

A k -linear triangulated category without a model

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Abstract

In this paper we give an example of a triangulated category, linear over a field of characteristic zero, which does not carry a DG-enhancement. The only previous examples of triangulated categories without a model have been constructed by Muro, Schwede and Strickland. These examples are however not linear over a field.

1. Introduction

1.1. *Main result.* The only known examples of triangulated categories without model (not even topological) are given in [9]. The examples in loc. cit. are not linear over a field and furthermore they depend on some special properties of the number 2. In particular, they satisfy $2 \neq 0$ but $4 = 0$.

In this paper we discuss triangulated categories over a field k of characteristic zero.¹ In this case the appropriate notion of a model is a DG-enhancement [3], [4], [8], or what amounts to the same thing²: an A_∞ -enhancement (see Section 10). Our main result is an example of a *k -linear triangulated category which does not carry an A_∞ -enhancement*. This, in particular, answers positively what is described as a challenging question in the survey [4] by Canonaco and Stellari, namely, Question 3.8. Our example also provides a negative answer to Question 3.3 of their survey.

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¹Notwithstanding what we say here, almost everything we do is valid in arbitrary characteristic. However in finite characteristic we would also have to consider topological enhancements, and we do not discuss these in the current paper.

²We can always transform an A_∞ -enhancement into a DG-enhancement by taking its DG-hull. See [7, p. 127] or [12, App. C].

To describe the example we have to introduce some notation. Fix a natural number $n \geq 3$ and let k be either a field of characteristic zero or an infinite field of characteristic $> n$. Let $R = k[x_1, \dots, x_n]$, and let K be the quotient field of R . Furthermore let $R[\varepsilon]$ be the R -linear DG-algebra with $|\varepsilon| = -n+2$, $\varepsilon^2 = 0$, $d\varepsilon = 0$. Let $C(R, R)$ be the Hochschild cochain complex of R and let $\mathrm{HH}^n(R, R) = H^n(C(R, R))$. Let $T_{R/k}^n = \wedge_R^n \mathrm{Der}_k(R, R)$. The HKR theorem furnishes an inclusion $T_{R/k}^n \subset Z^n C(R, R)$ which induces an isomorphism $T_{R/k}^n \cong \mathrm{HH}^n(R, R)$. For $\eta \in T_{R/k}^n$ we let R_η be the $k[\varepsilon]$ -linear A_∞ -deformation of $R[\varepsilon]$ whose only non-trivial higher multiplication is given by $\varepsilon\eta$.

THEOREM 1.1 (see [Section 10.3](#)). *Assume $n \geq 14$ and $\eta \neq 0$. Then there exists a triangulated category without A_∞ -enhancement with semi-orthogonal decomposition $\langle D(K), D(R_\eta) \rangle$.*

In the next few sections we discuss in more detail the ingredients that go into the construction of this example.

1.2. Pre-triangulated A_n -categories. An A_∞ -category [\[7\]](#) is a DG-graph equipped with higher compositions $(m_i)_{i \geq 1}$ which satisfy certain natural quadratic relations.³ If only m_i with $i \leq n$ are defined, then we obtain the corresponding notion of an A_n -category. As a general principle, for any A_∞ -notion, there is a corresponding A_n -notion in which we consider only operations with $\leq n$ arguments, and we require the axioms to only hold for expressions with $\leq n$ arguments. Facts about A_∞ -categories remain valid for A_n -categories as long as they only involve such expressions. It is useful to note that if \mathfrak{a} is an A_n -category for $n \geq 3$, then its “homotopy category” $H^0(\mathfrak{a})$ is an honest category.

A DG-category is an A_∞ -category with $m_i = 0$ for $i > 2$. In their seminal paper [\[3\]](#) Bondal and Kapranov introduced *pre-triangulated DG-categories* which, in particular, have the property that their homotopy category is canonically triangulated. Their most striking insight is that, whereas a triangulated category is an additive category with extra structure, a pre-triangulated DG-category is a DG-category with extra properties.

It is well understood how to define the analogous notion of a *pre-triangulated A_∞ -category* (see [\[1\]](#)). An A_∞ -category is pre-triangulated if the natural functor $\mathfrak{a} \rightarrow \mathrm{Tw} \mathfrak{a}$ is a quasi-equivalence, where $\mathrm{Tw} \mathfrak{a}$ is the category of twisted complexes over \mathfrak{a} . It is easy to see that this is equivalent to \mathfrak{a} being closed under suspensions, desuspensions and cones of closed maps, up to isomorphism

³We also have to specify the compatibility with units. As specified in [Section 4](#), throughout in this paper we will use A_∞ -categories (and A_n -categories) that are strictly unital.

in $H^0(\mathrm{Tw} \mathfrak{a})$. Stating these properties explicitly requires only a finite number of higher operations on \mathfrak{a} and so they make sense for A_n -categories for $n \gg 0$.

For any A_∞ -category \mathfrak{a} , $H^0(\mathrm{Tw} \mathfrak{a})$ is canonically triangulated. Hence if \mathfrak{a} is pre-triangulated, then $H^0(\mathfrak{a})$ is also canonically triangulated. (Note: In this introduction we will follow tradition by viewing a triangulated category as an additive category. However in the main body of the paper we will equip a triangulated category with its canonical graded enrichment. This means, in particular, that we use $H^*(\mathfrak{a})$ rather than $H^0(\mathfrak{a})$. See [Section 2](#) for the rationale for this choice.) Now it is intuitively clear that it should be possible to prove this using only a finite number of the higher operations on \mathfrak{a} . It then follows that it must be possible to define for $n \gg 0$ a notion of a pre-triangulated A_n -category which induces a canonical triangulation on its homotopy category.

Unfortunately, carrying out this program naively using explicit equations seems to be a nightmare. Therefore we are forced carry over some more advanced technology from the A_∞ -context. This is done in [Sections 4](#) and [5](#). The main difficulty we face is that the definition of $\mathrm{Tw} \mathfrak{a}$ depends on higher compositions in \mathfrak{a} of unbounded arity and therefore does not generalize to A_n -categories. Luckily this issue can be solved by considering twisted complexes of uniformly bounded length. In fact we only need $\mathrm{Tw}_{\leq 1} \mathfrak{a}$, which consists of twisted complexes of length two. This leads to our first main result.

THEOREM 1.2 ([Lemma 5.6](#), [Definition 5.10](#), [Theorem 7.3](#)). *If \mathfrak{a} is an A_n -category, then $\mathrm{Tw}_{\leq 1} \mathfrak{a}$ is an $A_{\lfloor (n-1)/2 \rfloor}$ -category. If $n \geq 7$, then we say that \mathfrak{a} is pre-triangulated if $H^*(\mathfrak{a}) \rightarrow H^*(\mathrm{Tw}_{\leq 1} \mathfrak{a})$ is a graded equivalence. If \mathfrak{a} is pre-triangulated and $n \geq 13$, then $H^0(\mathfrak{a})$ is canonically triangulated.*

The number 13 seems quite high. We are rather curious if it can be reduced.

1.3. Gluing. We have already pointed out that if \mathfrak{a} is an A_n -category, then its “pre-triangulated hull” $\mathrm{Tw} \mathfrak{a}$ is not well defined. So while we have a satisfactory theory of pre-triangulated A_n -categories, it is unclear how to actually construct non-trivial examples of them. Luckily there is one approach which works very well. It turns out that pre-triangulated A_n -categories admit a “gluing” procedure, and starting from pre-triangulated A_∞ -categories we can in this way produce pre-triangulated A_n -categories which are not themselves A_∞ -categories.

Let us first review gluing in the context of triangulated categories. If \mathcal{A}, \mathcal{B} are triangulated categories and \mathcal{M} is a $\mathcal{B} - \mathcal{A}$ -bimodule (an additive bifunctor $\mathcal{A}^\circ \times \mathcal{B} \rightarrow \mathbf{Ab}$), then a *gluing* of \mathcal{A}, \mathcal{B} across \mathcal{M} is a triangulated category \mathcal{C} together with a semi-orthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ such that $\mathcal{C}(A, B) = \mathcal{M}(A, B)$ for $A \in \mathrm{Ob}(\mathcal{A}), B \in \mathrm{Ob}(\mathcal{B})$. The data $(\mathcal{A}, \mathcal{B}, \mathcal{M})$ determines the objects of \mathcal{C} up to isomorphism and there is a long exact sequence relating the

Hom-spaces in \mathcal{C} to those in \mathcal{A} , \mathcal{B} and the elements of \mathcal{M} . However this is as far as it goes. Triangulated categories are too flabby to allow one to fully construct \mathcal{C} from the triple $(\mathcal{A}, \mathcal{B}, \mathcal{M})$.

On the other hand, if \mathfrak{a} , \mathfrak{b} are A_∞ -categories and M is an A_∞ - \mathfrak{b} - \mathfrak{a} -module, then it is a routine matter to define an A_∞ -gluing category $\mathfrak{c} = \mathfrak{a} \coprod_M \mathfrak{b}$ such that if \mathfrak{a} , \mathfrak{b} are pre-triangulated, then so is \mathfrak{c} and there is a semi-orthogonal decomposition $H^0(\mathfrak{c}) = \langle H^0(\mathfrak{a}), H^0(\mathfrak{b}) \rangle$ with associated bimodule $H^0(M)$.

To prove that \mathfrak{c} is pre-triangulated we have to prove it is closed under cones of closed maps. Again it is clear that this will only involve a finite number of higher operations. Hence the theory can be developed for A_n -categories. This leads to our next main result.

THEOREM 1.3 (Theorem 8.5). *Assume that $n \geq 13$, that \mathfrak{a} , \mathfrak{b} are pre-triangulated A_n -categories, and that M is an A_n - \mathfrak{b} - \mathfrak{a} -bimodule. Then $\mathfrak{a} \coprod_M \mathfrak{b}$ is a pre-triangulated A_{n-1} category. If $n \geq 14$, so that $H^0(\mathfrak{a} \coprod_M \mathfrak{b})$ is triangulated by Theorem 1.2, then we have a semi-orthogonal decomposition $H^0(\mathfrak{a} \coprod_M \mathfrak{b}) = \langle H^0(\mathfrak{a}), H^0(\mathfrak{b}) \rangle$ whose associated bimodule is $H^0(M)$.*

1.4. The counterexample. The counterexample we describe in Theorem 1.1 will be more specifically of the form $\mathcal{D} = H^0(\mathfrak{a} \coprod_M \mathfrak{b})$, where \mathfrak{a} , \mathfrak{b} are pre-triangulated A_∞ -categories and M is an A_n - \mathfrak{b} - \mathfrak{a} -bimodule. We will in fact assume that M is obtained from an A_{n-1} -functor $F : \mathfrak{a} \rightarrow \mathfrak{b}$ via $M(A, B) = \mathfrak{b}(FA, B)$. By Theorems 1.2 and 1.3, \mathcal{D} is canonically triangulated for $n \gg 0$. Moreover any A_∞ -enhancement on \mathcal{D} induces A_∞ -enhancements \mathfrak{a}' , \mathfrak{b}' on $H^0(\mathfrak{a})$, $H^0(\mathfrak{b})$ as well as an A_∞ -functor $F' : \mathfrak{a}' \rightarrow \mathfrak{b}'$ such that $H^0(F') = H^0(F)$. One may hope to be able to prove that such F' does not exist. This then implies that an A_∞ -enhancement on \mathcal{D} does not exist.

We carry out this program with \mathfrak{a} , \mathfrak{b} being the standard A_∞ -enhancements of $D(K)$, $D(R_\eta)$ for $\eta \neq 0$ (see Section 10). The exact functor

$$f : D(K) \rightarrow D(R_\eta) : K \rightarrow K_\eta$$

(defined using the fact that $D(K)$ is the category of graded K -vector spaces, equipped with its unique triangulation) lifts to an A_{n-1} -functor F : by [12, Lemma 7.2.1] this follows from the fact that $H^i(K_\eta) = 0$ for $i = 0, \dots, -n+3$. However, using the fact that $\eta \neq 0$ one deduces that f does not lift to an A_∞ -functor, even if we are allowed to change enhancements. This follows from the fact that the enhancement on $D(R_\eta)$ is actually unique in a weak, but sufficient, sense. This is proved using higher Toda brackets (see Proposition 10.8). This finishes the proof that an A_∞ -enhancement on \mathcal{D} does not exist.

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2. Notation and conventions

Below k is an arbitrary field, except in [Section 10.3](#) where it will be subject to some restrictions. Unless otherwise specified, categories are pre-additive (enriched in abelian groups), except when we are in an A_n -context. In that case we assume all objects and constructions are k -linear.

Triangulated categories will be equipped with their canonical graded enhancement (see [Section 3.4](#)). The motivation for this is that the principal “homotopy invariant” associated to an A_n -category \mathfrak{a} is $H^*(\mathfrak{a})$ as $H^0(\mathfrak{a})$ loses too much information in general. If \mathfrak{a} is pre-triangulated, then $H^*(\mathfrak{a})$ can be recovered from $H^0(\mathfrak{a})$ together with a “shift functor” but, since the shift functor is not canonical (despite being unique up to unique isomorphism), this extra step creates some complications, notably with signs, which are often unnecessary. In any case, not all A_n -categories we will encounter will be pre-triangulated.

In situations where the shift functor is canonical we will use it. The most obvious case is graded objects over an abelian category \mathcal{A} . If $A^\bullet = (A_i)_{i \in \mathbb{Z}}$ is such an object, then we put $\Sigma^n(A^\bullet)_i = A_{i+n}$. If $f : A^\bullet \rightarrow B^\bullet$ has degree i , then we put $\Sigma^n f = (-1)^{ni} f$. If A^\bullet is a graded object over **Ab** and $x \in A_i$, then we write sx for x considered as an element of $(\Sigma A^\bullet)_{i-1}$. The “degree change operator” s makes it easy to find the correct sign in formulas using the Koszul convention.

3. Preliminaries on triangulated categories

3.1. Graded categories. For us a *graded category* is a category enriched in \mathbb{Z} -graded abelian groups. Assume that \mathfrak{a} is a graded category, and let $X \in \text{Ob}(\mathfrak{a})$. A *suspension* of X is a pair (Y, η) where $Y \in \text{Ob}(\mathfrak{a})$ and $\eta \in \mathfrak{a}(X, Y)_{-1}$ is invertible. Conversely we call (X, η) a *desuspension* of Y . (De)suspensions are clearly functorial if they exist. So if every object X has a suspension (Y, η) , we may define a functor $\Sigma : \mathfrak{a} \rightarrow \mathfrak{a}$ by putting $\Sigma X = Y$ and requiring for maps $f \in \mathfrak{a}(X, X')$ that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \Sigma X \\ f \downarrow & & \downarrow \Sigma f \\ X' & \xrightarrow{\eta} & \Sigma X \end{array}$$

commutes up to a sign $(-1)^{|f|}$. It is clear that Σ is unique up to unique equivalence. We say that \mathfrak{a} has a *shift functor* Σ if every object has a suspension and a desuspension and Σ is as above. In this case Σ is an auto-equivalence.

3.2. Graded categories from pre-additive categories with shift functor. Now assume that \mathfrak{a} is a *pre-additive category* (i.e., a category enriched in abelian groups) equipped with an auto-equivalence Σ . Then we can make \mathfrak{a} into a

graded category $\tilde{\mathfrak{a}}$ with the same objects by putting, for $n \in \mathbb{Z}$,

$$\tilde{\mathfrak{a}}(A, B)_n := \mathfrak{a}(A, \Sigma^n B)$$

and with compositions

$$\tilde{\mathfrak{a}}(B, C)_m \times \tilde{\mathfrak{a}}(A, B)_n \rightarrow \tilde{\mathfrak{a}}(A, C)_{m+n} : (g, f) \mapsto (-1)^{nm} \Sigma^n g \circ f.$$

We obtain that Σ is a shift functor on $\tilde{\mathfrak{a}}$ in the sense of [Section 3.1](#).

3.3. Triangles. A *triangle* in a graded category \mathfrak{a} is a diagram

$$\begin{array}{ccc} & C & \\ h \swarrow & & \nwarrow g \\ A & \xrightarrow{(1)} & B \end{array}$$

with $A, B, C \in \text{Ob}(\mathfrak{a})$ and $|f| = |g| = 0$, $|h| = 1$. To save space a triangle will usually be written in linear form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{(1)} A.$$

If \mathfrak{a} is equipped with a shift functor, then a triangle can also be written in “traditional” form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A.$$

A morphism of triangles is given by three degree zero morphisms fitting into the obvious commutative diagram.

3.4. Triangulated categories as graded categories. We will assume that the reader is familiar with the standard axioms for triangulated categories [\[14\]](#). If (\mathcal{T}, Σ) is triangulated category in the traditional sense, then it can be made into a graded category as in [Section 3.2](#). In this section we will reformulate the usual axioms of triangulated categories in such a way that they do not explicitly refer to a shift functor.

Definition 3.1. A *triangulated* category \mathcal{T} is a graded category equipped with a collection of “distinguished” triangles such that⁴

(TR0) \mathcal{T} admits (possibly empty) finite direct sums and every object has a suspension and a desuspension.

(TR1) • For any object $X \in \text{Ob}(\mathcal{T})$, the following triangle is distinguished:

$$X \xrightarrow{\text{id}_X} X \xrightarrow{0_0} 0 \xrightarrow{(1)} X,$$

⁴Morphisms in a graded category whose degree is not specified are assumed to have degree zero. This convention is maintained throughout this document.

where 0 is a zero object (which exists by (TR0)) and where 0_i is the zero morphism in $\mathcal{T}(U, V)_i$.

- For any morphism $u : X \rightarrow Y$ in \mathcal{T} of degree zero, there is an object Z (called a mapping cone of the morphism u) fitting into a distinguished triangle

$$X \xrightarrow{(1)} Y \rightarrow Z \xrightarrow{u} X.$$

- Any triangle isomorphic to a distinguished triangle is distinguished.

(TR2) If

$$\begin{array}{ccc} & Z & \\ w \swarrow & & \searrow v \\ X & \xrightarrow{(1)} & Y \\ & u \searrow & \end{array}$$

is a distinguished triangle, then so are the two “rotated triangles”

$$\begin{array}{ccc} & Z & \\ \eta w \swarrow & & \searrow v \\ X' & \xrightarrow{(1)} & Y, \\ & -u\eta^{-1} \searrow & \end{array} \quad \begin{array}{ccc} & Z' & \\ -w\gamma \swarrow & & \searrow \gamma^{-1}v \\ X & \xrightarrow{(1)} & Y, \\ & u \searrow & \end{array}$$

where $X \xrightarrow{\eta} X'$ is a suspension of X and $Z' \xrightarrow{\gamma} Z$ is a desuspension of Z .

(TR3) A commutative diagram of solid arrows

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow{(1)} & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \xrightarrow{(1)} & X' \end{array}$$

in which the rows are distinguished can be completed with the dotted arrow.

(TR4) For every upper cap of an octahedron (drawn on the left) there is a corresponding lower cap (drawn on the right)

$$(3.1) \quad \begin{array}{ccc} \begin{array}{ccc} Z & \xrightarrow{\quad} & X' \\ \uparrow & \swarrow \text{d}^{(1)} & \searrow \text{d} \\ Y & & \end{array} & \quad & \begin{array}{ccc} Z & \xrightarrow{\quad} & X' \\ \uparrow & \swarrow \text{d}^{(1)} & \searrow \text{d} \\ Y' & & \end{array} \\ \begin{array}{ccc} X & \xleftarrow{(1)} & Z' \\ \uparrow & \swarrow \text{d} & \searrow \text{d} \\ Y & & \end{array} & \quad & \begin{array}{ccc} X & \xleftarrow{(1)} & Z' \\ \uparrow & \swarrow \text{d}^{(1)} & \searrow \text{d} \\ Y' & & \end{array} \end{array}$$

such that in addition the compositions $Y \rightarrow Z \rightarrow Y'$ and $Y \rightarrow Z' \rightarrow Y'$ are the same and similarly the compositions $Y' \rightarrow X \rightarrow Y$ and $Y' \rightarrow X' \rightarrow Y$ are the same. In the diagram the triangles marked \mathbf{d} are distinguished and those marked with \circlearrowleft are commutative,

4. Preliminaries about A_n -categories

Let $n \geq 0$. As a general principle, for any A_∞ -notion there is a corresponding A_n -notion in which we consider only operations with $\leq n$ arguments and we require the axioms to only hold for expressions with $\leq n$ arguments. Facts about A_∞ -categories remain valid for A_n -categories as long as they only involve such expressions. We discuss this below. Throughout we place ourselves in the strictly unital context.

4.1. A_n -categories and functors.

Definition 4.1 ([7]). An A_n -category \mathfrak{a} is the data of the following:

- A set of objects $\text{Ob}(\mathfrak{a})$.
- For each couple (A, A') of objects of \mathfrak{a} , a graded vector space of morphisms $\mathfrak{a}(A, A')$. We call $\mathfrak{a}(A, A')$ the Hom-space between A and A' . A (homogeneous) element of $\mathfrak{a}(A, A')$ is called a morphism (or sometimes an arrow).
- For each sequence (A_0, \dots, A_i) of objects of \mathfrak{a} with $1 \leq i \leq n$, “higher” compositions

$$b_i : \Sigma \mathfrak{a}(A_{i-1}, A_i) \otimes \dots \otimes \Sigma \mathfrak{a}(A_0, A_1) \rightarrow \Sigma \mathfrak{a}(A_0, A_i)$$

of degree 1 verifying $(*)_i$ of [7, Def. 1.2.1.1].

- For each object A , an *identity (or unit) element* $\text{id}_A \in \mathfrak{a}(A, A)_0$ satisfying

$$\begin{aligned} b_i(\dots, s \text{id}_A, \dots) &= 0 && \text{(for } i = 1 \text{ and } 3 \leq i \leq n\text{),} \\ b_2(sf, s \text{id}_A) &= (-1)^{|f|} sf && \text{if } n \geq 2, \\ b_2(s \text{id}_A, sg) &= sg && \text{if } n \geq 2. \end{aligned}$$

If the identities hold for every i , then we get the notion of an A_∞ -category. Below an A_n -category will be silently considered as an A_m -category for all $m \leq n$.

As for A_∞ -categories, it is sometimes more convenient to express the higher compositions as operations

$$m_i : \mathfrak{a}(A_{i-1}, A_i) \otimes \dots \otimes \mathfrak{a}(A_0, A_1) \rightarrow \mathfrak{a}(A_0, A_i)$$

of degree $2 - n$ where $(m_n)_n$ and $(b_n)_n$ are related by $b_n = s^{-n+1}m_n$ so that, in particular, using the Koszul convention we obtain

$$(4.1) \quad \begin{aligned} b_1(sf) &= -sm_1(f), \\ b_2(sg, sf) &= (-1)^{|g|}sm_2(g, f), \\ b_3(sh, sg, sf) &= (-1)^{|g|+1}sm_3(h, g, f). \end{aligned}$$

Sometimes we write $df = m_1(f)$ and $gf = m_2(g, f)$. It is useful to consider the case of low n .

- (1) An A_0 -category is simply a directed graph (with distinguished “identity arrows”) whose Hom-spaces are graded vector spaces. We call this a *graded graph*.
- (2) An A_1 -category is a graded graph whose arrows form complexes of vector spaces. (The differential is given by m_1 and it annihilates identity arrows.) We call this a *DG-graph*. A DG-graph \mathfrak{a} has an associated graded graph $H^*(\mathfrak{a})$ obtained by replacing the Hom-spaces in \mathfrak{a} by their cohomology. A morphism f in \mathfrak{a} is called *closed* if $m_1(f) = 0$. We denote by $Z^0\mathfrak{a}$ the k -linear graph which has the same objects as \mathfrak{a} and whose morphisms are the closed morphisms of degree zero.
- (3) An A_2 category is a DG-graph equipped with a bilinear composition of arrows given by m_2 (for which the identity arrows behave as unit elements) which is compatible with m_1 . In particular, m_2 descends to well-defined operations on $H^*(\mathfrak{a})$ and $Z^0\mathfrak{a}$.
- (4) For $n \geq 3$, the composition on $H^*(\mathfrak{a})$ induced by m_2 is associative and hence, in particular, $H^*(\mathfrak{a})$ is a graded category.

Definition 4.2. An A_n -functor $f : \mathfrak{a} \rightarrow \mathfrak{b}$ between two A_n -categories \mathfrak{a} and \mathfrak{b} is the data of

- a map on objects $f : \text{Ob}(\mathfrak{a}) \rightarrow \text{Ob}(\mathfrak{b})$;
- for each sequence (A_0, \dots, A_i) of objects of \mathfrak{a} with $i \leq n$, compositions

$$f_i : \Sigma\mathfrak{a}(A_{i-1}, A_i) \otimes \dots \otimes \Sigma\mathfrak{a}(A_0, A_1) \rightarrow \Sigma\mathfrak{b}(f(A_0), f(A_i))$$

of degree zero verifying $(**)_i$ of [7, Def. 1.2.1.2] for $i = 1, \dots, n$;

- if $n \geq 1$, then for each $A \in \text{Ob}(\mathfrak{a})$ we have $f_1(s\text{id}_A) = s\text{id}_{f(A)}$ and $f_n(\dots, s\text{id}_A, \dots) = 0$ for $n \geq 2$.

Again it is instructive to unravel this definition for small values of n .

- (1) An A_0 -functor is just a map between sets of objects (there is no compatibility with morphisms).
- (2) An A_1 -functor $f : \mathfrak{a} \rightarrow \mathfrak{b}$ is a morphism of DG-graphs. In particular, we have an induced morphism of graded graphs $H^*(f) := H^*(f_1)$.

(3) If f is an A_n -functor for $n \geq 2$, then $H^*(f)$ is compatible with compositions. In particular, if f is an A_2 -functor between A_3 -categories, then $H^*(f)$ is a graded functor.

Like A_∞ -notions, one may also approach A_n -notions via cocategories. Let \mathfrak{a} be a graded graph. Then $(B\mathfrak{a})_{\leq n}$ is the graded cocategory with Hom-spaces

$$(4.2) \quad \begin{aligned} (B\mathfrak{a})_{\leq n}(A, B) &= \bigoplus_{i=1}^n (\Sigma\mathfrak{a})^{\otimes i}(A, B), \\ (\Sigma\mathfrak{a})^{\otimes i}(A, B) &= \bigoplus_{A=A_0, \dots, A_i=B} \Sigma\mathfrak{a}(A_{i-1}, A_i) \otimes \dots \otimes \Sigma\mathfrak{a}(A_0, A_1) \end{aligned}$$

equipped with the usual *bar coproduct*. That is, if $(sf_{i-1} | \dots | sf_0) := sf_{i-1} \otimes \dots \otimes sf_0 \in (\Sigma\mathfrak{a})^{\otimes i}$, then

$$\Delta(sf_{i-1} | \dots | sf_0) = \sum_{j=1}^{i-1} (sf_{i-1} | \dots | sf_j) \otimes (sf_{j-1} | \dots | sf_0).$$

If we ignore the compatibility with units, then an A_n -structure on \mathfrak{a} is the same as a *codifferential* on $(B\mathfrak{a})_{\leq n}$, i.e., a coderivation b of degree one satisfying $b \circ b = 0$. Similarly, ignoring units, an A_n -functor $f : \mathfrak{a} \rightarrow \mathfrak{b}$ is the same as a cofunctor $(B\mathfrak{a})_{\leq n} \rightarrow (B\mathfrak{b})_{\leq n}$ commuting with the codifferentials on $(B\mathfrak{a})_{\leq n}$ and $(B\mathfrak{b})_{\leq n}$. With this observation one may define the composition of A_n -functors simply as the composition of the corresponding cofunctors.

4.2. Some auxilliary definitions.

Definition 4.3. Let $f : \mathfrak{a} \rightarrow \mathfrak{b}$ be an A_m -functor between A_n -categories, for $m \leq n$. Then

- (1) f is *strict* provided $m \geq 1$ and $f_i = 0$ for $i \geq 2$. Equivalently, f_1 commutes with higher compositions with arity at most m .
- (2) f is *fully faithful* if it is strict, and for all $A, A' \in \text{Ob}(\mathfrak{a})$, we have that $\mathfrak{a}(A, A') \rightarrow \mathfrak{b}(fA, fA')$ is an isomorphism of graded vector spaces.
- (3) f is a *quasi-fully faithful* if $m \geq 2$, $n \geq 3$ and $H^*(f) : H^*(\mathfrak{a}) \rightarrow H^*(\mathfrak{b})$ is fully faithful.
- (4) f is a *quasi-isomorphism* if $m \geq 2$, $n \geq 3$ and $H^*(f) : H^*(\mathfrak{a}) \rightarrow H^*(\mathfrak{b})$ is an isomorphism.
- (5) f is a *quasi-equivalence* if $m \geq 2$, $n \geq 3$ and $H^*(f) : H^*(\mathfrak{a}) \rightarrow H^*(\mathfrak{b})$ is an equivalence.

4.3. The category of functors between A_n -categories. Here we discuss some concepts from [7, Ch. 8]. As indicated above, the (decomposable) arrows of $(B\mathfrak{a})_{\leq n}$ are usually written as $(sf_{i-1} | \dots | sf_0)$ for a path of $1 \leq i \leq n$ composable arrows f_0, \dots, f_{i-1} in \mathfrak{a} . We let $(B^+\mathfrak{a})_{\leq n}$ be the coaugmented cocategory

obtained by also admitting empty paths $()_A$ starting and ending in $A \in \text{Ob}(\mathfrak{a})$ (see Section 2.1.2 in loc. cit.). More precisely, we have

$$(B^+ \mathfrak{a})_{\leq n}(A, B) = \begin{cases} (B\mathfrak{a})_{\leq n}(A, B) & \text{if } A \neq B, \\ k()_A \oplus (B\mathfrak{a})_{\leq n}(A, A) & \text{if } A = B \end{cases}$$

with $|()_A| = 0$. The coproduct $\Delta^+(t)$ for $t \in (B\mathfrak{a})_{\leq n}(A, B)$ is defined as

$$\Delta^+(t) = ()_B \otimes t + t \otimes ()_A + \Delta(t),$$

where Δ is the coproduct on $(B\mathfrak{a})_{\leq n}$ and furthermore $\Delta^+(()_A) = ()_A \otimes ()_A$. If $(B\mathfrak{a})_{\leq n}$ is equipped with a codifferential b , then we extend it to $(B^+ \mathfrak{a})_{\leq n}$ by putting $b(()_A) = 0 \in \mathfrak{a}(A, A)$.

Given two A_n -categories \mathfrak{c} and \mathfrak{d} , denote by $A_n(\mathfrak{c}, \mathfrak{d})$ the set of A_n -functors $\mathfrak{c} \rightarrow \mathfrak{d}$. Now assume that $\mathfrak{a}, \mathfrak{b}$ are respectively A_m, A_n -categories for $m \leq n-1$. We will equip $A_m(\mathfrak{a}, \mathfrak{b})$ with the structure of an A_{n-m} -category as follows.

Definition 4.4 (Morphisms in $A_m(\mathfrak{a}, \mathfrak{b})$). Assume $m \leq n-1$. Let $f_1, f_2 : \mathfrak{a} \rightarrow \mathfrak{b}$ be A_m -functors. We view these as cofunctors $(B^+ \mathfrak{a})_{\leq m} \rightarrow B\mathfrak{b}_{\leq n}$ by putting $f_{i,0}(()_A) := f_{i,0}(()_A) = 0$. Then

$$\begin{aligned} \Sigma \text{Hom}(f_1, f_2) \\ = \{h \in \text{coDer}_{f_1, f_2}(B^+ \mathfrak{a}_{\leq m}, B\mathfrak{b}_{\leq n}) \mid \forall A \in \text{Ob}(\mathfrak{a}) : h(\cdots \otimes s \text{id}_A \otimes \cdots) = 0\}. \end{aligned}$$

Here $\text{coDer}_{f_1, f_2}((B^+ \mathfrak{a})_{\leq m}, B\mathfrak{b}_{\leq n})$ consists of collections k -linear morphisms $h(A, A') : B^+ \mathfrak{a}_{\leq m}(A, A') \rightarrow B\mathfrak{b}_{\leq n}(f_1(A), f_2(A'))$ such that $h = h(A, A')_{A, A'}$ satisfies the following identity for $u \in (B^+ \mathfrak{a})_{\leq m}$:

$$\Delta(h(u)) = \sum_{(u)} (f_2 \otimes h + h \otimes f_1)(u_{(1)} \otimes u_{(2)}),$$

where (using the Sweedler notation) $\Delta^+(u) = \sum_{(u)} u_{(1)} \otimes u_{(2)}$. It follows that $h \in \Sigma \text{Hom}(f_1, f_2)$ is determined by the “Taylor coefficients”

(4.3)

$$h_k : \Sigma \mathfrak{a}(A_{k-1}, A_k) \otimes \Sigma \mathfrak{a}(A_{k-2}, A_{k-1}) \otimes \cdots \otimes \Sigma \mathfrak{a}(A_0, A_1) \rightarrow \Sigma \mathfrak{b}(f_1(A_0), f_2(A_k))$$

for $1 \leq k \leq m$ as well as for each $A \in \text{Ob}(\mathfrak{a})$ an element $h()_A := h_0(()_A) \in \Sigma \mathfrak{b}(f_1(A), f_2(A))$. The corresponding coderivation is given by

$$(4.4) \quad h = \sum_{\sum_{t=1}^q j_t + \sum_{s=1}^p i_s + k \leq m} f_{2,j_q} \otimes \cdots \otimes f_{2,j_1} \otimes h_k \otimes f_{1,i_p} \otimes \cdots \otimes f_{1,i_1},$$

where the right-hand side is restricted to terms which have $\leq m$ arguments. Note that h sends $(B^+ \mathfrak{a})_{\leq m}$ to $(B\mathfrak{b})_{\leq m+1}$ (as the f 's take at least one argument but h_0 takes zero arguments). So since $m \leq n-1$, h is indeed well defined.

Definition 4.5 (The differential on $A_m(\mathfrak{a}, \mathfrak{b})$). If $m \leq n - 1$ and $h \in \Sigma \text{Hom}(f_1, f_2)$, then $b_1(h) = [b, h] = b \circ h - (-1)^{|h|} h \circ b$. Concretely,

$$(4.5) \quad \begin{aligned} b_1(h)_k = & \sum_{\sum_{t=1}^q j_t + \sum_{s=1}^p i_s + l = k} b_{p+q+1} \circ (f_{2,j_q} \otimes \cdots \otimes f_{2,j_1} \otimes h_l \otimes f_{1,i_p} \otimes \cdots \otimes f_{1,i_1}) \\ & - (-1)^{|h|} \sum_{a_0 + a_1 + l = k} h_{1+a_0+a_1} \circ (\text{id}^{\otimes a_0} \otimes b_l \otimes \text{id}^{\otimes a_1}). \end{aligned}$$

Definition 4.6 (The higher multiplications on $A_m(\mathfrak{a}, \mathfrak{b})$). Assume we have morphisms

$$f_0 \xrightarrow{h_1} f_1 \xrightarrow{h_2} \cdots \xrightarrow{h_k} f_k$$

represented by

$$h_i \in \text{coDer}_{f_{i-1}, f_i}(B^+ \mathfrak{a}_{\leq m}, B\mathfrak{b}_{\leq n}),$$

and assume $2 \leq k \leq n - m$. Then we put

$$(4.6) \quad \begin{aligned} h_k \cup \cdots \cup h_1 = & \sum f_{k,i_{k,p_k}} \otimes \cdots \otimes f_{k,i_{k1}} \otimes h_{k,u_k} \otimes f_{k-1,i_{k-1,p_{k-1}}} \otimes \cdots \\ & \cdots \otimes f_{k-1,i_{k-1,1}} \otimes \cdots \otimes f_{1,i_{1p_1}} \otimes \cdots \otimes f_{1,i_{11}} \otimes h_{1,u_1} \otimes f_{0,i_{0p_0}} \otimes \cdots \otimes f_{0,i_{01}} \end{aligned}$$

and $b_k(h_k, \dots, h_1)_l = (b \circ (h_k \cup \cdots \cup h_1))_l$, where $(-)_l$ denotes Taylor coefficients.

Note that on the right-hand side of (4.6) the f 's take at least one argument but the h 's may take zero arguments. It follows that $h_k \cup \cdots \cup h_1$ maps $B^+ \mathfrak{a}_{\leq m}$ to $B\mathfrak{b}_{\leq m+k}$, and hence by the hypothesis $k \leq n - m$ is a well-defined element of $\text{Hom}(B^+ \mathfrak{a}_{\leq m}, B\mathfrak{b}_{\leq n})$. It is however not a coderivation. Instead is it inductively characterized by the following property for $u \in (B^+ \mathfrak{a})_{\leq m}$ (using again the Sweedler notation):

$$\begin{aligned} \Delta((h_k \cup \cdots \cup h_1)(u)) = & \sum_{(u)} \left(f_k \otimes (h_k \cup \cdots \cup h_1) + (h_k \cup \cdots \cup h_1) \otimes f_1 \right. \\ & \left. + \sum_{1 \leq j \leq k} (h_k \cup \cdots \cup h_{j+1}) \otimes (h_j \cup \cdots \cup h_1) \right) (u_{(1)} \otimes u_{(2)}). \end{aligned}$$

One checks

LEMMA 4.7. *The collection of maps $(b_i)_{i=1, \dots, n-m}$ makes $A_m(\mathfrak{a}, \mathfrak{b})$ into an A_{n-m} -category.*

4.4. Homotopies and homotopic functors. Let $\mathfrak{a}, \mathfrak{b}$ be A_n -categories, let $1 \leq m \leq n - 1$ (thus $n \geq 2$), and let $h \in Z^0 \Sigma A_m(\mathfrak{a}, \mathfrak{b})(f_1, f_2)$. Then $h \in \text{coDer}_{f_1, f_2}(B^+ \mathfrak{a}_{\leq m}, B\mathfrak{b}_{\leq n})_{-1}$ and $[b, h] = 0$. Let $(h_k)_{k=0, \dots, m}$ be the Taylor coefficients of h . Specializing (4.5) to $k = 0, 1$ we find that $h_0()_A \in (\Sigma \mathfrak{b}(f_1 A, f_2 A))_{-1} = \mathfrak{b}(f_1 A, f_2 A)_0$ satisfies $d(h_0()_A) = 0$ and

$$b_1 \circ h_1 + b_2 \circ (h_0 \otimes f_1 + f_2 \otimes h_0) + h_1 \circ b_1 = 0.$$

Evaluating this on st for $t \in \mathfrak{a}(A, B)$ we find

$$(4.7) \quad \begin{aligned} & b_1(h_1(st)) + b_2(h_0(), f_1(st)) \\ & + (-1)^{|t|+1} b_2(f_2(st), h_0(), A) + h_1(b_1(st)) = 0. \end{aligned}$$

Put $h_0()_A = sh_{0,A}$. Using the usual sign convention $h_1(st) = -sh_1(t)$, etc. together with (4.1), this may be rewritten as

$$m_1(h_1(t)) + m_2(h_{0,B}, f_1(t)) + (-1)^{|t|+1} (-1)^{|t|} m_2(f_2(t), h_{0,A}) + h_1(m_1(t)) = 0.$$

So we find, in particular, that $H^*(h_0)$ defines a natural transformation $H^*(f_1) \rightarrow H^*(f_2)$.

Definition 4.8. Let h, f_1, f_2 be as above but assume $n \geq 3$. We say that h is a *homotopy* $h : f_1 \rightarrow f_2$ if $H^*(h_0)$ is a natural isomorphism, i.e., if for all $A \in \text{Ob}(\mathfrak{a})$, $H^*(h_{0,A}) \in H^*(\mathfrak{a})(A, A)$ is invertible. We say that f_1, f_2 are *homotopic* if there exists a homotopy $h : f_1 \rightarrow f_2$.

LEMMA 4.9. *Assume $1 \leq m \leq n - 3$. Then $h : f_1 \rightarrow f_2$ is a homotopy if and only if $H^*(h)$ is invertible in $H^*(A_m(\mathfrak{a}, \mathfrak{b}))$. (The latter is a genuine category because of the restriction on m, n .) In particular, the relation of being homotopic is an equivalence relation.*

Proof. We have $(hh')_0 = h_0h'_0$. So if h is invertible, then it is a homotopy. Assume now h_0 is invertible. Consider the morphism of complexes

$$S : A_m(\mathfrak{a}, \mathfrak{b})(f_2, f_1) \rightarrow A_m(\mathfrak{b}, \mathfrak{b})(f_2, f_2) : h' \mapsto m_2(h, h').$$

Using an appropriate spectral sequence one finds that S is a quasi-isomorphism. Hence there exists $h' \in Z^0 A_m(\mathfrak{a}, \mathfrak{b})(f_2, f_1)$ such that $m_2(h, h') - \text{id}_{f_2}$ has zero image in $H^*(A_m(\mathfrak{a}, \mathfrak{b})(f_2, f_2))$. \square

4.5. Inverting quasi-equivalences. We prove some A_n -versions of results which are well known in the A_∞ -setting (e.g., [7, Th. 9.2.0.4]).

LEMMA 4.10. *Let $\mathfrak{a}, \mathfrak{b}$ be A_n categories for $n \geq 3$, and let $f : \mathfrak{a} \rightarrow \mathfrak{b}$ be an A_n -functor which is a quasi-equivalence. There exists an A_{n-1} -quasi-equivalence $g : \mathfrak{b} \rightarrow \mathfrak{a}$ such that fg and $\text{id}_{\mathfrak{b}}$ are homotopic. Moreover the quasi-inverse $H^*(g_1)$ to $H^*(f_1)$ may be chosen freely.*

Proof. If \mathfrak{a} is an A_n -category, then we define $\bar{\mathfrak{a}}$ as the DG-graph obtained from \mathfrak{a} by dividing out identities. That is,

$$\bar{\mathfrak{a}}(A, B) = \begin{cases} \mathfrak{a}(A, B) & \text{if } A \neq B, \\ \mathfrak{a}(A, A)/k \text{id}_A & \text{if } A = B. \end{cases}$$

Note that formally f is a cofunctor $B\mathfrak{a}_{\leq n} \rightarrow B\mathfrak{b}_{\leq n}$ such that $[b, f] = 0$. Likewise g should be a cofunctor $B\mathfrak{b}_{\leq n-1} \rightarrow B\mathfrak{a}_{\leq n-1}$ satisfying $[b, g] = 0$ and the homotopy $h : fg \rightarrow \text{id}_{\mathfrak{b}}$ should be an element of $\text{coDer}_{fg, \text{id}_{\mathfrak{b}}}(B^+ \mathfrak{b}_{\leq n-1}, B\mathfrak{b}_{\leq n})$

of degree -1 satisfying $[b, h] = 0$ such that $H^*(h_0)$ is a natural isomorphism $H^*(fg) \rightarrow \text{id}_{H^*(\mathfrak{b})}$.

We will construct g and h step by step. The existence of g_1 and h_0, h_1 follows simply from the fact that f is a quasi-equivalence: we choose a unit preserving graded graph homomorphism $g_1 : \mathfrak{b} \rightarrow \mathfrak{a}$ commuting with differentials such that there is a natural isomorphism $H^*(f_1)H^*(g_1) \rightarrow \text{id}_{H^*(\mathfrak{b})}$. We choose $h_0 : (B\mathfrak{b})_0 \rightarrow \mathfrak{b}$ in such away that this natural isomorphism is of the form $H^*(h_0)$, and then we choose h_1 such that [equation \(4.7\)](#) holds.

Assume that for $1 \leq m < n - 1$, we have constructed a cofunctor $g_{\leq m} : (B\mathfrak{b})_{\leq m} \rightarrow (B\mathfrak{b})_{\leq m}$ satisfying $[b, g_{\leq m}] = 0$ and a homotopy $h_{\leq m} : fg_{\leq m} \rightarrow \text{id}_{\mathfrak{b}}$. We will extend the maps $(g_{\leq m}, h_{\leq m})$ to maps $(g_{\leq m+1}, h_{\leq m+1})$ with the same properties.

As a first approximation we extend $g_{\leq m}, h_{\leq m}$ to respectively a cofunctor $g_{\leq m+1} : B\mathfrak{b}_{\leq m+1} \rightarrow B\mathfrak{a}_{\leq m+1}$ and a $(fg_{\leq m+1}, \text{id}_{\mathfrak{b}})$ -coderivation $h_{\leq m+1} : B^+\mathfrak{b}_{\leq m+1} \rightarrow B\mathfrak{b}_{m+1}$ by setting $g_{m+1}, h_{m+1} : (\Sigma\bar{\mathfrak{b}})^{\otimes m+1} \rightarrow \Sigma\mathfrak{a}$ equal to zero; see [\(4.2\)](#) for the definition of $\Sigma\bar{\mathfrak{b}}^{\otimes m+1}$. Here $\pi = [b, g_{\leq m+1}]$ is zero on $(\Sigma\bar{\mathfrak{b}})^{\otimes i}$, $i \leq m$ and hence it may be regarded as a map $(\Sigma\bar{\mathfrak{b}})^{\otimes m+1} \rightarrow \Sigma\mathfrak{a}$. Moreover $0 = [b, \pi] = [b_1, \pi]$. So π is closed for the b_1 -differential, and since $f\pi = [b, fg_{\leq m+1}]$ is zero on cohomology and f is a quasi-isomorphism, π is equal to zero in cohomology as well. In other words, there exists $\delta_{m+1} : \bar{\mathfrak{b}}^{\otimes m+1} \rightarrow \mathfrak{a}$ such that $\pi = [b_1, \delta_{m+1}]$. We now replace g_{m+1} by $g_{m+1} - \delta_{m+1}$. Then $[b, g_{\leq m+1}] = 0$. In other words, $g_{\leq m+1}$ is an A_{m+1} -morphism.

Put $D = [b, h_{\leq m+1}]$ (see [\(4.5\)](#)). Then D is a $(fg_{\leq m+1}, \text{id}_{\mathfrak{b}})$ -coderivation $(B^+\mathfrak{b})_{\leq m+1} \rightarrow B\mathfrak{b}_{\leq n}$ which is zero on $(B\mathfrak{b})_{\leq m}$, and hence it can be considered as a map $(\Sigma\bar{\mathfrak{b}})^{\otimes m+1} \rightarrow \Sigma\mathfrak{b}$. Hence we have

$$(4.8) \quad [b_1, D] = [b, D] = 0.$$

We will now try to choose $\sigma_{m+1} : (\Sigma\bar{\mathfrak{b}})^{\otimes m+1} \rightarrow \Sigma\mathfrak{a}$, $\tau_{m+1} : (\Sigma\bar{\mathfrak{b}})^{\otimes m+1} \rightarrow \Sigma\mathfrak{b}$ such that for $g'_{m+1} = g_{m+1} + \sigma_{m+1}$, $h'_{m+1} = h_{m+1} + \tau_{m+1}$, $g'_i = g_i$, $h'_i = h_i$ for $i \leq m$, we have $[b, g'_{\leq m+1}] = 0$, $[b', h'_{\leq m+1}] = 0$, where here $[b', -]$ is the differential [\(4.5\)](#) computed with $f_1 = fg'_{\leq m+1}$ and $f_2 = \text{id}_{\mathfrak{b}}$. The conditions we have to satisfy are

$$(4.9) \quad 0 = [b, g'_{\leq m+1}] = [b_1, \sigma_{m+1}],$$

$$(4.10) \quad 0 = [b', h'_{\leq m+1}] = D + b_2 \circ (h_0 \otimes f_1(g_{m+1} + \sigma_{m+1})) + [b_1, \tau_{m+1}].$$

We claim these equations have a solution. First note that [\(4.10\)](#) may be written as

$$(4.11) \quad b_2 \circ (h_0 \otimes f_1\sigma_{m+1}) = -D - b_2 \circ (h_0 \otimes f_1g_{m+1}) \quad \text{mod } \text{im}[b_1, -].$$

Recall that here we have $[b_1, D] = 0$, $b_1 \circ h_0 = 0$ (see [Section 4.4](#)) and $[b_1, f_1g_{m+1}] = 0$. Hence if we have a solution σ_{m+1} to [\(4.9\)](#) and [\(4.11\)](#) and we replace σ_{m+1} by $\sigma_{m+1} + [b_1, s]$, then it is still a solution.

It follows that we may combine (4.9) and (4.11) into a single equation

$$(4.12) \quad \bar{b}_2 \circ (\bar{h}_0 \otimes \bar{f}_1 \bar{\sigma}_{m+1}) = \overline{-D - b_2 \circ (h_0 \otimes f_1 g_{m+1})}$$

in

$$H^*(\text{Hom}((\Sigma \bar{\mathfrak{b}})^{\otimes m+1}, \Sigma \mathfrak{b})) = \text{Hom}(\Sigma H^*(\bar{\mathfrak{b}})^{\otimes m+1}, \Sigma H^*(\mathfrak{b})),$$

where $\bar{?}$ denotes cohomology classes or actions on cohomology. Using the fact that $\bar{f}_1 = H^*(f_1)$ is an equivalence and $H^*(h_0)$ is a natural isomorphism, one easily sees that (4.12) has a (unique) solution. \square

We will need the following variant of [Lemma 4.10](#), which is proved in a similar way.

LEMMA 4.11. *Let $\mathfrak{a}, \mathfrak{b}$ be A_n categories for $n \geq 3$, and let $f : \mathfrak{a} \rightarrow \mathfrak{b}$ be a fully faithful A_n -functor which is also a quasi-equivalence. Then there exists an A_{n-1} -quasi-equivalence $g : \mathfrak{b} \rightarrow \mathfrak{a}$ such that fg and $\text{id}_{\mathfrak{b}}$ are homotopic and such that $gf = \text{id}_{\mathfrak{a}}$. Moreover the quasi-inverse $H^*(g_1)$ to $H^*(f_1)$ may be chosen freely.*

4.6. The category $\text{Free}(\mathfrak{a})$.

Definition 4.12. Given an A_n -category \mathfrak{a} , $\text{Free}(\mathfrak{a})$ is obtained from \mathfrak{a} by formally adding finite (possibly empty) direct sums and shifts of objects in \mathfrak{a} ; i.e., an object of $\text{Free}(\mathfrak{a})$ is given by

$$(4.13) \quad A = \bigoplus_{i \in I} \Sigma^{a_i} A_i,$$

where $A_i \in \text{Ob}(\mathfrak{a})$, $a_i \in \mathbb{Z}$, $|I| < \infty$. We allow $|I| = \emptyset$. Morphisms in \mathfrak{a} are defined as

$$\text{Free}(\mathfrak{a})(\bigoplus_i \Sigma^{a_i} A_i, \bigoplus_j \Sigma^{b_j} B_j) = \bigoplus_{i,j} \Sigma^{b_j - a_i} \mathfrak{a}(A_i, B_j).$$

An element $f \in \mathfrak{a}(A, B)$ considered as an element of $\text{Free}(\mathfrak{a})(\Sigma^a A, \Sigma^b B)$ will be written as $\sigma^{b-a} f$ such that $|\sigma^{b-a} f| = |f| - (b - a)$.

If \mathfrak{a} is an A_n -category, we can then make $\text{Free}(\mathfrak{a})$ into an A_n -category. We need to define the higher compositions between morphisms between objects of the form $\Sigma^a A$. (The case of more complicated objects is done by linear extension.) So if we have maps in \mathfrak{a} ,

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n,$$

and corresponding maps in $\text{Free}(\mathfrak{a})$,

$$\Sigma^{a_0} A_0 \xrightarrow{\sigma^{a_1 - a_0} f_1} \Sigma^{a_1} A_1 \xrightarrow{\sigma^{a_2 - a_1} f_2} \dots \xrightarrow{\sigma^{a_n - a_{n-1}} f_n} \Sigma^{b_n} A_n,$$

then

$$b_n(s\sigma^{a_n - a_{n-1}} f_n, \dots, s\sigma^{a_2 - a_1} f_2, s\sigma^{a_1 - a_0} f_1) = \pm \sigma^{a_n - a_0} b_n(sf_n, \dots, sf_2, sf_1),$$

where the sign is determined by the usual Koszul sign convention (used with the rule $s\sigma = -\sigma s$).

The A_n -category $\text{Free}(\mathfrak{a})$ is equipped with a strict A_n -endo functor Σ such that on objects we have

$$(4.14) \quad \Sigma(\bigoplus_i \Sigma^{a_i} A_i) = \bigoplus_i \Sigma^{a_i+1} A_i$$

and on morphisms Σ is given by $\Sigma(\sigma^a f) = (-1)^a \sigma^a f$ for f a morphism in \mathfrak{a} . We will call Σ the shift functor on $\text{Free}(\mathfrak{a})$. Likewise $\text{Free}(\mathfrak{a})$ is equipped with an (associative) operation \oplus with an obvious definition. We will call it the “direct sum.” Finally if $I = \emptyset$ in (4.13), then the resulting object is denoted by 0 and is called the “zero object.”

5. Truncated twisted complexes

From now on let \mathfrak{a} be an A_n -category.

5.1. Higher cone categories. Let $\mathfrak{a}^{\oplus m}$ be the graded graph whose objects are formal direct sums of precisely m objects in \mathfrak{a} :

$$(5.1) \quad A = A_0 \oplus A_1 \oplus \dots \oplus A_{m-1}.$$

Morphisms are given by

$$(5.2) \quad \mathfrak{a}^{\oplus m}(A, B) = \bigoplus_{i,j=0}^{m-1} \mathfrak{a}(A_i, B_j).$$

We extend the higher operations on \mathfrak{a} linearly to $\mathfrak{a}^{\oplus m}$ so that $\mathfrak{a}^{\oplus m}$ becomes an A_n -category.

Remark 5.1. Below we usually think of objects in $\mathfrak{a}^{\oplus m}$ as column vectors and similarly of morphisms in $\mathfrak{a}^{\oplus m}$ as matrices acting on those column vectors.

Definition 5.2 (Higher cone categories). Assume $m \leq n+1$. The graded graph \mathfrak{a}^{*m} is defined as follows.

- Objects are given by couples (A, δ_A) such that $A \in \text{Ob}(\mathfrak{a}^{\oplus m})$ and $\delta_A \in \mathfrak{a}^{\oplus m}(A, A)_1$ is a “Maurer-Cartan element” with strictly lower triangular matrix; i.e., it satisfies

$$(5.3) \quad \sum_{i \leq m-1} b_i(s\delta_A, \dots, s\delta_A) = 0.$$

- Morphisms are given by

$$(5.4) \quad \mathfrak{a}^{*m}((A, \delta_A), (B, \delta_B)) = \mathfrak{a}^{\oplus m}(A, B).$$

LEMMA 5.3. Assume $m \leq n + 1$. The graded graph \mathfrak{a}^{*m} has the structure of an $A_{\lfloor \frac{n-m+1}{m} \rfloor}$ -category with higher multiplications given by

(5.5)

$$b_{\mathfrak{a}^{*m},i}(sg_i, \dots, sg_1) = \sum_{\substack{l_0, \dots, l_i \\ h=i+\sum l_j \leq n}} b_{\mathfrak{a}^{\oplus m},h} \underbrace{s\delta_i, \dots, s\delta_i}_{l_i}, sg_i, \underbrace{s\delta_{i-1}, \dots, s\delta_{i-1}}_{l_{i-1}}, \dots, \underbrace{s\delta_1, \dots, s\delta_1}_{l_1}, sg_1, \underbrace{s\delta_0, \dots, s\delta_0}_{l_0}$$

for any set

$$(B_0, \delta_0) \xrightarrow{g_1} (B_1, \delta_1) \xrightarrow{g_2} \dots \xrightarrow{g_i} (B_i, \delta_i)$$

of $i \leq \lfloor (n-m+1)/m \rfloor$ composable arrows in \mathfrak{a}^{*m} .

Proof. We need to check $b_{\mathfrak{a}^{*m}} \circ b_{\mathfrak{a}^{*m}} = 0$ on i composable arrows for $i \leq \lfloor (n-m+1)/m \rfloor$ as well as the correct behavior of identities. We will concentrate on the first condition as it is the most interesting one. As we will use similar facts several times below we present the argument in some detail.

If we expand $(b_{\mathfrak{a}^{*m}} \circ b_{\mathfrak{a}^{*m}})_i$, then it becomes the sum of multilinear expressions evaluated on lists of arguments of the form

$$(5.6) \quad \underbrace{s\delta_i, \dots, s\delta_i}_{l_i}, sg_i, \underbrace{s\delta_{i-1}, \dots, s\delta_{i-1}}_{l_{i-1}}, \dots, \underbrace{s\delta_1, \dots, s\delta_1}_{l_1}, sg_1, \underbrace{s\delta_0, \dots, s\delta_0}_{l_0}.$$

The crucial point is that those multilinear expression are obtained by linear expansion of the corresponding expressions evaluated on composable arrows in \mathfrak{a} . Now for each element $(A, \delta_A) \in \mathfrak{a}^{*m}$, the Maurer-Cartan element δ_A is a strictly lower triangular $m \times m$ -matrix, and hence such extended expressions are zero on (5.6) whenever one of the l_j is $\geq m$.

By the assumption

$$i \leq \left\lfloor \frac{n-m+1}{m} \right\rfloor$$

we obtain that the length of the relevant lists of arguments in (5.6) is

$$\begin{aligned} &\leq (m-1)(i+1) + i \\ &= mi + m - 1 \\ &\leq m \left\lfloor \frac{n-m+1}{m} \right\rfloor + m - 1 \\ &\leq n - m + 1 + m - 1 \\ &= n. \end{aligned}$$

Now the condition $b_{\mathfrak{a}^{*m}} \circ b_{\mathfrak{a}^{*m}} = 0$ combined with (5.3) becomes $b_{\mathfrak{a}^{\oplus m}} \circ b_{\mathfrak{a}^{\oplus m}} = 0$ when evaluated on lists of $\leq n$ arguments. This holds since $\mathfrak{a}^{\oplus m}$ is an A_n -category. \square

Below we call \mathfrak{a}^{*m} a higher cone category. This is motivated by [Definition 5.8](#) below.

LEMMA 5.4 (Functoriality of $*$). *Given A_n -categories \mathfrak{a} and \mathfrak{b} and $t \leq m+1 \leq n+1$, we obtain a strict A_p -functor*

$$*^t : A_m(\mathfrak{a}, \mathfrak{b}) \rightarrow A_{\lfloor \frac{m-t+1}{t} \rfloor}(\mathfrak{a}^{*t}, \mathfrak{b}^{*t})$$

for $p = \lfloor (n-t+1)/t \rfloor - \lfloor (m-t+1)/t \rfloor$. Moreover $*^t$ is strictly compatible with the compositions

$$A_m(\mathfrak{b}, \mathfrak{c}) \times A_m(\mathfrak{a}, \mathfrak{b}) \rightarrow A_m(\mathfrak{a}, \mathfrak{c}).$$

Proof. Since we are defining a strict functor, we only need to define $(*)^t_1$. We will write $(-)^{*t}$ for $(*)^t_1(-)$.

First of all we define the functor on “objects.” For an element $f \in \text{Ob}(A_m(\mathfrak{a}, \mathfrak{b}))$ and $(A, \delta_A) \in \text{Ob}(\mathfrak{a}^{*t})$, define

$$f^{*t}(A, \delta_A) = (f(A), \sum_{i \leq t-1} f(s\delta_A, \dots, s\delta_A)),$$

where f is understood to be extended linearly to direct sums. For a sequence of composable arrows

$$(5.7) \quad (A_0, \delta_0) \xrightarrow{a_1} (A_1, \delta_1) \xrightarrow{a_2} \dots \xrightarrow{a_d} (A_d, \delta_d),$$

put

$$(f^{*t})_d(sa_d, \dots, sa_1) = \sum f_{d+i_0+\dots+i_d}(s\delta_d^{\otimes i_d}, sa_d, s\delta_{d-1}^{\otimes i_{d-1}}, \dots, sa_1, s\delta_0^{\otimes i_0}).$$

To show that $*^t$ sends an A_m -functor to an $A_{\lfloor (m-t+1)/t \rfloor}$ -functor (i.e., an element of $\text{Ob}(A_{\lfloor (m-t+1)/t \rfloor}(\mathfrak{a}^{*t}, \mathfrak{b}^{*t}))$, one proceeds in the same way as in the proof of [Lemma 5.3](#).

Now we define $(*)^t_1$ on Hom-spaces in $A_m(\mathfrak{a}, \mathfrak{b})$. Given $f, g \in A_m(\mathfrak{a}, \mathfrak{b})$ and $h \in A_m(\mathfrak{a}, \mathfrak{b})(f, g)$ we define $h^{*t} \in \text{Hom}(f^{*t}, g^{*t})$ as follows: for a sequence of composable arrows as in (5.7), we have

$$(h^{*t})_d(sa_d, \dots, sa_1) = \sum h_{d+i_0+\dots+i_d}(s\delta_d^{\otimes i_d}, sa_d, s\delta_{d-1}^{\otimes i_{d-1}}, \dots, sa_1, s\delta_0^{\otimes i_0}).$$

One verifies that $(*)^t_1$ commutes with the higher operations on $A_m(\mathfrak{a}, \mathfrak{b})$ and $A_{\lfloor (m-t+1)/t \rfloor}(\mathfrak{a}^{*t}, \mathfrak{b}^{*t})$ (see [Lemma 4.7](#)) and hence defines a strict functor. It is an A_p -functor since $A_{\lfloor \frac{m-t+1}{t} \rfloor}(\mathfrak{a}^{*t}, \mathfrak{b}^{*t})$ is an A_p -category by [Lemmas 5.3](#) and [4.7](#). The strict compatibility with compositions is also a standard verification. \square

5.2. *Truncated twisted complexes.* In the A_n -category setting, untruncated twisted complexes are not well behaved as they form only a graded graph. Indeed even the definition of the differential on morphisms between twisted complexes involves higher operations of unbounded arity. Therefore in this section we introduce *truncated twisted complexes* over an A_n -category. In this case the resulting object is still an A_p -category for some p , although p is much smaller than n .

Definition 5.5 (Truncated twisted complexes). Assume $m \leq n$. We define the truncated twisted complexes over \mathfrak{a} as

$$\mathrm{Tw}_{\leq m} \mathfrak{a} = \mathrm{Free}(\mathfrak{a})^{*m+1}.$$

The map

$$(A, \delta_A) \mapsto (A \oplus 0, (\delta_A, 0))$$

defines a fully faithful functor $\mathrm{Tw}_{\leq m} \mathfrak{a} \rightarrow \mathrm{Tw}_{\leq m+1} \mathfrak{a}$, which we will treat as an inclusion. With this convention we write $\mathrm{Tw} \mathfrak{a}$ for $\bigcup_m \mathrm{Tw}_{\leq m} \mathfrak{a}$ in case \mathfrak{a} is an A_∞ -category. In a similar vein we define the fully faithful functor $\Phi : \mathfrak{a} \rightarrow \mathrm{Tw}_{\leq m} \mathfrak{a} : A \mapsto (A \oplus 0 \oplus \cdots, 0)$, which again we will treat as an inclusion.

From [Lemma 5.3](#) we obtain

LEMMA 5.6. *Assume $m \leq n$. The category of truncated twisted complexes $\mathrm{Tw}_{\leq m} \mathfrak{a}$ has a structure of an $A_{\lfloor \frac{n-m}{m+1} \rfloor}$ -category.*

LEMMA 5.7 (Functionality of Tw). *Let $F : \mathfrak{a} \rightarrow \mathfrak{b}$ be an A_m -functor between two A_n -categories with $a \leq m \leq n$. Then we obtain a corresponding $A_{\lfloor \frac{m-a}{a+1} \rfloor}$ functor*

$$\mathrm{Tw}_{\leq a} F : \mathrm{Tw}_{\leq a} \mathfrak{a} \rightarrow \mathrm{Tw}_{\leq a} \mathfrak{b}.$$

Moreover $\mathrm{Tw}_{\leq a}(-)$ is strictly compatible with compositions of A_n -functors.

Proof. This follows immediately from [Lemma 5.4](#). \square

5.3. Distinguished triangles.

Definition 5.8. Assume $f : A \rightarrow B$ is a closed morphism in \mathfrak{a} of degree zero. Then $C(f)$ is the object $(\Sigma A \oplus B, \delta_{C(f)}) \in \mathrm{Tw}_{\leq 1} \mathfrak{a}$ such that

$$\delta_{C(f)} = \begin{pmatrix} 0 & 0 \\ \sigma^{-1} f & 0 \end{pmatrix}.$$

(Recall that we write objects as column vectors and morphisms as matrices; see [Remark 5.1](#).)

Definition 5.9. Let $f : A \rightarrow B$ be a morphism in $Z^0\mathfrak{a}$. The associated *standard distinguished triangle* δ_f in $\text{Tw}_{\leq 1}\mathfrak{a}$ is given by

$$(5.8) \quad A \xrightarrow{f} B \xrightarrow{i} (C(f), \delta_{C(f)}) \xrightarrow[\substack{p \\ (1)}]{} A,$$

where

$$i = \begin{pmatrix} 0 \\ \text{id}_B \end{pmatrix} \quad p = (\sigma^{-1} \text{id}_A \quad 0).$$

The image of δ_f in $H^0(\text{Tw}_{\leq 1}\mathfrak{a})$ is written as $\bar{\delta}_f$. It is also called a standard distinguished triangle.

Definition 5.10. Let \mathfrak{a} be an A_n -category with $n \geq 7$. A triangle in $H^*(\mathfrak{a})$ is said to be *distinguished* if its image under $H^*(\Phi)$ is isomorphic to a standard distinguished triangle in $H^*(\text{Tw}_{\leq 1}\mathfrak{a})$.

From this definition we immediately obtain

THEOREM 5.11. *Let $\rho : \mathfrak{a} \rightarrow \mathfrak{b}$ be an A_m -functor between A_n -categories for $m \geq 5$, $n \geq 7$. Then $H^*(\rho)$ preserves distinguished triangles.*

Proof. It is clear that there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{\Phi} & \text{Tw}_{\leq 1}\mathfrak{a} \\ \rho \downarrow & & \downarrow \text{Tw}_{\leq 1}\rho \\ \mathfrak{b} & \xrightarrow[\Phi]{} & \text{Tw}_{\leq 1}\mathfrak{b}. \end{array}$$

By [Lemma 5.6](#), $\text{Tw}_{\leq 1}\mathfrak{a}$ and $\text{Tw}_{\leq 1}\mathfrak{b}$ are A_3 -categories, and by [Lemma 5.7](#), $\text{Tw}_{\leq 1}\rho$ is an A_2 -functor. Hence $H^*(\text{Tw}_{\leq 1}\rho)$ is a graded functor (see [Section 4.1](#)). One checks $H^*(\text{Tw}_{\leq 1}\rho)(\bar{\delta}_f) = \bar{\delta}_{\rho_1(f)}$. This implies what we want. \square

6. DG-categories

6.1. Generalities. Recall that a DG-category is an A_∞ -category such that $m_i = 0$ for $i \geq 3$. In that case $\text{Tw}\mathfrak{a}$ is also a DG-category. We recall the following definition.

Definition 6.1 ([3]). A DG-category is *pre-triangulated* if the DG-functor $\Phi : \mathfrak{a} \rightarrow \text{Tw}\mathfrak{a}$ is a quasi-equivalence.

The main result concerning pre-triangulated DG-categories is

THEOREM 6.2 ([3]). *If \mathfrak{a} is pre-triangulated, then $H^*(\mathfrak{a})$, when equipped with distinguished triangles as in [Definition 5.10](#), is triangulated.*

Proof. Assume first that \mathfrak{a} is a general DG-category. Then $\mathrm{Tw} \mathfrak{a}$ is equipped with a natural cone functor $C(f)$ and a notion of standard triangles δ_f for any closed map $f : A \rightarrow B$:

$$A \xrightarrow{f} B \xrightarrow{i} C(f) \xrightarrow{(1)} A.$$

A triangle in $\mathrm{Tw} \mathfrak{a}$ is called distinguished if it is isomorphic to a standard triangle. In [3] it is proved that $H^*(\mathrm{Tw} \mathfrak{a})$ is triangulated when equipped with this class of distinguished triangles. If \mathfrak{a} is pre-triangulated, then $H^*(\mathfrak{a})$ inherits the triangulated structure from $H^*(\mathrm{Tw} \mathfrak{a})$. We have to prove that the distinguished triangles are the same as those in [Definition 5.10](#). Assume that

$$\bar{\delta} : A \xrightarrow{f} B \rightarrow C \xrightarrow{(1)} A$$

is a triangle in $H^*(\mathfrak{a})$ distinguished in the sense of [3]; i.e., $\Phi(\bar{\delta})$ is distinguished in $H^*(\mathrm{Tw} \mathfrak{a})$. Now $\bar{\delta}_{\Phi(f)} \in H^*(\mathrm{Tw}_{\leq 1} \mathfrak{a})$ is a distinguished triangle in $H^*(\mathrm{Tw} \mathfrak{a})$ which has the same base as $\bar{\delta}_f$. By the axioms for triangulated categories we conclude that $\Phi(\bar{\delta}) \cong \bar{\delta}_{\Phi(f)}$. Hence $\bar{\delta}$ is distinguished in the sense of [Definition 5.10](#). The opposite direction is similar. \square

6.2. Some small DG-categories.

Definition 6.3. Let $n \geq 0$. Then I_n is the DG-category with objects $(x_i)_{i=0}^n$ such that

$$I_n(x_i, x_j) = \begin{cases} ka_{ij} & \text{if } i < j, \\ k \mathrm{id}_{x_i} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

with $|a_{ij}| = 0$, $a_{jl}a_{ij} = a_{il}$ and $da_{ij} = 0$. We will write $a_i = a_{i,i+1}$ for $i = 0, \dots, n-1$.

LEMMA 6.4. $\mathrm{Tw}_{\leq 1} I_n$ is pre-triangulated.

Proof. Since $\mathrm{Tw} I_n$ is pre-triangulated [3], it is sufficient to prove that $H^*(\mathrm{Tw}_{\leq 1} I_n) \rightarrow H^*(\mathrm{Tw} I_n)$ is essentially surjective. This is essentially [13, Prop. 7.27]. For the convenience of the reader we repeat the argument. The Yoneda embedding realizes $H^*(\mathrm{Tw} I_n)$ as the bounded derived category $D^b(\mathrm{rep}(I_n))$ of the representations of I_n , viewed as a quiver. Since $\mathrm{rep}(I_n)$ is a hereditary category, every object in $D^b(\mathrm{rep}(I_n))$ is the direct sum of its (shifted) cohomology objects which are in $\mathrm{rep}(I_n)$. Moreover every object in $\mathrm{rep}(I_n)$ has projective dimension one, and so it is isomorphic to a single cone of objects in $\mathrm{Free}(I_n)$. In other words, it is in the essential image of $H^*(\mathrm{Tw}_{\leq 1} I_n)$. \square

Remark 6.5. Assume $n = 0$. Then $\mathrm{rep}(I_0)$ has global dimension zero and we have in fact that $\mathrm{Free}(I_0) = \mathrm{Tw}_{\leq 0} I_0$ is pre-triangulated.

7. Pre-triangulated A_n -categories

From now on let \mathfrak{a} be an A_n -category. The purpose of this section is to define what it means for \mathfrak{a} to be pre-triangulated and to show that this definition implies that $H^*(\mathfrak{a})$ is triangulated.

Definition 7.1. An A_n -category \mathfrak{a} , with $n \geq 7$, is said to be *pre-triangulated* if the inclusion $\mathfrak{a} \xrightarrow{\Phi} \mathrm{Tw}_{\leq 1} \mathfrak{a}$ is a quasi-equivalence.

Remark 7.2. The lower bound $n \geq 7$ comes from the fact that we want $H^*(\mathrm{Tw}_{\leq 1} \mathfrak{a})$ to be an honest category. This happens when $\mathrm{Tw}_{\leq 1} \mathfrak{a}$ is an A_3 -category. For this to be true, \mathfrak{a} needs to be at least an A_7 -category by Lemma 5.6.

THEOREM 7.3. *Let \mathfrak{a} be a pre-triangulated A_n -category for $n \geq 13$. When equipped with the collection of distinguished triangles as in Definition 5.10, $H^*(\mathfrak{a})$ is a triangulated category.*

Proof. Here is the “strategy”: we have to prove that $H^*(\mathfrak{a})$ satisfies (TR0)–(TR4) as in Section 3.4. For the (TR1)–(TR4) axioms, we will translate their input into a suitable A_n -functor $\mu : I_m \rightarrow \mathfrak{a}$, for $m \leq 2$, which is then extended to an $A_{\lfloor (n-1)/2 \rfloor}$ -functor $\mathrm{Tw}_{\leq 1} \mu : \mathrm{Tw}_{\leq 1} I_m \rightarrow \mathrm{Tw}_{\leq 1} \mathfrak{a}$. Then we use that $\mathrm{Tw}_{\leq 1} I_m$ is pre-triangulated by Lemma 6.4 and hence, in particular, $H^*(\mathrm{Tw}_{\leq 1} I_m)$ is triangulated by Theorem 6.2. Roughly speaking we then transfer the output of the (TR1)–(TR4)-axioms for $H^*(\mathrm{Tw}_{\leq 1} I_m)$ to $H^*(\mathfrak{a})$ by using Theorem 5.11.

To accomplish the last step we will pick an A_p -functor $\pi : \mathrm{Tw}_{\leq 1} \mathfrak{a} \rightarrow \mathfrak{a}$, for $p = \lfloor (n-1)/2 \rfloor - 1 = \lfloor (n-3)/2 \rfloor$, which is a homotopy inverse to Φ such that $\pi \Phi$ is the identity (see Lemmas 5.6 and 4.11). In particular, we have that $H^*(\Phi)$ and $H^*(\pi)$ are quasi-inverses to each other. Since $n \geq 13$, $\mathrm{Tw}_{\leq 1} \mu$ is at least an A_6 -functor and π is at least an A_5 -functor. So $H^*(\pi \mathrm{Tw}_{\leq 1} \mu)$ preserves distinguished triangles by Theorem 5.11. To avoid making some arguments needlessly cumbersome we will in fact also use that $H^*(\mathrm{Tw}_{\leq 1} \mu)$ preserves standard distinguished triangles and that $H^*(\pi)$ sends a standard distinguished triangle in $H^*(\mathrm{Tw}_{\leq 1} \mathfrak{a})$ to a distinguished triangle in $H^*(\mathfrak{a})$. The latter follows easily from the fact that $H^*(\pi)$ is a quasi-inverse to $H^*(\Phi)$. Note that the intermediate category $\mathrm{Tw}_{\leq 1} \mathfrak{a}$ may be only an A_6 -category so, with our current definitions, we cannot talk about distinguished triangles in⁵ $H^*(\mathrm{Tw}_{\leq 1} \mathfrak{a})$.

(TR0) Like $\mathrm{Free}(\mathfrak{a})$ (see Section 4.6), $\mathrm{Tw}_{\leq 1} \mathfrak{a}$ is equipped with canonical operations Σ and \oplus . These descend to operations on $H^*(\mathrm{Tw}_{\leq 1} \mathfrak{a})$ which one

⁵We could have eliminated this minor technical complication by simply requiring $n \geq 15$.

easily checks to be to be the categorical direct sum and shift functor. Since $H^*(\mathfrak{a}) \rightarrow H^*(\mathrm{Tw}_{\leq 1} \mathfrak{a})$ is an equivalence, the direct sum and shift functor defined on $H^*(\mathrm{Tw}_{\leq 1} \mathfrak{a})$ descend to $H^*(\mathfrak{a})$.

(TR1) First we note that the triangle

$$(7.1) \quad A \xrightarrow{\mathrm{id}_A} A \rightarrow 0 \xrightarrow{(1)} A$$

is distinguished. Indeed the functor $\mu : I_0 \rightarrow \mathfrak{a} : x_0 \mapsto A$ extends to a functor⁶ $\mu : \mathrm{Tw}_{\leq 1} I_0 \rightarrow \mathrm{Tw}_{\leq 1} \mathfrak{a}$, and (7.1) is the image under $H^*(\pi \mathrm{Tw}_{\leq 1} \mu)$ of the distinguished triangle in $H^*(\mathrm{Tw}_{\leq 1} I_0)$ (which satisfies (TR1))

$$x_0 \xrightarrow{\mathrm{id}_{x_0}} x_0 \xrightarrow{0} 0 \xrightarrow{(1)} x_0.$$

Now we prove the second part of the (TR1) conditions: the existence of distinguished triangles with a given base. Consider a map $A \xrightarrow{f} B$ in $H^*(\mathfrak{a})$, and put $\bar{\delta} = H^*(\pi)(\bar{\delta}_f)$. Since $\bar{\delta}_f$ is a standard distinguished triangle in $\mathrm{Tw}_{\leq 1} \mathfrak{a}$, $\bar{\delta}$ is distinguished.

Finally, the fact that any triangle isomorphic to a distinguished triangle is distinguished follows immediately from [Definition 5.10](#).

TR2 Let $\bar{\delta}$ be a distinguished triangle in $H^*(\mathfrak{a})$. Then there exists an isomorphism with a standard triangle $H^*(\Phi)(\bar{\delta}) \cong \bar{\delta}_f$ and hence, in particular, $\bar{\delta} \cong H^*(\pi)(\bar{\delta}_f) := \bar{\delta}'$. There is a strict A_n -functor $\mu : I_1 \rightarrow \mathfrak{a}$ which sends a_0 to f , and $\bar{\delta}_f$ is the image of $\bar{\delta}_{a_0} \in H^*(\mathrm{Tw}_{\leq 1} I_1)$ under the morphism $H^*(\mathrm{Tw}_{\leq 1} \mu)$. Since $H^*(\mathrm{Tw}_{\leq 1} I_1)$ satisfies (TR2), the rotated versions of $\bar{\delta}_{a_0}$ are distinguished in $H^*(\mathrm{Tw}_{\leq 1} I_1)$ and we obtain rotated versions of $\bar{\delta}'$ by applying $H^*(\pi \mathrm{Tw}_{\leq 1} \mu)$. (Note that a graded functor preserves suspensions and desuspensions.) By (TR1) the corresponding rotated versions of $\bar{\delta}$ are also distinguished.

(TR3) Suppose we have a diagram of distinguished triangles in $H^0(\mathfrak{a})$:

$$(7.2) \quad \begin{array}{ccccccc} A & \xrightarrow{f} & B & \longrightarrow & C & \xrightarrow{(1)} & A \\ u \downarrow & & v \downarrow & & & & \downarrow u \\ A' & \xrightarrow{f'} & B' & \longrightarrow & C' & \xrightarrow{(1)} & A' \end{array}$$

Up to composing with an isomorphism of triangles, we can assume that the two distinguished triangles in the diagram are standard distinguished triangles in $\mathrm{Tw}_{\leq 1} \mathfrak{a}$ so that $C = C(f)$, $C' = C(f')$. Hence

⁶The reader will note that here the literal execution of our “strategy” is a bit uneconomical and that by [Remark 6.5](#) we could have used $\mathrm{Tw}_{\leq 0} I_0$.

we have to construct the dotted arrow in

$$\begin{array}{ccccccc}
 A & \xrightarrow{f} & B & \longrightarrow & C(f) & \xrightarrow{(1)} & A \\
 u \downarrow & & v \downarrow & & \downarrow w & & \downarrow u \\
 A' & \xrightarrow{f'} & B' & \longrightarrow & C(f') & \xrightarrow{(1)} & A'.
 \end{array}$$

It is easy to give a formula for w . Alternatively one may lift the square on the left to an A_n -functor $I \otimes I \rightarrow \mathfrak{a}$ and then proceed by considering the induced functor $\text{Tw}_{\leq 1}(I \otimes I) \rightarrow \text{Tw}_{\leq 1} \mathfrak{a}$.

We will give instead a proof compatible with our “strategy.” By writing the solid square as a composition of 2 squares it is sufficient to consider the case in which either u or v is the identity. The two cases are similar, so we will consider the first one. Now the diagram is

$$(7.3) \quad
 \begin{array}{ccccccc}
 A & \xrightarrow{f} & B & \longrightarrow & C(f) & \xrightarrow{(1)} & A \\
 \parallel & & v \downarrow & & \downarrow w & & \parallel \\
 A & \xrightarrow{f'u} & B' & \longrightarrow & C(f'u) & \xrightarrow{(1)} & A.
 \end{array}$$

We may construct an A_n -morphism $\mu : I_2 \rightarrow \mathfrak{a}$ such that $\mu_1(a_0) = f$, $\mu_1(a_1) = v$, $\mu_1(a_1 a_0) = f'u$. (Note that we need a non-trivial μ_2 as $v f$ is not necessarily equal to $f'u$ in \mathfrak{a} .) Inside $H^*(\text{Tw}_{\leq 1} I_2)$ we have the diagram

$$\begin{array}{ccccccc}
 x_0 & \xrightarrow{a_0} & x_1 & \longrightarrow & C(a_0) & \xrightarrow{(1)} & x_0 \\
 \parallel & & a_1 \downarrow & & \downarrow & & \parallel \\
 x_0 & \xrightarrow{a_1 a_0} & x_2 & \longrightarrow & C(a_1 a_0) & \xrightarrow{(1)} & x_0,
 \end{array}$$

where now the dotted arrow exists as $H^*(\text{Tw}_{\leq 1} I_2)$ satisfies (TR3). Applying $H^*(\text{Tw}_{\leq 1} \mu)$ we obtain (7.3).

(TR4) Since we have shown (TR1)–(TR3), by [2, 1.1.6] it suffices to show that any composable pair of degree zero morphisms $X \rightarrow Y \rightarrow Z$ in $H^*(\mathfrak{a})$ can be completed to an octahedron as in (3.1).

A composable pair of degree zero morphisms in $H^*(\mathfrak{a})$ can be lifted to an A_n -functor $\mu : I_2 \rightarrow \mathfrak{a}$. The image of the octahedron in $H^*(\text{Tw}_{\leq 1} I_2)$ built on $x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} x_2$ under $H^*(\pi \text{Tw}_{\leq 1} \mu)$ is now the sought octahedron in $H^*(\mathfrak{a})$. \square

8. Gluing A_n -categories

8.1. *Bimodules.* Let $\mathfrak{a}, \mathfrak{b}$ be A_n -categories. For $m \leq n$, an A_{m+1} - \mathfrak{b} - \mathfrak{a} -bimodule is a collection of graded vector spaces $M(A, B)$, $A \in \text{Ob}(\mathfrak{a})$, $B \in \text{Ob}(\mathfrak{b})$ together with a codifferential on $(B^+ \mathfrak{b} \otimes M \otimes B^+ \mathfrak{a})_{\leq m+1}$, where the latter is regarded as a DG- $(B^+ \mathfrak{a})_{\leq m} - (B^+ \mathfrak{b})_{\leq m}$ -bicomodule. In other words, such a bimodule is equipped with higher operations of degree one

$$(8.1) \quad \begin{aligned} b_M : \Sigma \mathfrak{b}(B_{p-1}, B_p) \otimes \cdots \otimes \Sigma \mathfrak{b}(B_{a+1}, B_{a+2}) \otimes M(A_a, B_{a+1}) \\ \otimes \Sigma \mathfrak{b}(A_{a-1}, A_a) \otimes \cdots \otimes \Sigma \mathfrak{b}(A_0, A_1) \rightarrow M(A_0, B_p) \end{aligned}$$

for $(A_i)_{i=0, \dots, a} \in \text{Ob}(\mathfrak{a})$, $(B_j)_{j=a+1, \dots, p} \in \text{Ob}(\mathfrak{b})$, $p \leq m+1$ such that $b \circ b = 0$. In addition we require that the higher operations vanish on identities, when appropriate. If $\mathfrak{a} = \mathfrak{b}$, then the *identity A_n - \mathfrak{a} -bimodule* is given by $M(A, A') = \mathfrak{a}(A, A')$ and the higher operations are those of \mathfrak{a} .

If $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2$ are A_n -categories, $f_i : \mathfrak{a}_i \rightarrow \mathfrak{b}_i$ are A_n -functors and M is an A_{m+1} - \mathfrak{b}_2 - \mathfrak{b}_1 -bimodule for some $m \leq n$, then we write ${}_{f_1}M_{f_2}$ for the \mathfrak{a}_2 - \mathfrak{a}_1 -bimodule which is the pullback of M along (f_1, f_2) . For $A_1 \in \text{Ob}(\mathfrak{a}_1)$, $A_2 \in \text{Ob}(\mathfrak{a}_2)$, we have ${}_{f_1}M_{f_2}(A_1, A_2) = M(f_1(A_1), f_2(A_2))$, and the higher operations on ${}_{f_1}M_{f_2}$ are schematically given by the following formula for $m \in {}_{f_1}M_{f_2}(A_1, A_2)$:

$$b_{{}_{f_1}M_{f_2}}(\dots, m, \dots) = \sum \pm b_M(f_2(\dots), \dots, f_2(\dots), m, f_1(\dots), \dots, f_1(\dots)).$$

(The sign is given by the Koszul convention.) It is easy to see that ${}_{f_1}M_{f_2}$ is an A_{m+1} -bimodule. If f_1 or f_2 is the identity, then we omit it from the notation.

8.2. The arrow category.

Definition 8.1 (The arrow category). Let $\mathfrak{a}, \mathfrak{b}$ be A_n -categories, and let M be a \mathfrak{b} - \mathfrak{a} - A_n -bimodule. The arrow category $\mathfrak{c} = \mathfrak{a} \xrightarrow{M} \mathfrak{b}$ has $\text{Ob}(\mathfrak{c}) = \text{Ob}(\mathfrak{b}) \coprod \text{Ob}(\mathfrak{a})$ and morphisms for $B, B' \in \text{Ob}(\mathfrak{b})$, $A, A' \in \text{Ob}(\mathfrak{a})$ given by $\mathfrak{c}(A, A') = \mathfrak{a}(A, A')$, $\mathfrak{c}(B, B') = \mathfrak{b}(B, B')$ and $\mathfrak{c}(A, B) = M(A, B)$, $\mathfrak{c}(B, A) = 0$.

It is easy to see that $\mathfrak{a} \xrightarrow{M} \mathfrak{b}$ becomes an A_n -category by combining the higher multiplications on \mathfrak{a} , \mathfrak{b} and M (as in (8.1)).

Assume that we have A_n -categories $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}', \mathfrak{b}'$ and \mathfrak{b} - \mathfrak{a} and \mathfrak{b}' - \mathfrak{a}' bimodules M and M' . Below it will be convenient to consider the category $A_m^\circ(\mathfrak{a} \xrightarrow{M} \mathfrak{b}, \mathfrak{a}' \xrightarrow{M'} \mathfrak{b}')$ of A_m -functors $F : (\mathfrak{a} \xrightarrow{M} \mathfrak{b}) \rightarrow (\mathfrak{a}' \xrightarrow{M'} \mathfrak{b}')$ such that $F(\text{Ob}(\mathfrak{a})) \subset \text{Ob}(\mathfrak{a}')$, $F(\text{Ob}(\mathfrak{b})) \subset \text{Ob}(\mathfrak{b}')$. It is easy to see that F contains the same data as A_m -functors $F_{\mathfrak{a}} : \mathfrak{a} \rightarrow \mathfrak{a}'$, $F_{\mathfrak{b}} : \mathfrak{b} \rightarrow \mathfrak{b}'$ together with an A_m -bimodule morphism $F_M : M \rightarrow {}_{F_{\mathfrak{a}}}M'_{F_{\mathfrak{b}}}$. Sometimes we will write $F = (F_{\mathfrak{a}}, F_M, F_{\mathfrak{b}})$.

8.3. The gluing category.

Definition 8.2 (The gluing category). Assume that $n \geq 1$. Let $\mathfrak{a}, \mathfrak{b}$ be A_n -categories, and let M be a \mathfrak{b} - \mathfrak{a} - A_n -bimodule. The gluing category $\mathfrak{a} \coprod_M \mathfrak{b}$ is the full graded subgraph of $(\mathfrak{a} \xrightarrow{M} \mathfrak{b})^{*2}$ given by objects of the form $(A \oplus B, \delta)$ with $A \in \text{Ob}(\mathfrak{a})$ and $B \in \text{Ob}(\mathfrak{b})$. (Note that δ is simply an element of $Z^1 M(A, B)$.)

LEMMA 8.3. *The gluing category $\mathfrak{a} \coprod_M \mathfrak{b}$ has the structure of an A_{n-1} category with higher multiplications given by (5.5).*

Proof. The proof is as in [Lemma 5.3](#) except that now in the relevant argument lists in (5.6), we can have at most one δ , as the $(g_j)_j$ are now represented by lower triangular 2×2 -matrices. \square

Remark 8.4. An alternative way of defining $\mathfrak{a} \coprod_M \mathfrak{b}$ is as follows. Let J_1 be defined like I_1 (see [Section 6.2](#)) except that we put $|a_0| = 1$. Then $\mathfrak{a} \coprod_M \mathfrak{b}$ may be identified with the full subcategory of $A_1(J_1, \mathfrak{a} \xrightarrow{M} \mathfrak{b})$ consisting of A_1 -functors $F : J_1 \rightarrow (\mathfrak{a} \xrightarrow{M} \mathfrak{b})$ such that $F(x_0) \in \text{Ob}(\mathfrak{a})$, $F(x_1) \in \text{Ob}(\mathfrak{b})$. It then follows from [Lemma 4.7](#) that $\mathfrak{a} \coprod_M \mathfrak{b}$ is indeed an A_{n-1} -category.

The following will be our main result in this section.

THEOREM 8.5. *Assume that $n \geq 13$, that $\mathfrak{a}, \mathfrak{b}$ are pre-triangulated A_n -categories and that M is an A_n - \mathfrak{b} - \mathfrak{a} -bimodule. Then $\mathfrak{a} \coprod_M \mathfrak{b}$ is a pre-triangulated A_{n-1} category. Moreover the obvious fully faithful functors $\varphi_{\mathfrak{a}} : H^*(\mathfrak{a}) \rightarrow H^*(\mathfrak{a} \coprod_M \mathfrak{b})$, $\varphi_{\mathfrak{b}} : H^*(\mathfrak{b}) \rightarrow H^*(\mathfrak{a} \coprod_M \mathfrak{b})$ preserve distinguished triangles. If $n \geq 14$ so that $H^*(\mathfrak{a} \coprod_M \mathfrak{b})$ is triangulated by [Theorem 7.3](#) and [Lemma 8.3](#), then $\varphi_{\mathfrak{a}}, \varphi_{\mathfrak{b}}$ give rise to a semi-orthogonal decomposition*

$$(8.2) \quad H^*(\mathfrak{a} \coprod_M \mathfrak{b}) = \langle H^*(\mathfrak{a}), H^*(\mathfrak{b}) \rangle$$

whose associated bimodule (see [Section 1.3](#)) is $H^*(M)$.

The proof of this theorem requires some preparation. We start with

PROPOSITION 8.6 (Functionality of gluing). *Assume we have A_n -categories $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}', \mathfrak{b}'$ and \mathfrak{b} - \mathfrak{a} and \mathfrak{b}' - \mathfrak{a}' bimodules M and M' . Then for $m \leq n$, there is a strict A_{n-m} -functor*

$$\phi : A_m^{\circ}(\mathfrak{a} \xrightarrow{M} \mathfrak{b}, \mathfrak{a}' \xrightarrow{M'} \mathfrak{b}') \rightarrow A_{m-1}(\mathfrak{a} \coprod_M \mathfrak{b}, \mathfrak{a}' \coprod_{M'} \mathfrak{b}').$$

Moreover ϕ is strictly compatible with compositions.

Proof. This is proved like [Lemma 5.4](#), which also gives the relevant formulas (where we take into account that in this case at most one δ can appear in the relevant arguments lists in (5.6)). \square

COROLLARY 8.7. *Let $3 \leq m \leq n - 3$, and let $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}', \mathfrak{b}', M, M'$ be as in [Proposition 8.6](#). Let $F \in A_m^\circ(\mathfrak{a} \xrightarrow{M} \mathfrak{b}, \mathfrak{a}' \xrightarrow{M'} \mathfrak{b}')$. If F is a quasi-equivalence, then so is $\phi(F)$.*

Proof. Note that F is a quasi-equivalence if and only if $F_{\mathfrak{a}}, F_{\mathfrak{b}}$ are quasi-equivalences and F_M is a quasi-isomorphism. By [Lemma 4.10](#) we may choose an inverse $G \in A_{m-1}^\circ(\mathfrak{a}' \xrightarrow{M'} \mathfrak{b}', \mathfrak{a} \xrightarrow{M} \mathfrak{b})$ to F , up to homotopy (making use of the fact that the quasi-inverse to $H^*(F_1)$ may be chosen freely). Note that $H^*(G)$ is a functor as $m - 1 \geq 2$.

Since $H^*(\phi)$ also being a functor (as $n - m \geq 3$) preserves invertible maps, we conclude by [Lemma 4.9](#) that it preserves homotopies. Hence $\phi(G)$ is an inverse to $\phi(F)$ up to homotopy. It follows that $H^*(\phi(F))$ is an equivalence $H^*(\mathfrak{a} \coprod_M \mathfrak{b}) \rightarrow H^*(\mathfrak{a}' \coprod_{M'} \mathfrak{b}')$. \square

For the next few results we assume that $\mathfrak{a}, \mathfrak{b}$ are A_n -categories and that M is an A_n - \mathfrak{b} - \mathfrak{a} -bimodule. We define M^{*2} as the $\mathfrak{b}*\mathfrak{b}$ - $\mathfrak{a}*\mathfrak{a}$ bimodule such that

$$\begin{aligned} M^{*2}((A_0 \oplus A_1, \delta_A), (B_0 \oplus B_1, \delta_B)) \\ = M(A_0, B_0) \oplus M(A_0, B_1) \oplus M(A_1, B_0) \oplus M(A_1, B_1), \end{aligned}$$

where the higher operations on M^{*2} are obtained from those of M by “inserting Maurer-Cartan elements” like in [Lemma 5.3](#). In a similar way as [Lemma 5.4](#) one proves

LEMMA 8.8. *M^{*2} is a $A_{\lfloor(n-1)/2\rfloor}$ -bimodule.*

LEMMA 8.9. *Let $n \geq 3$, and let $\mathfrak{a}, \mathfrak{b}, M$ be as above. We have a fully faithful functor of $A_{\lfloor(n-1)/2\rfloor-1}$ -categories*

$$(8.3) \quad (\mathfrak{a} \coprod_M \mathfrak{b})*(\mathfrak{a} \coprod_M \mathfrak{b}) \rightarrow \mathfrak{a}*\mathfrak{a} \coprod_{M^{*2}} \mathfrak{b}*\mathfrak{b}$$

Proof. An object in $(\mathfrak{a} \coprod_M \mathfrak{b})*(\mathfrak{a} \coprod_M \mathfrak{b})$ is of the form

$$((A_0 \oplus B_0, \delta_0) \oplus (A_1 \oplus B_1, \delta_1), \delta),$$

where $\delta = (\delta_{00}, \delta_{10}, \delta_{11}) \in \mathfrak{a}(A_0, A_1)_1 \oplus M(A_0, B_1)_1 \oplus \mathfrak{b}(B_0, B_1)_1$ is such that

$$\delta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta_{00} & 0 & 0 & 0 \\ \delta_{01} & \delta_{11} & 0 & 0 \end{pmatrix} \text{ acting on } \begin{pmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \end{pmatrix}$$

is a Maurer-Cartan element in $(\mathfrak{a} \coprod_M \mathfrak{b})^{\oplus 2}$. One verifies that the matrix

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_0 & 0 & 0 & 0 \\ \delta_{00} & 0 & 0 & 0 \\ \delta_{01} & \delta_{11} & \delta_1 & 0 \end{pmatrix} \text{ acting on } \begin{pmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \end{pmatrix}$$

defines a Maurer-Cartan element in $(\mathfrak{a} \xrightarrow{M} \mathfrak{b})^{\oplus 4}$. Rearranging Δ we get a different Maurer-Cartan element in $(\mathfrak{a} \xrightarrow{M} \mathfrak{b})^{\oplus 4}$

$$\Delta^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_{00} & 0 & 0 & 0 \\ \delta_0 & 0 & 0 & 0 \\ \delta_{01} & \delta_1 & \delta_{11} & 0 \end{pmatrix} \text{ acting on } \begin{pmatrix} A_0 \\ A_1 \\ B_0 \\ B_1 \end{pmatrix},$$

which is a block-matrix representation for an object in $(\mathfrak{a} * \mathfrak{a}) \coprod_{M^{*2}} (\mathfrak{b} * \mathfrak{b})$. This construction defines and injection

$$\text{Ob} \left(\left(\mathfrak{a} \coprod_M \mathfrak{b} \right) * \left(\mathfrak{a} \coprod_M \mathfrak{b} \right) \right) \hookrightarrow \text{Ob} \left(\mathfrak{a} * \mathfrak{a} \coprod_{M^{*2}} \mathfrak{b} * \mathfrak{b} \right)$$

(but not a bijection) which is compatible with Hom-sets. It is now an easy verification (but messy to write down) that we also get compatibility with higher operations. \square

The bimodule M may be extended to a $\text{Free}(\mathfrak{b})$ - $\text{Free}(\mathfrak{a})$ - A_n -bimodule, which we denote by $\text{Free}(M)$.

LEMMA 8.10. *We have a fully faithful functor of A_{n-1} -categories*

$$\text{Free}(\mathfrak{a} \coprod_M \mathfrak{b}) \rightarrow \text{Free}(\mathfrak{a}) \coprod_{\text{Free}(M)} \text{Free}(\mathfrak{b}).$$

Proof. An object in $\text{Free}(\mathfrak{a} \coprod_M \mathfrak{b})$ is of the form $\bigoplus_{i \in I} \Sigma^{a_i} (A_i \oplus B_i, \delta_i)$. We send it to $(\bigoplus_i \Sigma^{a_i} A_i \oplus \bigoplus_i \Sigma^{a_i} B_i, \bigoplus_i \delta_i)$. It is easy to see that this operation is fully faithful. \square

Now we put $\text{Tw}_{\leq 1} M = (\text{Free } M)^{*2}$. From [Lemma 8.8](#) we obtain

LEMMA 8.11. $\text{Tw}_{\leq 1} M$ is a $A_{\lfloor (n-1)/2 \rfloor}$ -bimodule.

COROLLARY 8.12. *Assume $n \geq 3$. There is a fully faithful functor of $A_{\lfloor (n-1)/2 \rfloor - 1}$ -categories*

$$(8.4) \quad \text{Tw}_{\leq 1}(\mathfrak{a} \coprod_M \mathfrak{b}) \rightarrow \text{Tw}_{\leq 1} \mathfrak{a} \coprod_{\text{Tw}_{\leq 1} M} \text{Tw}_{\leq 1} \mathfrak{b}$$

whose restriction to $\mathfrak{a} \coprod_M \mathfrak{b}$ is (Φ, I, Φ^*) , where $\Phi : \mathfrak{a} \rightarrow \text{Tw}_{\leq 1} \mathfrak{a}$, is as in [Definition 5.5](#), $\Phi^* : \mathfrak{b} \rightarrow \text{Tw}_{\leq 1} \mathfrak{b}$ is the related map $B \mapsto (0 \oplus B, 0)$ and $I : M \rightarrow \Phi \text{Tw}_{\leq 1} M \Phi^*$ is the obvious inclusion.

Proof. The existence of (8.4) follows by combining [Lemmas 8.9](#) and [8.10](#). The fact that the restriction to $\mathfrak{a} \coprod_M \mathfrak{b}$ has the indicated form follows from the construction of the map. \square

Proof of Theorem 8.5. If $n \geq 13$, then $\text{Tw}_{\leq 1} \mathfrak{a}$, $\text{Tw}_{\leq 1} \mathfrak{b}$ are at least A_6 -categories by [Lemma 5.6](#). By [Lemma 8.11](#), $\text{Tw}_{\leq 1} M$ is at least an A_6 -bimodule.

We can use [Corollary 8.7](#) with $n = 6$ and $m = 3$, together with [Lemma 8.13](#) below, to conclude that the composition

$$\mathfrak{a} \coprod_M \mathfrak{b} \rightarrow \mathrm{Tw}_{\leq 1}(\mathfrak{a} \coprod_M \mathfrak{b}) \rightarrow \mathrm{Tw}_{\leq 1} \mathfrak{a} \coprod_{\mathrm{Tw}_{\leq 1} M} \mathrm{Tw}_{\leq 1} \mathfrak{b}$$

(which is equal to (Φ, I, Φ^*) by [Corollary 8.12](#)) is a quasi-equivalence. Since both functors are fully faithful (the second one by [Corollary 8.12](#)), the first one must be a quasi-equivalence as well.

Put $\mathfrak{c} = \mathfrak{a} \coprod_M \mathfrak{b}$. The claim about the exactness of $\varphi_{\mathfrak{a}}, \varphi_{\mathfrak{b}}$ follows from [Theorem 5.11](#). We clearly also have $H^*(\mathfrak{c})(H^*(\mathfrak{b}), H^*(\mathfrak{a})) = 0$. So to show that we have a semi-orthogonal decomposition as in [\(8.2\)](#) we have to show that every object C in $H^*(\mathfrak{c})$ is of the form $C \cong \mathrm{cone}(C_{\mathfrak{a}} \rightarrow C_{\mathfrak{b}})$ with $C_{\mathfrak{a}} \in \mathrm{Ob}(\mathfrak{a}), C_{\mathfrak{b}} \in \mathrm{Ob}(\mathfrak{b})$. Assume $C = (A \oplus B, \delta)$. We have a fully faithful functor $\mathfrak{a} \coprod_M \mathfrak{b} \subset \mathrm{Free} \mathfrak{a} \coprod_{\mathrm{Free} M} \mathrm{Free} \mathfrak{b}$, and the latter category is also pre-triangulated (as “Free” preserves A -ness). Again by [Theorem 5.11](#) this functor is exact. The following triangle

$$\Sigma^{-1} A \xrightarrow{\sigma\delta} B \xrightarrow{i} C \xrightarrow[(1)]{p} \Sigma^{-1} A$$

is distinguished in $H^*(\mathrm{Free} \mathfrak{a} \coprod_{\mathrm{Free} M} \mathrm{Free} \mathfrak{b})$ as it is trivially isomorphic to the standard triangle $\bar{\delta}_{\sigma\delta}$ in $H^*(\mathrm{Tw}_{\leq 1}(\mathrm{Free} \mathfrak{a} \coprod_{\mathrm{Free} M} \mathrm{Free} \mathfrak{b}))$. Choose $A' \in \mathrm{Ob}(\mathfrak{a})$ such that $A' \cong \Sigma^{-1} A$ in $\mathrm{Free} \mathfrak{a}$ (A' is a desuspension of A). Then by the axioms of triangulated categories we obtain $\mathrm{cone}(A' \rightarrow B) \cong C$ in $H^*(\mathrm{Free} \mathfrak{a} \coprod_{\mathrm{Free} M} \mathrm{Free} \mathfrak{b})$. By fully faithfulness this isomorphism also holds in $H^*(\mathfrak{a} \coprod_M \mathfrak{b})$.

The fact that the corresponding bimodule is as given is clear. \square

LEMMA 8.13. *Let \mathfrak{a} be an A_n -category. The strict $A_{\lfloor(n-1)/2\rfloor}$ functors $\Phi, \Phi^* : \mathfrak{a} \rightarrow \mathrm{Tw}_{\leq 1} \mathfrak{a}$ given by $\Phi(A) = (A \oplus 0, 0)$, $\Phi^*(A) = (0 \oplus A, 0)$ are homotopic.*

Proof. The homotopy h is such that $h_n = 0$ for $n \geq 1$ and h_0 is the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. \square

9. Higher Toda brackets in triangulated and A_{∞} -categories

9.1. Postnikov systems.

Let

$$(9.1) \quad X^{\bullet} : X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n$$

be a complex in a triangulated category \mathcal{T} , i.e., a sequence of composable morphisms in \mathcal{T} such that the composition of any two consecutive morphisms

is zero. A *Postnikov system* for X^\bullet is any exact diagram in \mathcal{T} of the form (9.2)

where the triangles marked with \circlearrowleft are commutative and the triangles marked with \mathbf{d} are distinguished. This means that we should have the distinguished triangles

$$(9.3) \quad Y_i \rightarrow X_{i+1} \rightarrow Y_{i+1} \xrightarrow{(1)} Y_i$$

with $X_0 = Y_0$. A Postnikov system need not exist, and if it exists, it may not be unique. If a Postnikov system exists, then the object Y_n will be called a *convolution* of X^\bullet .

Remark 9.1. Sometimes it is helpful to think of a convolution Y_n as an object with an ascending filtration with subquotients (starting from the bottom) $X_n, \Sigma X_{n-1}, \Sigma^2 X_{n-2}, \dots, \Sigma^n X_0$. In particular, the convolution Y_n comes with maps

$$(9.4) \quad \begin{array}{ccc} & Y_n & \\ p \swarrow & (n) & \uparrow i \\ X_0 & & X_n \end{array}$$

where i is as (9.2) and p is the composition $Y_n \rightarrow Y_{n-1} \rightarrow \cdots \rightarrow Y_0 = X_0$ in that same diagram. Note that $pi = 0$.

9.2. Existence. Some existence and functoriality results for Postnikov systems are stated in [10, Lemmas 1.5, 1.6] but since they require the vanishing of arbitrary negatives Ext's between suitable objects, they are not completely sufficient for our purposes. So we give some slightly strengthened versions in the next two sections.

LEMMA 9.2. *Assume X^\bullet is a complex in a triangulated category \mathcal{T} such that*

$$(9.5) \quad \mathcal{T}(X_a, X_b)_{-(b-a)+2} = 0 \quad \text{for } b \geq a + 3.$$

Then X^\bullet may be extended to a Postnikov system. Moreover if the condition

$$(9.6) \quad \mathcal{T}(X_a, X_b)_{-(b-a)+1} = 0 \quad \text{for } b \geq a+2$$

holds, then such an extension is unique, up to non-unique isomorphism.

Proof. The Postnikov system built on X^\bullet will be constructed inductively. Assume we have constructed the part involving $X_0, X_1, \dots, X_i, Y_0, Y_1, \dots, Y_i$ (so this is a Postnikov system on $X_0 \rightarrow \dots \rightarrow X_i$). To lift the map $X_i \rightarrow X_{i+1}$ to a map $Y_i \rightarrow X_{i+1}$ we need that the composition

$$Y_{i-1} \rightarrow X_i \rightarrow X_{i+1}$$

is zero. Since the composition of $X_{i-1} \rightarrow X_i \rightarrow X_{i+1}$ is zero by definition, it follows from (9.3) that it is sufficient to have $\mathcal{T}(Y_{i-2}, X_{i+1})_{-1} = 0$. Using Remark 9.1 we see that this condition is implied by (9.5).

Once we have lifted to $X_i \rightarrow X_{i+1}$ to $Y_i \rightarrow X_{i+1}$ we may construct Y_{i+1} via the distinguished triangle (9.3).

To obtain uniqueness we note that if X^\bullet can be extended to two Postnikov systems, then by Lemma 9.3 below the identity on X^\bullet can be extended to a morphism between these Postnikov systems. It is then easy to see that this extension must be an isomorphism. \square

9.3. Weak functoriality.

LEMMA 9.3. *Assume we have a morphism of complexes in a triangulated category \mathcal{T} ,*

$$(9.7) \quad \begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \dots \longrightarrow X_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'_0 & \longrightarrow & X'_1 & \longrightarrow & X'_2 & \longrightarrow & \dots \longrightarrow X'_n, \end{array}$$

such that there exist Postnikov systems for X^\bullet and $(X')^\bullet$ and the following conditions hold:

$$(9.8) \quad \mathcal{T}(X_a, X'_b)_{-(b-a)+1} = 0 \quad \text{for } b \geq a + 2.$$

Then, given a choice of Postnikov systems for X^\bullet and $(X')^\bullet$, the diagram (9.7) can be extended to a map of Postnikov systems (not necessarily uniquely).

Proof. We work inductively. Assume we have defined the extended map on Y_0, \dots, Y_i with the required commutativity holding on $Y_0, \dots, Y_i, X_0, \dots, X_i$. We perform the induction step. We have a diagram

$$(9.9) \quad \begin{array}{ccccccc} Y_i & \longrightarrow & X_{i+1} & \longrightarrow & Y_{i+1} & \xrightarrow{(1)} & Y_i \\ \downarrow & \swarrow \delta & \downarrow & & \downarrow & & \downarrow \\ Y'_i & \longrightarrow & X'_{i+1} & \longrightarrow & Y'_{i+1} & \xrightarrow{(1)} & Y'_i. \end{array}$$

We do not know that the left most square is commutative, so let the dotted arrow denote the difference of the two compositions. From the following

diagram,

$$\begin{array}{ccccc}
 X_i & \longrightarrow & Y_i & \longrightarrow & X_{i+1} \\
 \downarrow & & \downarrow & \searrow \delta & \downarrow \\
 X'_i & \longrightarrow & Y'_i & \longrightarrow & X'_{i+1},
 \end{array}$$

we obtain that the composition of δ with $X_i \rightarrow Y_i$ is zero. So in view of the distinguished triangle

$$Y_{i-1} \xrightarrow{(1)} X_i \rightarrow Y_i \longrightarrow Y_{i-1},$$

δ will be zero provided $\mathcal{T}(Y_{i-1}, X'_{i+1})_{-1} = 0$. This follows from [Remark 9.1](#) and the hypothesis [\(9.8\)](#).

So $\delta = 0$ and the square in [\(9.9\)](#) is commutative. We now finish by invoking [\(TR3\)](#). \square

9.4. Higher Toda brackets. In this section we define higher Toda brackets. One may verify that they are the same as those defined in [\[5\]](#).

Definition 9.4. Let $X^\bullet = ((X_i)_{i=0}^n, (d_i)_{i=0}^{n-1})$ for $n \geq 3$ be a complex in a triangulated category \mathcal{T} . The (higher) *Toda bracket* $\langle X^\bullet \rangle \subset \mathcal{T}(X_0, X_n)_{-n+2}$ of X^\bullet is the collection of compositions $\beta\alpha$ where α, β fit in the following commutative diagram:

$$\begin{array}{ccccc}
 (9.10) & & & & \\
 & \alpha & & & \beta \\
 & \nearrow (-n+2) & & & \searrow \\
 X_0 & \xrightarrow{d_0} & X_1 & \xrightarrow{p} & Y \xrightarrow{i} X_{n-1} \xrightarrow{d_{n-1}} X_n,
 \end{array}$$

where Y is a convolution of $(X_i)_{i=1}^{n-1}$ and p, i are as in [\(9.4\)](#).

Note that if $n > 3$, then $\langle X^\bullet \rangle$ may be empty.

THEOREM 9.5. *Let X^\bullet be as in [Definition 9.4](#).*

- (1) *If $t \in \langle X^\bullet \rangle$, then $t + d_{n-1}\mathcal{T}(X_0, X_{n-1})_{-n+2} + \mathcal{T}(X_1, X_n)_{-n+2}d_0 \subset \langle X^\bullet \rangle$.*
- (2) *If*

$$(9.11) \quad \mathcal{T}(X_a, X_b)_{-(b-a)+2} = 0 \quad \text{for } b-a \in [3, n-1],$$

then $\langle X^\bullet \rangle \neq \emptyset$.

- (3) *If, moreover,*

$$(9.12) \quad \mathcal{T}(X_a, X_b)_{-(b-a)+1} = 0 \quad \text{for } b-a \in [2, n-2],$$

then $\langle X^\bullet \rangle$ is a coset of $d_{n-1}\mathcal{T}(X_0, X_{n-1})_{-n+2} + \mathcal{T}(X_1, X_n)_{-n+2}d_0$.

Proof.

- (1) If $\phi \in \mathcal{T}(X_0, X_{n-1})_{-n+2}$, then as $pi\phi = 0$, adding to $i\phi$ to α still keeps the diagram (9.10) commutative. Since $\beta i\phi = d_{n-1}\phi$, we obtain that $t + d_{n-1}\phi \in \langle X^\bullet \rangle$. A similar reasoning applies if we start with $\phi \in \mathcal{T}(X_1, X_n)_{-n+2}$.
- (2) Note that (9.11) implies, in particular, (9.5) for $(X_i)_{i=1}^{n-1}$. So a convolution Y as in (9.10) exists, and we have to verify the existence of α and β . We will now introduce notation similar to [Section 9.1](#). So we will denote the Postnikov systems giving rise to Y by Y_1, \dots, Y_{n-1} , where $Y_{n-1} = Y$ and $Y_1 = X_1$.

We first consider the existence of β . We have a distinguished triangle

$$(9.13) \quad Y_{n-2} \rightarrow X_{n-1} \xrightarrow{i} Y_{n-1} \rightarrow .$$

Thus in order for the map $d_{n-1} : X_{n-1} \rightarrow X_n$ to factor through Y_{n-1} we have to prove that the composition $Y_{n-2} \rightarrow X_{n-1} \rightarrow X_n$ is zero. Since we already know that the composition $X_{n-2} \rightarrow Y_{n-2} \rightarrow X_{n-1} \rightarrow X_n$ is zero and there is a distinguished triangle

$$X_{n-2} \rightarrow Y_{n-2} \rightarrow \Sigma Y_{n-3} \rightarrow ,$$

it is sufficient to show that $\mathcal{T}(\Sigma Y_{n-3}, X_n)_0 = 0$. Now by [Remark 9.1](#), ΣY_{n-3} has subquotients $\Sigma X_{n-3}, \dots, \Sigma^{n-3} X_1$. The conclusion now follows from (9.11).

Now we look at the existence of α . We will successively lift $X_0 \xrightarrow{d_0} X_1 = Y_1$ to maps $X_0 \xrightarrow{(-1)} Y_2, \dots, X_0 \xrightarrow{(-n+2)} Y_{n-1}$. The last map is the sought α . First we look at the distinguished triangle

$$X_1 \rightarrow X_2 \rightarrow Y_2 \rightarrow .$$

Since the composition $X_0 \rightarrow X_1 \rightarrow X_2$ is zero, the map d_0 factors through $\Sigma^{-1} Y_2$. To continue we use the distinguished triangles

$$Y_{i-1} \rightarrow X_i \rightarrow Y_i \rightarrow$$

for $3 \leq i \leq n-1$. Assume we have constructed the map $X_0 \rightarrow \Sigma^{-i+2} Y_{i-1}$. From (9.11), we obtain that the composition $X_0 \rightarrow \Sigma^{-i+2} Y_{i-1} \rightarrow \Sigma^{-i+2} X_i$ is zero, and hence $X_0 \rightarrow \Sigma^{-i+2} Y_{i-1}$ factors through $\Sigma^{-i+1} Y_i$ and we can continue.

- (3) First we observe that (9.12) implies, in particular, (9.6) and hence the Postnikov system built on $(X_i)_{i=1}^{n-1}$ is unique. To prove the asserted statement we have to investigate the freedom in choosing α and β .

Again we will discuss β first. The map β is determined up to an element of the kernel of $\mathcal{T}(Y_{n-1}, X_n)_0 \rightarrow \mathcal{T}(X_{n-1}, X_n)_0$. Using the distinguished triangle (9.13) we see that β is determined up to a composition of the form $Y_{n-1} \rightarrow \Sigma Y_{n-2} \xrightarrow{\gamma} X_n$. Using [Remark 9.1](#) we see that ΣY_{n-2} has

subquotients $\Sigma X_{n-2}, \Sigma^2 X_{n-3}, \dots, \Sigma^{n-2} X_1$. Hence by (9.12) any morphism $\Sigma Y_{n-2} \rightarrow X_n$ factors through $\Sigma^{n-2} X_1$. It follows that β is determined up to a composition of the form $Y_{n-1} \xrightarrow{p} \Sigma^{n-2} X_1 \xrightarrow{\gamma'} X_n$. Composing with α we see as in (1) that changing β in this way changes $\beta\alpha$ by an element of $\mathcal{T}(X_1, X_n)_{-n+2} d_0$.

Now we discuss α , which is determined up to an element of

$$\ker(\mathcal{T}(X_0, Y_{n-1})_{-n+2} \rightarrow \mathcal{T}(X_0, X_1)_0).$$

Define $Y'_i = \Sigma^{-1} \text{cone}(Y_i \rightarrow \Sigma^{i-1} X_1)$ so that, in particular, $Y'_1 = 0, Y'_2 = X_2$. Using the octahedral axiom we may construct commutative diagrams for $i = 2, \dots, n-1$,

$$\begin{array}{ccccccc} & & & \uparrow & & & \\ & & & \Sigma Y'_{i-1} & \longrightarrow & \Sigma Y_{i-1} & \longrightarrow \Sigma^{i-1} X_1 \longrightarrow \\ & & & \uparrow & & \uparrow & \\ & & & Y'_i & \longrightarrow & Y_i & \longrightarrow \Sigma^{i-1} X_1 \longrightarrow \\ & & & \uparrow & & \uparrow & \\ & & & X_i & \xlongequal{\quad} & X_i & \end{array}$$

with rows and columns that are distinguished triangles, where the maps not involving Y' 's are taken from the Postnikov system. Hence similar to Remark 9.1, Y'_i has subquotients $X_i, \Sigma X_{i-1}, \dots, \Sigma^{i-2} X_2$.

We have a distinguished triangle

$$Y'_{n-1} \rightarrow Y_{n-1} \rightarrow \Sigma^{n-2} X_1 \rightarrow,$$

and hence α is determined up to a composition $X_0 \xrightarrow{\delta} \Sigma^{-n+2} Y'_{n-1} \rightarrow \Sigma^{-n+2} Y_{n-1}$. Now $\Sigma^{-n+2} Y'_{n-1}$ has subquotients $\Sigma^{-n+2} X_{n-1}, \dots, \Sigma^{-1} X_2$, and hence by (9.12) we obtain that any map $X_0 \xrightarrow{\delta} \Sigma^{-n+2} Y'_{n-1}$ factors through $\Sigma^{-n+2} X_{n-1}$. Hence we obtain that α is determined up to a composition $X_0 \xrightarrow{\delta'} \Sigma^{-n+2} X_{n-1} \rightarrow \Sigma^{-n+2} Y'_{n-1} \rightarrow \Sigma^{-n+2} Y_{n-1}$, which by construction is the same as a composition $X_0 \xrightarrow{\delta'} \Sigma^{-n+2} X_{n-1} \xrightarrow{\Sigma^{-n+2} i} Y_{n-1}$. We now finish as for β . \square

9.5. Postnikov systems associated to twisted complexes. In this section \mathfrak{a} is an A_∞ -category.

9.5.1. More on the category $\text{Free}(\mathfrak{a})$. Recall that in Section 4.6 we introduced the strict endo-functor Σ of $\text{Free}(\mathfrak{a})$. Below we introduce some more notation concerning the category $\text{Free}(\mathfrak{a})$. If $X \in \text{Ob}(\mathfrak{a})$, then we let $\eta_{X,a,b} :$

$\Sigma^a X \rightarrow \Sigma^b X$ be given by $\sigma^{b-a} \text{id}_X$. We similarly define $\eta_{X,a,b} : \Sigma^a X \rightarrow \Sigma^b X$ for $X \in \text{Free}(\mathfrak{a})$: for each summand $\Sigma^{x_i} X_i$ of X , $\eta_{X,a,b} : \Sigma^{a+x_i} X_i \rightarrow \Sigma^{b+x_i} X_i$ is given by $\sigma^{b-a} \text{id}_{X_i}$. Note that $m_2(\eta_{X,b,c}, \eta_{X,a,b}) = \eta_{X,a,c}$. All operations on \mathfrak{a} , except m_2 , vanish when one of its arguments is of the form $\eta_{X,b,c}$. Moreover we have formulas

$$\begin{aligned} m_n(\dots, f, m_2(\eta_{X,a,b}, g), \dots) &= m_n(\dots, m_2(f, \eta_{X,a,b}), g, \dots), \\ m_n(m_2(\eta_{X,a,b}, f), \dots) &= (-1)^{(-2+n)(b-a)} m_2(\eta_{X,a,b}, m_n(f, \dots)), \\ m_n(\dots, m_2(g, \eta_{X,a,b})) &= m_2(m_n(\dots, g), \eta_{X,a,b}), \end{aligned}$$

and their b -versions which are useful for computations:

$$\begin{aligned} b_n(\dots, sf, sm_2(\eta_{X,a,b}, g), \dots) &= (-1)^{b-a} b_n(\dots, sm_2(f, \eta_{X,a,b}), sg, \dots), \\ (9.14) \quad b_n(sm_2(\eta_{X,a,b}, f), \dots) &= m_2(\eta_{X,a,b}, b_n(sf, \dots)), \\ b_n(\dots, sm_2(g, \eta_{X,a,b})) &= m_2(b_n(\dots, sg), \eta_{X,a,b}). \end{aligned}$$

Below we usually write $\eta_{X,a,b}g$ for $m_2(\eta_{X,a,b}, g)$ and similarly $m_2(g, \eta_{X,a,b})$. By the vanishing of m_3 on arguments involving $\eta_{X,a,b}$ this will not lead to confusion. Sometimes we also write $\eta_{X,a,b}^{-1}$ for $\eta_{X,b,a}$. One verifies using the definition of the functor Σ (see [Section 4.6](#)) that for $f : \Sigma^a X \rightarrow \Sigma^b Z$, one has

$$(9.15) \quad \Sigma^n f = (-1)^{|f|} \eta_{Z,b,b+n} f \eta_{X,a+n,a}.$$

Finally we put $\eta_X = \eta_{X,0,1}$.

9.5.2. More on the triangulated structure of $\text{Tw } \mathfrak{a}$. Let $f : (A, \delta_A) \rightarrow (B, \delta_B)$ be a closed morphism of degree 0 in $\text{Tw } \mathfrak{a}$. To f we associate a triangle in $H^*(\text{Tw } \mathfrak{a})$,

$$(9.16) \quad (A, \delta_A) \xrightarrow{f} (B, \delta_B) \xrightarrow{i} (C(f), \delta_{C(f)}) \xrightarrow[\text{(1)}]{p} (A, \delta_A),$$

where $C(f) = \Sigma A \oplus B$ and

$$\delta_{C(f)} = \begin{pmatrix} \Sigma \delta_A & 0 \\ f \eta_A^{-1} & \delta_B \end{pmatrix}$$

and, furthermore,

$$(9.17) \quad i = \begin{pmatrix} 0 \\ \text{id}_B \end{pmatrix}, \quad p = (\eta_A^{-1} \ 0).$$

The following lemma is an easy verification:

LEMMA 9.6. *The triangles (9.16) are distinguished according to [Definition 5.10](#).*

9.5.3. Postnikov systems from objects in $\text{Tw } \mathfrak{a}$.

PROPOSITION 9.7. *A twisted complex in $\text{Tw } \mathfrak{a}$,*

$$Y_n = (\Sigma^n X_0 \oplus \Sigma^{n-1} X_1 \oplus \cdots \oplus X_n, \delta)$$

with $X_i \in \text{Free}(\mathfrak{a})$, gives rise to a Postnikov system in $H^*(\text{Tw } \mathfrak{a})$ built on the complex

$$X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} X_n$$

with

$$d_{j-1} = (-1)^{n-j} \eta_{X_j, n-j, 0} \cdot \delta_{j, j-1} \cdot \eta_{X_{j-1}, 0, n-j+1},$$

where $\delta_{j, j-1} : \Sigma^{n-j+1} X_{j-1} \rightarrow \Sigma^{n-j} X_j$ is the $(j, j-1)$ entry of the matrix δ ; the \cdot 's are for easier reading.

In the Postnikov system we also have

$$Y_j = (\Sigma^j X_0 \oplus \Sigma^{j-1} X_1 \oplus \cdots \oplus X_j, \delta_{Y_j})$$

such that $\Sigma^{n-j} \delta_{Y_i}$ is given by the upper left $j+1 \times j+1$ -square in the matrix representing δ .

Finally the maps $p : Y_n \xrightarrow{(n)} X_0$, $i : X_n \rightarrow Y_n$ as in (9.4) are given by

$$(9.18) \quad i = \begin{pmatrix} 0 \\ \vdots \\ \text{id}_{X_n} \end{pmatrix}, \quad p = (\eta_{X_0, n, 0}, 0, \dots, 0).$$

Proof. We may write

$$\delta_{Y_j} = \begin{pmatrix} \Sigma \delta_{Y_{j-1}} & 0 \\ f_j \eta_{Y_{j-1}}^{-1} & 0 \end{pmatrix},$$

where $f_j : (Y_{j-1}, \delta_{Y_{j-1}}) \rightarrow X_j$ is the closed map in $\text{Tw } \mathfrak{a}$ with matrix

$$((\delta_{Y_j})_{j, 0} \eta_{\Sigma^{j-1} X_0}, \dots, (\delta_{Y_j})_{j, j-1} \eta_{X_{j-1}}).$$

Clearly $Y_j = C(f_j)$ so that we have standard triangles

$$(9.19) \quad Y_{j-1} \xrightarrow{f_j} X_j \xrightarrow{i_j} Y_j \xrightarrow{(1)} Y_{j-1},$$

where (i_j, p_j) are as in (9.17). In particular, $i = i_n$ is given by the formula (9.18). We compute the composition

$$X_{j-1} \xrightarrow{i_{j-1}} Y_{j-1} \xrightarrow{f_j} X_j.$$

It is given by the matrix multiplications

$$((\delta_{Y_j})_{j,0}\eta_{\Sigma^{j-1}X_0}, \dots, (\delta_{Y_j})_{j,j-1}\eta_{X_{j-1}}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \text{id}_{X_{j-1}} \end{pmatrix} = (\delta_{Y_j})_{j,j-1}\eta_{X_{j-1}},$$

which is equal to $(\Sigma^{-(n-j)}(\delta_{Y_n})_{j,j-1})\eta_{X_{j-1}}$. One computes using (9.15) that the latter expression is equal to d_{j-1} .

Finally to show p is as in (9.18), we use $p = p_1 \cdots p_{n-1}p_n$ by the description in Remark 9.1. Then we use the formula (9.17) for p_j . \square

9.6. *Higher Toda brackets in A_∞ -categories.* We prove the following result:

THEOREM 9.8. *Let \mathfrak{a} be a pre-triangulated A_∞ -category, and let $X_0 \xrightarrow{d_0}$ $X_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} X_n$ be a complex in $\mathcal{T} = H^*(\mathfrak{a})$. Assume the following conditions hold:*

- (1) *The A_∞ -subcategory of \mathfrak{a} spanned by the objects $(X_i)_i$ is minimal (i.e., $b_1 = 0$).*
- (2) *$\mathfrak{a}(X_i, X_j)_u = 0$ for $-n+2 < u < 0$.*

Using (1) we may regard d_i as closed arrows in \mathfrak{a} . With this convention we have that $\langle X^\bullet \rangle$ is the coset for $d_{n-1}\mathcal{T}(X_0, X_{n-1})_{-n+2} + \mathcal{T}(X_1, X_n)_{-n+2}d_0$ given by $\overline{s^{-1}b_n(sd_{n-1}, \dots, sd_0)}$.

Proof. Since (9.11) and (9.12) hold, it is sufficient to produce a single element of $\langle X^\bullet \rangle$. Since higher Toda brackets are obviously invariant under equivalences of triangulated categories, we may perform the calculation in $\text{Tw } \mathfrak{a}$. We start with the Postnikov system built on $X_1 \xrightarrow{d_1} X_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-2}} X_{n-1}$. By Proposition 9.7 it is obtained from the twisted complex

$$Y = (\Sigma^{n-2}X_1 \oplus \cdots \oplus X_{n-1}, \delta),$$

where the only non-zero entries of δ are $\delta_{j,j-1}$ for $j = 2, \dots, n-1$ and $\delta_{j,j-1}$ is given by

$$\delta_{j,j-1} = (-1)^{n-1-j}\eta_{X_j, 0, n-1-j} \cdot d_{j-1} \cdot \eta_{X_{j-1}, n-j, 0}.$$

Using the formulas for i and p (see (9.18)) it is then easy to see that we may take

$$\alpha = \begin{pmatrix} \eta_{X_1, 0, n-2}d_0 \\ \vdots \\ 0 \end{pmatrix}, \quad \beta = (0, 0, \dots, d_{n-1}).$$

Then $\overline{m_{\text{Tw } \mathfrak{a}, 2}(\beta, \alpha)} \in \langle X^\bullet \rangle$. It will be more convenient to compute $b_{\text{Tw } \mathfrak{a}, 2}(s\beta, s\alpha) = sm_{\text{Tw } \mathfrak{a}, 2}(\beta, \alpha)$. We have

$$\begin{aligned} b_{\text{Tw } \mathfrak{a}, 2}(s\beta, s\alpha) &= b_{\mathfrak{a}, n}(s\beta, \underbrace{s\delta, \dots, s\delta}_{n-2}, s\alpha) \\ &= b_{\mathfrak{a}, n}(sd_{n-1}, sd_{n-2} \cdot \eta_{X_{n-2}, 1, 0}, \dots \\ &\quad \dots, (-1)^{n-1-j} s\eta_{X_j, 0, n-1-j} \cdot d_{j-1} \cdot \eta_{X_{j-1}, n-j, 0}, \dots \\ &\quad \dots, (-1)^{n-3} s\eta_{X_2, 0, n-3} \cdot d_1 \cdot \eta_{X_1, n-2, 0}, s\eta_{X_1, 0, n-2} d_0) \\ &= b_{\mathfrak{a}, n}(sd_{n-1}, sd_{n-2}, \dots, sd_1, sd_0), \end{aligned}$$

where in the last line we have used (9.14). \square

10. Triangulated categories without models

If \mathcal{A} is a triangulated category, then an A_∞ -enhancement on \mathcal{A} is a pair consisting of a pre-triangulated A_∞ -category \mathfrak{a} such that $\text{Ob}(\mathfrak{a}) = \text{Ob}(\mathcal{A})$ and an isomorphism of triangulated categories $H^*(\mathfrak{a}) \rightarrow \mathcal{A}$ inducing the identity on objects. The following proposition will be the basis for constructing a triangulated category that does not admit an A_∞ -enhancement.

PROPOSITION 10.1. *Let $\mathfrak{a}, \mathfrak{b}$ be pre-triangulated A_∞ -categories. Suppose we have an A_n -functor $F : \mathfrak{a} \rightarrow \mathfrak{b}$ for $n \geq 13$ such that $H^*(F)$ does not lift to an A_∞ -functor for any A_∞ -enhancements on $H^*(\mathfrak{a}), H^*(\mathfrak{b})$. Let \mathfrak{c} be the gluing category $\mathfrak{c} = \mathfrak{a} \coprod_M \mathfrak{b}$, where $M = {}_F \mathfrak{b}$ (see Section 8.1). Then $H^*(\mathfrak{c})$ is a triangulated category which does not admit an A_∞ -enhancement.*

Proof. By the discussion in Section 8.1, M is an A_{14} -bimodule. Therefore by Theorem 8.5, \mathfrak{c} is a pre-triangulated A_{13} -category. Hence by Theorem 7.3, $H^*(\mathfrak{c})$ is triangulated.

Suppose that an A_∞ -enhancement \mathfrak{d} on $H^*(\mathfrak{c})$ exists. Since $H^*(\mathfrak{a}), H^*(\mathfrak{b})$ are full exact subcategories of $H^*(\mathfrak{c})$ (see Theorem 8.5), it follows that the A_∞ -structure on \mathfrak{d} induces A_∞ -enhancements $\mathfrak{a}', \mathfrak{b}'$, on $H^*(\mathfrak{a})$ and $H^*(\mathfrak{b})$. By $H^*(\mathfrak{d}) \cong H^*(\mathfrak{c})$ it follows that $\mathfrak{d}(A, B)_{A, B}$ for $A \in \text{Ob}(H^*(\mathfrak{a})), B \in \text{Ob}(H^*(\mathfrak{b}))$ defines an A_∞ - \mathfrak{b}' - \mathfrak{a}' -bimodule which is a co-quasi-functor in the sense of Section 10.1 below. Hence by Lemma 10.4, \mathfrak{d} induces an A_∞ -functor $F' : \mathfrak{a}' \rightarrow \mathfrak{b}'$ such that $H^*(F') \cong F$. This contradicts the hypotheses on F . \square

Remark 10.2. The idea of creating a triangulated category without model by gluing a non-enhanceable functor was suggested to us by Bondal and Orlov on a number of occasions. In fact, the idea of translating an enhancement of the glued category into an A_∞ -enhancement of the gluing functor, thereby obtaining a contradiction, was specifically suggested to us by Orlov.

10.1. *Co-quasi-functors.* To fill in a missing ingredient in the proof of [Proposition 10.1](#) we use an A_∞ -version of the notion of a (co)-quasi-functor (see [\[6\]](#)). In the rest of this section we assume that $\mathfrak{a}, \mathfrak{b}$ are A_∞ -categories.

Definition 10.3. An A_∞ - \mathfrak{b} - \mathfrak{a} -bimodule M is a *co-quasi-functor* $\mathfrak{a} \rightarrow \mathfrak{b}$ if for every object $A \in \text{Ob}(\mathfrak{a})$, there exists $fA \in \text{Ob}(\mathfrak{b})$ together with an element $\bar{\phi}_A \in (H^*M)(A, fA)_0$ inducing an isomorphism for all $B \in \text{Ob}(\mathfrak{b})$: $\tilde{\phi}_A : H^*(\mathfrak{b})(fA, B) \rightarrow (H^*M)(A, B) : u \mapsto u\bar{\phi}_A$.

It is clear from the definition that being a co-quasi-functor depends only on the structure of H^*M as graded $H^*(\mathfrak{a}) - H^*(\mathfrak{b})$ -bimodule. A co-quasi-functor induces an actual functor $f^\circ : H^*(\mathfrak{a}) \rightarrow H^*(\mathfrak{b})$. Indeed for $u : A \rightarrow A'$ in $H^*(\mathfrak{a})$, $f^\circ u : fA \rightarrow fA'$ is defined to be the unique morphism such that $H^*(\mathfrak{b})(f^\circ u, -)$ is the composition

$$H^*(\mathfrak{b})(fA', -) \xrightarrow[\cong]{\tilde{\phi}_{A'}} (H^*M)(A', -) \xrightarrow{H^*M(f, -)} (H^*M)(A, -) \xrightarrow[\cong]{\tilde{\phi}_A^{-1}} H^*(\mathfrak{b})(fA, -).$$

Moreover it is clear that different choices of $(\phi_A, f^\circ A)$ lead to naturally isomorphic functors.

LEMMA 10.4. *Assume that M is a co-quasi-functor $\mathfrak{a} \rightarrow \mathfrak{b}$, and let $f^\circ : H^*(\mathfrak{a}) \rightarrow H^*(\mathfrak{b})$ be the induced functor as explained above. Then there exists an A_∞ -functor $f : \mathfrak{a} \rightarrow \mathfrak{b}$ such that $H^*(f) = f^\circ$.*

Proof. Let $\mathcal{C}_\infty^l(\mathfrak{b})$ be the DG-category of strictly unital left A_∞ - \mathfrak{b} -modules [\[7, Ch. 5\]](#), and let $Y : \mathfrak{b} \rightarrow \mathcal{C}_\infty^l(\mathfrak{b})^\circ : B \mapsto \mathfrak{b}(B, -)$ be the Yoneda embedding. Furthermore let $\tilde{\mathfrak{b}} \subset \mathcal{C}_\infty^l(\mathfrak{b})^\circ$ be the full subcategory spanned by A_∞ -modules M which are A_∞ -quasi-isomorphic to some \mathfrak{b} -module of the form $\mathfrak{b}(B, -)$. Clearly we have that Y corestricts to an A_∞ -quasi-equivalence $Y^c : \mathfrak{b} \rightarrow \tilde{\mathfrak{b}}$. Since M is a co-quasi-functor, the image of the A_∞ -functor $F : \mathfrak{a} \rightarrow \mathcal{C}_\infty^l(\mathfrak{b})^\circ : A \mapsto M(A, -)$ lies in $\tilde{\mathfrak{b}}$. Let $F^c : \mathfrak{a} \rightarrow \tilde{\mathfrak{b}}$ be the corestriction of F .

Choose an A_∞ -quasi-inverse $W : \tilde{\mathfrak{b}} \rightarrow \mathfrak{b}$ to the quasi-equivalence $Y^c : \mathfrak{b} \rightarrow \tilde{\mathfrak{b}}$ which sends $M(A, -)$ to fA for $A \in \text{Ob} \mathfrak{a}$ and u to (a representative of) $f^\circ u$ for $u : A \rightarrow A'$ a closed map in \mathfrak{a} . By [Lemma 4.10](#) this is possible. Then one easily verifies that $f^\circ = H^*(F^c W)$. \square

Remark 10.5. It is also easy to prove that we have a quasi-isomorphism of A_∞ -bimodules ${}_f\mathfrak{b} \cong M$. However we will not need this.

10.2. *Localization of triangulated categories.* The following result is well known, although we did not find the precise statement we require. Since the proof is short, we include it for the convenience of the reader.

PROPOSITION 10.6. *Let \mathcal{T} be a triangulated category admitting arbitrary coproducts, and let $T \in \text{Ob}(\mathcal{T})$ be a compact generator for \mathcal{T} . Let $S \subset \mathcal{T}(T, T)$*

be a graded right Ore set, and let \mathcal{T}_S be the full subcategory of \mathcal{T} spanned by the objects X such that $\mathcal{T}(s, X)$ is an isomorphism for all $s \in S$, or equivalently the objects for which

$$(10.1) \quad \mathcal{T}(T, X) \rightarrow \mathcal{T}(T, X)_S$$

is an isomorphism. Then \mathcal{T}_S is a triangulated subcategory of \mathcal{T} and moreover the inclusion functor $\mathcal{T}_S \rightarrow \mathcal{T}$ has a left adjoint, denoted by $(-)_S$ such that for $Y \in \text{Ob}(\mathcal{T})$, the induced map

$$(-)_S : \mathcal{T}(T, Y) \rightarrow \mathcal{T}_S(T_S, Y_S)$$

factors uniquely through an isomorphism

$$(10.2) \quad \mathcal{T}(T, Y)_S \cong \mathcal{T}_S(T_S, Y_S).$$

Proof. The fact that \mathcal{T}_S is triangulated follows trivially from the 5-lemma. Let us now discuss the existence of the adjoint. Let \mathcal{C} be the full subcategory of \mathcal{T} spanned by objects X such that all morphisms $T \rightarrow X$ (not necessarily of degree zero) are annihilated after composing with some $s : T \rightarrow T \in S$, or equivalently

$$(10.3) \quad \mathcal{T}(T, X)_S = 0.$$

It is clear that \mathcal{C} is triangulated and closed under arbitrary coproducts (the latter by the compactness of T).

For $s \in S$, let $C(s)$ be the cone of the morphism $s : T \rightarrow \Sigma^{|s|}T$. It is clear that $\mathcal{T}_S = \langle C(s)_{s \in S} \rangle^\perp$. By the Ore condition on S the objects $C(s)$ are in \mathcal{C} . Moreover as $\langle C(s)_{s \in S} \rangle^\perp \cap \mathcal{C} = \mathcal{T}_S \cap \mathcal{C}$ and it is easy to see that $\mathcal{T}_S \cap \mathcal{C} = 0$, we obtain that \mathcal{C} is in fact generated by $\langle C(s)_{s \in S} \rangle$. This yields $\mathcal{C}^\perp = \mathcal{T}_S$.

Hence, in particular, \mathcal{C} is compactly generated, and using the Brown representability theorem we obtain that the inclusion functor $\mathcal{C} \rightarrow \mathcal{T}$ has a right adjoint $U : \mathcal{T} \rightarrow \mathcal{C}$ such that every $X \in \mathcal{T}$ fits in a unique distinguished triangle

$$(10.4) \quad UX \rightarrow X \rightarrow VX \rightarrow,$$

where $VX \in \mathcal{C}^\perp = \mathcal{T}_S$. It follows easily that $X \rightarrow VX$ is a functor $\mathcal{T} \rightarrow \mathcal{C}^\perp = \mathcal{T}_S$. Applying $\mathcal{T}(-, Z)$ for $Z \in \mathcal{T}_S$ to (10.4) we obtain that V is the sought left adjoint $(-)_S$ to the inclusion $\mathcal{T}_S \rightarrow \mathcal{T}$.

Finally we discuss the formula (10.2). As $\text{cone}(Y \rightarrow Y_S) = \Sigma UY \in \mathcal{C}$ we have $\mathcal{T}(T, \text{cone}(Y \rightarrow Y_S))_S = 0$ by (10.3). Hence $(-)_S$ induces an isomorphism $\mathcal{T}(T, Y)_S \xrightarrow{\cong} \mathcal{T}(T, Y_S)_S \xrightarrow{(10.1)} \mathcal{T}(T, Y_S) = \mathcal{T}_S(T_S, Y_S)$ where the last equality is adjointness. \square

10.3. *A non-enhanceable functor.* Now let k be either a field of characteristic zero or an infinite field of characteristic $> n \geq 3$. Put $R = k[x_1, \dots, x_n]$, and let K be the quotient field of R . Furthermore let $R[\varepsilon]$ be the R -linear DG-algebra with $|\varepsilon| = -n + 2$, $\varepsilon^2 = 0$, $d\varepsilon = 0$. Let $C(R, R)$ be the Hochschild complex of R , and let $\mathrm{HH}^n(R, R) = H^n(C(R, R))$. Let $T_{R/k}^n = \wedge_R^n \mathrm{Der}_k(R, R)$. The HKR theorem gives an inclusion $T_{R/k}^n \subset Z^n C(R, R)$ which induces an isomorphism $T_{R/k}^n \cong \mathrm{HH}^n(K, K)$. For $\eta \in T_{R/k}^n$, we let R_η be the $k[\varepsilon]$ -linear A_∞ -deformation of $R[\varepsilon]$ whose only higher multiplication is given by $\varepsilon\eta$.

As above, for an A_∞ -algebra A , let $\mathcal{C}_\infty^r(A)$ be the DG-category of strictly unital right A_∞ -modules over A [7]. We put $D(A) = H^*(\mathcal{C}_\infty^r(A))$. This is one of the many realizations for the derived category of an A_∞ -algebra (see [7, Th. 4.1.3.1(D2)]) for which we consider $\mathcal{C}_\infty^r(A)$ to be its standard enhancement.

Remark 10.7. A is an A - A -bimodule, and hence the left A -action on A defines A_∞ -quasi-isomorphism (see [7, Lemma 5.3.0.1]) $A \rightarrow \mathcal{C}_\infty^r(A)(A, A)$ which is however not an isomorphism.

PROPOSITION 10.8. *Assume \mathcal{T} is a triangulated category with arbitrary coproducts and T is compact generator of \mathcal{T} such that $\mathcal{T}(T, T) = R[\varepsilon]$. Assume \mathfrak{a} is some A_∞ -enhancement of \mathcal{T} . Then there is an A_∞ -quasi-equivalence $\mathfrak{a} \cong \mathcal{C}_\infty^r(R_\eta)$ for a suitable $\eta \in T_{R/k}^n$ which sends T to an object isomorphic to R_η in $H^*(\mathcal{C}_\infty^r(R_\eta)) = D(R_\eta)$ such that the induced map $R[\varepsilon] = \mathcal{T}(T, T) \cong D(R_\eta)(R_\eta, R_\eta) = R[\varepsilon]$ is the identity. Moreover η is uniquely determined by the triangulated structure on \mathcal{T} and, in particular, is independent of the chosen quasi-equivalence.*

Proof. Let $R = \mathfrak{a}(T, T)$. By [6, §4.3] as formulated in the work of Porta [11], the A_∞ -functor

$$Y : \mathfrak{a} \rightarrow \mathcal{C}_\infty^r(R) : X \mapsto \mathfrak{a}(T, X)$$

is a quasi-equivalence which sends T to R . Indeed $\mathcal{C}_\infty^r(R)$ is pre-triangulated and so is \mathfrak{a} by the definition of enhancement. So $H^*(Y)$ is exact. Since the essential image of $H^*(Y)$ contains a generator of $H^*(\mathcal{C}_\infty^r(R))$ (namely, R), it is sufficient to show that $L := H^*(Y)$ is fully faithful. By the Brown representability theorem L has a right adjoint R which moreover commutes with coproducts. (This follows from the fact that L send the compact generator T to the compact object R .) Hence the full subcategory of $H^*(\mathfrak{a})$ spanned by objects X such that $X \rightarrow RLX$ is an isomorphism is closed under shifts, cones, summands and arbitrary coproducts. Moreover applying $H^*(\mathfrak{a})(T, -)$ we see that it contains T . Hence it must be $H^*(\mathfrak{a})$ itself. From this one deduces that L is fully faithful.

Now \mathbf{R} is a DG-algebra with cohomology $R[\varepsilon]$, so it is A_∞ -isomorphic to a minimal A_∞ -structure on $R[\varepsilon]$ with m_2 being the usual multiplication. For degree reasons, the only such A_∞ -structures are (up to A_∞ -isomorphism) of the form R_η . Hence after choosing an A_∞ -quasi-isomorphism $R_\eta \rightarrow \mathbf{R}$ we obtain a quasi-equivalence $\mathcal{C}_\infty^r(\mathbf{R}) \rightarrow \mathcal{C}_\infty^r(R_\eta)$ which sends \mathbf{R} to an object quasi-isomorphic to R_η in a way which induces the identity on cohomology. Composing with Y completes the proof of the first part of the proposition.

For $\lambda \in k^n$, let K_λ^\bullet be the R -Koszul complex on $(x_1 - \lambda_1, \dots, x_n - \lambda_n)$. This is a resolution of $R_\lambda := R/((x_i - \lambda_i)_i)$. Put $K_{\lambda,T}^\bullet = K_\lambda^\bullet \otimes_R T$. This is a complex in \mathcal{T} . Conditions (9.11) and (9.12) hold for $K_{\lambda,T}^\bullet$, and hence the higher Toda bracket $\langle K_{\lambda,T}^\bullet \rangle$ is a coset of $\sum_i \mathcal{T}(T, T)_{-n+2}(x_i - \lambda_i)$ in $\mathcal{T}(T, T)_{-n+2} = R\varepsilon$. We define $\eta_{\lambda,T} \in R_\lambda$ such that $\eta_{\lambda,T}\varepsilon$ is the sole element of the image of $\langle K_{\lambda,T}^\bullet \rangle$ in R_λ .

By the constructed quasi-equivalence we have $\eta_{\lambda,T} = \eta_{\lambda,R_\eta}$. Alas we cannot immediately apply [Theorem 9.8](#) to the right-hand side of this equality as the A_∞ -category spanned by the terms of the complex $K_{\lambda,R_\eta}^\bullet$ (finite direct sums of R_η) is not minimal (see [Remark 10.7](#)). To work around this let $\mathbf{S} = \mathcal{C}_\infty^r(R_\eta)(R_\eta, R_\eta)$, which we regard as a one object A_∞ -category (\mathbf{S}, \bullet) . As in [Remark 10.7](#) we obtain an A_∞ -quasi-isomorphism $R_\eta \rightarrow \mathbf{S}$. Composing with $(\mathbf{S}, \bullet) \rightarrow \mathcal{C}_\infty^r(R_\eta) : \bullet \mapsto R_\eta$ we obtain a quasi-fully faithful A_∞ -functor $(R_\eta, \bullet) \rightarrow \mathcal{C}_\infty^r(R_\eta) : \bullet \mapsto R_\eta$ which gives rise to a quasi-fully faithful A_∞ -functor

$$\mathrm{Tw} R_\eta \rightarrow \mathrm{Tw} \mathcal{C}_\infty^r(R_\eta) \cong \mathcal{C}_\infty^r(R_\eta)$$

which sends $K_{\lambda,R_\eta}^\bullet \in \mathrm{Tw} R_\eta$ to $K_{\lambda,R_\eta}^\bullet \in \mathcal{C}_\infty^r(R_\eta)$. It follows that we may perform the calculation of η_{λ,R_η} in $\mathrm{Tw} R_\eta$. As the A_∞ -subcategory of $\mathrm{Tw} R_\eta$ spanned by direct sums of R_η is minimal, we are now in a position to apply [Theorem 9.8](#). We obtain that, up to a global sign, η_{λ,R_η} is the image of $\sum_{\sigma \in S_n} (-1)^\sigma \eta(x_{\sigma(1)} - \lambda_{\sigma(1)}, \dots, x_{\sigma(n)} - \lambda_{\sigma(n)})$ in R_λ . Since $T_{R/k}^n = R \bigwedge_i \partial/\partial x_i$, this is the same as the image of $n! \eta(x_1, \dots, x_n)$. We obtain by varying λ that η is uniquely determined. \square

THEOREM 10.9. *Choose $0 \neq \eta \in T_{R/k}^n$, and put $\mathbf{a} = \mathcal{C}_\infty^r(K)$, $\mathbf{b} = \mathcal{C}_\infty^r(R_\eta)$. After extending η to $T_{K/k}^n = T_{R/k}^n \otimes_R K$, we consider K_η as an object in \mathbf{b} . There is an A_{n-1} -functor*

$$F : \mathbf{a} \rightarrow \mathbf{b}$$

which sends K to K_η . The corresponding functor

$$H^*(F) : D(K) \rightarrow D(R_\eta)$$

does not lift to an A_n -functor, even after changing the enhancements on $D(K)$ and $D(R_\eta)$.

Before giving the proof of this theorem we show that it implies [Theorem 1.1](#) in the introduction.

Proof of Theorem 1.1. From [Theorem 10.9](#) we obtain that the hypotheses of [Proposition 10.1](#) are satisfied for $(\mathfrak{a}, \mathfrak{b}, F)$ (with n replaced by $n - 1$). Thus for $n \geq 14$, we obtain a triangulated category $\mathcal{D} = H^*(\mathfrak{a} \coprod_{F\mathfrak{b}} \mathfrak{b})$ without A_∞ -enhancement with a semi-orthogonal decomposition

$$\mathcal{D} = \langle D(K), D(R_\eta) \rangle. \quad \square$$

Proof of Theorem 10.9. We first discuss the construction of the functor F . To be compatible with [Propositions 10.6](#) and [10.8](#), put $\mathcal{T} = D(R_\eta) = H^*(\mathcal{C}_\infty^r(R_\eta))$ and let T be the object R_η . Put $S = R - \{0\}$. It is easy to see that $T_S = K_\eta$. Indeed K_η is in \mathcal{T}_S and $\text{cone}(R_\eta \rightarrow K_\eta)$ is in \mathcal{C} by [\(10.3\)](#). In particular, it follows by [\(10.2\)](#) that $\mathcal{T}(K_\eta, K_\eta) = K[\varepsilon]$.

Choosing homotopies we obtain an A_2 -functor

$$(10.5) \quad F : K \rightarrow \mathcal{C}_\infty^r(R_\eta) : K \mapsto K_\eta,$$

and the obstructions against extending μ to an A_i -functor are in

$$\text{HH}^j(K, \mathcal{T}(K_\eta, K_\eta)_{-j+2})$$

for $3 \leq j \leq i$; see, e.g., [\[12, Lemma 7.2.1\]](#). Since $\mathcal{T}(K_\eta, K_\eta) = K[\varepsilon]$, the obstructions vanish for $j < n$. So F extends to an A_{n-1} -functor. Let $\text{Free}^\sim(-)$ be defined as $\text{Free}(-)$ but allowing arbitrary formal direct sums. If \mathfrak{a} is an A_n -category, then so is $\text{Free}^\sim(\mathfrak{a})$, and a similar statement is true for functors. We then obtain an A_{n-1} -functor $\text{Free}^\sim(F) : \text{Free}^\sim(K) \rightarrow \text{Free}^\sim(\mathcal{C}_\infty^r(R_\eta))$. Since $\text{Free}^\sim(K)$ is quasi-equivalent to $\mathcal{C}_\infty(K)$ (both are models for $D(K)$ which is semi-simple) and the direct sum defines an A_∞ -functor $\text{Free}^\sim(\mathcal{C}_\infty^r(R_\eta)) \rightarrow \mathcal{C}_\infty^r(R_\eta)$, after choosing a suitable A_∞ -quasi-inverse to the first functor we obtain the sought A_{n-1} -functor $F : \mathcal{C}_\infty^r(K) \rightarrow \mathcal{C}_\infty^r(R_\eta)$ which sends K to K_η .

We claim that F does not lift to an A_n -functor, even if we change enhancements. If it did, the A_2 -functor [\(10.5\)](#) would also lift to an A_n -functor, as by [Proposition 10.8](#) the enhancement on $\mathcal{C}_\infty^r(R_\eta)$ is (weakly) unique and (as we have shown in the first paragraph) the object K_η is determined by the triangulated structure. If this were possible, then it would induce the structure of an A_n -functor on the corestriction

$$K \xrightarrow{\mu} \mathfrak{c} \subset \mathcal{C}_\infty^r(R_\eta),$$

where \mathfrak{c} is the full subcategory of $\mathcal{C}_\infty^r(R_\eta)$ spanned by the single object K_η . Put $\mathsf{K} = \mathcal{C}_\infty^r(R_\eta)(K_\eta, K_\eta)$. Since K_η is an K_η - R_η -bimodule, the left K_η -action on K_η induces an A_∞ -quasi-isomorphism $K_\eta \rightarrow \mathsf{K} = \mathfrak{c}$. Taking an A_∞ -quasi-inverse and composing with $K \xrightarrow{\mu} \mathfrak{c}$ we obtain an A_n -morphism $K \rightarrow K_\eta$ such that $H^*(K) \rightarrow H^*(K_\eta) = K[\varepsilon]$ is the natural inclusion. Such an A_n -morphism does not exist as $\eta \neq 0$ [\[7, Ch. B\]](#). \square

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