properties of $\mathcal{F}^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau \otimes \sigma})$ are established in Section 6, which are similar to those in the automorphic descents of Ginzburg, Rallis and Soudry ([23]) and in our previous work joint with Liu and Xu ([40]). As a consequence, we obtain results towards another direction of the global Gan-Gross-Prasad conjecture (Theorem 6.10).

Diagram (6.7) illustrates the main idea and process of the construction of concrete modules for irreducible cuspidal automorphic representations of G that have generic global Arthur parameters. In the framework of the construction given by diagram (6.7), we state the main conjecture of the theory (Conjecture 6.7). As one of the main results of this paper, we prove in Theorem 7.1 that Conjecture 6.7 holds, assuming that Conjectures 2.3 and 3.1 hold. In two special cases, Conjecture 2.3 is trivial or can be easily verified. Hence we can have stronger results for those two special cases. The one attached to the regular partition (Corollary 7.2) is essentially the automorphic descents in [23], and the other attached to the subregular partition (Corollary 7.4) is new, and is a generalization of the construction considered in [40].

There are two appendices following the main body of this paper. Appendix A proves Proposition 5.5 in a more general setting. We put this as one of the two appendices so as to ensure a smoother logic flow in the main body of this paper. Appendix B proves Theorem 5.1. We leave this out of the main body because the proof needs different preparation, although there is a possibility to put it in Section 3.

Finally, we would like to thank J. Arthur and W. Schmid for asking and posing this very interesting and important problem in 2013, which stimulates and encourages us to carry out the work in this paper. We hope the main results and conjectures in this paper to be helpful towards the understanding of the nature of their problem. We are grateful to D. Soudry for his help in finding the proof presented in Appendix A, which works uniformly for all

local places. We would also like to thank C. Mœglin, F. Shahidi, D. Vogan, and B. Xu for very helpful conversations about the proof of the results in Appendix B, and thank W. T. Gan for his helpful comments and suggestions on several issues on the theory considered here. Last, but not least, we would like to thank P. Sarnak for his interest in and encouraging comments on the theory and results developed in this paper, and we thank the referee for very important and useful comments and suggestions, which greatly improved the exposition of the paper.

2. **Discrete spectrum and Fourier coefficients**

2.1. *Certain classical groups.* The classical groups considered in this paper are unitary groups and special orthogonal groups that are explicitly defined below.

Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ be the ring of adeles of F. Let $F(\sqrt{\varsigma})$ be a quadratic field extension of F, with ς a non-square in $F^\times$. Let E be either F or $F(\sqrt{\varsigma})$, and consider the Galois group $\Gamma_{E/F} = \mathrm{Gal}(E/F)$. It is trivial if $E = F$, and it has a unique non-trivial element ι if $E = F(\sqrt{\varsigma})$. Let (V, q) be an $\mathfrak{n}$-dimensional non-degenerate vector space over E, which is Hermitian if $E = F(\sqrt{\varsigma})$ and is symmetric (or quadratic) if $E = F$. Denote by $G_n = \mathrm{Isom}(V, q)^\circ$ the identity connected component of the isometry group of the space (V, q), with $n = [\frac{\mathfrak{n}}{2}]$. Let $G_n^* = \mathrm{Isom}(V^*, q^*)^\circ$ be an F-quasisplit group of the same type, so that G_n is a pure inner form of G_n^* over the field F, following [80] and [16].

Let (V_0, q) be the F-anisotropic kernel of (V, q) with dimension $\mathfrak{d}_0 = \mathfrak{n} - 2\mathfrak{r}$, where the F-rank $\mathfrak{r} = \mathfrak{r}_\mathfrak{n} = \mathfrak{r}(G_n)$ of G_n is the same as the Witt index of (V, q). Let V^+ be a maximal totally isotropic subspace of (V, q), with $\{e_1, \ldots, e_\mathfrak{r}\}$ being its basis. Choose E-linearly independent vectors $\{e_{-1}, \ldots, e_{-\mathfrak{r}}\}$ in (V, q) such that

$$q(e_i, e_{-j}) = \delta_{i,j}$$

for all $1 \leq i, j \leq \mathfrak{r}$. Denote by $V^- = \mathrm{Span}\{e_{-1}, \ldots, e_{-\mathfrak{r}}\}$ the dual space of V^+. Then (V, q) has the following polar decomposition,

$$V = V^+ \oplus V_0 \oplus V^-,$$

where $V_0 = (V^+ \oplus V^-)^\perp$ is an F-anisotropic kernel of (V, q). We choose an orthogonal basis $\{e_1', \ldots, e_{\mathfrak{d}_0}'\}$ of V_0 with the property that

$$q(e_i', e_i') = d_i,$$

where d_i is non-zero for all $1 \leq i \leq \mathfrak{d}_0$. Set $G_{d_0} = \mathrm{Isom}(V_0, q)^\circ$ with $d_0 = [\frac{\mathfrak{d}_0}{2}]$, which is anisotropic over F and is regarded as an F-subgroup of G_n.

We put the above bases together in the following order to form a basis of (V, q):

(2.1) $$e_1, \ldots, e_{\mathfrak{r}}, e_1', \ldots, e_{\mathfrak{d}_0}', e_{-\mathfrak{r}}, \ldots, e_{-1}.$$

We fix the following full isotropic flag in (V, q),

$$\mathrm{Span}\{e_1\} \subset \mathrm{Span}\{e_1, e_2\} \subset \cdots \subset \mathrm{Span}\{e_1, \ldots, e_{\mathfrak{r}}\},$$

which defines a minimal parabolic F-subgroup P_0. Moreover, P_0 contains a maximal F-split torus S, consisting of elements

$$\mathrm{diag}\{t_1, \ldots, t_{\mathfrak{r}}, 1, \ldots, 1, t_{\mathfrak{r}}^{-1}, \ldots, t_1^{-1}\},$$

with $t_i \in F^\times$ for $i = 1, 2, \ldots, \mathfrak{r}$. Then the centralizer $Z(S)$ in G_n is $\mathrm{Res}_{E/F}S \times G_{d_0}$, the Levi subgroup of P_0, where $\mathrm{Res}_{E/F}S$ is the Weil restriction of S from E to F. Then P_0 has the Levi decomposition

$$P_0 = (\mathrm{Res}_{E/F}S \times G_{d_0}) \ltimes N_0,$$

where N_0 is the unipotent radical of P_0. Also, with respect to the order of the basis in (2.1), the group G_n is also defined by the following symmetric matrix:

(2.2) $$J_{\mathfrak{r}}^{\mathfrak{n}} = \begin{pmatrix} & & 1 \\ & J_{\mathfrak{r}-1}^{\mathfrak{n}-2} & \\ 1 & & \end{pmatrix}_{\mathfrak{n} \times \mathfrak{n}} \quad \text{and } J_0^{\mathfrak{d}_0} = \mathrm{diag}\{d_1, \ldots, d_{\mathfrak{d}_0}\}$$

as defined inductively.

Let $_F\Phi(G_n, S)$ be the root system of G_n over F. Let $_F\Phi^+(G_n, S)$ be the positive roots corresponding to the minimal parabolic F-subgroup P_0, and let $_F\Delta = \{\alpha_1, \ldots, \alpha_{\mathfrak{r}}\}$ be a set of simple roots in $_F\Phi^+(G_n, S)$. When G_n is an orthogonal group, the root system $_F\Phi(G_n, S)$ is of type $B_{\mathfrak{r}}$ unless $\mathfrak{n} = 2\mathfrak{r}$, in which case it is of type $D_{\mathfrak{r}}$. When G_n is a unitary group, the root system $_F\Phi(G_n, S)$ is non-reduced of type $BC_{\mathfrak{r}}$ if $2\mathfrak{r} < \mathfrak{n}$; otherwise, $_F\Phi(G_n, S)$ is of type $C_{\mathfrak{r}}$.

For a subset $J \subset \{1, \ldots, \mathfrak{r}\}$, let $_F\Phi_J$ be the root subsystem of $_F\Phi(G_n, S)$ generated by the simple roots $\{\alpha_j : j \in J\}$. Let $P_J = M_J U_J$ be the standard parabolic F-subgroup of G_n, whose Lie algebra consists of all roots spaces $\mathfrak{g}_\alpha$ with $\alpha \in {}_F\Phi^+(G_n, S) \cup {}_F\Phi_J$. For instance, if we set $\hat{i} := \{1, \ldots, \mathfrak{r}\} \setminus \{i\}$, then $P_{\hat{i}} = M_{\hat{i}} U_{\hat{i}}$ is the standard maximal parabolic F-subgroup of G_n, which stabilizes the rational isotropic space V_i^+, where $V_i^\pm := \mathrm{Span}\{e_{\pm 1}, \ldots, e_{\pm i}\}$. Here $U_{\hat{i}}$ is the unipotent radical of $P_{\hat{i}}$ and the Levi subgroup $M_{\hat{i}}$ is isomorphic to $G_{E/F}(i) \times G_{n-i}$. Following the notation of [3] and [70], $G_{E/F}(i) := \mathrm{Res}_{E/F}\mathrm{GL}_i$ denotes the Weil restriction of E-group GL_i restricted to F. Write $V_{(i)} = (V_i^+ \oplus V_i^-)^\perp$; hence $V_{(\mathfrak{r})} = V_0$ is the F-anisotropic kernel of (V, q).

We recall simply from [16] the classification of pure inner F-forms of F-quasisplit classical groups G_n^* for a local field and then for a number field.

For a local field F of characteristic zero, we recall the notion of a *relevant pair* of classical groups. As above, we let $G_n := \mathrm{Isom}(V, q)^\circ$ be defined for an $\mathfrak{n}$-dimensional non-degenerate space (V, q) with $n = [\frac{\mathfrak{n}}{2}]$. Take an $\mathfrak{m}$-dimensional non-degenerate subspace (W, q) of (V, q) with the property that the orthogonal complement $(W^\perp, q)$ is F-split and has an odd dimension. Define $H_m :=$ $\mathrm{Isom}(W, q)^\circ$ with $m = [\frac{\mathfrak{m}}{2}]$. By [16, §2], the pair (G_n, H_m) forms a *relevant* pair.

If $G'_n := \mathrm{Isom}(V', q')^\circ$ and $H'_m := \mathrm{Isom}(W', q')^\circ$ form another relevant pair, and if G'_n and H'_m are pure inner F-form of G_n and H_m, respectively, the product $G'_n \times H'_m$ is defined to be *relevant* to the product $G_n \times H_m$ if the orthogonal complement $((W')^\perp, q')$ is equivalent to the orthogonal complement $(W^\perp, q)$, as Hermitian vector spaces. From [16, Lemma 2.2, part (i)], one can have an easy list of all F-relevant pairs (G_n, H_m) whose product $G_n \times H_m$ is relevant to the F-quasisplit product $G_n^* \times H_m^*$.

For a number field F, G_n is a pure inner F-form of an F-quasisplit G_n^* if it is obtained by inner twisting by elements in the pointed set $H^1(F, G_n)$. It follows that at every local place ν, G_n is a pure inner F_ν-form of G_n^*. The notion of *relevance* is defined in the same way. We will come back to this in Section 3 when we discuss Vogan packets and the Gan-Gross-Prasad conjectures.

2.2. *Discrete spectrum and Arthur packets.* For a reductive algebraic group G defined over F, denote by $\mathcal{A}_{\mathrm{disc}}(G)$ the set of equivalence classes of irreducible unitary representations π of $G(\mathbb{A})$ occurring in the discrete spectrum $L^2_{\mathrm{disc}}(G)$ of $L^2(G(F)\backslash G(\mathbb{A})^1)$, when π is restricted to $G(\mathbb{A})^1$. Also denote by $\mathcal{A}_{\mathrm{cusp}}(G)$ the subset of $\mathcal{A}_{\mathrm{disc}}(G)$, whose elements occur in the cuspidal spectrum $L^2_{\mathrm{cusp}}(G)$. The theory of endoscopic classification for classical groups G_n is to parametrize the set $\mathcal{A}_{\mathrm{disc}}(G_n)$ by means of the global Arthur parameters, which can be realized as certain automorphic representations of general linear groups. We recall from the work of Arthur ([3]), the work of Mok ([70]) and the work of Kaletha, Minguez, Shin, and White ([52]) the theory for the (special) orthogonal groups and the unitary groups considered in this paper.

First, we take an F-quasisplit classical group G_n^*, of which G_n is a pure inner F-form. Both G_n^* and G_n share the same L-group $^L G_n^* = {}^L G_n$. Define $\mathfrak{n}^\vee$ to be $\mathfrak{n}$ if G_n is a unitary group or an even special orthogonal group, and to be $\mathfrak{n} - 1$ if G_n is an odd special orthogonal group. This number $\mathfrak{n}^\vee$ is denoted by N in [3], [70] and [52].

Following [3], [70] and [52], we denote by $\widetilde{\mathcal{E}}_{\mathrm{sim}}(N)$ (with $N = \mathfrak{n}^\vee$) the set of the equivalence classes of simple twisted endoscopic data. Each member in $\widetilde{\mathcal{E}}_{\mathrm{sim}}(N)$ is represented by a triple (G, s, ξ), where G is an F-quasisplit classical group, s is a semi-simple element as described in [3, p. 11] and [70, p. 16], and ξ is the L-embedding

$$^L G \to {}^L G_{E/F}(N).$$

Note that when G is an F-quasisplit unitary group, the L-embedding $\xi = \xi_{\chi_\kappa}$ depends on $\kappa = \pm 1$. As in [70, p. 18], for a simple twisted endoscopic datum $(U_{E/F}(N), \xi_{\chi_\kappa})$ of $G_{E/F}(N)$, the sign $(-1)^{N-1} \cdot \kappa$ is called the *parity* of the datum. The set of global Arthur parameters for G_n^* is denoted by $\widetilde{\Psi}_2(G_n^*, \xi)$, or simply by $\widetilde{\Psi}_2(G_n^*)$ if the L-embedding ξ is well understood in the discussion.

In order to explicate the structure of the parameters in $\widetilde{\Psi}_2(G_n^*, \xi)$, we first recall from [3] and [70] the description of the conjugate self-dual, elliptic, global Arthur parameters for $G_{E/F}(N)$, the set of which is denoted by $\widetilde{\Psi}_{\mathrm{ell}}(N)$. We refer to [3], [70] and also [52] for detailed discussion about general global Arthur parameters. The elements of $\widetilde{\Psi}_{\mathrm{ell}}(N)$ are denoted by ψ^N, which have the form

$$(2.3) \qquad \psi^N = \psi_1^{N_1} \boxplus \cdots \boxplus \psi_r^{N_r}$$

with $N = \sum_{i=1}^r N_i$. The formal summands $\psi_i^{N_i}$ are *simple* parameters of the form

$$\psi_i^{N_i} = \mu_i \boxtimes \nu_i$$

with $N_i = a_i b_i$, where $\mu_i = \tau_i \in \mathcal{A}_{\mathrm{cusp}}(G_{E/F}(a_i))$ and ν_i is a b_i-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. Following the notation used in our previous paper [36], we also denote

$$\psi_i^{N_i} = (\tau_i, b_i)$$

for $i = 1, 2, \ldots, r$. A global parameter ψ^N is called *conjugate self-dual* if each simple parameter $\psi_i^{N_i}$ that occurs in the decomposition of ψ^N is conjugate self-dual in the sense that τ_i is conjugate self-dual. An irreducible cuspidal automorphic representation τ of $G_{E/F}(a)$ is called *conjugate self-dual* if $\tau \cong \tau^*$, where $\tau^* = \iota(\tau)^\vee$ is the contragredient of $\iota(\tau)$, with ι being the non-trivial element in $\Gamma_{E/F}$ if $E \neq F$; otherwise, $\iota = 1$. The global parameter ψ^N is called *elliptic* if it is conjugate self-dual and its decomposition into the simple parameters is multiplicity free, i.e., $\psi_i^{N_i}$ and $\psi_j^{N_j}$ are not equivalent if $i \neq j$ in the sense that either τ_i is not equivalent to τ_j, or $b_i \neq b_j$. A global parameter ψ^N in $\widetilde{\Psi}_{\mathrm{ell}}(N)$ is called *generic* if $b_i = 1$ for $i = 1, 2, \ldots, r$. The set of generic, elliptic, global Arthur parameters for $G_{E/F}(N)$ is denoted by $\widetilde{\Phi}_{\mathrm{ell}}(N)$. Hence elements ϕ in $\widetilde{\Phi}_{\mathrm{ell}}(N)$ are of the form

$$(2.4) \qquad \phi^N = (\tau_1, 1) \boxplus \cdots \boxplus (\tau_r, 1).$$

When $r = 1$, the parameters are called *simple*. The corresponding sets are denoted by $\widetilde{\Psi}_{\mathrm{sim}}(N)$ and $\widetilde{\Phi}_{\mathrm{sim}}(N)$, respectively. It is clear that the set $\widetilde{\Phi}_{\mathrm{sim}}(N)$ is in one-to-one correspondence with the set of equivalence classes of the conjugate self-dual, irreducible cuspidal automorphic representations of $G_{E/F}(N)(\mathbb{A}_F)$. By [3, Th. 1.4.1] and [70, Th. 2.4.2], for a simple parameter $\phi = \phi^a = (\tau, 1)$ in $\widetilde{\Phi}_{\mathrm{sim}}(a)$, there exists a unique endoscopic datum $(G_\phi, s_\phi, \xi_\phi)$,

such that the parameter ϕ^a descends to a global parameter for (G_ϕ, ξ_ϕ) in the sense that there exists an irreducible automorphic representation π in $\mathcal{A}_2(G_\phi)$, whose Satake parameters are determined by the Satake parameters of ϕ^a.

When $E \neq F$, $G_\phi = U_{E/F}(a)$ is a unitary group, the L-embedding carries a sign κ_a, which determines the nature of the *base change* from the unitary group $U_{E/F}(a)$ to $G_{E/F}(a)$. By [70, Th. 2.5.4], the (partial) L-function

$$L(s, (\tau, 1), \mathrm{As}^{\eta_{(\tau,1)}})$$

has a (simple) pole at $s = 1$ with the sign $\eta_{(\tau,1)} = \kappa_a \cdot (-1)^{a-1}$ (see also [16, Th. 8.1] and [70, Lemma 2.2.1]). Then the irreducible cuspidal automorphic representation τ or equivalently the simple generic parameter $(\tau, 1)$ is called *conjugate orthogonal* if $\eta_{(\tau,1)} = 1$ and *conjugate symplectic* if $\eta_{(\tau,1)} = -1$, following the terminology of [16, §3] and [70, §2]. Here $L^S(s, (\tau, 1), \mathrm{As}^+)$ is the (partial) Asai L-function of τ and $L^S(s, (\tau, 1), \mathrm{As}^-)$ is the (partial) Asai L-function of $\tau \otimes \omega_{E/F}$, where $\omega_{E/F}$ is the quadratic character associated to E/F by the global class field theory.

The sign of a simple global Arthur parameter $\psi = \psi^{ab} = (\tau, b) \in \widetilde{\Psi}_2(ab)$ can be calculated following [70, §2.4]. Fix the sign κ_a as before for the endoscopic datum $(U_{E/F}(a), \xi_{\chi_{\kappa_a}})$. The sign of $(\tau, 1)$ is $\eta_{(\tau,1)} = \eta_\tau = \kappa_a(-1)^{a-1}$. Hence the sign of (τ, b) is given by

$$\eta_{(\tau,b)} = \kappa_a(-1)^{a-1+b-1} = \kappa_a(-1)^{a+b} = \eta_\tau(-1)^{b-1}.$$

As in [70, eq. (2.4.9)], define $\kappa_{ab} := \kappa_a(-1)^{ab-a-b+1}$. Then we have $\kappa_{ab}(-1)^{ab-1} = \eta_\tau(-1)^{b-1} = \eta_{(\tau,b)}$ and hence $\kappa_{ab} = \eta_\tau(-1)^{(a-1)b}$, which gives the endoscopic datum $(U_{E/F}(ab), \xi_{\kappa_{ab}})$. More generally, for an elliptic parameter ψ^N as in (2.3), following from [70, §2.1], each simple parameter $\psi_i^{N_i}$ determines the simple twisted endoscopic datum $(U_{E/F}(N_i), \xi_{\chi_{\kappa_i}})$ with $\kappa_i = (-1)^{N-N_i} = \eta_{\tau_i}(-1)^{(a_i-1)b_i}$, and hence determines the parity of the $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(G_{E/F}(a_i))$ for the simple parameter $\psi_i^{N_i} = (\tau_i, b_i)$.

When $E = F$, the notion of conjugate self-dual becomes just self-dual in the usual sense. A self-dual $\tau \in \mathcal{A}_{\mathrm{cusp}}(a)$ is called *of symplectic type* if the (partial) exterior square L-function $L^S(s, \tau, \wedge^2)$ has a (simple) pole at $s = 1$; otherwise, τ is called *of orthogonal type*. In the latter case, the (partial) symmetric square L-function $L^S(s, \tau, \mathrm{sym}^2)$ has a (simple) pole at $s = 1$.

More generally, from [3, §1.4] and [70, §2.4], for any parameter ψ^N in $\widetilde{\Psi}_{\mathrm{ell}}(N)$, there is a twisted elliptic endoscopic datum $(G, s, \xi) \in \widetilde{\mathcal{E}}_{\mathrm{ell}}(N)$ such that the set of the global parameters $\widetilde{\Psi}_2(G, \xi)$ can be identified as a subset of $\widetilde{\Psi}_{\mathrm{ell}}(N)$. We refer to [3, §1.4], [70, §2.4], and [52, §1.3] for more constructive description of the parameters in $\widetilde{\Psi}_2(G, \xi)$. The elements of $\widetilde{\Psi}_2(G_n^*, \xi)$, with

$N = \mathfrak{n}^\vee$ and $n = [\frac{\mathfrak{n}}{2}]$, are of the form

(2.5) $$\psi = (\tau_1, b_1) \boxplus \cdots \boxplus (\tau_r, b_r).$$

Here $N = N_1 + \cdots + N_r$ and $N_i = a_i \cdot b_i$, and $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(G_{E/F}(a_i))$ and b_i represents the b_i-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. Note that each simple parameter $\psi_i = (\tau_i, b_i)$ belongs to $\widetilde{\Psi}_2(G^*_{n_i}, \xi_i)$ with $n_i = [\frac{\mathfrak{n}_i}{2}]$ and $N_i = \mathfrak{n}_i^\vee$, for $i = 1, 2, \ldots, r$; and for $i \neq j$, ψ_i is not equivalent to ψ_j. The parity for τ_i and b_i is discussed as above. The subset of generic elliptic global Arthur parameters in $\widetilde{\Psi}_2(G^*_n, \xi)$ is denoted by $\widetilde{\Phi}_2(G^*_n, \xi)$, whose elements are in the form of (2.4).

Without lose of generality and for convenience, we choose ξ with sign $\kappa = 1$ *throughout this paper*, which is consistent with the choices in the Gan-Gross-Prasad conjecture ([15, p. 35]) and in the automorphic descents of Ginzburg-Rallis-Soudry ([23, p. 55]). That is, when G^*_n is an odd unitary group, its parameters are conjugate orthogonal; when G^*_n is an even unitary group, its parameters are conjugate symplectic.

The following is a simplified version of the endoscopic classification for classical groups established in [3], [70], and [52].

THEOREM 2.1 (Endoscopic Classification). *For any $\pi \in \mathcal{A}_{\mathrm{disc}}(G_n)$, there is a G_n-relevant global Arthur parameter $\psi \in \widetilde{\Psi}_2(G^*_n, \xi)$, such that π belongs to the global Arthur packet, $\widetilde{\Pi}_\psi(G_n)$, attached to the global Arthur parameter ψ.*

Following [3], [70] and [52], when G_n is *not* an even special orthogonal group, the multiplicity of $\pi \in \mathcal{A}_{\mathrm{disc}}(G_n)$ realizing in the discrete spectrum $L^2_{\mathrm{disc}}(G_n)$ is expected to be one. However, when G_n is an even special orthogonal group, the discrete multiplicity of $\pi \in \mathcal{A}_{\mathrm{disc}}(G_n)$ could be two. The multiplicities of the discrete automorphic representations of classical groups depend on the multiplicity property of local Arthur packets, which is known for the p-adic and complex cases for general local Arthur parameters. However, for the general local Arthur packets, which is what we need in this paper, the multiplicity property holds for all local fields. Hence the expected multiplicities of the automorphic representations in generic global Arthur packets are known. In the following, we may fix a realization of $\pi \in \mathcal{A}_{\mathrm{disc}}(G_n)$ in the discrete spectrum $L^2_{\mathrm{disc}}(G_n)$, which will be denoted by $\mathcal{C}_\pi$, especially when the discrete multiplicity of π is two.

Recall the notation from the definition of [3, Ch. 8] that

(2.6) $$\widetilde{\mathrm{O}}(G_n) := \widetilde{\mathrm{Out}}_N(G_n) := \widetilde{\mathrm{Aut}}_N(G_n)/\widetilde{\mathrm{Int}}_N(G_n)$$

is regarded as the diagonal subgroup of $\widetilde{\mathrm{Out}}_N(G_n(\mathbb{A}))$. When G_n is an even special orthogonal group, one may take $\varepsilon \in \mathrm{O}_{2n}(F)$ with $\det \varepsilon = -1$ and $\varepsilon^2 = I_{2n}$, such that the action of $\widetilde{\mathrm{O}}(G_n)$ on π can be realized as the ε-conjugate on π, i.e., $\pi^\varepsilon(g) = \pi(\varepsilon g \varepsilon^{-1})$. Hence the $\widetilde{\mathrm{O}}(G_n)$-orbit of π has one or two

elements. If $\widetilde{\mathrm{O}}(G_n)$ acts freely on π, following the notation in [3], we denote the $\widetilde{\mathrm{O}}(G_n)$-torsor of π by $\{\pi, \pi_\star\}$. When G_n is not an even special orthogonal group, the group $\widetilde{\mathrm{O}}(G_n)$ is trivial, and so is its action. Hence in this case, the $\widetilde{\mathrm{O}}(G_n)$-orbit of π contains only π itself.

When G_n is an even special orthogonal group, an elliptic global Arthur parameter ψ^N as in (2.3) may descend to two different global Arthur parameters ψ and $\psi_\star$ for G_n, which form an $\widetilde{\mathrm{O}}(G_n)$-orbit. If the $\widetilde{\mathrm{O}}(G_n)$-orbit of ψ^N is an $\widetilde{\mathrm{O}}(G_n)$-torsor $\{\psi, \psi_\star\}$, then they define different global Arthur packets and different global Vogan packets. However, following [3], their tensor product L-functions with any cuspidal automorphic representations of general linear groups are the same. We refer to Chapter 8 of [3] and Section 6 of [5] for a more detailed discussion.

In the rest of this paper, when we say that ψ^N is a global Arthur parameter of an even special orthogonal group G_n, we really mean that ψ^N is identified with either ψ or $\psi_\star$, through a specific twisted endoscopic datum.

2.3. *Fourier coefficients and partitions.* For an F-quasisplit classical group G_n^* defined by an $\mathfrak{n}$-dimensional non-degenerate space (V^*, q^*) with the Witt index $n = [\frac{\mathfrak{n}}{2}]$, the relation between Fourier coefficients of automorphic forms φ of $G_n^*(\mathbb{A})$ and the partitions of type $(\mathfrak{n}, G_n^*)$ has been discussed with details in [36] and also in [37]. We denote by $\mathcal{P}(\mathfrak{n}, G_n^*)$ the set of all partitions of type $(\mathfrak{n}, G_n^*)$. The set $\mathcal{P}(\mathfrak{n}, G_n^*)$ parametrizes the set of all F-stable nilpotent adjoint orbits in the Lie algebra $\mathfrak{g}_n^*(F)$ of $G_n^*(F)$, and hence each partition $\underline{p} \in \mathcal{P}(\mathfrak{n}, G_n^*)$ defines an F-stable nilpotent adjoint orbit $\mathcal{O}_{\underline{p}}^{\mathrm{st}}$. For an F-rational orbit $\mathcal{O}_{\underline{p}} \in \mathcal{O}_{\underline{p}}^{\mathrm{st}}$, the datum $(\underline{p}, \mathcal{O}_{\underline{p}})$ determines a datum $(V_{\underline{p}}, \psi_{\mathcal{O}_{\underline{p}}})$ for defining Fourier coefficients as explained in [36] and [37]. Here $V_{\underline{p}}$ is a unipotent subgroup of G_n^* and $\psi_{\mathcal{O}_{\underline{p}}}$ is a non-degenerate character of $V_{\underline{p}}(\mathbb{A})$, which is trivial on $V_{\underline{p}}(F)$ and determined by a given non-trivial character ψ_F of $F \backslash \mathbb{A}$.

For an automorphic form φ on $G_n^*(\mathbb{A})$, the $\psi_{\mathcal{O}_{\underline{p}}}$-Fourier coefficient of φ is defined by the following integral:

$$(2.7) \qquad \mathcal{F}^{\psi_{\mathcal{O}_{\underline{p}}}}(\varphi)(g) := \int_{V_{\underline{p}}(F) \backslash V_{\underline{p}}(\mathbb{A})} \varphi(vg) \psi_{\mathcal{O}_{\underline{p}}}^{-1}(v) dv.$$

Let $N_{G_n^*}(V_{\underline{p}})^{\mathrm{ss}}$ be the connected component of the semi-simple part of the normalizer of the subgroup $V_{\underline{p}}$ in G_n^*. Define

$$(2.8) \qquad H^{\mathcal{O}_{\underline{p}}} := \mathrm{Cent}_{N_{G_n^*}(V_{\underline{p}})^{\mathrm{ss}}}(\psi_{\mathcal{O}_{\underline{p}}})^\circ,$$

the identity connected component of the stabilizer. It is clear that the $\psi_{\mathcal{O}_{\underline{p}}}$-Fourier coefficient of φ, $\mathcal{F}^{\psi_{\mathcal{O}_{\underline{p}}}}(\varphi)(g)$, is left $H^{\mathcal{O}_{\underline{p}}}(F)$-invariant, smooth when restricted on $H^{\mathcal{O}_{\underline{p}}}(\mathbb{A})$, and of moderate growth on a Siegel set of $H^{\mathcal{O}_{\underline{p}}}(\mathbb{A})$.

For any $\pi \in \mathcal{A}_{\mathrm{disc}}(G_n^*)$, we denote by $\mathcal{C}_\pi$ a realization of π in the discrete spectrum $L^2_{\mathrm{disc}}(G_n^*)$. We define $\mathcal{F}^{\mathcal{O}_{\underline{p}}}(\mathcal{C}_\pi)$ (or simply $\mathcal{F}^{\mathcal{O}_{\underline{p}}}(\pi)$ when no confusion is caused) to be the space spanned by all $\mathcal{F}^{\psi_{\mathcal{O}_{\underline{p}}}}(\varphi_\pi)$ with φ_π running in the space of $\mathcal{C}_\pi$, and we call $\mathcal{F}^{\mathcal{O}_{\underline{p}}}(\mathcal{C}_\pi)$ a $\psi_{\mathcal{O}_{\underline{p}}}$-*Fourier module* of π. We note that if the discrete multiplicity of π is one, it has a unique $\psi_{\mathcal{O}_{\underline{p}}}$-Fourier module. For a given $\pi \in \mathcal{A}_{\mathrm{disc}}(G_n^*)$, we denote by $\mathfrak{p}(\mathcal{C}_\pi)$ (or simply $\mathfrak{p}(\pi)$) the subset of $\mathcal{P}(\mathfrak{n}, G_n^*)$ consisting of all partitions $\underline{p}$ with the property that the $\psi_{\mathcal{O}_{\underline{p}}}$-Fourier module, $\mathcal{F}^{\mathcal{O}_{\underline{p}}}(\mathcal{C}_\pi)$, is non-zero for some choice of the F-rational orbit $\mathcal{O}_{\underline{p}}$ in the F-stable orbit $\mathcal{O}_{\underline{p}}^{\mathrm{st}}$, and we denote by $\mathfrak{p}^m(\pi)$ (short for $\mathfrak{p}^m(\mathcal{C}_\pi)$) the subset of all maximal members in $\mathfrak{p}(\pi)$. In the rest of this paper, we may write $\mathcal{F}^{\mathcal{O}_{\underline{p}}}(\pi)$ to be $\mathcal{F}^{\mathcal{O}_{\underline{p}}}(\mathcal{C}_\pi)$ and $\mathfrak{p}^m(\pi)$ to be $\mathfrak{p}^m(\mathcal{C}_\pi)$ for a discrete realization $\mathcal{C}_\pi$ of π.

For a pure inner F-form G_n of G_n^*, a partition $\underline{p}$ in the set $\mathcal{P}(\mathfrak{n}, G_n^*)$ is called G_n-*relevant* if the unipotent subgroup $V_{\underline{p}}$ of G_n as algebraic groups over the algebraic closure $\overline{F}$ is actually defined over F. We denote by $\mathcal{P}(\mathfrak{n}, G_n^*)_{G_n}$ the subset of the set $\mathcal{P}(\mathfrak{n}, G_n^*)$ consisting of all G_n-relevant partitions of type $(\mathfrak{n}, G_n^*)$. It is easy to see that the above discussion about Fourier coefficients and Fourier modules can be applied to all $\pi \in \mathcal{A}_{\mathrm{disc}}(G_n)$ and all $\underline{p} \in \mathcal{P}(\mathfrak{n}, G_n^*)_{G_n}$, without change.

Following R. Howe ([32] and [31]), N. Kawanaka ([53]), Mœglin and Waldspurger ([64]), and Mœglin ([60]), one expects that the partitions $\underline{p}$ in $\mathfrak{p}^m(\pi)$, the F-rational orbits $\mathcal{O}_{\underline{p}}$ in the F-stable orbits $\mathcal{O}_{\underline{p}}^{\mathrm{st}}$, and the automorphic spectrum of the Fourier modules $\mathcal{F}^{\mathcal{O}_{\underline{p}}}(\pi)$ as representations of $H^{\mathcal{O}_{\underline{p}}}(\mathbb{A})$ carry fundamental information about the given automorphic representation π of $G_n(\mathbb{A})$. However, it is usually not easy to obtain explicit information about those data from the given π. In reality, we may consider certain special pieces of those data that may already carry enough information for us to understand the given representation π in the theory discussed in this paper.

We consider a family of partitions of type $(\mathfrak{n}, G_n^*)$, which leads to the so called Bessel-Fourier coefficients of automorphic forms on $G_n(\mathbb{A})$. These partitions are of the form

$$(2.9) \qquad \underline{p}_\ell = [(2\ell + 1)1^{\mathfrak{n}-2\ell-1}].$$

They are of type $(\mathfrak{n}, G_n^*)$. The partition $\underline{p}_\ell$ is G_n-relevant if ℓ is less than or equal to the F-rank $\mathfrak{r}$ of G_n. For example, if G_n is F-anisotropic, then the only G_n-relevant partition is the trivial partition $\underline{p}_0 = [1^{\mathfrak{n}}]$. For $\pi \in \mathcal{A}_{\mathrm{disc}}(G_n)$, and for a partition $\underline{p}_\ell \in \mathcal{P}(\mathfrak{n}, G_n^*)_{G_n}$, the Fourier module $\mathcal{F}^{\mathcal{O}_{\underline{p}_\ell}}(\pi)$ will be called the ℓ-*th Bessel module* of π. As explained before, the ℓ-th Bessel module $\mathcal{F}^{\mathcal{O}_{\underline{p}_\ell}}(\pi)$ consists of moderately increasing automorphic functions on $H^{\mathcal{O}_{\underline{p}_\ell}}(\mathbb{A})$ and is a representation of $H^{\mathcal{O}_{\underline{p}_\ell}}(\mathbb{A})$ by the right translation.

To simplify the notation, we set $\psi_{\mathcal{O}_\ell} := \psi_{\mathcal{O}_{\underline{p}_\ell}}$, $H^{\mathcal{O}_\ell} := H^{\mathcal{O}_{\underline{p}_\ell}}$, and $\mathcal{F}^{\mathcal{O}_\ell}(\pi) := \mathcal{F}^{\mathcal{O}_{\underline{p}_\ell}}(\pi)$. In this case, the F-algebraic group $H^{\mathcal{O}_\ell}$ is the classical group $H_{\ell^-}^{\mathcal{O}_\ell} = \mathrm{Isom}(W^{\mathcal{O}_\ell}, q)^\circ$, where $(W^{\mathcal{O}_\ell}, q)$ is an $\mathfrak{l}^-$-dimensional non-degenerate subspace of (V, q) with the following properties:

- $\mathfrak{l}^- = \mathfrak{n} - 2\ell - 1$ and $\ell^- = [\frac{\mathfrak{l}^-}{2}]$,
- the product $G_n \times H_{\ell^-}^{\mathcal{O}_\ell}$ is relevant in the sense of the Gan-Gross-Prasad conjecture ([16]), and
- the product $G_n \times H_{\ell^-}^{\mathcal{O}_\ell}$ is a pure inner F-form of an F-quasisplit $G_n^* \times H_{\ell^-}^*$.

We refer to Section 2.4 for more a detailed discussion. One may extend the proof of [23, Th. 7.3] to the current case and prove the cuspidality of the maximal Bessel module of π.

PROPOSITION 2.2 (Cuspidality of Bessel Modules). *For any π belonging to $\mathcal{A}_{\mathrm{cusp}}(G_n)$ with a cuspidal realization $\mathcal{C}_\pi$, the ℓ-th Bessel module $\mathcal{F}^{\mathcal{O}_\ell}(\mathcal{C}_\pi)$ of $\mathcal{C}_\pi$ enjoys the following property: There exists an integer ℓ_0 in $\{0, 1, \ldots, \mathfrak{r}\}$, where $\mathfrak{r}$ is the F-rank of G_n, such that*

(1) *the ℓ_0-th Bessel module $\mathcal{F}^{\mathcal{O}_{\ell_0}}(\mathcal{C}_\pi)$ of $\mathcal{C}_\pi$ is non-zero, but for any $\ell \in \{0, 1, \ldots, \mathfrak{r}\}$ with $\ell > \ell_0$, the ℓ-th Bessel module $\mathcal{F}^{\mathcal{O}_\ell}(\mathcal{C}_\pi)$ is identically zero; and*

(2) *the ℓ_0-th Bessel module $\mathcal{F}^{\mathcal{O}_{\ell_0}}(\mathcal{C}_\pi)$ is cuspidal in the sense that its constant terms along all the parabolic subgroups of $H_{\ell_0^-}^{\mathcal{O}_{\ell_0}}$ are zero.*

We note that when the cuspidal multiplicity of π is two, the index ℓ_0 of π in Proposition 2.2 may depend on a particular realization $\mathcal{C}_\pi$ of π in the cuspidal spectrum $L_{\mathrm{cusp}}^2(G_n)$. Hence we write $\ell_0 = \ell_0(\mathcal{C}_\pi)$ to be a *first occurrence index* of π. Of course, if the cuspidal multiplicity of π is one, then π has the unique first occurrence index, which may be written as $\ell_0 = \ell_0(\pi)$.

By Proposition 2.2, for any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, the ℓ_0-th Bessel module $\mathcal{F}^{\mathcal{O}_{\ell_0}}(\pi)$, or more precisely, $\mathcal{F}^{\mathcal{O}_{\ell_0}}(\mathcal{C}_\pi)$, as a representation of $H_{\ell_0^-}^{\mathcal{O}_{\ell_0}}(\mathbb{A})$, is non-zero and can be embedded as a submodule in the cuspidal spectrum $L_{\mathrm{cusp}}^2(H_{\ell_0^-}^{\mathcal{O}_{\ell_0}})$, and hence can be written as the following Hilbert direct sum of irreducible cuspidal automorphic representations of $H_{\ell_0^-}^{\mathcal{O}_{\ell_0}}(\mathbb{A})$,

$$(2.10) \qquad \mathcal{F}^{\mathcal{O}_{\ell_0}}(\pi) = \sigma_1 \oplus \sigma_2 \oplus \cdots \oplus \sigma_t \oplus \cdots,$$

where all $\sigma_i \in \mathcal{A}_{\mathrm{cusp}}(H_{\ell_0^-}^{\mathcal{O}_{\ell_0}})$. By the uniqueness of local Bessel models for classical groups ([2], [77], [16] and [44]), it is easy to deduce that the decomposition (2.10) is multiplicity free. Furthermore, we have the following conjecture.

CONJECTURE 2.3 (Generic Summand). *Assume that $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ has a G_n-relevant, generic global Arthur parameter $\phi \in \widetilde{\Phi}_2(G_n^*)$. Then there exists*

a cuspidal realization $\mathcal{C}_\pi$ of π in $L^2_{\mathrm{cusp}}(G_n)$ with the first occurrence index $\ell_0 = \ell_0(\mathcal{C}_\pi)$, such that there exists an F-rational orbit $\mathcal{O}_{\ell_0} = \mathcal{O}_{\underline{p}_{\ell_0}}$ in the F-stable orbits $\mathcal{O}^{\mathrm{st}}_{\underline{p}_{\ell_0}}$ associated to the partition $\underline{p}_{\ell_0}$ with the

GENERIC SUMMAND PROPERTY: *There exists at least one σ in $\mathcal{A}_{\mathrm{cusp}}(H^{\mathcal{O}_{\ell_0}}_{\ell_0^-})$ with an $H^{\mathcal{O}_{\ell_0}}_{\ell_0^-}$-relevant, generic global Arthur parameter ϕ_σ in $\widetilde{\Phi}_2(H^*_{\ell_0^-})$, and with a cuspidal realization $\mathcal{C}_\sigma$ of σ in $L^2_{\mathrm{cusp}}(H^{\mathcal{O}_{\ell_0}}_{\ell_0^-})$, such that the L^2-inner product*

$$\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_0}}}(\varphi_\pi), \varphi_\sigma \right\rangle_{H^{\mathcal{O}_{\ell_0}}_{\ell_0^-}}$$

in the Hilbert space $L^2_{\mathrm{cusp}}(H^{\mathcal{O}_{\ell_0}}_{\ell_0^-})$ is non-zero for some $\varphi_\pi \in \mathcal{C}_\pi$ and $\varphi_\sigma \in \mathcal{C}_\sigma$.

It is clear that the Generic Summand Conjecture seeks a refined structure of the generalized branching law for automorphic representations with help of the endoscopic classification theory. We introduce such a property of invariant theoretic nature into the explicit construction of cuspidal automorphic modules. Some interesting examples of this nature are obtained through a simple relative trace formula approach by W. Zhang in [88]. In Section 7.3, we consider the situation that a cuspidal automorphic member π in $\widetilde{\Pi}_\phi(G_n)$ has the property that $\mathfrak{p}^m(\pi) = \{\underline{p}_{\mathrm{subr}}\}$, where $\underline{p}_{\mathrm{subr}}$ is the partition associated to the subregular nilpotent orbit. We prove in Proposition 7.3 that Conjecture 2.3 holds for this case. Further discussions on the Generic Summand Conjecture, its variants, and applications can be found in our work ([46] and [49]). In [47], we establish the local analogy of the Generic Summand Conjecture for orthogonal groups defined over p-adic local fields of characteristic zero.

2.4. *Rationality of $H^{\mathcal{O}_\ell}_{\ell^-}$.* We are going to make more explicit the parametrization of the F-rational orbits $\mathcal{O}_\ell$ in the F-stable orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$ for the family of partitions $\underline{p}_\ell$, which define the family of Bessel modules. This yields more explicit structure about the groups $H^{\mathcal{O}_\ell}_{\ell^-}$.

For the partition $\underline{p}_\ell = [(2\ell+1)1^{\mathfrak{n}-2\ell-1}]$ of type $(\mathfrak{n}, G_n^*)$, which is G_n-relevant, the unipotent subgroup $V_\ell = V_{\underline{p}_\ell}$ of G_n can be chosen to consist of all unipotent elements of the form

$$(2.11) \qquad V_\ell = \left\{ v = \begin{pmatrix} z & y & x \\ & I_{\mathfrak{n}-2\ell} & y' \\ & & z^* \end{pmatrix} \in G_n \mid z \in Z_\ell \right\},$$

where Z_ℓ is the standard maximal (upper-triangular) unipotent subgroup of $G_{E/F}(\ell)$. It follows that the F-rational nilpotent orbits $\mathcal{O}_\ell$ in the F-stable

nilpotent orbit $\mathcal{O}_{\underline{p}_\ell}^{\mathrm{st}}$ are in one-to-one correspondence with the $G_{E/F}(1) \times G_{n-\ell}$-orbits of F-anisotropic vectors in $(E^{\mathfrak{n}-2\ell}, q)$, viewed as a subspace of (V, q). Hence the generic character $\psi_{\mathcal{O}_\ell}$ of $V_\ell(\mathbb{A})$ may also be explicitly defined as follows. Fix a nontrivial character ψ_F of $F\backslash\mathbb{A}$, and define a character ψ_E of $E\backslash\mathbb{A}_E$ by

$$\psi_E(x) := \begin{cases} \psi_F(x) & \text{if } E = F, \\ \psi_F(\frac{1}{2}\mathrm{tr}_{E/F}(\frac{x}{\sqrt{\varsigma}})) & \text{if } E = F(\sqrt{\varsigma}). \end{cases}$$

Consider the following identification:

$$V_\ell/[V_\ell, V_\ell] \cong \oplus_{i=1}^{\ell-1}\mathfrak{g}_{\alpha_i} \oplus E^{\mathfrak{n}-2\ell}.$$

Let w_0 be an anisotropic vector in $(E^{\mathfrak{n}-2\ell}, q)$, and define a character ψ_{ℓ,w_0} of $V_\ell(\mathbb{A}_F)$ by

$$(2.12) \qquad \psi_{\mathcal{O}_\ell}(v) = \psi_{\ell,w_0}(v) := \psi_E(\sum_{i=1}^{\ell-1} z_{i,i+1} + q(y_\ell, w_0)),$$

where y_ℓ is the last row of y as defined in (2.11). The Levi subgroup of $\mathcal{P}_{\{\ell+1,\ldots,\mathfrak{r}\}}$ normalizes the unipotent subgroup V_ℓ and acts on the set of such defined characters ψ_{ℓ,w_0}. The group $H_{\ell^-}^{\mathcal{O}_\ell} = H_{\ell^-}^{w_0}$ is the identity connected component of the stabilizer of ψ_{ℓ,w_0}, which is given by

$$(2.13) \qquad \left\{ \begin{pmatrix} I_\ell & & \\ & \gamma & \\ & & I_\ell \end{pmatrix} \in G_n \mid \gamma J_{\mathfrak{n}-2\ell}w_0 = J_{\mathfrak{n}-2\ell}w_0 \right\},$$

where $\ell^- = [\frac{\mathfrak{l}^-}{2}]$ with $\mathfrak{l}^- := \mathfrak{n} - 2\ell - 1$. As introduced in Section 2.1, we may write $V_{(\ell)} = E^{\mathfrak{n}-2\ell}$ and view $(V_{(\ell)}, q)$ as a non-degenerate subspace of (V, q) under the natural embedding. Hence the group $H_{\ell^-}^{\mathcal{O}_\ell} = H_{\ell^-}^{w_0}$ can also be identified as $\mathrm{Isom}(V_{(\ell)} \cap w_0^\perp, q)^\circ$. Write $W_{\mathfrak{l}^-} := V_{(\ell)} \cap w_0^\perp$ so that $(W_{\mathfrak{l}^-}, q)$ is an $\mathfrak{l}^-$-dimensional non-degenerate subspace of (V, q). It follows that the dimension $\mathfrak{d}_0^-$ of its anisotropic kernel of the space $(W_{\mathfrak{l}^-}, q)$ is $\mathfrak{d}_0 \pm 1$, depending on the choice of w_0. Note that $(W_{\mathfrak{l}^-}, q)$ is isometric to $(W^{\mathcal{O}_\ell}, q)$ as introduced in Section 2.3. Define $\mathfrak{r}^-$ to be the Witt index of $(W_{\mathfrak{l}^-}, q)$, which equals $\mathfrak{r} - \ell$ or $\mathfrak{r} - \ell - 1$, depending on $\mathfrak{d}_0^- = \mathfrak{d}_0 - 1$ or $\mathfrak{d}_0^- = \mathfrak{d}_0 + 1$, respectively.

For further explicit calculations, we may take the representative w_0 of the F-anisotropic vectors corresponding to the F-rational nilpotent orbits $\mathcal{O}_\ell$ in the F-stable nilpotent orbit $\mathcal{O}_{\underline{p}_\ell}^{\mathrm{st}}$ as follows. The representative w_0 is an F-anisotropic vector in the space $(E^{\mathfrak{n}-2\ell}, q)$, which defines the character ψ_{ℓ,w_0}. Under the action of the product $G_{E/F}(1) \times G_{n-\ell}$ on the space $(E^{\mathfrak{n}-2\ell}, q)$, in particular, on the set of F-anisotropic vectors w_0, if $\ell < \mathfrak{r}$, we may choose

$$(2.14) \qquad w_0 = y_\kappa = e_{\mathfrak{r}} + (-1)^{\mathfrak{n}+1}\frac{\kappa}{2}e_{-\mathfrak{r}}$$

for some $\kappa \in F^\times$, using the following lemma.

LEMMA 2.4. *If the Witt index of $(V_{(\ell)}, q)$ is not zero, i.e., if $\ell < \mathfrak{r}$, then there exists an element g in $G_{\mathfrak{n}-2\ell}(F) = \mathrm{Isom}(V_{(\ell)}, q)^{\circ}(F)$ such that*

$$g \cdot w_0 = e_{\mathfrak{r}} + (-1)^{\mathfrak{n}+1} \frac{\kappa}{2} e_{-\mathfrak{r}}$$

for some $\kappa \in F^{\times}$.

Proof. The proof is straightforward. We omit the details here. $\square$

It is clear that if $\ell = \mathfrak{r}$, then the subspace $(E^{\mathfrak{n}-2\mathfrak{r}}, q)$ is F-anisotropic and hence is not sensitive to the choice of the F-anisotropic vector w_0. The structure of $H_{\ell^-}^{\mathcal{O}_\ell}$ is summarized in the following proposition.

PROPOSITION 2.5. *For the partition $\underline{p}_\ell$, let $\mathcal{O}_\ell$ in $\mathcal{O}_{\underline{p}_\ell}^{\mathrm{st}}$ be determined by the F-anisotropic vector w_0 as in Lemma 2.4. Then the classical group $H_{\ell^-}^{\mathcal{O}_\ell} = H_{\ell^-}^{w_0}$ is defined by an $\mathfrak{l}^-$-dimensional non-degenerate subspace $(W_{\mathfrak{l}^-}, q)$ of (V, q) with a $(\mathfrak{d}_0 - 1)$-dimensional F-anisotropic kernel if $y_{-\kappa}$ belongs to the $G_{E/F}(1) \times G_{n-\ell}$-orbit of a non-zero vector in the F-anisotropic kernel (V_0, q) of (V, q); it is defined by an $\mathfrak{l}^-$-dimensional non-degenerate subspace $(W_{\mathfrak{l}^-}, q)$ of (V, q) with a $(\mathfrak{d}_0 + 1)$-dimensional F-anisotropic kernel if $y_{-\kappa}$ does not belong to the $G_{E/F}(1) \times G_{n-\ell}$-orbit of any non-zero vector in the F-anisotropic kernel (V_0, q).*

Following the explicit discussions on pure inner forms of F-quasisplit classical groups in [16], it is easy to obtain the following proposition.

PROPOSITION 2.6. *Let H_m^* be an F-quasisplit classical group as introduced in Section 2.1. For any pure inner F-form H_m of H_m^*, there exist*
- *a classical group G_n defined over F that is a pure inner form of an F-quasi-split classical group G_n^*; and*
- *a datum $(\underline{p}_\ell, \mathcal{O}_\ell)$ for the Fourier coefficients for automorphic forms on $G_n(\mathbb{A})$*

such that $m = \ell^-$ and $H_m \cong H_{\ell^-}^{\mathcal{O}_\ell}$. Moreover, the product $G_n \times H_m$ is a relevant pure inner form of the F-quasisplit $G_n^* \times H_m^*$ in the sense of the Gan-Gross-Prasad conjecture.

We will recall the Gan-Gross-Prasad conjecture and related notions in Section 3.

3. The local Gan-Gross-Prasad conjecture

We recall the local Gan-Gross-Prasad conjecture from [16] for the cases considered in this paper. The version of the local Gan-Gross-Prasad conjecture, which will be stated as Conjecture 3.1, was proved by Waldspurger and by Mœglin and Waldspurger in a series of papers (see [82] and [67], for instance) for orthogonal groups over p-adic local fields. Over archimedean local fields, it is proved by Z. Luo for tempered local L-parameters in [59], but the case

of general generic local L-parameters is still in progress. For unitary groups, Beuzart-Plessis ([9] and [8]) proves the conjecture (Conjecture 3.1) for tempered local L-parameters over all local fields, and in [28], H. He proves the conjecture for discrete representations over $\mathbb{R}$ via a different approach. The extension to the generic local L-parameters was obtained by Gan and Ichino ([18]) for p-adic local fields, but over archimedean local fields, such an extension remains an open problem, as far as the authors knew. In the proof of the main conjecture (Conjecture 6.7), we need the local Gan-Gross-Prasad conjecture (as in Conjecture 3.1) for generic local parameters at all local places as an input. In the process towards the proof of Conjecture 6.7, we are able to prove the global Gan-Gross-Prasad conjecture (with one direction having an extra assumption). This will be explained in Sections 5.5 and 6.3.

3.1. *Generic Arthur parameters.* We consider generic local Arthur parameters for the classical groups considered in this paper. This has been extensively discussed in [16] and in [67]. We recall the basics for the case of orthogonal groups, and we refer to [18] for the case of unitary groups. Let $G_n^* = \mathrm{SO}(V^*, q^*)$ be the special orthogonal group defined by a non-degenerate, $\mathfrak{n}$-dimensional quadratic space (V^*, q^*) with $n = [\frac{\mathfrak{n}}{2}]$, which is F-quasisplit. We recall that the generic global Arthur parameters for G_n^* are of the form

$$(3.1) \qquad \phi = (\tau_1, 1) \boxplus \cdots \boxplus (\tau_r, 1)$$

as in (1.2), where $\tau_1, \ldots, \tau_r$ are irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{a_1}(\mathbb{A}), \ldots, \mathrm{GL}_{a_r}(\mathbb{A})$, respectively, with required constraints to make ϕ a global Arthur parameter of G_n^*. As before, the set of generic global Arthur parameters of G_n^* is denoted by $\widetilde{\Phi}_2(G_n^*)$. It is known that the global Arthur packet $\widetilde{\Pi}_\phi(G_n^*)$ associated to a generic global Arthur parameter ϕ contains a generic member. We refer to [37, Th. 3.3] for details.

With the assumption of the Ramanujan conjecture for general linear groups, at each local place ν of F, the localization ϕ_ν of the generic global Arthur parameter ϕ must be a tempered local L-parameter for $G_n^*(F_\nu)$. Hence with possible failure of the Ramanujan conjecture for general linear groups, one has to figure out the possible structure of the localization ϕ_ν of the generic global Arthur parameter ϕ. We recall a work of Mœglin and Waldspurger ([67]) for special orthogonal groups and refer to [18] for the unitary group case.

For each local place ν of F, we denote by $\mathcal{W}_{F_\nu}$ the local Weil group of F_ν. The local Langlands group of F_ν, which is denoted by $\mathcal{L}_{F_\nu}$, is equal to the local Weil-Deligne group. Hence the local Langlands group $\mathcal{L}_{F_\nu}$, as usual, may be taken to be $\mathcal{W}_{F_\nu} \times \mathrm{SL}_2(\mathbb{C})$ or equivalently $\mathcal{W}_{F_\nu} \times \mathrm{SU}(2)$ if ν is a finite local place, and to be the local Weil group $\mathcal{W}_{F_\nu}$ if ν is an archimedean local place.

The local L-parameters for $G_n^*(F_\nu)$ are of the form

$$(3.2) \qquad \phi_\nu \ : \ \mathcal{L}_{F_\nu} \to {}^L G_n^*$$

with the property that the restriction of ϕ_ν to the local Weil group $\mathcal{W}_{F_\nu}$ is Frobenius semi-simple and trivial on an open subgroup of the inertia group $\mathcal{I}_{F_\nu}$ of F_ν, and the restriction to $\mathrm{SL}_2(\mathbb{C})$ is algebraic. By the local Langlands conjecture for general linear groups ([55], [30], [27], and [71]), the localization ϕ_ν at a local place ν of F of a generic global Arthur parameter ϕ is a local L-parameter, for which there exists a datum $(L_\nu^*, \phi_\nu^{L^*}, \underline{\beta})$ with the following properties:

(1) L_ν^* is a Levi subgroup of $G^*(F_\nu)$ of the form

$$L_\nu^* = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_t} \times G_{n_0}^*,$$

where $\mathrm{GL}_{n_1}, \ldots, \mathrm{GL}_{n_t}$ and $G_{n_0}^*$ depend on the local place ν;

(2) $\phi^{L_\nu^*}$ is a local L-parameter of L^* given by

$$\phi^{L_\nu^*} := \phi_1 \oplus \cdots \oplus \phi_t \oplus \phi_0 \ : \ \mathcal{L}_{F_\nu} \to {}^L L^*,$$

where ϕ_j is a local tempered L-parameter of GL_{n_j} for $j = 1, 2, \ldots, t$, and ϕ_0 is a local tempered L-parameter of $G_{n_0}^*$, with dependence on the local place ν;

(3) $\underline{\beta} := (\beta_1, \ldots, \beta_t) \in \mathbb{R}^t$, such that $\beta_1 > \beta_2 > \cdots > \beta_t > 0$, which is also dependent of the local place ν.

With the given datum, following [67], which is expressed in terms of the parabolic induction, one can write

$$\phi_\nu = (\phi_1 \otimes |\cdot|_\nu^{\beta_1} \oplus \phi_1^\vee \otimes |\cdot|_\nu^{-\beta_1}) \oplus \cdots \oplus (\phi_t \otimes |\cdot|_\nu^{\beta_t} \oplus \phi_t^\vee \otimes |\cdot|_\nu^{-\beta_t}) \oplus \phi_0.$$

Following [3] and also [67], the local L-packets can be formed for all local L-parameters ϕ_ν as displayed above, and they are denoted by $\widetilde{\Pi}_{\phi_\nu}(G_n^*)$. A local L-parameter ϕ_ν is called *generic* if the associated local L-packet $\widetilde{\Pi}_{\phi_\nu}(G_n^*)$ contains a generic member, i.e., a member with a non-zero Whittaker model with respect to a certain Whittaker data for G_n^*. Using the notation of [3], the set of all generic local L-parameters is denoted by $\widetilde{\Phi}_{\mathrm{unit}}^+(G_n^*(F_\nu))$. All the members in any generic local L-packet are irreducible and unitary. It is clear that the localization ϕ_ν of a generic global Arthur parameter ϕ is a generic local L-parameter according the definition in [67] since there exists a generic member in the local L-packet $\widetilde{\Pi}_{\phi_\nu}(G_n^*)$. Hence, following [67], the local Gan-Gross-Prasad conjecture can be formulated for the localization ϕ_ν of all generic global Arthur parameters ϕ in $\widetilde{\Phi}_2(G_n^*)$, which will be discussed in the following section.

3.2. *The local Gan-Gross-Prasad conjecture.* We are going to recall the local Gan-Gross-Prasad conjecture that was explicitly formulated in [16] for general classical groups. We discuss the case of orthogonal groups with details and refer the unitary group case to [16] and [9], [8], and [18] for the details.

Assume that in this section F is a local field of characteristic zero. Recall that an F-quasisplit special orthogonal group $G_n^* = \mathrm{SO}(V^*, q^*)$ and its pure inner F-forms $G_n = \mathrm{SO}(V, q)$ share the same L-group ${}^L G_n^*$. As explained in [16, §7], if the dimension $\mathfrak{n} = \dim V = \dim V^*$ is odd, one may take $\mathrm{Sp}_{\mathfrak{n}-1}(\mathbb{C})$ to be the L-group ${}^L G_n^*$, and if the dimension $\mathfrak{n} = \dim V = \dim V^*$ is even, one may take $\mathrm{O}_{\mathfrak{n}}(\mathbb{C})$ to be ${}^L G_n^*$ when $\mathrm{disc}(V^*)$ is not a square in $F^\times$ and take $\mathrm{SO}_{\mathfrak{n}}(\mathbb{C})$ to be ${}^L G_n^*$ when $\mathrm{disc}(V^*)$ is a square in $F^\times$.

For a relevant pair $G_n = \mathrm{SO}(V, q)$ and $H_m = \mathrm{SO}(W, q)$, and for an F-quasisplit relevant pair $G_n^* = \mathrm{SO}(V^*, q^*)$ and $H_m^* = \mathrm{SO}(W^*, q^*)$ as recalled in Section 2.1 from [16], we are going to discuss the local Langlands parameters for the group $G_n^* \times H_m^*$ and its relevant pure inner F-form $G_n \times H_m$. As in Section 3.1, we use $\mathcal{L}_F$ to denote the local Langlands group associated to F. We only consider the local Langlands parameters that satisfy the three properties in Section 3.1:

$$(3.3) \qquad \phi \; : \; \mathcal{L}_F \to {}^L G_n^* \times {}^L H_m^*.$$

Hence they are the localization of the generic global Arthur parameters for the product of the F-quasisplit relevant pair G_n^* and H_m^*. The set of such local Langlands parameters is denoted by $\widetilde{\Phi}_{\mathrm{unit}}^+(G_n^* \times H_m^*)$. As in Section 3.1, each local L-parameter ϕ in $\widetilde{\Phi}_{\mathrm{unit}}^+(G_n^* \times H_m^*)$ defines a local L-packet $\widetilde{\Pi}_\phi(G_n^* \times H_m^*)$. For any relevant pure inner F-form $G_n \times H_m$, if a parameter $\phi \in \widetilde{\Phi}_{\mathrm{unit}}^+(G_n^* \times H_m^*)$ is $G_n \times H_m$-relevant, it defines a local L-packet $\widetilde{\Pi}_\phi(G_n \times H_m)$, as in Section 3.1, following [3] and [67]. If a parameter $\phi \in \widetilde{\Phi}_{\mathrm{unit}}^+(G_n^* \times H_m^*)$ is not $G_n \times H_m$-relevant, the corresponding local L-packet $\widetilde{\Pi}_\phi(G_n \times H_m)$ is defined to be the empty set. The local Vogan packet for a local Langlands parameter ϕ belonging to $\widetilde{\Phi}_{\mathrm{unit}}^+(G_n^* \times H_m^*)$ is defined to be the union of the local L-packets $\widetilde{\Pi}_\phi(G_n \times H_m)$ over all pure inner F-forms $G_n \times H_m$ of the F-quasisplit group $G_n^* \times H_m^*$, and it is denoted by

$$(3.4) \qquad \widetilde{\Pi}_\phi[G_n^* \times H_m^*].$$

In order to state the local Gan-Gross-Prasad conjecture and relevant progress, we have to introduce the local analogue of the Fourier coefficients as introduced in Section 2.3, which is usually called the local Bessel models. For a given relevant pair (G_n, H_m), take a partition of the form $\underline{p}_\ell = [(2\ell+1)1^{\mathfrak{n}-2\ell+1}]$, where $2\ell + 1 = \dim W^\perp = \mathfrak{n} - \mathfrak{m}$. The F-stable nilpotent orbit $\mathcal{O}_{\underline{p}_\ell}^{\mathrm{st}}$ corresponding to the partition $\underline{p}_\ell$ defines a unipotent subgroup $V_{\underline{p}_\ell}$ and a generic character $\psi_{\mathcal{O}_\ell}$ associated to any F-rational orbit $\mathcal{O}_\ell$ in the F-stable orbit $\mathcal{O}_{\underline{p}_\ell}^{\mathrm{st}}$. According

to the discussion in Section 2.4, there is an F-rational orbit $\mathcal{O}_\ell$ in the F-stable orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$, such that the subgroup $H_m = H^{\mathcal{O}_\ell}_{\ell^-}$ normalizes the unipotent subgroup $V_{\underline{p}_\ell}$ and stabilizes the character $\psi_{\mathcal{O}_\ell}$. We define the following subgroup of G_n:

$$R_{\mathcal{O}_\ell} := H_m \ltimes V_{\underline{p}_\ell} = H^{\mathcal{O}_\ell}_{\ell^-} \ltimes V_{\underline{p}_\ell}.$$

Let π be an irreducible admissible representation of $G_n(F)$ and σ be an irreducible admissible representation of $H_m(F)$. The local functionals we considered belong to the following Hom-space:

$$(3.5) \qquad \mathrm{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma, \psi_{\mathcal{O}_\ell}).$$

This is usually called the space of local Bessel functionals. The uniqueness of local Bessel functionals asserts that

$$\dim \mathrm{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma, \psi_{\mathcal{O}_\ell}) \leq 1.$$

This was proved in [2], [77], [16], and [44]. The stronger version in terms of local Vogan packets for more general classical groups is given as follows, which will be called as the *local Gan-Gross-Prasad conjecture* in the rest of the paper.

CONJECTURE 3.1. *Let G_n^* and H_m^* be a relevant pair of F-quasisplit classical groups. For a given local L-parameter ϕ in $\widetilde{\Phi}^+_{\mathrm{unit}}(G_n^* \times H_m^*)$, the following identity holds*:

$$(3.6) \qquad \sum_{\pi \otimes \sigma \in \widetilde{\Pi}_\phi[G_n^* \times H_m^*]} \dim \mathrm{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma, \psi_{\mathcal{O}_\ell}) = 1.$$

The known cases of Conjecture 3.1 can be summarized as follows.

THEOREM 3.2. *Conjecture 3.1 holds for the following cases*:

(1) *the relevant orthogonal group pair G_n^* and H_m^* over a p-adic local field F, by Mœglin and Waldspurger in [67] for generic local L-parameters*;
(2) *the relevant orthogonal group pair G_n^* and H_m^* over an archimedean local field F, by Zhilin Luo in his Ph.D. thesis [59], for tempered local L-parameters*;
(3) *the relevant unitary group pair G_n^* and H_m^* over a p-adic local field F or over the real number field $\mathbb{R}$, by Beuzart-Plessis in [9] and [8] for tempered local L-parameters; and*
(4) *the relevant unitary group pair G_n^* and H_m^* over a p-adic local field F, extended by Gan and Ichino in [18] to generic local L-parameters*.

We remark that over the real number field $\mathbb{R}$, H. He proves in [28] the local Gan-Gross-Prasad conjecture for discrete series representations of unitary groups via a different approach.

4. **Bessel periods and global zeta integrals**

The automorphic analog of the local Bessel functionals is the notion of Bessel periods for a pair of cuspidal automorphic forms or representations. When one of cuspidal automorphic forms is replaced by a certain Eisenstein series, the Bessel periods become the global zeta integrals that represent the tensor product L-functions. We extend such a construction of the global zeta integrals considered in our previous work ([45]) to a more general setting that is needed for the main results of this paper. For quasi-split orthogonal groups, a special family of the global zeta integrals was first investigated by Ginzburg, Piatetski-Shapiro and Rallis in [19].

4.1. *Global zeta integrals.* The global zeta integrals that we are going to study are defined for the following three families of classical groups:

(1) $G_b = \mathrm{SO}_{2b+1}(V, q)$ and $H_c = \mathrm{SO}_{2c}(W, q)$, such that the product $G_b \times H_c$ is a relevant pure inner form of an F-quasisplit $G_b^* \times H_c^*$ over F;
(2) $G_b = \mathrm{SO}_{2b}(V, q)$ and $H_c = \mathrm{SO}_{2c+1}(W, q)$, such that the product $G_b \times H_c$ is a relevant pure inner form of an F-quasisplit $G_b^* \times H_c^*$ over F;
(3) $G_b = \mathrm{U}_\mathfrak{b}(V, q)$ and $H_c = \mathrm{U}_\mathfrak{c}(W, q)$ with $\mathfrak{b}$ and $\mathfrak{c}$ being of different parity and $b = [\frac{\mathfrak{b}}{2}]$ and $c = [\frac{\mathfrak{c}}{2}]$, such that the product $G_b \times H_c$ is a relevant pure inner form of an F-quasisplit $G_b^* \times H_c^*$.

In the following, we use the notation that $G_\star = \mathrm{Isom}(V, q)^\circ$ and $H_\square = \mathrm{Isom}(W, q)^\circ$, such that $G_\star \times H_\square$ is a relevant pure inner form of an F-quasisplit $G_\star^* \times H_\square^*$. Because the global zeta integrals considered in this paper extend what were studied for F-quasisplit groups by the authors in [45], which generalizes the work of Ginzburg, Piatetski-Shapiro and Rallis in [19], we will try to follow the arguments and proofs used in [45] and provide explanation only for the steps that are necessary for understanding of the main results of this section.

In order to formulate the families of global zeta integrals, we take τ to be an irreducible unitary automorphic representation of $G_{E/F}(a)(\mathbb{A}_F)$ of the following isobaric type,

$$(4.1) \qquad \tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r,$$

where $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(G_{E/F}(a_i))$, $\sum_{i=1}^r a_i = a$, and $\tau_i \not\cong \tau_j$ if $i \neq j$.

We note that as in [45], the global unfolding of the global zeta integrals and the unramified calculation for the unramified local zeta integrals in this section only require the assumption that τ is a generic isobaric automorphic representation, which means that some τ_i and τ_j could be equivalent.

Take $H_m = \mathrm{Isom}(W, q)^\circ$ with $\dim W = \mathfrak{m}$ and $m = [\frac{\mathfrak{m}}{2}]$. Let σ be an irreducible automorphic representation of $H_m(\mathbb{A})$. Note that we need not assume the cuspidality of σ in this section.

Let $M_{\hat{a}} = G_{E/F}(a) \times H_m$ be an F-Levi subgroup of H_{a+m}, so that $P_{\hat{a}} = M_{\hat{a}} U_{\hat{a}}$ is a standard parabolic F-subgroup of H_{a+m}. Following Section I.1.4 in [66], we denote $X_{M_{\hat{a}}}$ to be the group of continuous homomorphisms of $M_{\hat{a}}(\mathbb{A})$ into $\mathbb{C}^\times$ that are trivial on $M_{\hat{a}}^1 := \cap_{\chi \in \mathrm{Hom}(M_{\hat{a}}, \mathbb{G}_m)} \ker |\chi|$. Since the parabolic subgroup $P_{\hat{a}}$ is maximal, the $\mathbb{C}$-vector space $X_{M_{\hat{a}}}$ is one-dimensional. As in Section 2.2 in [45], for any $s \in \mathbb{C}$, we define $\lambda_s(m, h) := |\det m|_{\mathbb{A}_E}^s$ for $(m, h) \in G_{E/F}(a)(\mathbb{A}) \times H_m(\mathbb{A})$. It is clear that $\lambda_s \in X_{M_{\hat{a}}}$. Via the Iwasawa decomposition, we may make the trivial extension of λ_s to be a function on $H_{a+m}(\mathbb{A})$.

For any

(4.2) $$\phi = \phi_{\tau \otimes \sigma} \in \mathcal{A}(U_{\hat{a}}(\mathbb{A}) M_{\hat{a}}(F) \backslash H_{a+m}(\mathbb{A}))_{\tau \otimes \sigma},$$

we may set $\phi_s := \lambda_s \cdot \phi$ and form the associated Eisenstein series

(4.3) $$E(h, \phi, s) = E(h, \phi_{\tau \otimes \sigma}, s) = \sum_{\delta \in P_{\hat{a}}(F) \backslash H_{a+m}(F)} \phi_s(\delta g).$$

Note that the character λ_s is normalized as in [72]. The theory of Langlands on Eisenstein series ([57] and [66]) shows that $E(h, \phi, s)$ converges absolutely for $\mathrm{Re}(s)$ large, has meromorphic continuation to the complex plane $\mathbb{C}$, and defines an automorphic form on $H_{a+m}(\mathbb{A})$ when s is not a pole.

Take a family of H_{a+m}-relevant partitions $\underline{p}_\ell = [(2\ell+1)1^{\mathfrak{m}+2a-2\ell-1}]$ of type $(\mathfrak{m}+2a, H_{a+m}^*)$, with $\ell \le a + \mathfrak{r}_{\mathfrak{m}}$, where $\mathfrak{r}_{\mathfrak{m}} := \mathfrak{r}(H_m)$ is the F-rank of H_m and is the Witt index of $\mathfrak{m}$-dimensional non-degenerate space (W, q) that defines H_m. We define the Bessel-Fourier coefficient of the Eisenstein series $E(h, \phi, s)$ on $H_{a+m}(\mathbb{A})$:

(4.4) $$\mathcal{F}^{\psi_\ell, w_0}(E(\cdot, \phi, s))(h) := \int_{N_\ell(F) \backslash N_\ell(\mathbb{A})} E(nh, \phi, s) \psi_{\ell, w_0}^{-1}(n) \, \mathrm{d}n,$$

where the unipotent subgroup N_ℓ of H_{a+m} determined by the partition $\underline{p}_\ell$ is similar to the unipotent subgroup V_ℓ of G_n considered in Section 2.3. We use N_ℓ in this section in order to match the notation used in [45], since we have to recall from there some technical computations of the global zeta integrals.

As in Lemma 2.4, one may choose the representative w_0 that defines the character ψ_{ℓ, w_0} for Fourier coefficient, as in (2.14):

(4.5) $$w_0 = y_\kappa = e_{a+\mathfrak{r}_{\mathfrak{m}}} + (-1)^{\mathfrak{m}+1} \frac{\kappa}{2} e_{-(a+\mathfrak{r}_{\mathfrak{m}})}$$

for some $\kappa \in F^\times$. Following Proposition 2.6, we have that $m^- = [\frac{\mathfrak{m}^-}{2}]$ and $\mathfrak{m}^- := 2a + \mathfrak{m} - 2\ell - 1$ and $\ell < a + \mathfrak{r}_{\mathfrak{m}}$, and that the stabilizer $G_{m^-}^{w_0}$ and the subgroup H_m form a relevant pair in the sense of the Gan-Gross-Prasad conjecture (see Section 3). Of course, when $\ell = a + \mathfrak{r}_{\mathfrak{m}}$, the representative w_0 is any F-anisotropic vector in the F-anisotropic kernel (W_0, q) of (W, q). It is

clear that the pair $(G_{m^-}^{w_0}, H_m)$ is relevant in this case. Note that in this case, we must have

$$\mathfrak{m}^- = 2a + \mathfrak{m} - 2\ell - 1 = \mathfrak{d}_0(W) - 1,$$

and the group $G_{m^-}^{w_0}$ is F-anisotropic.

As in [45] we define the following semi-direct product of subgroups:

$$(4.6) \qquad\qquad R_\ell^{w_0} := G_{m^-}^{w_0} \ltimes N_\ell.$$

For an automorphic form $\varphi_{2a+\mathfrak{m}}$ on $H_{a+m}(\mathbb{A})$, and an automorphic form $\varphi_{\mathfrak{m}^-}$ on $G_{m^-}^{w_0}(\mathbb{A})$, we define the Bessel period by

$$(4.7) \quad \mathcal{P}^{\psi_\ell, w_0}(\varphi_{2a+\mathfrak{m}}, \varphi_{\mathfrak{m}^-}) := \int_{G_{m^-}^{w_0}(F)\backslash G_{m^-}^{w_0}(\mathbb{A})} \mathcal{F}^{\psi_\ell, w_0}(\varphi_{2a+\mathfrak{m}})(g)\varphi_{\mathfrak{m}^-}(g)\,\mathrm{d}g.$$

It is absolutely convergent if one of the automorphic forms $\varphi_{2a+\mathfrak{m}}$ and $\varphi_{\mathfrak{m}^-}$ is cuspidal. In fact, following [66, Ch. 2], the Fourier coefficients of an automorphic form with moderate growth is still of moderate growth on the stabilizer of the character that defines the Fourier coefficients. Moreover, if an automorphic form is rapidly decreasing on a Siegel set, then its Bessel-Fourier coefficient is also rapidly decreasing on a Siegel set of the stabilizer of the character that defines the Bessel-Fourier coefficient. We refer to [23, Lemma 10.1], [6, Lemma 2.1] and [7] (and also [45, Prop. 2.1]) for details. Otherwise, some regularization may be needed to define this integral in (4.7). We define the L^2-inner product of automorphic functions φ_1 and φ_2 over $G_{m^-}^{w_0}(\mathbb{A})$ by

$$(4.8) \qquad\qquad \langle \varphi_1, \varphi_2 \rangle_{G_{m^-}^{w_0}} := \int_{G_{m^-}^{w_0}(F)\backslash G_{m^-}^{w_0}(\mathbb{A})} \varphi_1(x)\overline{\varphi}_2(x)\,\mathrm{d}x$$

assuming it converges, where $\overline{\varphi}(x) = \overline{\varphi(x)}$ defines the complex conjugation of the function φ.

For $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_{m^-}^{w_0})$, the *global zeta integral* $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell, w_0})$, as in (2.17) of [45], is defined by the following Bessel period,

$$(4.9) \qquad\qquad \mathcal{Z}(s, \varphi_\pi, \phi_{\tau\otimes\sigma}, \psi_{\ell, w_0}) := \mathcal{P}^{\psi_\ell, w_0}(E(\phi_{\tau\otimes\sigma}, s), \varphi_\pi),$$

where the Bessel period is written in our convention in this paper by means of the L^2-inner product as follows:

$$\mathcal{P}^{\psi_\ell, w_0}(E(\phi_{\tau\otimes\sigma}, s), \varphi_\pi) = \left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_\ell, w_0}(E(\phi_{\tau\otimes\sigma}, s))} \right\rangle_{G_{m^-}^{w_0}}.$$

As given in Proposition 2.1 of [45], $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell, w_0})$ converges absolutely and hence is holomorphic at s where the Eisenstein series $E(h, \phi, s)$ has no poles.

From the theory of Eisenstein series and induced representations, it is clear that the Eisenstein series is an automorphic realization of the following

induced representation:

$$(4.10) \qquad \mathrm{I}_s(\tau, \sigma) := \mathrm{Ind}_{P_{\hat{a}}(\mathbb{A})}^{H_{a+m}(\mathbb{A})}(\tau |\det|^s \otimes \sigma).$$

The global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ when s is away from the pole of $E(\phi_{\tau \otimes \sigma}, s)$ defines a global Bessel functional ℓ^{aut} belonging to the following Hom-space,

$$(4.11) \qquad \mathrm{Hom}_{R_\ell^{w_0}(\mathbb{A})}(\mathrm{I}_s(\tau, \sigma), \pi^\vee \otimes \psi_{\ell, w_0}),$$

which has the following restricted tensor product decomposition:

$$(4.12) \qquad \otimes_\nu \mathrm{Hom}_{R_\ell^{w_0}(F_\nu)}(\mathrm{I}_s(\tau_\nu, \sigma_\nu), \pi_\nu^\vee \otimes \psi_{\ell, w_0, v}).$$

By the local uniqueness of the Bessel models as proved in [2], [77], [16], and [44], this Hom-space has dimension at most one, when $\mathrm{I}_s(\tau_\nu, \sigma_\nu)$ is irreducible. Hence we expect that this global functional ℓ^{aut} in (4.11) can be written as an Euler product of the local Bessel functionals

$$(4.13) \qquad \ell^{\mathrm{aut}} = c \cdot \prod_\nu \ell_\nu$$

with certain normalization on ℓ_ν when the data are unramified. The global unfolding process (or the global calculation) of the global zeta integral

$$\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}),$$

when $\mathrm{Re}(s)$ is large, is to find explicitly the Euler product factorization in (4.13) and an explicit formula for the local Bessel functionals ℓ_ν. This global calculation contains two main steps. The first is to calculate the Fourier coefficient of the Eisenstein series $\mathcal{F}^{\psi_{\ell, w_0}}(E(\phi_{\tau \otimes \sigma}, s))$ and by using the cuspidality of π to show that $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ is equal to an integration associated to the Zariski open dense double coset from $P_{\hat{a}} \backslash H_{a+m} / R_\ell^{w_0}$. The second step is to show that this remaining integration is in fact an Euler product of local zeta integrals.

4.2. *Fourier coefficients of Eisenstein series.* The goal of this subsection is to calculate the Fourier coefficient of the Eisenstein series as defined in (4.4). Since the calculation is very similar to that in the proof of [45, Prop. 3.3], we will not repeat every detail from there, but point out the key steps in the proof.

For $\mathrm{Re}(s)$ sufficiently large, we are able to unfold the Eisenstein series in the integral defining the Fourier coefficient $\mathcal{F}^{\psi_{\ell, w_0}}(E(\phi_{\tau \otimes \sigma}, s))$. This leads to the calculation of the double coset decomposition $P_{\hat{a}} \backslash H_{a+m} / R_\ell^{w_0}$. First, we consider the generalized Bruhat decomposition $P_{\hat{a}} \backslash H_{a+m} / P_{\hat{\ell}}$, as a preliminary step towards the calculation. This decomposition corresponds to the double coset decomposition $W_{\hat{a}} \backslash W_{F\Delta} / W_{\hat{\ell}}$. Here $W_{F\Delta}$ is the Weyl group of H_{a+m} relative to F, which is generated by the simple reflections s_α associated to the

roots $\alpha \in {}_F\Delta$. Similarly, $W_{\hat{a}}$ is the subgroup of $W_{F\Delta}$ generated by the simple reflections s_α for $\alpha \in {}_F\Delta \smallsetminus \{\alpha_a\}$, and so is $W_{\hat{\ell}}$.

We discuss the group H_{a+m} in the following four cases:

(1) If H_{a+m} is the quasi-split even unitary group (i.e., $\mathfrak{m} = 2\mathfrak{r}_{\mathfrak{m}}$ and $E = F(\sqrt{\varsigma})$), then $W_{F\Delta} = W(C_{a+\mathfrak{r}_{\mathfrak{m}}})$.
(2) If H_{a+m} is a unitary group, but not a quasi-split even unitary group (i.e., $2\mathfrak{r}_{\mathfrak{m}} < \mathfrak{m}$ and $E = F(\sqrt{\varsigma})$), then $W_{F\Delta} = W(B_{a+\mathfrak{r}_{\mathfrak{m}}})$.
(3) If H_{a+m} is the split even special orthogonal group (i.e., $\mathfrak{m} = 2\mathfrak{r}_{\mathfrak{m}}$ and $E = F$), then $W_{F\Delta} = W(D_{a+\mathfrak{r}_{\mathfrak{m}}})$.
(4) If H_{a+m} is a special orthogonal group but not a split even special orthogonal group (i.e., $2\mathfrak{r}_{\mathfrak{m}} < \mathfrak{m}$ and $E = F$), then $W_{F\Delta} = W(B_{a+\mathfrak{r}_{\mathfrak{m}}})$.

Here $W(X_{a+\mathfrak{r}_{\mathfrak{m}}})$ is the Weyl group of the split classical group of type X with rank $a + \mathfrak{r}_{\mathfrak{m}}$. Following from [45, §3.1], we put the double coset decomposition $W_{\hat{a}} \backslash W_{F\Delta} / W_{\hat{\ell}}$ into three cases for discussions, i.e., *Case* (1-1), *Case* (2-1) and *Case* (2-2). Both *Case* (2-1) and *Case* (2-2) in [45, §3.1] are only for the split even special orthogonal groups. The result that we are to prove here has already been proved in [45]. Hence, we assume that H_{a+m} *is not the split even special orthogonal group*, which is *Case* (1-1) in [45, §3.1]. We extend below the proof for *Case* (1-1) in [45, §3.1] to the current general case considered in this section.

In this situation, the double coset decomposition $P_{\hat{a}} \backslash H_{a+m} / P_{\hat{\ell}}$ is in bijection parametrized by the set of pairs of nonnegative integers

$$\mathfrak{E}_{a,\ell} = \{\epsilon_{\alpha,\beta} \mid 0 \leq \alpha \leq \beta \leq a \text{ and } a \leq \ell + \beta - \alpha \leq a + \mathfrak{r}_{\mathfrak{m}}\}.$$

The representatives $\epsilon_{\alpha,\beta}$ are chosen as in [23, §4.2]. For each double coset $P_{\hat{a}} \epsilon_{\alpha,\beta} P_{\hat{\ell}}$, we take a further decomposition $P_{\hat{a}} \backslash P_{\hat{a}} \epsilon_{\alpha,\beta} P_{\hat{\ell}} / R_\ell^{w_0}$, where the group $R_\ell^{w_0}$ is defined as in (4.6). It is equivalent to considering the decomposition $P_{\hat{\ell}}^{\epsilon_{0,\beta}} \backslash P_{\hat{\ell}} / R_\ell^{w_0}$ with $P_{\hat{\ell}}^{\epsilon_{0,\beta}} := \epsilon_{0,\beta}^{-1} P_{\hat{a}} \epsilon_{0,\beta} \cap P_{\hat{\ell}}$. Let $\mathcal{N}_{\beta,\ell,w_0}$ be the set of representatives of $P_{\hat{\ell}}^{\epsilon_{0,\beta}}(F) \backslash P_{\hat{\ell}}(F) / R_\ell^{w_0}(F)$, and set

(4.14) $$W_{\ell,i}^{\pm} = \text{Span}_E \{e_{\pm(\ell+1)}, e_{\pm(\ell+2)}, \ldots, e_{\pm(\ell+i)}\}$$

for $1 \leq i \leq a + \mathfrak{r}_{\mathfrak{m}} - \ell$, which are totally isotropic subspaces of $(W_{\mathfrak{m}+2a}, q)$. Following the same argument in [45, Lemmas 3.1 and 3.2], we can prove that Proposition 3.3 in [45] also holds for the more general cases in this paper that H_{a+m} may not be F-quasisplit.

PROPOSITION 4.1. *For* $\text{Re}(s)$ *large, the Bessel-Fourier coefficient of the Eisenstein series as in* (4.4), $\mathcal{F}^{\psi_\ell, w_0}(E(\cdot, \phi_{\tau \otimes \sigma}, s))(h)$, *is equal to*

$$\sum_{\epsilon_\beta} \sum_\eta \sum_\delta \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \int_{N_\ell^\eta(F) \backslash N_\ell^\eta(\mathbb{A})} \phi_s(\epsilon_\beta \eta \delta unh) \psi_{\ell,w_0}^{-1}(un) \, du \, dn,$$

where $N_\ell^\eta = N_\ell \cap \eta^{-1} P_{\hat{\ell}}^{\epsilon_{0,\beta}} \eta$ and $G_{m^-}^\eta := G_{m^-}^{w_0} \cap \gamma^{-1} P_w' \gamma$, and the summations are over the following representatives:

- $\epsilon_\beta = \epsilon_{0,\beta} \in \mathfrak{E}_{a,\ell}^0$, which is the subset of $\mathfrak{E}_{a,\ell}$ consisting of elements with $\alpha = 0$;

- $\eta = \mathrm{diag}(\epsilon, \gamma, \epsilon^*)$ belongs to $\mathcal{N}_{\beta,\ell,w_0}^0$, which is the subset of $\mathcal{N}_{\beta,\ell,w_0}$ consisting of elements with $\alpha = 0$, $\epsilon = \left(\begin{smallmatrix} 0 & I_{\ell-t} \\ I_t & 0 \end{smallmatrix} \right)$, and $t = a - \beta$, and which has the property that if $\beta > \max\{a - \ell, 0\}$, then γw_0 is orthogonal to $W_{\ell,\beta}^-$ for $\gamma \in P_w'(F) \backslash H_{a+m-\ell}(F) / G_{m^-}^{w_0}(F)$ where $P_w' = H_{a+m-\ell} \cap \epsilon_{0,\beta}^{-1} P_{\hat{a}} \epsilon_{0,\beta}$; and

- δ belongs to $G_{m^-}^\eta(F) \backslash G_{m^-}^{w_0}(F)$.

4.3. *Euler product decomposition.* Next, we apply the expression in Proposition 4.1 to the further calculation of the global zeta integral (4.9) and have

$$(4.15) \quad \begin{aligned} &\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell,w_0}) \\ &= \sum_{\epsilon_\beta; \eta} \int_h \varphi_\pi(h) \int_n \int_{[N_\ell^\eta]} \phi_s(\epsilon_\beta \eta u n h) \psi_{\ell,w_0}^{-1}(un) \, \mathrm{d}u \, \mathrm{d}n \, \mathrm{d}h, \end{aligned}$$

where the summation over ϵ_β and η is a finite sum as in Proposition 4.1, and the integration $\int_h$ is over $G_{m^-}^\eta(F) \backslash G_{m^-}^{w_0}(\mathbb{A})$ and $\int_n$ is over $N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})$. Similar to [45, Lemma 3.4], for each $\eta = \mathrm{diag}(\epsilon, \gamma, \epsilon^*)$, if the stabilizer $G_{m^-}^\eta$ is a proper maximal F-parabolic subgroup of $G_{m^-}^{w_0}$, then the summand over such η vanishes due to the cuspidality of φ_π.

To proceed with the calculation, we need to study the double coset decomposition $P_w' \backslash H_{a+m-\ell} / G_{m^-}^{w_0}$ as given in Proposition 4.1 and extend the calculation in [45, §3.2] to the current setting. With the choice of the w_0, it is easy to see that the group $H_{a+m-\ell}$ has its F-rank greater than or equal to one. We may apply [23, Prop. 4.4] to the current situation and show that only one integral associated to the Zariski open dense double coset in $P_{\hat{a}} \backslash H_{a+m} / R_\ell^{w_0}$ remains and all other integrals in the summation are zero for any choice of data. In other words, similar to Proposition 3.6 in [45], we still obtain the following expression for the global zeta integrals, which have two different forms according to the two cases: $a \leq \ell$ and $a > \ell$.

If $a \leq \ell$, we must have that $\beta = 0$, $\eta = \mathrm{diag}\{\epsilon, I_{\mathfrak{m}}, \epsilon^*\}$ with $\epsilon = \left(\begin{smallmatrix} 0 & I_{\ell-a} \\ I_a & 0 \end{smallmatrix} \right)$ and

$$(4.16) \quad \begin{aligned} \mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell,w_0}) = &\int_{G_{m^-}^{w_0}(F) \backslash G_{m^-}^{w_0}(\mathbb{A})} \varphi_\pi(h) \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \\ &\int_{[N_\ell^\eta]} \phi_s(\epsilon_{0,\beta} \eta u n h) \psi_{\ell,w_0}^{-1}(un) \, \mathrm{d}u \, \mathrm{d}n \, \mathrm{d}h, \end{aligned}$$

where $[N_\ell^\eta] = N_\ell^\eta(F) \backslash N_\ell^\eta(\mathbb{A})$.

If $a > \ell$, we must have that $\beta = a - \ell$, $\eta = \mathrm{diag}\{I_\ell, \gamma_0, I_\ell\}$ and

(4.17)
$$
\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) = \int_{G^\eta_{m^-}(F) \backslash G^{w_0}_{m^-}(\mathbb{A})} \varphi_\pi(h) \int_{N^\eta_\ell(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \\
\int_{[N^\eta_\ell]} \phi_s(\epsilon_{0,\beta} \eta u n h) \psi^{-1}_{\ell, w_0}(un) \, \mathrm{d}u \, \mathrm{d}n \, \mathrm{d}h,
$$

where γ_0 is a representative in the open double coset of

$$
P'_w(F) \backslash H_{a+m-\ell}(F) / G^{w_0}_{m^-}(F),
$$

with the property that $\gamma_0 w_0$ is not orthogonal to $W^-_{\ell, \beta}$.

It remains to show that those global integrals are in fact integrals over adelic domains and can be written as Euler products of local zeta integrals. In order to continue the calculation, we have to recall the relevant calculations in [45] with replacement of notation used here. Section 4.4 will deal with the case of $a > \ell$ and hence is for the integral in (4.17). Section 4.5 will deal with the case of $a \le \ell$ and hence is for the integral in (4.16).

4.4. *Case* $a > \ell$. We are studying the integral in (4.17). For convenience, we recall the open coset representative $\epsilon_{0,\beta}$ with $\beta = a - \ell$ in equation (4.14) in [23],

(4.18)
$$
\epsilon_{0,\beta} = w^\ell_q \cdot \begin{pmatrix} 0 & I_{a-\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_\ell \\ 0 & 0 & I_\mathfrak{m} & 0 & 0 \\ I_\ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{a-\ell} & 0 \end{pmatrix}.
$$

Note that when $E = F$, w_q is defined by

$$
\begin{cases}
-I_{\mathfrak{m}+2a} & \text{if } \mathfrak{m} \text{ is odd}, \\
\mathrm{diag}\{I_{\frac{\mathfrak{m}}{2}+a-1}, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), I_{\frac{\mathfrak{m}}{2}+a-1}\} & \text{if } \mathfrak{m} \text{ is even and } H_{a+m} \text{ is not split}, \\
\mathrm{diag}\{I_{\mathfrak{r}_\mathfrak{m}+a-1}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), I_{\mathfrak{r}_\mathfrak{m}+a-1}\} & \text{if } \mathfrak{m} = 2\mathfrak{r}_\mathfrak{m} \text{ and } H_{a+m} \text{ is split},
\end{cases}
$$

and when $E = F(\sqrt{\varsigma})$, $w_q = I_{\mathfrak{m}+2a}$.

First, we write the integral in (4.17) as

(4.19)
$$
\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) = \int_{G^\eta_{m^-}(F) \backslash G^{w_0}_{m^-}(\mathbb{A})} \varphi_\pi(h) \Phi_s(h) \, \mathrm{d}h,
$$

where the function $\Phi_s(h)$ is defined, as in [45, (3.34)], to be

(4.20)
$$
\Phi_s(h) := \int_{N^\eta_\ell(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \int_{[N^\eta_\ell]} \phi_s(\epsilon_{0,\beta} \eta u n h) \psi^{-1}_{\ell, w_0}(un) \, \mathrm{d}u \, \mathrm{d}n.
$$

To calculate the function $\Phi_s(h)$, we first consider $(\epsilon_{0,\beta}\eta)u(\epsilon_{0,\beta}\eta)^{-1}$, similar to Section 3.3 of [45]. Note that N_ℓ^η consists of elements of the form

$$u = \begin{pmatrix} c & 0 & 0 & 0 & y_6 & 0 & 0 \\ & I_{\mathfrak{r}_\mathfrak{m}} & 0 & 0 & 0 & 0 & 0 \\ & & I_{a-\ell} & 0 & 0 & 0 & y_6' \\ & & & I_{\mathfrak{m}-2\mathfrak{r}_\mathfrak{m}} & 0 & 0 & 0 \\ & & & & I_{a-\ell} & 0 & 0 \\ & & & & & I_{\mathfrak{r}_\mathfrak{m}} & 0 \\ & & & & & & c^* \end{pmatrix} \in N_\ell,$$

where $c \in Z_\ell$. Then the conjugation $(\epsilon_{0,\beta}\eta)u(\epsilon_{0,\beta}\eta)^{-1}$ is of the form

$$\begin{pmatrix} \begin{matrix} I_{a-\ell} & y_6' \\ 0 & c^* \end{matrix} & & \\ & I_\mathfrak{m} & \\ & & \begin{matrix} c & y_6 \\ 0 & I_{a-\ell} \end{matrix} \end{pmatrix}.$$

Hence the stabilizer $(\epsilon_{0,\beta}\eta)N_\ell^\eta(\epsilon_{0,\beta}\eta)^{-1}$ as a subgroup of $P_{\hat{a}}$ is in fact contained in the $G_{E/F}(a)$-part of the Levi subgroup of $P_{\hat{a}}$. We denote it by Z_ℓ'. We may write elements of Z_ℓ' as $\hat{z}'$ with $z' = \begin{pmatrix} I_{a-\ell} & y \\ 0 & z \end{pmatrix}$. Accordingly, the character $\psi_{\ell,w_0}^{-1}(u)$ becomes

$$\psi_{Z_\ell',\kappa}(z') := \psi_E\left((-1)^{\mathfrak{m}+1}\frac{\kappa}{2}z_{\beta,\beta+1} + z_{\beta+1,\beta+2} + \cdots + z_{a-1,a}\right).$$

Hence we obtain

$$(4.21) \qquad \Phi_s(h) = \int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \phi_s^{\psi_{Z_\ell',\kappa}}(\epsilon_{0,\beta}\eta nh)\psi_{\ell,w_0}^{-1}(n)\,\mathrm{d}n,$$

with

$$(4.22) \qquad \phi_s^{\psi_{Z_\ell',\kappa}}(h) := \int_{[Z_\ell']} \phi_s(\hat{z}'h)\psi_{Z_\ell',\kappa}(z')\,\mathrm{d}z'.$$

Next, we need to calculate the integration over $G_{m^-}^\eta(F)\backslash G_{m^-}^{w_0}(\mathbb{A})$ in (4.17) and in (4.19), in order to show that the global zeta integral is in fact an integration over an adelic domain. Similar to the decomposition in [45, eq. (3.33)], we also have the decomposition

$$G_{m^-}^\eta = (G_{E/F}(W_{\mathfrak{r}_\mathfrak{m}+a-1,\beta-1}^+) \times H(q_{\eta^{-1}W_{(a)}})) \ltimes V_{\beta-1,\eta},$$

where $W_{\mathfrak{r}_\mathfrak{m}+a-1,\beta-1}^+$ is defined in (4.14), $W_{(a)} = (W_a^+ \oplus W_a^-)^\perp$, and $V_{\beta-1,\eta}$ is the unipotent radical of the stabilizer $G_{m^-}^\eta$, as described on page 573 of [45].

More precisely, in Proposition 4.1, take $\eta = \eta_{I,\gamma_0}$. Then $V_{\beta-1,\eta}$ consists of the elements of the form

$$(4.23) \qquad \eta^{-1} \begin{pmatrix} I_\ell & & & & & & \\ & I_{\beta-1} & d_1 & u & v_1 & v \\ & & 1 & 0 & 0 & v'_1 \\ & & & I_\mathfrak{m} & 0 & u' \\ & & & & 1 & d'_1 \\ & & & & & I_{\beta-1} \\ & & & & & & I_\ell \end{pmatrix} \eta$$

with $d_1 + (-1)^{\mathfrak{m}+1}\frac{\kappa}{2}v_1 = 0$, where d_1 and v_1 are column vectors of dimension $\beta - 1$. Let $Z^\eta_{\ell,\beta-1}$ be the maximal unipotent subgroup of $G_{E/F}(W^+_{\mathfrak{r}_\mathfrak{m}+a-1,\beta-1})$, consisting of elements of following type:

$$\eta^{-1} \cdot \mathrm{diag}\{I_\ell, d, I_{\mathfrak{m}+2}, d^*, I_\ell\} \cdot \eta$$

with $d \in Z_{\beta-1}$.

Write $N^\eta_{\ell,\beta-1} := Z^\eta_{\ell,\beta-1} V_{\beta-1,\eta}$. It is a unipotent subgroup of $G^{w_0}_{\mathfrak{m}^-}$ associated to the nilpotent orbit with partition $[(2(a-\ell-1)+1)1^\mathfrak{m}]$. Fixing the anisotropic vector $y_{-\kappa}$ that defines the character of $N^\eta_{\ell,\beta-1}$, we deduce that the corresponding stabilizer in $G^{w_0}_{\mathfrak{m}^-}$ is $\mathrm{Isom}(\eta^{-1}W_{(a)}, q)^\circ$. Hence $\mathrm{Isom}(\eta^{-1}W_{(a)}, q)^\circ = \eta^{-1}H_m\eta$. The elements of $N^\eta_{\ell,\beta-1}$ have the form

$$(4.24) \qquad (\epsilon_{0,\beta}\eta)^{-1} \begin{pmatrix} d & d_1 & 0 & u & 0 & v_1 & v \\ & 1 & 0 & 0 & 0 & 0 & v'_1 \\ & & I_\ell & 0 & 0 & 0 & 0 \\ & & & I_\mathfrak{m} & 0 & 0 & u' \\ & & & & I_\ell & 0 & 0 \\ & & & & & 1 & d'_1 \\ & & & & & & d^* \end{pmatrix} (\epsilon_{0,\beta}\eta),$$

where $d \in Z_{\beta-1}$. Remark that $Z^\eta_{\ell,\beta-1}$ is the set of all matrices of the form (4.24) with all entries 0 except d. Denote $Z_{\beta,\eta}$ (resp. $C_{\beta-1,\eta}$) to be the subgroup of $N^\eta_{\ell,\beta-1}$ consisting of all matrices in (4.24) with all entries 0 except d and d_1 (resp. with $d = I_{\beta-1}$ and $d_1 = 0$). Then $N^\eta_{\ell,\beta-1} = Z_{\beta,\eta}C_{\beta-1,\eta}$.

Similar to [45, p. 575], we have the following isomorphism,

$$(4.25) \qquad C_{\beta-1,\eta}\backslash G^\eta_{\mathfrak{m}^-} \cong P^1_\beta \times H^\eta_m \qquad (H^\eta_m := \eta^{-1}H_m\eta),$$

where H^η_m is a subgroup of $G^{w_0}_{\mathfrak{m}^{-1}}$ and P^1_β is the mirabolic subgroup of $G_{E/F}(\beta)$ containing $Z_{\beta,\eta}$. Continuing with the global zeta integral as displayed in (4.19), we obtain that the global zeta integral $\mathcal{Z}(s,\cdot)$ is equal to

$$(4.26) \qquad \int_{P^1_\beta(F)H^\eta_m(F)C_{\beta-1,\eta}(\mathbb{A})\backslash G^{w_0}_{\mathfrak{m}^-}(\mathbb{A})} \Phi_s(h) \int_{[C_{\beta-1,\eta}]} \varphi_\pi(ch)\, \mathrm{d}c\, \mathrm{d}h.$$

This is the integral similar to that displayed in (3.36) of [45]. We note that there is a typo in the integration domain in equation (3.36) of [45], and the integral in (4.26) gives the correct version.

Following closely the argument in [45], we apply the Fourier expansion on φ_π along the mirabolic subgroup P_β^1 repeatedly and obtain the same expansion as that displayed in equation (3.38) in [45]. Plugging the so obtained expansion into (4.26) and combining the integrals, we obtain

$$
\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})
$$
$$
= \int_{Z_{\beta,\eta}(F) H_m^\eta(F) C_{\beta-1,\eta}(\mathbb{A}) \backslash G_{m-}^{w_0}(\mathbb{A})} \Phi_s(h) \mathcal{F}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi)(h) \, \mathrm{d}h,
$$

where $\mathcal{F}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi)$ is the $(\beta-1)$-th Bessel coefficient with respect to $\psi_{\beta-1,y_{-\kappa}}$ (with $\beta = a - \ell$), as defined in (2.7), by

$$
(4.27) \qquad \mathcal{F}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi)(h) = \int_{N_{\ell,\beta-1}^\eta(F) \backslash N_{\ell,\beta-1}^\eta(\mathbb{A})} \varphi_\pi(nh) \psi_{\beta-1,y_{-\kappa}}(n) \, \mathrm{d}n.
$$

Since $\mathcal{F}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi)$ is left $(Z_{\beta,\eta}, \psi_{\beta-1,y_{-\kappa}}^{-1})$-equivariant, $\mathcal{Z}(s, \cdot)$ is equal to (see [45, (3.40)])
$$
(4.28)
$$
$$
\int_{H_m^\eta(F) N_{\ell,\beta-1}^\eta(\mathbb{A}) \backslash G_{m-}^{w_0}(\mathbb{A})} \mathcal{F}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi)(h) \int_{[Z_{\beta,\eta}]} \Phi_s(zh) \psi_{\beta-1,y_{-\kappa}}^{-1}(z) \, \mathrm{d}z \, \mathrm{d}h.
$$

Let us now focus on the inner integral $\int_{[Z_{\beta,\eta}]} \Phi_s(zh) \psi_{\beta-1,y_{-\kappa}}^{-1}(z) \, \mathrm{d}z$, which by definition (as in (4.21)) is equal to

$$
\int_{[Z_{\beta,\eta}]} \int_{N_\ell^\eta(\mathbb{A}) \backslash N_\ell(\mathbb{A})} \phi_s^{\psi_{Z_\ell',\kappa}}(\epsilon_{0,\beta}\eta nzh) \psi_{\ell,w_0}^{-1}(n) \, \mathrm{d}n \, \psi_{\beta-1,y_{-\kappa}}^{-1}(z) \, \mathrm{d}z.
$$

By the definition of $\phi_s^{\psi_{Z_\ell',\kappa}}$ (see (4.22)), we may combine the two integrals over $[Z_{\beta,\eta}]$ and $[N_\ell^\eta]$. As a subgroup of $P_{\hat{a}}$, $(\epsilon_{0,\beta}\eta) N_\ell^\eta Z_{\beta,\eta}(\epsilon_{0,\beta}\eta)^{-1}$ consists of elements of the form

$$
\begin{pmatrix}
d & d_1 & (y_6)'_{*,*} & & & & \\
0 & 1 & (y_6)'_{\beta,*} & & & & \\
0 & 0 & c^* & & & & \\
& & & I_\mathfrak{m} & & & \\
& & & & c & (y_6)_{*,\beta} & (y_6)_{*,*} \\
& & & & 0 & 1 & d_1' \\
& & & & 0 & 0 & d
\end{pmatrix},
$$

where the notation is the same as in [45, (3.42)]. It follows that $N_\ell^\eta Z_{\beta,\eta} \cong Z_a$, where Z_a is the maximal upper-triangular unipotent subgroup of $G_{E/F}(a)$,

which is regarded canonically as a subgroup of $P_{\hat{a}}$. Combining the integrals over $[Z_{\beta,\eta}]$ and $[N_\ell^\eta]$, we define

$$(4.29) \qquad \phi_s^{Z_a,\kappa}(h) := \int_{[Z_a]} \phi_s(zh)\psi_{Z_a,\kappa}(z)\,\mathrm{d}z,$$

where the character $\psi_{Z_a,\kappa}(z)$ is given by

$$(4.30)\quad \psi_E(-z_{1,2} - \cdots - z_{\beta-1,\beta} + (-1)^{\mathfrak{m}+1}\frac{\kappa}{2}z_{\beta,\beta+1} + z_{\beta+1,\beta+2} + \cdots + z_{a-1,a})$$

with $\beta = a - \ell$, which is a non-degenerate character of Z_a. Hence $\phi_s \mapsto \phi_s^{Z_a,\kappa}$ can be regarded as an $H_{a+m}(\mathbb{A})$-equivariant isomorphism from the induced representation $\mathrm{I}_s(\tau,\sigma)$ onto the induced representation $\mathrm{I}_s(\mathcal{W}_\tau,\sigma)$, where $\mathcal{W}_\tau := \mathcal{W}_\tau^{\overline{\psi}_{Z_a,\kappa}}$ is the global Whittaker model of τ with respect to the non-degenerate character $\overline{\psi}_{Z_a,\kappa}$.

For $\mathrm{Re}(s)$ sufficiently large, the integrals we considered here are absolutely convergent, which allow us to switch the order of integration. After combining the integrals $[Z_{\beta,\eta}]$ and $[N_\ell^\eta]$ and by (4.21), similar to [45, (3.41)], we obtain that

$$(4.31)$$
$$\int_{[Z_{\beta,\eta}]} \Phi_s(zh)\psi_{\beta-1,y_{-\kappa}}^{-1}(z)\,\mathrm{d}z = \int_{U_{a,\eta}^-(\mathbb{A})} \phi_s^{Z_a,\kappa}(n\epsilon_{0,\beta}\eta h)\psi_{(\mathfrak{m}+a+\ell,a-\ell)}(n)\,\mathrm{d}n.$$

Here $U_{a,\eta}^-$ consists of matrices of the form

$$(4.32) \qquad \begin{pmatrix} I_{a-\ell} & & & & \\ 0 & I_\ell & & & \\ 0 & x_2' & I_\mathfrak{m} & & \\ x_1 & x_3 & x_2 & I_\ell & \\ 0 & x_1' & 0 & 0 & I_{a-\ell} \end{pmatrix},$$

which is a section for the domain of integration, $N_\ell^\eta \backslash N_\ell$, under the adjoint action of $\epsilon_{0,\beta}\eta$. The character $\psi_{(\mathfrak{m}+a+\ell,a-\ell)}$ of $U_{a,\eta}^-$ is given by

$$\psi_{(\mathfrak{m}+a+\ell,a-\ell)}(n) = \psi_E(n_{\mathfrak{m}+a+\ell,a-\ell}),$$

where $n_{\mathfrak{m}+a+\ell,a-\ell} = (x_1)_{\ell,a-\ell}$.

Note that the adelic integration over $U_{a,\eta}^-$ in (4.33) converges absolutely due to the same reason as that of the quasi-split orthogonal group case considered in Appendix II to Section 5 of [19], and also that in [73] and [75, Th. 3.1], for instance. Another way to confirm the absolute convergence is that after taking the absolute value of the integrand, the integral is the product of local intertwining operators, which converges absolutely for $\mathrm{Re}(s)$ sufficiently large.

From (4.31), for $h \in G_{m_-}^{w_0}(\mathbb{A})$, we define

$$(4.33) \qquad \mathrm{J}_s(\phi_s)(h) := \int_{U_{a,\eta}^-(\mathbb{A})} \phi_s^{Z_a,\kappa}(n\epsilon_{0,\beta}\eta h)\psi_{(\mathfrak{m}+a+\ell,a-\ell)}(n)\,\mathrm{d}n.$$

Following a similar argument as in Theorem 3.1 in [75] for split special orthogonal groups, we verify the absolute convergence of $J_s(\phi_s)$ for $\mathrm{Re}(s)$ sufficiently large in the part of the proof of the absolute convergence of local zeta integrals for more general groups over all local fields in [43]. Moreover, the function $J_s(\phi_s)$ enjoys the following property.

PROPOSITION 4.2. *For $\mathrm{Re}(s)$ sufficiently large, the mapping*

$$J_s \ : \ \phi_s \mapsto J_s(\phi_s)$$

composing with the restriction to $G_{m^-}^{w_0}(\mathbb{A})$ gives the $G_{m^-}^{w_0}(\mathbb{A})$-equivariant homomorphism from $I_s(\tau, \sigma)$ as defined in (4.10) to $I_s^{w_0}(\psi_{\beta-1, y_{-\kappa}}, \sigma^{w_q^\ell})$, which is the following smooth induction,

$$I_s^{w_0}(\psi_{\beta-1, y_{-\kappa}}, \sigma^{w_q^\ell}) := \mathrm{Ind}_{N_{\ell, \beta-1}^\eta(\mathbb{A}) H_m^\eta(\mathbb{A})}^{G_{m^-}^{w_0}(\mathbb{A})} (\psi_{\beta-1, y_{-\kappa}} \otimes \sigma^{w_q^\ell}, s),$$

where the character $\psi_{\beta-1, y_{-\kappa}}$ is given as in (4.27).

Proof. For $g \in G_{m^-}^{w_0}(\mathbb{A})$, the function $J_s(\phi_s)(g)$ is smooth on $G_{m^-}^{w_0}(\mathbb{A})$. The left quasi-invariance with respect to $(N_{\ell, \beta-1}^\eta, \psi_{\beta-1, y_{-\kappa}})$ is clear from the calculation above Proposition 4.2. It remains to check the left equivariant property for $x \in H_m^\eta(\mathbb{A})$. By definition, we have

$$J_s(\phi_s)(xg) = \int_{U_{a,\eta}^-(\mathbb{A})} \phi_s^{Z_a, \kappa}(n\epsilon_{0,\beta}\eta xg)\psi_{(\mathfrak{m}+a+\ell, a-\ell)}(n) \, \mathrm{d}n$$

$$= \int_{U_{a,\eta}^-(\mathbb{A})} \phi_s^{Z_a, \kappa}(n\epsilon_{0,\beta}\eta x\eta^{-1}\eta g)\psi_{(\mathfrak{m}+a+\ell, a-\ell)}(n) \, \mathrm{d}n.$$

Since $\eta x \eta^{-1}$ belongs to $H_m(\mathbb{A})$, it is enough to understand the group $\epsilon_{0,\beta} H_m \epsilon_{0,\beta}^{-1}$. According to (4.18), where $\epsilon_{0,\beta}$ is explicitly given,

$$\epsilon_{0,\beta} H_m \epsilon_{0,\beta}^{-1} = w_q^\ell H_m w_q^{-\ell}.$$

It is clear that $J_s(\phi_s)(xg) = \sigma^{w_q^\ell}(x) \cdot J_s(\phi_s)(g)$. We are done.   □

Note that by (4.18), the adjoint action of w_q on σ is trivial except when H_m is an even special orthogonal group. In this case, $\det(w_q) = -1$ and the adjoint action of w_q is the non-trivial action of $O(W_\mathfrak{m}, q)/H_m$ on σ. In other words, w_q restricted to $O(W_\mathfrak{m}, q)$ is a choice of ε as defined right below (2.6) on page 753. For simplicity, denote

(4.34) $\sigma' := \sigma^{w_q^\ell}.$

We note that if H_m is an even special orthogonal group and ℓ is odd, then $\{\sigma, \sigma^{w_q}\}$ is an $\widetilde{O}(G)$-orbit of σ as discussed on page 754. Therefore, for any fixed $h \in G_{m^-}^{w_0}(\mathbb{A})$, the function $J_s(\phi_s)(xh)$, as a function in x, belongs to the space $V_{\sigma'}$ of cuspidal automorphic forms, which is the space of the cuspidal

automorphic representation σ, up to a twist by $\epsilon_{0,\beta}$. Hence we obtain the following composition of $G_{m^-}^{w_0}(\mathbb{A})$-equivariant mappings:

$$(4.35) \qquad I_s(\tau, \sigma) \to I_s(\mathcal{W}_\tau, \sigma) \to I_s^{w_0}(\psi_{\beta-1,y_{-\kappa}}, \sigma').$$

We summarize the calculation above and state the formula for the global zeta integral in the following

PROPOSITION 4.3. *With the notation above and for* $\mathrm{Re}(s)$ *large, the global zeta integral* $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ *has the following expression,*

$$(4.36) \quad \mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) = \int_g \int_{[H_m^\eta]} \mathcal{F}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi)(xg) \mathrm{J}_s(\phi_s)(xg) \, \mathrm{d}x \, \mathrm{d}g,$$

where $\mathrm{d}g$ *is over* $R_{\ell,\beta-1}^\eta(\mathbb{A}) \backslash G_{m^-}^{w_0}(\mathbb{A})$ *with* $R_{\ell,\beta-1}^\eta := H_m^\eta \ltimes N_{\ell,\beta-1}^\eta$, *and* $[H_m^\eta] := H_m^\eta(F) \backslash H_m^\eta(\mathbb{A})$, *as defined in* (4.25).

Note that the pairing

$$\mathcal{P}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi, \varphi_{\sigma'}) = \int_{[H_m^\eta]} \mathcal{F}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi)(x)\varphi_{\sigma'}(x) \, \mathrm{d}x$$

defines a Bessel period for the pair (π, σ'), where $\varphi_{\sigma'}$ is a cuspidal automorphic form in $\mathcal{C}_\sigma$ under the conjugation of w_q^ℓ, and belongs to the space

$$\mathrm{Hom}_{R_{\ell,\beta-1}^\eta(\mathbb{A})}(\pi \otimes \sigma', \psi_{\beta-1,y_{-\kappa}}^{-1}).$$

In this way, the inner integral of the integration formula (4.36) for the global zeta integral $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ can be written as
(4.37)
$$\int_{[H_m^\eta]} \mathcal{F}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(\varphi_\pi)(xg) \mathrm{J}_s(\phi_s)(xg) \, \mathrm{d}x = \mathcal{P}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}(R(g)\varphi_\pi, \mathrm{J}_s(\phi_s)(g)),$$

where $R(g)\varphi_\pi(x) := \varphi_\pi(xg)$ is the right translation and $\mathrm{J}_s(\phi_s)(g)$ is in σ' by Proposition 4.2. From this expression, we deduce the following easy, but important vanishing result.

COROLLARY 4.4. *If the Bessel period for* (π, σ') *is zero, then the global zeta integral* $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ *is zero for all choices of data.*

From now on, it is meaningful to assume that the Bessel period $\mathcal{P}^{\psi_{\beta-1,y_{-\kappa}}^{-1}}$ for (π, σ') is non-zero. By the uniqueness of local Bessel functionals, which is proved in [2], [77], [16], and [44], we have the Euler factorization: $\mathcal{P}^{\psi_{\beta-1,y_{-\kappa}}^{-1}} = \otimes_\nu \mathcal{P}_\nu^{\psi_{\beta-1,y_{-\kappa}}^{-1}}$. It follows that the integral in (4.37) can be written as an Euler product of local Bessel functionals when φ_π and $\phi_s = \phi_{\tau\otimes\sigma,s}$ are factorizable vectors. More precisely, we take $\varphi_\pi = \otimes_\nu \varphi_{\pi_\nu}$ and $\phi_{\tau\otimes\sigma,s} = \otimes_\nu \phi_{\tau_\nu\otimes\sigma_\nu,s}$. Then

$$(4.38) \qquad \phi_s^{Z_a,\kappa}(h) = \prod_\nu f_{\mathcal{W}_{\tau_\nu}^\kappa \otimes \sigma_\nu, s}(h_\nu),$$

where $f_{\mathcal{W}^{\kappa}_{\tau_\nu} \otimes \sigma_\nu, s}$ belongs to the space of induced representation

$$(4.39) \qquad \mathrm{I}_{s,\nu}(\mathcal{W}_{\tau_\nu}, \sigma_\nu) = \mathrm{Ind}_{P_{\hat{a}}(F_\nu)}^{H_{a+m}(F_\nu)}(|\cdot|^s \mathcal{W}_{\tau_\nu} \otimes \sigma_\nu).$$

It follows that

$$(4.40) \quad \begin{aligned} &\int_{[H_m^\eta]} \mathcal{F}^{\psi_{\beta-1}^{-1}, y_{-\kappa}}(\varphi_\pi)(xg) \mathrm{J}_s(\phi_s)(xg)\,\mathrm{d}x \\ &= \prod_\nu \mathcal{P}_\nu^{\psi_{\beta-1}^{-1}, y_{-\kappa}}(\pi_\nu(g_\nu)\varphi_{\pi_\nu}, \mathrm{J}_{s,\nu}(\phi_{s,\nu})(g_\nu)), \end{aligned}$$

where at each local place ν, $\mathcal{P}_\nu^{\psi_{\beta-1}^{-1}, y_{-\kappa}}$ is the unique local Bessel functional up to scalar, and $\mathrm{J}_{s,\nu}$ is the ν-local twisted Jacquet module associated to the adelic integration over $U_{a,\eta}^-(\mathbb{A})$ that defines J_s in (4.33).

Now, for $\mathrm{Re}(s)$ sufficiently large, we define the local zeta integral for this case by

$$(4.41) \quad \mathcal{Z}_\nu(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) := \int_{g_\nu} \mathcal{P}_\nu^{\psi_{\beta-1}^{-1}, y_{-\kappa}}(\pi_\nu(g_\nu)\varphi_{\pi_\nu}, \mathrm{J}_{s,\nu}(\phi_{s,\nu})(g_\nu))\,\mathrm{d}g_\nu,$$

where the integration is taken over $R_{\ell, \beta-1}^\eta(F_\nu) \backslash G_{m^-}^{w_0}(F_\nu)$.

THEOREM 4.5 ($a > \ell$). *Let $E(\phi_{\tau \otimes \sigma}, s)$ be the Eisenstein series on $H_{m+a}(\mathbb{A})$ as in* (4.3) *and let π belong to $\mathcal{A}_{\mathrm{cusp}}(G_{m^-}^{w_0})$. Then the global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ converges absolutely and is holomorphic at s where the Eisenstein series $E(h, \phi, s)$ has no poles.*

Assume that $\varphi_\pi = \otimes_\nu \varphi_{\pi_\nu}$ and $\phi_s = \otimes_\nu \phi_{\tau_\nu \otimes \sigma_\nu, s} = \otimes_\nu \phi_{s,\nu}$ are factorizable vectors, which yield factorization in (4.38), *and that the pair (π, σ') has a non-zero Bessel period. Then for the real part of s sufficiently large, it can be written as an Euler product*

$$\mathcal{Z}(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}) = \prod_\nu \mathcal{Z}_\nu(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0}),$$

where the local zeta integral $\mathcal{Z}_\nu(s, \phi_{\tau \otimes \sigma}, \varphi_\pi, \psi_{\ell, w_0})$ is defined in (4.41).

Note that this Euler decomposition of the global zeta integral in terms of the local zeta integrals is a more explicit realization of the abstract Euler decomposition as in (4.13); further properties of the local and global zeta integrals will be discussed in Section 5.

4.5. *Case $a \leq \ell$.* Although not necessary for the current paper, for completeness and future applications, we briefly study the global zeta integral as given in (4.16). In principle, it is similar to the case of $a > \ell$. We follow the discussion in Section 3.4 in [45] to give necessary steps in order to show that the global zeta integral can be factorized as an Euler product of local zeta integrals.

First, we have

$$N_\ell^\eta = \left\{ \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ & b & y_4 & z_4 & 0 \\ & & I_{\mathfrak{m}+2a-2\ell} & y_4' & 0 \\ & & & b^* & 0 \\ & & & & c^* \end{pmatrix} : c \in Z_a,\ b \in Z_{\ell-a} \right\}.$$

Write

$$N_{a,\ell-a} = \left\{ \begin{pmatrix} I_a & 0 & 0 & 0 & 0 \\ & b & y_4 & z_4 & 0 \\ & & I_{\mathfrak{m}+2a-2\ell} & y_4' & 0 \\ & & & b^* & 0 \\ & & & & I_a \end{pmatrix} : b \in Z_{\ell-a} \right\} \subset N_{\ell-a}.$$

Denote $\psi_{m,\ell-a;y_\kappa}$ to be the restriction to the subgroup $N_{a,\ell-a}$ of the character ψ_{ℓ,y_κ}. By the decomposition $N_\ell^\eta = Z_a N_{a,\ell-a}$, the inner integration over $[N_\ell^\eta]$ in (4.16) can be written as

(4.42) $$\int_{[N_\ell^\eta]} \phi_s(\epsilon_{0,0}\eta u h)\psi_{\ell,w_0}^{-1}(u)\,\mathrm{d}u = \mathcal{F}^{\psi_{m,\ell-a;y_\kappa}^{-1}}(\phi_s^{\psi_{Z_a,\kappa}})(\epsilon_{0,0}\eta h),$$

where

$$\phi_s^{\psi_{Z_a,\kappa}}(h) = \int_{[Z_a]} \phi_s(z)\psi_{Z_a,\kappa}(z)\,\mathrm{d}z$$

with

$$\psi_{Z_a,\kappa}(z) = \psi_E(z_{1,2} + z_{2,3} + \cdots + z_{a-1,a}).$$

Here $\mathcal{F}^{\psi_{m,\ell-a;y_\kappa}^{-1}}$ defines a Bessel-Fourier coefficient of σ. As in (4.38), we have

(4.43) $$\phi_s^{\psi_{Z_a,\kappa}}(h) = \prod_\nu f_{W_{\tau_\nu}^\kappa \otimes \sigma_\nu, s}(h_\nu),$$

with $f_{W_{\tau_\nu}^\kappa \otimes \sigma_\nu, s}$ belonging to the space of induced representation

$$\mathrm{I}_{s,\nu}(\mathcal{W}_{\tau_\nu}, \sigma_\nu) = \mathrm{Ind}_{P_{\hat{a}}(F_\nu)}^{H_{a+m}(F_\nu)}(|\cdot|^s \mathcal{W}_{\tau_\nu} \otimes \sigma_\nu).$$

After changing variables, we obtain

$$\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$$

(4.44) $$= \int_{N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})} \int_{[G_{m-}^{w_0}]} \varphi_\pi(h)\mathcal{F}^{\psi_{m,\ell-a;y_\kappa}^{-1}}(\phi_s^{\psi_{Z_a,\kappa}})(\epsilon_{0,0}\eta h n)\psi_{\ell,w_0}^{-1}(n)\,\mathrm{d}h\,\mathrm{d}n.$$

Note that the integral (4.44) is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large. The inner integration over $[G_{m-}^{w_0}]$ converges absolutely because of rapid decay of the cuspidal automorphic form φ_π. The outer integration over the quotient $N_\ell^\eta(\mathbb{A})\backslash N_\ell(\mathbb{A})$ converges absolutely due to the reason that we used to explain (4.33). For convenience, we write down explicitly the quotient $N_\ell^\eta\backslash N_\ell$

and the restriction of ψ_{ℓ,y_κ}. The quotient $N_\ell^\eta \backslash N_\ell$ is isomorphic to the subgroup consisting of elements

$$
\begin{pmatrix}
I_a & x_1 & x_2 & x_3 & x_4 \\
 & I_{\ell-a} & 0 & 0 & x_3' \\
 & & I_{\mathfrak{m}+2a-2\ell} & 0 & x_2' \\
 & & & I_{\ell-a} & x_1' \\
 & & & & I_a
\end{pmatrix}.
$$

The restriction of ψ_{ℓ,y_κ} is $\psi_E((x_1)_{a,1})$.

It is clear that the inner integration in the variable h in (4.44) gives a Bessel period for the pair (σ, π). Hence we obtain the following.

COROLLARY 4.6. *If the Bessel period for (σ, π) is zero, then the global zeta integral $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ is zero for all choices of data.*

By the uniqueness of the local Bessel models, this Bessel period may be written as an Euler product of local Bessel functionals for factorizable input data. More precisely, we take $\varphi_\pi = \otimes_\nu \varphi_{\pi_\nu}$ and $\phi_s = \otimes_\nu \phi_{s,\nu}$ and write

$$
(4.45) \qquad \int_{[G_{m^-}^{w_0}]} \varphi_\pi(h) \mathrm{J}_s(\phi_s)(hn)\, \mathrm{d}h = \prod_\nu \mathcal{P}_\nu^{\psi_{m,\ell-a;y_\kappa,\nu}^{-1}} (R(n_\nu) f_{W_{\tau_\nu}^\kappa \otimes \sigma_\nu, s}, \varphi_{\pi_\nu}),
$$

where $\mathrm{J}_s(\phi_s)(hn) := \mathcal{F}^{\psi_{m,\ell-a;y_\kappa}^{-1}}(\phi_s^{\psi_{Z_a,\kappa}})(\epsilon_{0,0}\eta hn)$, $R(\cdot)$ is the right translation, and for each local place ν, $\mathcal{P}_\nu^{\psi_{m,\ell-a;y_\kappa,\nu}^{-1}}$ is the unique functional, up to scalar, in the space

$$
\mathrm{Hom}_{G_{m^-}^{w_0}(F_\nu) \ltimes N_{a,\ell-a}(F_\nu)}(\pi_\nu \otimes \sigma_\nu, \psi_{m,\ell-a;y_\kappa,\nu}^{-1}).
$$

In this way, we define the local zeta integral by

$$
(4.46)
$$
$$
\mathcal{Z}_\nu(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) := \int_{n_\nu} \mathcal{P}_\nu^{\psi_{m,\ell-a;y_\kappa,\nu}^{-1}}(R(n_\nu) f_{W_{\tau_\nu}^\kappa \otimes \sigma_\nu, s}, \varphi_{\pi_\nu}) \psi_{\ell,w_0,\nu}^{-1}(n_\nu)\, \mathrm{d}n_\nu,
$$

where the integration is taken over $N_\ell^\eta(F_\nu) \backslash N_\ell(F_\nu)$, and obtain the following.

THEOREM 4.7 $(a \leq \ell)$. *With the notation as in Theorem 4.5, the global zeta integral $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ converges absolutely and is holomorphic at s where the Eisenstein series $E(h, \phi, s)$ has no poles.*

Assume that $\varphi_\pi = \otimes_\nu \varphi_{\pi_\nu}$ and $\phi_s = \otimes_\nu \phi_{\tau_\nu\otimes\sigma_\nu,s} = \otimes_\nu \phi_{s,\nu}$ are factorizable vectors, which yield the factorization in (4.43), and that the pair (σ, π) has a non-zero Bessel period. Then for the real part of s sufficiently large, it can be written as an Euler product

$$
\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) = \prod_\nu \mathcal{Z}_\nu(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}),
$$

where the local zeta integral $\mathcal{Z}_\nu(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ is defined in (4.46).

Note that this Euler decomposition of the global zeta integral in terms of the local zeta integrals is a more explicit realization of the abstract Euler decomposition as in (4.13). Since this case is not directly used in this paper, we refer to Section 3.4 of [45] for a more detailed explanation of a special case.

4.6. *Unramified local zeta integrals and local L-factors.* We define the local L-factors for the cases under consideration and recall the results from the unramified computations of the local zeta integrals as considered in [45], [75], [76] and [43].

Note that the group $G_{m^-}^{w_0}$ from the construction in Section 2.4 yields all the groups G_n as listed in the beginning of this section. Hence there exists a datum such that $G_{m^-}^{w_0}$ is isomorphic to a given G_n over F. From now on, we assume that $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_{m^-}^{w_0})$ and $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ have generic global Arthur parameters, respectively.

As in (4.1), we have $\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$, which is an irreducible generic isobaric automorphic representation of $G_{E/F}(a)(\mathbb{A}_F)$. We define

$$(4.47) \qquad \mathcal{L}(s, \tau_\nu, \pi_\nu, \sigma_\nu; \rho) = \frac{L(s + \frac{1}{2}, \tau_\nu \times \pi_\nu)}{L(s + 1, \tau_\nu \times \sigma_\nu)L(2s + 1, \tau_\nu, \rho)},$$

where

$$\rho = \begin{cases} \Lambda^2 & \text{if } H_{m+a} \text{ is an even orthogonal group;} \\ \mathrm{sym}^2 & \text{if } H_{m+a} \text{ is an odd orthogonal group; and} \\ \mathrm{As} \otimes \xi^m & \text{if } H_{m+a} \text{ is a unitary group.} \end{cases}$$

Some remarks on the local L-functions are in order. At archimedean local places or at unramified local places, the local L-functions in (4.47) are well defined. The main concern here is at the ramified finite local places. Formally, one may take the greatest common denominator of the ramified local zeta integrals as the definition or take the one from the normalization of the local intertwining operators from the Eisenstein series in the global zeta integrals. This of course needs the full theory of the local zeta integrals, which is not available at this moment for general representations π and σ. On the other hand, since both π and σ are assumed to be cuspidal and to have generic global Arthur parameters ([3, Ch. 9] and [52]), in this paper we follow [3], [70] and [52] to define the local L-functions in (4.47) at ramified finite local places in terms of the local L-functions of the corresponding localization of the global Arthur parameters. We refer to [63] for discussion with more general parameters when the groups are F-quasisplit.

We note that only when H_m is an even special orthogonal group, the twisted representation σ_ν' (see (4.34)) may not be equivalent to σ_ν if $w_q^\ell \neq I$. However, their corresponding local L-parameters are $O_{\mathfrak{m}}(\mathbb{C})$-conjugate, since $H_m^\vee(\mathbb{C}) = \mathrm{SO}_{\mathfrak{m}}$ is the complex dual group of H_m. Therefore $\mathcal{L}(s, \tau_\nu, \pi_\nu, \sigma_\nu; \rho)$

and the local L-functions $L(s, \tau_\nu \times \sigma_\nu)$ are the same when the local factors of σ_ν replaced by those of σ'_ν.

Recall that $\sigma' = \sigma^{w_q^\ell}$ in (4.34) when $a > \ell$, and also denote $\sigma' = \sigma$ when $a \leq \ell$ for notational consistency. The Euler products in Theorems 4.5 and 4.7 can be uniformly rewritten as

$$\mathcal{Z}(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\ell, w_0}) = \prod_\nu \mathcal{Z}_\nu(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\ell, w_0}).$$

Next, we state the result of unramified calculation for the local zeta integrals. The full detail of the computation in this generality will appear in our joint work with D. Soudry ([43]), based on the idea of Soudry as developed in his work ([74], [75], [76]). Many special cases have been treated in [19] and [45], for instance.

THEOREM 4.8 (Unramified Computation). *With all data being unramified, the local unramified zeta integral $\mathcal{Z}_\nu(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\ell, w_0})$ has the following expression,*

$$(4.48) \qquad \mathcal{Z}_\nu(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\ell, w_0}) = \mathcal{L}(s, \tau_\nu, \pi_\nu, \sigma_\nu; \rho),$$

where $f_{W_{\tau_\nu}^\kappa \otimes \sigma_\nu, s}$, ϕ_{σ_ν} and φ_{π_ν} are the spherical vectors, which are so normalized that the corresponding spherical functions are equal to 1 at the identity element.

Let S be a finite set of places consisting of all ramified places of relevant data and all archimedean places such that for $\nu \notin S$, all data are unramified. Following Theorem 4.8, we obtain that

$$(4.49) \qquad \begin{aligned} \mathcal{Z}(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\ell, w_0}) &= \prod_{\nu \in S} \mathcal{Z}_\nu(s, \cdot) \cdot \prod_{\nu \notin S} \mathcal{Z}_\nu(s, \cdot) \\ &= \mathcal{Z}_S(s, \cdot) \cdot \mathcal{L}^S(s, \tau, \pi, \sigma; \rho). \end{aligned}$$

Here we set $\mathcal{Z}_\nu(s, \cdot) := \mathcal{Z}_\nu(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\ell, w_0})$, $\mathcal{Z}_S(s, \cdot) := \prod_{\nu \in S} \mathcal{Z}_\nu(s, \cdot)$, and $\mathcal{L}^S(s, \cdot) := \prod_{\nu \notin S} \mathcal{L}(s, \cdot_\nu)$.

4.7. *On even special orthogonal groups.* We explain with more details the twists that we get in the case of even special orthogonal groups. We follow the notation from Section 2 of [45]. First, P_j is the standard parabolic subgroup of SO_{4a+2m} with Levi subgroup isomorphic to $\mathrm{GL}_\ell \times \mathrm{SO}_{4n+2m-2\ell}(W_\ell)$. Here $\mathrm{SO}_{4n+2m-2\ell}(W_\ell)$ preserves the quadratic space

$$W_\ell = \mathrm{Span}\{e_{\ell+1}^\pm, \ldots, e_{\mathfrak{r_m}-1}^\pm, e_{\mathfrak{r_m}}^\pm\} \oplus V_0.$$

G is the stabilizer of y_κ preserving the quadratic space

$$W_\ell \cap y_\kappa^\perp = \mathrm{Span}\{e_{\ell+1}^\pm, \ldots, e_{\mathfrak{r_m}-1}^\pm, y_{-\kappa}\} \oplus V_0.$$

The anisotropic kernel of $W_\ell \cap y_\kappa^\perp$ is a subspace of $F y_{-\kappa} \oplus V_0$. The inner period over π is arisen from the open double coset of $P_j \backslash \mathrm{SO}_{4n+2m} / G \cdot N_\ell$. Recall that

we choose the following representative η for this coset:

$$
\begin{pmatrix}
I_\ell & & & & & & \\
 & 0 & I_{j-\ell} & & & & \\
 & I_{\mathfrak{r_m}-j} & 0 & & & & \\
 & & & I_{V_0} & & & \\
 & & & & 0 & I_{\mathfrak{r_m}-j} & \\
 & & & & I_{j-\ell} & 0 & \\
 & & & & & & I_\ell
\end{pmatrix}.
$$

Recall that $G^\eta = G \cap (\eta^{-1} P_j \eta)$. Then under the conjugation of η, $\mathrm{SO}_{2m}(\eta^{-1} W_j)$ is the subgroup of G^η, which preserves

$$
\mathrm{Span}\{e_{\ell+1}^{\pm}, \ldots, e_{\mathfrak{r_m}-j+\ell}^{\pm}\} \oplus V_0.
$$

For example, when $j = \ell + 1$, then $(G, \mathrm{SO}_{2m}(\eta^{-1} W_j))$ is the Gross-Prasad pair. That is, $\mathrm{SO}_{2m}(\eta^{-1} W_j)$ is the stabilizer of the anisotropic vector $y_{-\kappa}$. Thus $\mathrm{SO}_{2m}(\eta^{-1} W_j)$ is isomorphic to $\mathrm{SO}_{2m}(W_j)$.

5. **Reciprocal non-vanishing of Bessel periods**

The purpose of this section is to address the non-vanishing property of the Bessel periods for the pair $(\mathcal{E}_{\tau \otimes \sigma}, \pi)$ and for the pair (π, σ), where $\mathcal{E}_{\tau \otimes \sigma}$ is the iterated residue at $s = \frac{1}{2}$ of the Eisenstein series $E(\cdot, \phi_{\tau \otimes \sigma}, s)$ as defined in (4.3), and σ may have to be replaced by σ' as in (4.34).

5.1. *Residue of the Eisenstein series.* Here we recall the Eisenstein series $E(\cdot, \phi_{\tau \otimes \sigma}, s)$ from (4.3). Assume as before that $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ has a generic global Arthur parameter ϕ_σ, and a cuspidal realization $\mathcal{C}_\sigma$ in the case when the cuspidal multiplicity is not one. Let $\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$ be the irreducible unitary generic isobaric automorphic representation of $G_{E/F}(a)(\mathbb{A}_F)$ associated to distinct $\tau_1, \tau_2, \ldots, \tau_r$, as given in (4.1). Assume that the generic global Arthur parameter ϕ_τ determined by τ has a different parity with ϕ_σ. It follows that the L-function

$$
L(s, \tau \times \sigma) = L(s, \phi_\tau \times \phi_\sigma),
$$

as in [3], is holomorphic at $s = \frac{1}{2}$.

We calculate the constant terms of $E(\cdot, \phi_{\tau \otimes \sigma}, s)$. According to the cuspidal support of $E(\cdot, \phi_{\tau \otimes \sigma}, s)$, among all of the constant terms that are not identically zero, the term that carries the highest order of the pole at $s = \frac{1}{2}$ is given by the global intertwining operator integral

$$
(5.1) \qquad \mathcal{M}(\omega_0, \tau \otimes \sigma, s)(\phi_{\tau \otimes \sigma})(g) := \int_{U_{\hat{a}}(\mathbb{A})} \lambda_s \phi_{\tau \otimes \sigma}(\omega_0^{-1} n g) \, dn,
$$

where $U_{\hat{a}}$ is the unipotent radical of the standard maximal parabolic subgroup $P_{\hat{a}} = M_{\hat{a}} U_{\hat{a}}$ with $M_{\hat{a}} = G_{E/F}(a) \times H_m$, and the Weyl group element ω_0 takes $U_{\hat{a}}$

to its opposite $U_{\hat{a}}^{-}$. Following the calculation of Langlands ([56] and also [72]), one may choose the factorizable section $\phi = \phi_{\tau \otimes \sigma}$ so that $\mathcal{M}(\omega_0, \tau \otimes \sigma, s)(\phi)$ can be written as

$$(5.2) \quad \mathcal{M}(\omega_0, \tau \otimes \sigma, s)_S(\phi_S) \cdot \frac{L^S(s, \tau \times \sigma) L^S(2s, \tau, \rho)}{L^S(1 + s, \tau \times \sigma) L^S(1 + 2s, \tau, \rho)} \lambda_{-s} \phi^S_{\omega_0(\tau \otimes \sigma)},$$

where $\mathcal{M}(\omega_0, \phi_{\tau \otimes \sigma}, s)_S$ is the finite product of the local intertwining operators over $\nu \in S$, and $\phi_S = \prod_{\nu \in S} \phi_{\tau_\nu \otimes \sigma_\nu}$ and $\phi^S = \otimes_{\nu \notin S} \phi_{\tau_\nu \otimes \sigma_\nu}$. Since the cuspidal automorphic representation σ is assumed to have a generic global Arthur parameter, we define, following [3], the local L-factors at $\nu \in S$ in terms of τ and the generic global Arthur parameter of σ. Then we take the Shahidi normalization by defining, for each $\nu \in S$,

$$(5.3) \qquad \mathcal{N}(\omega_0, \tau \otimes \sigma, s)_\nu := \beta_\nu(s, \tau, \sigma, \psi_F; \rho) \cdot \mathcal{M}(\omega_0, \tau \otimes \sigma, s)_\nu,$$

where the local normalizing factor $\beta_\nu(s, \tau, \sigma, \psi_F; \rho)$ is defined to be

$$(5.4) \qquad \frac{L_\nu(1 + s, \tau \times \sigma) L_\nu(1 + 2s, \tau, \rho) \epsilon_\nu(s, \tau \times \sigma, \psi_F) \epsilon_\nu(2s, \tau, \rho, \psi_F)}{L_\nu(s, \tau \times \sigma) L_\nu(2s, \tau, \rho)}.$$

Hence we obtain the following:

$$\mathcal{M}(\omega_0, \tau \otimes \sigma, s) = \frac{\mathcal{N}(\omega_0, \tau \otimes \sigma, s) \cdot L(s, \tau \times \sigma) L(2s, \tau, \rho)}{L(1 + s, \tau \times \sigma) L(1 + 2s, \tau, \rho) \epsilon(s, \tau \times \sigma) \epsilon(2s, \tau, \rho)}.$$

We call $\mathcal{N}(\omega_0, \tau \otimes \sigma, s)_\nu$ the normalized local intertwining operators.

THEOREM 5.1. *Let $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ be the irreducible isobaric automorphic representation of $G_{E/F}(a)(\mathbb{A})$ as in (4.1), and let $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ with a generic global Arthur parameter ϕ_σ. Then, for each local place ν of F, the normalized local intertwining operator $\mathcal{N}(\omega_0, \tau \otimes \sigma, s)_\nu$ from the induced space $\mathrm{Ind}_{P_{\hat{a}}(F_\nu)}^{H_{a+m}(F_\nu)} |\cdot|^s \tau_\nu \otimes \sigma_\nu$ to $\mathrm{Ind}_{P_{\hat{a}}(F_\nu)}^{H_{a+m}(F_\nu)} |\cdot|^{-s} \tau_\nu^* \otimes \sigma_\nu$ is holomorphic and non-zero for $\mathrm{Re}(s) \geq \frac{1}{2}$, where $\tau_\nu^* = \iota(\tau_\nu)^\vee$ is the contragredient of $\iota(\tau)$.*

We remark that when H_m is F-quasisplit, a much stronger result than what was stated in Theorem 5.1 can be proved when σ is also assumed to be generic; see [11], for instance, and also see [63]. We will prove Theorem 5.1 in Appendix B.

By Theorem 5.1, we have that the normalized global intertwining operator $\mathcal{N}(\omega_0, \tau \otimes \sigma, s)$ is holomorphic and non-zero for $\mathrm{Re}(s) \geq \frac{1}{2}$. We are able to study the order of the pole at $s = \frac{1}{2}$ of the global intertwining operator $\mathcal{M}(\omega_0, \tau \otimes \sigma, s)$.

In fact, it is easy to write

$$L(2s, \tau, \rho) = \prod_{j=1}^{r} L(2s, \tau_j, \rho) \cdot \prod_{1 \leq i < j \leq r} L(2s, \tau_i \times \tau_j^\iota).$$

Since $\tau_1, \ldots, \tau_r$ are conjugate self-dual and distinct for all $1 \leq i < j \leq r$, $L(2s, \tau_i \times \tau_j^\iota)$ is holomorphic and non-zero at $s = \frac{1}{2}$. It follows that the L-function $L(2s, \tau, \rho)$ has a pole at $s = \frac{1}{2}$ of order r. Since the generic global Arthur parameter ϕ_τ associated to τ and the generic global Arthur parameter ϕ_σ associated to σ are in different parity, the L-function $L(s, \tau \times \sigma)$ must be of symplectic type ([16]), and it is holomorphic at $s = \frac{1}{2}$, but may have zero at $s = \frac{1}{2}$. It follows that the global intertwining operator $\mathcal{M}(\omega_0, \tau \otimes \sigma, s)$ has a pole at $s = \frac{1}{2}$ of order at most r, and has the pole of order exactly r if and only if the L-function $L(s, \tau \times \sigma)$ is non-zero at $s = \frac{1}{2}$. This implies that the Eisenstein series $E(\cdot, \phi_{\tau \otimes \sigma}, s)$ has a pole at $\frac{1}{2}$ of order at most r, and it has a pole of order r at $s = \frac{1}{2}$ if and only if $L(s, \tau \times \sigma)$ is non-zero at $s = \frac{1}{2}$. We summarize this result as follows.

PROPOSITION 5.2. *Assume that $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ has a generic global Arthur parameter ϕ_σ. Let $\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$ be the irreducible unitary generic isobaric automorphic representation of $G_{E/F}(a)(\mathbb{A}_F)$ associated to distinct $\tau_1, \tau_2, \ldots, \tau_r$, as given in (4.1). Assume that the generic global Arthur parameter ϕ_τ determined by τ has a different parity with ϕ_σ. Then the L-function $L(s, \tau \times \sigma)$ is holomorphic at $s = \frac{1}{2}$, and the Eisenstein series $E(\cdot, \phi_{\tau \otimes \sigma}, s)$ has a pole at $s = \frac{1}{2}$ of order at most r. Moreover, $E(\cdot, \phi_{\tau \otimes \sigma}, s)$ has a pole at $s = \frac{1}{2}$ of order r if and only if $L(s, \tau_i, \rho)$ has a pole at $s = 1$ for $i = 1, 2, \ldots, r$, and $L(s, \tau \times \sigma)$ is non-zero at $s = \frac{1}{2}$.*

When the Eisenstein series $E(\cdot, \phi_{\tau \otimes \sigma}, s)$ has a pole at $s = \frac{1}{2}$ of order r, we denote by $\mathcal{E}_{\tau \otimes \sigma}$ the r-th iterated residue at $s = \frac{1}{2}$ of $E(\cdot, \phi_{\tau \otimes \sigma}, s)$.

5.2. *Special data for Bessel periods.* We are going to choose a set of special data in order to establish the reciprocal non-vanishing of the Bessel periods for the pair $(\mathcal{E}_{\tau \otimes \sigma}, \pi)$ and for the pair (π, σ).

Take as before the classical group $G_n = \mathrm{Isom}(V_{\mathfrak{n}}, q)^\circ$. The group G_n is a pure inner F-form of an F-quasisplit classical group $G_n^* = \mathrm{Isom}(V_{\mathfrak{n}}^*, q^*)^\circ$ of the same type. Here $\mathfrak{n} = \dim_E V_{\mathfrak{n}} = \dim_E V_{\mathfrak{n}}^*$ and $n = [\frac{\mathfrak{n}}{2}]$. Recall from Section 2.2 that $N = \mathfrak{n}^\vee$ is $\mathfrak{n}$ if G_n is a unitary group or an even special orthogonal group, and it is $\mathfrak{n} - 1$ if G_n is an odd special orthogonal group.

Assume that $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ has a G_n-relevant, generic global Arthur parameter $\phi \in \widetilde{\Phi}_2(G_n^*)$. As in (1.2), the generic global Arthur parameter ϕ determines an irreducible unitary generic isobaric automorphic representation $\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$ of $G_{E/F}(N)(\mathbb{A}_F)$, as given in (4.1). Recall that the sign κ of ξ is $+1$ for the global A-parameters of unitary groups, as explained in Section 2.2. Take a G_n-relevant partition

$$(5.5) \qquad \underline{p}_{\ell_*} = [(2\ell_* + 1)1^{\mathfrak{n} - 2\ell_* - 1}],$$

and consider the ℓ_*-th Bessel module $\mathcal{F}^{\mathcal{O}_{\ell_*}}(\pi)$ of π, or $\mathcal{F}^{\mathcal{O}_{\ell_*}}(\mathcal{C}_\pi)$ for a cuspidal realization $\mathcal{C}_\pi$ of π. Since π is irreducible and cuspidal, $\mathcal{F}^{\mathcal{O}_{\ell_*}}(\pi)$ consists of rapidly decreasing automorphic functions on $H_{\ell_*^-}^{\mathcal{O}_{\ell_*}}(\mathbb{A})$, and hence is a sub-representation of $H_{\ell_*^-}^{\mathcal{O}_{\ell_*}}(\mathbb{A})$ in the space of L^2-automorphic functions on $H_{\ell_*^-}^{\mathcal{O}_{\ell_*}}(\mathbb{A})$. Note that the group $H_{\ell_*^-}^{\mathcal{O}_{\ell_*}}$ is a pure inner F-form of an F-quasisplit $H_{\ell_*^-}^*$ with $\ell_*^- = [\frac{\mathfrak{l}_*^-}{2}]$ and $\mathfrak{l}_*^- = \mathfrak{n} - 2\ell_* - 1$.

We further assume that $\mathcal{F}^{\mathcal{O}_{\ell_*}}(\mathcal{C}_\pi)$ is non-zero and has the property that there exists a $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_{\ell_*^-}^{\mathcal{O}_{\ell_*}})$ with a generic global Arthur parameter, such that the inner product

$$(5.6) \qquad \mathcal{P}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi, \varphi_\sigma) = \left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_{\ell_*^-}^{\mathcal{O}_{\ell_*}}} \neq 0$$

for some $\varphi_\pi \in \mathcal{C}_\pi$ and $\varphi_\sigma \in \mathcal{C}_\sigma$, where $\mathcal{C}_\sigma$ is a cuspidal realization of σ.

Note that the index ℓ_* may not be the *first occurrence index* as described in Conjecture 2.3. In this generality, the discussion in this section can also be applied to the proof of one of the directions of the global Gan-Gross-Prasad conjecture in Section 5.5.

In this section we take that $m := \ell_*^-$ and $\mathfrak{m} := \mathfrak{l}_*^- = \mathfrak{n} - 2\ell_* - 1$. In the definition of global zeta integrals in Section 4.1, we take

$$(5.7) \qquad H_m = H_{\ell_*^-}^{\mathcal{O}_{\ell_*}} \quad \text{and} \quad a = N = \mathfrak{n}^\vee$$

and take H_{a+m} to be the classical group containing the Levi subgroup $G_{E/F}(a) \times H_m$. To define the global zeta integrals, we take the partition

$$(5.8) \qquad \underline{p}_{\kappa_*} = [(2\kappa_* + 1)1^{2a+\mathfrak{m}-2\kappa_*-1}]$$

with $\kappa_* := a - \ell_* - 1$. It is a partition of type $(2a + \mathfrak{m}, H_{a+m})$.

Since $a - \kappa_* = \ell_* + 1 > 0$, we are in the situation of Section 4.4. For any F-rational orbit $\mathcal{O}_{\kappa_*}$ in the F-stable orbit $\mathcal{O}_{\underline{p}_{\kappa_*}}^{\mathrm{st}}$, we have the stabilizer $G_{m^-}^{\mathcal{O}_{\kappa_*}} = G_{m^-}^{w_*}$ as in Proposition 4.3, with $(\kappa_*)^- = m^-$. The integer $\mathfrak{m}^-$, such that $m^- = [\frac{\mathfrak{m}^-}{2}]$, can be calculated as follows. By definition, we have

$$(5.9) \qquad \mathfrak{m}^- = 2a + \mathfrak{m} - 2\kappa_* - 1 = \mathfrak{m} + 2(a - \kappa_*) - 1.$$

Since $a - \kappa_* = \ell_* + 1$, we have $\mathfrak{m}^- = \mathfrak{m} + 2\ell_* + 1$. Since $\mathfrak{m} = \mathfrak{n} - 2\ell_* - 1$, we must have that $\mathfrak{m}^- = \mathfrak{n}$ and hence that $m^- = n$. By Proposition 2.6, and the relation of the three groups (H_{a+m}, G_n, H_m), it is not hard to find the F-anisotropic vector w_* corresponding to the F-rational orbit $\mathcal{O}_{\kappa_*}$ such that G_n can be identified with the stabilizer $G_{m^-}^{\mathcal{O}_{\kappa_*}} = G_{m^-}^{w_*}$.

Recall that $\mathcal{E}_{\tau \otimes \sigma}$ is the iterated residue at $s = \frac{1}{2}$ of the Eisenstein series $E(\cdot, \phi_{\tau \otimes \sigma}, s)$. The reciprocal non-vanishing of the Bessel periods for the pair $(\mathcal{E}_{\tau \otimes \sigma}, \pi)$ and for the pair (π, σ') is given below.

THEOREM 5.3 (Reciprocal non-vanishing of Bessel periods). *Assume that* $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ *has a generic global Arthur parameter* ϕ_σ. *Let* $\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$ *be the irreducible unitary generic isobaric automorphic representation of* $G_{E/F}(a)(\mathbb{A}_F)$ *with* $a = N = \mathfrak{n}^\vee$, *which determines a generic global Arthur parameter* ϕ_τ *of* G_n^*. *Assume that the residue* $\mathcal{E}_{\tau \otimes \sigma'}$ *is non-zero and* $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ *has a generic global Arthur parameter* ϕ_τ. *Then the Bessel period* $\left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa_*}}}(\mathcal{E}_{\tau \otimes \sigma'})} \right\rangle_{G_n}$ *for the pair* $(\mathcal{E}_{\tau \otimes \sigma'}, \pi)$ *is non-zero for some choice of data if and only if the Bessel period* $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m}$ *for the pair* (π, σ) *is non-zero for some choice of data.*

By using Corollary 4.4, it is easy to prove that if the Bessel period $\left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa_*}}}(\mathcal{E}_{\tau \otimes \sigma'})} \right\rangle_{G_n}$ is non-zero for some choice of data, then the Bessel period $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m}$ is non-zero for some choice of data. In fact, if $\left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa_*}}}(\mathcal{E}_{\tau \otimes \sigma'})} \right\rangle_{G_n}$ is not identically zero, then by replacing the residue $\mathcal{E}_{\tau \otimes \sigma'}$ by the Eisenstein series $E(\cdot, \phi_{\tau \otimes \sigma'}, s)$, we obtain that the global zeta integral $\mathcal{Z}(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_*}})$ is not identically zero for $\mathrm{Re}(s)$ large. Hence by Corollary 4.4, the Bessel period $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m}$ is non-zero for some choice of data.

The proof of the opposite direction is more technical. We have to know enough analytic properties of the local zeta integrals at the ramified and the archimedean local places.

5.3. *Normalization of local zeta integrals.* We continue our discussion of the global and local zeta integrals from Section 4, with special data as given in Section 5.2. In particular, we will deal with the case where $a - \kappa_* = \ell_* + 1 > 0$, which is the case of Section 4.4. Recall from (4.49) that the global zeta integral has the following expression,

$$\mathcal{Z}(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_*}}) = \mathcal{Z}_S(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_*}}) \cdot \mathcal{L}^S(s, \tau, \pi, \sigma; \rho),$$

where $\mathcal{Z}_S(s, \cdot) = \prod_{\nu \in S} \mathcal{Z}_\nu(s, \cdot)$ is the finite Euler product with the local zeta integral $\mathcal{Z}_\nu(s, \cdot)$ as in (4.41), and

$$\mathcal{L}^S(s, \tau, \pi, \sigma; \rho) = \prod_{\nu \notin S} \mathcal{L}(s, \tau_\nu, \pi_\nu, \sigma_\nu; \rho)$$

with $\mathcal{L}(s, \tau_\nu, \pi_\nu, \sigma_\nu; \rho)$ as in (4.47). Recall that S consists of all ramified places of relevant data and all archimedean places such that for $\nu \notin S$, all data are unramified. Hence at $\nu \notin S$ we only consider spherical vectors in the discussion.

We normalize the local zeta integrals by

$$(5.10) \qquad \mathcal{Z}_\nu^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_*}}) := \frac{\mathcal{Z}_\nu(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_*}})}{\mathcal{L}(s, \tau_\nu, \pi_\nu, \sigma_\nu; \rho)}.$$

By taking the finite product, we have $\mathcal{Z}_S(s, \cdot) = \mathcal{Z}_S^*(s, \cdot) \cdot \mathcal{L}_S(s, \cdot)$. Hence the formula in (4.49) becomes

$$(5.11) \qquad \mathcal{Z}(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}}) = \mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}}) \cdot \mathcal{L}(s, \tau, \pi, \sigma; \rho).$$

PROPOSITION 5.4. *The assumption on* (π, τ, σ) *is taken as in Theorem 5.3. Then the following hold*:

(1) $\mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}})$ *is meromorphic in* $s \in \mathbb{C}$ *for any choice of the smooth sections* $\phi_{\tau \otimes \sigma', s}$.

(2) $\mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}})$ *is holomorphic at* $s = \frac{1}{2}$ *for any choice of the smooth sections* $\phi_{\tau \otimes \sigma', s}$.

Note that $\phi_{\tau \otimes \sigma', s}$ is called a *smooth section* if $\phi_{\tau_\nu \otimes \sigma'_\nu, s}$ is a smooth holomorphic section at archimedean places and is a flat section at non-archimedean places.

Proof. Recall from (4.9) that we have

$$\mathcal{Z}(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}}) = \left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa*}}}(E(\cdot, \phi_{\tau \otimes \sigma'}, s))} \right\rangle_{G_n},$$

and hence by (5.11), we obtain
$$(5.12)$$
$$\left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa*}}}(E(\cdot, \phi_{\tau \otimes \sigma'}, s))} \right\rangle_{G_n} = \mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}}) \cdot \mathcal{L}(s, \tau, \pi, \sigma; \rho).$$

We first consider the right-hand side of the identity in (5.12). In the L-function part, we have

$$(5.13) \qquad \mathcal{L}(s, \tau, \pi, \sigma; \rho) = \frac{L(s + \frac{1}{2}, \tau \times \pi)}{L(s + 1, \tau \times \sigma) L(2s + 1, \tau, \rho)}.$$

Since the cuspidal automorphic representations π and σ are assumed to have generic global Arthur parameters, the complete L-functions $L(s, \tau \times \pi)$ and $L(s, \tau \times \sigma)$ are defined in terms of the global Arthur parameters of π and σ, respectively. Hence $\mathcal{L}(s, \tau, \pi, \sigma; \rho)$ is meromorphic in s over $\mathbb{C}$.

In the left-hand side of the identity in (5.12), the Fourier coefficient $\mathcal{F}^{\psi_{\mathcal{O}_{\kappa*}}}(E(\cdot, \phi_{\tau \otimes \sigma'}, s))$ is meromorphic in s over $\mathbb{C}$ and the inner product

$$\left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa*}}}(E(\cdot, \phi_{\tau \otimes \sigma'}, s))} \right\rangle_{G_n}$$

is well defined when s is away from the poles of $\mathcal{F}^{\psi_{\mathcal{O}_{\kappa*}}}(E(\cdot, \phi_{\tau \otimes \sigma'}, s))$, since φ_π is cuspidal. Hence the left-hand side of the identity is meromorphic in s over $\mathbb{C}$. It follows that the finite product of the normalized local zeta integrals, $\mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}})$, is a meromorphic function in s over $\mathbb{C}$ for any choice of $\phi_{\tau \otimes \sigma'}$ with the given φ_σ and for the given φ_π. This proves part (1).

In order to prove part (2), we need more specific information from both sides. In the expression (5.13), the product

$$L\left(s + \frac{1}{2}, \tau \times \pi\right) = \prod_{i=1}^{r} L\left(s + \frac{1}{2}, \tau_i \times \pi\right)$$

has a pole at $s = \frac{1}{2}$ of order r, since the cuspidal automorphic representation π has the generic global Arthur parameter ϕ_τ with $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$, as in (4.1). It is clear that the product

$$L(s + 1, \tau \times \sigma) = \prod_{i=1}^{r} L(s + 1, \tau_i \times \sigma)$$

and the product

$$L(2s + 1, \tau, \rho) = \prod_{i=1}^{r} L(2s + 1, \tau_i, \rho) \times \prod_{1 \leq i < j \leq r} L(2s + 1, \tau_i \times \tau_j^\iota)$$

are holomorphic and non-zero at $s = \frac{1}{2}$. It follows that the L-function part $\mathcal{L}(s, \tau, \pi, \sigma; \rho)$ has a pole at $s = \frac{1}{2}$ of order r.

In order to show that the finite product of the normalized local zeta integrals, $\mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_*}})$, is holomorphic at $s = \frac{1}{2}$, it is enough to show that the inner product in the left-hand side of (5.12),

$$\left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa_*}}}(E(\cdot, \phi_{\tau \otimes \sigma'}, s))} \right\rangle_{G_n},$$

has a pole at $s = \frac{1}{2}$ of order at most r for any smooth sections $\phi_{\tau \otimes \sigma', s}$.

By Proposition 5.2, the Fourier coefficient $\mathcal{F}^{\psi_{\mathcal{O}_{\kappa_*}}}(E(\cdot, \phi_{\tau \otimes \sigma'}, s))$ of the Eisenstein series $E(\cdot, \phi_{\tau \otimes \sigma'}, s)$ has a pole at $s = \frac{1}{2}$ of order at most r for any smooth sections $\phi_{\tau \otimes \sigma', s}$. The inner product of the Fourier coefficient with the cuspidal automorphic form φ_π cannot increase the order of the pole at $s = \frac{1}{2}$. It follows that the left-hand side of (5.12) has a pole at $s = \frac{1}{2}$ of order at most r. Therefore, we obtain that the finite product of the normalized local zeta integrals, $\mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_*}})$, must be holomorphic at $\frac{1}{2}$. This proves part (2). $\square$

In order to obtain more properties of the local zeta integrals or the normalized ones for the global purpose in this paper, we have to introduce the global condition that the Bessel period for (π, σ) is non-zero. For $\varphi_\pi \in \mathcal{C}_\pi$ and $\varphi_\sigma \in \mathcal{C}_\sigma$, the Bessel period $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m}$, as in (5.6) with the given data, defines a non-zero element in the one-dimensional space

$$\otimes_\nu \mathrm{Hom}_{H_m(F_\nu)}(\mathcal{J}_{\psi_{\mathcal{O}_{\ell_*}}}(\pi_\nu) \otimes \sigma_\nu, \mathbb{C}),$$

where $\mathcal{J}_{\psi_{\mathcal{O}_{\ell_*}}}(\pi_\nu)$ is the local twisted Jacquet module of π_ν with respect to $(V_{\underline{p}_{\ell_*}}, \psi_{\mathcal{O}_{\ell_*}})$. Let $\mathfrak{b}_\nu^{\psi_{\mathcal{O}_{\ell_*}}}$ be a non-zero functional in

$$\mathrm{Hom}_{H_m(F_\nu)}(\mathcal{J}_{\psi_{\mathcal{O}_{\ell_*}}}(\pi_\nu) \otimes \sigma_\nu, \mathbb{C}),$$

which is unique, up to scalar. We normalize the functional $\mathfrak{b}_\nu^{\psi_{\mathcal{O}_{\ell_*}}}$, so that at the unramified local places,

$$\mathfrak{b}_\nu^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_{\pi_\nu}, \varphi_{\sigma_\nu}) = 1,$$

where φ_{π_ν} and φ_{σ_ν} are normalized spherical vectors in π_ν and σ_ν, respectively. Here a spherical vector is normalized if its corresponding spherical function has value 1 at the identity. And at the ramified local places $\nu \in S$, the local functional $\mathfrak{b}_\nu^{\psi_{\mathcal{O}_{\ell_*}}}$ will be normalized according to (A.9) in Appendix A. Hence we obtain the following identity: for factorizable factors $\varphi_\pi = \otimes_\nu \varphi_{\pi_\nu}$ and $\varphi_\sigma = \otimes_\nu \varphi_{\sigma_\nu}$,

$$(5.14) \qquad \left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m} = c_{\pi,\sigma} \cdot \prod_\nu \mathfrak{b}_\nu^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_{\pi_\nu}, \varphi_{\sigma_\nu}),$$

by the uniqueness of the local Bessel models ([2], [77], [16] and [44]).

PROPOSITION 5.5. *The assumption on* (π, τ, σ) *is taken as in Theorem* 5.3. *Fix any given* $s = s_0 \in \mathbb{C}$. *If for every* $\nu \in S$, *the local pairing* $\mathfrak{b}_\nu^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_{\pi_\nu}, \varphi_{\sigma_\nu})$ *is non-zero for some* $\varphi_{\sigma_\nu} \in \sigma_\nu$ *and* $\varphi_{\pi_\nu} \in \pi_\nu$, *then there exists a collection of sections* $\phi_{\tau_\nu \otimes \sigma_\nu'}$ *with* ν *running in* S *such that the finite product of the local zeta integrals,* $\mathcal{Z}_S(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_*}})$, *is a non-zero constant at* $s = s_0$.

The proof of Proposition 5.5 will be given in Appendix A.

PROPOSITION 5.6. *The assumption on* (π, τ, σ) *is taken as in Theorem* 5.3. *Assume further that* (π, σ) *has a non-zero Bessel period. Then there exist factorizable data* φ_π, φ_σ, *and* $\phi_{\tau \otimes \sigma'}$ *such that* $\mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_*}})$ *at* $s = \frac{1}{2}$ *and the inner product* $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m}$ *are simultaneously non-zero.*

Proof. By assumption, the Bessel period $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m}$ is not zero for the pair (π, σ). By (5.14), for factorizable vectors $\varphi_\pi = \otimes_\nu \varphi_{\pi_\nu}$ and $\varphi_\sigma = \otimes_\nu \varphi_{\sigma_\nu}$, we have

$$\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m} = c_{\pi,\sigma} \cdot \prod_\nu \mathfrak{b}_\nu^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_{\pi_\nu}, \varphi_{\sigma_\nu}).$$

Since $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m}$ is not zero, it follows that $c_{\pi,\sigma} \neq 0$ and $\mathfrak{b}_\nu(\varphi_{\pi_\nu}, \varphi_{\sigma_\nu})$ is non-zero for every ν. By Proposition 5.5, there exists a smooth factorizable section $\phi_{\tau \otimes \sigma'}$ such that the finite product of the local zeta integral $\mathcal{Z}_S(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_*}})$ is non-zero at $s = \frac{1}{2}$.

Recall from (5.10) that we have

$$\mathcal{Z}_S(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}}) = \mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}}) \cdot \mathcal{L}_S(s, \tau, \pi, \sigma; \rho).$$

By using the structures of generic unitary dual of τ_ν, π_ν, and σ_ν for each $\nu \in S$, respectively, and by following similar arguments with Appendix B, we obtain that the normalizing factors $\mathcal{L}(s, \tau_\nu, \pi_\nu, \sigma_\nu; \rho)$ as defined in (4.47) for each $\nu \in S$ is holomorphic for $\mathrm{Re}(s) \geq \frac{1}{2}$. Hence the finite Euler product $\mathcal{L}_S(s, \tau, \pi, \sigma; \rho)$ is holomorphic for $\mathrm{Re}(s) \geq \frac{1}{2}$. This proves that the normalized $\mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}})$ is also non-zero at $\frac{1}{2}$. $\square$

5.4. *Proof of Theorem 5.3.* We already proved one direction of Theorem 5.3. Now we are ready to prove the other direction of Theorem 5.3.

By the assumptions in Theorem 5.3, the equation in (5.12) reads

$$\left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa*}}}(E(\cdot, \phi_{\tau \otimes \sigma'}, s))} \right\rangle_{G_n} = \mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}}) \cdot \mathcal{L}(s, \tau, \pi, \sigma; \rho).$$

By Proposition 5.4, $\mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}})$ is meromorphic in s and is holomorphic at $s = \frac{1}{2}$ for any section $\phi_{\tau \otimes \sigma'}$ depending on the choice of φ_π and φ_σ with property that the Bessel period $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m}$ is non-zero. Furthermore, by Proposition 5.6, there exists a choice of factorizable φ_π, φ_σ, and $\otimes_\nu f_{\mathcal{W}_{\tau_\nu}^\kappa \otimes \sigma'_\nu}$ that occur in the definition of the local zeta integrals (see (4.38)), such that both $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell*}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m}$ is non-zero and $\mathcal{Z}_S^*(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa*}})$ is non-zero at $s = \frac{1}{2}$.

With such a choice of data, and with a factorizable $\phi_{\tau \otimes \sigma'} = \otimes_\nu \phi_{\tau_\nu \otimes \sigma'_\nu}$ corresponding to the above $\otimes_\nu f_{\mathcal{W}_{\tau_\nu}^\kappa \otimes \sigma'_\nu}$, the right-hand side of (5.12) has a pole at $s = \frac{1}{2}$ of order r. It follows that the left-hand side of (5.12), i.e., $\left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa*}}}(E(\cdot, \phi_{\tau \otimes \sigma'}, s))} \right\rangle_{G_n}$, has a pole at $s = \frac{1}{2}$ of order r. Since the Eisenstein series $E(\cdot, \phi_{\tau \otimes \sigma'}, s)$ has a pole at $s = \frac{1}{2}$ of order at most r, we must have that $E(\cdot, \phi_{\tau \otimes \sigma'}, s)$ has a pole at $s = \frac{1}{2}$ of order r. Since φ_π is cuspidal, by taking the iterated residue of $\left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa*}}}(E(\cdot, \phi_{\tau \otimes \sigma'}, s))} \right\rangle_{G_n}$ at $s = \frac{1}{2}$, we obtain that the Bessel period $\left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa*}}}(\mathcal{E}_{\tau \otimes \sigma'})} \right\rangle_{G_n}$ is non-zero with such a chosen data where, as before, $\mathcal{E}_{\tau \otimes \sigma'}$ denotes the iterated residue of the Eisenstein series $E(\cdot, \phi_{\tau \otimes \sigma'}, s)$ at the pole $s = \frac{1}{2}$ of order exactly equal to r. This completes the proof of Theorem 5.3.

5.5. *Global Gan-Gross-Prasad conjecture*: *one direction.* We are ready to derive the proof of one of the two directions of the global Gan-Gross-Prasad conjecture (Conjectures 24.1 and 26.1 in [16]), as a continuation of the proof of Theorem 5.3 in Section 5.4.

Assume that the Bessel period $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \varphi_\sigma \right\rangle_{H_m}$ for the pair (π, σ) is non-zero for some φ_π and φ_σ as in (5.6). By the same proof as in Section 5.4, we obtain that the Fourier coefficient $\mathcal{F}^{\psi_{\mathcal{O}_{\kappa_*}}}(\mathcal{E}_{\tau \otimes \bar{\sigma}'})$ is non-zero, where $\bar{\sigma}'$ is the complex conjugate of σ'. As a consequence, we obtain that the iterated residual representation $\mathcal{E}_{\tau \otimes \bar{\sigma}'}$ is non-zero. By Proposition 5.2, we obtain that $L(s, \tau \times \bar{\sigma}) = L(s, \tau \times \bar{\sigma}')$ is holomorphic and non-zero at $s = \frac{1}{2}$.

Because $\bar{\sigma}$ is isomorphic to the contragredient $\sigma^\vee$ of σ, it follows that $L(s, \tau \times \sigma^\vee)$ is holomorphic and non-zero at $s = \frac{1}{2}$, and so is $L(s, \tau \times \sigma)$. This proves one direction of the global Gan-Gross-Prasad conjecture ([16]) in the full generality for the classical groups considered in this paper.

THEOREM 5.7 (Global Gan-Gross-Prasad conjecture: one direction). *For any* $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ *with a* G_n-*relevant, generic global Arthur parameter in* $\widetilde{\Phi}_2(G_n^*)_{G_n}$, *and with a cuspidal realization* $\mathcal{C}_\pi$ *of* π, *assume that the Bessel period*

$$\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_*}}}(\varphi_\pi), \varphi_\sigma \right\rangle_{H_m}$$

is non-zero with a choice of $\varphi_\pi \in \mathcal{C}_\pi$ *and* $\varphi_\sigma \in \mathcal{C}_\sigma$, *for some* $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ *with an* H_m-*relevant, generic global Arthur parameter in* $\widetilde{\Phi}_2(H_m^*)_{H_m}$, *and with a cuspidal realization* $\mathcal{C}_\sigma$ *of* σ. *Then the tensor product* L-*function* $L(s, \pi \times \sigma) = L(s, \tau \times \sigma)$ *must be holomorphic and non-zero at* $s = \frac{1}{2}$.

Some remarks are in order. First of all, the original global Gan-Gross-Prasad conjecture in [16] assumes that the cuspidal multiplicity of $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ should be one. Theorem 5.7 takes care of the even special orthogonal group case, where the cuspidal multiplicity of π could be two.

When $G_n = G_n^*$ and $H_m = H_m^*$ are F-quasisplit, and when both π and σ are generic, i.e., have non-zero Whittaker-Fourier coefficients, and have simple, generic global Arthur parameters, i.e., their Langlands functorial transfers to the corresponding general linear groups are cuspidal, Theorem 5.7 was considered in [20], [21], and [22] by an approach mixing the Arthur truncation method and the Rankin-Selberg method. Recently, it was noticed by experts that there exists a gap in the proof of Proposition 5.3 in [20], which was duplicated in [21] and [22]. This technical gap is crucial to the complete proof of the special case of Theorem 5.7 considered in those papers, and it needs to be filled up.

Meanwhile, the assumption of the genericity of both π and σ and the assumption of the cuspidality of the functorial transfer of π and σ to general linear groups are critical to make the arguments and proofs work before Proposition 5.3 in [20], and the same in [21] and [22]. Those restrictions disappear in the approach taken in this paper. It seems to the authors of this paper that the approach taken up using the general framework (including the

twisted automorphic descents and the reciprocal non-vanishing for Bessel periods) considered in this paper is a more natural and conceptual way to attack the global Gan-Gross-Prasad conjecture.

It is also very important to mention that W. Zhang proved the global Gan-Gross-Prasad conjecture ([88] and [87]) for unitary groups $U_n \times U_{n-1}$, with certain local assumptions, and with the global assumption on cuspidality of the global Langlands functorial transfers from unitary groups to general linear groups. His approach is based on the Jacquet-Rallis relative trace formula originally developed in [34] for unitary groups. The progress to extend the approach of Zhang to a more general situation has been picked up by Y. Liu ([58]) and by H. Xue ([84]). However, this relative trace formula approach is so far not known to be available for classical groups that are not unitary groups. The approach taken up in this paper treats both unitary groups and orthogonal groups uniformly. The same approach is expected to work for symplectic groups and metaplectic groups with replacement of Bessel models by Fourier-Jacobi models. We refer to our work ([46]) for more details.

The other direction of the global Gan-Gross-Prasad conjecture ([16]) is more delicate and will be discussed in Section 6.3 with an assumption on the structure of Fourier coefficients of the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ on $H_{a+m}(\mathbb{A})$. See Theorem 6.10 for details.

6. Twisted automorphic descents

We develop here a basic theory of twisted automorphic descents and point out two relevant applications. One is discussed in Section 6.2 on the explicit construction of cuspidal automorphic modules for any irreducible cuspidal automorphic representations of G_n and another is discussed in Section 6.3 on the other direction of the global Gan-Gross-Prasad conjecture.

6.1. *Automorphic descents and certain Arthur packets.* For a given $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ with a G_n-relevant, generic global Arthur parameter $\phi = \phi_\tau \in \widetilde{\Phi}_2(G_n^*)$, we recall that ϕ_τ has the form

$$(\tau_1, 1) \boxplus (\tau_2, 1) \boxplus \cdots \boxplus (\tau_r, 1) \in \widetilde{\Phi}_2(G_n^*).$$

Remark that we choose the sign $\kappa = +1$ for the unitary group case as in Section 2.2. Take a $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ with an H_m-relevant, generic global Arthur parameter $\phi_\sigma \in \widetilde{\Phi}_2(H_m^*)$, and define a non-generic global Arthur parameter by

$$(6.1) \qquad \psi_{\tau,\sigma} := (\tau_1, 2) \boxplus (\tau_2, 2) \boxplus \cdots \boxplus (\tau_r, 2) \boxplus \phi_\sigma.$$

Clearly $\psi_{\tau,\sigma}$ belongs to $\widetilde{\Psi}_2(H_{a+m}^*)$ and is H_{a+m}-relevant. Let $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$ be the global Arthur packet attached to the global Arthur parameter $\psi_{\tau,\sigma}$ in (6.1). As in [41], one may easily verify the following property.

PROPOSITION 6.1. *The residual representation $\mathcal{E}_{\tau \otimes \sigma}$ is square integrable and, if non-zero, belongs to the global Arthur packet $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$ with the global Arthur parameter $\psi_{\tau,\sigma}$ given in* (6.1).

From the endoscopic classification of Arthur in [3], it is expected that the global Arthur packet $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$ contains some members that belong to $\mathcal{A}_{\mathrm{disc}}(H_{a+m})$. If these automorphic members are not of residue type, they are cuspidal. The twisted automorphic descent is an approach to understand the structures and the properties of the global packet $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$, instead of a certain individual member in the global packet $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$.

We assume that a $\Sigma \in \mathcal{A}_{\mathrm{disc}}(H_{a+m})$ has the global Arthur parameter $\psi_{\tau,\sigma}$ as given in (6.1), and has a discrete realization $\mathcal{C}_{\Sigma}$. Consider Fourier coefficients associated to the partitions of the form

$$\underline{p}_{\kappa} = [(2\kappa + 1)1^{2a+\mathfrak{m}-2\kappa-1}]$$

of $2a + \mathfrak{m}$ with $0 \leq \kappa \leq a + \mathfrak{r}_{\mathfrak{m}}$, where $\mathfrak{r}_{\mathfrak{m}}$ is the F-rank of H_m. It is clear that the partition $\underline{p}_{\kappa}$ is of type $(2a + \mathfrak{m}, H_{a+m}^*)$. As in Section 2.3, we study the $\psi_{\underline{p}_{\kappa},\mathcal{O}_{\kappa}}$-Fourier coefficient of $f_{\Sigma} \in \mathcal{C}_{\Sigma}$ and denote by $\mathcal{F}_{\kappa^-}^{\mathcal{O}_{\kappa}}(\Sigma)$ the κ-th Bessel module of $G_{\kappa^-}^{\mathcal{O}_{\kappa}}(\mathbb{A})$ generated by all the Fourier coefficients $\mathcal{F}^{\psi_{\mathcal{O}_{\kappa}}}(f_{\Sigma})$ with all $f_{\Sigma} \in \mathcal{C}_{\Sigma}$. As in [23] for the case $m = 0$ and in [40] for $m = 1$, we prove the following proposition by investigating the local structure at one unramified place of the global Arthur parameter $\psi_{\tau,\sigma}$ given in (6.1).

PROPOSITION 6.2. *Assume that a $\Sigma \in \mathcal{A}_{\mathrm{disc}}(H_{a+m})$ belongs to the global Arthur packet $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$ with the parameter $\psi_{\tau,\sigma}$ given in* (6.1). *Set $\ell_0 := \frac{\mathfrak{n}-\mathfrak{m}-1}{2}$. For any integer κ with $a - \ell_0 - 1 < \kappa \leq a + \mathfrak{r}_{\mathfrak{m}}$, the κ-th Bessel modules $\mathcal{F}_{\kappa^-}^{\mathcal{O}_{\kappa}}(\Sigma)$ are zero for all F-rational nilpotent orbits $\mathcal{O}_{\kappa}$ in the F-stable orbit $\mathcal{O}_{\underline{p}_{\kappa}}^{\mathrm{st}}(F)$.*

Proof. First, the κ-th Bessel module $\mathcal{F}_{\kappa^-}^{\mathcal{O}_{\kappa}}(\Sigma)$ produces the corresponding local Jacquet module of Σ_{ν} with respect to $(V_{\underline{p}_{\kappa}}, \psi_{\mathcal{O}_{\kappa}})$ at any finite local place ν. At almost all finite local places, Σ_{ν} is unramified and is completely determined by the ν-component of the global Arthur parameter $\psi_{\tau,\sigma}$. Taking one such unramified finite local place ν, the generic unramified representation τ_{ν} of $G_{E/F}(a)(F_{\nu})$ is conjugate self-dual and hence is completely determined by $[\frac{a}{2}]$ unramified characters $\mu_1, \ldots, \mu_{[\frac{a}{2}]}$, and σ_{ν} is also an irreducible generic unramified representation of F_{ν}-quasisplit $H_m(F_{\nu})$. As in [23, Ch. 5] and in the proof of Proposition 2.3 of [40], the unramified local component Σ_{ν} can be realized as the unique irreducible unramified subquotient of the following induced representation,

$$(6.2) \qquad\qquad \mathrm{Ind}_{P_{\hat{a}}(F_{\nu})}^{H_{a+m}(F_{\nu})}(\tau_{\nu}' \otimes \sigma_{\nu}),$$

where $\tau_\nu' = \mathrm{Ind}_{Q_{[2^{[\frac{a}{2}]}]}(F_\nu)}^{G_{E/F}(a)(F_\nu)}(\mu_1 \circ \det_2) \otimes \cdots \otimes (\mu_{[\frac{a}{2}]} \circ \det_2)$ in most cases, and $\det_2$ is the determinant of $G_{E/F}(2)$. Refer to the discussions in the proof of Lemma 6.5 for all cases of τ_ν'. In the rest of the proof, the argument works for all cases of τ_ν', although we only discuss the situation as in (6.2).

In [23, Ch. 5], the calculation of the local Jacquet modules of the induced representation as in (6.2) with respect to $(V_{\underline{p}_\kappa}, \psi_{\mathcal{O}_\kappa})$ and for general κ has been explicitly carried out. See [23, Th. 5.1], in particular. Hence it is not hard to figure out, as in [40, §2], that for κ with $a - \ell_0 - 1 < \kappa \le a + \mathfrak{r}_\mathfrak{m}$, such a local Jacquet module is always zero for the induced representation as in (6.2), and so is always zero for the unramified local component Σ_ν at the fixed local place ν. This proves that for all κ with $a - \ell_0 - 1 < \kappa \le a + \mathfrak{r}_\mathfrak{m}$, the κ-th Bessel module $\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma)$ must be zero for all such orbits F-rational $\mathcal{O}_\kappa$ in the F-stable orbit $\mathcal{O}_{\underline{p}_\kappa}^{\mathrm{st}}(F)$. $\qquad\square$

The proof uses the structure of unramified local components of Σ and hence is independent of the discrete realization of Σ if Σ has high discrete multiplicity. The same happens to the proof of the following proposition, which considers the κ_0-Bessel modules $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ for the case where $\kappa_0 = a - \ell_0 - 1$ and hence $\kappa_0^- = m^- = n$.

PROPOSITION 6.3. *Let τ and σ be as in Proposition 6.2, and set $\kappa_0 = a - \ell_0 - 1$ with $\ell_0 = \frac{\mathrm{n}-\mathrm{m}-1}{2}$. Let Σ be any automorphic member in the global Arthur packet $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$. For $n = \kappa_0^-$, and for all F-rational nilpotent orbits $\mathcal{O}_{\kappa_0}$ in the F-stable orbit $\mathcal{O}_{\underline{p}_{\kappa_0}}^{\mathrm{st}}(F)$, the κ_0-Bessel modules $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ are cuspidal, as sub-representations of $G_n^{\mathcal{O}_{\kappa_0}}(\mathbb{A})$ in the cuspidal spectrum $L_{\mathrm{cusp}}^2(G_n^{\mathcal{O}_{\kappa_0}})$.*

Proof. It suffices to show that the constant term of $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ along every standard parabolic subgroup of $G_n^{\mathcal{O}_{\kappa_0}}$ is zero. The proof uses essentially the tower property developed in [23, Ch. 7]. We take, in particular, Theorem 7.3 of [23]. As in the proof of Proposition 2.5 of [40], it is enough to show the conditions in [23, Th. 7.3] hold. Because of Proposition 6.2, the terms in [23, eq. (7.35)] are all zero. If Σ is cuspidal, the conditions in [23, Th. 7.3] are automatic. Hence in this case, all the constant terms are zero, and therefore, κ_0-th Bessel modules $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ are cuspidal.

When Σ in the global Arthur packet $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$ is not cuspidal, it must be a residual representation with the global Arthur parameter $\psi_{\tau,\sigma}$. According to [61, Th. §1.3] and [62, Th. B], among the residual representations in the global Arthur packet $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$, $\Sigma = \mathcal{E}_{\tau\otimes\sigma}$ has the least cuspidal support in the sense that among the cuspidal supports of those residual representations, the Levi subgroup in the cuspidal support of $\mathcal{E}_{\tau\otimes\sigma}$ is the smallest one. It is

enough to consider the case when $\Sigma = \mathcal{E}_{\tau \otimes \sigma}$. The same argument will be applicable to the other residual representations.

For $\Sigma = \mathcal{E}_{\tau \otimes \sigma}$, in formula (7.35) of [23], it follows from Proposition 6.2 that all the summands in summation are zero. Hence it is enough to check the assumption of Theorem 7.3 of [23]. By the cuspidal support of $\mathcal{E}_{\tau \otimes \sigma}$, if the constant term $f^{U_{p-i}}$ is zero (using the notation of [23, Th. 7.3], with $f \in \mathcal{E}_{\tau \otimes \sigma}$), we are done. It remains to consider the cases when the constant terms are not zero. To this end, we may consider the first non-zero constant term, which reduces to the case with $\tau = \tau_2 \boxplus \cdots \boxplus \tau_r$ of $G_{E/F}(a - a_1)(\mathbb{A})$. Here we refer to (4.1) for notation. In this case, the index for the Fourier coefficient is $\kappa_0 + i$ with $i = 0, 1, \ldots, p-1$, following the notation of [23, Th. 7.3]. Since $\kappa_0 = a - \ell_0 - 1$ and $\ell_0 = \frac{\mathfrak{n} - \mathfrak{m} - 1}{2}$, we must have

$$\kappa_0 + i = \left(a - \frac{\mathfrak{n}}{2} \right) + \frac{\mathfrak{m} - 1}{2} + i.$$

On the other hand, the term $(f^{U_{p-i}})^{\psi_{\kappa_0+i,\alpha}}$ is a Fourier coefficient on H_{a-a_1+m} with index

$$\frac{a - a_1 + \mathfrak{m} + \epsilon - 1}{2},$$

where $\epsilon = -1$ if G_n is an odd special orthogonal group; otherwise, $\epsilon = 0$. It follows that

$$\kappa_0 + i > \frac{a - a_1 + \mathfrak{m} + \epsilon - 1}{2}.$$

This is because $\frac{a - \epsilon}{2} = \frac{\mathfrak{n}}{2}$ and $\frac{a_1}{2} + i > 0$. According to the structure of the global Arthur parameter $\psi_{\tau,\sigma}$ as in (6.1), the term $(f^{U_{p-i}})^{\psi_{\kappa_0+i,\alpha}}$ must be zero. Namely, the condition in [23, Th. 7.3] holds in this reduced case because of Proposition 6.2. Hence by induction, we obtain that the κ_0-th Bessel modules $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ must also be cuspidal. This completes the proof. $\square$

It is clear that the κ_0-th Bessel modules $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ could be zero. We assume that the cuspidal sub-representation $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ of $G_n^{\mathcal{O}_{\kappa_0}}(\mathbb{A})$, occurring in the cuspidal spectrum $L^2_{\mathrm{cusp}}(G_n^{\mathcal{O}_{\kappa_0}})$, is non-zero, and we write it as a Hilbert direct sum

(6.3) $$\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma) = \pi_1 \oplus \cdots \oplus \pi_k \oplus \cdots,$$

where all π_i are irreducible cuspidal automorphic representations of $G_n^{\mathcal{O}_{\kappa_0}}(\mathbb{A})$. In fact, the κ_0-th Bessel modules $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ are also $\widetilde{\mathrm{O}}(G_{\kappa_0^-}^{\mathcal{O}_{\kappa_0}})$-stable, as indicated in the following proposition.

PROPOSITION 6.4. *For $\Sigma \in \mathcal{A}_{\mathrm{disc}}(H_{a+m})$, the κ-th Bessel module $\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma)$ is $\widetilde{\mathrm{O}}(G_{\kappa^-}^{\mathcal{O}_\kappa})$-stable.*

Proof. To prove this, it suffices to consider the case when H_{a+m} is an odd special orthogonal group. In fact, for all the other cases, the stabilizer $G^{\mathcal{O}_\kappa}_{\kappa^-}$ is not an even special orthogonal group and hence $\widetilde{O}(G^{\mathcal{O}_\kappa}_{\kappa^-})$ is trivial.

Suppose that H_{a+m} is an odd special orthogonal group. Then $\mathfrak{m}$ is odd and $\mathfrak{m} = 2m + 1$. In this case, the discrete multiplicity of Σ is one. We may take the unique discrete realization $\mathcal{C}_\Sigma$ of Σ in this proof.

By the definition in Section 2.4, $G^{\mathcal{O}_\kappa}_{\kappa^-}$ is identified as the connected component group $\mathrm{Isom}(V_{(\kappa)} \cap w_0^\perp, q)^\circ$, where the anisotropic vector w_0 is of form (2.14), namely,

$$w_0 = e_{a+\mathfrak{r}_\mathfrak{m}} + (-1)^{\mathfrak{n}+1}\frac{x}{2}e_{-(a+\mathfrak{r}_\mathfrak{m})}$$

for some $x \in F^\times$, where $\mathfrak{r}_\mathfrak{m} = \mathfrak{r}(H_m)$ is the F-rank of H_m. Recall that $\mathrm{Isom}(V_{(\kappa)}, q)^\circ = H_{a+m-\kappa}$ is a subgroup of the Levi subgroup $M_{\hat{\kappa}}$ of H_{a+m}. Assume that $\kappa > 0$. Take the element

$$\varepsilon = \mathrm{diag}\{-I_\kappa, I_{a+\mathfrak{r}_\mathfrak{m}-\kappa-1}, -1, I_{\mathfrak{m}-2\mathfrak{r}_\mathfrak{m}}, -1, I_{a+\mathfrak{r}_\mathfrak{m}-\kappa-1}, -I_\kappa\}.$$

It is easy to check that $\varepsilon \in H_{a+m}$ and it stabilizes w_0. Note that the stabilizer of w_0 is $\mathrm{SO}(V_{(\kappa)} \cap w_0^\perp, q) \rtimes \langle\varepsilon\rangle$, which is isomorphic to $\mathrm{O}(V_{(\kappa)} \cap w_0^\perp, q)$. The adjoint action of $\langle\varepsilon\rangle$ on $G^{\mathcal{O}_\kappa}_{\kappa^-} = \mathrm{SO}(V_{(\kappa)} \cap w_0^\perp, q)$ is the same as the action of $\widetilde{O}(G^{\mathcal{O}_\kappa}_{\kappa^-})$.

Consider the action of $\varepsilon \in H_{a+m}(F)$ on a discrete realization $\mathcal{C}_\Sigma$ of Σ, defined by $f^\varepsilon(g) := f(\varepsilon^{-1}g\varepsilon)$ for $f \in \mathcal{C}_\Sigma$. Since $\varepsilon \in H_{a+m}(F)$, $f^\varepsilon(g)$ also belongs to $\mathcal{C}_\Sigma$. By the definition in (2.7), since ε stabilizes ψ_{κ,w_0},

$$\mathcal{F}^{\psi_\kappa,w_0}(f^\varepsilon)(h) = \mathcal{F}^{\psi_\kappa,w_0}(f)(\varepsilon^{-1}h\varepsilon) := \mathcal{F}^{\psi_\kappa,w_0}(f)^\varepsilon(h),$$

where the action of ε on $h \in G^{\mathcal{O}_\kappa}_{\kappa^-}$ is given as above. It follows that if $f \in \mathcal{C}_\Sigma$, then $\mathcal{F}^{\psi_\kappa,w_0}(f)^\varepsilon \in \mathcal{F}^{\psi_\kappa,w_0}(\mathcal{C}_\Sigma)$. As explained on page 753, the action of ε on $\mathcal{F}^{\psi_\kappa,w_0}(f)$ coincides the action of $\widetilde{O}(G^{\mathcal{O}_\kappa}_{\kappa^-})$ on $\mathcal{F}^{\mathcal{O}_\kappa}_{\kappa^-}(\mathcal{C}_\Sigma)$. That is, $\mathcal{F}^{\mathcal{O}_\kappa}_{\kappa^-}(\mathcal{C}_\Sigma)$ is $\widetilde{O}(G^{\mathcal{O}_\kappa}_{\kappa^-})$-stable.

When $\kappa = 0$, the κ-th Bessel module $\mathcal{F}^{\mathcal{O}_\kappa}_{\kappa^-}(\mathcal{C}_\Sigma)$ is the restriction of $\mathcal{C}_\Sigma$ into the even special orthogonal group $\mathrm{SO}_{2a+\mathfrak{m}-1}(w_0^\perp)$. Let us extend the representation $\mathcal{C}_\Sigma$ as the representation of $H_{a+m} \times \langle-I_{2a+\mathfrak{m}}\rangle = \mathrm{O}_{2a+\mathfrak{m}}(V)$, by letting the action be trivial on $\langle-I_{2a+\mathfrak{m}}\rangle$. We may choose

$$\varepsilon = \{I_{a+m}, -1, I_{a+m}\}.$$

Then the rest of the proof is the same as that for the case $\kappa > 0$. We complete the proof. $\square$

The general calculation of the local Jacquet module of the induced representation of type (6.2) (as explained in [23, Ch. 5], or more precisely, in [23, Ths. 5.4 and 5.6], and also as in [40, §4.1]) can be adopted to prove that those irreducible summands π_i in (6.3) are actually nearly equivalent to each

other. At almost all local finite places ν, the unramified local component $\pi_{i,\nu}$ of π_i shares the same Satake parameter with the unramified local component τ_ν under the unramified local Langlands functorial transfer from $G_n^{\mathcal{O}_{\kappa_0}}(F_\nu)$ to $G_{E/F}(a)(F_\nu)$ except that $G_n^{\mathcal{O}_{\kappa_0}}(F_\nu)$ is a split even special orthogonal group. In this case, the unramified local component $\pi_{i,\nu}$ belongs to the $\widetilde{\mathrm{O}}(G_n^{\mathcal{O}_{\kappa_0}}(F_\nu))$-orbit of the Satake parameters of $G_n^{\mathcal{O}_{\kappa_0}}(F_\nu)$, which are the descents of the Satake parameters of τ_ν under the local Langlands functorial transfer.

For the sake of completeness and also for future applications, we apply Theorems 5.4 and 5.6 in [23] to elaborate with some details the above discussions. We summarize the results on the local descent at the unramified places as the following lemma.

LEMMA 6.5. *Let ν be a finite place such that all data are unramified. Assume that Σ_ν is the unique irreducible unramified constituent of*

$$\mathrm{Ind}_{P_{\hat{a}}(F_\nu)}^{H_{a+m}(F_\nu)} \tau_\nu |\det|^{\frac{1}{2}} \otimes \sigma_\nu,$$

where τ_ν and σ_ν are irreducible, generic and unramified local components of τ and σ in (6.1). Then the unramified constituents of $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma_\nu)$ have the same Satake parameter with τ_ν under the local Langlands functorial transfer from $G_n^{\mathcal{O}_{\kappa_0}}(F_\nu)$ to $G_{E/F}(F_\nu)$.

Proof. We proceed the proof for the following two different cases:

(1) H_m is special even orthogonal, or H_m is split odd orthogonal and $G_{\kappa^-}^{\mathcal{O}_\kappa}$ is split, or H_m is quasi-split odd unitary; and
(2) H_m is split special odd orthogonal and $G_{\kappa^-}^{\mathcal{O}_\kappa}$ is non-split, or H_m is quasi-split even unitary.

Case (1). Under the assumption, if H_m is split special even orthogonal, then the assumption that the Witt index of $E_\nu y_{-\alpha} + V_0$ is zero in [23, Th. 5.4 (1)] holds; if H_m is split special odd orthogonal and $G_{\kappa^-}^{\mathcal{O}_\kappa}$ is split, then the Witt index of $E_\nu y_{-\alpha} + V_0$ is one, $\delta^1_{h(V),\alpha} = 0$ and $\delta^2_{h(V),\alpha} = 1$ in the notation of [23, Th. 5.4 (2)]; otherwise, the Witt index of $E_\nu y_{-\alpha} + V_0$ is one, and $\delta^1_{h(V),\alpha} = 1$ and $\delta^2_{h(V),\alpha} = 0$ in [23, Th. 5.4 (2)]. In Case (1), a is even by the parity of the dimension of the Arthur parameters involved.

As in (6.2), we consider the unramified local component Σ_ν as the unramified subquotient of

(6.4) $\qquad \mathrm{Ind}_{P_{[2^{\frac{a}{2}}]}(F_\nu)}^{H_{a+m}(F_\nu)} (\mu_1 \circ \det_2) \otimes \cdots \otimes (\mu_{\frac{a}{2}} \circ \det_2) \otimes \delta_1 \otimes \cdots \otimes \delta_{\mathfrak{r}_m}.$

Here $\mathfrak{r}_m = \mathfrak{r}(H_m)$ is the F_ν-rank of H_m. It follows that $\mathfrak{r}_m = m - 1$ if $H_m(F_\nu)$ is quasi-split and non-split even orthogonal; and $\mathfrak{r}_m = m$ otherwise. As before, $P_{[2^{\frac{a}{2}}]}$ is the standard parabolic subgroup of H_{a+m} whose Levi part is isomorphic

to $G_{E/F}(2)^{\times \frac{a}{2}} \times G_{E/F}(1)^{\times \mathfrak{r}_m}$; and $\sigma_\nu = \mathrm{Ind}_{B_m}^{H_m} \delta_1 \otimes \cdots \otimes \delta_{\mathfrak{r}_m}$. Let us substitute the following representation for τ in [23, Th. 5.4],

$$(6.5) \qquad \mathrm{Ind}_{Q_{[2^{\frac{a}{2}}]}(F_\nu)}^{G_{E/F}(a+\mathfrak{r}_m)(F_\nu)} (\mu_1 \circ \det_2) \otimes \cdots \otimes (\mu_{\frac{a}{2}} \circ \det_2) \otimes \delta_1 \otimes \cdots \otimes \delta_{\mathfrak{r}_m},$$

where $Q_{[2^{\frac{a}{2}}]}$ is the standard parabolic subgroup $G_{E/F}(a + \mathfrak{r}_m) \cap P_{[2^{\frac{a}{2}}]}$. We regard $G_{E/F}(a + \mathfrak{r}_m)$ as the subgroup of the standard parabolic subgroup of $P_{(a+\mathfrak{r}_m)^\wedge}$. The symbols $\tilde{m}$ and ℓ in [23, Th. 5.4] are replaced by $a + \mathfrak{r}_m$ and κ respectively in our case. Then $0 \leq \kappa < a + \mathfrak{r}_m$, which is a part of the conditions in [23, Th. 5.4, (1) and (2)].

If H_m is split special even orthogonal, by [23, Th. 5.4 (1)], one has $\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu) = 0$ for $\kappa > \frac{a}{2} + \mathfrak{r}_m - 1$, and for $\kappa = \frac{a}{2} + \mathfrak{r}_m - 1$,

$$\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu) \prec \mathrm{Ind}_{B_{G,a}}^{G_{\kappa^-}^{\mathcal{O}_\kappa}} \mu_1 \otimes \mu_2 \cdots \otimes \mu_{\frac{a}{2}},$$

where $B_{G,a}$ is the Borel subgroup of $G_{\kappa^-}^{\mathcal{O}_\kappa}$, and $\pi_1 \prec \pi_2$ means that π_1 is a subquotient of π_2. Since $\mathfrak{r}_m = m$, we have that $G_{\kappa^-}^{\mathcal{O}_\kappa}$ is isomorphic to the split odd orthogonal SO_{a+1}.

If H_m is split odd orthogonal and $G_{\kappa^-}^{\mathcal{O}_\kappa}$ is split, by [23, Th. 5.4 (2)], one has $\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu) = 0$ for $\kappa > \frac{a}{2} + \mathfrak{r}_m$, and for $\kappa = \frac{a}{2} + \mathfrak{r}_m - 1$,

$$\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu) \prec \mathrm{Ind}_{B_{G,a}}^{G_{\kappa^-}^{\mathcal{O}_\kappa}} \mu_1 \otimes \mu_2 \cdots \otimes \mu_{\frac{a}{2}-1} \otimes \mu_{\frac{a}{2}} \oplus \mathrm{Ind}_{B_{G,a}}^{G_{\kappa^-}^{\mathcal{O}_\kappa}} \mu_1 \otimes \mu_2 \cdots \otimes \mu_{\frac{a}{2}-1} \otimes \mu_{\frac{a}{2}}^{-1}.$$

Since $\mathfrak{r}_m = m$, we have that $G_{\kappa^-}^{\mathcal{O}_\kappa}$ is isomorphic to the split special even orthogonal SO_a and that the two unramified representations are $\widetilde{\mathrm{O}}(G_{\kappa^-}^{\mathcal{O}_{\kappa_0}}(F_\nu))$-conjugate.

If H_m is quasi-split, but non-split, special even orthogonal or odd unitary, following [23, Th. 5.4 (2)], one has $\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu) = 0$ for $\kappa > \frac{a}{2} + \mathfrak{r}_m$, and for $\kappa = \frac{a}{2} + \mathfrak{r}_m$,

$$\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu) \prec \mathrm{Ind}_{B_{G,a}}^{G_{\kappa^-}^{\mathcal{O}_\kappa}} \mu_1 \otimes \mu_2 \cdots \otimes \mu_{\frac{a}{2}}.$$

More precisely, if H_m is quasi-split, but non-split, special even orthogonal, then $\mathfrak{r}_m = m - 1$ and $G_{\kappa^-}^{\mathcal{O}_\kappa}$ is isomorphic to the split odd orthogonal SO_{a+1}; and if H_m is odd unitary, then $\mathfrak{r}_m = m$ and $G_{\kappa^-}^{\mathcal{O}_\kappa}$ is isomorphic to the quasi-split odd unitary U_{a+1}.

Case (2). Denote $\omega_{\tau,\nu}$ to be the central character of τ_ν. Since τ_ν is (conjugate) self-dual, $\omega_{\tau,\nu}$ is a quadratic character — that is, $\omega_{\tau,\nu} = 1$ or λ_0. Here λ_0 is the unique non-trivial unramified quadratic character of $E_\nu^\times$.

Assume that H_m is split odd orthogonal and $G_{\kappa^-}^{\mathcal{O}_\kappa}$ is non-split. In this case, the assumption that the Witt index of $F_\nu y_{-\alpha} + V_0$ is zero in [23, Th. 5.4(1)] holds and a is even.

If $\omega_{\tau,\nu} = 1$, then the unramified local component Σ_ν is the unramified subquotient of the unramified induced representation as in (6.4). We substitute the representation (6.5) for τ in [23, Th. 5.4 (1)]. Then $\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu) = 0$ for $\kappa \geq \frac{a}{2} + m$, and the descent $\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu)$ to the orthogonal group SO_a is zero when $\kappa = \frac{a}{2} + m$. Remark that over the inert finite places, the determinant of the local L-parameter of the quasi-split, but non-split SO_a is not 1. This verifies that if $\omega_{\tau,\nu} = 1$, then the descent $\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu)$ at this rational orbit $\mathcal{O}_\kappa$ is zero.

Assume that $\omega_{\tau,\nu} = \lambda_0$. The unramified local component Σ_ν is isomorphic to the unramified subquotient of

$$\mathrm{Ind}_{P_{[2^{\frac{a}{2}-1}]}(F_\nu)}^{H_{a+m}(F_\nu)}(\mu_1 \circ \det_2) \otimes \cdots \otimes (\mu_{\frac{a}{2}-1} \circ \det_2) \otimes 1 \otimes \lambda_0 \otimes \delta_1 \otimes \cdots \otimes \delta_m.$$

We may replace the above representation by $\mathrm{Ind}_{P_{(a+m-2)^\wedge}(F_\nu)}^{H_{a+m}(F_\nu)} \tau_1 \otimes \sigma_1$, where σ_1 is the representation of split $\mathrm{SO}_5(F_\nu)$ induced from the parabolic subgroup that preserves an isotropic line and the character $\lambda_0 |\cdot|^{\frac{1}{2}} \otimes 1$, and

$$\tau_1 = \mathrm{Ind}_{Q_{[2^{\frac{a}{2}}]}(F_\nu)}^{G_{E/F}(a+m-2)(F_\nu)}(\mu_1 \circ \det_2) \otimes \cdots \otimes (\mu_{\frac{a}{2}-1} \circ \det_2) \otimes \delta_1 \otimes \cdots \otimes \delta_m.$$

Applying [23, Th. 5.1 (1)], after the same calculation with page 104 in [23], one has $\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu) = 0$ for $\kappa > \frac{a}{2} + m$, and for $\kappa = \frac{a}{2} + m - 1$,

$$\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu) \prec \mathrm{Ind}_{B_{G,a}}^{G_{\kappa^-}^{\mathcal{O}_\kappa}} \mu_1 \otimes \mu_2 \cdots \otimes \mu_{\frac{a}{2}-1} \otimes 1,$$

where 1 is the trivial representation of the anisotropic part of the torus of $B_{G,a}$.

It remains to treat the case that H_m is quasi-split even unitary. In our setting suppose that H_m is quasi-split even unitary. Then in our setting a is odd, τ_ν is conjugate orthogonal and $\omega_{\tau,\nu} = 1$. The unramified local component Σ_ν is isomorphic to the unramified subquotient of

$$\mathrm{Ind}_{P_{[2^{[\frac{a}{2}]}]}(F_\nu)}^{H_{a+m}(F_\nu)}(\mu_1 \circ \det_2) \otimes \cdots \otimes (\mu_{[\frac{a}{2}]} \circ \det_2) \otimes |\cdot|^{\frac{1}{2}} \otimes \delta_1 \otimes \cdots \otimes \delta_m.$$

We may replace the above induced representation by

$$\mathrm{Ind}_{P_{(a+m-1)^\wedge}(F_\nu)}^{H_{a+m}(F_\nu)} \tau_1 \otimes 1,$$

where 1 is the trivial character of quasi-split $\mathrm{U}_2(F_\nu)$ and

$$\tau_1 = \mathrm{Ind}_{Q_{[2^{[\frac{a}{2}]}]}(F_\nu)}^{G_{E/F}(a+m-1)(F_\nu)}(\mu_1 \circ \det_2) \otimes \cdots \otimes (\mu_{[\frac{a}{2}]} \circ \det_2) \otimes \delta_1 \otimes \cdots \otimes \delta_m.$$

Let us apply [23, Th. 5.1 (1)] and follow the same calculation as on page 105 in the proof of Theorem 5.6 in [23] for the case $\omega_{\tau,\nu} = 1$. Then one has $\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu) = 0$ for $\kappa > \frac{a}{2} + m$. For $\kappa = [\frac{a}{2}] + m$, if $\omega_{\tau,\nu} = 1$, then

(6.6) $$\mathcal{F}_{\kappa^-}^{\mathcal{O}_\kappa}(\Sigma_\nu) \prec \mathrm{Ind}_{B_{G,a}}^{G_{\kappa^-}^{\mathcal{O}_\kappa}} \mu_1 \otimes \mu_2 \cdots \otimes \mu_{[\frac{a}{2}]-1} \otimes 1,$$

where 1 is the trivial character of $\mathrm{U}_1(F_\nu)$.

Therefore, we complete all cases involved in our discussion in this paper and verify that $\pi_{i,\nu}$ shares the same Satake parameter with τ_ν under the unramified local Langlands functorial transfer from $G_n^{\mathcal{O}_{\kappa_0}}(F_\nu)$ to $G_{E/F}(a)(F_\nu)$. $\qquad\square$

Now we go back to the decomposition in (6.3). By the local uniqueness of Bessel models at all local places ([2], [77], [16], and [44]), it is easy to deduce that π_i is not equivalent to π_j if $i \neq j$; that is, the decomposition in (6.3) is multiplicity free. Of course, in the situation that the cuspidal spectrum is multiplicity free, the decomposition in (6.3) will be automatically multiplicity free. We summarize the discussion as the following theorem.

THEOREM 6.6. *Assume that τ and σ are as given above. For an automorphic member Σ in the global Arthur packet $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$, assume that the κ_0-th Bessel module $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ is non-zero for some F-rational nilpotent orbit $\mathcal{O}_{\kappa_0}$ in the F-stable orbit $\mathcal{O}_{\underline{p}_{\kappa_0}}^{\mathrm{st}}(F)$ with $\ell_0 = \frac{\mathrm{n-m}-1}{2}$, $\kappa_0 = a - \ell_0 - 1$, and $\kappa_0^- = m^- = n$. Then the following hold:*

(1) *The κ_0-th Bessel module $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ is cuspidal and can be regarded as a sub-representation of $G_n^{\mathcal{O}_{\kappa_0}}(\mathbb{A})$ in the cuspidal spectrum $L^2_{\mathrm{cusp}}(G_n^{\mathcal{O}_{\kappa_0}})$.*

(2) *In the cuspidal spectrum $L^2_{\mathrm{cusp}}(G_n^{\mathcal{O}_{\kappa_0}})$, $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ has a multiplicity free, Hilbert direct sum decomposition*

$$\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma) = \pi_1 \oplus \cdots \oplus \pi_k \oplus \cdots,$$

where all π_i belong to $\mathcal{A}_{\mathrm{cusp}}(G_n^{\mathcal{O}_{\kappa_0}})$ and have a generic global Arthur parameter belonging to the $\widetilde{\mathrm{O}}(G_n^{\mathcal{O}_{\kappa_0}})$-orbit of ϕ_τ, which is $G_n^{\mathcal{O}_{\kappa_0}}$-relevant and is determined by τ. Moreover, $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ is $\widetilde{\mathrm{O}}(G_n^{\mathcal{O}_{\kappa_0}})$-stable.

Note that in part (2) of Theorem 6.6, the $\widetilde{\mathrm{O}}(G_n^{\mathcal{O}_{\kappa_0}})$-orbit of ϕ_τ contains only one parameter unless $G_n^{\mathcal{O}_{\kappa_0}}$ is an even special orthogonal group. In this case, the $\widetilde{\mathrm{O}}(G_n^{\mathcal{O}_{\kappa_0}})$-orbit of ϕ_τ may contain two parameters $\{\phi, \phi_\star\}$, which are the descents of ϕ_τ, as explained on page 754. It is worthwhile to remember that in this case, the global Arthur packets $\Pi_\phi(G_n^{\mathcal{O}_{\kappa_0}})$ and $\Pi_{\phi_\star}(G_n^{\mathcal{O}_{\kappa_0}})$ are different. We may identify the parameter ϕ_τ with either ϕ or $\phi_\star$, as on page 754.

6.2. *Construction of cuspidal automorphic modules.* The main issue remaining from Theorem 6.6 is the non-vanishing assumption that the κ_0-th Bessel module $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ is non-zero for some automorphic member Σ in the global Arthur packet $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$. We refer to Section 6.3 and Conjecture 6.8 in particular for more details.

We are instead going to discuss the impact of Conjecture 2.3 in the theory of twisted automorphic descents. To this end, we recall the specific data suggested by Conjecture 2.3. By Proposition 2.2, for a given $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ with

a G_n-relevant, generic global Arthur parameter $\phi = \phi_\tau \in \widetilde{\Phi}_2(G_n^*)$ and with the cuspidal realization $\mathcal{C}_\pi$, there exists the first occurrence index $\ell_0 = \ell_0(\mathcal{C}_\pi)$, such that the (maximal) ℓ_0-Bessel module $\mathcal{F}^{\mathcal{O}_{\ell_0}}(\mathcal{C}_\pi)$ associated to the partition $\underline{p}_{\ell_0} = [(2\ell_0 + 1)1^{\mathfrak{n}-2\ell_0-1}]$ is cuspidal and non-zero, as a representation of $H_m^{\mathcal{O}_{\ell_0}}(\mathbb{A})$ occurring in the cuspidal spectrum $L^2_{\mathrm{cusp}}(H_m^{\mathcal{O}_{\ell_0}})$. In this situation, we take the data that $m = \ell_0^-$, $\mathfrak{m} = \mathfrak{l}_0^- = \mathfrak{n} - 2\ell_0 - 1$, and $H_m = H_m^{\mathcal{O}_{\ell_0}}$. By Conjecture 2.3, there exists a $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ with an H_m-relevant, generic global Arthur parameter ϕ_σ in $\widetilde{\Phi}_2(H_m^*)$ and with the cuspidal realization $\mathcal{C}_\sigma$, such that the inner product $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_0}}}(\varphi_\pi), \overline{\varphi}_{\sigma'} \right\rangle_{H_m}$ is non-zero for some $\varphi_\pi \in \mathcal{C}_\pi$ and $\varphi_{\sigma'} \in \mathcal{C}_{\sigma'}$, where $\sigma' = \sigma^{w_q^\ell}$ and $\mathcal{C}_{\sigma'} = \mathcal{C}_\sigma^{w_q^\ell}$ as defined in (4.34).

With the data associated to Conjecture 2.3, Theorem 6.6 may be illustrated by the following diagram:

$$
(6.7) \quad
\begin{array}{ccccc}
\widetilde{\Phi}_2(G_n^*)_{G_n} & \phi_\sigma \in \widetilde{\Phi}_2(H_m) & & \widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m}) \\[4pt]
\phi_\tau & (H_m, \sigma) & \Longrightarrow & \Sigma \\[4pt]
\Updownarrow & \Uparrow & & \Downarrow \\[4pt]
\widetilde{\Pi}_{\phi_\tau}(G_n) & \ni \pi & \overset{(?)}{\longleftrightarrow} & \mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma) \subset L^2_{\mathrm{cusp}}(G_n^{\mathcal{O}_{\kappa_0}}).
\end{array}
$$

In this diagram, we starts with a generic global Arthur parameter ϕ_τ of G_n^*, which is G_n-relevant. It gives the global Arthur packet $\widetilde{\Pi}_{\phi_\tau}(G_n)$ by the endoscopic classification theory. Now take any cuspidal member π in $\widetilde{\Pi}_{\phi_\tau}(G_n)$. By the Generic Summand Conjecture (Conjecture 2.3), it produces the pair (H_m, σ), where $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ with a generic global Arthur parameter ϕ_σ in $\widetilde{\Phi}_2(H_m^*)_{H_m}$. Then σ and τ together produce the global Arthur packet $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$. Finally, we take the Bessel-Fourier coefficient $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ for any automorphic member Σ in $\widetilde{\Pi}_{\psi_{\tau,\sigma}}(H_{a+m})$, which is a cuspidal automorphic $G_n^{\mathcal{O}_{\kappa_0}}(\mathbb{A})$-module contained in $L^2_{\mathrm{cusp}}(G_n^{\mathcal{O}_{\kappa_0}})$. The *big question* in the construction is the following: what can we say about π and $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ as representations of $G_n(\mathbb{A})$ and $G_n^{\mathcal{O}_{\kappa_0}}(\mathbb{A})$, respectively? Theorem 6.6 gives an answer to this question with a non-vanishing assumption.

Without the participation of σ and H_m, diagram (6.7) may be reduced to the following diagram:

$$
(6.8) \quad
\begin{array}{ccc}
\widetilde{\Phi}_2(G_n^*) \ni \phi_\tau & \Longrightarrow & \mathcal{E}_\tau \in \widetilde{\Pi}_{\psi_\tau}(H_a^*) \\[4pt]
\Downarrow & & \Downarrow \\[4pt]
\widetilde{\Pi}_{\phi_\tau}(G_n) \ni \pi & \cong & \mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_\tau) \subset L^2_{\mathrm{cusp}}(G_n^{\mathcal{O}_{\kappa_0}}).
\end{array}
$$

When $G_n = G_n^{\mathcal{O}_{\kappa_0}} = G_n^*$ is F-quasisplit, and $\mathcal{E}_\tau$ is the residual representation of $H_a(\mathbb{A})$ (with $a = N = \mathfrak{n}^\vee$) having the global Arthur parameter

$$\psi_\tau = (\tau_1, 2) \boxplus \cdots \boxplus (\tau_r, 2).$$

The reduced diagram (6.8) yields the automorphic descents of Ginzburg-Rallis-Soudry ([23]) that construct certain generic cuspidal automorphic representations of an F-quasisplit classical group $G_n^*(\mathbb{A})$.

By Proposition 2.6, G_n and $G_n^{\mathcal{O}_{\kappa_0}}$ are pure inner forms. If one of G_n and $G_n^{\mathcal{O}_{\kappa_0}}$ is not equal to G_n^*, then the relation between π and $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\Sigma)$ is the generalized Jacquet-Langlands correspondence between G_n and $G_n^{\mathcal{O}_{\kappa_0}}$. However, as shown in [40], this will not cover the general situation as the F-ranks of G_n and $G_n^{\mathcal{O}_{\kappa_0}}$ must satisfy the condition given in Proposition 2.5. The introduction of σ and H_m in the construction is to avoid such restriction.

With Conjecture 2.3 and the participation of σ and H_m in the construction as displayed in diagram (6.7), which is essential, the proposed construction may (in principle) produce all irreducible cuspidal automorphic representations of the classical groups G_n that are pure inner F-forms of an F-quasisplit classical group G_n^*.

In fact, we are going to show in Section 7.1 that the κ_0-th Bessel module $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma})$ is non-zero, assuming Conjecture 2.3. If we assume that the stronger uniqueness of the local Bessel models over a local Vogan packet holds at all local places (Conjecture 3.1, the known cases of which is given in Theorem 3.2), then $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma})$ is in fact irreducible, when $G_n = G_n^{\mathcal{O}_{\kappa_0}}$ is *not* an even special orthogonal group. However, if $G_n = G_n^{\mathcal{O}_{\kappa_0}}$ is an even special orthogonal group, then $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma})$ could be a direct sum of two irreducible cuspidal automorphic representations that belong to the $\widetilde{\mathrm{O}}(G_n)$-orbit. In any situation, we set

$$(6.9) \qquad \mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau;\sigma) := \mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma})$$

and call $\mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau;\sigma)$ a σ-*twisted automorphic descent* of τ from $G_{E/F}(N)$ to $G_n^{\mathcal{O}_{\kappa_0}}$, or simply a *twisted automorphic descent* of τ, where $N = a = \mathfrak{n}^\vee$. The main result in the theory of the *cuspidal automorphic modules* outlined in diagram (6.7), by means of *twisted automorphic descents*, is to confirm that the constructed module $\mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau;\sigma)$ in (6.9) is in principle isomorphic to the given irreducible cuspidal automorphic representation π.

In general, we may state the *main conjecture* in the theory of the cuspidal automorphic modules via the twisted automorphic descents as follows.

CONJECTURE 6.7 (Main Conjecture). *Let $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ be an irreducible isobaric representation of $G_{E/F}(a)(\mathbb{A})$ that defines a generic global*

Arthur parameter $\phi = \phi_\tau \in \widetilde{\Phi}_2(G_n^*)$ as in (3.1). Assume that ϕ is G_n-relevant. For any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ belonging to the global Arthur packet $\widetilde{\Pi}_\phi(G_n)$, there exists a datum (H_m, σ) with the following properties:

(1) H_m is a classical group defined over F and is a pure inner F-form of an F-quasisplit classical group H_m^* such that the pairs (G_n, H_m) and (G_n^*, H_m^*) are relevant and the product $G_n \times H_m$ is a relevant pure inner form of the product $G_n^* \times H_m^*$; and

(2) $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ belongs to the global Arthur packet $\widetilde{\Pi}_{\phi'}(H_m)$ associated to an H_m-relevant generic global Arthur parameter $\phi' \in \widetilde{\Phi}_2(H_m^*)$,

such that

(a) if $G_n = G_n^{\mathcal{O}_{\kappa_0}}$ is not an even special orthogonal group, or if $G_n = G_n^{\mathcal{O}_{\kappa_0}}$ is an even special orthogonal group, but the $\widetilde{\mathrm{O}}(G_n)$-orbit of π contains only π, then the automorphic module $\mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau; \sigma)$ that is constructed via the twisted automorphic descent (6.9) is isomorphic to the given π,

$$\mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau; \sigma) \cong \pi;$$

(b) if $G_n = G_n^{\mathcal{O}_{\kappa_0}}$ is an even special orthogonal group, and the $\widetilde{\mathrm{O}}(G_n)$-orbit of π is equal to $\{\pi, \pi_\star\}$, then

$$\mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau; \sigma) \cong \pi \oplus \pi_\star.$$

It is not hard to see that the constructed cuspidal automorphic module $\mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau; \sigma)$ in Conjecture 6.7 is the special realization of the cuspidal automorphic module $\mathcal{M}(\psi, \mathcal{F}(\pi, G))$ in Principle 1.1 in the particular case under consideration. We remark that the construction outlined in diagram (6.7) only uses a piece of information from the data $\mathcal{F}(\pi, G)$. We will come back to the discussion of Conjecture 6.7 with more details in Section 7.

6.3. *Global Gan-Gross-Prasad conjecture: another direction.* Let $\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$ be the irreducible isobaric automorphic representation of $G_{E/F}(a)(\mathbb{A})$ as in (4.1), with $a = \mathfrak{n}^\vee = N$, which defines a generic global Arthur parameter $\phi = \phi_\tau$ in $\widetilde{\Phi}_2(G_n^*)$. Let ϕ' be a generic global Arthur parameter of H_m^*. Assume that $L(\frac{1}{2}, \phi \times \phi') \neq 0$. For any member σ in the global Vogan packet $\widetilde{\Pi}_{\phi'}[H_m^*]$, in which all the automorphic members are cuspidal ([37, §3]), we have

$$L\left(\frac{1}{2}, \tau \times \sigma\right) = L\left(\frac{1}{2}, \tau \times \sigma^{w_q^\ell}\right) = L\left(\frac{1}{2}, \phi \times (\phi')^{w_q^\ell}\right) = L\left(\frac{1}{2}, \phi \times \phi'\right) \neq 0.$$

The *direction of the global Gan-Gross-Prasad conjecture* to be considered in this section asserts that under the above assumptions, there exists a unique pair (π, σ) in the global Vogan packet $\widetilde{\Pi}_{\phi \times \phi'}[G_n^* \times H_m^*]$ with property that (π, σ) with the cuspidal realization $(\mathcal{C}_\pi, \mathcal{C}_\sigma)$ admits a non-zero Bessel period (depending

on the F-rational structure of the unipotent orbits as discussed in Section 2.4). The uniqueness follows from the local Gan-Gross-Prasad conjecture at all local places (Conjecture 3.1). Hence the key point is the existence of such a pair with a non-zero Bessel period.

We are going to prove this direction of the global Gan-Gross-Prasad conjecture by constructing such a pair via the twisted automorphic descent developed in the early sections of this paper. For a technical reason, we have to take an *assumption*, which we are only able to verify for some special situation for the time being.

Take a member $\sigma \in \widetilde{\Pi}_{\phi'}[H_m^*]$. There is an F-inner form H_m of H_m^* such that $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ with a cuspidal realization $\mathcal{C}_\sigma$. By Proposition 5.2, the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ of $H_{a+m}(\mathbb{A})$ is non-zero. As discussed in [41], $\mathcal{E}_{\tau \otimes \sigma}$ is square-integrable. By [62, Th. B], $\mathcal{E}_{\tau \otimes \sigma}$ is irreducible. Following from [41, §6], $\mathcal{E}_{\tau \otimes \sigma}$ has the global Arthur parameter

$$\psi_{\tau,\sigma} = (\tau_1, 2) \boxplus (\tau_2, 2) \boxplus \cdots \boxplus (\tau_r, 2) \boxplus \phi'.$$

It is expected that the structure of the Fourier coefficients of $\mathcal{E}_{\tau \otimes \sigma}$ has significant impact to the understanding of the global Vogan packet $\widetilde{\Pi}_{\phi \times \phi'}[G_n^* \times H_m^*]$.

As in [36, §4] and as recalled in Section 2.3, the Fourier coefficients of $\mathcal{E}_{\tau \otimes \sigma}$ are defined in terms of the H_{a+m}-relevant partitions of $(2a + \mathfrak{m}, H_{a+m}^*)$. We denote by $\mathfrak{p}(\mathcal{E}_{\tau \otimes \sigma})$ the set of the H_{a+m}-relevant partitions with which the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ has a non-zero Fourier coefficient. To the pair of the generic global Arthur parameters (ϕ, ϕ') as given above, we define the following partition:

$$\underline{p}_{\phi,\phi'} := \begin{cases} [(a + \mathfrak{m} - 1)(a + 1)] & \text{if } H_m^* = \mathrm{SO}_{2m}, \mathfrak{m} = 2m, \\ [(a + \mathfrak{m})(a - 1)1] & \text{if } H_m^* = \mathrm{SO}_{2m+1}, \mathfrak{m} = 2m + 1, \\ [(a + \mathfrak{m})a] & \text{if } H_m^* \text{ is a unitary group.} \end{cases}$$

Note that $a = \mathfrak{n}^\vee$, and the integers $a + \mathfrak{m} - 1$ and $a + \mathfrak{m}$ are odd, in the respective cases. The main conjecture in [36, §4] asserts that for all $\sigma \in \widetilde{\Pi}_{\phi'}[H_m^*]$, every partition $\underline{p} \in \mathfrak{p}(\mathcal{E}_{\tau \otimes \sigma})$ has the property that $\underline{p} \leq \underline{p}_{\phi,\phi'}$. If we get back to the construction of cuspidal automorphic modules as illustrated in diagram (6.7), then we need the following partition:

$$\underline{p}_{\phi,\phi'}^1 := \begin{cases} [(a + \mathfrak{m} - 1)1^{a+1}] & \text{if } H_m^* = \mathrm{SO}_{2m}, \\ [(a + \mathfrak{m})1^a] & \text{otherwise.} \end{cases}$$

CONJECTURE 6.8. *With notation and assumptions as above, for the given pair of parameters* (ϕ, ϕ'), *there exists a* $\sigma \in \widetilde{\Pi}_{\phi'}[H_m^*]$ *with a cuspidal realization* $\mathcal{C}_\sigma$ *on* $H_m(\mathbb{A})$, *such that* $\underline{p}_{\phi,\phi'}^1$ *belongs to* $\mathfrak{p}(\mathcal{E}_{\tau \otimes \sigma})$, *where* $\mathcal{E}_{\tau \otimes \sigma}$ *on* $H_{a+m}(\mathbb{A})$ *is defined through* $\mathcal{C}_\sigma$.

For $m = 0$, Conjecture 6.8 was proved in [23]. For $m = 1$ and H_1 an F-form of SO_2, it is proved in [40]. Similar results can be checked for unitary groups, but we do not discuss them here with further details.

PROPOSITION 6.9. *Conjecture* 6.8 *holds when* $m = 0$ *and for all* F-*quasi-split classical groups* H_a^*, *and when* $m = 1$ *for even special orthogonal groups* $H_{2n+1} = \mathrm{SO}_{2n+2,2n}$.

We refer to [36], [35], [37] and [38] for more discussions of Fourier coefficients of automorphic representations occurring in the discrete spectrum of classical groups, and of residual representations in particular.

THEOREM 6.10 (Global Gan-Gross-Prasad Conjecture: another direction). *For* $a = \mathfrak{n}^\vee = N$, *take* τ *to be the irreducible isobaric automorphic representation of* $G_{E/F}(a)(\mathbb{A})$ *as in* (4.1). *Let* $\phi = \phi_\tau$ *be a generic global Arthur parameter in* $\widetilde{\Phi}_2(G_n^*)$ *and* ϕ' *be a generic global Arthur parameter of* H_m^*. *Assume that*

$$L(\tfrac{1}{2}, \phi \times \phi') \neq 0.$$

Assume that Conjecture 6.8 *holds for the pair of parameters* $(\phi, (\phi')^{w_q^\ell})$. *Then there exist a cuspidal automorphic member* π *in the global Vogan packet* $\widetilde{\Pi}_\phi[G_n^*]$ *with a cuspidal realization* $\mathcal{C}_\pi$, *and a cuspidal automorphic member* σ *in the global Arthur packet* $\widetilde{\Pi}_{\phi'}(H_m)$ *with a cuspidal realization* $\mathcal{C}_\sigma$, *such that the pair* (π, σ) *belongs to the global Vogan packet* $\widetilde{\Pi}_{\phi \times \phi'}[G_n^* \times H_m^*]$ *and the inner product*

$$\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_0}}}(\varphi_\pi), \varphi_\sigma \right\rangle_{H_m} \neq 0$$

for some $\varphi_\pi \in \mathcal{C}_\pi$ *and* $\varphi_\sigma \in \mathcal{C}_\sigma$, *where* $\ell_0^- = m$, *and the* $\psi_{\mathcal{O}_{\ell_0}}$-*Fourier coefficient* $\mathcal{F}^{\psi_{\mathcal{O}_{\ell_0}}}(\varphi_\pi)$ *is defined by an* F-*rational nilpotent orbit* $\mathcal{O}_{\ell_0}$ *in the* F-*stable nilpotent orbit* $\mathcal{O}_{\underline{p}_{\ell_0}}^{\mathrm{st}}$, *associated to the partition* $\underline{p}_{\ell_0} = [(2\ell_0 + 1)1^{\mathfrak{n}-2\ell_0-1}]$.

Proof. By assumption, $L(\tfrac{1}{2}, \phi \times \phi') = L(\tfrac{1}{2}, \phi \times (\phi')^{w_q^\ell}) \neq 0$. Let σ_0 be the member in the global Vogan packet $\widetilde{\Pi}_{(\phi')^{w_q^\ell}}[H_m^*]$, with which Conjecture 6.8 holds, and let $\sigma_0 \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ for some pure inner F-form of H_m^*, having the cuspidal realization $\mathcal{C}_{\sigma_0}$. By Proposition 5.2, the Eisenstein series $E(h, \phi_{\tau \otimes \sigma_0}, s)$ produces the non-zero iterated residual representation $\mathcal{E}_{\tau \otimes \sigma_0}$ on $H_{a+m}(\mathbb{A})$, with a non-zero Fourier coefficient associated to the partition $\underline{p}_{\phi,\phi'}^1$. In other words, take

$$\underline{p}_{\kappa_0} := [(2\kappa_0 + 1)1^{2a+\mathfrak{m}-2\kappa_0-1}]$$

with $\kappa_0 = a - \ell_0 - 1$, $\ell_0^- = m$, and $\kappa_0^- = n$. Then the $\psi_{\mathcal{O}_{\kappa_0}}$-Fourier coefficient $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau \otimes \sigma_0})$ is non-zero and cuspidal as a sub-representation of $G_n^{\mathcal{O}_{\kappa_0}}(\mathbb{A})$ in

the cuspidal spectrum $L^2_{\mathrm{cusp}}(G_n^{\mathcal{O}_{\kappa_0}})$, where $\mathcal{O}_{\kappa_0}$ is an F-rational nilpotent orbit in the F-stable nilpotent orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_{\kappa_0}}$ associated to the partition $\underline{p}_{\kappa_0}$. Note that the group $G_n^{\mathcal{O}_{\kappa_0}}$ is a pure inner F-form of G_n^*, and by Theorem 6.6, the κ_0-th Bessel module $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma_0})$ is $\widetilde{\mathrm{O}}(G_n^{\mathcal{O}_{\kappa_0}})$-stable, and every irreducible summand of $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma_0})$ has a global Arthur parameter belonging to the $\widetilde{\mathrm{O}}(G_n^{\mathcal{O}_{\kappa_0}})$-orbit $\{\phi = \phi_\tau, \phi_\star\}$ of ϕ_τ.

Take $(\pi, \mathcal{C}_\pi)$ to be one of the irreducible summands, such that π belongs to the global Arthur packet $\Pi_\phi(G_n^{\mathcal{O}_{\kappa_0}})$. Then for some $\varphi_\pi \in \mathcal{C}_\pi$, the Bessel period

$$\left\langle \varphi_\pi, \mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma_0}) \right\rangle_{G_n^{\mathcal{O}_{\kappa_0}}}$$

is non-zero. By replacing the residue $\mathcal{E}_{\tau\otimes\sigma_0}$ by the corresponding Eisenstein series, we obtain that the global zeta integral

$$\left\langle \varphi_\pi, \mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(E(\cdot, \phi_{\tau\otimes\sigma_0}, s)) \right\rangle_{G_n^{\mathcal{O}_{\kappa_0}}}$$

is non-zero for $\mathrm{Re}(s)$ large. By Corollary 4.4, the pair $(\pi, \sigma_0^{w_q^\ell})$ admits a non-zero Bessel period. It is clear that $\sigma_0^{w_q^\ell}$ belongs to the global Vogan packet $\widetilde{\Pi}_{\phi'}[H_m^*]$. We take $\sigma := \sigma_0^{w_q^\ell}$. Then the pair (π, σ) belongs to the global Vogan packet $\widetilde{\Pi}_{\phi\times\phi'}[G_n^* \times H_m^*]$ and has the desired property. We are done. $\qquad\square$

We note that Theorem 6.10 does not assume that the cuspidal multiplicity of π should be one, while the global Gan-Gross-Prasad conjecture takes this cuspidal multiplicity one assumption in [16].

Also, for F-quasisplit classical groups G, a special case of Theorem 6.10 was also considered in [20] and [21]. It is clear that within the theory of the construction via twisted automorphic descents of concrete modules for irreducible cuspidal automorphic representations, the proof of Theorem 6.10 is more transparent than that in [20] or [21].

By Proposition 6.9 and [40], the assumption in Theorem 6.10 is verified for the case of $m = 1$ and H_1 is an F-form of SO_2. Hence, Theorem 6.10 holds *without the assumption of Conjecture* 6.8 for this special case. Combining with Theorem 5.7, the global Gan-Gross-Prasad Conjecture holds for this case.

COROLLARY 6.11 (Global Gan-Gross-Prasad Conjecture: special case). *Let G_n^* be the F-split SO_{2n+1} and $\phi = \phi_\tau$ be a generic global Arthur parameter in $\widetilde{\Phi}_2(G_n^*)$ determined by the irreducible isobaric automorphic representation τ of $G_{E/F}(a)(\mathbb{A})$ as given in (4.1). Let ϕ' be a generic global Arthur parameter of H_1^*, which is an anisotropic SO_2 over F. Then the following statements are equivalent*:

(1) *There exist an automorphic member π in $\widetilde{\Pi}_{\phi_\tau}[G_n^*]$ and an automorphic member σ in $\widetilde{\Pi}_{\phi'}[H_1^*]$ such that the inner product*

$$\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_0}}}(\varphi_\pi), \varphi_\sigma \right\rangle_{H_1}$$

is non-zero for some $\varphi_\pi \in \pi$ and $\varphi_\sigma \in \sigma$;

(2) *The central L-value $L(\frac{1}{2}, \tau \times \phi')$ is non-zero.*

Note that the cuspidal multiplicities of π and σ in Corollary 6.11 are one. Hence the cuspidal realizations of π and σ are unique. Also we would like to mention that Corollary 6.11 can be proved for unitary groups, but we will not discuss the details here. We also note that Corollary 6.11 with trivial σ was considered in [13], via a different approach.

7. **On the main conjecture**

7.1. *The main conjecture*: *general case.* We are going to prove the main conjecture (Conjecture 6.7), assuming Conjectures 2.3 and 3.1. More precisely, we show, assuming the conjectures, that for any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ with a G_n-relevant, generic global Arthur parameter ϕ in $\widetilde{\Phi}_2(G_n^*)$, the cuspidal automorphic module $\mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau; \sigma) = \mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau \otimes \sigma})$ as constructed through diagram (6.7) is a direct sum of the two irreducible cuspidal representations in the $\widetilde{\mathrm{O}}(G_n)$-orbit of π in the cuspidal spectrum of G_n. If we assume further that the $\widetilde{\mathrm{O}}(G_n)$-orbit of π contains only π, then we have

$$\mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau; \sigma) \cong \pi.$$

By Proposition 2.6, the F-rational orbit $\mathcal{O}_{\kappa_0}$ can be chosen such that $G_n^{\mathcal{O}_{\kappa_0}} = G_n$. We note that the proof of Conjecture 3.1 has been well developed, the known cases of which were explained in Theorem 3.2.

THEOREM 7.1 (Cuspidal automorphic modules). *Conjectures 2.3 and 3.1 imply Conjecture 6.7.*

Proof. Take any cuspidal automorphic member $\pi \in \widetilde{\Pi}_\phi(G_n)$ with a cuspidal realization $\mathcal{C}_\pi$ satisfying the conditions in Conjecture 2.3. It follows that $m := \ell_0^-$, $H_m := H_{\ell_0^-}^{\mathcal{O}_{\ell_0}}$, and $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H_m)$ with a generic, H_m-relevant global Arthur parameter $\phi' \in \widetilde{\Phi}_2(H_m^*)$ and with a cuspidal realization $\mathcal{C}_\sigma$. They have the property that the inner product $\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_0}}}(\varphi_\pi), \overline{\varphi}_\sigma \right\rangle_{H_m}$ is non-zero for some $\varphi_\pi \in \mathcal{C}_\pi$ and $\varphi_\sigma \in \mathcal{C}_\sigma$. As proved in Section 2.4, for each local place ν of F, the group $G_n(F_\nu) \times H_m(F_\nu)$ is relevant in the sense of the local Gan-Gross-Prasad conjecture as discussed in Section 3.2, and the local parameter $\phi_\nu \otimes \phi'_\nu$ belongs to $\widetilde{\Phi}_{\mathrm{unit},\nu}^+(G_n \times H_m)$. By Conjecture 3.1, the pair (π_ν, σ_ν) must be the unique distinguished member in the local Vogan packet $\widetilde{\Pi}_{\phi_\nu \otimes \phi'_\nu}[G_n^* \times H_m^*]$ as defined

in (3.4), such that the following space, as defined in (3.5),

$$\mathrm{Hom}_{R_{\ell_0},\mathcal{O}_{\ell_0}(F_\nu)}(\pi_\nu \otimes \sigma_\nu, \psi_{\mathcal{O}_{\ell_0},\nu})$$

is non-zero. Hence the pair (π,σ) is the unique distinguished member in the global Vogan packet $\widetilde{\Pi}_{\phi\otimes\phi'}[G_n^* \times H_m^*]$.

We apply the reciprocal non-vanishing for Bessel periods (Theorem 5.3) to the data $(G_n, H_m; \tau, \pi, \sigma)$, following the choice in Section 5.2 and obtain that the Bessel period

$$\left\langle \varphi_\pi, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa_0}}}(\mathcal{E}_{\tau\otimes\sigma'})} \right\rangle_{G_n} \neq 0,$$

for some choice of data. In particular, this implies that the $\psi_{\mathcal{O}_{\kappa_0}}$-Fourier coefficient $\mathcal{F}^{\psi_{\mathcal{O}_{\kappa_0}}}(\mathcal{E}_{\tau\otimes\sigma'})$ is non-zero.

On the other hand, by Theorem 6.6, $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma'})$ with $n = \kappa_0^-$ is non-zero and cuspidal as a sub-representation of $G_n(\mathbb{A})$ in the cuspidal spectrum $L^2_{\mathrm{cusp}}(G_n)$, with $G_n = G_n^{\mathcal{O}_{\kappa_0}}$, and hence can be written as a multiplicity free, Hilbert direct sum

$$\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma'}) = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k \oplus \cdots,$$

where $\pi_i \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ for all $i = 1, 2, \ldots$. Each irreducible summand π_i has a generic global Arthur parameter belonging to the $\widetilde{\mathrm{O}}(G_n)$-orbit $\{\phi = \phi_\tau, \phi_\star\}$ of ϕ_τ. We apply Theorem 5.3 to π_i for all i. The non-vanishing of the Bessel period $\left\langle \varphi_{\pi_i}, \overline{\mathcal{F}^{\psi_{\mathcal{O}_{\kappa_0}}}(\mathcal{E}_{\tau\otimes\sigma'})} \right\rangle_{G_n}$ implies the inner product on the right-hand side

$$\left\langle \mathcal{F}^{\psi_{\mathcal{O}_{\ell_0}}}(\varphi_{\pi_i}), \overline{\varphi}_\sigma \right\rangle_{H_m}$$

is non-zero for some choice of data. Following Section 2.4, the product $G_n^{\mathcal{O}_{\kappa_0}} \times H_m$ constructed as in diagram (6.7) is a pure inner F-form of an F-quasisplit $G_n^* \times H_m^*$. Then by Theorem 6.6 again, the pair (π_i, σ) belongs to either the global Vogan packet $\widetilde{\Pi}_{\phi\otimes\phi'}[G_n^* \times H_m^*]$ or the global Vogan packet $\widetilde{\Pi}_{\phi_\star\otimes\phi'}[G_n^* \times H_m^*]$. Since the pair (π, σ) is the unique distinguished member in the global Vogan packet $\widetilde{\Pi}_{\phi\otimes\phi'}[G_n^* \times H_m^*]$, and the pair $(\pi_\star, \sigma)$ is the unique distinguished member in the global Vogan packet $\widetilde{\Pi}_{\phi_\star\otimes\phi'}[G_n^* \times H_m^*]$, we must have that for each index i, π_i is isomorphic to either π or $\pi_\star$, under the assumption of Conjecture 3.1.

Because the direct sum decomposition of $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma})$ is multiplicity free, it follows that $\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma})$ must be of the form

$$\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma}) \cong \pi \oplus \pi_\star,$$

if the $\widetilde{\mathrm{O}}(G_n)$-orbit of π has two members π and $\pi_\star$. If the $\widetilde{\mathrm{O}}(G_n)$-orbit of π contains only π, then we must have

$$\mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_{\tau\otimes\sigma}) \cong \pi.$$

We are done. $\square$

7.2. *The main conjecture: regular orbit case.* In this section, we assume that the group $G_n = G_n^*$ is F-quasisplit, and $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n^*)$ is generic, i.e., has a non-zero Whittaker-Fourier coefficient. In this case, the global Arthur parameter of π can be taken in the form (3.1). Then the Langlands functorial transfer of π from G_n^* to $G_{E/F}(N)$ is τ, which is of the form (4.1). This is essentially proved by the work of Cogdell, Kim, Piatetski-Shapiro and Shahidi in [11], with combination of the automorphic descent of Ginzburg-Rallis-Soudry ([23]). We refer to [37, §3.1] for detailed discussions of this and some related issues.

In this case, Conjecture 2.3 holds automatically without (H_m, σ). The residual representation is $\mathcal{E}_\tau$ on the F-quasisplit $H_a^*(\mathbb{A})$. The automorphic descent of Ginzburg-Rallis-Soudry in [23] shows that

$$\mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau; \emptyset) = \mathcal{F}_n^{\mathcal{O}_{\kappa_0}}(\mathcal{E}_\tau)$$

is a non-zero cuspidal automorphic representation of $G_n^*(\mathbb{A})$. As proved in [42], the descent $\mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau; \emptyset)$ is in fact irreducible for G_n^*, which is an F-split odd special orthogonal group. In general, the structure of $\mathcal{D}_n^{\mathcal{O}_{\kappa_0}}(\tau; \emptyset)$ follows from Conjecture 3.1. Hence Conjecture 6.7 is proved under Conjecture 3.1 as a consequence of the proof of Theorem 7.1.

COROLLARY 7.2 (Regular orbit). *Let G_n^* be F-quasisplit. For any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n^*)$ to be generic with its global Arthur parameter (3.1) and τ as in (4.1), then Conjecture 6.7 holds for π under the assumption of Conjecture 3.1.*

7.3. *The main conjecture: subregular orbit case.* We consider in this subsection the irreducible cuspidal automorphic representations π of $G_n(\mathbb{A})$ such that the set $\mathfrak{p}^m(\pi)$ contains the partition $\underline{p}_{\mathrm{subr}}$ corresponding to the subregular nilpotent orbit of G_n^*. In this situation, it is clear that $\mathfrak{p}^m(\pi) = \{\underline{p}_{\mathrm{subr}}\}$. Conjecture 2.3 can be verified as follows. The group H_m constructed via Diagram (6.7) can be determined as below.

If G_n^* is an F-quasisplit SO_{2n}, the subregular partition $\underline{p}_{\mathrm{subr}}$ is $[(2n-3)3]$. The partition with the first occurrence index ℓ_0 is $\underline{p}_{\ell_0} = [(2n-3)1^3]$ with $\ell_0 = n-2$. Hence H_m is a pure inner F-form of SO_3, where $m = \ell_0^-$. According to [38, Th. 11.2], because $\mathfrak{p}^m(\pi) = \{\underline{p}_{\mathrm{subr}} = [(2n-3)3]\}$, the ℓ_0-th Bessel module $\mathcal{F}^{\mathcal{O}_{\ell_0}}(\pi)$ associated to the F-rational orbit $\mathcal{O}_{\ell_0}$ must be non-zero if $H_m = H_{\ell_0^-}^{\mathcal{O}_{\ell_0}}$ is the split SO_3. Hence Conjecture 2.3 holds for this case.

If G_n^* is an F-split SO_{2n+1}, then $\underline{p}_{\mathrm{subr}}$ is $[(2n-1)1^2]$, which is the partition with the first occurrence index $\ell_0 = n-1$. In this case, the group H_m is an F-form of SO_2, and hence Conjecture 2.3 holds.

If G_n^* is an F-quasisplit U_{2n}, then $\underline{p}_{\mathrm{subr}}$ is $[(2n-1)1]$, which is the partition with the first occurrence index $\ell_0 = n-1$. In this case, the group H_m is equal to U_1, and hence Conjecture 2.3 holds.

If G_n^* is an F-quasisplit U_{2n+1}, then the subregular partition $\underline{p}_{\mathrm{subr}}$ is $[(2n)1]$. The partition with the first occurrence index ℓ_0 is $\underline{p}_{\ell_0} = [(2n-1)1^2]$ with $\ell_0 = n-1$. Hence H_m is an F-form of U_2. It is clear that Conjecture 2.3 also holds for this case, following the proof for the case of F-quasisplit SO_{2n}.

We summarize this discussion as

PROPOSITION 7.3. *Let* $\phi = \phi_\tau$ *be the generic global Arthur parameter of* G_n^* *as given in* (3.1) *with* τ *as defined in* (4.1). *If a cuspidal automorphic member* π *in the global Vogan packet* $\widetilde{\Pi}_\phi[G_n^*]$ *has the property that* $\mathfrak{p}^m(\pi) = \{\underline{p}_{\mathrm{subr}}\}$, *then Conjecture* 2.3 *holds for* π.

As a consequence of the proof of Theorem 7.1, we have the following result.

COROLLARY 7.4 (Subregular orbit). *Assume that* $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ *has a* G_n-*relevant, generic global Arthur parameter in* $\widetilde{\Phi}_2(G_n^*)$ *and the set* $\mathfrak{p}^m(\pi)$ *contains the subregular partition* $\underline{p}_{\mathrm{subr}}$ *of type* $(\mathfrak{n}, G_n^*)$. *Conjecture* 6.7 *holds for* π *under the assumption of Conjecture* 3.1.

Appendix A. Non-vanishing of local zeta integrals

In this appendix, we prove Proposition 5.5. It is a purely local non-vanishing property of the finite product of the local zeta integrals

$$\mathcal{Z}_S(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_0}}).$$

However, the local data have constraints from the global assumption for (π, τ, σ) from Theorem 5.3. From Proposition 5.4, $\mathcal{Z}_S(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_0}})$ converges absolutely for $\mathrm{Re}(s)$ large, has a meromorphic continuation to $s \in \mathbb{C}$, and is holomorphic at $s = \frac{1}{2}$. What we need to prove Theorem 5.3 is the non-vanishing at $s = \frac{1}{2}$ for a choice of data with certain global constraints as described in Proposition 5.5. In fact, we are going to show a more general non-vanishing property for the local zeta integral $\mathcal{Z}_\nu(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\mathcal{O}_{\kappa_0}})$ for every $\nu \in S$. These local zeta integrals converge absolutely for $\mathrm{Re}(s)$ large and have a meromorphic continuation to $s \in \mathbb{C}$. We give the proof in [43] and refer to [75] and [76] for the case of the split special orthogonal groups.

Throughout this appendix, all algebraic groups X are defined over F_ν. The F_ν-rational points of X is simply denoted by $X = X(F_\nu)$ when no confusion is caused.

For $\mathrm{Re}(s)$ large, the local zeta integral in Theorem 4.5 is defined as in (4.41) by
(A.1)
$$\mathcal{Z}_\nu(s, \phi_{\tau \otimes \sigma'}, \varphi_\pi, \psi_{\ell, w_0}) = \int_{R_{\ell, \beta-1}^\eta \backslash G_{m^-}^{w_0}} \mathcal{P}_\nu^{\psi_{\beta-1, y-\kappa}^{-1}} (\pi_\nu(g_\nu)\varphi_{\pi_\nu}, \mathrm{J}_{s,\nu}(\phi_{s,\nu})(g_\nu)) \, \mathrm{d}g_\nu,$$

where $\mathcal{P}_\nu^{\psi_{\beta-1,y-\kappa}^{-1}}$ is the unique local Bessel functional, up to scalar, in the space

$$\text{(A.2)} \qquad \text{Hom}_{R_{\ell,\beta-1}^\eta(F_\nu)}(\pi_\nu \otimes \sigma_\nu, \psi_{\beta-1,y-\kappa}^{-1}).$$

This Hom-space is at most one-dimensional, by the uniqueness of local Bessel models for classical groups ([2], [77], [16] and [44]). Alternatively, for a local Bessel functional $\mathfrak{b}_\nu$ in (A.2), we may rewrite $\mathcal{Z}_\nu(s, \phi_{\tau\otimes\sigma'}, \varphi_\pi, \psi_{\ell,w_0})$ as

$$\text{(A.3)}$$
$$\int_{R_{\ell,\beta-1}^\eta \backslash G_{m^-}^{w_0}} \int_{U_{a,\eta}^-(F_\nu)} \mathfrak{b}_\nu(\pi_\nu(g_\nu)\varphi_{\pi_\nu}, f_{\mathcal{W}_{\tau_\nu}\otimes\sigma'_\nu,s}(u\epsilon_\beta\eta g_\nu))\psi_{\mathfrak{m}+a+\ell,a-\ell}(u)\,\mathrm{d}u\,\mathrm{d}g_\nu,$$

where $f_{\mathcal{W}_{\tau_\nu}\otimes\sigma'_\nu,s}$ is determined by $\phi_{\tau\otimes\sigma'}$ given in (4.38).

Our goal is to construct a section $f_{\mathcal{W}_{\tau_\nu}\otimes\sigma'_\nu,s}$ belonging to $\mathrm{I}_{s,\nu}(\mathcal{W}_{\tau_\nu}, \sigma'_\nu)$ such that the following non-vanishing holds.

PROPOSITION A.1. *Suppose that a non-zero local Bessel functional $\mathfrak{b}_\nu$ in the $\text{Hom}_{R_{\ell,\beta-1}^\eta(F_\nu)}$-space (A.2) is not zero at some $\varphi_{\pi_\nu} = v_{\pi_\nu} \in \pi_\nu$ and $v_{\sigma_\nu} \in \sigma_\nu$, i.e., $\mathfrak{b}_\nu(v_{\pi_\nu}, v_{\sigma_\nu}) \neq 0$. Then, for any given $s = s_0 \in \mathbb{C}$, there exists a holomorphic section $f_{\mathcal{W}_{\tau_\nu}\otimes\sigma'_\nu,s}$ in (A.3) belonging to $\mathrm{I}_{s,\nu}(\mathcal{W}_{\tau_\nu}, \sigma'_\nu)$ such that the local zeta integral $\mathcal{Z}_\nu(s, \phi_{\tau\otimes\sigma'}, \varphi_\pi, \psi_{\ell,w_0})$ is non-zero at $s = s_0$.*

It is clear that Proposition A.1 for split orthogonal groups over p-adic fields is just Proposition 4.1 of [75]. In the proof of Proposition A.1, one of the technical issues is to construct the section $f_{\mathcal{W}_{\tau_\nu}\otimes\sigma'_\nu,s}$ in the space of the induced representation $\mathrm{I}_{s,\nu}(\mathcal{W}_{\tau_\nu}, \sigma'_\nu)$ with the given constraints. Soudry in his proof of [75, Prop. 4.1] uses the Iwasawa decomposition to explicitly construct such sections $f_{\mathcal{W}_{\tau_\nu}\otimes\sigma'_\nu,s}$. We are going to use the Bruhat decomposition to proceed the explicit construction, which works for more general groups over local fields of characteristic 0.

We recall from Section 4.1 that H_{a+m} is either a special orthogonal group or unitary group. When H_{a+m} is unitary and ν splits in the number field E, $H_{a+m}(F_\nu) = \mathrm{U}_{2a+\mathfrak{m}}(F_\nu)$ is isomorphic to $\mathrm{GL}_{2a+\mathfrak{m}}(F_\nu)$. We defer the discussion on this case to the end of this proof. We first consider the case that $H_{a+m}(F_\nu)$ is not isomorphic to $\mathrm{GL}_{2a+\mathfrak{m}}(F_\nu)$. For convenience, we consider $\mathrm{J}_{s,\nu}$ as a map

$$\text{(A.4)} \qquad \mathrm{I}_{s,\nu}(\mathcal{W}_{\tau_\nu}, \sigma'_\nu) \to \mathrm{I}_{s,\nu}^{w_0}(\psi_{\beta-1,y-\kappa}, \sigma_\nu),$$

which is given by the following $U_{a,\eta}^-(F_\nu)$-integration,

$$\text{(A.5)} \qquad \mathrm{J}_{s,\nu}(f_{\mathcal{W}_{\tau_\nu}\otimes\sigma'_\nu,s})(g) := \int_{U_{a,\eta}^-(F_\nu)} f_{\mathcal{W}_{\tau_\nu}\otimes\sigma'_\nu,s}(n\epsilon_\beta\eta g)\psi_{\mathfrak{m}+a+\ell,a-\ell}(n)\,\mathrm{d}n,$$

as in (4.33).

It is not hard to show that the integration in (A.5) converges absolutely for $\mathrm{Re}(s)$ large. It is a little bit more technical to show that it admits a meromorphic continuation to $s \in \mathbb{C}$ in general, which will be treated in [43].

However, for the purpose of this appendix, we are able to prove this easily for the particular sections $f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}$ that will be constructed below for the proof of Proposition A.1.

Remark A.2. For further refined applications of the global zeta integrals considered in this paper, one may be interested in the characterization of the image of $\mathrm{J}_{s,\nu}$ in (A.4). However, for the purpose of this paper, we do not need this. Hence we will leave this interesting question to be considered in our future work.

Let $\mathfrak{b}_\nu$ be a non-zero local Bessel functional in the Hom-space (A.2). Take some $v_{\pi_\nu} \in \pi_\nu$ and $v_{\sigma_\nu} \in \sigma_\nu$, such that $\mathfrak{b}_\nu(v_{\pi_\nu}, v_{\sigma_\nu}) \neq 0$. We are going to construct a section $f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}$ in

$$\mathrm{I}_{s,\nu}(\mathcal{W}_{\tau_\nu}, \sigma'_\nu) = \mathrm{Ind}_{P_{\hat{a}}}^{H_{a+m}}(|\cdot|^s \mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu),$$

which is compactly supported in the open cell $P_{\hat{a}} U_{\hat{a}}^-$ of H_{a+m}, modulo $P_{\hat{a}}$ from the left. Recall that $U_{\hat{a}}^-$ is the unipotent subgroup opposite to the unipotent radical $U_{\hat{a}}$ of $P_{\hat{a}}$, as defined in Section 4.1. We define

$$(\text{A.6}) \quad f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}\left(\begin{pmatrix} g & & \\ & h & \\ & & g^* \end{pmatrix} u \bar{n}' \epsilon_\beta \eta\right) := |\det g|^{s+\rho_a} W_{\tau_\nu}^\kappa(g) f_\nu(\bar{n}') \sigma(h) v_{\sigma_\nu}$$

with $g \in \mathrm{GL}_a(E_\nu)$, $h \in H_m(F_\nu)$, $u \in U_{\hat{a}}(F_\nu)$, and $\bar{n}' \in U_{\hat{a}}^-(F_\nu)$. Here $W_{\tau_\nu}^\kappa(g)$ is a Whittaker function in $\mathcal{W}_{\tau_\nu}$, $f_\nu(\bar{n}')$ is a smooth, compactly supported function defined on $U_{\hat{a}}^-(F_\nu)$, and $|\cdot|^{2\rho_a}$ is the modular character of the parabolic subgroup $P_{\hat{a}}$. Over archimedean places, we may take $f_\nu(\bar{n}')$ also to be a positive real-valued function. Since $H_{a+m}(F_\nu) \neq \mathrm{GL}_{2a+\mathfrak{m}}(F_\nu)$, $G_{E_\nu/F_\nu}(a)(F_\nu) \neq \mathrm{GL}_a(F_\nu) \times \mathrm{GL}_a(F_\nu)$. Hence the subgroup $(\mathrm{Res}_{E/F}\mathrm{GL}_a)(F_\nu)$ of the Levi part of $P_{\hat{a}}$ can be written as $\mathrm{GL}_a(E_\nu)$, where E_ν is either F_ν or a quadratic field extension over F_ν. Remark that because of the conjugation by w_q^ℓ, it is v_σ on the right-hand side of (A.6), instead of $v_{\sigma'}$. It is clear that the section $f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}$ defined in (A.6) is a smooth section in $\mathrm{I}_{s,\nu}(\mathcal{W}_{\tau_\nu}, \sigma'_\nu)$. It is clear that for such a constructed section $f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}$, the integration in (A.5) is over a compact set. Hence the integral converges absolutely for every $s \in \mathbb{C}$ and admits meromorphic continuation to all $s \in \mathbb{C}$.

We may assume that the value of $f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}(g)$ at $g = \epsilon_\beta \eta$ is

$$(\text{A.7}) \qquad\qquad f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}(\epsilon_\beta \eta) = W_{\tau_\nu}^\kappa(\mathrm{I}_a) v_{\sigma_\nu},$$

where I_a is the identity matrix of GL_a. It is clear that the functional $\mathfrak{b}_\nu$ evaluated at $(v_\pi, f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}(\epsilon_\beta \eta))$ is given by

$$(\text{A.8}) \qquad \mathfrak{b}_\nu(v_\pi, f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}(\epsilon_\beta \eta)) = W_{\tau_\nu}^\kappa(\mathrm{I}_a) \cdot \mathfrak{b}_\nu(v_{\pi_\nu}, v_{\sigma_\nu}).$$

We may take the value of $W^\kappa_{\tau_\nu}(\mathrm{I}_a)$, so that

$$(A.9) \qquad W^\kappa_{\tau_\nu}(\mathrm{I}_a) \cdot \mathfrak{b}_\nu(v_{\pi_\nu}, v_{\sigma_\nu}) = 1.$$

This gives the normalization of the local Bessel functional $\mathfrak{b}_\nu$ at all $\nu \in S$.

To finish the proof of Proposition A.1, we have to calculate explicitly the relation between $R^\eta_{\ell,\beta-1}\backslash G^{w_0}_{m^-}$ and the open dense set $P_{\hat a} U^-_{\hat a}$ — in particular, the following domain:

$$(A.10) \qquad R^\eta_{\ell,\beta-1}\backslash G^{w_0}_{m^-} \cap (\epsilon_\beta\eta)^{-1}(P_{\hat a}U^-_{\hat a})(\epsilon_\beta\eta).$$

The group $G^{w_0}_{m^-}$ is identified as a subgroup of the Levi subgroup of $P_{\hat\ell}$. According to the structure of the stabilizer of the open cell $P_{\hat a}\epsilon_\beta P_{\hat\ell}$ as given in Section 4.4, the intersection (A.10) can be written as the following intersection:

$$(A.11) \qquad R^\eta_{\ell,\beta-1}\backslash G^{w_0}_{m^-} \cap \mathrm{Ad}(\eta^{-1})(P'_w U^-_{a-\ell}).$$

Recall that $P'_w = H_{a+m-\ell}\cap\epsilon^{-1}_{0,\beta}P_{\hat a}\epsilon_{0,\beta}$ defined in Proposition 4.1 is the standard parabolic subgroup of $H_{a+m-\ell}$ with Levi decomposition $(G_{E_\nu/F_\nu}(a-\ell)\times H_m)\ltimes U_{a-\ell}$, where $U_{a-\ell}$ is the unipotent radical. More details can be found in [45, §3.1]. Because of

$$R^\eta_{\ell,\beta-1} = G^{w_0}_{m^-} \cap \eta^{-1}P'_w\eta,$$

the intersection set $G^{w_0}_{m^-} \cap \mathrm{Ad}(\eta^{-1})(P'_w U^-_{a-\ell})$ is $R^\eta_{\ell,\beta-1}$-left stable. Thus the intersection modulo $R^\eta_{\ell,\beta-1}$ in (A.11) is well defined. It is clear that $P'_w U^-_{a-\ell}$ is an open subset of $H_{a+m-\ell}$.

With the above choice of $f_{\mathcal{W}_{\tau_\nu}\otimes\sigma'_\nu,s}$, the integral (A.1) can be taken over the set (A.11). To proceed with the integral (A.1), we explicitly describe the intersection $G^{w_0}_{m^-} \cap \mathrm{Ad}(\eta^{-1})(P'_w U^-_{a-\ell})$. It is enough to describe the set $\mathrm{Ad}(\eta)G^{w_0}_{m^-} \cap (P'_w U^-_{a-\ell})$. Take

$$(A.12) \qquad p = \begin{pmatrix} g & -Y\cdot h^{-1} & \iota(\hat Z)g^* \\ & h^{-1} & -Y'g^* \\ & & g^* \end{pmatrix} \in P'_w$$

and

$$(A.13) \qquad \bar n = \begin{pmatrix} I_{a-\ell} & & \\ X' & I_{\mathfrak m} & \\ A & X & I_{a-\ell} \end{pmatrix} \in U^-_{a-\ell},$$

with $g \in \mathrm{GL}_{a-\ell}(E_\nu)$, $h \in H_m$, $Y \in \mathrm{Mat}_{(a-\ell)\times\mathfrak m}$, $\hat Z := \omega_{a-\ell}Z^t\omega_{a-\ell}$, $Y' := -\omega_{a-\ell}\cdot\iota(Y)^t\cdot(J^{\mathfrak m}_{\hat{\mathfrak m}})^{-1}$, and $g^* = \iota(\hat g)^{-1}$. Here $\omega_{a-\ell}$ is the anti-diagonal matrix with the unit entry of the size $(a-\ell)$-by-$(a-\ell)$, $J^{\mathfrak m}_{\hat{\mathfrak m}}$ is defined in (2.2), and ι is the Galois element in Γ_{E_ν/F_ν} (as on page 748).

Since $G_{m^-}^{w_0}$ fixes the anisotropic vector $w_0 = y_\kappa$ defined in (4.5), $\mathrm{Ad}(\eta)G_{m^-}^{w_0}$ stabilizes the vector

$$\mathrm{Ad}(\eta)w_0 = \begin{pmatrix} E_1 \\ 0_{\mathfrak{m}} \\ E_2 \end{pmatrix}_{(2(a-\ell)+\mathfrak{m})\times 1},$$

where $0_{\mathfrak{m}}$ is the $\mathfrak{m}$-dimensional zero column vector,

$$E_1 = (0,\ldots,0,1)^t \text{ and } E_2 = ((-1)^{\mathfrak{m}+1}\frac{\kappa}{2},0,\ldots,0)^t \text{ in } \mathrm{Mat}_{(a-\ell)\times 1}.$$

Here we consider $\mathrm{Ad}(\eta)w_0$ as an anisotropic vector in the Hermitian space defining $H_{a+m-\ell}$. Then $p\cdot\bar{n} \in P_w' U_{a-\ell}^-$ is in $\mathrm{Ad}(\eta)G_{m^-}^{w_0}$ if and only if $p\cdot\bar{n}$ fixes the vector $\mathrm{Ad}(\eta)w_0$. That is to say that both p and $\bar{n}$ satisfy the following equations:

(A.14) $ZE_2 = (I_{a-\ell} - g)E_1,\ AE_1 = (\iota(\hat{g}) - I_{a-\ell})E_2,\ X'E_1 = h\cdot Y'E_2.$

Since our integral domain is a set of $R_{\ell,\beta-1}^\eta$-right cosets, we identify the quotient set (A.11) by choosing $h = I_{\mathfrak{m}}$, $g \in Z_{a-\ell}(E_\nu)\backslash\mathrm{GL}_{a-\ell}(E_\nu)$,

$$Y = \begin{pmatrix} 0_{(\mathfrak{m}-1)\times(a-\ell)} \\ y \end{pmatrix} \text{ and } \iota(\hat{Z})g^* = \begin{pmatrix} z_1 & 0_{(a-\ell-1)\times(a-\ell-1)} \\ z_2 & z_3 \end{pmatrix}.$$

Due to (A.14) and the above choice, the vector y in Y, and z_i in Z for $1 \le i \le 3$ are determined by X and g, respectively. Because of this, we write Y_X and Z_g for Y and Z, respectively.

To separate variables, we choose

(A.15) $$f_\nu \begin{pmatrix} I_{a-\ell} & & & & \\ 0 & I_\ell & & & \\ X' & x_2' & I_{\mathfrak{m}} & & \\ x_1 & x_3 & x_2 & I_\ell & \\ A & x_1' & X & 0 & I_{a-\ell} \end{pmatrix} = f_1(x_1,x_2,x_3)f_2(X,A),$$

where f_1 and f_2 are smooth, compactly supported functions, and the size of matrices X and x_i are indicated by the matrix in (A.15). With the above choices, we are able to evaluate more explicitly the function $\mathrm{J}_{s,\nu}(f_{\mathcal{W}_{\tau_\nu}\otimes\sigma_\nu',s})(\mathrm{g})$ as defined in (A.5) for g in the set (A.11). We decompose $\mathrm{Ad}(\eta)\mathrm{g} = p\cdot\bar{n}$ as given in (A.12) and (A.13). Let us conjugate $p\cdot\bar{n}$ by ϵ_β. Referring to (3.6) in [45],

as elements in H_{a+m}, we have

$$\epsilon_\beta p \epsilon_\beta^{-1} = u(Y_X, Z_g) m(g) = \begin{pmatrix} g & 0 & -Y_X & 0 & \iota(\hat{Z}_g)g^* \\ & I_\ell & 0 & 0 & 0 \\ & & I_{\mathrm{m}} & 0 & -Y_X' g^* \\ & & & I_\ell & 0 \\ & & & & g^* \end{pmatrix},$$

(A.16)

$$\epsilon_\beta \bar{n} \epsilon_\beta^{-1} = \begin{pmatrix} I_{a-\ell} & & & & \\ 0 & I_\ell & & & \\ X' & 0 & I_{\mathrm{m}} & & \\ 0 & 0 & 0 & I_\ell & \\ A & 0 & X & 0 & I_{a-\ell} \end{pmatrix},$$

where

$$u(Y_X, Z_g) = \begin{pmatrix} I_{a-\ell} & 0 & -Y_X & 0 & \iota(\hat{Z}_g) \\ & I_\ell & 0 & 0 & 0 \\ & & I_{\mathrm{m}} & 0 & -Y_X' \\ & & & I_\ell & 0 \\ & & & & I_{a-\ell} \end{pmatrix} \quad \text{and} \quad m(g) = \begin{pmatrix} g & & \\ & I_{\mathrm{m}+2\ell} & \\ & & g^* \end{pmatrix}.$$

By the definition of $\mathrm{J}_{s,\nu}(f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s})(\mathrm{g})$ in (A.5), using the above decomposition of g in the set (A.11), we have

(A.17) $\mathrm{J}_{s,\nu}(f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s})(\mathrm{g})$

$$= \int_{U_{a,\eta}^-(F_\nu)} f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}(nu(Y_X, Z_g)m(g) \cdot \epsilon_\beta \bar{n} \epsilon_\beta^{-1} \cdot \epsilon_\beta \eta) \psi_{\mathrm{m}+a+\ell, a-\ell}(n) \, \mathrm{d}n.$$

Recall that the element in $U_{a,\eta}^-$ (see (4.32)) is of form

$$n(x_1, x_2, x_3) := \begin{pmatrix} I_{a-\ell} & & & & \\ 0 & I_\ell & & & \\ 0 & x_2' & I_{\mathrm{m}} & & \\ x_1 & x_3 & x_2 & I_\ell & \\ 0 & x_1' & 0 & 0 & I_{a-\ell} \end{pmatrix}.$$

By simple manipulations, one has

$$n(x_1, x_2, x_3) u(Y_X, Z_g) m(g)$$
$$= m(g) \cdot u_a(-\iota(\hat{B})) \cdot n(x_1 g, x_2 - x_1 Y_X, x_3 - B x_1'),$$

where $B = x_1 \iota(\hat{Z}_g) - x_2 Y_X'$ and $u_a(-\iota(\hat{B})) = \begin{pmatrix} I_{a-\ell} & -\iota(\hat{B}) \\ 0 & I_\ell \end{pmatrix}$ is considered as an element in GL_a as the subgroup of the Levi subgroup of $P_{\hat{a}}$. Continuing with (A.17), by the definition of $f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}$ in (A.6) and f_ν in (A.15), after changing variables we have

$$\mathrm{J}_{s,\nu}(f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s})(\mathrm{g}) = |\det g|^{s+\rho_a} W_{\tau_\nu}^\kappa \left(\begin{pmatrix} g & \\ & I_\ell \end{pmatrix} \right) f_2(X, A) v_\sigma$$

(A.18)

$$\times \int_{U_{a,\eta}^-} f_1(x_1, x_2, x_3) \psi_E((x_1 g^{-1})_{\ell, a-\ell}) \psi_{Z_a, \kappa}^{-1}(u_a(-\iota(\hat{B}_1))) |\det g|^{-\ell} \, \mathrm{d}x_i,$$

where the matrices x_i define the element $n(x_1, x_2, x_3)$ in $U_{a,\eta}^-$ and $B_1 = x_1 g^{-1}(\iota(\hat{Z}_g) - Y_X Y_X') - x_2 Y_X'$. Although the term B_1 is complicated, after we choose suitable X and A defining $\bar{n}(X, A)$, the matrices Y_X and Z_g are zero, so is B_1. Since the function $f_1(x_1, x_2, x_3)$ is chosen to be a smooth and

compactly supported function and is independent of complex variable s, the integral $J_{s,\nu}$ is well defined over the whole complex plane for such choice of the section $f_{\mathcal{W}_{\tau_\nu} \otimes \sigma'_\nu, s}$, and so is the local zeta integral (A.1).

Finally, by plugging the formula (A.18) into (A.1), we obtain that $\mathcal{Z}_\nu(, \cdot)$ equals

(A.19)
$$
\int_{X,A} \int_g |\det g|^{s+\rho_a} W^\kappa_{\tau_\nu} \left(\begin{pmatrix} g & \\ & I_\ell \end{pmatrix} \right) f_2(X,A) \mathfrak{b}_\nu(\pi(\eta^{-1}p(g,X)\bar{n}(X,A)\eta)v_\pi, v_\sigma)
$$
$$
\int_{U_{a,\eta}^-} |\det g|^{-\ell} f_1(x_1,x_2,x_3) \psi_E((x_1 g^{-1})_{\ell,a-\ell}) \psi_{Z_a,\kappa}^{-1}(u_a(-\iota(\hat{B}_1))) \, \mathrm{d}x_i \, \mathrm{d}g \, \mathrm{d}X \, \mathrm{d}A.
$$

The notation in the formula is explained in order. The integration $\int_g$ is over $Z_{a-\ell}(E_\nu) \backslash \{g \in \mathrm{GL}_{a-\ell}(E_\nu) : \iota(\hat{g})E_2 = AE_1 + E_2\}$, with constraints given in (A.14). Rewrite $\bar{n}$ and p to be $\bar{n}(X,A)$ and $p(g,X)$ respectively to indicate their dependence on variables X, A and g, following (A.16). The integration $\int_{X,A}$ is over the set $U_{a-\ell}^-$ with $AE_1 \neq -E_2$, due to $AE_1 = (\iota(\hat{g}) - I_a)E_2$ in (A.14) and $\det(g) \neq 0$. Indeed, because $AE_1 + E_2 = \iota(\hat{g})E_2$, if $AE_1 = -E_2$, then $\iota(\hat{g})E_2 = 0_{a-\ell}$, which implies $\det(g) = 0$.

We are going to finish the proof based on the above expression for the local zeta integral $\mathcal{Z}_\nu(s, \cdot)$. Suppose that $\mathcal{Z}_\nu(s, \cdot)$ is identically zero for all choices of data f_1 and f_2 at the given $s = s_0$. We vary the function $f_2(X,A)$ first and consider the rest of the integral as a continuous function of X and A. Since the integral over $U_{a-\ell}^-$ is identically zero, the remaining integration in the variable $\bar{n}$ as given in (A.13) is identically zero; that is,

(A.20)
$$
\int_g |\det g|^{s+\rho_a-\ell} W^\kappa_{\tau_\nu} \left(\begin{pmatrix} g & \\ & I_\ell \end{pmatrix} \right) \mathfrak{b}_\nu(\pi(\eta^{-1}p\bar{n}\eta)v_\pi, v_\sigma)
$$
$$
\times \int_{U_{a,\eta}^-} f_1(x_1,x_2,x_3) \psi_E((x_1 g^{-1})_{\ell,a-\ell}) \psi_{Z_a,\kappa}^{-1}(u_a(-\iota(\hat{B}_1))) \, \mathrm{d}x_i \, \mathrm{d}g \equiv 0.
$$

Especially, the integral on the left-hand side of (A.20) is identically zero at $\bar{n} = I_{2(a-\ell)+\mathfrak{m}}$, equivalently, at $X = 0_{(a-\ell) \times \mathfrak{m}}$ and $A = 0_{(a-\ell) \times (a-\ell)}$. Because $M = I_\mathfrak{m}$ and $X'E_1 = Y'E_2$ in (A.14), we must have that $Y_X = 0_{(a-\ell) \times \mathfrak{m}}$ due to $X = 0_{(a-\ell) \times \mathfrak{m}}$, and similarly $Z_g = 0_{(a-\ell) \times (a-\ell)}$. It follows that $B_1 = 0_{\ell \times (a-\ell)}$ and the character $\psi_{Z_a,\kappa}^{-1}(u_a(-\iota(\hat{B}_1)))$ disappears. As $A = 0_{(a-\ell) \times (a-\ell)}$ and $AE_1 = (\iota(\hat{g}) - I_a)E_2$ in (A.14), g must belong to the standard mirabolic subgroup of $\mathrm{GL}_{a-\ell}(E_\nu)$; that is, $E_1^t g = E_1^t$. Since the integration domain of g is modulo $Z_{a-\ell}(E_\nu)$ and $E_1^t g = E_1^t$, the integral $\int_g$ is over $Z_{a-\ell-1}(E_\nu) \backslash \mathrm{GL}_{a-\ell-1}(E_\nu)$. Since g is in the standard mirabolic subgroup of $\mathrm{GL}_{a-\ell}(E_\nu)$, g^{-1} stabilizes the character $\psi_E((x_1)_{\ell,a-\ell})$ of $U_{a,\eta}^-$, that is, $(x_1 g^{-1})_{\ell,a-\ell} = (x_1)_{\ell,a-\ell}$.

Furthermore, one may choose a suitable smooth, compactly supported function f_1 such that

$$(\text{A.21}) \qquad \int_{U_{a,\eta}^-} f_1(x_1, x_2, x_3)\psi_E((x_1)_{\ell, a-\ell}) \, dx_1 \, dx_2 \, dx_3 = 1.$$

Plugging (A.21) into (A.20), we have

$$(\text{A.22}) \quad \int_g |\det g|^{s+\rho_a-\ell} W_{\tau_\nu}^\kappa \left(\left({}^g{}_{I_{\ell+1}} \right) \right) \mathfrak{b}_\nu \left(\pi \left(\left({}^g{}_{I_{\mathfrak{m}+1}} {}_{g^*} \right) \right) v_\pi, v_\sigma \right) dg \equiv 0,$$

where $\int_g$ is over $Z_{a-\ell-1}(E_\nu)\backslash \mathrm{GL}_{a-\ell-1}(E_\nu)$.

It is clear that the left-hand side of (A.22) is exactly the same with (4.7) in [75], up to a non-zero constant. We note that the reduction to this type of the integrals is a key step in the proof of such non-vanishing of the local Rankin-Selberg integrals. See [73] for instance. Applying the same inductive argument in Sections 6 and 7 of [73] and the Dixmier-Marlliavin Lemma ([12]), we obtain that

$$W_{\tau_\nu}^\kappa(I_a)\mathfrak{b}_\nu(v_\pi, v_\sigma) = 0.$$

However, this contradicts (A.9). Therefore, there must exist a choice of data such that $\mathcal{Z}_\nu(s, \cdot)$ is not zero at the given $s = s_0$. This completes the proof of Proposition A.1 when H_{a+m} is not isomorphic to $\mathrm{GL}_{2a+\mathfrak{m}}(F_\nu)$.

If $H_{a+m}(F_\nu)$ is isomorphic to $\mathrm{GL}_{2a+\mathfrak{m}}(F_\nu)$, due to the splitness of the group, the matrix calculation such as (A.12) and (A.13) is slight different. See [86] for instance. However, the proof for this case is completely same. Hence we omit the details here.

Appendix B. **On local intertwining operators**

Throughout this appendix, let F be a local field of characteristic 0. Recall that H_m^* is a quasi-split classical group defined over F and H_m is a pure inner F-form of H_m^*. Let ϕ be a local L-parameter of $H_m^*(F)$ and $\widetilde{\Pi}_\phi(H_m)$ the associated L-packet. Assume that ϕ is generic; that is, $\widetilde{\Pi}_\phi(H_m^*)$ contains a generic member, following [67]. Up to a conjugation, assume that ϕ is of form as in Section 3.1,

$$(\text{B.1}) \quad \phi = (\phi_1 \otimes |\cdot|^{\beta_1} \oplus \phi_1^\vee \otimes |\cdot|^{-\beta_1}) \oplus \cdots \oplus (\phi_t \otimes |\cdot|^{\beta_t} \oplus \phi_t^\vee \otimes |\cdot|^{-\beta_t}) \oplus \phi_0,$$

where $\beta_1 > \beta_2 > \cdots > \beta_t > 0$, all $\phi_i \colon \mathcal{L}_F \to {}^L G_{E/F}(n_i)$ for $1 \le i \le t$ and $\phi_0 \colon \mathcal{L}_F \to {}^L H_{n_0}$ are tempered local L-parameters. Then, the L-packet $\widetilde{\Pi}_\phi(H_m)$ is defined to be the set of the Langlands quotients of the induced representations

$$(\text{B.2}) \qquad \mathrm{Ind}_{P(F)}^{H_m(F)} \tau(\phi_1)|\det|^{\beta_1} \otimes \cdots \otimes \tau(\phi_t)|\det|^{\beta_t} \otimes \sigma_0,$$

where the parabolic subgroup P has the Levi subgroup isomorphic to $G_{E/F}(n_1) \times \cdots \times G_{E/F}(n_t) \times H_{n_0}$, σ_0 runs through the tempered L-packet $\widetilde{\Pi}_{\phi_0}(H_{n_0})$, and

$\tau(\phi_i)$ is the irreducible admissible unitary generic representation of $G_{E/F}(n_i)(F)$ given by the local Langlands correspondence for the general linear groups.

PROPOSITION B.1. *If ϕ is a generic L-parameter of H_m^* as given in* (B.1), *then all representations in $\widetilde{\Pi}_\phi(H_m)$ can be written as irreducible standard modules; that is, the induced representations displayed in* (B.2) *are irreducible for all pure inner forms H_m and $\sigma_0 \in \widetilde{\Pi}_\phi(H_{n_0})$.*

Proof. If F is non-archimedean, this proposition is proved by Mœglin and Waldspurger in [67] for orthogonal groups, by Gan and Ichino in [18, Prop. 9.1], and by Heiermann in [29] for general reductive groups. If F is archimedean, it is a special case of Theorem 1.24 in the book by Adams, Barbasch and Vogan ([1]). More details can be found in Chapters 14 and 15 of [1].   □

Proposition B.1 serves as a base for us to prove Theorem 5.1. Recall that the normalized local intertwining operator $\mathcal{N}(\omega_0, \tau \otimes \sigma, s)_\nu$ takes sections in the induced representation

$$\text{(B.3)} \qquad \text{Ind}_{P_{\hat{a}}(F_\nu)}^{H_{a+m}(F_\nu)}(\tau_\nu |\det|^s \otimes \sigma_\nu)$$

to sections in the induced representation

$$\text{(B.4)} \qquad \text{Ind}_{P_{\hat{a}}(F_\nu)}^{H_{a+m}(F_\nu)}(\tau_\nu^* |\det|^{-s} \otimes \sigma_\nu),$$

where τ_ν is the local ν-component of the irreducible isobaric automorphic representation τ as given in (4.1), and σ_ν is the local ν-component of the irreducible cuspidal automorphic representation σ in $\mathcal{A}_{\text{cusp}}(H_m)$ with an H_m-relevant, generic global Arthur parameter ϕ_σ as in Theorem 5.1. It is clear that Theorem 5.1 follows from the following theorem.

THEOREM B.2. *Let ϕ^+ be a local ν-component of an H_m-relevant, generic global Arthur parameter of H_m^*. If τ is an irreducible admissible unitary generic self-dual representation of $G_{E/F}(a)(F)$ and σ is an irreducible representation in the generic local L-packet $\widetilde{\Pi}_{\phi^+}(H_m)$, then the normalized local intertwining operator $\mathcal{N}(\omega_0, \tau \otimes \sigma, s)$ is holomorphic and non-zero for $\text{Re}(s) \geq \frac{1}{2}$.*

Proof. First of all, the local L-packet $\widetilde{\Pi}_{\phi^+}(H_m)$ has a generic member σ° ([3] and [70]) when $H_m = H_m^*$ is quasisplit. If σ is generic, the proposition follows from Theorem 11.1 in [11].

Assume now that σ is not generic. For such a generic local L-packet $\widetilde{\Pi}_{\phi^+}(H_m)$, by Proposition B.1, the standard modules as displayed in (B.2) are irreducible. This is the key point for us to apply the argument in [11] in the proof of this proposition.

According to the structure of the generic unitary dual of the general linear groups, given by Vogan in [79] for the archimedean case and by Tadić in [78] for

the non-archimedean case, any generic member σ° in $\widetilde{\Pi}_{\phi^+}(H_m^*)$ is isomorphic to the irreducible generic unitary induced representation

$$\mathrm{Ind}_{P(F)}^{H_m^*(F)} \tau(\phi_1)|\det|^{\beta_1} \otimes \cdots \otimes \tau(\phi_t)|\det|^{\beta_t} \otimes \sigma_0,$$

where

(B.5)
$$\frac{1}{2} > \beta_1 > \beta_2 > \cdots > \beta_t > 0,$$

and all $\tau(\phi_i)$ and σ_0 are irreducible, unitary, generic, and tempered. By Proposition B.1, each σ in $\widetilde{\Pi}_{\phi^+}(H_m)$ is of form (B.2) with the exponents satisfying (B.5).

Again, by the generic unitary dual of the general linear groups, τ is isomorphic to the irreducible induced representation

(B.6) $\mathrm{Ind}_{P'(E)}^{\mathrm{GL}_a(E)} \tau_1|\det|^{\alpha_1} \otimes \cdots \otimes \tau_d|\det|^{\alpha_d} \otimes \tau_0 \otimes \tau_d^*|\det|^{-\alpha_d} \otimes \cdots \otimes \tau_1^*|\det|^{-\alpha_1},$

where all τ_i are unitary, generic and tempered, and

$$\frac{1}{2} > \alpha_1 > \cdots > \alpha_d > 0.$$

Now we replace the representations τ_ν and σ_ν in (B.3) by their corresponding realizations in (B.6) and (B.2), respectively. By the transitivity of parabolic induction, the normalized local intertwining operator $\mathcal{N}(\omega_0, \tau_\nu \otimes \sigma_\nu, s)$ can be expressed as a composition of the local intertwining operators of rank one, which are of form

(B.7)
$$\mathcal{N}(w_{j,i}, \tau_j \otimes \tau(\phi_i), s \pm \alpha_j \pm \beta_i),$$

(B.8)
$$\mathcal{N}(w'_{j,i}, \tau_j \otimes \tau_i, 2s \pm \alpha_j \pm \alpha_i),$$

(B.9)
$$\mathcal{N}(w''_j, \tau(\phi_i) \otimes \sigma_0, s \pm \alpha_j),$$

where $w_{j,i}$, $w'_{j,i}$, and w''_j are the corresponding Weyl elements. We deal with these three types of the local intertwining operators separately.

The first two types (B.7) and (B.8) were studied by Mœglin and Waldspurger in [65]. For any unitary tempered τ and τ' of general linear groups, the normalized intertwining operator $\mathcal{N}(w, \tau \otimes \tau', s)$ is holomorphic and non-zero for $\mathrm{Re}(s) > -1$. Because of the bounds for the exponents, it follows that

$$\mathcal{N}(w_{j,i}, \tau_j \otimes \tau(\phi_i), s \pm \alpha_j \pm \beta_i) \text{ and } \mathcal{N}(w'_{j,i}, \tau_j \otimes \tau_i, 2s \pm \alpha_j \pm \alpha_i)$$

are holomorphic and non-zero for $\mathrm{Re}(s) \geq 0$.

For the remaining type (B.9), by the bound $0 < \alpha_j < \frac{1}{2}$, it is sufficient to show that the normalized intertwining operator $\mathcal{N}(w'', \tau \otimes \sigma_0, s)$ is holomorphic and non-zero for $\mathrm{Re}(s) > 0$, when τ and σ_0 are unitary tempered. Since ϕ_0 is a generic parameter, there is a generic representation σ_0° in the tempered local L-packet $\widetilde{\Pi}_{\phi_0}(H_{n_0})$. Hence we have the identity of local L-factors:

$$L(s, \tau \times \sigma_0) = L(s, \tau \times \sigma_0^\circ).$$

Referring to [54], $L(s, \tau \times \sigma_0^\circ)$ and $L(s, \tau, \rho)$ are holomorphic and non-zero for $\mathrm{Re}(s) > 0$, and so is the normalizing factor. In addition, following Proposition IV.2.1 in [81] for the non-archimedean case and Lemma 4.4 in [10] for the archimedean case, the non-normalized local intertwining operator $\mathcal{M}(w'', \tau \otimes \sigma_0, s)$ for tempered data is holomorphic and non-zero for $\mathrm{Re}(s) > 0$. It follows that both $\mathcal{M}(w'', \tau_j \otimes \sigma_0, s - \alpha_j)$ and $L(s - \alpha_j, \tau_j \times \sigma_0)$ are holomorphic and non-zero for $\mathrm{Re}(s) \geq \frac{1}{2}$ because $0 < \alpha_j < \frac{1}{2}$. Therefore, the normalized local intertwining operator $\mathcal{N}(w_j'', \tau(\phi_i) \otimes \sigma_0, s \pm \alpha_j)$ is holomorphic and non-zero for $\mathrm{Re}(s) \geq \frac{1}{2}$. Putting together the results for all three types, we complete the proof of this proposition. $\square$

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University of Minnesota, Minneapolis, MN
E-mail: dhjiang@math.umn.edu

National University of Singapore, Singapore
E-mail: matzhlei@nus.edu.sg