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Minimal surfaces and the Allen–Cahn equation on 3-manifolds: index, multiplicity, and curvature estimates

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Abstract

The Allen–Cahn equation is a semilinear PDE which is deeply linked to the theory of minimal hypersurfaces via a singular limit. We prove *curvature estimates* and strong *sheet separation estimates* for stable solutions (building on recent work of Wang–Wei) of the Allen–Cahn equation on a 3-manifold. Using these, we are able to show that for generic metrics on a 3-manifold, minimal surfaces arising from Allen–Cahn solutions with bounded energy and bounded Morse index are two-sided and occur with multiplicity one and the expected Morse index. This confirms, in the Allen–Cahn setting, a strong form of the *multiplicity one-conjecture* and the *index lower bound conjecture* of Marques–Neves in 3-dimensions regarding min-max constructions of minimal surfaces.

Allen–Cahn min-max constructions were recently carried out by Guaraco and Gaspar–Guaraco. Our resolution of the multiplicity-one and the index lower bound conjectures shows that these constructions can be applied to give a new proof of *Yau's conjecture on infinitely many minimal surfaces* in a 3-manifold with a generic metric (recently proven by Irie–Marques–Neves) with *new* geometric conclusions. Namely, we prove that a 3-manifold with a generic metric contains, for every $p = 1, 2, 3, \ldots$, a two-sided embedded minimal surface with Morse index p and area $\sim p^{\frac{1}{3}}$, as conjectured by Marques–Neves.

Contents

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1. **Introduction**

Minimal surfaces—critical points of the area functional with respect to local deformations—are fundamental objects in Riemannian geometry due to their intrinsic interest and richness, as well as deep and surprising applications to the study of other geometric problems. Because many manifolds do not contain *any* area-minimizing hypersurfaces, one is quickly led to the study of surfaces that are only critical points of the area functional. Such surfaces are naturally constructed by min-max (i.e., mountain-pass) type methods. To this end, Almgren and Pitts [Pit81] have developed a far-reaching theory of existence and regularity (cf. [SS81]) of min-max (unstable) minimal hypersurfaces. In particular, their work implies that any closed Riemannian manifold (M^n, g) contains at least one minimal hypersurface Σ^{n-1}. (In sufficiently high dimensions, Σ may have a thin singular set.) This result motivates a well-known question of Yau: "do all 3-manifolds contain infinitely many immersed minimal surfaces?" [Yau82].

Recently, there have been several amazing applications of Almgren–Pitts theory to geometric problems, including the proof of the Willmore conjecture by Marques–Neves [MN14] and the resolution of Yau's conjecture for generic metrics in dimensions 3 through 7 by Irie–Marques–Neves [IMN18]. In spite of this, certain basic questions concerning the Almgren–Pitts construction remain unresolved, including whether or not the limiting minimal surfaces can arise with multiplicity (for a generic metric) as well as whether or not one-sided minimal surfaces can arise as limits of an "oriented" min-max sequence (see, however, [KMN16], [MN16a]).[1]

Guaraco [Gua18] has proposed an alternative to Almgren–Pitts theory, later extended by Gaspar–Guaraco [GG18], which is based on study of a semi-linear PDE known as the Allen–Cahn equation

$$\varepsilon^2 \Delta_g u = W'(u) \tag{1.1}$$

and its singular limit as $\varepsilon \searrow 0$. There is a well-known expectation that, in $\varepsilon \searrow 0$ limit, solutions to (1.1) produce minimal surfaces whose regularity reflects the solutions' variational properties. In particular,

(1) It is known that the Allen–Cahn functional Γ-converges to the perimeter functional [Mod87], [Ste88], so minimizing solutions to (1.1) converge as $\varepsilon \searrow 0$ to minimizing hypersurfaces (and are thus regular away from a codimension 7 singular set).

(2) Under weaker assumptions on the sequence of solutions, one obtains different results. In general, solutions to (1.1) on a Riemannian manifold (M^n, g) have a naturally associated $(n-1)$-varifold obtained by "smearing out" their level sets of u, weighted by the gradient,

$$V[u](\varphi) \triangleq h_0^{-1} \int \varphi(x, T_x\{u = u(x)\})\, \varepsilon |\nabla u(x)|^2 \, d\mu_g(x), \ \varphi \in C_c^0(\mathrm{Gr}_{n-1}(M)).$$

Here, $h_0 > 0$ is a constant that is canonically associated with W (see Section 1.3). A deep result of Hutchinson–Tonegawa [HT00, Th. 1] ensures that V limits to a varifold with almost every integer density as $\varepsilon \searrow 0$. If, in addition, one assumes that the solutions are stable, Tonegawa–Wickramasekera [TW12] have shown that the limiting varifold is stable and satisfies the conditions of Wickramasekera's deep regularity theory [Wic14]; thus the limiting varifold is a smooth stable minimal hypersurface (outside of a codimension 7 singular set). In two dimensions, this was shown by Tonegawa [Ton05].

[1]Added in proof: There has been dramatic progress in Almgren–Pitts theory since we first posted this article. In particular, we note that A. Song [Son18] has proved the full Yau conjecture in dimensions 3 through 7, and X. Zhou [Zho19] proved the multiplicity-one conjecture in the Almgren–Pitts setting, also in dimensions 3 through 7.

Guaraco's approach has certain advantages when compared with Almgren–Pitts theory:

(1) A key difficulty in the work of Almgren–Pitts is a lack of a Palais–Smale condition, which is usually fundamental in mountain pass constructions. On the other hand, the Allen–Cahn equation does satisfy the usual Palais–Smale condition for each $\varepsilon > 0$ (see [Gua18, Prop. 4.4]), so this aspect of the theory is much simpler.

We note, however, that the bulk of the regularity theory in Guaraco's work is applied *after* taking the limit $\varepsilon \searrow 0$ and thus relies on the deep works of Wickaramsekera [Wic14] and Tonegawa–Wickramasekera [TW12]. This places a more serious burden on regularity theory than Almgren–Pitts.

(2) In Almgren–Pitts theory, there is no "canonical" approximation of the limiting min-max surface by nearby elements of a sweepout. On the other hand, Allen–Cahn provides a canonical approximation built out of the function u (which satisfies a PDE). It is thus natural to suspect that this might be useful when studying the geometric properties of the limiting surface.

For example, Hiesmayr [Hie18] and Gaspar [Gas17] have shown that index upper bounds for Allen–Cahn solutions directly pass to the limiting surface. (We note that the Almgren–Pitts version of this result has been proven by Marques–Neves [MN16a]). Moreover, the second-named author has recently shown [Man17] that one-parameter Allen–Cahn min-max on a surface produces a smooth immersed curve with at most one point of self-intersection; in general, Almgren–Pitts on a surface will only produce a geodesic net (cf. [Aie19]).

Our main contributions in this work are as follows:

(1) We show (see Theorem 1.3 below) that the individual level sets of stable solutions to the Allen–Cahn equation on a 3-manifold with energy bounds satisfy a priori curvature estimates (similar to stable minimal surfaces). Using this, we can avoid the regularity theory of Wickramasekera and Tonegawa–Wickramasekera entirely, making the whole theory considerably more self-contained.

(2) More fundamentally, our curvature estimates (and strong sheet separation estimates, which we will discuss below) allow us to study geometric properties of the limiting minimal surface using the "canonical" PDE approximations that exist *prior* to taking the $\varepsilon \searrow 0$ limit. In particular, we will prove the multiplicity-one conjecture of Marques–Neves [MN16a] in the Allen–Cahn setting (see Theorem 1.7 below) for min-max sequences on 3-manifolds. In fact, we prove a strengthened version of the conjecture by ruling out (generically) stable components and one-sided surfaces.

As an application of our multiplicity-one results we are able to give a new proof of Yau's conjecture on infinitely many minimal surfaces in a 3-manifold, when the metric is bumpy (see Corollary 1.10 below). This has been recently proven using Almgren–Pitts theory[2] by Irie–Marques–Neves [IMN18] for a slightly different class of metrics; their proof works in (M^n, g) for $3 \leq n \leq 7$ and proves, in addition, that the minimal surfaces are dense. Our proof establishes several new geometric properties of the surfaces; in particular, we show that they are two-sided and that their area and Morse index behave as one would expect, based on the theory of p-widths [Gro03], [Gut09], [MN17], [GG18].

We wish to emphasize two things:

(1) Our results work at the level of sequences of critical points of the Allen–Cahn energy functional with uniform energy and Morse index bounds. At no point do we use any min-max characterization of the limiting surface; min-max is merely used as a tool to construct non-trivial sequences of critical points with energy and index bounds.
(2) Our results highlight the philosophy that the solutions to Allen–Cahn provide a "canonical" approximation of the min-max surfaces.

1.1. *Notation.* In all that follows, (M^n, g) is a smooth Riemannian manifold.

Definition 1.1. A function $W \in C^\infty(\mathbf{R})$ is a *double-well potential* if

(1) W is non-negative and vanishes precisely at ± 1;
(2) W satisfies $W'(0) = 0$, $tW'(t) < 0$ for $|t| \in (0,1)$, and $W''(0) \neq 0$;
(3) $W''(\pm 1) = 2$;
(4) $W(t) = W(-t)$.

The standard double-well potential is $W(t) = \frac{1}{4}(1-t^2)^2$, in which case (1.1) becomes $\varepsilon^2 \Delta_g u = u^3 - u$.

The Allen–Cahn equation, (1.1), is the Euler–Lagrange equation for the energy functional

$$E_\varepsilon[u] = \int_M \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) d\mu_g.$$

Depending on what we wish to emphasize, we will go back and forth between saying that a function u is a solution of (1.1) on M (or in a domain $U \subset M$) or a critical point of E_ε (resp. of $E_\varepsilon \llcorner U$). The second variation of E_ε is easily

[2]We note that after the first version of this work was posted, Gaspar–Guaraco [GG19] gave a new proof of Yau's conjecture for generic metrics (in the spirit of Irie–Marques–Neves [IMN18]) by proving a Weyl law for their Allen–Cahn p-widths.

computed (for $\zeta, \psi \in C^\infty_c(M)$) to be

$$\delta^2 E_\varepsilon[u]\{\zeta, \psi\} = \int_M \left(\varepsilon \langle \nabla \zeta, \nabla \psi \rangle + \frac{W''(u)}{\varepsilon} \zeta \psi \right) d\mu_g. \tag{1.2}$$

We are thus led to the notion of stability and Morse index (with respect to Dirichlet eigenvalues).

Definition 1.2. For (M^n, g) a complete Riemannian manifold and $U \subset M \setminus \partial M$ open, we say that a critical point of $E_\varepsilon \llcorner U$ is *stable* on U if $\delta^2 E_\varepsilon[u]\{\zeta, \zeta\} \geq 0$ for all $\zeta \in C^\infty_c(U)$. More generally, we say u has Morse index k, denoted $\operatorname{ind}(u) = k$, if

$$\max\{\dim V : \delta^2 E_\varepsilon[u]\{\zeta, \zeta\} < 0 \text{ for all } \zeta \in V \setminus \{0\}\} = k,$$

where the maximum is taken over all subspaces $V \subset C^\infty_c(U)$. Sometimes we will write $\operatorname{ind}(u; U) = k$ to emphasize the underlying set. Note that $\operatorname{ind}(u; U) = 0$ if and only if u is stable on U.

When u is a solution of (1.1) and $\nabla u(x) \neq 0$, we will write

(1) $\nu(x) = \frac{\nabla u(x)}{|\nabla u(x)|}$ for the unit normal of the level set of u through x;
(2) $\mathrm{I\!I}(x)$ for the second fundamental form of the level set of u through x;
(3) $\mathcal{A}(x)$ for the "Allen–Cahn" or "enhanced" second fundamental form of the level set

$$\mathcal{A} = \frac{\nabla^2 u - \nabla^2 u(\cdot, \nu) \otimes \nu^\flat}{|\nabla u|} \left(= \nabla \left(\frac{\nabla u}{|\nabla u|} \right) (x) \right).$$

One may check that

$$|\mathcal{A}(x)|^2 = |\mathrm{I\!I}(x)|^2 + |\nabla_T \log |\nabla u(x)||^2,$$

where ∇_T represents the gradient in the directions orthogonal to ∇u; in other words, $|\mathcal{A}|$ strictly dominates the second fundamental form of the level sets.

Finally, we will often use Fermi coordinates centered on a hypersurface. To avoid confusion about which hypersurface the coordinates are associated to, we will define a function

$$Z_\Sigma(y, z) \triangleq \exp_y(z \nu_\Sigma(y)), \; y \in \Sigma, \; z \in \mathbf{R},$$

where ν_Σ will denote a distinguished normal vector to Σ. In this paper, ν_Σ is generally taken to be the upward pointing unit normal. Note that the pullback of the metric g along Z_Σ has the form $g_z + dz^2$, which is the setting that most of our analysis will take place below.

1.2. *Main results.*

1.2.1. *Curvature estimates for stable solutions of* (1.1) *on* 3*-manifolds.* We start this section by discussing the concept of stability applied to minimal surfaces, since that guides some aspects of our work in the Allen–Cahn setting.

We recall that a two-sided minimal surface $\Sigma^2 \subset (M^3, g)$ with normal vector ν is said to be *stable* if it satisfies

$$\int_\Sigma \left(|\nabla_\Sigma \zeta|^2 - (|\mathrm{I\!I}_\Sigma|^2 + \mathrm{Ric}_g(\nu,\nu))\zeta^2 \right) d\mu_g \geq 0 \tag{1.3}$$

for $\zeta \in C_c^\infty(\Sigma)$. Here, we briefly recall the well-known curvature estimates of Schoen [Sch83] for stable minimal surfaces. If $\Sigma^2 \subset (M^3, g)$ is a complete, two-sided stable minimal surface, then the second fundamental form of Σ, $\mathrm{I\!I}_\Sigma$, satisfies

$$|\mathrm{I\!I}_\Sigma|(x) d(x, \partial\Sigma) \leq C = C(M, g). \tag{1.4}$$

Observe that (1.4) readily implies a stable Bernstein theorem: "a complete two-sided stable minimal surfaces Σ in $\mathbf{R}^3$ without boundary must be a flat plane." On the other hand, the stable Bernstein theorem (proven in [FCS80], [dCP79], [Pog81]) implies (1.4) by a well-known blow-up argument: if (1.4) failed for a sequence of stable minimal surfaces Σ_j, then by choosing a point of (nearly) maximal curvature and rescaling appropriately (cf. [Whi16]), we can produce $\tilde{\Sigma}_j$ a sequence of minimal surfaces in manifolds (M_j^3, g_j) that are converging on compact sets to $\mathbf{R}^3$ with the flat metric, and so that $d_{g_j}(0, \partial\Sigma_j) \to \infty$, $|\mathrm{I\!I}_{\Sigma_j}|$ uniformly bounded on compact sets, and $|\mathrm{I\!I}_{\Sigma_j}|(0) = 1$. The second fundamental form bounds yield local C^2 bounds for the surfaces Σ_j, which may then be upgraded to C^k bounds for all k. Thus, passing to a subsequence, the surfaces Σ_j converge smoothly to a complete stable minimal surface Σ_∞ without boundary in $\mathbf{R}^3$. Because the convergence occurs in C^2, we see that $|\mathrm{I\!I}_{\Sigma_\infty}|(0) = 1$, so Σ_∞ is non-flat. This contradicts the stable Bernstein theorem.

As such, before discussing curvature estimates for stable solution to Allen–Cahn, we must discuss the stable Bernstein theorem for complete solutions on $\mathbf{R}^3$. In general, it is not known if there are stable solutions to Allen–Cahn $\Delta u = W'(u)$ on $\mathbf{R}^3$ with non-flat level sets. However, under the additional assumption of quadratic energy growth, i.e.,

$$(E_1 \llcorner B_R(0))[u] \leq \Lambda R^2,$$

it follows from the work of Ambrosio–Cabre [AC00] (see also [FMV13]) that u has flat level sets. We note that the corresponding stable Bernstein theorem on $\mathbf{R}^2$ is known to hold without any energy growth assumption; see the works of Ghoussoub–Gui [GG98] and Ambrosio–Cabre [AC00].

As such, one may expect that the blow-up argument described above may be used to prove curvature estimates. However, there is a fundamental difficulty present in the Allen–Cahn setting: if u_i are stable solutions of (1.1) on (M^3, g) whose curvature (we will make this precise below) is diverging, then rescaling by a factor $\lambda_i \to \infty$ in a blow-up argument changes ε_i to $\lambda_i\varepsilon_i$. If $\lambda_i\varepsilon_i$ converges to a non-zero constant, then standard elliptic regularity implies the rescaled functions limit smoothly to an entire stable solution of Allen–Cahn on $\mathbf{R}^3$. The smooth convergence guarantees that this solution will have non-flat level sets. If the original functions u_i has uniformly bounded energy, we can show that the limit has quadratic area growth, which contradicts the aforementioned Bernstein theorem. However, if $\lambda_i\varepsilon_i$ still converges to zero, we must argue differently. In this case, we have a sequence of solutions to Allen–Cahn whose level sets are uniformly bounded in a C^2-sense. This can be used to show that the level sets converge to a plane (possibly with multiplicity) in the $C^{1,\alpha}$-sense. If the level sets behaved precisely like minimal surfaces, we could upgrade this $C^{1,\alpha}$-convergence using elliptic regularity, to conclude that the limit was not flat. However, in this situation, the level sets themselves do not satisfy a good PDE, so this becomes a significant obstacle.

Recently, a fundamental step in understanding this issue has been undertaken by Wang–Wei [WW19a]. They have developed a technique for gaining geometric control of solutions to Allen–Cahn whose level sets are converging with Lipschitz bounds. Using this (and the 2-dimensional stable Bernstein theorem) they have proven curvature estimates for individual level sets of stable solutions on two-dimensional surfaces. Moreover, they have shown that if one cannot upgrade C^2 bounds to $C^{2,\alpha}$ convergence, then by appropriately rescaling the height functions of the nodal sets, one obtains a non-trivial solution to the a system of PDE's known as the Toda system (see [WW19a, Rem. 14.1]). Finally, their proof of curvature estimates in 2-dimensions points to the crucial observation that it is necessary to use stability to upgrade the regularity of the convergence of the level sets.

This brings us to our first main result here, which is an extension of the Wang–Wei curvature estimates to 3-dimensions. Our 3-dimensional curvature estimates can be roughly stated as follows (see Theorem 3.4 for a slightly more refined statement and the proof):

THEOREM 1.3. *For a complete Riemannian metric on $\overline{B_2}(0) \subset \mathbf{R}^3$ and a stable solution u to (1.1) with $E_\varepsilon(u) \leq E_0$, the enhanced second fundamental form of u satisfies*

$$\sup_{B_1(0)\cap\{|u|<1-\beta\}} |\mathcal{A}|(x) \leq C = C(g, E_0, W, \beta)$$

as long as $\varepsilon > 0$ is sufficiently small.

We emphasize that Wang–Wei's 2-dimensional estimates [WW19a, Th. 3.7] do not require the energy bound. (See also [Man17, Th. 4.13] for the Riemannian modifications of this result.) Note that we cannot expect to prove estimates with a constant that tends to 0 as $\varepsilon \searrow 0$ (which was the case in [WW19a]) since—unlike geodesics—minimal surfaces do not necessarily have vanishing second fundamental form.

We note that due to our curvature estimates, it is not hard to see that stable (and more generally, uniformly bounded index) solutions to the Allen–Cahn equation (with uniformly bounded energy) in a 3-manifold limit to a $C^{1,\alpha}$ surface that has vanishing (weak) mean curvature. Standard arguments thus show that the surface is smooth. Thus, our estimates show that it is possible to completely avoid the regularity results of Wickramasekera and Wickramasekera–Tonegawa [Wic14], [TW12] in the setting of Allen–Cahn min-max on a 3-manifold (cf. [Gua18]).

Remark 1.4. We briefly remark on the possibility of extending curvature estimates to higher dimensions:

(1) For $n \geq 8$, curvature estimates fail for stable (and even minimizing) solutions to the Allen–Cahn equation; see [PW13], [LWW17].
(2) For $4 \leq n \leq 7$, the Allen–Cahn stable Bernstein result is not known (even with an energy growth condition).

Even if the stable Bernstein theorem were to be established in dimensions $4 \leq n \leq 7$, we note that our proof currently uses the dimension restriction $n = 3$ in one other place: we use a logarithmic cutoff function in the proof of our sheet separation estimates (Propositions 3.1 and 3.2).[3]

On the other hand, we remark that the curvature estimate for minimizing solutions can be proven using the "multiplicity-one" nature of minimizers [HT00, Th. 2], together with [WW19a, §15] (or Remark 2.6).

We note that the case of complete minimizers is closely related to the well known "De Giorgi conjecture." See [GG98], [AC00], [Sav09], [dPKW11], [Wan17].

1.2.2. *Strong sheet separation estimates for stable solutions.* A key ingredient in the proof of our curvature estimates is showing that distinct sheets of the nodal set of a stable solution to the Allen–Cahn equation remain sufficiently far apart. This aspect was already present in the work of Wang–Wei. For our applications to the case of uniformly bounded Morse index (and thus min-max theory), we must go beyond the sheet separation estimates proven in [WW19a]. We prove in Proposition 3.2 that distinct sheets of nodal sets of a stable solution to the Allen–Cahn equation must be separated by a sufficiently large distance so that the location of the nodal sets becomes "mean curvature dominated."

[3] Added in proof: Wang–Wei have recently found [WW19b] the appropriate higher dimensional replacement for the log-cutoff argument used here. We note that the stable Bernstein problem for Allen–Cahn remains open in dimensions $4 \leq n \leq 7$.

In particular, as a consequence of these estimates, we show in Theorem 4.1 that if a sequence of stable solutions to the Allen–Cahn equation converge with multiplicity to a closed two-sided minimal surface Σ, then there is a positive Jacobi field along Σ (which implies that Σ is stable). It is interesting to compare this to the examples constructed by del Pino–Kowalczyk–Wei–Yang of minimal surfaces in 3-manifolds with positive Ricci curvature that are the limit with multiplicity of solutions to the Allen–Cahn equation [dPKWY10]. Note that such a minimal surface cannot admit a positive Jacobi field, so the point here is that the Allen–Cahn solutions are not stable. (In fact, our Theorem 4.1 implies that they have diverging Morse index.) Note that the separation D between the sheets of the examples constructed in [dPKWY10] satisfy, as $\varepsilon \searrow 0$,

$$D \sim \sqrt{2}\varepsilon|\log\varepsilon| - \frac{1}{\sqrt{2}}\varepsilon\log|\log\varepsilon|,$$

while we prove in Proposition 3.2 that stability implies that the separation satisfies

$$D - \left(\sqrt{2}\varepsilon|\log\varepsilon| - \frac{1}{\sqrt{2}}\varepsilon\log|\log\varepsilon|\right) \to -\infty.$$

We emphasize that the improved separation estimates here are not contained in the work of Wang–Wei [WW19a] and are fundamental for the subsequent applications of our results.

1.2.3. *The multiplicity one-conjecture for limits of the Allen–Cahn equation in 3-manifolds.* In their recent work [MN16a], Marques–Neves make the following conjecture:

CONJECTURE 1.5 (Multiplicity one conjecture). *For generic metrics on (M^n, g), $3 \leq n \leq 7$, two-sided unstable components of closed minimal hypersurfaces obtained by min-max methods must have multiplicity one.*

In [MN16a], Marques–Neves confirm this in the case of a one parameter Almgren–Pitts sweepout. The one parameter case had been previously considered for metrics of positive Ricci curvature by Marques–Neves [MN12] and subsequently by Zhou [Zho15]. See also [Gua18, Cor. E] and [GG18, Th. 1] for results comparing the Allen–Cahn setting to the Almgren–Pitts setting which establish multiplicity one for hypersurfaces obtained by a one parameter Allen–Cahn min-max method in certain settings. We also note that Ketover–Liokumovich–Song [KLS19] have proven multiplicity (and index) estimates for one parameter families in the Simon–Smith [Smi82] variant of Almgren–Pitts in 3-manifolds.[4]

We recall the following standard definition:

[4]Added in proof: As noted before, the full multiplicity-one conjecture for Almgren–Pitts (in dimensions 3 through 7) has now been proven by X. Zhou [Zho19].

Definition 1.6. We say that a metric g on a Riemannian manifold M^n is *bumpy* if there is no immersed closed minimal hypersurface Σ^{n-1} with a non-trivial Jacobi field.

By work of White [Whi91], [Whi17], bumpy metrics are generic in the sense of Baire category. Here, "generic" will always mean in the Baire category sense.

We are able to prove a strong version of the multiplicity-one conjecture (when $n = 3$) for minimal surfaces obtained by Allen–Cahn min-max methods with an *arbitrary* number of parameters. Such a method was set up by Gaspar–Guaraco [GG18].

Indeed, we prove that for *any* metric g on a closed 3-manifold, the unstable components of such a surface are multiplicity one. Moreover, for a generic metric, we show that *each* component of the surface occurs with multiplicity one (not just the unstable components). Finally, we are able to show for generic metrics on a 3-mainifold, the minimal surfaces constructed by Allen–Cahn min-max methods are two-sided. For a one-parameter Almgren–Pitts sweepout in an n-manifold $3 \leq n \leq 7$ with positive Ricci curvature, this was proven by Ketover–Marques–Neves [KMN16]. More precisely, our main results here are as follows. (See Theorem 4.1 and Corollary 6.1 for the full statements.)

Theorem 1.7 (Multiplicity and two-sidedness of minimal surfaces constructed via Allen–Cahn min-max). *Let $\Sigma^2 \subset (M^3, g)$ denote a smooth embedded minimal surface constructed as the $\varepsilon \searrow 0$ limit of solutions to the Allen–Cahn equation on a* 3*-manifold with uniformly bounded index and energy. If Σ occurs with multiplicity or is one-sided*, *then it carries a positive Jacobi field* (*on its two-sided double cover*, *in the second case*).

Note that positive Jacobi fields do not occur when g is bumpy or when g has positive Ricci curvature. Thus, *in either of these cases*, *each component of Σ is two-sided and occurs with multiplicity one.*

Remark 1.8. We re-emphasize that our theorem applies generally to sequences of Allen–Cahn solutions with uniformly bounded energy and Morse index. Thus, unlike the proofs in the Almgren–Pitts setting, we do not need to make use of any min-max characterization of the limiting surface to rule out multiplicity.

Our proof here is modeled on the study of bounded index minimal hypersurfaces in a Riemannian manifold. Indeed, Sharp has shown that minimal hypersurfaces in (M^n, g) for $3 \leq n \leq 7$ with uniformly bounded area and index are smoothly compact away from finitely many points where the index can concentrate [Sha17]. (See also White's proof [Whi87] of the Choi–Schoen compactness theorem [CS85].) A crucial point there is to prove that higher multiplicity of the limiting surface produces a positive Jacobi field (even across the points of index concentration (where the convergence of the hypersurfaces need not occur smoothly). This can be handled via an elegant argument of White, based on the construction of a local foliation by minimal surfaces to use as a barrier for the limiting surfaces (cf. [Whi18]).

In the minimal surface setting, the existence of the foliation is a simple consequence of the implicit function theorem. However, in the Allen–Cahn setting, the singular limit $\varepsilon \searrow 0$ limit complicates this argument. Instead, we construct barriers by a more involved fixed point method in Theorem 7.4. Once that theorem is proven, we show how the barriers can be used to bound the Jacobi fields along the points of index concentration in the process of the proof of Theorem 4.1 by carrying out a new sliding plane type argument for the Allen–Cahn equation on Riemannian manifolds. Our proof of Theorem 7.4 is modeled on the work of Pacard [Pac12] (with appropriate extension to the case of Dirichlet boundary conditions), but there is a significant technical obstruction here: we do not know that the level sets of the solution Allen–Cahn converge smoothly, but only in $C^{2,\alpha}$. To apply the fixed point argument, we need some control on higher derivatives. By an observation of Wang–Wei [WW19a, Lemma 8.1], we control one higher derivative of the level sets, but only by a constant that is $O(\varepsilon^{-1})$ (see (7.4)). This complicates the proof of Theorem 7.4.

1.2.4. *Index lower bounds.* Lower semicontinuity of the Morse index along the singular limit $\varepsilon \searrow 0$ of a sequence of solutions to the Allen–Cahn equation is proven by Hiesmayr [Hie18] (for two-sided surfaces) and Gaspar [Gas17] without assuming two-sidedness (see also [Le11]). On the other hand, upper semicontinuity of the index does not hold in general (cf. Example 5.2). Here, we establish upper semicontinuity of the index, in all dimensions, under the a priori assumption that the limiting surface is multiplicity one.[5] In particular, we prove (see Theorem 5.11 for the full statement)

THEOREM 1.9 (Upper semicontinuity of the index in the multiplicity one case). *Suppose that a smooth embedded minimal hypersurface $\Sigma^{n-1} \subset (M^n, g)$ is the multiplicity-one limit as $\varepsilon \searrow 0$ of a sequence of solutions u to the Allen–Cahn equation. Then for $\varepsilon > 0$ sufficiently small,*

$$\mathrm{nul}(\Sigma) + \mathrm{ind}(\Sigma) \geq \mathrm{nul}(u) + \mathrm{ind}(u).$$

To prove this upper semicontinuity, we need to delve deeper into the equation that controls the level sets of u and obtain a more accurate approximation. What was done for Theorem 1.3—while well suited to understanding the phenomenon of multiplicity—does not suffice for Theorem 1.9.

[5] We note that Marques–Neves had previously announced the analogous index uppper-semicontinuity result for multiplicity-one Almgren–Pitts limits and that their proof [MN18] appeared shortly after the first version of this paper.

1.2.5. *Applications related to Yau's conjecture on infinitely many minimal surfaces.* A well-known conjecture of Yau posits that any closed 3-manifold admits infinitely many immersed minimal surfaces [Yau82]. By considering the p-widths introduced by Gromov [Gro03] (see also [Gut09]), Marques–Neves proved [MN17] that a closed Riemannian manifold (M^n, g) (for $3 \leq n \leq 7$) with positive Ricci curvature admits infinitely many minimal surfaces. Moreover, by an ingenious application of the Weyl law for the p-widths proven by Liokumovich–Marques–Neves [LMN18], Irie–Marques–Neves [IMN18] (see also the recent work of Gaspar–Guaraco [GG19] that appeared after the first version of this paper was posted) have recently shown that the set of metrics on a closed Riemannian manifold (M^n, g) (with $3 \leq n \leq 7$) with the property that the set of minimal surfaces is dense in the manifold is generic; see also [MNS19].

We note that the arguments in each of [MN17], [IMN18], [GG19] to prove the existence of infinitely many minimal surfaces are *necessarily indirect*, as they do not rule out the p-widths being achieved with higher multiplicity. Having overcome this obstacle, we may give a "direct" proof (for $n = 3$) of Yau's conjecture for bumpy metrics[6] with some new geometric conclusions; see Corollaries 6.1 and 6.2 for proofs.

Corollary 1.10 (Yau's conjecture for bumpy metrics and geometric properties of the minimal surfaces). *Let (M^3, g) denote a closed 3-manifold with a bumpy metric. Then, there are $C = C(M, g, W) > 0$ and a smooth embedded minimal surfaces Σ_p for each positive integer $p > 0$ so that*

- *each component of Σ_p is two-sided,*
- *the area of Σ_p satisfies $C^{-1}p^{\frac{1}{3}} \leq \text{area}_g(\Sigma_p) \leq Cp^{\frac{1}{3}}$,*
- *the index of Σ_p is satisfies $\text{ind}(\Sigma_p) = p$, and*
- *the genus of Σ_p satisfies $\text{genus}(\Sigma_p) \geq \frac{p}{6} - Cp^{\frac{1}{3}}$.*

In particular, thanks to the index estimate, all of the Σ_p are geometrically distinct.

We emphasize that each of the bullet points in the preceding corollary do not follow from the work of Irie–Marques–Neves [IMN18]. Some of these properties were conjectured by Marques and Neves in [Mar14, p. 24], [Nev14, p. 17] and [MN16b, Conj. 6.2]. In particular, they conjectured that a generic Riemannian manifold contains an embedded two-sided minimal surface of each positive Morse index.

Remark 1.11 (Yau's conjecture for 3-manifolds with positive Ricci curvature). We note that because the multiplicity-one property also holds even

[6] We note that [IMN18] and [GG19] prove Yau's conjecture for a different (also generic) set of metrics.

for non-bumpy metrics of positive Ricci curvature, we may also give a "direct" proof of Yau's conjecture for a 3-manifold with positive Ricci curvature. (This was proven by Marques–Neves [MN17] in dimensions $3 \leq n \leq 7$ using Almgren–Pitts theory.) We obtain, exactly as in Corollary 6.2, the new conclusions that the surfaces Σ_p are two-sided, have $\operatorname{area}(\Sigma_p) \sim p^{\frac{1}{3}}$, $\operatorname{ind}(\Sigma_p) \leq p$ and $\operatorname{nul}(\Sigma_p) + \operatorname{ind}(\Sigma_p) \geq p$. 1 Moreover, approximating the metric by a sequence of bumpy metrics and passing to the limit (the limit occurs smoothly and with multiplicity one due to the positivity of the Ricci curvature; cf. [Sha17]), we find that there is a sequence Σ_p' (we do not know if this is the same sequence as Σ_p) with these properties and additionally satisfies the genus bound (note that Σ_p is connected by Frankel's theorem) for possibly a larger constant C

$$\operatorname{genus}(\Sigma_p') \geq \frac{p}{6} - Cp^{\frac{1}{3}}.$$

It is interesting to observe that when (M^3, g) is the round 3-sphere, combining our bound $\operatorname{ind}(\Sigma_p') \leq p$ with work of Savo [Sav10] implies that

$$\operatorname{genus}(\Sigma_p') \leq 2p - 8$$

as long as p is sufficiently large to guarantee that $\operatorname{genus}(\Sigma_p') \geq 1$. Similar conclusions can be derived in certain other 3-manifolds embedded in Euclidean spaces by [ACS18].

There has been significant activity concerning the index of the minimal surfaces constructed in [MN17], but before the present work, all that was known was that for a bumpy metric of positive Ricci curvature, there are closed embedded minimal surfaces of arbitrarily large Morse index [LZ16], [CKM17], [Car17], albeit without information on their area.

Remark 1.12 (Connected components in Corollary 1.10). Unless (M, g) has the Frankel property (e.g., when it has positive Ricci curvature), the minimal surfaces Σ_p obtained in Corollary 1.10 may be disconnected. In this case, every connected component Σ_p' of Σ_p must satisfy

- Σ_p' is two-sided and has $\operatorname{area}_g(\Sigma_p') \leq Cp^{\frac{1}{3}}$

and, by a counting argument, there will exist at least one component Σ_p' of Σ_p such that

- $\operatorname{genus}(\Sigma_p') \geq C^{-1} \operatorname{ind}(\Sigma_p') \geq C^{-1} p^{\frac{2}{3}}$.

See Corollary 6.4.

It is not clear that the component Σ_p' will have unbounded area. In a follow up paper [CM19] we prove the following dichotomy: Either

(1) (M, g) contains a sequence of connected closed embedded stable minimal surfaces with unbounded area, or

(2) some connected component Σ_p'' of the surfaces Σ_p obtained in Corollary 1.10 has $\text{area}_g(\Sigma_p'') \geq Cp^{\frac{1}{3}}$.

We note that by [CKM17], [Car17], when (M^3, g) is a bumpy metric with positive scalar curvature the prior condition cannot hold, so the latter alternative holds and, moreover, $\text{ind}(\Sigma_p'') \to \infty$. It would be interesting to determine if one can find a connected component Σ_p'' with arbitrarily large area and $\text{ind}(\Sigma_p'') \geq cp$ for some $c \in (0, 1)$.

1.3. *One-dimensional heteroclinic solution*, $\mathbb{H}$. Recall that the one-dimensional Allen–Cahn equation with $\varepsilon = 1$ is $u'' = W'(u)$ for a function $u = u(t)$ of one variable. It is not hard to see that this ODE admits a unique bounded solution with the properties

$$u(0) = 0, \ \lim_{t \to -\infty} u(t) = -1, \ \lim_{t \to \infty} u(t) = 1.$$

We call this the one-dimensional heteroclinic solution and denote it as $\mathbb{H} : \mathbf{R} \to (-1, 1)$. It is also standard to see that the heteroclinic solution satisfies

$$\mathbb{H}(\pm t) = \pm 1 \mp A_0 \exp(-\sqrt{2}t) + O(\exp(-2\sqrt{2}t)), \tag{1.5}$$

$$\mathbb{H}'(\pm t) = \sqrt{2} A_0 \exp(-\sqrt{2}t) + O(\exp(-2\sqrt{2}t)), \tag{1.6}$$

$$\mathbb{H}''(\pm t) = -2A_0 \exp(-\sqrt{2}t) + O(\exp(-2\sqrt{2}t)) \tag{1.7}$$

as $t \to \infty$, for some fixed $A_0 > 0$ that depends on W. Moreover,

$$\int_{-\infty}^{\infty} (\mathbb{H}'(t))^2 \, dt = h_0,$$

where $h_0 > 0$ also depends on W; it is explicitly given by

$$h_0 = \int_{-1}^{1} \sqrt{2W(t)} \, dt.$$

Finally, we also define

$$\mathbb{H}_\varepsilon(t) \triangleq \mathbb{H}(\varepsilon^{-1} t), \ t \in \mathbf{R}, \tag{1.8}$$

which is clearly a solution of $\varepsilon^2 \mathbb{H}_\varepsilon'' = W'(\mathbb{H}_\varepsilon)$.

1.4. *Organization of the paper*. In Section 2 we make precise *the dependence of the regularity* of the nodal set $\{u = 0\}$ of bounded energy and bounded curvature solutions of (1.1) *on the distance* between its different sheets. The dependence is essentially modeled by a Toda system; see, e.g., (2.18) and Remark 2.6. Restricting to $n = 3$-dimensions, in Section 3 we use the stability of Allen–Cahn solutions to bootstrap the distance estimates from Section 2 until they become sharp. In Section 4 we study solutions of (1.1) with bounded energy and Morse index in $n = 3$-dimensions. We use our strong sheet separation estimates from Section 3 to construct, in the presence of multiplicity, positive

Jacobi fields on the limiting minimal surface away from finitely many points. Then, a "sliding plane" argument (modulo a barrier construction deferred to Section 7) allows us to extend the Jacobi field to the entire limiting surface.

In Section 5 we return to the arbitrary dimensional setting and prove the Morse index is lower semicontinuous for smooth multiplicity-one limits. In Section 6 we apply all our tools to prove a strong form of Marques' and Neves' multiplicity-one conjecture, and Yau's conjecture for generic metrics. In Section 7 we construct curved sliding plane barriers for (1.1) that resemble multiplicity-one heteroclinic solutions with prescribed Dirichlet data centered on non-degenerate minimal submanifolds-with-boundary $\Sigma^{n-1} \subset (M^n, g)$, $n \geq 3$.

In Appendix A, we recall several expressions related to the mean curvature and second fundamental form of graphical hypersurfaces in a Riemannian manifold. In Appendix B we recall several auxiliary results from [WW19a]. In Appendix C, we prove Lemma 2.8 relating regularity of the "centering" functions h_ℓ to that of the function ϕ with improved error estimates. In Appendix D, we derive the Toda-system stability inequality with improved error estimates (3.2). In Appendix E we recall an interpolation inequality for Hölder norms.

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2. From phase transitions to Jacobi-Toda systems

2.1. *Approximation by superimposed heteroclinics.* In this section we follow Wang-Wei's [WW19a] investigation of local properties of solutions to the Allen–Cahn equation,

$$\varepsilon^2 \Delta_g u = W'(u),$$

whose nodal set $\{u = 0\}$ can be (locally) decomposed as a union of graphs over a fixed hypersurface (to be denoted Σ), whose height functions (to be denoted $f_1, \ldots, f_Q$) are bounded in C^2 and small in C^1. The ultimate goal is to deduce, in a quantitative sense, that the height functions approximately satisfy a Jacobi-Toda system.

The reason we rework the setup is twofold:

(1) First, most of the analysis in [WW19a] was performed in $\mathbf{R}^n$, while here we include the details necessary to handle the Riemannian setting (cf. [WW19a, §16]).
(2) Secondly (and more fundamentally), we combine the argument from [WW19a] with a further bootstrap argument based on improved error estimates. This allows us to prove much sharper separation estimates than were obtained in [WW19a]. Indeed, we will show that the behavior of the transition layers is dominated by mean curvature, rather than interaction between the layers. This will be crucial for our subsequent applications in Section 4.

Let us set things up. Suppose that D^{n-1} is an $(n-1)$-dimensional disk, over which we take a topological cylinder $\Omega \triangleq D \times [-1,1]$, whose coordinates we label $X = (y,z) \in D \times [-1,1]$. Consider a smooth metric g on Ω, which we assume to be in Fermi coordinate form with respect to Σ; in (y,z) coordinates,

$$g = g_z + dz^2.$$

For convenience, we denote $\Sigma \triangleq D \times \{0\} \subset \Omega$. Let us require that

$$\sum_{\ell=0}^{3} |\nabla_\Sigma^\ell \mathrm{I\!I}_\Sigma| \leq \eta. \tag{2.1}$$

We additionally assume that Σ is covered by C^4-coordinate charts so that the induced metric on Σ, g_0 is C^3-close to the Euclidean metric in the charts, i.e.,

$$\sum_{\ell=0}^{3} |\partial_y^{(\ell)}((g_0)_{ij} - \delta_{ij})| \leq \eta. \tag{2.2}$$

We make no assumptions on the mean curvature of Σ beyond what follows automatically from (2.1). Notice that, as a consequence of (2.1)–(2.2), Fermi coordinates with respect to Σ are a C^4 diffeomorphism.

In all that follows, for $y_0 \in \Sigma \setminus \partial\Sigma$ and $0 < r < \mathrm{dist}_{g_0}(y_0, \partial\Sigma)$, we denote

$$B_r^{n-1}(y_0) \triangleq \{y \in \Sigma : \mathrm{dist}_{g_0}(y, y_0) < r\},$$

where dist_{g_0} is the intrinsic distance on Σ. We assume, without loss of generality, that $\Sigma = \overline{B}_2^{n-1}(0)$.

Remark 2.1. We have chosen to work at the original scale, rather than rescaling by ε as in [WW19a]. This does not affect our subsequent analysis, but certain expressions will change by appropriate multiples of ε.

Let $u : \Omega \to (-1, 1)$ be a critical point of $E_\varepsilon \llcorner \Omega$, with

$$\varepsilon \leq \varepsilon_0, \tag{2.3}$$

$$(E_\varepsilon \llcorner \Omega)[u_i] \leq E_0, \tag{2.4}$$

$$\varepsilon|\nabla u| \geq c_0^{-1} > 0 \text{ on } \Omega \cap \{|u| \leq 1 - \beta\}, \tag{2.5}$$

$$|\mathcal{A}| \leq c_0 \text{ on } \Omega \cap \{|u| \leq 1 - \beta\}. \tag{2.6}$$

By (2.5), (2.6), and elliptic regularity, we automatically also get

$$\varepsilon|\nabla \mathcal{A}| + \varepsilon^2|\nabla^2 \mathcal{A}| \leq c_0 \text{ on } \Omega \cap \{|u| \leq 1 - \beta\} \tag{2.7}$$

for a possibly larger $c_0 > 0$; see [WW19a, Lemma 8.1]. With regard to the nodal set of u, we require

$$\{u = 0\} \cap \Omega = \bigcup_{\ell=1}^{Q} \Gamma_\ell, \tag{2.8}$$

where $\Gamma_\ell = \operatorname{graph}_\Sigma f_\ell$ denote normal graphs over Σ ordered so that $f_1 < f_2 < \cdots < f_Q$, and the graphing functions $f_\ell : \Sigma \to \mathbf{R}$ are assumed to satisfy

$$|f_\ell| + |\nabla_\Sigma f_\ell| \leq \eta, \tag{2.9}$$

and (this alternatively follows automatically from (2.1) and (2.6))

$$|\nabla^2_\Sigma f_\ell| \leq c_0. \tag{2.10}$$

Finally, after possibly sending $z \mapsto -z$, we can assume that for $z \approx -1$, $u(y, z) \approx -1$. The constants that appear above are to be considered independent of $\varepsilon \leq \varepsilon_0$ and fixed so that

$$c_0 \gg 1, \ 0 < \varepsilon_0, \beta, \eta \ll 1, \ Q \in \{1, 2, \ldots\}. \tag{2.11}$$

For $\ell \in \{1, \ldots, Q\}$, $y_0 \in \Sigma$, $r > 0$, denote

(1) $\Pi : \Omega \to \Sigma$ to be the closest point projection onto Σ with respect to g;
(2) $C_r(y_0) \triangleq \{X \in \Omega : \Pi(X) \in B_r^{n-1}(y_0)\}$;
(3) $\Gamma_\ell(r) \triangleq \Gamma_\ell \cap C_r(0)$;
(4) $Z_{\Gamma_\ell} : \Gamma_\ell(3/2) \times [-1, 1] \to \Omega$ to be the normal exponential map with respect to Γ_ℓ;
(5) $\Pi_\ell : \Omega \to \Gamma_\ell$ to be the closest point projection onto Γ_ℓ with respect to g;
(6) $d_\ell : \Omega \to \mathbf{R}$ to be the signed distance from Γ_ℓ (with respect to g), which is positive above it and negative below it;
(7) $D_\ell \triangleq \min\{|d_{\ell-1}|, |d_{\ell+1}|\}$.

Let us agree once and for all regarding Sections 2–3 that each Γ_ℓ is endowed with the same coordinates $(y^1, \ldots, y^{n-1})$ as Σ via the diffeomorphism $\Pi|_{\Gamma_\ell} : \Gamma_\ell \xrightarrow{\approx} \Sigma$.

Set $\Omega' \triangleq B_1^{n-1}(0) \times [-2\eta, 2\eta] \subset \Omega$. Consider arbitrary C^2 functions

$$h_\ell : \Gamma_\ell \cap C_1(0) \to (-\tfrac{\eta}{2}, \tfrac{\eta}{2}),\ \ell \in \{1, \ldots, Q\}.$$

Let $\mathbf{h} = (h_1, \ldots, h_n)$. From $\mathbf{h}$, we construct an approximate critical point $U(\mathbf{h})$ of $E_\varepsilon \llcorner \Omega'$,

$$U[\mathbf{h}] \triangleq \frac{(-1)^{Q+1} - 1}{2} + \sum_{\ell=1}^{Q} \overline{\mathbb{H}}_{\varepsilon,\ell}. \tag{2.12}$$

Here, each $\overline{\mathbb{H}}_{\varepsilon,\ell}$ is given by

$$\begin{aligned} ((Z_{\Gamma_\ell})^* \overline{\mathbb{H}}_{\varepsilon,\ell})(y,z) &\triangleq \overline{\mathbb{H}}^{3|\log\varepsilon|}\big((-1)^{\ell-1}\varepsilon^{-1}(z - h_\ell(y))\big) \\ &\iff \overline{\mathbb{H}}_{\varepsilon,\ell} = \overline{\mathbb{H}}^{3|\log\varepsilon|}((-1)^{\ell-1}\varepsilon^{-1}(d_\ell - h_\ell \circ \Pi_\ell)), \end{aligned} \tag{2.13}$$

with $\overline{\mathbb{H}}^\Lambda : \mathbf{R} \to [-1, 1]$ (here, $\Lambda = 3|\log\varepsilon|$) being

$$\overline{\mathbb{H}}^\Lambda(t) \triangleq \chi(\Lambda^{-1}t)\mathbb{H}(t) \pm (1 - \chi(\Lambda^{-1}t)) \tag{2.14}$$

($\pm$ depending on $t > 0$ or $t < 0$). Here, $\chi(t) = 1$ for $t \in (-1, 1)$ and $\operatorname{spt}\chi \subset (-2, 2)$ is a fixed cutoff function. These functions, $\overline{\mathbb{H}}^{3|\log\varepsilon|}$, are truncations of $\mathbb{H}$ that coincide with it on $(-3|\log\varepsilon|, 3|\log\varepsilon|)$, with ± 1 outside $(-6|\log\varepsilon|, 6|\log\varepsilon|)$, and such that

$$|(\overline{\mathbb{H}}^{3|\log\varepsilon|})'' - W'(\overline{\mathbb{H}}^{3|\log\varepsilon|})|_{C^2(\mathbf{R})} = O(\varepsilon^3). \tag{2.15}$$

See [WW19a, §9.1] for more details.

Remark 2.2. The components of $\mathbf{h}$ represent the vertical offset of the heteroclinic solutions we are superimposing relative to the nodal set of u.

One can show (see [WW19a, §9.1]) that there exists $\mathbf{h}$ such that for every $\ell \in \{1, \ldots, Q\}$, $y \in \Gamma_\ell$, we have the orthogonality relation

$$\int_{-\eta}^{\eta} ((Z_{\Gamma_\ell})^*(u - U[\mathbf{h}]))(y,z)\partial_z((Z_{\Gamma_\ell})^*\overline{\mathbb{H}}_{\varepsilon,\ell})(y,z)\,dz = 0. \tag{2.16}$$

Moreover (see [WW19a, Rem. 9.2]),

$$\sum_{j=0}^{3} \varepsilon^{j-1} \|\nabla^j \mathbf{h}\|_{C^0(B_1^{n-1}(0))} = o(1) \text{ as } \varepsilon \to 0.$$

It will prove useful to introduce the notation

$$\phi \triangleq u - U[\mathbf{h}], \tag{2.17}$$

seeing as to how we can conveniently bound $\mathbf{h}$ in terms of ϕ, as Lemma 2.3 below shows.

LEMMA 2.3 ([WW19a, Lemma 9.6]). *For* $\ell \in \{1, \dots, Q\}$, $y \in \Gamma_\ell(\frac{9}{10})$,

$$\begin{aligned}
\varepsilon^{-1}|h_\ell(y)| &\le c\left(|\phi|_{\Gamma_\ell}(y)| + \exp(-\sqrt{2}\varepsilon^{-1}D_\ell(y))\right),\\
|\nabla_{\Gamma_\ell} h_\ell(y)| &\le c\Big(\varepsilon|\nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y)| + o(1)\exp(-\sqrt{2}\varepsilon^{-1}D_\ell(y))\Big),\\
\varepsilon|\nabla^2_{\Gamma_\ell} h_\ell(y)| &\le c\Big(\varepsilon^2|\nabla^2_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y)|\\
&\qquad + \varepsilon^2|\nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y)|^2 + o(1)\exp(-\sqrt{2}\varepsilon^{-1}D_\ell(y))\Big),\\
\varepsilon^{1+\theta}[\nabla^2_{\Gamma_\ell} h_\ell]_\theta &\le c'\Big(\varepsilon^{2+\theta}[\nabla^2_{\Gamma_\ell}(\phi|_{\Gamma_\ell})]_\theta + \varepsilon^{2+\theta}\|\nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})\|_{C^0}[\nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})]_\theta\\
&\qquad + \|\exp(-\sqrt{2}\varepsilon^{-1}D_\ell)\|_{C^0}\Big),
\end{aligned}$$

where $c = c(n, c_0, E_0, \eta, \beta)$, $c' = c'(n, c_0, E_0, \eta, \beta, \theta)$, *and* $o(1)$ *is taken as* $\varepsilon \to 0$ *with the remaining parameters held fixed. In the last inequality, the Hölder seminorms and the* C^k *norms are taken over all* $y' \in \Gamma_\ell \cap C_\varepsilon(\Pi(y))$.

Wang–Wei deduce (see [WW19a, (10.2)]) the following Jacobi-Toda-like system: for $y \in \Gamma_\ell(\frac{9}{10})$,

(2.18)
$$\begin{aligned}
&\varepsilon(\Delta_{\Gamma_\ell} h_\ell(y) - H_{\Gamma_\ell}(y))\\
&= \frac{4(A_0)^2}{h_0}\left(\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-1}(y)|) - \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+1}(y)|)\right)\\
&+ O\Big(\varepsilon^{-1}|h_\ell(y)| + \varepsilon^{-1}\|(h_{\ell-1}\circ\Pi_{\ell-1}\circ Z_{\Gamma_\ell})(y,\cdot)\|_{C^0} + \varepsilon^{\frac{1}{3}}\Big)\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-1}(y)|)\\
&+ O\Big(\varepsilon^{-1}|h_\ell(y)| + \varepsilon^{-1}\|(h_{\ell+1}\circ\Pi_{\ell+1}\circ Z_{\Gamma_\ell})(y,\cdot)\|_{C^0} + \varepsilon^{\frac{1}{3}}\Big)\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+1}(y)|)\\
&+ O(\exp(-(\tfrac{3}{2}\sqrt{2})\varepsilon^{-1}|d_{\ell-1}(y)|)) + O(\exp(-(\tfrac{3}{2}\sqrt{2})\varepsilon^{-1}|d_{\ell+1}(y)|))\\
&+ O(\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-2}(y)|)) + O(\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+2}(y)|))\\
&+ \sum_{m\neq\ell}\varepsilon^{-1}|d_m(y)|\exp(-\sqrt{2}\varepsilon^{-1}|d_m(y)|)\Big[\varepsilon\|\Delta_{\Gamma_m} h_m - H_{\Gamma_m}\|_{C^0} + \|\nabla_{\Gamma_m} h_m\|^2_{C^0}\Big]\\
&\quad + \sup_{|t|<6\varepsilon|\log\varepsilon|}\Big[\varepsilon^4|(\nabla^2_{\Gamma_{\ell,t}}(\phi|_{\Gamma_{\ell,t}}))(Z_{\Gamma_\ell}(y,t))|^2\\
&\qquad + \varepsilon^2|(\nabla_{\Gamma_{\ell,t}}(\phi|_{\Gamma_{\ell,t}}))(Z_{\Gamma_\ell}(y,t))|^2 + |\phi(Z_{\Gamma_\ell}(y,t))|^2\Big] + O(\varepsilon^2).
\end{aligned}$$

The C^0 norms appearing in the second and third term of the right-hand side are taken over $|t| < 6\varepsilon|\log\varepsilon|$, and the C^0 norms appearing in the third term from the end are taken over $\Gamma_m \cap C_{\varepsilon^{4/3}}(\Pi(y))$.

Remark 2.4. $\Gamma_{\ell,t}$ denote t-level sets in Fermi coordinates (y, t) relative to Γ_ℓ, i.e., $\Gamma_{\ell,t} = \{d_\ell = t\}$.

Remark 2.5. Notice the sign difference in the mean curvature terms between (2.18) and [WW19a, (10.2)]. For us, the mean curvature is the divergence of the upper pointing unit normal. For instance, the ambient Laplace-Beltrami operator expands as

$$\Delta_g = \Delta_{\Gamma_{\ell,z}} + \partial_z^2 + H_{\Gamma_{\ell,z}}\partial_z.$$

For this reason, all instances of the mean curvature in this work must have the opposite sign relative to [WW19a].

It will also be convenient to introduce the notation

$$A_\ell(r) \triangleq \sup\Big\{ \exp(-\sqrt{2}\varepsilon^{-1}D_\ell(y)) : y \in \Gamma_\ell(r)\Big\}. \tag{2.19}$$

We record [WW19a, (12.4)], which will help estimate terms involving h, ϕ, and the mean curvature,

$$\|\phi\|_{C^{2,\theta}_\varepsilon(\mathcal{M}_\ell(r))} + \varepsilon\|\Delta_{\Gamma_\ell}h_\ell - H_{\Gamma_\ell}\|_{C^{0,\theta}_\varepsilon(\Gamma_\ell(r))} \le c'\varepsilon^2 + c'\sum_{m=1}^{Q} A_m(r + K\varepsilon|\log\varepsilon|), \tag{2.20}$$

where we are using the weighted Hölder space notation from (7.1) (see Section 7), and

$$\mathcal{M}_\ell(r) \triangleq \{X \in C_r(0) : |d_\ell(X)| < 1, -d_{\ell-1}(X) < d_\ell(X) < -d_{\ell+1}(X)\}.$$

Likewise, we record [WW19a, (13.6)]:

$$\begin{aligned}\varepsilon\|((Z_{\Gamma_\ell})_*\partial_{y_i})\phi\|_{C^{1,\theta}_\varepsilon(\mathcal{M}_\ell(r))} \le c'\varepsilon^2 + c'\sum_{m=1}^{Q} A_m(r + 2K\varepsilon|\log\varepsilon|)^{1+\kappa} \\ + c'\varepsilon^\kappa \sum_{m=1}^{Q} A_m(r + 2K\varepsilon|\log\varepsilon|),\end{aligned} \tag{2.21}$$

with $\kappa > 0$.

The expressions above, (2.20)–(2.21), are true for all $\ell \in \{1, \ldots, Q\}$, $r \le 8/10$, $\theta \in (0, 1)$, $\varepsilon \le \varepsilon'$, where c', ε', K, κ, depend on n, c_0, E_0, η, β, θ.

Remark 2.6. In the remainder of Sections 2–3, we will be actively interested in estimating the vertical distances D_ℓ from below. This is because Lemma 2.3, (2.19), (2.20), and interior Schauder estimates together imply that, with r, θ as above,

$$\min_{\ell\in\{1,\ldots,Q\}} \inf_{\Gamma_\ell(r)} D_\ell \ge \tfrac{1+\theta}{2}\sqrt{2}\varepsilon|\log\varepsilon| \implies \Gamma_\ell(r') \text{ is uniformly } C^{2,\theta} \tag{2.22}$$

for all $\ell \in \{1, \ldots, Q\}$, $r' \le \sigma r$, $\sigma \in (0, 1)$, $\varepsilon \le \varepsilon' = \varepsilon'(n, c_0, E_0, \eta, \beta, \theta, \sigma)$.

2.2. *Bootstrapping regularity via sheet distance lower bounds.* We recall the following lemma from [WW19a]. (See [Man17, App. C] for necessary modifications for the Riemannian setting.)

LEMMA 2.7 ([WW19a, §14]). *If* $\ell \in \{1, \ldots, Q\}$, $y \in \Gamma_\ell(\frac{8.5}{10})$, *and* $\varepsilon \leq \varepsilon_1$, *then*

$$D_\ell(y) \geq \tfrac{1}{2}\sqrt{2}\varepsilon|\log\varepsilon| - c_1\varepsilon,$$

where $\varepsilon_1 = \varepsilon_1(n, c_0, E_0, \eta, \beta)$, $c_1 = c_1(n, c_0, E_0, \eta, \beta)$.

As a corollary of Lemma 2.7, we can bootstrap the proof of Lemma 2.3 and obtain the following *improved* estimates:

LEMMA 2.8. *For* $\ell \in \{1, \ldots, Q\}$, $y \in \Gamma_\ell(\frac{8}{10})$,

$$\begin{aligned}
\varepsilon^{-1}|h_\ell(y)| &\leq c\left(|\phi|_{\Gamma_\ell}(y)| + \exp(-\sqrt{2}\varepsilon^{-1}D_\ell(y))\right), \\
|\nabla_{\Gamma_\ell} h_\ell(y)| &\leq c\left(\varepsilon|\nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y)| + \varepsilon^\kappa \exp(-\sqrt{2}\varepsilon^{-1}D_\ell(y))\right), \\
\varepsilon|\nabla^2_{\Gamma_\ell} h_\ell(y)| &\leq c\Big(\varepsilon^2|\nabla^2_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y)| \\
&\qquad + \varepsilon^2|\nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y)|^2 + \varepsilon^\kappa \exp(-\sqrt{2}\varepsilon^{-1}D_\ell(y))\Big), \\
\varepsilon^{1+\theta}[\nabla^2_{\Gamma_\ell} h_\ell]_\theta &\leq c'\Big(\varepsilon^{2+\theta}[\nabla^2_{\Gamma_\ell}(\phi|_{\Gamma_\ell})]_\theta \\
&\qquad + \varepsilon^{2+\theta}\|\nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})\|[\nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})]_\theta + \varepsilon^{\kappa'}\|\exp(-\sqrt{2}\varepsilon^{-1}D_\ell)\|_{C^0}\Big),
\end{aligned}$$

where $c = c(n, c_0, E_0, \eta, \beta)$, $c' = c'(n, c_0, E_0, \eta, \beta, \theta)$, $\kappa = \kappa(n, c_0, E_0, \eta, \beta)$, $\kappa' = \kappa'(n, c_0, E_0, \eta, \beta, \theta)$. *The norms and seminorms in the last inequality are taken over all* $y' \in \Gamma_\ell$ *with* $\Pi(y') \in B^{n-1}_\varepsilon(\Pi(y))$.

Proof. See Appendix C. □

We now indicate how the enhanced second fundamental form tensor is affected by these estimates.

Fix $\ell \in \{1, \ldots, Q\}$. We see from (2.21) that

$$\varepsilon\|\nabla\phi - \langle\nabla\phi, \nabla d_\ell\rangle\nabla d_\ell\|_{C^0(\mathcal{M}_\ell(r))} \leq c'\varepsilon^2 + c'\sum_{m=1}^{Q} A_m(r + 2K\varepsilon|\log\varepsilon|)^{1+\kappa} \tag{2.23}$$

for some $\kappa = \kappa(n, c_0, E_0, \eta, \beta) > 0$. Likewise, from (2.12), (2.13), Lemmas 2.7–2.8, and (2.21),

(2.24)

$$\varepsilon\|\nabla U[\mathbf{h}] - \langle\nabla U[\mathbf{h}], \nabla d_\ell\rangle\nabla d_\ell\|_{C^0(\mathcal{M}_\ell(r))} \leq c'\varepsilon^2 + c'\sum_{m=1}^{Q} A_m(r + 2K\varepsilon|\log\varepsilon|)^{1+\kappa}.$$

Combining (2.17), (2.23), and (2.24), we get

$$\varepsilon\|\nabla u-\langle\nabla u,\nabla d_\ell\rangle\nabla d_\ell\|_{C^0(\mathcal{M}_\ell(r))}\leq c'\varepsilon^2+c'\sum_{m=1}^{Q}A_m(r+2K\varepsilon|\log\varepsilon|)^{1+\kappa}. \tag{2.25}$$

Combining (2.5) and (2.25), we get

$$\|\nu-(-1)^{\ell-1}\nabla d_\ell\|_{C^0(\mathcal{M}_\ell(r)\cap\{|u|\leq 1-\beta\})}\leq c'\varepsilon^2+c'\sum_{m=1}^{Q}A_m(r+2K\varepsilon|\log\varepsilon|)^{1+\kappa}, \tag{2.26}$$

where $\nu=|\nabla u|^{-1}\nabla u$ denotes the normal to the level set of u through each point. (The level set is smooth on $\{|u|\leq 1-\beta\}$ in view of (2.5).)

For the remainder of this section, we choose to work in Fermi coordinates (y,t) relative to Γ_ℓ; note that $t=d_\ell$. It is not hard to see that the only non-trivial Christoffel symbols in this coordinate system are Γ^t_{ij}, Γ^i_{jt}, Γ^i_{tj}, and Γ^k_{ij}. Set

$$\widehat{\Gamma}_\ell(r)\triangleq\sup_{\mathcal{M}_\ell(r)\cap\{|u|\leq 1-\beta\}}|\Gamma^t_{ij}|+|\Gamma^i_{jt}|+|\Gamma^i_{tj}|+|\Gamma^k_{ij}|. \tag{2.27}$$

By arguing as above, and relying on (2.21), we find that

$$\begin{aligned}\varepsilon^2\|\nabla^2 u-\nabla^2 u(\partial_t,\partial_t)\,dt^2\|_{C^0(\mathcal{M}_\ell(r)\cap\{|u|\leq 1-\beta\})}&\\ \leq \varepsilon^2\sum_{i=1}^{n-1}\|\nabla(((Z_{\Gamma_\ell})_*\partial_{y_i})u)\|_{C^0(\mathcal{M}_\ell(r)\cap\{|u|\leq 1-\beta\})}&\\ +\varepsilon^2\widehat{\Gamma}_\ell(r)\|\nabla u\|_{C^0(\mathcal{M}_\ell(r)\cap\{|u|\leq 1-\beta\})}&\\ \leq c'\varepsilon^2+c'\sum_{m=1}^{Q}A_m(r+2K\varepsilon|\log\varepsilon|)^{1+\kappa}+c'\varepsilon\widehat{\Gamma}_\ell(r).&\end{aligned} \tag{2.28}$$

Using (2.26) (note that $\partial_t=\nabla d_\ell$),

$$\begin{aligned}&\varepsilon^2\|\nabla^2 u(\partial_t,\partial_t)\,dt\otimes(dt-\langle dt,\nu\rangle\nu^\flat)\|_{C^0(\mathcal{M}_\ell(r)\cap\{|u|\leq 1-\beta\})}\\ &\quad\leq c'\|\nu-\partial_t\|_{C^0(\mathcal{M}_\ell(r)\cap\{|u|\leq 1-\beta\})}\leq c'\varepsilon^2+c'\sum_{m=1}^{Q}A_m(r+2K\varepsilon|\log\varepsilon|)^{1+\kappa},\end{aligned} \tag{2.29}$$

where $\nu^\flat$ denotes ν's dual 1-form. Finally, (2.5), (2.28), and (2.29) give

$$\|\mathcal{A}\|_{C^0(\mathcal{M}_\ell(r)\cap\{|u|\leq 1-\beta\})}\leq c'\varepsilon+c'\varepsilon^{-1}\sum_{m=1}^{Q}A_m(r+2K\varepsilon|\log\varepsilon|)^{1+\kappa}+c'\widehat{\Gamma}_\ell(r). \tag{2.30}$$

Now, we turn to estimating H_{Γ_ℓ}. From Lemma 2.8 and (2.21) we have, for $y \in \Gamma_\ell(\frac{8}{10})$,

$$\begin{aligned}\varepsilon|\Delta_{\Gamma_\ell} h_\ell(y)| &\leq \varepsilon^2 |\nabla^2_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y)| + \varepsilon^\kappa \exp(-\sqrt{2}\varepsilon^{-1} D_\ell(y)) \\ &\leq c'\varepsilon^2 + c'\varepsilon^\kappa \sum_{m=1}^{Q} A_m(|y| + 2K\varepsilon|\log\varepsilon|) + c' \sum_{m=1}^{Q} A_m(|y| + 2K\varepsilon|\log\varepsilon|)^{1+\kappa}.\end{aligned} \tag{2.31}$$

We are going to estimate the terms in (2.18) from above by a function of ε and the quantities in (2.19). Fix $\ell \in \{1, \ldots, Q\}$, $y \in \Gamma_\ell(\frac{7}{10})$.

From Lemmas 2.7 and 2.8 and (2.20), we have

$$\begin{aligned}&(\varepsilon^{-1}|h_\ell| + \varepsilon^{-1}|h_{\ell-1} \circ \Pi_{\ell-1} \circ Z_{\Gamma_\ell}| + \varepsilon^{\frac{1}{3}}) \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-1}(y)|) \\ &\quad + (\varepsilon^{-1}|h_\ell| + \varepsilon^{-1}|h_{\ell+1} \circ \Pi_{\ell+1} \circ Z_{\Gamma_\ell}| + \varepsilon^{\frac{1}{3}}) \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+1}(y)|) \\ &\quad + \exp(-(\tfrac{3}{2}\sqrt{2})\varepsilon^{-1}|d_{\ell-1}(y)|) + \exp(-(\tfrac{3}{2}\sqrt{2})\varepsilon^{-1}|d_{\ell+1}(y)|) \\ &\quad + \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-2}(y)|)) + \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+2}(y)|) \\ &\leq c'\varepsilon^2 + c'\varepsilon^\kappa \sum_{m=1}^{Q} A_m(|y| + K\varepsilon|\log\varepsilon|) + c' \sum_{m=1}^{Q} A_m(|y| + K\varepsilon|\log\varepsilon|)^{1+\kappa}.\end{aligned} \tag{2.32}$$

By Lemma 2.3, (2.20), and (2.21), every $m \neq \ell$ satisfies

$$\begin{aligned}&\varepsilon^{-1}|d_m(y)| \exp(-\sqrt{2}\varepsilon^{-1}|d_m(y)|)\Big[\varepsilon\|\Delta_{\Gamma_m} h_m - H_{\Gamma_m}\|_{C^0} + \|\nabla_{\Gamma_m} h_m\|^2_{C^0}\Big] \\ &\leq c' A_m(|y| + 2K\varepsilon|\log\varepsilon|)^{1-\rho} \sum_{m' \neq m} A_m(|y| + 2K\varepsilon|\log\varepsilon|) \\ &\quad + c'\varepsilon^2 A_m(|y| + 2K\varepsilon|\log\varepsilon|)^{1-\rho}\end{aligned} \tag{2.33}$$

for small $\rho > 0$, $\varepsilon \leq \varepsilon'$. The C^0 norms are taken over $\Gamma_m \cap C_{\varepsilon^{4/3}}(\Pi(y))$. By Lemma 2.7 and (2.20),

$$\begin{aligned}\sup_{|t|<6\varepsilon|\log\varepsilon|} &\Big[\varepsilon^4 |(\nabla^2_{\Gamma_{m,t}}(\phi|_{\Gamma_{m,t}}))(Z_{\Gamma_m}(y,t))|^2 \\ &\qquad + \varepsilon^2|(\nabla_{\Gamma_{m,t}}(\phi|_{\Gamma_{\ell,t}})(Z_{\Gamma_m}(y,z))|^2 + |\phi(Z_{\Gamma_m}(y,z))|^2\Big] \\ &\leq c'\varepsilon^2 + c' \sum_{m'=1}^{Q} A_{m'}(|y| + K\varepsilon|\log\varepsilon|)^2.\end{aligned} \tag{2.34}$$

Combined, (2.18) and (2.31)–(2.34) give

$$-\varepsilon H_{\Gamma_\ell}(y) = \frac{4(A_0)^2}{h_0}\Big(\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-1}(y)|) - \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+1}(y)|)\Big) + \mathcal{R}_\ell \tag{2.35}$$

for all $y \in \Gamma_\ell(\frac{7}{10})$, where

$$\begin{aligned} |\mathcal{R}_\ell(y)| \le c'\varepsilon^2 + c'\varepsilon^\kappa \sum_{m=1}^{Q} A_m(|y| + 2K\varepsilon|\log\varepsilon|) \\ + c' \sum_{m=1}^{Q} A_m(|y| + 2K\varepsilon|\log\varepsilon|)^{1+\kappa}. \end{aligned} \tag{2.36}$$

LEMMA 2.9. *Let $f : B_1^{n-1}(0) \to \mathbf{R}$ be as in (2.9)–(2.10). If $G[f]$ is the normal graph of f over Γ_ℓ, i.e., $G[f] = \{Z_{\Gamma_\ell}(y, f(y)) : y \in B_1^{n-1}(0)\}$, then*

$$H_{G[f]} - H_{\Gamma_\ell} = -(\mathcal{L} + |\mathrm{I\!I}_{\Gamma_\ell}|^2 + \mathrm{Ric}_g(\nu_{\Gamma_\ell}, \nu_{\Gamma_\ell})|_{\Gamma_\ell})f + \mathcal{Q}(f),$$

where $\mathcal{L}$ is the linear uniformly elliptic operator

$$\mathcal{L}\varphi = \mathcal{L}_{\Gamma_\ell, G[f]}\varphi \triangleq a(y)^{-1}\,\mathrm{div}_{\Gamma_\ell}\left(a(y)\langle (Z_{\Gamma_\ell})_*\nu_{\Gamma_\ell}, \nu_{G[f]}\rangle \nabla_{G[f]}\varphi\right), \tag{2.37}$$

with

$$a(y) = a_{\Gamma_\ell, G[f]}(y) \triangleq \frac{\sqrt{g_{\Gamma_\ell}}}{\sqrt{g_{f(y)}}}. \tag{2.38}$$

Here $(Z_{\Gamma_\ell})_*\nu_{\Gamma_\ell}$, $\nu_{G[f]}$ are upward pointing unit normal in Fermi coordinates and the upward pointing unit normal to $G[f]$, both evaluated at $Z_{\Gamma_\ell}(y, f(y))$. Note that the elliptic symbol coefficients are uniformly bounded away from 0 and ∞ depending on (2.9). The (non-linear) error term $\mathcal{Q}(f)$ satisfies

$$|\mathcal{Q}(f)| \le c'(|f|^2 + |\nabla_{\Gamma_\ell} f|^2).$$

Proof. This is a restatement of Lemma A.1 from Appendix A. □

Notice that, by (2.9)–(2.10), $\Gamma_{\ell+1}$ can be viewed as a normal graph of some function $f_{\ell,\ell+1}$ over Γ_ℓ that satisfies the conditions of Lemma 2.9. Let

$$y' \triangleq Z_{\Gamma_\ell}(y, f_{\ell,\ell+1}(y)) \in \Gamma_{\ell+1}.$$

Applying (2.35) to y at Γ_ℓ and to y' at $\Gamma_{\ell+1}$, subtracting, and invoking Lemma 2.9, we see that

$$
\begin{aligned}
&\varepsilon(\mathcal{L} + |\mathrm{I\!I}_{\Gamma_\ell}|^2 + \mathrm{Ric}_g(\nu,\nu)|_{\Gamma_\ell} + \mathcal{Q}) f_{\ell,\ell+1}(y) \\
&\quad = \varepsilon(H_{\Gamma_\ell}(y) - H_{\Gamma_{\ell+1}}(y')) \\
&\quad = \frac{4(A_0)^2}{h_0}\Big(\exp(-\sqrt{2}\varepsilon^{-1} f_{\ell,\ell+1}(y)) - \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+2}(y')|) \\
&\qquad\qquad - \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-1}(y)|) + \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+1}(y)|)\Big) \\
&\qquad - \mathcal{R}_\ell(y) + \mathcal{R}_{\ell+1}(y').
\end{aligned}
\tag{2.39}
$$

Here, $\mathcal{L}$ is the second order linear operator defined in (2.37), which depends on Γ_ℓ, $\Gamma_{\ell+1}$. Note that (see Lemma B.1):

$$
\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+1}(y)|) = \exp(-\sqrt{2}\varepsilon^{-1} f_{\ell,\ell+1}(y)) + O(\varepsilon^{\frac{1}{3}}) \exp(-\sqrt{2}\varepsilon^{-1} D_\ell(y)).
\tag{2.40}
$$

Absorbing the last term above into $\mathcal{R}_\ell$ in view of (2.36), we conclude that

$$
\begin{aligned}
&\varepsilon(\mathcal{L} + |\mathrm{I\!I}_{\Gamma_\ell}|^2 + \mathrm{Ric}_g(\nu,\nu)|_{\Gamma_\ell} + \mathcal{Q}) f_{\ell,\ell+1}(y) \\
&\quad = \frac{4(A_0)^2}{h_0}\Big(2\exp(-\sqrt{2}\varepsilon^{-1} f_{\ell,\ell+1}(y)) - \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+2}(y')|) \\
&\qquad\qquad - \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-1}(y)|)\Big) \\
&\qquad - \mathcal{R}_\ell(y) + \mathcal{R}_{\ell+1}(y').
\end{aligned}
\tag{2.41}
$$

Finally, dropping the negative terms gives

$$
\begin{aligned}
&\varepsilon(\mathcal{L} + |\mathrm{I\!I}_{\Gamma_\ell}|^2 + \mathrm{Ric}_g(\nu,\nu)|_{\Gamma_\ell} + \mathcal{Q}) f_{\ell,\ell+1}(y) \\
&\quad \leq \frac{8(A_0)^2}{h_0} \exp(-\sqrt{2}\varepsilon^{-1} f_{\ell,\ell+1}(y)) + c'|\mathcal{R}_\ell(y)| + |\mathcal{R}_{\ell+1}(y')|;
\end{aligned}
\tag{2.42}
$$

the error terms $\mathcal{R}_\ell$, $\mathcal{R}_{\ell+1}$ are still as in (2.36).

3. Stable phase transitions ($n = 3$)

In this section, we use the Allen–Cahn stability inequality and bootstrap the distance estimates from the previous section until they become sufficiently sharp. Specifically, we combine three things: (i) an L^2 estimate on the height function of $\{u = 0\}$ (following an observation of Wang–Wei [WW19a, (19.7)]), (ii) a subtle application of Moser's Harnack inequality, and (iii) the non-existence of non-trivial entire stable critical points of the Toda system on $\mathbf{R}^2$; cf. the stable Bernstein problem for minimal surfaces in $\mathbf{R}^3$.

3.1. *Strong sheet distance lower bounds.* We continue to adopt the conventions and notation laid out in Section 2. In particular, we emphasize that we continue to assume (2.1)–(2.6) as well as assuming that u is a stable critical point of $E_\varepsilon \llcorner \Omega$ (cf. Definition 1.2).

In [WW19a, (19.7)], Wang–Wei derive the following stability inequality (in a slightly different setting) from the usual Allen–Cahn stability inequality:

$$\begin{aligned}(3.1)\qquad &\int_{\Gamma_\ell(7/10)} \zeta^2 \Big[\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-1}|) + \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+1}|)\Big] \\ &\quad \leq c' \int_{\Gamma_\ell(7/10)} \varepsilon^2 |\nabla_{\Gamma_\ell}\zeta|^2 + c'\varepsilon^{1+\kappa} \int_{\Gamma_\ell(7/10)} \zeta^2\end{aligned}$$

for all $\ell \in \{1, \ldots, Q\}$, $\zeta \in C_c^\infty(\Gamma_\ell(\frac{7}{10}))$, $\varepsilon \leq \varepsilon'$, where ε', c', κ depend on c_0, E_0, η, β. In fact, by a careful inspection of Wang–Wei's derivation of (3.1) from [WW19a, §19], we see that the following stronger inequality is true here:

$$\begin{aligned}(3.2)\qquad &\int_{\Gamma_\ell(7/10)} \zeta^2 \Big[\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-1}|) + \exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+1}|)\Big] \\ &\quad \leq c' \int_{\Gamma_\ell(7/10)} \varepsilon^2 |\nabla_{\Gamma_\ell}\zeta|^2 + |\mathcal{E}_\zeta| \int_{\Gamma_\ell(7/10)} \zeta^2,\end{aligned}$$

with

$$\begin{aligned}(3.3)\qquad &|\mathcal{E}_\zeta| \leq c'\varepsilon^2 \\ &+ c' \sum_{m=1}^{Q} \sup\Big\{\exp(-\sqrt{2}(1+\kappa)\varepsilon^{-1} D_m(y')) : y' \in \Gamma_m \cap \Pi_\ell^{-1}(B^2_{2K\varepsilon|\log\varepsilon|}(\operatorname{spt} h))\Big\}\,;\end{aligned}$$

here, c', κ are independent of ζ. We prove (3.2) in Appendix D in a general n-dimensional setting, $n \geq 3$. (Below, we use it for $n = 3$.) Note that, by Lemma 2.7, this recovers (3.1).

Our first main result is the following sheet-distance estimate. (cf. Remark 2.6.)

Proposition 3.1 (Stable sheet distances, I). *If u is a stable critical point of $E_\varepsilon \llcorner \Omega$, $\varepsilon \leq \varepsilon_3$, and $\nu \in (0, \frac{1}{2})$, then*

$$D_\ell \geq (1-\nu)\sqrt{2}\varepsilon|\log\varepsilon| \text{ on } \Gamma_\ell(\tfrac{1}{3})$$

for all $\ell \in \{1, \ldots, Q\}$, where $\varepsilon_3 = \varepsilon_3(c_0, E_0, \eta, \beta, \nu)$.

Proof. Take $\nu \in (0, \frac{1}{2})$ and assume, for contradiction, that

$$(3.4)\qquad A_{\ell_0}(r) \geq A_{\ell_0}(\tfrac{1}{3}) > \varepsilon^{2(1-\nu)} \text{ for all } r \in [\tfrac{1}{3}, \tfrac{1}{2}] \text{ and some } \ell_0 \in \{1, \ldots, Q\}.$$

We will aim to prove

$$(3.5)\qquad \max_{\ell\in\{1,\ldots,Q\}} A_\ell(r - K_\nu\varepsilon^\nu) < \tfrac{1}{2} \max_{\ell\in\{1,\ldots,Q\}} A_\ell(r) \text{ for all } r \in [\tfrac{1}{3}, \tfrac{1}{2}],$$

where $K_\nu = K_\nu(c_0, E_0, \eta, \beta, \nu) > 0$; this will in turn prove our claim by a simple iteration. (We denote the dependence of K_ν on ν explicitly to disambiguate with the previous constant K. Let us assume $K_\nu > 2K$.)

Let $r \in [\frac{1}{3}, \frac{1}{2}]$, $\alpha \triangleq \max\{A_\ell(r) : \ell \in \{1, \ldots, Q\}\}$. Since

$$\alpha > \varepsilon^{2(1-\nu)} \tag{3.6}$$

by (3.4), it follows that to prove (3.5) it will suffice to prove

$$A_\ell(r - \varepsilon K_\nu \alpha^{-\frac{1}{2}}) < \tfrac{1}{2}\alpha \text{ for all } \ell \in \{1, \ldots, Q\}. \tag{3.7}$$

Suppose, by way of contradiction, that (3.7) is violated at some $\ell_0 \in \{1, \ldots, Q\}$ and $y \in \Gamma_{\ell_0}(r - \varepsilon K_\nu \alpha^{-\frac{1}{2}})$. From now on let us work in the coordinate chart induced on Γ_{ℓ_0} by $\Pi|_{\Gamma_{\ell_0}} \approx \Sigma$. For $\widetilde{y} \in B^2_{K_\nu/2}(0)$, define

$$\widetilde{f}(\widetilde{y}) \triangleq \varepsilon^{-1} f_{\ell_0,\ell_0+1}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y}) - \frac{1}{\sqrt{2}}|\log\alpha|. \tag{3.8}$$

If $\widetilde{\mathcal{L}}$ denotes the translation and rescaling of $\mathcal{L}$ that respects the stretched coordinate, $\widetilde{y}$, then from (2.42) we find

$$\begin{aligned}
\widetilde{\mathcal{L}}\widetilde{f}(\widetilde{y}) &= \varepsilon\alpha^{-1}\mathcal{L} f_{\ell_0,\ell_0+1}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y}) \\
&\le \frac{8(A_0)^2\alpha^{-1}}{h_0}\exp(-\sqrt{2}\varepsilon^{-1} f_{\ell_0,\ell_0+1}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y})) \\
&\quad + \alpha^{-1}c'|\mathcal{R}_{\ell_0}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y})| + \alpha^{-1}|\mathcal{R}_{\ell_0+1}((y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y})')| \\
&\quad - \varepsilon\alpha^{-1}(|\mathrm{I\!I}_{\Gamma_{\ell_0}}|^2 + \mathrm{Ric}_g(\nu,\nu)|_{\Gamma_{\ell_0}} + \mathcal{Q}) f_{\ell_0,\ell_0+1}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y}).
\end{aligned}$$

Recalling (3.8), the computation above readily implies that

$$\begin{aligned}
\widetilde{\mathcal{L}}\widetilde{f}(\widetilde{y}) \le{} & \frac{8(A_0)^2}{h_0}\exp(-\sqrt{2}\widetilde{f}(\widetilde{y})) \\
& + \alpha^{-1}c'|\mathcal{R}_{\ell_0}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y})| + \alpha^{-1}|\mathcal{R}_{\ell_0+1}((y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y})')| \\
& - \varepsilon\alpha^{-1}(|\mathrm{I\!I}_{\Gamma_{\ell_0}}|^2 + \mathrm{Ric}_g(\nu,\nu)|_{\Gamma_{\ell_0}} + \mathcal{Q}) f_{\ell_0,\ell_0+1}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y}).
\end{aligned} \tag{3.9}$$

From (2.36) and (3.6) we have

$$\alpha^{-1}c'|\mathcal{R}_{\ell_0}| + \alpha^{-1}|\mathcal{R}_{\ell_0+1}| \le c'(\varepsilon^2\alpha^{-1} + \varepsilon^\kappa + \alpha^\kappa) \le c'(\alpha^{\frac{\nu}{1-\nu}} + \alpha^{\frac{\kappa}{2(1-\nu)}}) \le c'. \tag{3.10}$$

Now define the auxiliary function $\psi \triangleq \exp(-\sqrt{2}\widetilde{f}) > 0$. From the chain rule, (3.9), and (3.10), we have

(3.11)
$$\begin{aligned}\widetilde{\mathcal{L}}\psi &= -\sqrt{2}(\widetilde{\mathcal{L}}\widetilde{f})\psi + 2|\widetilde{\nabla}\widetilde{f}|^2\psi \\ &\geq -\frac{8\sqrt{2}(A_0)^2}{h_0}\psi^2 - c'\psi + \sqrt{2}\varepsilon\alpha^{-1}(|\mathrm{I\!I}_{\Gamma_{\ell_0}}|^2 + \mathrm{Ric}_g(\nu,\nu)|_{\Gamma_{\ell_0}})f_{\ell_0,\ell_0+1}\psi \\ &\quad + \alpha^{-1}(\sqrt{2}\varepsilon\mathcal{Q}(f_{\ell_0,\ell_0+1}) + |\nabla_{\Gamma_{\ell_0}} f_{\ell_0,\ell_0+1}|^2)\psi \\ &\geq -\frac{8\sqrt{2}(A_0)^2}{h_0}\psi^2 - c'\psi + \sqrt{2}\varepsilon\alpha^{-1}(|\mathrm{I\!I}_{\Gamma_{\ell_0}}|^2 + \mathrm{Ric}_g(\nu,\nu)|_{\Gamma_{\ell_0}})f_{\ell_0,\ell_0+1}\psi \\ &\quad - \alpha^{-1}\left[\sqrt{2}\varepsilon\mathcal{Q}(f_{\ell_0,\ell_0+1}) + |\nabla_{\Gamma_{\ell_0}} f_{\ell_0,\ell_0+1}|^2\right]_-\psi\end{aligned}$$

on $B^2_{K_\nu}(0)$. Here, $[\cdot]_-$ denotes the negative part of a real number (and is a non-negative quantity). Using a logarithmic cutoff function in (3.1), which is 1 on $B^2_{\varepsilon\alpha^{-1/2}\sqrt{K_\nu}}(0)$ and 0 outside $B^2_{\varepsilon\alpha^{-1/2}K_\nu/2}(0)$, we get

$$\int_{B^2_{\sqrt{K_\nu}}(0)} \psi \leq c'(\log K_\nu)^{-1} + c'\alpha^{\frac{\kappa+2\nu-1}{2(1-\nu)}}K_\nu^2 \tag{3.12}$$

in the scale of ψ. By Moser's weak maximum principle on B_1 for (3.11) (see, e.g., [HL97, Th. 4.1]), the L^1 bound in (3.12) implies the L^∞ bound

$$\psi(0) \leq C_\star \int_{B^2_1(0)} \psi \leq C_\star\left((\log K_\nu)^{-1} + \alpha^{\frac{\kappa+2\nu-1}{2(1-\nu)}}K_\nu^2\right) \tag{3.13}$$

for a constant $C_\star$ that depends on the constants in (2.11) and the L^∞ norm of the coefficients in the differential inequality (3.11) on $B^2_1(0)$. We are assuming that (3.7) fails at y, so together with (2.1), (2.9), (2.10), and Lemma 2.9, we have

(3.14)
$$\begin{aligned}&\sup_{\widetilde{y}\in B^2_1(0)} |\varepsilon\alpha^{-1}(|\mathrm{I\!I}_{\Gamma_{\ell_0}}|^2 + \mathrm{Ric}_g(\nu,\nu)|_{\Gamma_{\ell_0}})f_{\ell_0,\ell_0+1}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y})| \\ &\qquad \leq c'\varepsilon\alpha^{-1}(|f_{\ell_0,\ell_0+1}(y)| + \operatorname{osc}\{f_{\ell_0,\ell_0+1} : \Gamma_{\ell_0} \cap C_{\varepsilon\alpha^{-1/2}}(\Pi(y))\}) \leq c'\varepsilon^2\alpha^{-\frac{3}{2}} \\ &\qquad \leq c'\alpha^{\frac{3\nu-1}{2(1-\nu)}}.\end{aligned}$$

Likewise, using Lemma 2.9, we can estimate

$$\sqrt{2}\varepsilon|\mathcal{Q}(f_{\ell_0,\ell_0+1})| \leq c'\varepsilon(|f_{\ell_0,\ell_0+1}|^2 + |\nabla_{\Gamma_{\ell_0}} f_{\ell_0,\ell_0+1}|^2).$$

By absorbing the gradient term and estimating f_{ℓ_0,ℓ_0+1} with the same argument as in (3.14), we also estimate

$$\begin{aligned}(3.15)\qquad &\alpha^{-1}\left[2|\nabla_{\Gamma_{\ell_0}} f_{\ell_0,\ell_0+1}|^2 + \sqrt{2}\varepsilon\mathcal{Q}f_{\ell_0,\ell_0+1}(y+\varepsilon\alpha^{-\frac{1}{2}}\widetilde{y})\right]_- \\ &\leq c'\varepsilon\alpha^{-1}f^2_{\ell_0,\ell_0+1} \leq c'\varepsilon^3\alpha^{-2} \leq c'\alpha^{\frac{4\nu-1}{2(1-\nu)}}.\end{aligned}$$

Thus, ignoring the unimportant dependencies on (2.11), we have

$$(3.16)\qquad C_\star = C_\star(1+\alpha^{\frac{\kappa+2\nu-1}{2(1-\nu)}} + \alpha^{\frac{3\nu-1}{2(1-\nu)}} + \alpha^{\frac{4\nu-1}{2(1-\nu)}}),$$

which, as long as $\nu > \max\{\frac{1}{3}, \frac{1-\kappa}{2}\}$, can be taken to be uniformly bounded independently of α—though certainly depending on the constants in (2.11)—since $\alpha \leq 1$ by definition.

Since $C_\star$ is uniformly bounded per (3.16), it follows from (3.13) that by choosing suitably large $K_\nu = K_\nu(c_0, E_0, \eta, \beta, \nu) > 0$, $\psi(0)$ will become less than $\frac{1}{2}$ for small α, contradicting our assumption that (3.7) is violated. Specifically, recalling (3.13), we may simply pick K_ν large enough that $C_\star(\log K_\nu)^{-1} < \frac{1}{4}$, in which case $\psi(0) < \frac{1}{2}$ as long as α is small enough that $C_\star\alpha^{\frac{\kappa+2\nu-1}{2(1-\nu)}}K_\nu^2 < \frac{1}{4}$.

This concludes the proof of Proposition 3.1 for

$$\nu \in (\nu_0, \tfrac{1}{2}),\ \text{where } \nu_0 = \min\{\tfrac{1}{3}, \tfrac{1-\kappa}{2}\}.$$

The next step is to show that ν_0 can be taken to be arbitrarily small, at the expense of possibly having to rescale our domain a finite number of times.

Retracing the proof above, it is not hard to see that what one needs to improve are

(1) the exponent of α in (3.12) and (3.13), and
(2) the oscillation bounds in (3.14) and (3.15).

For the prior, we may use (3.2) and (3.3) instead of (3.1); we get

$$\psi(0) \leq C_\star\left((\log K_\nu)^{-1} + (\alpha^{\frac{\nu}{1-\nu}} + \alpha^\kappa)K_\nu^2\right),$$

a sufficient bound.

For the latter, we need to use a Harnack-type inequality on the elliptic *equation* satisfied by f_{ℓ_0,ℓ_0+1}, (2.41). Recalling (2.36), and using the fact that we now know Proposition 3.1 to hold for $\nu' \in (\nu_0, \frac{1}{2})$, we see that the right-hand side of (2.41) can be bounded in L^∞ by

$$c'\varepsilon^2 + c'\sum_{m=1}^{Q} A_m(|y| + 2K\varepsilon|\log\varepsilon|) \leq c'\varepsilon^{2(1-\nu')}$$

for some $\nu' \in (\nu_0, \frac{1}{2})$. Diving (2.41) through by ε, we thus get a uniformly elliptic equation

$$(3.17)\qquad (\mathcal{L} + |\,\mathrm{I\!I}_{\Gamma_\ell}|^2 + \mathrm{Ric}_g(\nu,\nu)|_{\Gamma_\ell} + \mathcal{Q}) f_{\ell,\ell+1}(y) = O(\varepsilon^{1-2\nu'}).$$

Now we apply the inhomogeneous Harnack-type inequality found in [GT01, Ths. 8.17, 8.18] to (2.41), multiplied through by the $a(y)$ in (2.38), with some $q > 2$, $R = \varepsilon\alpha^{-\frac{1}{2}}$, and $g = O(\varepsilon^{1-2\nu'})$ (in the L^∞ sense). We get

$$\begin{aligned}&\sup\{f_{\ell_0,\ell_0+1} : \Gamma_{\ell_0} \cap C_{\varepsilon\alpha^{-1/2}}(\Pi(y))\} \\ &\qquad \le c'\Big(f_{\ell_0,\ell_0+1}(y) + \varepsilon^2\alpha^{-1}\cdot\varepsilon^{1-2\nu'}\Big) = c'\Big(f_{\ell_0,\ell_0+1}(y) + \varepsilon^{3-2\nu'}\alpha^{-1}\Big).\end{aligned}$$

Recall that we are assuming, by contradiction, that (3.7) is violated at our y, implying that $f_{\ell_0,\ell_0+1}(y)$ is an error term relative to the last term of the right-hand side. Together with (3.6), this gives

$$(3.18)\qquad \begin{aligned}&\sup\{\varepsilon\alpha^{-1} f_{\ell_0,\ell_0+1} : \Gamma_{\ell_0} \cap C_{\varepsilon\alpha^{-1/2}}(\Pi(y))\} \\ &\qquad \le c'\varepsilon^{4-2\nu'}\alpha^{-2} \le c'\alpha^{\frac{2-\nu'}{1-\nu}-2} = c'\alpha^{\frac{2\nu-\nu'}{1-\nu}}.\end{aligned}$$

This is $\le c'\alpha^\delta$ for some $\delta > 0$ as long as $\nu > \nu_0' \triangleq \frac{1}{2}\nu'$. This gives the improved oscillation bound that we sought in place of (3.14), and Proposition 3.1 follows in full by iteratively pushing ν, ν_0' down to zero. ☐

PROPOSITION 3.2 (Stable sheet distances, II). *If u is as in Proposition 3.1, then*

$$\lim_{\varepsilon\to 0} \frac{\exp(-\sqrt{2}\varepsilon^{-1}D_\ell)}{\varepsilon^2|\log\varepsilon|} = 0 \text{ on } \Gamma_\ell(\tfrac{1}{6})$$

for all $\ell \in \{1,\ldots,Q\}$, uniformly in terms of c_0, E_0, η, β.

Proof. The proof follows along the same lines as the bootstrap portion of the proof of Proposition 3.1. However, the modifications are somewhat delicate so we give the argument here.

We first prove a weaker bound. We argue by contradiction, assuming that there exists $\ell \in \{1,\ldots,Q\}$ such that

$$(3.19)\qquad A_\ell(r) \ge A_\ell(1/5) > \varepsilon^2|\log\varepsilon|^2 \text{ for all } r \in [\tfrac{1}{5},\tfrac{1}{4}].$$

Let $r \in [\frac{1}{5},\frac{1}{4}]$, and then let $\alpha \triangleq \max\{A_\ell(r) : \ell \in \{1,\ldots,Q\}\}$. Then

$$(3.20)\qquad \alpha > \varepsilon^2|\log\varepsilon|^2.$$

We claim that

$$(3.21)\qquad \max_{\ell\in\{1,\ldots,Q\}} A_\ell(r - \varepsilon K_0\alpha^{-\frac{1}{2}}) < \tfrac{1}{2}\alpha$$

for a constant $K_0 = K_0(c_0, E_0, \eta, \beta) > 0$.

Suppose that (3.21) fails for $\ell_0 \in \{1, \ldots, Q\}$ and $y \in \Gamma_{\ell_0}(\varepsilon K_0 \alpha^{-\frac{1}{2}})$. Define

$$\widetilde{f}(\widetilde{y}) \triangleq \varepsilon^{-1} f_{\ell_0,\ell_0+1}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y}) - \tfrac{1}{\sqrt{2}}|\log \alpha| \tag{3.22}$$

for $\widetilde{y} \in B^2_{K_0/2}(0)$. Proceeding as in (3.9), we find that

$$\begin{aligned}\widetilde{\mathcal{L}}\widetilde{f}(\widetilde{y}) \leq{}& \frac{8A_0}{h_0}\exp(-\sqrt{2}\widetilde{f}(\widetilde{y}))\\ &+ \alpha^{-1}c'|\mathcal{R}_{\ell_0}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y})| + \alpha^{-1}|\mathcal{R}_{\ell_0+1}((y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y})')|\\ &- \varepsilon\alpha^{-1}(|\,\mathrm{I\!I}_{\Gamma_{\ell_0}}|^2 + \mathrm{Ric}_g(\nu,\nu)|_{\Gamma_{\ell_0}} + \mathcal{Q}) f_{\ell_0,\ell_0+1}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y}).\end{aligned} \tag{3.23}$$

We also still have an estimate of the form

$$\alpha^{-1}c'|\mathcal{R}_{\ell_0}| + \alpha^{-1}|\mathcal{R}_{\ell_0+1}| \leq c', \tag{3.24}$$

and the function $\psi \triangleq \exp(-\sqrt{2}\widetilde{f})$ still satisfies a differential inequality of the form

$$\begin{aligned}\widetilde{\mathcal{L}}\psi \geq{}& -\frac{8\sqrt{2}(A_0)^2}{h_0}\psi^2 - c'\psi\\ &+ \sqrt{2}\varepsilon\alpha^{-1}(|\,\mathrm{I\!I}_{\Gamma_{\ell_0}}|^2 + \mathrm{Ric}_g(\nu,\nu)|_{\Gamma_{\ell_0}} + \mathcal{Q}) f_{\ell_0,\ell_0+1}(y + \varepsilon\alpha^{-\frac{1}{2}}\widetilde{y})\psi.\end{aligned} \tag{3.25}$$

Applying the same inhomogeneous Harnack-type inequality that led to (3.18) before, we obtain

$$\begin{aligned}\sup\{f_{\ell_0,\ell_0+1} : \Gamma_{\ell_0} \cap C_{\varepsilon\alpha^{-1/2}}(\Pi(y))\} &\leq c'\Big(f_{\ell_0,\ell_0+1}(y) + \varepsilon^2\alpha^{-1}\cdot\varepsilon^{-1}(\varepsilon^2 + \alpha)\Big)\\ &\leq c'\Big(\varepsilon|\log\alpha| + \varepsilon^3\alpha^{-1} + \varepsilon\Big).\end{aligned}$$

Thus, we have the following L^∞ estimate on the coefficient in front of ψ in the last term of (3.25) on the domain $B^2_{\varepsilon\alpha^{-1/2}}(y)$:

$$\begin{aligned}&\sup\{\varepsilon\alpha^{-1} f_{\ell_0,\ell_0+1} : \Gamma_{\ell_0} \cap C_{\varepsilon\alpha^{-1/2}}(\Pi(y))\}\\ &\quad\leq c'\Big(\varepsilon^2\alpha^{-1}|\log\alpha| + \varepsilon^4\alpha^{-2} + \varepsilon^2\alpha^{-1}\Big) \leq c',\end{aligned} \tag{3.26}$$

where we have used the simple fact that

$$(3.20) \iff \alpha > \varepsilon^2|\log\varepsilon|^2 \implies \varepsilon^2\alpha^{-1}|\log\alpha| = o(1). \tag{3.27}$$

Thus, (3.25) implies the uniformly elliptic partial differential inequality

$$\widetilde{\mathcal{L}}\psi \geq -\frac{8\sqrt{2}(A_0)^2}{h_0}\psi^2 - c'\psi. \tag{3.28}$$

From Moser's weak maximum principle (see, e.g., [HL97, Th. 4.1]) applied to (3.28) on $B_1^2(0)$, combined with (3.3), we get the L^∞ bound

$$\psi(0) \le c' \int_{B_1^2(0)} \psi \le c'\Big((\log K_0)^{-1} + (\varepsilon^2\alpha^{-1} + \alpha^\kappa)K_0^2\Big) \le c'\Big((\log K_0)^{-1} + (o(|\log\alpha|^{-1}) + \alpha^\kappa)K_0^2\Big),$$

violating the assumption that (3.21) fails, provided we take K_0 large and α small.

Thus, (3.21) holds true with a fixed K_0. Then notice that

$$\varepsilon K_0 \alpha^{-\frac{1}{2}} \le K_0 |\log\varepsilon|^{-1}.$$

A backward iteration of (3.21) from $r = \frac{1}{4}$ to $r = \frac{1}{5}$, followed by an application of Proposition 3.1 at radius $r = 1/4$ with $\nu < \frac{\log 2}{20K_0}$, yields

$$\log A_{\ell_0}(\tfrac{1}{5}) \le \log A_{\ell_0}(\tfrac{1}{4}) - \tfrac{\log 2}{20K_0}|\log\varepsilon| \le 2(\nu-1)|\log\varepsilon| - \tfrac{\log 2}{20K_0}|\log\varepsilon| < -2|\log\varepsilon| = \log\varepsilon^2,$$

violating (3.19).

We now prove the main claim. We argue by contradiction again assuming that there exists $\ell \in \{1,\ldots,Q\}$ such that

$$A_\ell(r) \ge A_\ell(\tfrac{1}{5}) > \mu\varepsilon^2|\log\varepsilon| \text{ for all } r \in [\tfrac{1}{6}, \tfrac{1}{5}] \tag{3.29}$$

for some $\mu > 0$. Let $r \in [\frac{1}{6}, \frac{1}{5}]$, $\alpha \triangleq \max\{A_\ell(r) : \ell \in \{1,\ldots,Q\}\}$. Then

$$\alpha > \mu\varepsilon^2|\log\varepsilon|. \tag{3.30}$$

We claim that

$$A_\ell(r - \varepsilon K_0'\alpha^{-\frac{1}{2}}) < \tfrac{1}{2}\alpha \text{ for every } \ell \in \{1,\ldots,Q\} \tag{3.31}$$

for a constant $K_0' = K_0'(c_0, E_0, \eta, \beta) > 0$. This indeed follows from the same argument as above, modulo the fact that one needs to replace (3.27) with

$$(3.30) \iff \alpha > \mu\varepsilon^2|\log\varepsilon| \implies \varepsilon^2\alpha^{-1}|\log\alpha| \le \mu^{-1}(2 + o(1)).$$

Notice, now, that

$$\varepsilon K_0'\alpha^{-\frac{1}{2}} \le \mu^{-\frac{1}{2}} K_0' |\log\varepsilon|^{-\frac{1}{2}},$$

so that a backward iteration of (3.31) from $r = 1/5$ to $r = 1/6$, together with the weaker assertion verified above, yields

$$\log A_{\ell_0}(\tfrac{1}{6}) \le \log A_{\ell_0}(\tfrac{1}{5}) - \mu^{\frac{1}{2}}\tfrac{\log 2}{20K_0'}|\log\varepsilon|^{\frac{1}{2}} \le -2|\log\varepsilon| + 2\log|\log\varepsilon| - \mu^{\frac{1}{2}}\tfrac{\log 2}{20K_0'}|\log\varepsilon|^{\frac{1}{2}}.$$

However,

$$\lim_{\varepsilon\to 0}\left(\log|\log\varepsilon| - \mu^{\frac{1}{2}}\tfrac{\log 2}{20K_0'}|\log\varepsilon|^{\frac{1}{2}}\right) = -\infty$$

so, for sufficiently small ε (depending on K_0, μ), this quantity is $< \log\mu$. Thus, for small ε,

$$\log A_{\ell_0}(\tfrac{1}{6}) \le \log\mu - 2|\log\varepsilon| + \log|\log\varepsilon| = \log(\mu\varepsilon^2|\log\varepsilon|),$$

which violates (3.29). The result follows. □

In fact, Proposition 3.2 and (2.35)–(2.36) establish the following:

COROLLARY 3.3. *If u is as in Proposition 3.2, then for all $\ell \in \{1,\ldots,Q\}$,*

$$\frac{H_{\Gamma_\ell}}{\varepsilon|\log\varepsilon|} \to 0 \text{ uniformly on } \Gamma_\ell(\tfrac{1}{6})$$

as $\varepsilon \to 0$.

This estimate is key for our geometric applications, since it says that the mean curvature of the zero sets u dominates the effect of interactions between the sheets. This will allow us to treat the sheets (essentially) like disjoint minimal surfaces.

3.2. *Curvature estimates.* In what follows, we let $B^n_r(0)$ be a smooth n-ball equipped with a Riemannian metric g so that $B^n_r(0)$ is a geodesic r-ball centered at 0 (with respect to g).

THEOREM 3.4. *Suppose* $\mathrm{inj}_g \ge 3$ *and* $|\mathrm{Rm}_g| + |\nabla_g \mathrm{Rm}_g| \le 1$ *on* $B^3_1(0)$. *If $\varepsilon \le \varepsilon_1$, $u \in C^\infty(B^3_1(0);(-1,1))$ is a stable critical point of $E_\varepsilon \llcorner B^3_1(0)$, and $(E_\varepsilon \llcorner B^3_1(0))[u] \le E_0$, then*

$$|\mathcal{A}(x)| \le c_1 \text{ for all } x \in B^3_{1/2}(0) \cap \{|u| \le 1-\beta\},$$

where $\varepsilon_1 = \varepsilon_1(n, E_0, \beta, W)$, $c_1 = c_1(n, E_0, \beta, W)$.

Remark 3.5. We emphasize that, in one dimension lower, Wang–Wei have proven [WW19a] that stable critical points of E_ε satisfy curvature bounds even without the assumption of uniformly bounded energy.

Remark 3.6. It is not immediately obvious that the enhanced second fundamental form $\mathcal{A}$ is well defined on $B^3_{3/4}(0) \cap \{|u| \le 1-\beta\}$. This can be seen, for instance, by applying the following proposition with $n = 3$. Its "non-existence" condition, when $n = 3$, is guaranteed in view of the work of Ambrosio–Cabré [AC00]. (See also the work of Farina–Mari–Valdinoci [FMV13].)

PROPOSITION 3.7. *Let $u : B^n_1(0) \to (-1,1)$ be a stable critical point of $E_\varepsilon \llcorner B^n_1(0)$ with $(E_\varepsilon \llcorner B^n_1(0))[u] \le E_0$. If $\varepsilon \le \varepsilon_0$ and $\mathbf{R}^n$ with the standard* *metric does not carry any non-trivial (i.e., heteroclinic or* ± 1*) entire stable solutions with Euclidean energy growth, then*

$$\varepsilon|\nabla u_i| \geq c_0^{-1} > 0 \text{ for all } x \in B^n_{3/4}(0) \cap \{|u| \leq 1-\beta\},$$

where ε_0, c_0 *depend on* E_0, β, W.

Proof. We argue by contradiction. If the assertion were false, there would exist a sequence

$$\{(u_i, \varepsilon_i)\}_{i=1,2,\ldots} \subset C^\infty(B^n_1(0); (-1,1)) \times (0,\infty), \ \lim_i \varepsilon_i = 0,$$

where each $u_i : B^n_1(0) \to [-1,1]$ is a stable critical point for $E_{\varepsilon_i} \llcorner B^n_1(0)$, with $(E_{\varepsilon_i} \llcorner B^n_1(0))[u_i] \leq E_0$, and so that $\lim_i \varepsilon_i \nabla u_i(q_i) = 0$ along some $\{q_i\}_{i=1,2,\ldots} \subset B^n_{3/4}(0)$. The rescaled functions

$$v_i(x) \triangleq u_i(\varepsilon_i(x-q_i))$$

are all stable critical points of $E_1 \llcorner B^n_{(1-|q_i|)/\varepsilon_i(0)}$ with Euclidean energy growth. Since the ellipticity constants are uniform at this scale, we may pass to a subsequence with $\lim_i v_i = v_\infty$ in $C^\infty_{\mathrm{loc}}(\mathbf{R}^n)$, where v_∞ is a stable critical point of $E_1 \llcorner \mathbf{R}^n$ with Euclidean area growth, $|v_\infty(0)| \leq 1-\beta$, and $\nabla v_\infty(0) = 0$. No such v_∞ exists; the only entire stable solutions on $\mathbf{R}^n$ with Euclidean energy growth are the constants ± 1 and the one-dimensional heteroclinic solution. □

We are now in a position to prove Theorem 3.4.

Proof of Theorem 3.4. If the assertion were false, there would exist a sequence

$$\{(u_i, \varepsilon_i)\}_{i=1,2,\ldots} \subset C^\infty(B^3_1(0); (-1,1)) \times (0,\infty), \ \lim_i \varepsilon_i = 0,$$

where each $u_i : B^3_1(0) \to (-1,1)$ is a stable critical point for $E_{\varepsilon_i} \llcorner B^3_1(0)$, with $(E_{\varepsilon_i} \llcorner B^3_1(0))[u_i] \leq E_0$, and so that the maximum value

$$\max \left\{ \mathrm{dist}(x, \mathbf{R}^3 \setminus B^3_{3/4}(0)) |\mathcal{A}(x)| : x \in B^3_1(0) \cap \{|u| \leq 1-\beta\} \right\}$$

is attained at some $q_i \in B^3_{3/4}(0)$ with

$$\lim_i \mathrm{dist}(q_i, \partial B^3_{3/4}(0)) |\mathcal{A}_i(q_i)| = \infty.$$

Next, let $\lambda_i \triangleq |\mathcal{A}_i(q_i)|$, which we note also satisfies $\lim_i \lambda_i = \infty$.

Claim. $\liminf_i \varepsilon_i \lambda_i = 0$.

Proof of claim. Rescale to $v_i(x) \triangleq u_i(\varepsilon_i(x-q_i))$, a stable critical point of $E_1 \llcorner B^3_{(1-|q_i|)/\varepsilon_i}(0)$ with quadratic energy growth and $|v_i| \leq 1-\beta$. Since our ellipticity constants are uniform at this scale, we can pass to a subsequence such that $\lim_i v_i = v_\infty$ in $C^\infty_{\mathrm{loc}}(\mathbf{R}^3)$, where v_∞ is a stable critical point of $E_1 \llcorner \mathbf{R}^3$ with $|v_\infty(0)| \leq 1-\beta$. The only such v_∞ is the one-dimensional heteroclinic solution, for which $\mathcal{A}_\infty \equiv 0$, and therefore $\liminf_i \varepsilon_i \lambda_i = |\mathcal{A}_\infty(0)| = 0$. This completes the proof of the claim. □

Pass to a subsequence for which $\liminf_i \varepsilon_i \lambda_i = 0$ is attained, and rescale to $\widetilde{u}_i(x) \triangleq u_i(\lambda_i^{-1}(x - q_i))$. This is a critical point of $E_{\varepsilon_i \lambda_i} \llcorner B^3_{(3/4-|q_i|)\lambda_i}(0)$. We note that

$$\lim_i \varepsilon_i \lambda_i = 0, \ \lim_i (3/4 - |q_i|)\lambda_i = \infty. \tag{3.32}$$

Moreover, for every $R \geq 1$,

$$(E_{\varepsilon_i \lambda_i} \llcorner B^3_{(3/4-|q_i|)\lambda_i}(0))(B^3_R(0)) \leq cR^2 \tag{3.33}$$

for all sufficiently large i. Here, $c > 0$ is fixed. Combining (3.32), (3.33), together with the works of [HT00, Th. 1] and [Gua18, App. B] for the Riemannian modifications, the 2-varifolds

$$V_{\varepsilon_i \lambda_i}[\widetilde{u}_i](\varphi) \triangleq \int \varphi(x, T_x\{\widetilde{u}_i = \widetilde{u}_i(x)\})\, \varepsilon_i \lambda_i |\nabla \widetilde{u}_i(x)|^2,$$

for $\varphi \in C^0_c(\mathrm{Gr}_2(B^3_{(3/4-|q_i|)\lambda_i}(0)))$, converge weakly to a stationary integral varifold $V_\infty \in \mathbf{I}_2(\mathbf{R}^3)$. The enhanced second fundamental form estimates, moreover, imply that $\operatorname{spt} \|V_\infty\|$ is $C^{1,1}$ and, therefore, a smooth minimal surface.

Remark 3.8. We note that the most technical elements of [HT00], such as proving that the limit varifold is integral, can be proven (in the setting at hand) in a much more direct manner given the curvature estimates we now know to be true.

The stability of $\widetilde{u}_i$ is also known to imply stability of $\operatorname{spt} \|V_\infty\|$. Indeed, one may plug $\zeta = \psi |\nabla \widetilde{u}_i|$, $\psi \in C^\infty_c(\mathbf{R}^3)$, into the second variation operator $\delta^2 E_{\varepsilon_i \lambda_i}|_{\widetilde{u}_i}$ and let $i \to \infty$, and recover the second variation operator for $\operatorname{spt} \|V_\infty\|$ with $\psi|_{\operatorname{spt} \|V_\infty\|}$ being the test function. See also [Ton05].

Summarizing, $\operatorname{spt} \|V_\infty\|$ is a smooth, stable, embedded minimal surface in $\mathbf{R}^3$ (in fact, with quadratic area growth). Therefore, the limit is a a disjoint union of planes $P_1, \ldots, P_k \subset \mathbf{R}^3$ with integer multiplicities $m_1, \ldots, m_k \in \{1, 2, \ldots\}$. Without loss of generality, $P_j = \mathbf{R}^2 \times \{z_j\}$, with $0 = z_1 < z_2 < \cdots < z_k$.

We will only need to focus on one of these planes, e.g., P_1. Writing

$$\{\widetilde{u}_i = 0\} \cap (B^2_1(0) \times [-z_2/2, z_2/2]) = \bigcup_{\ell=1}^{m_1} \operatorname{graph} f_{i,\ell},$$

it follows from our rescaling that $f_{i,\ell} : B^2_1(0) \to \mathbf{R}$ all converge, in the $C^{1,\alpha}$ sense on $B^2_{1/2}(0)$, to the zero function as $i \to \infty$. In fact, by dilating as needed, we find ourselves in the setup of Sections 2.1–3.1.

Therefore, by employing Proposition 3.2 (in fact, Proposition 3.1 suffices), it follows from (2.30) that

$$\limsup_{i\to\infty} \|\widetilde{\mathcal{A}}_i\|_{C^0(\mathscr{M}_\ell(1/6)\cap\{|\widetilde{u}_i|\leq 1-\beta\})} \leq c' \limsup_{i\to\infty} \widehat{\Gamma}_\ell(1/6), \tag{3.34}$$

for all $\ell \in \{1,\ldots,Q\}$, where $\widehat{\Gamma}_\ell$ is as in (2.27).

Claim. The right-hand side of (3.34) is zero.

Notice that this claim violates the fact that our dilations were such that $|\widetilde{\mathcal{A}}_i(0)| = 1$ for all $i = 1, 2, \ldots$, and Theorem 3.4 follows.

Proof of claim. From the Riccati equation, (A.2), it suffices to check that the second fundamental form of $\{|\widetilde{u}_i| = 0\}$ converges to zero. This follows from our Hölder estimate on the mean curvatures from (2.21), Lemma 2.8, and the fact that our graphing functions converge to zero in C^1. ☐

This concludes the proof of the curvature estimates. ☐

Corollary 3.9. *Let (M,g), u, ε, ε_1 be as in Theorem 3.4, and let $\theta \in (0,1)$. Then,*

$$[\mathrm{II}_{\{u=t\}}]_{\theta,\{u=t\}\cap B^3_{1/3}(0)} \leq c_1' \text{ for all } |t| \leq 1-\beta,$$

where $c_1' = c_1'(n, E_0, \beta, \theta, W)$.

Proof. Formulas (2.33) and (2.34), Lemma 2.8, and Proposition 3.2 together give C^θ bounds on the mean curvatures of $\{u = 0\}$. The improvement to $C^{2,\theta}$ bounds on the level sets comes from (quasilinear) Schauder theory and Theorem 3.4. ☐

4. Phase transitions with bounded Morse index ($n = 3$)

4.1. *Multiplicity and Jacobi fields.* In this section we prove that uniform bounds on the Morse index generically prevent multiplicity from occurring in the Allen–Cahn setting. Specifically,

Theorem 4.1. *Suppose (M^3, g) is a compact Riemannian 3-manifold possibly with $\partial M \neq \emptyset$, and suppose that $u_i \in C^\infty(M; [-1,1])$, $\varepsilon_i > 0$, where each u_i is a critical point of E_{ε_i}, and*

$$E_{\varepsilon_i}[u_i] \leq E_0,\ \operatorname{ind}(u_i) \leq I_0 \text{ for all } i = 1, 2, \ldots . \tag{4.1}$$

Suppose $\lim_i \varepsilon_i = 0$. Passing to a subsequence, write $V \triangleq \lim_i h_0^{-1} V_{\varepsilon_i}[u_i]$ for the limit 2-varifold. Then V is a stationary integral varifold, $\operatorname{spt}\|V\|$ is smooth in the interior of M, and if Σ denotes a connected component of $\operatorname{spt}\|V\|$ that is a compact submanifold without boundary, then one of the following is true:

(1) *Σ is two-sided and $\Theta^2(V,\cdot) = 1$ on Σ (i.e., Σ has multiplicity 1);*

(2) Σ *is two-sided,* $\Theta^2(V,\cdot) \geq 2$ *on* Σ (*i.e., multiple interfaces have converged*), *it is stable* (*see* (1.3)), *and it carries a smooth positive Jacobi field*; *or*
(3) Σ *is one-sided, and the two-sided double cover of* Σ *is stable and carries a smooth positive Jacobi field.*

Proof. For $p \in M$, $i = 1, 2, \ldots$, define the index concentration scale by

$$\mathcal{R}(p,i) \triangleq \inf\{r > 0 : \operatorname{ind}(u_i; B_r(p)) \geq 1\}, \tag{4.2}$$

and then further let

$$\mathring{\Sigma} \triangleq \{p \in M : \liminf_{i\to\infty} \mathcal{R}(p,i) > 0\}.$$

By passing to an appropriate subsequence at the beginning of the proof, an elementary covering argument allows us to assume that $\mathcal{H}^0(\Sigma \setminus \mathring{\Sigma}) \leq I_0$.

The curvature estimates from Theorem 3.4 combined with the varifold convergence of $V_{\varepsilon_i}[u_i]$ (from[7] [HT00, Th. 1], and [Gua18, App. B]) show that along $\mathring{\Sigma}$, the limit varifold is supported with integer multiplicity (possibly greater than one) on a $C^{1,1}$ (and thus smooth) minimal surface. At this point, we may argue that Σ extends smoothly across the index concentration set $\Sigma \setminus \mathring{\Sigma}$ exactly as in [Gua18, Prop. 3.10]. We emphasize here that by using our curvature estimates, we give a proof of the regularity of Σ that does not rely on the deep works of Wickramasekera [Wic14] and Tonegawa–Wickramasekera [TW12] (cf. [Gua18], [Hie18]).

We now assume that Σ is connected (in general, one can apply the following argument to each component of the support of the limit varifold V).

First, suppose Σ is two-sided and denote

$$U \triangleq \text{ tubular neighborhood of } \Sigma \text{ such that } (\Sigma \cup \partial M) \cap U = \emptyset.$$

We may suppose that $U = Z_\Sigma(\Sigma \times (-1,1))$.

By the Constancy Theorem [Sim83, Th. 41.1], $\Theta^2(V,\cdot)$ is constant on Σ. If $\Theta^2(V,\cdot) = 1$ somewhere on Σ, then Σ occurs entirely with multiplicity one as claimed.

In what follows we may assume, then, that $\Theta^2(V,\cdot) \equiv m \in \{2,3,\ldots\}$ on Σ.

Let us assume, for the time being, that $I_0 = 0$, i.e., that the critical points u_i are all stable. The general case will be dealt with later.

It follows from (4.1), Corollary 3.9, and the two-sidedness of Σ, that the level sets $\{u_i = 0\} \cap U$ converge graphically in $C^{2,\theta}$ to Σ. In the case that $\{u_i = 0\} \cap U$ were minimal surfaces, it is standard to produce a positive Jacobi field on Σ out of this setup. We recall the argument here, with the necessary modifications for our lower regularity situation.

[7] See Remark 3.8.

Since Σ is two-sided, the level sets $\{u_i = 0\} \cap U$ (which are *smoothly* embedded) can be ordered by their signed distance to Σ in a fashion that is consistent across Σ. Without loss of generality, we may assume that there are $Q = 2$ level sets.[8] To stay consistent with Section 2, let us label the level sets

$$\Gamma_{i,1}, \Gamma_{i,2} \subset \{u_i = 0\} \cap U.$$

Denote their corresponding height functions (over Σ) as $f_{i,1},\ f_{i,2} : \Sigma \to \mathbf{R}$, $\ell \in \{1,2\}$, so that $f_{i,1} < f_{i,2}$ on Σ. We recall (A.13) from Appendix A, which tells us that

$$\begin{aligned} H_{\Gamma_{i,\ell}} = &- \operatorname{div}_{g_{f_{i,\ell}}} \left(\frac{\nabla_{g_{f_{i,\ell}}} f_{i,\ell}}{(1 + g^{pq}_{f_{i,\ell}} (f_{i,\ell})_p (f_{i,\ell})_q)^{1/2}} \right) \\ &- \frac{\mathrm{I\!I}^{pq}_{f_{i,\ell}} (f_{i,\ell})_p (f_{i,\ell})_q}{(1 + g^{pq}_{f_{i,\ell}} (f_{i,\ell})_p (f_{i,\ell})_q)^{1/2}} + (1 + g^{pq}_{f_{i,\ell}} (f_{i,\ell})_p (f_{i,\ell})_q)^{1/2} H_{f_{i,\ell}} \end{aligned} \tag{4.3}$$

for $\ell = 1,\ 2$. Here we are using notation from the appendix, where, e.g., $g = g_z + dz^2$ on U.

We now claim that $H_{\Gamma_{i,2}} - H_{\Gamma_{i,1}}$ satisfies a *linear* uniformly elliptic equation in $f_{i,2} - f_{i,1}$, whose parameters (obviously) depend on $f_{i,1},\ f_{i,2}$. Indeed, (4.3) tells us that

$$H_{\Gamma_{i,\ell}} = -A(f_{i,\ell}) \operatorname{div}_\Sigma \left(\mathscr{B}(f_{i,\ell}, \nabla_\Sigma f_{i,\ell}) \nabla_\Sigma f_{i,\ell} \right) + C(f_{i,\ell}, \nabla_\Sigma f_{i,\ell}) \tag{4.4}$$

for *smooth* functions (for each $p \in \Sigma$)

$$\begin{aligned} A &= A_p : \mathbf{R} \to \mathbf{R}, \\ \mathscr{B} &= \mathscr{B}_p : \mathbf{R} \times T_p\Sigma \to \operatorname{End}(T_p\Sigma), \\ C &= C_p : \mathbf{R} \times T_p\Sigma \to \mathbf{R}, \end{aligned}$$

which, additionally, satisfy that $A > 0$, $\mathscr{B}$ is positive definite. More specifically, at each point $p \in \Sigma$,

$$\begin{aligned} A(z) &\triangleq \frac{\sqrt{g_0}}{\sqrt{g_z}},\ z \in \mathbf{R}, \\ \mathscr{B}(z, \mathbf{v})\mathbf{w} &\triangleq \frac{\sqrt{g_z}}{\sqrt{g_0}} \frac{g_z^{ij} g^0_{jk} \mathbf{w}^j \partial_{y_k}}{(1 + g_z^{pq} g^0_{pk} g^0_{q\ell} \mathbf{v}^k \mathbf{v}^\ell)^{1/2}},\ z \in \mathbf{R},\ \mathbf{v}, \mathbf{w} \in T_p\Sigma, \\ C(z, \mathbf{v}) &\triangleq - \frac{\mathrm{I\!I}_z^{pq}\, g^0_{ip} g^0_{jq} \mathbf{v}^i \mathbf{v}^j}{(1 + g_z^{pq} g^0_{pk} g^0_{q\ell} \mathbf{v}^k \mathbf{v}^\ell)^{1/2}} \\ &\qquad + (1 + g_z^{pq} g^0_{pk} g^0_{q\ell} \mathbf{v}^k \mathbf{v}^\ell)^{1/2} H_z,\ z \in \mathbf{R},\ \mathbf{v} \in T_p\Sigma. \end{aligned}$$

[8] Otherwise, we apply the same argument verbatim to the *top* and *bottom* level sets, ignoring all intermediate ones.

From the fundamental theorem of calculus, as well as the fact that the two divergences (for the two cases $\ell = 1, 2$) are *pointwise* bounded (because the two mean curvatures are bounded), it follows that

$$H_{\Gamma_{i,2}} - H_{\Gamma_{i,1}} = -A \operatorname{div}_\Sigma(\widehat{\mathscr{B}} \nabla_\Sigma f_i + f_i \widehat{\mathbf{C}}) + \langle \widehat{\mathbf{D}}, \nabla_\Sigma f_i \rangle_\Sigma + \widehat{E} f_i \text{ on } \Sigma, \tag{4.5}$$

where $f_i \triangleq f_{i,2} - f_{i,1} > 0$ on Σ, with coefficients

$$\begin{aligned} \widehat{\mathscr{B}} &= \widehat{\mathscr{B}}_p : \mathbf{R}^2 \times (T_p\Sigma)^2 \to \operatorname{End}(T_p\Sigma), \\ \widehat{\mathbf{C}} &= \widehat{\mathbf{C}}_p, \widehat{\mathbf{D}} = \widehat{\mathbf{D}}_p : \mathbf{R}^2 \times (T_p\Sigma)^2 \to T_p\Sigma, \\ \widehat{E} &= \widehat{E}_p : \mathbf{R}^2 \times (T_p\Sigma)^2 \to \mathbf{R}, \end{aligned}$$

whose arguments are $(f_{i,1}, f_{i,2}, \nabla_\Sigma f_{i,1}, \nabla_\Sigma f_{i,2}) \in \mathbf{R}^2 \times (T_p\Sigma)^2$. These coefficients are *uniformly bounded* and satisfy

$$A \geq \mu, \ \langle B\mathbf{v}, \mathbf{v} \rangle_\Sigma \geq \mu \|\mathbf{v}\|_\Sigma^2, \ \mathbf{v} \in T_p\Sigma,$$

for a fixed $\mu > 0$, provided

$$\limsup_{i \to \infty} \|f_{i,1}\|_{C^1(\Sigma)} + \|f_{i,2}\|_{C^1(\Sigma)} < \infty.$$

It will be convenient to carry out the exact computation, as that will allow us to study a particular rescaled limit as $i \to \infty$. It will also be convenient to denote

$$\zeta_i^{(t)} \triangleq f_{i,1} + t(f_{i,2} - f_{i,1}) \equiv f_{i,2} + t f_i, \ t \in [0, 1].$$

Note that

$$\zeta_i^{(0)} \equiv f_{i,1}, \ \zeta_i^{(1)} \equiv f_{i,2}, \text{ and } \tfrac{\partial}{\partial t} \zeta_i^{(t)} \equiv f_i \text{ on } \Sigma.$$

Let us define $\widehat{\mathscr{B}}$, $\widehat{\mathbf{C}}$, $\widehat{\mathbf{D}}$, $\widehat{E}$. The easiest term to deal with in (4.4) is the low order term, C. Indeed

$$\begin{aligned} C(f_{i,2}, \nabla_\Sigma f_{i,2}) - C(f_{i,1}, \nabla_\Sigma f_{i,1}) = &\underbrace{\left[\int_0^1 D_z C(\zeta_i^{(t)}, \nabla_\Sigma \zeta_i^{(t)})\, dt\right]}_{\widehat{E}, \text{ term 1 out of 2}} f_i \\ &+ \left\langle \underbrace{\int_0^1 D_\mathbf{v} C(\zeta_i^{(t)}, \nabla_\Sigma \zeta_i^{(t)})\, dt}_{\widehat{\mathbf{D}}}, \nabla_\Sigma f_i \right\rangle. \end{aligned}$$

We study the higher order term in two steps. First,

$$\begin{aligned}&\mathscr{B}(f_{i,2}, \nabla_\Sigma f_{i,2})\nabla_\Sigma f_{i,2} - \mathscr{B}(f_{i,1}, \nabla_\Sigma f_{i,1})\nabla_\Sigma f_{i,1} \\ &\quad = \underbrace{\left(\left[\int_0^1 D_z \mathscr{B}(\zeta_i^{(t)}, \nabla_\Sigma \zeta_i^{(t)}) \nabla_\Sigma \zeta_i^{(t)} \, dt \right] \right)}_{\widehat{\mathbf{C}}} f_i \\ &\qquad + \left\langle \underbrace{\left[\int_0^1 D_{\mathbf{v}} \mathscr{B}(\zeta_i^{(t)}, \nabla_\Sigma \zeta_i^{(t)}) \nabla_\Sigma \zeta_i^{(t)} \, dt \right]}_{\widehat{\mathscr{B}}, \text{ term 1 out of 2}}, \nabla_\Sigma f_i \right\rangle \\ &\qquad + \underbrace{\left[\int_0^1 \mathscr{B}(\zeta_i^{(t)}, \nabla_\Sigma \zeta_i^{(t)}) \, dt \right]}_{\widehat{\mathscr{B}}, \text{ term 2 out of 2}} \nabla_\Sigma f_i.\end{aligned}$$

Second,

$$\begin{aligned}&(A(f_{i,2}) - A(f_{i,1})) \operatorname{div}_\Sigma(\mathscr{B}(f_{i,2}, \nabla_\Sigma f_{i,2}) \nabla_\Sigma f_{i,2}) \\ &\quad = \underbrace{\left(\left[\int_0^1 D_z A(\zeta_i^{(t)}) \, dt \right] \operatorname{div}_\Sigma(\mathscr{B}(f_{i,1}, \nabla_\Sigma f_{i,1}) \nabla_\Sigma f_{i,1}) \right)}_{\widehat{E}, \text{ term 2 out of 2}} f_i.\end{aligned}$$

We now return to the qualitative study of f_i. Applying the Harnack inequality in divergence form to (4.5) (after multiplying through by A^{-1}), we get

$$\sup_\Sigma f_i \le c \inf_\Sigma f_i \text{ for } i = 1, 2, \ldots \tag{4.6}$$

with a constant $c > 0$ that does not depend on i. From Proposition 3.2 and Corollary 3.3, we know that

$$\lim_{i \to \infty} \frac{\|H_{\Gamma_{i,\ell}}\|_{C^0(\Gamma_{i,\ell})}}{\varepsilon_i |\log \varepsilon_i|} = 0 \text{ for } \ell = 1, 2, \tag{4.7}$$

$$\liminf_{i \to \infty} \frac{\inf_\Sigma f_i}{\varepsilon_i |\log \varepsilon_i|} > 0. \tag{4.8}$$

Define the normalizations

$$\widehat{f_i} \triangleq (\sup_\Sigma f_i)^{-1} f_i : \Sigma \to [\tfrac{1}{c}, 1], \tag{4.9}$$

where c is as in (4.6). In view of (4.5), $\widehat{f_i}$ satisfies the linear, uniformly elliptic equation (note that we have multiplied through by A^{-1}, which is uniformly bounded):

$$\frac{H_{\Gamma_{i,2}} - H_{\Gamma_{i,1}}}{A \cdot \sup_\Sigma f_i} = -\operatorname{div}_\Sigma(\widehat{\mathscr{B}} \nabla_\Sigma \widehat{f}_i + \widehat{f}_i \widehat{\mathbf{C}}) + \langle A^{-1} \widehat{\mathbf{D}}, \nabla_\Sigma \widehat{f}_i \rangle_\Sigma + A^{-1} \widehat{E} \widehat{f}_i \text{ on } \Sigma. \tag{4.10}$$

We will *test* this PDE by multiplying through by some $\zeta \in C^\infty(\Sigma)$ and integrating by parts. By testing, first with $\zeta = \widehat{f}_i$, we get uniform energy estimates

$$\limsup_{i\to\infty} \int_\Sigma |\nabla_\Sigma \widehat{f}_i|^2 < \infty.$$

Moreover, since $\widehat{f}_i$ is (trivially) bounded, it follows from Rellich's theorem that there exist $\widehat{f} \in W^{1,2}(\Sigma)$ and a subsequence such that

$$\widehat{f}_i \rightharpoonup \widehat{f} \text{ in } W^{1,2}(\Sigma),\ \widehat{f}_i \to \widehat{f} \text{ in } L^2(\Sigma).$$

Therefore, since the coefficients in (4.10) are all uniformly bounded as $i \to \infty$, it follows that we can test (4.10) with arbitrary $\zeta \in C^\infty(\Sigma)$ and pass to a subsequential limit $i \to \infty$.

The left-hand side of (4.10) converges to zero uniformly as $i \to \infty$ because of (4.7)–(4.8) above. Thus, $\widehat{f}$ is a $W^{1,2}$-weak solution of

$$-\operatorname{div}_\Sigma(\widehat{\mathscr{B}}_\infty \nabla_\Sigma \widehat{f} + \widehat{f}\widehat{\mathbf{C}}_\infty) + \langle A_\infty^{-1} \widehat{\mathbf{D}}_\infty, \nabla_\Sigma \widehat{f} \rangle + A_\infty^{-1} \widehat{E}_\infty \widehat{f} = 0 \text{ on } \Sigma, \tag{4.11}$$

where A_∞, $\widehat{\mathscr{B}}_\infty$, $\widehat{\mathbf{C}}_\infty$, $\widehat{\mathbf{D}}_\infty$, $\widehat{E}_\infty$ are just the same coefficients, except now they are evaluated at the limiting configuration of $(0, 0, \mathbf{0}, \mathbf{0})$. It is not hard to see, using the evolutions in Appendix A, that

$$A_\infty \equiv 1,\ \widehat{\mathscr{B}}_\infty \equiv \mathrm{Id},\ \widehat{\mathbf{C}}_\infty \equiv \widehat{\mathbf{D}}_\infty \equiv \mathbf{0},\ \text{and}\ \widehat{E}_\infty \equiv -(|\,\mathrm{I\!I}_\Sigma\,|^2 + \mathrm{Ric}_g(\nu_\Sigma, \nu_\Sigma)).$$

Thus, $\widehat{f}$ is $W^{1,2}$-weak solution of the Jacobi equation

$$(\Delta_\Sigma + |\,\mathrm{I\!I}_\Sigma\,|^2 + \mathrm{Ric}_g(\nu_\Sigma, \nu_\Sigma)|_\Sigma) h = 0 \text{ on } \Sigma. \tag{4.12}$$

It must be smooth—and thus classically a solution—by elliptic regularity. Moreover,

$$\tfrac{1}{c} \le \widehat{f}_i \le 1 \text{ for all } i = 1, 2, \ldots \implies \tfrac{1}{c} \le \widehat{f} \le 1.$$

In particular, the function is positive. It follows that the principal eigenvalue of the Jacobi operator is zero, so Σ is stable.[9] The result follows.

We now drop the stability assumption and proceed to the general case of $I_0 \in \{0, 1, 2, \ldots\}$. We continue to assume that Σ is two-sided. Without loss of generality, we will assume $I_0 = 1$ from this point on. The general case is similar.

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[9]The fact that $\mathrm{ind}(u_i) = 0$ for all $i = 1, 2, \ldots$ implies the stability of Σ is not new; see [Ton05], [TW12], [Hie18], [Gas17]. Nonetheless, by appropriately generalizing the argument given here, we are going to be able to extend the conclusion that Σ is stable in the case where $\mathrm{ind}(u_i) \le I_0$ for $i = 1, 2, \ldots$, $I_0 \in \{0, 1, \ldots\}$ and convergence occurs with multiplicity ≥ 2.

The index concentration set is either empty (in which case, we can argue as in the previous case) or satisfies $\mathring{\Sigma} = \Sigma \setminus \{P_\star\}$ for some $P_\star \in \Sigma$, and the convergence of $\{u_i = 0\} \cap U$ to $\mathring{\Sigma}$ is graphical $C^{2,\theta}_{\mathrm{loc}}$ on $\Sigma \setminus \{P_\star\}$. Notice that by definition, for every $r > 0$, there exists a subsequence along which

$$\operatorname{ind}(u_i; M \setminus B_{r/2}(P_\star)) = 0. \tag{4.13}$$

Our previous discussion regarding the stable case applies verbatim to $M \setminus C_r(P_\star)$ where, in exponential normal coordinates,

$$C_\rho(P_\star) \triangleq B^2_\rho(P_\star) \times (-1, 1),$$

and yields functions $f_{i,1},\ f_{i,2} : \Sigma \setminus B^2_r(P_\star) \to \mathbf{R}$ representing the *incomplete* properly embedded surfaces-*with-boundary*

$$\Gamma_{i,1}, \Gamma_{i,2} \subset \{u_i = 0\} \cap U \setminus C_r(P_\star). \tag{4.14}$$

Remark 4.2. Recall that we assumed U is the image of the normal exponential map of Σ restricted to $\Sigma \times (-1, 1)$. Then, $\partial C_\rho(P_\star) \cap U = \partial B^2_\rho(P_\star) \times (-1, 1)$ for every sufficiently small $\rho > 0$.

All of (4.3)–(4.8) continue to hold over $\Sigma \setminus B^2_r(P_\star)$ instead of Σ. All the constants inevitably depend on our choice of $r > 0$, which is yet to be determined. We note that, trivially, the energy estimate

$$\limsup_{i \to \infty} \int_{\Sigma \setminus B^2_{2r}(P_\star)} |\nabla_\Sigma \widehat{f}_i|^2 < \infty$$

holds true for any fixed $r > 0$ by our previous discussion. In fact, because $\Gamma_{i,1} \setminus C_r(P_\star)$, $\Gamma_{i,2} \setminus C_r(P_\star)$ converge in $C^{2,\theta}$ to $\Sigma \setminus B^2_r(P_\star)$ as $i \to \infty$, a subset of the fixed surface Σ, the coefficients of (4.5) will satisfy

$$\limsup_{r \to 0} \Big[\limsup_{i \to \infty} \|A\|_{C^0(\Sigma \setminus B^2_{3r/2}(P_\star))} + \|\widehat{\mathscr{B}}\|_{C^0(\Sigma \setminus B^2_{3r/2}(P_\star))} + \|\widehat{\mathbf{C}}\|_{C^0(\Sigma \setminus B^2_{3r/2}(P_\star))} + \|\widehat{\mathbf{D}}\|_{C^0(\Sigma \setminus B^2_{3r/2}(P_\star))} + \|\widehat{E}\|_{C^0(\Sigma \setminus B^2_{3r/2}(P_\star))} \Big] < \infty,$$

and therefore, we will have the *uniform* energy estimate

$$\limsup_{r \to 0} \left[\limsup_{i \to \infty} \int_{\Sigma \setminus B^2_{2r}(P_\star)} |\widehat{f}_i|^2 \right] < \infty.$$

This means we can pass to a limiting $\widehat{f}$ in the following sense:

$$\widehat{f}_i \rightharpoonup \widehat{f} \text{ in } W^{1,2}_{\mathrm{loc}}(\mathring{\Sigma}),\ \widehat{f}_i \to \widehat{f} \text{ in } L^2_{\mathrm{loc}}(\mathring{\Sigma}). \tag{4.15}$$

Now, (4.6)–(4.8) also hold for each fixed $r > 0$, with the sup and inf taken over $\Sigma \setminus B^2_r(P_\star)$ and the C^0 norm of $H_{\Gamma_{i,\ell}}$ taken over $\Gamma_{i,\ell} \setminus C_r(P_\star)$; the constant c and rates of convergence of the limits, a priori, depend on r. Nonetheless, $\widehat{f} \in W^{1,2}_{\mathrm{loc}}(\mathring{\Sigma})$ is a weak solution of (4.12) on $\mathring{\Sigma}$. By elliptic regularity, $\widehat{f}$ is smooth and solves (4.12) classically on $\mathring{\Sigma}$.

Proposition 4.3. *$\widehat{f} \in L^\infty(\mathring{\Sigma})$, $\widehat{f} \not\equiv 0$ almost everywhere on $\mathring{\Sigma}$.*

We defer the proof of Proposition 4.3 to the next section, since the argument is of independent interest.

This proposition, once verified, completes the proof of Theorem 4.1: By standard removable singularity results for elliptic PDE, $\widehat{f}$ must extend to a smooth non-negative solution of (4.12) on Σ, which is not identically zero, and the result follows as it did in the stable setting.

Finally, we explain the necessary modifications when Σ is one-sided. Assume, as above, that $I_0 = 1$. (The general case is similar.) As before, we can define $\mathring{\Sigma}$ to be the complement of the index concentration set. Considering a tubular neighborhood U of Σ, we can use the normal exponential map to lift Σ and $u : U \to \mathbf{R}$ to $\check{\Sigma} \subset \check{U}$, where $\check{\Sigma}$ is the orientable double cover of Σ and $\check{U}$ is the associated lift of U. We can assume that $\check{U}$ is diffeomorphic (via the normal exponential map) to $\check{\Sigma} \times (-1, 1)$. Let $\mathring{\check{\Sigma}}$ be the lift of $\mathring{\Sigma}$. Observe that $\check{\Sigma} \setminus \mathring{\check{\Sigma}}$ contains at most two points (more generally $2I_0$ points).

Note that the covering map $\pi : \check{U} \to U$ admits an deck transformation $\tau : \check{U} \to \check{U}$ with τ^2 equal to the identity. Define $\check{u} \triangleq u \circ \pi$, which is still a critical point of E_{ε_i}. Clearly $\check{u} \circ \tau = \check{u}$.

We claim that the convergence of $\check{u}$ to $\check{\Sigma}$ occurs with even multiplicity. If not, (up to switching the normal vector) we can assume that $\check{u} \to -1$ on $\check{\Sigma} \times (-1, 0)$ and $\check{u} \to 1$ on $\check{\Sigma} \times (0, 1)$. (This is clear on $\mathring{\check{\Sigma}}$, which then implies that it holds for all $p \in \check{\Sigma}$.) Note, however, that $\tau(\{p\} \times (0, 1)) = \{\tau(p)\} \times (-1, 0)$; otherwise, we would find that Σ was two-sided. This contradicts the fact that $\check{u}$ is invariant under τ.

Thus, the convergence of $\check{u}$ occurs with even multiplicity (and thus multiplicity at least two). Now, the argument above can be applied verbatim to $\check{\Sigma}$ and $\check{u}$ to produce a smooth positive Jacobi field on $\check{\Sigma}$. (We emphasize that it is not clear what the index of $\check{u}$ is; here, we rely on the index bounds of u to bound the cardinality of $\check{\Sigma} \setminus \mathring{\check{\Sigma}}$; after this step, the definition of $\mathring{\check{\Sigma}}$, rather than the index bounds is all that is used.) As above, this implies that $\check{\Sigma}$ is stable. $\square$

4.2. *Sliding heteroclinic barriers.* For the reader's convenience we start by recalling the following important result of White on local foliations by minimal surfaces.

Proposition 4.4 ([Whi87, Appendix]). *Let Φ be an even elliptic integrand, where Φ and $D_2\Phi$ are $C^{2,\theta}$. Let Φ_r be the integrand defined by $\Phi_r(x, v) = \Phi(rx, v)$. There is an $\eta > 0$ such that if $r < \eta$ and if*

$$w : B_1 \subset \mathbf{R}^2 \to \mathbf{R},\ \|w\|_{C^{2,\theta}} < \eta,$$

then for each $t \in [-1, 1]$, *there is a* $C^{2,\theta}$ *function* $v_t : B_1 \to \mathbf{R}$ *whose graph is* Φ_r*-stationary and such that*

$$v_t(x) = w(x) + t \quad \text{if } x \in \partial B_1.$$

Furthermore, v_t *depends in a* C^1 *way on* t *so that the graphs of the* v_t *foliate a region of* $\mathbf{R}^3$. *If* M *is a* C^1 *properly immersed* Φ_r*-stationary surface in* $B_{1/2}(0)$ *with* $\partial M \subset \operatorname{graph} v_t$, *then* $M \subset \operatorname{graph} v_t$.

We will use the minimal disks constructed by this proposition to construct barriers (using Theorem 7.4) that will allow us to control the height of the top and bottom $\{u_i = 0\}$ sheets near $P_\star$. This can be thought of as a variant of the moving planes method adapted to the Riemannian Allen–Cahn setting.

Proof of Proposition 4.3. We continue with the same notation as in the previous section. Let $\rho > 2r$ be sufficiently small so that (4.13)–(4.14) still apply.

Let $w_i : B^2_{2\rho}(P_\star) \to \mathbf{R}$ be a harmonic function (defined on $B^2_{2\rho}(P_\star) \subset \Sigma$) with boundary data

$$w_i = f_{i,2} \text{ on } \partial B^2_\rho(P_\star).$$

Recalling, from Corollary 3.9, that $f_{i,2} \to 0$ in $C^{2,\theta}(B^2_{2\rho}(P_\star) \setminus B^2_{\rho/2}(P_\star))$, it follows that (by potentially going farther down the sequence of $i = 1, 2, \ldots$) $\|w_i\|_{C^{2,\theta}}$ — suitably scaled — is small enough for Proposition 4.4 to apply. Once we are in that regime, Proposition 4.4 guarantees a foliation

$$t \mapsto D_{i,\rho}(t), \ t \in [-\delta, \delta],$$

consisting of minimal disks that all project to $B^2_\rho(P_\star) \subset \Sigma$. Without loss of generality, we may suppose that the foliated region $\cup_{|t|<\delta} D_{i,\rho}(t)$ lies entirely within U. Note that

(1) the curves $t \mapsto \partial D_{i,\rho}(t)$ move at unit vertical speed in $\partial C_\rho(P_\star)$;
(2) the second fundamental form of the disks $D_{i,\rho}(t)$ is bounded in $C^{0,\theta}$ uniformly over $i = 1, 2, \ldots$, $t \in [-\delta, \delta]$,

$$|\,\mathrm{I\!I}_{D_{i,\rho}(t)}\,| + [\mathrm{I\!I}_{D_{i,\rho}(t)}]_\theta \leq \eta, \tag{4.16}$$

and $\eta > 0$ can be made arbitrarily small.

As a consequence of (4.16), (2.7), and minimal surface curvature estimates, the disks also satisfy the following weaker $C^{3,\theta}$ bound uniformly over $i = 1, 2, \ldots$, $t \in [-\delta, \delta]$,

$$\varepsilon|\nabla_{D_{i,\rho}(t)}\, \mathrm{I\!I}_{D_{i,\rho}(t)}\,| + \varepsilon^{1+\theta}[\nabla_{D_{i,\rho}(t)}\, \mathrm{I\!I}_{D_{i,\rho}(t)}]_\theta \leq \eta, \tag{4.17}$$

after possibly relaxing $\eta > 0$, which can still nevertheless be made arbitrarily small.

We will now use a sliding/moving planes argument that relies on the barrier construction in Section 7, adopting relevant notation from therein. We assume, without loss of generality, that

$$u > 0 \text{ above } \Gamma_{i,2} \text{ in } U \setminus C_{\rho/2}(P_\star). \tag{4.18}$$

Our constructions below will take place for $\varepsilon = \varepsilon_i$, $i = 1, 2, \ldots$, but we suppress the dependence on i for the sake of notational brevity.

Define $\hat{\chi} : \mathbf{R} \to [0, 1]$ to be a cutoff function such that

$$\hat{\chi}(s) = \begin{cases} 1 & |s| \leq B\varepsilon|\log\varepsilon|, \\ 0 & |s| \geq 2B\varepsilon|\log\varepsilon|, \end{cases} \tag{4.19}$$

where $B \gg 1$ is to be chosen later. This can be constructed so that

$$|\hat{\chi}^{(k)}| = O((\varepsilon|\log\varepsilon|)^{-k}) \text{ for } k \geq 1,\ \varepsilon \to 0.$$

Let us very briefly run through some notation which is introduced later, in Section 7; we will need to use some of it here in invoking that section's main theorem. In Section 7 we consider $\delta_* \in (0, 1)$ fixed and a Hölder exponent $\alpha \in (0, 1)$, $\alpha \leq \theta$, where θ is as in (4.16), (4.17). (We will eventually choose α near 0 and θ near 1.) In (7.11), cutoff functions χ_j are introduced that are supported on strips of width $O(\varepsilon^{\delta_*})$ (while $\hat{\chi}$ is supported on a thinner strip of size $O(\varepsilon|\log\varepsilon|)$). Finally, in (7.12), χ_1 is used to define a suitably truncated approximate heteroclinic solution $\widetilde{\mathbb{H}}_\varepsilon$ that is constant outside a strip of size $O(\varepsilon^{\delta_*})$; see Remark 7.2.

Given this notation, let us set

$$\hat{v}^\sharp(s) \triangleq \gamma\hat{\chi}(s)\mathbb{H}'(\varepsilon^{-1}s) + (1 - \hat{\chi}(s)) \begin{cases} 1 - \varepsilon^3 - \widetilde{\mathbb{H}}_\varepsilon(s) & s > 0, \\ -1 - \widetilde{\mathbb{H}}_\varepsilon(s) & s < 0, \end{cases}$$

where $\gamma \in \mathbf{R}$ is chosen so that the orthogonality constraint

$$\int_{-\infty}^{\infty} \hat{v}^\sharp(s)\mathbb{H}'(\varepsilon^{-1}s)\,ds = 0 \tag{4.20}$$

holds. Recalling (1.5) and (1.6), and that $\delta_* \in (0, 1)$, (4.20) is equivalent to

(4.21)

$$\begin{aligned}
\gamma(h_0 - o(1)) &= O(\varepsilon^{-1}) \int_{B\varepsilon|\log\varepsilon|}^{\infty} (1 - \mathbb{H}(\varepsilon^{-1}s))\mathbb{H}'(\varepsilon^{-1}s)\,ds \\
&\quad + O(\varepsilon^2) \int_{2B\varepsilon|\log\varepsilon|}^{\frac{47}{50}\varepsilon^{\delta_*}} |\mathbb{H}'(\varepsilon^{-1}s)|\,ds \\
&= O(1) \int_{B|\log\varepsilon|}^{\infty} (1 - \mathbb{H}(s))\mathbb{H}'(s)\,ds + O(\varepsilon^3) \int_{2B|\log\varepsilon|}^{\frac{47}{50}\varepsilon^{\delta_* - 1}} |\mathbb{H}'(s)|\,ds \\
&= O(1)\exp(-2\sqrt{2}B|\log\varepsilon|) \\
&\quad + O(\varepsilon^3)\exp(-2\sqrt{2}B|\log\varepsilon|) = O(\varepsilon^{2\sqrt{2}B}).
\end{aligned}$$

Also,

$$\|\hat{\chi}(s)\mathbb{H}'(\varepsilon^{-1}s)\|_{C^{2,\alpha}_{\varepsilon}(\mathbf{R})} = O(1) \text{ as } \varepsilon \to 0. \tag{4.22}$$

Taking B sufficiently large, (4.21) and (4.22) together imply

$$\|\hat{v}^{\sharp}(s)\|_{C^{2,\alpha}_{\varepsilon}(\mathbf{R})} = O(\varepsilon^3). \tag{4.23}$$

Next, for $(y,s) \in \partial(B^2_\rho(P_\star) \times [-\frac{1}{2}, \frac{1}{2}])$, define

$$\hat{v}^{\flat}(y,s) \triangleq (1-\chi_4(s)) \begin{cases} 1 - \varepsilon^3 - \widetilde{\mathbb{H}}_\varepsilon(s) & s > 0, \\ -1 - \widetilde{\mathbb{H}}_\varepsilon(s) & s < 0. \end{cases}$$

Recall that χ_4 is defined in (7.11). It is easy to see that $\|\hat{v}^{\flat}\|_{C^{2,\alpha}_\varepsilon} = O(\varepsilon^3)$. In fact, $\chi_5 \hat{v}^{\flat} = 0$, so

$$\|\hat{v}^{\flat}\|_{\widetilde{C}^{2,\alpha}_\varepsilon} = O(\varepsilon^3) \tag{4.24}$$

as well. (See (7.16) for the definition of $\widetilde{C}^{2,\alpha}_\varepsilon$.)

We emphasize that everything from (4.19) to (4.24) above is *agnostic* of our particular solutions with bounded Morse index. They will serve as prescribed boundary data for solutions of the Allen–Cahn equation on the fixed product manifold $B^2_\rho \times [-\frac{1}{2}, \frac{1}{2}]$, albeit with varying interior metric that will depend on g, $i = 1, 2, \ldots, \rho$, and $t \in [-\delta, \delta]$. Indeed, we let

$$\begin{aligned} g_{i,\rho}(t) \triangleq {} & \text{pullback metric from } Z_{D_{i,\rho}(t)}(D_{i,\rho}(t) \times [-\tfrac{1}{2}, \tfrac{1}{2}]) \subset U \\ & \text{to } B^2_\rho \times [-\tfrac{1}{2}, \tfrac{1}{2}] \text{ under Fermi coordinates } (y,s) \text{ with respect to } D_{i,\rho}(t). \end{aligned} \tag{4.25}$$

We may apply Theorem 7.4 with $\hat{v}^{\sharp}$, $\hat{v}^{\flat}$ as above, $\hat{\zeta} \equiv 0$, and with the Hölder exponents α near 0 and θ near 1 per the theorem, to $\Omega \triangleq B^2_\rho \times [-\frac{1}{2}, \frac{1}{2}]$ and the non-constant Riemannian metric $g_{i,\rho}(t)$. Note that the conditions of the theorem are met trivially for $\hat{\zeta}$ and are also met for $\hat{v}^{\sharp}$, $\hat{v}^{\flat}$ by (4.23)–(4.24).

The theorem yields $\mathfrak{b}_{i,\rho,t} : \Omega \to \mathbf{R}$ such that

$$\varepsilon_i^2 \Delta_{g_{i,\rho}(t)} \mathfrak{b}_{i,\rho,t} = W'(\mathfrak{b}_{i,\rho,t}) \tag{4.26}$$

and, for all $(y,s) \in \partial\Omega$,

$$\mathfrak{b}_{i,\rho,t}(y,s) = \widetilde{\mathbb{H}}_\varepsilon(s) + \chi_4(s)\hat{v}^{\sharp}(s) + \hat{v}^{\flat}(y,s). \tag{4.27}$$

We constructed $\hat{v}^{\sharp}$, $\hat{v}^{\flat}$ specifically so that

$$\mathfrak{b}_{i,\rho,t}(y,s) = \begin{cases} 1 - \varepsilon_i^3 & (y,s) \in \partial\Omega,\ s \geq 2B\varepsilon_i |\log \varepsilon_i|, \\ -1 & (y,s) \in \partial\Omega,\ s \leq -2B\varepsilon_i |\log \varepsilon_i|. \end{cases} \tag{4.28}$$

Claim. For every $\beta > 0$, $\varepsilon_i \leq 1$, we have

$$|\mathfrak{b}_{i,\rho,t}(y,s)| \leq 1 - \beta \implies |s| \leq c'\varepsilon_i, \tag{4.29}$$

where $c' = c'(W, \beta, \eta, c_0) > 0$.

Proof of Claim. This is a straightforward consequence of the ansatz $\mathfrak{b}_{i,\rho,t} = (\widetilde{\mathbb{H}}_\epsilon + \chi_4 v^\sharp + v^\flat) \circ D_\zeta$, $\|v^\sharp\|_{C^0}$, $\|v^\flat\|_{C^0} = o(1)$, $\|\zeta\| = O(\varepsilon_i^{2-2\alpha})$, and (1.5), at least provided we take α small enough. ☐

Claim. For sufficiently large i,

$$\mathfrak{b}_{i,\rho,\delta} < (Z_{D_{i,\rho}(\delta)})^* u_i \text{ on } \Omega. \tag{4.30}$$

Recall that $\delta > 0$ represents the "top" of the foliation $D_{i,\rho}(\delta)$.

Proof of Claim. Let us agree, for the remainder of the proof of this claim, to write u_i instead of $(Z_{D_{i,\rho}(\delta)})^* u_i$. We seek to show that $\mathcal{G} \triangleq \{x \in \Omega : \mathfrak{b}_{i,\rho,\delta}(x) < u_i(x)\}$ coincides with Ω. (Recall that we are assuming (4.18).)

Fix $\beta \in (0,1)$ so that $W'' \geq 2\kappa^2 > 0$ on $[-1,-1+\beta] \cup [1-\beta,1]$ for some $\kappa > 0$. From (4.29), $\{|\mathfrak{b}_{i,\rho,\delta}| \leq 1-\beta\}$ is contained in an $O(\varepsilon_i)$-neighborhood of $D_{i,\rho}(\delta)$. From [HT00, Th. 1], $\{|u_i| \leq 1-\beta\}$ converges, in the Hausdorff sense, to Σ. In particular, for sufficiently large i,

$$(\Omega \cap \{|u_i| \leq 1-\beta\}) \cup \{|\mathfrak{b}_{i,\rho,\delta}| \leq 1-\beta\} \subset \mathcal{G}.$$

Note that

$$\varepsilon_i^2 \Delta_g (1-u_i) = -W'(u_i) = \tfrac{W'(1)-W'(u_i)}{1-u_i}(1-u_i) \geq 2\kappa^2(1-u_i) \text{ on } \{u_i > 1-\beta\},$$

$$\varepsilon_i^2 \Delta_g (1+u_i) = W'(u_i) = \tfrac{W'(u_i)-W'(-1)}{u_i-(-1)}(1+u_i) \geq 2\kappa^2(1+u_i) \text{ on } \{u_i < -1+\beta\},$$

so by an application of the barrier principle together with the saddle property of W at zero (see [KLP12, Lemma 4.1]), we get

$$|u_i^2 - 1| = O\big(\exp(-\kappa \varepsilon_i^{-1} \operatorname{dist}_g(\cdot, \{u_i = 0\}))\big). \tag{4.31}$$

Combined with (4.28), this shows $\partial\Omega \subset \mathcal{G}$ for sufficiently large i. Thus,

$$\Omega \setminus \mathcal{G} \subset \Omega \setminus (\partial\Omega \cup \{|u_i| \leq 1-\beta\} \cup \{|\mathfrak{b}_{i,\rho,\delta}| \leq 1-\beta\}). \tag{4.32}$$

Subtracting from (4.26) the PDE satisfied by u_i, we see that

$$\varepsilon_i^2 \Delta_g (\mathfrak{b}_{i,\rho,\tau} - u_i) = c(x)(\mathfrak{b}_{i,\rho,\tau} - u_i)$$

for $c(x) \triangleq (W'(\mathfrak{b}_{i,\rho,t}(x)) - W'(u_i(x)))/(\mathfrak{b}_{i,\rho,\tau}(x) - u_i(x))$. This is *negative* on $\Omega \setminus \mathcal{G}$ by (4.32), and this violates the maximum principle unless $\mathcal{G} = \Omega$. The claim follows. ☐

Next, since

(1) $\mathfrak{b}_{i,\rho,t}$ and $(Z_{D_{i,\rho}(t)})^* u_i$ both vary continuously in $t \in [-\delta, \delta]$ by Theorem 7.4,
(2) (4.30) holds true, and
(3) $\mathfrak{b}_{i,\rho,-\delta} \not\leq (Z_{D_{i,\rho}(-\delta)})^* u_i$,

there will exist exactly one $\tau_i \in (-\delta, \delta)$, and at least one $Q_i^\star \in \Omega$, such that

(4.33)
$$\mathfrak{b}_{i,\rho,t} < (Z_{D_{i,\rho}(t)})^* u_i \text{ on } \Omega \text{ for all } t \in (\tau_i, \delta], \text{ and } \mathfrak{b}_{i,\rho,\tau_i}(Q_i^\star) = [(Z_{D_{i,\rho}(\tau_i)})^* u](Q_i^\star).$$

Our goal is to estimate τ_i. Abusing notation again, we will write u_i instead of $(Z_{D_{i,\rho}(\tau_i)})^* u_i$, and g instead of $g_{i,\rho}(\tau_i)$. Thus,

$$u_i - \mathfrak{b}_{i,\rho,\tau_i} \geq 0 \text{ on } \Omega, \ (u_i - \mathfrak{b}_{i,\rho,\tau_i})(Q_i^\star) = 0. \tag{4.34}$$

Subtracting (4.26) from the PDE satisfied by u, we see that

$$\varepsilon_i^2 \Delta_g (u_i - \mathfrak{b}_{i,\rho,\tau_i}) = c(x)(u_i - \mathfrak{b}_{i,\rho,\tau_i})$$

for $c(x) \triangleq (W'(u_i(x)) - W'(\mathfrak{b}_{i,\rho,\tau_i}(x)))/(u_i(x) - \mathfrak{b}_{i,\rho,\tau_i}(x))$. Then the maximum principle tells us that

(1) either $Q_i^\star \in \partial\Omega$, or
(2) $u_i \equiv \mathfrak{b}_{i,\rho,\tau_i}$ on Ω.

We only consider the first case here, since the second reduces to it by replacing $Q_i^\star$ with another point on $\partial\Omega$. Note that (4.33), the fact that $\mathfrak{b}_{i,\rho,0}|_{\partial D_{i,\rho}(0)} \equiv 0$, and the uniqueness of τ_i give a lower bound on τ_i:

$$\tau_i \geq 0. \tag{4.35}$$

The upper bound is more subtle. We claim that

$$\tau_i < 7B\varepsilon_i |\log \varepsilon_i|, \tag{4.36}$$

provided B is chosen (independently of i) such that

$$\operatorname{dist}_g(x, \{u_i = 0\}) > 3B\varepsilon_i |\log \varepsilon_i| \implies |u_i(x)| > 1 - \varepsilon_i^3. \tag{4.37}$$

The existence of a B that satisfies (4.37) is guaranteed by (4.31).

It will be convenient to introduce the following notation (here, $\lambda \geq 0$ is some parameter):

$$\overline{\partial\Omega}[\lambda] \triangleq \{(y, s) \in \partial\Omega : s \in [\lambda, \tfrac{1}{2}]\},$$
$$\underline{\partial\Omega}[\lambda] \triangleq \{(y, s) \in \partial\Omega : s \in [-\tfrac{1}{2}, -\lambda]\}.$$

To start, let us estimate the height of $Q_i^\star$ from below. We have

$$u_i > -1 \text{ on } (M^n, g) \implies \mathfrak{b}_{i,\rho,\tau_i}(Q_i^\star) = u_i(Q_i^\star) > -1$$

so, from (4.28),

$$Q_i^\star \in \partial\Omega \setminus \underline{\partial\Omega}[2B\varepsilon_i |\log \varepsilon_i|].$$

Equivalently, the image $\widetilde{Q}_i^\star$ of $Q_i^\star$ to (M^n, g) under $Z_{D_{i,\rho}(\tau_i)}$ satisfies

$$\widetilde{Q}_i^\star \in Z_{D_{i,\rho}(\tau_i)}(\partial\Omega \setminus \underline{\partial\Omega}[2B\varepsilon_i |\log \varepsilon|]).$$

In particular, $\widetilde{Q}_i^\star$ belongs to the open tubular neighborhood of the image $Z_{D_{i,\rho}(\tau_i)}(\overline{\partial\Omega}[0]) \subset (M^n, g)$ with radius $3B\varepsilon_i|\log\varepsilon_i|$:

$$\widetilde{Q}_i^\star \in B_{3B\varepsilon_i|\log\varepsilon_i|}\big(Z_{D_{i,\rho}(\tau_i)}(\overline{\partial\Omega}[0])\big). \tag{4.38}$$

We now prove (4.36) by contradiction. We will show that

$$\operatorname{dist}_g(Z_{D_{i,\rho}(\tau_i)}(\overline{\partial\Omega}[0]), \{u_i = 0\}) > 6B\varepsilon_i|\log\varepsilon_i| \tag{4.39}$$

when (4.36) fails, i.e., when $\tau_i \geq 7B\varepsilon_i|\log\varepsilon_i|$.

Since $D_{i,\rho}(\tau_i)$ is an $o(1)$-Lipschitz graph over Σ (note that the argument used to prove (4.30) shows that $\tau_i \to 0$) and $Z_{D_{i,\rho}(\tau_i)}(\partial\Omega) \perp D_{i,\rho}(\tau_i)$, there will exist $\eta > 0$ (independent of i) such that and

$$Z_{D_{i,\rho}(\tau_i)}(\overline{\partial\Omega}[0] \setminus \overline{\partial\Omega}[\eta]) \subset C_{3\rho/2}(P_\star) \setminus C_{\rho/2}(P_\star)$$

for sufficiently large i. Moreover, $\lim_{i\to\infty}\{u_i = 0\} = \Sigma$ in the Hausdorff topology ([HT00, Th. 1], [Gua18, App. B]), so

$$\liminf_{i\to\infty} \operatorname{dist}_g(Z_{D_{i,\rho}(\tau_i)}(\overline{\partial\Omega}[\eta]), \{u_i = 0\}) > 0$$

because $\tau_i \geq 0$ by (4.35). Thus, (4.39) will follow from

$$\operatorname{dist}_g(Z_{D_{i,\rho}(\tau_i)}(\overline{\partial\Omega}[0] \setminus \overline{\partial\Omega}[\eta]), \{u_i = 0\} \cap C_{2\rho}(P_\star) \setminus C_{\rho/2}(P_\star)) > 6B\varepsilon_i|\log\varepsilon_i|$$

when $\tau_i \geq 7B\varepsilon_i|\log\varepsilon_i|$. Since the components of $\{u_i = 0\} \cap C_{2\rho}(P_\star) \setminus C_{\rho/2}(P_\star)$ are well-ordered $o(1)$-Lipschitz graphs over Σ, with $\Gamma_{i,2}$ being the topmost, we may equivalently show

$$\operatorname{dist}_g(Z_{D_{i,\rho}(\tau_i)}(\overline{\partial\Omega}[0] \setminus \overline{\partial\Omega}[\eta]), \Gamma_{i,2}) > 6B\varepsilon_i|\log\varepsilon_i|.$$

Because $D_{i,\rho}(t)$, $t \in [0, \tau_i]$, are all $o(1)$-Lipschitz graphs over Σ as well, we have

$$\nabla_g(\operatorname{dist}_g^\pm(\cdot; \Gamma_{i,2})), \nabla_g(\operatorname{dist}_g^\pm(\cdot; D_{i,\rho}(t))\rangle \geq 1 - o(1),\ t \in [0, \tau_i]$$

in a small (but definite) neighborhood of Σ in $C_{2\rho}(P_\star) \setminus C_{\rho/2}(P_\star)$. Here $\operatorname{dist}_g^\pm$ denotes the signed distance. From it follows that for every $P \in Z_{D_{i,\rho}(\tau_i)}(\overline{\partial\Omega}[0] \setminus \overline{\partial\Omega}[\eta])$,

$$\begin{aligned}
\operatorname{dist}_g^\pm(P; \Gamma_{i,2}) &\geq (1 - o(1)) \operatorname{dist}_g^\pm(P; D_{i,\rho}(0)) \\
&\geq (1 - o(1))(\tau_i + \operatorname{dist}_g^\pm(P; D_{i,\rho}(\tau_i))) \\
&\geq (1 - o(1))\tau_i > (1 - o(1))7B\varepsilon_i|\log\varepsilon_i| > 6B\varepsilon_i|\log\varepsilon_i|,
\end{aligned}$$

as claimed, and (4.39) follows. It is now an automatic consequence of (4.38)–(4.39) that

$$\operatorname{dist}_g(\widetilde{Q}_i^\star, \{u_i = 0\}) > 3B\varepsilon_i|\log\varepsilon_i|.$$

Recalling (4.37), we find that $|u_i(Q_i^\star)| > 1 - \varepsilon_i^3$. Combined with $\operatorname{dist}_g^\pm(\widetilde{Q}_i^\star; \Gamma_{i,2}) > 6B\varepsilon_i|\log\varepsilon_i| > 0$, which guarantees that $u_i(Q_i^\star) > 0$, we conclude that $u_i(Q_i^\star) > 1 - \varepsilon_i^3$. This contradicts (4.28). Thus, (4.36) is true.

Summarizing (4.35) and (4.36), we have $0 \leq \tau_i < 7B\varepsilon_i|\log \varepsilon_i|$. Combined with the defining property (4.34) of τ_i, we get the following height estimate over Σ:

$$f_{i,2} \leq h^{D_{i,\rho}(\tau_i)} \leq h^{D_{i,\rho}(7B\varepsilon_i|\log \varepsilon_i|)} \text{ on } \Sigma \cap B^2_{2\rho}(P_\star) \setminus B^2_r(P_\star),$$

where

(1) $f_{i,2} : \Sigma \setminus B^2_r(P_\star) \to \mathbf{R}$ is the height of $\Gamma_{i,2}$ over Σ, with $r \in (0, \rho/2)$ as in (4.13) and (4.14), and
(2) $h^{D_{i,\rho}(t)}$ denotes the height of the minimal disk $D_{i,\rho}(t)$ over Σ.

The same sliding argument, carried out below the bottom-most sheet $\Gamma_{i,1}$ of $\{u_i = 0\}$, similarly gives

$$f_{i,1} \geq h^{D'_{i,\rho}(-7B\varepsilon_i|\log \varepsilon_i|)} \text{ on } \Sigma \cap B^2_{2\rho}(P_\star) \setminus B^2_r(P_\star).$$

Notice that we are denoting the disks by $D'_{i,\rho}(-7B\varepsilon_i|\log \varepsilon_i|)$, since they come from a different foliation, namely, the one generated by applying Proposition 4.4 to $w_i = f_{i,1}$. Therefore, by the regularity of the foliation guaranteed by Proposition 4.4, we have

$$\begin{aligned} f_i = f_{i,2} - f_{i,1} &\leq h^{D_{i,\rho}(7B\varepsilon_i|\log \varepsilon_i|)} - h^{D'_{i,\rho}(-7B\varepsilon_i|\log \varepsilon_i|)} \\ &\leq c\left(7B\varepsilon_i|\log \varepsilon_i| + h^{D_{i,\rho}(0)} - h^{D'_{i,\rho}(0)}\right) \\ &\leq c'\left(\varepsilon_i|\log \varepsilon_i| + \max_{B^2(\rho)(P_\star)} (h^{D_{i,\rho}} - h^{D'_{i,\rho}})\right) \\ &\leq c'\left(\varepsilon_i|\log \varepsilon_i| + \max_{\partial B^2_\rho(P_\star)} (f_{i,2} - f_{i,1})\right) \end{aligned}$$

on $\Sigma \cap B^2_{2\rho}(P_\star) \setminus B^2_r(P_\star)$. The last inequality follows from the maximum principle. We emphasize that c' is independent of i and r.

The proof of Proposition 4.3 is essentially done. Indeed, fix $0 < r < \rho/2$. By what we have shown so far, we have

$$\sup_{\Sigma \setminus B^2_r(P_\star)} f_i \leq c'\left(\varepsilon_i|\log \varepsilon_i| + \sup_{\Sigma \setminus B^2_\rho(P_\star)} f_i\right).$$

By the Harnack inequality (4.6) and sheet separation lower bound (4.8) on $\Sigma \setminus B^2_{2\rho}(P_\star)$,

$$\sup_{\Sigma \setminus B^2_r(P_\star)} f_i \leq c'' \inf_{\Sigma \setminus B^2_\rho(P_\star)} f_i.$$

This holds independently of i, r, so the renormalized limit $\widehat{f}$ taken in (4.15) (first with $i \to \infty$ and then with $r \to 0$) is nontrivial. This completes the proof of Proposition 4.3. ☐

5. Phase transitions with multiplicity one

In this section we return to working in arbitrary dimension $n \geq 3$, and we consider a compact Riemannian manifold (M^n, g) and a sequence $u_i \in C^\infty(M; (-1,1))$ of critical points of E_{ε_i}, $\varepsilon_i > 0$, $E_{\varepsilon_i}[u_i] \leq E_0$, for all $i = 1, 2, \ldots$, with $\lim_i \varepsilon_i = 0$. Let $V \triangleq \lim_i h_0^{-1} V_{\varepsilon_i}[u_i]$ denote the limit stationary integral varifold, which exists by [HT00, Th. 1]; see [Gua18, App. B] for Riemannian modifications. In this section we will *assume* that

$$\Theta^{n-1}(V, \cdot) = 1 \text{ on } \Sigma \triangleq \operatorname{spt} \|V\|, \qquad \text{which is a smooth minimal surface} \subset M \setminus \partial M. \tag{5.1}$$

In other words, we assume that the limit V is *smooth* and that it occurs with *multiplicity one*. (We are not assuming any bounds on the Morse index.)

Remark 5.1. We recall that this is *automatically* the case when

(i) $3 \leq n \leq 7$ and each u_i minimizes E_{ε_i} among compact perturbations in M (by [HT00, Th. 2]); or

(ii) $n = 3$, $\limsup_i \operatorname{ind}(u_i) < \infty$, and Σ carries no positive Jacobi fields (this follows from Theorem 4.1).

We emphasize that this section requires only the multiplicity one assumption (5.1), not (i) or (ii).

The main goal of this section is to prove Theorem 5.11. Roughly, it says that the Morse index is *upper semicontinuous*. Note that, in general, one only expects the index to be *lower* semicontinuous. This has been recently confirmed in the work of Hiesmayr [Hie18]; see also Gaspar's generalization to one-sided limit surfaces [Gas17]. Upper semicontinuity hinges strongly on the multiplicity one assumption, as the following example suggests:

Example 5.2. Let (u_i, ε_i), $i = 1, 2, \ldots$, $\lim_i \varepsilon_i = 0$, be a sequence constructed by [dPKWY10] to converge, with multiplicity ≥ 2, to a two-sided minimal surface Σ in a closed Riemannian 3-manifold (M^3, g) with positive Ricci curvature. Then, by Theorem 4.1, $\liminf_i \operatorname{ind}(u_i) = \infty$, because Σ cannot be stable and there there are no stable two-sided minimal surfaces in the presence of positive Ricci curvature.

In order to study the semicontinuity of the Morse index, we need to obtain a detailed understanding of the convergence of the u_i and their level sets to Σ. Somewhat surprisingly, the regularity estimates in Section 2 (or [WW19a, §15]) do not seem to suffice for our purposes. Instead, we must upgrade the estimates so that we have an explicit understanding of the $O(\varepsilon^2)$ term in (2.18). We use an ansatz inspired by the work of del Pino–Kowalczyk–Wei [dPKW13], although our setting is different: rather than having constructed u, we are *given* an arbitrary solution u converging with multiplicity one. This technique does not seem to have been previously considered in the context of regularity in Allen–Cahn.

5.1. *Improved convergence.* Note that by scaling M, we can arrange that (2.1)–(2.2) hold; we will do so without further remark in the sequel. Note that then, due Lemma 5.3 below, (2.3)–(2.7) hold as well. Thus, Section 2 applies (as does [WW19a, §15] in the flat setting).

LEMMA 5.3. *Let $U \subset\subset M \setminus \partial M$ be a neighborhood of Σ, and let $\beta \in (0, 1)$. Then, for sufficiently large i, $\varepsilon_i|\nabla u_i| \geq c > 0$ on $U \cap \{|u_i| \leq 1 - \beta\}$.*

Proof. We argue by contradiction. If the result were false, we would be able to pick a subsequence (labeled the same) along which there would exist $x_i \in U \cap \{|u_i| \leq 1 - \beta\}$ with $\varepsilon_i|\nabla u_i(x_i)| \to 0$. After rescaling by ε_i^{-1} around x_i, the rescaled critical points $\widetilde{u}_i$ converge to a non-trivial critical point of E_1 on $\mathbf{R}^n$ with $|\widetilde{u}(0)| \leq 1 - \beta$, $\nabla\widetilde{u}(0) = 0$. By the monotonicity formula (see [HT00, §3] and [Gua18, App. B]) and multiplicity-one convergence at the original scale, we see that the tangent cone at infinity of $\widetilde{u}$ is a multiplicity-one plane. Hence, by [Wan17, Th. 11.1] (cf. [Man17, Th. 3.6]), $\widetilde{u}$ has flat level sets. This contradicts $|\widetilde{u}(0)| \leq 1 - \beta$, $\nabla\widetilde{u}(0) = 0$. □

Combined with the multiplicity-one analysis in [WW19a, §15] (cf. Section 2 and Remark 2.6 above), we may argue as in the proof of Theorem 4.1 to conclude that $\Sigma = \operatorname{spt} V$ is a smooth two-sided embedded minimal hypersurface and the convergence of the level sets of u_i to Σ occurs in $C^{2,\theta}$. (Of course, convergence in the Hausdorff sense follows immediately from [HT00, Th. 1].)

LEMMA 5.4. *If $U \subset\subset M \setminus \partial M$ is a neighborhood of Σ, and $\theta, \beta \in (0, 1)$, then $U \cap \{u_i = t\}$ converges uniformly in $C^{2,\theta}$ to Σ for every $t \in (-1+\beta, 1-\beta)$.*

Proof. By Section 2, it suffices to check that the level sets are bounded in C^2. One uses a blow-up argument again, as in the proof of Theorem 3.4. Suppose that the enhanced second fundamental form were not bounded. Pick $x_i \in U \cap \{|u_i| \leq 1 - \beta\}$ such that $\lambda_i \triangleq |\mathcal{A}_i(x_i)|$ are within a factor of $\frac{1}{2}$ from $\sup_{U\cap\{|u_i|\leq 1-\beta\}} |\mathcal{A}_i|$; thus, $\lambda_i \to \infty$. Note that $\limsup_i \lambda_i\varepsilon_i < \infty$ by elliptic regularity. Moreover, we in fact have that $\limsup_i \lambda_i\varepsilon_i = 0$ because (by [Wan17, Th. 11.1] and monotonicity) there are no non-trivial (i.e., non-constant and non-heteroclinic) entire critical points of E_1 in $\mathbf{R}^n$ with a planar tangent cone at infinity. In particular, rescaling by λ_i^{-1} around x_i, we get a sequence $(\widetilde{u}_i, \widetilde{\varepsilon}_i)$ with $\widetilde{\varepsilon}_i \to 0$ and uniformly bounded enhanced second fundamental form, $|\widetilde{\mathcal{A}}_i(0)| = 1$, and which therefore converges to a $C^{1,1}$ minimal surface in $\mathbf{R}^n$. However, by monotonicity, this minimal surface is a plane; this contradicts $|\widetilde{\mathcal{A}}_i(0)| = 1$ by Remark 2.6. □

Let us return to the notation and conventions used in Section 2. Also, we drop the subscript i.

Because of the multiplicity one assumption, we have reasonably strong estimates on ϕ, h, and H_Γ; see (2.17). We will write h for $\mathbf{h}$, U for $U[\mathbf{h}]$, Γ for Γ_1, and d for d_1, since $Q = 1$. We record the specialization of (2.20) and Lemma 2.3 here (cf. [WW19a, §15], and [Man17, Th. 3.6]):

$$\|\phi\|_{C^{2,\theta}_\varepsilon(\mathcal{M})} + \varepsilon\|\Delta_\Gamma h - H_\Gamma\|_{C^{0,\theta}_\varepsilon(\Gamma)} + \varepsilon^{-1}\|h\|_{C^{2,\theta}_\varepsilon(\Gamma)} \le c'\varepsilon^2, \tag{5.2}$$

where $\mathcal{M} \triangleq \{X \in M : |d(X)| < 1\}$. As we have already indicated, we must upgrade our estimates for $\Delta_\Gamma h - H_\Gamma$ in (5.2) as well as determine the $O(\varepsilon^2)$ behavior of ϕ.

Let us work in Fermi coordinates around Γ so as not to write the diffeomorphism Z_Γ explicitly below. We will also denote $\Gamma_z \triangleq \{X \in \mathcal{M} : d(X) = z\}$ and will write $\overline{\mathbb{H}}$ for $\overline{\mathbb{H}}^{3|\log\varepsilon|}$.

We can compute the equation for ϕ as follows. Using (A.2), (A.3), (A.7), as well as (2.3)–(2.7), (2.15), and (5.2), one computes the following in $\mathcal{M}$ (see [WW19a, (9.4)]):

(5.3)
$$\begin{aligned}
\varepsilon^2\Delta_g\phi &= \varepsilon^2\Delta_{\Gamma_z}\phi + \varepsilon^2 H_{\Gamma_z}\partial_z\phi + \varepsilon^2\partial_z^2\phi \\
&= W'(u) - \varepsilon^2\Delta_{\Gamma_z}U - \varepsilon^2 H_{\Gamma_z}\partial_z U - \varepsilon^2\partial_z^2 U \\
&= W'(U+\phi) - (W'(U) + O(\varepsilon^3)) \\
&\quad + \varepsilon(\Delta_{\Gamma_z}h - H_{\Gamma_z})\cdot\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) - |\nabla_{\Gamma_z}h|^2\cdot\overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y))) \\
&= W''(U)\phi + \varepsilon((\Delta_\Gamma h - H_\Gamma)\circ\Pi_\Gamma)\cdot\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) \\
&\quad + \varepsilon((|\mathrm{I\!I}_\Gamma|^2 + \mathrm{Ric}_g(\partial_z,\partial_z))\circ\Pi_\Gamma)\cdot z\cdot\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) \\
&\quad + \varepsilon^2 O(|z|)\cdot\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) + O(\varepsilon^3).
\end{aligned}$$

By using (5.2), (5.3), and the multiplicity one assumption, one may revisit [WW19a, App. B] and establish the following bounds:

LEMMA 5.5. *We can improve the estimate in* (5.2) *to*

$$\varepsilon\|\Delta_\Gamma h - H_\Gamma\|_{C^0(\Gamma)} \le c'\varepsilon^3.$$

Proof. Multiply (5.3) by $\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))$ and integrate over $z \in [-\eta,\eta]$. We find (at $y \in \Sigma$ fixed)

$$\begin{aligned}
&\int_{-\eta}^{\eta}(\varepsilon^2(\Delta_{\Gamma_z}\phi + H_{\Gamma_z}\partial_z\phi + \partial_z^2\phi) - W''(U)\phi)\cdot\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))))\,dz \\
&\qquad = \varepsilon^2(h_0 - o(1))(\Delta_\Gamma h - H_\Gamma) \\
&\qquad\quad + \varepsilon(|\mathrm{I\!I}_\Gamma|^2 + \mathrm{Ric}_g(\partial_z,\partial_z))\int_{-\eta}^{\eta} z\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))^2\,dz
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\eta}^{\eta} \varepsilon^2 O(|z|)\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))^2\,dz + O(\varepsilon^4)\\
&= \varepsilon^2(h_0 - o(1))(\Delta_\Gamma h - H_\Gamma) + O(\varepsilon^4).
\end{aligned}
$$

We have used (1.6) together with $\int_{-\infty}^{\infty} t\mathbb{H}'(t)^2 dt = 0$ (which holds by parity).

Twice differentiating the orthogonality relation (2.16) used to define h (see Section 2.1 and [WW19a, App. B]) and using (5.2), we have

$$
\int_{-\eta}^{\eta} \varepsilon^2(\Delta_{\Gamma_z}\phi)\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\,dz = O(\varepsilon^4).
$$

From (5.2), we have

$$
\int_{-\eta}^{\eta} \varepsilon^2 H_{\Gamma_z}\partial_z\phi\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\,dz = O(\varepsilon^5).
$$

Finally, an integration by parts shows that

$$
\begin{aligned}
&\int_{-\eta}^{\eta}\Big(\varepsilon^2\partial_z^2\phi\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) - W''(u)\phi\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\Big)\,dz\\
&= \int_{-\eta}^{\eta}\Big(\overline{\mathbb{H}}'''(\varepsilon^{-1}(z-h(y))) - W''(u)\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\Big)\,\phi\,dz.
\end{aligned}
$$

Using (2.15) here, combined with the previous expressions, we conclude the proof. □

Thus, returning to (5.3) we find that in $\mathcal{M}$, we have

(5.4)
$$
\varepsilon^2\Delta_g\phi - W''(U)\phi = \varepsilon((|\mathrm{I\!I}_\Gamma|^2 + \mathrm{Ric}_g(\partial_z,\partial_z))\circ\Pi_\Gamma)\cdot z\cdot\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) + O(\varepsilon^3).
$$

We have used the fact that $z\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) = O(\varepsilon)$.

Observe that the right-hand side of (5.4) is only bounded in $O(\varepsilon^2)$. Thus, we expect this to represent the leading term of ϕ, after inverting $\varepsilon^2\Delta_g - W''(U)$. To make this precise, we first define (cf. [dPKW13, §3.2]) a function $\mathbb{J}(t)$ to be the unique bounded solution of the ODE

$$
\mathbb{J}''(t) = W''(\mathbb{H}(t))\mathbb{J}(t) + t\mathbb{H}'(t), \text{ with } \mathbb{J}(0) = 0. \tag{5.5}
$$

Indeed, we even have the explicit expression (cf. [dPKW13, p. 82])

$$
\mathbb{J}(t) = \mathbb{H}'(t)\int_0^t\int_{-\infty}^s \tau\mathbb{H}'(s)^{-2}\,\mathbb{H}'(\tau)^2 d\tau ds,
$$

which shows that $\mathbb{J}$ is well defined and decays exponentially as $t\to\pm\infty$. It will be important in the sequel to observe that $\mathbb{J}(-t) = -\mathbb{J}(t)$, which follows from the parity of $\mathbb{H}(t)$ and either the uniqueness of solutions to the ODE, or the explicit integral expression.

Observe that $|\mathrm{I\!I}_\Gamma|^2 + \mathrm{Ric}_g(\partial_z,\partial_z)$ converges to $|\mathrm{I\!I}_\Sigma|^2 + \mathrm{Ric}_g(\nu,\nu)$ in $C^{0,\theta}$ because Γ converges to Σ in $C^{2,\theta}$ by Lemma 5.4. We fix functions $V:\Gamma\to\mathbf{R}$

with the property that V still converges to $|\mathrm{II}_\Sigma|^2 + \mathrm{Ric}_g(\nu,\nu)$ in C^0 and $\|V\|_{C^2(\Gamma)} \le C$. For definiteness, we choose $V(y) = (|\mathrm{II}_\Sigma|^2 + \mathrm{Ric}_g(\nu,\nu)) \circ \Pi_\Sigma(y)$, where Π_Σ is the nearest point projection to Σ.

We claim that $\varepsilon^2 V(y)\mathbb{J}(\varepsilon^{-1}(z-h(y)))$ represents the leading order term in ϕ. To this end, in $\mathcal{M}$, we define a refined discrepancy function

$$\widetilde{\phi}(y,z) \triangleq \phi(y,z) - \varepsilon^2 (V\circ \Pi_\Gamma)(y,z)\cdot \mathbb{J}(\varepsilon^{-1}(z-h(y))).$$

We compute (using the C^2 bounds for V, as well as (5.2) and Lemma 5.5) that on $\mathcal{M}$, we have

$$\begin{aligned}
&\varepsilon^2\Delta_g\widetilde{\phi} - W''(U)\widetilde{\phi}\\
&= \varepsilon((|\mathrm{II}_\Gamma|^2 + \mathrm{Ric}_g(\partial_z,\partial_z))\circ\Pi_\Gamma)\cdot z\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\\
&\quad - \varepsilon^2(V\circ\Pi_\Gamma)\big[\mathbb{J}''(\varepsilon^{-1}(z-h(y))) - W''(U)\cdot\mathbb{J}(\varepsilon^{-1}(z-h(y)))\big] + O(\varepsilon^3)\\
&= \varepsilon\big[(|\mathrm{II}_\Gamma|^2 + \mathrm{Ric}_g(\partial_z,\partial_z))\circ\Pi_\Gamma - V\circ\Pi_\Gamma\big]\cdot z\cdot\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\\
&\quad - \varepsilon^2\big[W''(\mathbb{H}(\varepsilon^{-1}(z-h(y)))) - W''(U)\big](V\circ\Pi_\Gamma)\\
&\quad \cdot\mathbb{J}(\varepsilon^{-1}(z-h(y))) + O(\varepsilon^3)\\
&= o(\varepsilon^2).
\end{aligned}$$

We again used that $z\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) = O(\varepsilon)$ as well as the definition of V. We now use the defining property of h to invert $\varepsilon^2\Delta_g - W''(U)$.

Proposition 5.6. *We have that $\widetilde{\phi} = o(\varepsilon^2)$ on $\mathcal{M}$.*

Proof. For contradiction, suppose that $\lambda \triangleq \sup_{\mathcal{M}}|\widetilde{\phi}| \ge \gamma\varepsilon^2$ for some $\gamma > 0$. Note that $\widetilde{\phi}$ is exponentially small at points that are uniformly bounded away from Γ, so it is clear that this supremum is achieved at some $X^* \in \mathcal{M}$ with $d(X^*) \to 0$. We can assume that $\widetilde{\phi}(X^*) = \lambda$. Write $X^* = (y^*, z^*)$ in Fermi coordinates over Γ. We split the argument into two cases: (i) $\varepsilon^{-1}|z^*|$ is uniformly bounded, or (ii) $\varepsilon^{-1}|z^*| \to \infty$.

First we consider case (i). We can assume that $\varepsilon^{-1}z^* \to z_\infty$. Define $\widehat{\phi}(\widehat{X}) = \lambda^{-1}\widetilde{\phi}(X^* + \varepsilon\widehat{X})$ which, in blown up Fermi coordinates $\widehat{X} = (\widehat{y}, \widehat{z})$, satisfies

$$\Delta_{\widehat{g}}\widehat{\phi}(\widehat{y},\widehat{z}) - W''(\overline{\mathbb{H}}(\varepsilon^{-1}z^* + \widehat{z} - \varepsilon^{-1}h(y^* + \varepsilon\widehat{y})))\widehat{\phi}(\widehat{y},\widehat{z}) = o(1)$$

for $\widehat{z} \in (-\varepsilon^{-1}\eta, \varepsilon^{-1}\eta)$ and $\widehat{y} \in \Sigma$, and where $\widehat{g}$ is converging smoothly to the Euclidean metric. Moreover, $\widehat{\phi}(0) = 1$ and $|\widehat{\phi}|$ is uniformly bounded on compact sets. Interior Schauder estimates yield uniform bounds for $\widehat{\phi}$ in $C^{1,\theta}_{\mathrm{loc}}$. Thus, $\widehat{\phi}$ converges in C^1 to a weak (and thus strong, by elliptic regularity) solution of

$$\Delta\widehat{\phi}(\widehat{y},\widehat{z}) - W''(\mathbb{H}(z_\infty + \widehat{z}))\widehat{\phi}(\widehat{y},\widehat{z}) = 0$$

on $\mathbf{R}^{n-1} \times \mathbf{R}$. By [Pac12, Lemma 3.7] (see also [PR03]), we have that

$$\widehat{\phi}(\widehat{y}, \widehat{z}) = \rho \mathbb{H}'(z_\infty + \widehat{z}) \text{ for some } \rho \in \mathbf{R},$$

because $\widehat{\phi} \in L^\infty(\mathbf{R}^{n-1} \times \mathbf{R})$. In fact, $\widehat{\phi}(0) = 1$ implies that $\rho = \mathbb{H}'(z_\infty)^{-1}$. At the original scale, write $X = (y, z)$ in Fermi coordinates over Γ. Then, for K fixed sufficiently large, if $|z| \leq K\varepsilon$, we have

$$\widetilde{\phi}(y, z) = \lambda\big[\mathbb{H}'(z_\infty)^{-1}\mathbb{H}'(z_\infty + \varepsilon^{-1}(z - z^*)) + o(1)\big].$$

Therefore,

$$\phi(y, z) = \lambda\big[\mathbb{H}'(z_\infty)^{-1}\mathbb{H}'(z_\infty + \varepsilon^{-1}(z - z^*)) + o(1)\big] + \varepsilon^2 V(y)\mathbb{J}(\varepsilon^{-1}(z - h(y))). \tag{5.6}$$

By estimating the exponential tail using (1.6), and then using the definition of ϕ and h, and also (5.2), we have

$$\int_{-K\varepsilon}^{K\varepsilon} \phi(y, z) \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z - h(y)))\, dz = O(\varepsilon^3 e^{-\sqrt{2}K}). \tag{5.7}$$

By parity ($\mathbb{H}'$ is even, $\mathbb{J}$ is odd) and similarly estimating an exponential tail, we also have

$$\int_{-K\varepsilon}^{K\varepsilon} \mathbb{J}(\varepsilon^{-1}(z - h(y))) \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z - h(y)))\, dz = O(\varepsilon e^{-\sqrt{2}K}). \tag{5.8}$$

Finally,

$$\int_{-K\varepsilon}^{K\varepsilon} \mathbb{H}'(z_\infty + \varepsilon^{-1}(z - z^*)) \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z - h(y)))\, dz \geq (h_0 - O(e^{-\sqrt{2}K}))\varepsilon. \tag{5.9}$$

Altogether, (5.6)–(5.9) imply $\lambda = h_0^{-1}O(\varepsilon^2 e^{-\sqrt{2}K})$, which (for large K) contradicts our assumption that $\lambda \geq \gamma\varepsilon^2$ for a fixed $\gamma > 0$. This is a contradiction, completing the proof of case (i).

We now turn to case (ii). The proof here is analogous (and simpler). By rescaling as above, we find a non-zero smooth function $\widehat{\phi} \in L^\infty(\mathbf{R}^{n-1} \times \mathbf{R})$ solving $\Delta\widehat{\phi} - W''(\pm 1)\widehat{\phi} = 0$. An integration by parts, using $W''(\pm 1) > 0$, shows that $\widehat{\phi} = 0$. This is a contradiction, completing the proof of case (ii). □

5.2. *Relating the second variations and index upper semicontinuity.* We now can give the fundamental computation linking the index of u as a critical point of E_ε with the index of Σ as a critical point of area. Our argument is closely related to the proof of [dPKW13, Lemma 9.2]. Recall from (1.2) that the second variation of E_ε is given by

$$\mathcal{Q}_u(\zeta, \psi) \triangleq \delta^2 E_\varepsilon[u]\{\zeta, \xi\} = \int_M \left(\varepsilon\langle\nabla\zeta, \nabla\xi\rangle + \frac{W''(u)}{\varepsilon}\zeta\xi\right) d\mu_g.$$

Similarly, we recall that the second variation of area at Σ is given by

$$\mathcal{Q}_\Sigma(\zeta,\xi) \triangleq \delta^2 \operatorname{Area}[\Sigma]\{\zeta,\xi\} = \int_\Sigma \left(\langle \nabla\zeta, \nabla\xi\rangle - (|\mathrm{I\!I}_\Sigma|^2 + \operatorname{Ric}_g(\nu,\nu))\zeta\xi\right)\, d\mu_\Sigma.$$

LEMMA 5.7. *For $f \in C^2(\Sigma)$, setting*

$$\psi(y,z) = f(y) \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))$$

for (y,z) Fermi coordinates with respect to Γ (the nodal set of u), and $\psi = 0$ far from Γ, we have that

$$\mathcal{Q}_u(\psi,\psi) = \varepsilon^2(h_0 - o(1)) \int_\Gamma \left(|\nabla_\Gamma f|^2 - \left((|\mathrm{I\!I}_\Gamma|^2 + \operatorname{Ric}_g(\partial_z,\partial_z))\circ \Pi_\Gamma\right) f^2\right) d\mu_\Gamma$$
$$+ o(\varepsilon^2)\int_\Gamma \left(|\nabla_\Gamma f|^2 + f^2\right) d\mu_\Gamma.$$

Here, Π_Γ denotes the nearest point projection onto Γ.

Proof. Using (2.15), we compute

$$\begin{aligned}
\mathcal{Q}_u(\psi,\psi) &= \int_M \left(-\varepsilon\psi\Delta_g\psi + \varepsilon^{-1}W''(u)\psi^2\right)\, d\mu_g\\
&= \int_{-\eta}^{\eta}\int_\Gamma \left(-\varepsilon\psi\Delta_{\Gamma_z}\psi - \varepsilon H_{\Gamma_z}\psi\partial_z\psi - \varepsilon\psi\partial_z^2\psi + \varepsilon^{-1}W''(u)\psi^2\right)\, d\mu_{g_z}dz\\
&= \int_{-\eta}^{\eta}\int_\Gamma \left(\varepsilon|\nabla_{\Gamma_z}\psi|^2 - \varepsilon H_{\Gamma_z}\psi\partial_z\psi - \varepsilon\psi\partial_z^2\psi + \varepsilon^{-1}W''(u)\psi^2\right)\, d\mu_{g_z}dz\\
&= \int_{-\eta}^{\eta}\int_\Gamma \Big(\varepsilon|(\nabla_{\Gamma_z} f)\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\\
&\quad - \varepsilon^{-1} f(y)(\nabla_{\Gamma_z} h)\cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y)))|^2\\
&\quad - H_{\Gamma_z} f(y)^2 \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y)))\\
&\quad - \varepsilon^{-1} f(y)^2 \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\overline{\mathbb{H}}'''(\varepsilon^{-1}(z-h(y)))\\
&\quad + \varepsilon^{-1}W''(u) f(y)^2 \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))^2\Big)\, d\mu_{g_z}dz.
\end{aligned}$$

Additionally, using (5.2), our C^2 bounds on Γ, (A.1), (A.2), and (A.3),

$$\begin{aligned}
&\mathcal{Q}_u(\psi,\psi)\\
&= \int_{-\eta}^{\eta}\int_\Gamma \Big(\varepsilon|(\nabla_{\Gamma_z} f)\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) - \varepsilon^{-1} f(y)(\nabla_{\Gamma_z} h)\cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y)))|^2\\
&\quad - H_\Gamma f(y)^2 \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y)))\\
&\quad + \left((|\mathrm{I\!I}_\Gamma|^2 + \operatorname{Ric}_g(\partial_z,\partial_z))\circ\Pi_\Gamma\right) f(y)^2\\
&\quad \cdot z \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y)))\\
&\quad + \varepsilon^{-1}(W''(U+\phi) - W''(U)) f(y)^2 \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))^2\Big)\, d\mu_{g_z}dz
\end{aligned}$$

$$
\begin{aligned}
&+ O(\varepsilon^3)\int_\Gamma f(y)^2 d\mu_\Gamma\\
&= \int_{-\eta}^{\eta}\int_\Gamma \Big(\varepsilon|\nabla_{\Gamma_z} f|^2\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))^2\\
&\quad + \big((|\mathrm{I\!I}_\Gamma|^2 + \mathrm{Ric}_g(\partial_z,\partial_z))\circ \Pi_\Gamma\big) f(y)^2\\
&\quad \cdot z\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y)))\\
&\quad + \varepsilon^{-1}W'''(U)\phi f(y)^2\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))^2\Big)\, d\mu_{g_z}\,dz\\
&\quad + O(\varepsilon^3)\int_\Gamma (|\nabla_\Gamma f|^2 + f^2)\, d\mu_\Gamma\\
&= \varepsilon^2(h_0 - o(1))\int_\Gamma \big(|\nabla_\Gamma f|^2 - \big((|\mathrm{I\!I}_\Gamma|^2 + \mathrm{Ric}_g(\partial_z,\partial_z))\circ\Pi_\Gamma\big) f(y)^2\big)\, d\mu_\Gamma\\
&\quad + o(\varepsilon^2)\int_\Gamma \big(|\nabla_\Gamma f|^2 + f^2\big)\, d\mu_\Gamma.
\end{aligned}
$$

In the final equality, we have used

$$\int_{-\infty}^{\infty} t\mathbb{H}'(t)\mathbb{H}''(t)dt = \tfrac{1}{2}\int_{-\infty}^{\infty} t\tfrac{d}{dt}\mathbb{H}'(t)^2 dt = -\tfrac{1}{2}h_0$$

on the second term. We have also used $\phi = \varepsilon^2 V(y)\mathbb{J}(\varepsilon^{-1}(z-h(y))) + o(\varepsilon^2)$, $V(y) = (|\mathrm{I\!I}_\Gamma|^2 + \mathrm{Ric}_g(\partial_z,\partial_z))\circ\Pi_\Gamma + o(1)$, and the following identity, which follows by differentiating (5.5) once and integrating by parts:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} W'''(\mathbb{H}(t))\mathbb{J}(t)\mathbb{H}'(t)^2 dt\\
&\quad = \int_{-\infty}^{\infty}\big(\mathbb{J}'''(t)\mathbb{H}'(t) - W''(\mathbb{H}(t))\mathbb{J}'(t)\mathbb{H}'(t) - \mathbb{H}'(t)^2 - t\mathbb{H}'(t)\mathbb{H}''(t)\big)\, dt\\
&\quad = \int_{-\infty}^{\infty}\big(\mathbb{J}'(t)\mathbb{H}'''(t) - W''(\mathbb{H}(t))\mathbb{J}'(t)\mathbb{H}'(t) - \mathbb{H}'(t)^2 - t\mathbb{H}'(t)\mathbb{H}''(t)\big)\, dt\\
&\quad = \int_{-\infty}^{\infty}\big(-\mathbb{H}'(t)^2 - t\mathbb{H}'(t)\mathbb{H}''(t)\big)\, dt = -\tfrac{1}{2}h_0.
\end{aligned}
$$

This completes the proof. □

Let Ω denote the η-tubular neighborhood of Γ, and consider the restriction $\mathcal{Q}_u^\Omega$ of $\mathcal{Q}_u$ to Ω:

$$
\begin{aligned}
\mathcal{Q}_u^\Omega(\zeta,\xi) &\triangleq \delta^2(E_\varepsilon \llcorner \Omega)[u]\{\zeta,\xi\}\\
&= \int_\Omega \left(\varepsilon\langle\nabla\zeta,\nabla\xi\rangle + \frac{W''(u)}{\varepsilon}\zeta\xi\right) d\mu_g,\ \zeta,\xi\in C^\infty(\Omega).
\end{aligned}
$$

Consider $w \in C^\infty(\Omega)$. We decompose w as

$$w(y,z) = f(y)\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) + w^\perp(y,z), \tag{5.10}$$

where

$$\int_{-\eta}^{\eta} w^{\perp}(y,z)\overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\,dz = 0. \tag{5.11}$$

It is useful to write

$$\psi(y,z) = f(y)\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))). \tag{5.12}$$

Note that

$$\begin{aligned}
\int_\Omega w^2 d\mu_g &= \int_{-\eta}^{\eta}\int_\Gamma f^2\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))^2\,d\mu_{g_z}\,dz + \int_\Omega (w^\perp)^2\,d\mu_g \\
&\quad + 2\int_{-\eta}^{\eta}\int_\Gamma f w^\perp \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))\,d\mu_{g_z}\,dz \\
&= (1+o(1))\int_{-\eta}^{\eta}\int_\Gamma f^2\cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y)))^2\,d\mu_\Gamma\,dz \\
&\quad + (1+o(1))\int_\Omega (w^\perp)^2\,d\mu_g \\
&= \varepsilon(h_0-o(1))\int_\Gamma f^2\,d\mu_\Gamma + (1+o(1))\int_\Omega (w^\perp)^2\,d\mu_g.
\end{aligned} \tag{5.13}$$

We now use this decomposition to estimate $\mathcal{Q}_u^\Omega(w,w)$.

LEMMA 5.8. *For $w^\perp$ as in (5.11), there is $\gamma > 0$ so that for $\varepsilon > 0$ sufficiently small*,

$$\mathcal{Q}_u^\Omega(w^\perp,w^\perp) \geq \gamma\int_\Omega \varepsilon|\nabla w^\perp|^2 + \varepsilon^{-1}(w^\perp)^2\,d\mu_g.$$

Proof. Recall that there is some $\gamma = \gamma(W) > 0$ so that if $f(t)$ satisfies $\int_{-\infty}^{\infty} f(t)\mathbb{H}'(t)dt = 0$, then

$$\int_{-\infty}^{\infty} f'(t)^2 + W''(\mathbb{H}(t))f(t)^2\,dt \geq 4\gamma\int_{-\infty}^{\infty} f'(t)^2 + f(t)^2\,dt.$$

(See, e.g., [dPKW13, (9.28)].) A change of variables and a compactness argument imply that

$$\begin{aligned}
&\int_{-\eta}^{\eta} \varepsilon(\partial_z w^\perp(y,z))^2 + \varepsilon^{-1}W''(U)(w^\perp(y,z))^2\,dz \\
&\qquad\qquad \geq 3\gamma\int_{-\eta}^{\eta}\varepsilon(\partial_z w^\perp(y,z))^2 + \varepsilon^{-1}(w^\perp(y,z))^2\,dz
\end{aligned}$$

as long as $\varepsilon > 0$ is sufficiently small. From this, and (5.2), we find

$$\begin{aligned}
\mathcal{Q}_u^\Omega(w^\perp,w^\perp) &= \int_{-\eta}^{\eta}\int_\Gamma \left(\varepsilon|\nabla_{\Gamma_z} w^\perp|^2 + \varepsilon(\partial_z w^\perp)^2 + \varepsilon^{-1}W''(u)(w^\perp)^2\right) d\mu_{g_z}dz \\
&= \int_{-\eta}^{\eta}\int_\Gamma \left(\varepsilon|\nabla_{\Gamma_z} w^\perp|^2 + \varepsilon(\partial_z w^\perp)^2 + \varepsilon^{-1}W''(U)(w^\perp)^2\right) d\mu_{g_z}dz
\end{aligned}$$

$$
\begin{aligned}
&+ O(\varepsilon) \int_\Omega (w^\perp)^2 d\mu_g \\
&\geq 2\gamma \int_{-\eta}^{\eta} \int_\Gamma \left(\varepsilon(\partial_z w^\perp)^2 + \varepsilon^{-1}(w^\perp)^2 \right) d\mu_\Gamma dz \\
&\quad + \int_{-\eta}^{\eta} \int_\Gamma \varepsilon |\nabla_{\Gamma_z} w^\perp|^2 d\mu_{g_z} dz + O(\varepsilon) \int_\Omega (w^\perp)^2 d\mu_g \\
&\geq \gamma \int_\Omega \varepsilon |\nabla w^\perp|^2 + \varepsilon^{-1}(w^\perp)^2 \, d\mu_g.
\end{aligned}
$$

This completes the proof. □

LEMMA 5.9. *For ψ, f, $w^\perp$ as in (5.10)–(5.12), we have*

$$
\mathcal{Q}_u^\Omega(\psi, w^\perp) \geq -o(\varepsilon^2) \int_\Gamma |\nabla_\Gamma f|^2 + f^2 \, d\mu_\Gamma - o(1) \int_\Omega \varepsilon |\nabla w^\perp|^2 + \varepsilon^{-1}(w^\perp)^2 \, d\mu_g.
$$

Proof. Repeatedly using (1.6), (1.7), (5.2), Lemma 5.5, (A.3), and (A.4),

$$
\begin{aligned}
\mathcal{Q}_u^\Omega(\psi, w^\perp) &= \int_\Omega \left(-\varepsilon(\Delta_g \psi) w^\perp + \varepsilon^{-1} W''(u) \psi w^\perp \right) d\mu_g \\
&= \int_{-\eta}^{\eta} \int_\Gamma \Big(- \varepsilon(\Delta_{\Gamma_z} \psi) w^\perp - H_{\Gamma_z} f w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z - h(y))) \\
&\quad - \varepsilon^{-1} f w^\perp \cdot \overline{\mathbb{H}}'''(\varepsilon^{-1}(z - h(y))) + \varepsilon^{-1} W''(u) \psi w^\perp \Big) d\mu_{g_z} dz \\
&= \int_{-\eta}^{\eta} \int_\Gamma \Big(- \varepsilon(\Delta_{\Gamma_z} f) w^\perp \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z - h(y))) \\
&\quad + 2\langle \nabla_{\Gamma_z} f, \nabla_{\Gamma_z} h \rangle w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z - h(y))) \\
&\quad + (\Delta_{\Gamma_z} h - H_{\Gamma_z}) f w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z - h(y))) \\
&\quad - \varepsilon^{-1} f w^\perp |\nabla_{\Gamma_z} h|^2 \cdot \overline{\mathbb{H}}'''(\varepsilon^{-1}(z - h(y))) \\
&\quad - \varepsilon^{-1} f w^\perp \cdot \overline{\mathbb{H}}'''(\varepsilon^{-1}(z - h(y))) + \varepsilon^{-1} W''(u) \psi w^\perp \Big) d\mu_{g_z} dz \\
&= \int_{-\eta}^{\eta} \int_\Gamma \Big(- \varepsilon(\Delta_{\Gamma_z} f) w^\perp \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z - h(y))) \\
&\quad + 2\langle \nabla_{\Gamma_z} f, \nabla_{\Gamma_z} h \rangle w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z - h(y))) \\
&\quad + (\Delta_{\Gamma_z} h - H_{\Gamma_z}) f w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z - h(y))) \\
&\quad + \varepsilon^{-1}(W''(U + \phi) - W''(U) + O(\varepsilon^3)) \psi w^\perp \Big) d\mu_{g_z} dz \\
&= \int_{-\eta}^{\eta} \int_\Gamma \Big(- \varepsilon(\Delta_{\Gamma_z} f) w^\perp \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z - h(y)))
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\langle \nabla_{\Gamma_z} f, \nabla_{\Gamma_z} h\rangle w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y))) \\
&+ (\Delta_{\Gamma_z} h - H_{\Gamma_z}) f w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y)))) d\mu_{g_z} dz - O(\varepsilon) \int_\Omega |f w^\perp| \, d\mu_g \\
= &\int_{-\eta}^{\eta} \int_\Gamma \Big(\varepsilon \langle \nabla_{\Gamma_z} f, \nabla_{\Gamma_z} w^\perp \rangle \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) \\
&+ \langle \nabla_{\Gamma_z} f, \nabla_{\Gamma_z} h\rangle w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y))) \\
&+ (\Delta_{\Gamma_z} h - H_{\Gamma_z}) f w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y)))\Big) d\mu_{g_z} dz - O(\varepsilon) \int_\Omega |f w^\perp| \, d\mu_g \\
= &\int_{-\eta}^{\eta} \int_\Gamma \Big(\varepsilon \langle \nabla_{\Gamma_z} f, \nabla_{\Gamma_z} w^\perp \rangle \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) \\
&+ (\Delta_{\Gamma_z} h - H_{\Gamma_z}) f w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y)))\Big) d\mu_{g_z} dz \\
&- O(\varepsilon) \int_\Omega |f w^\perp| \, d\mu_g - O(\varepsilon^2) \int_\Omega |\nabla_{\Gamma_z} f| |w^\perp| \, d\mu_g \\
= &\int_{-\eta}^{\eta} \int_\Gamma \Big(\varepsilon \langle \nabla_{\Gamma_z} f, \nabla_{\Gamma_z} w^\perp \rangle \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) \\
&+ (\Delta_{\Gamma_z} h - H_{\Gamma_z}) f w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z-h(y)))\Big) d\mu_{g_z} dz \\
&- o(\varepsilon^3) \int_\Gamma |\nabla_\Gamma f|^2 + f^2 \, d\mu_\Gamma - o(1) \int_\Omega \varepsilon^{-1} (w^\perp)^2 \, d\mu_g.
\end{aligned}
$$

In the last inequality, we estimated, using Cauchy–Schwarz, $2ab \leq \varepsilon^{1-\sigma} a^2 + \varepsilon^{-1+\sigma} b^2$ for $\sigma \in (0,1)$, $(a,b) = (|f|, |w^\perp|)$, $(|\nabla_{\Gamma_z} f|, |w^\perp|)$. Using (5.2), (A.3), (A.4), and (A.7), we can further estimate

$$
\Delta_{\Gamma_z} h - H_{\Gamma_z} = \Delta_\Gamma h - H_\Gamma + O(|z|) = O(\varepsilon + |z|)
$$

and

$$
\langle \nabla_{\Gamma_z} f, \nabla_{\Gamma_z} w^\perp \rangle = \langle \nabla_\Gamma f, \nabla_\Gamma w^\perp \rangle + O(\varepsilon + |z|) |\nabla_\Gamma f| |\nabla_\Gamma w^\perp|.
$$

By the same Cauchy–Schwarz estimate applied to

$$
(a,b) = (|f|, |w^\perp|), \quad (|\nabla_\Gamma f|, |\nabla_\Gamma w^\perp|),
$$

we get

$$
\begin{aligned}
\mathcal{Q}_u^\Omega(\psi, w^\perp) = &\int_{-\eta}^{\eta} \int_\Gamma \varepsilon \langle \nabla_\Gamma f, \nabla_\Gamma w^\perp \rangle \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) \, d\mu_{g_z} dz \\
&- o(\varepsilon^2) \int_\Gamma |\nabla_\Gamma f|^2 + f^2 \, d\mu_\Gamma - o(1) \int_\Omega \varepsilon |\nabla w^\perp|^2 + \varepsilon^{-1} (w^\perp)^2 \, d\mu_g.
\end{aligned}
$$

Estimating $|d\mu_{g_z} - d\mu_\Gamma| = O(|z|) d\mu_\Gamma$ and using the same Cauchy–Schwarz inequality, we deduce that

$$
\mathcal{Q}_u^\Omega(\psi, w^\perp) = \int_{-\eta}^{\eta} \int_\Gamma \varepsilon \langle \nabla_\Gamma f, \nabla_\Gamma w^\perp \rangle \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z-h(y))) \, d\mu_\Gamma dz
$$

$$- o(\varepsilon^2) \int_\Gamma |\nabla_\Gamma f|^2 + f^2 \, d\mu_\Gamma - o(1) \int_\Omega \varepsilon |\nabla w^\perp|^2 + \varepsilon^{-1}(w^\perp)^2 \, d\mu_g$$

$$= \int_\Gamma \int_{-\eta}^{\eta} \varepsilon \langle \nabla_\Gamma f, \nabla_\Gamma w^\perp \rangle \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z - h(y))) \, dz \, d\mu_\Gamma$$

$$- o(\varepsilon^2) \int_\Gamma |\nabla_\Gamma f|^2 + f^2 \, d\mu_\Gamma - o(1) \int_\Omega \varepsilon |\nabla w^\perp|^2 + \varepsilon^{-1}(w^\perp)^2 \, d\mu_g.$$

Since $\langle \nabla_\Gamma f, \nabla_\Gamma w^\perp \rangle = g_\Gamma^{ij} \partial_{y_i} f \partial_{y_j} w^\perp$, whose first two factors are independent of z, we can use

$$\int_{-\eta}^{\eta} \partial_{y_j} w^\perp \overline{\mathbb{H}}'(\varepsilon^{-1}(z - h(y))) \, dz = \varepsilon^{-1} \int_{-\eta}^{\eta} (\partial_{y_j} h) w^\perp \overline{\mathbb{H}}''(\varepsilon^{-1}(z - h(y))) \, dz,$$

which follows from differentiating (5.11) once horizontally. We thus have

$$\mathcal{Q}_u^\Omega(\psi, w^\perp) = \int_{-\Gamma} \int_{-\eta}^{\eta} \langle \nabla_\Gamma f, \nabla_\Gamma h \rangle w^\perp \cdot \overline{\mathbb{H}}''(\varepsilon^{-1}(z - h(y))) \, dz \, d\mu_\Gamma$$

$$- o(\varepsilon^2) \int_\Gamma |\nabla_\Gamma f|^2 + f^2 \, d\mu_\Gamma - o(1) \int_\Omega \varepsilon |\nabla w^\perp|^2 + \varepsilon^{-1}(w^\perp)^2 \, d\mu_g.$$

This completes the proof, since we have already estimated terms of this form with the correct error term. ☐

LEMMA 5.10. *There is $\sigma = \sigma(M, g, W, \Sigma) > 0$ so that for $\varepsilon > 0$ sufficiently small and any $w \in C^\infty(\Omega)$, we have*

$$\mathcal{Q}_u^\Omega(w, w) \geq -\varepsilon \sigma \int_\Omega w^2 d\mu_g.$$

Proof. Because Γ converges to Σ in $C^{2,\theta}$, we find that for $\delta = \delta(M, g, \Sigma) \in (0, 1)$ and $\varepsilon > 0$ sufficiently small, we have

$$\int_\Gamma |\nabla_\Gamma f|^2 - \left((|\,\mathrm{I\!I}_\Gamma|^2 + \mathrm{Ric}_g(\partial_z, \partial_z)) \circ \Pi_\Gamma \right) f^2 d\mu_\Gamma \geq \int_\Gamma \delta |\nabla_\Gamma f|^2 - \delta^{-1} f^2 \, d\mu_\Gamma.$$

Thus, using (5.10)–(5.12), Lemmas 5.7, 5.8, and 5.9, we find that

$$\begin{aligned}
\mathcal{Q}_u^\Omega(w, w) &= \mathcal{Q}_u^\Omega(\psi, \psi) + \mathcal{Q}_u^\Omega(w^\perp, w^\perp) + 2\mathcal{Q}_u^\Omega(\psi, w^\perp) \\
&\geq \varepsilon^2(h_0 - o(1)) \int_\Gamma \delta |\nabla_\Gamma f|^2 - \delta^{-1} f^2 \, d\mu_\Gamma \\
&\quad + \gamma \int_\Omega \varepsilon |\nabla w^\perp|^2 + \varepsilon^{-1}(w^\perp)^2 \, d\mu_g \\
&\quad - o(\varepsilon^2) \int_\Gamma |\nabla f|^2 + f^2 \, d\mu_\Gamma - o(1) \int_\Omega \varepsilon |\nabla w^\perp|^2 + \varepsilon^{-1}(w^\perp)^2 \, d\mu_g \\
&\geq -\varepsilon^2 \delta^{-1}(h_0 - o(1)) \int_\Gamma f^2 d\mu_\Gamma \geq -\varepsilon \delta^{-1}(1 + o(1)) \int_\Omega w^2 d\mu_g.
\end{aligned}$$

In the last inequality we used (5.13). This completes the proof. ☐

We are now able to prove the main theorem. In what follows,

- ind(u), nul(u) denote the index and nullity of the second variation of Allen–Cahn energy functional (see (1.2)); and
- ind(Σ), nul(Σ) denote the index and nullity of the second variation of the area functional for the limiting multiplicity-one smooth minimal surface (recall (5.1)).

For simplicity, we will assume that $\partial M = \emptyset$, although we expect that the general strategy used here should extend to Dirichlet or Neumann boundary conditions with appropriate modifications.

THEOREM 5.11. *If* (M^n, g), u, *and* Σ *are as above, and if* $\partial M = \emptyset$, *then for sufficiently small* $\varepsilon > 0$,

$$\operatorname{ind}(\Sigma) + \operatorname{nul}(\Sigma) \geq \operatorname{ind}(u) + \operatorname{nul}(u).$$

Proof. For brevity, let us set $I_\Sigma \triangleq \operatorname{ind}(\Sigma) + \operatorname{nul}(\Sigma)$, $I_0 \triangleq \operatorname{ind}(u) + \operatorname{nul}(u)$. First, we show

Claim. There are smooth functions $f_1, \ldots, f_{I_\Sigma} : \Gamma \to \mathbf{R}$ and a constant $\delta > 0$ so that if $f \in C^1(\Gamma)$ satisfies $\langle f, f_i \rangle_{L^2(\Gamma)} = 0$ for all $i = 1, \ldots, I_\Sigma$, then

$$\mathcal{Q}_\Gamma(f, f) \geq \delta \int_\Gamma |\nabla_\Gamma f|^2 + f^2 \, d\mu_\Gamma. \tag{5.14}$$

Proof of claim. As the nodal set Γ converges to Σ in $C^{2,\theta}$ (by Lemma 5.4), it is not hard to see that there is a lower bound $\nu > 0$ for the first *positive* eigenvalue of the second variation of area of Γ. Take $f_1, \ldots, f_{I_\Sigma}$ to be the first I_Σ eigenfunctions of $\mathcal{Q}_\Gamma$. Then,

$$\begin{aligned}\mathcal{Q}_\Gamma(f, f) &= \int_\Gamma |\nabla_\Gamma f|^2 - (|\,\mathrm{I\!I}_\Gamma|^2 + \operatorname{Ric}_g(\partial_z, \partial_z)) f^2 \, d\mu_\Gamma \\ &\geq \nu \int_\Gamma f^2 \, d\mu_\Gamma\end{aligned}$$

for $f \in C^1(\Gamma)$, $\langle f, f_1 \rangle_{L^2(\Gamma)} = \cdots = \langle f, f_{I_\Sigma} \rangle_{L^2(\Gamma)} = 0$. If $|\,\mathrm{I\!I}_\Gamma|^2 + \operatorname{Ric}_g(\partial_z, \partial_z) \leq C$, then

$$\begin{aligned}\tfrac{\nu}{2C} \mathcal{Q}_\Gamma(f, f) &= \tfrac{\nu}{2C} \int_\Gamma |\nabla_\Gamma f|^2 - (|\,\mathrm{I\!I}_\Gamma|^2 + \operatorname{Ric}_g(\partial_z, \partial_z)) f^2 \, d\mu_\Gamma \\ &\geq \int_\Gamma \tfrac{\nu}{2C} |\nabla_\Gamma f|^2 - \tfrac{\nu}{2} f^2 \, d\mu_\Gamma.\end{aligned}$$

The claim follows by adding these two inequalities. □

We define the linear functional $\Pi_\varepsilon : L^2(M) \to L^2(\Gamma)$,

$$\Pi_\varepsilon(w)(y) \triangleq \varepsilon^{-1} \int_{-\eta}^{\eta} w(y, z) \cdot \overline{\mathbb{H}}'(\varepsilon^{-1}(z - h(y))) \, dz,$$

and another linear functional $\mathcal{I}_\Gamma : C^1(\Gamma) \to \mathbf{R}^{I_\Sigma}$,

$$\mathcal{I}_\Gamma(f) \triangleq \big(\langle f, f_1\rangle_{L^2(\Gamma)}, \ldots, \langle f, f_{I_\Sigma}\rangle_{L^2(\Gamma)}\big),$$

so that $f \in \ker \mathcal{I}_\Gamma$ precisely implies (5.14). We note one more property of elements of $\ker \mathcal{I}_\Gamma$:

Claim. Let $w \in C^\infty(\Omega)$ be such that $\Pi_\varepsilon(w) \in \ker \mathcal{I}_\Gamma$. Then,

$$\mathcal{Q}_u^\Omega(w, w) \geq \varepsilon\sigma' \int_\Omega w^2 \, d\mu_g \tag{5.15}$$

for $\sigma' = \sigma'(M, g, W, \Sigma) > 0$ and $\varepsilon > 0$ sufficiently small.

Proof of claim. We proceed as in Lemma 5.10 but we use the improved lower bound for $\mathcal{Q}_\Gamma(f, f)$, (5.14) for $f = \Pi_\varepsilon(w)$. Write

$$\psi(y, z) = \Pi_\varepsilon(w)\overline{\mathbb{H}}'(\varepsilon^{-1}(z - h(y))).$$

Then, using Lemmas 5.7, 5.8, and 5.9,

$$\begin{aligned}
\mathcal{Q}_u^\Omega(w, w) &= \mathcal{Q}_u^\Omega(\psi, \psi) + \mathcal{Q}^\Omega(w^\perp, w^\perp) + 2\mathcal{Q}_u^\Omega(\psi, w^\perp) \\
&\geq \varepsilon^2\delta(h_0 - o(1)) \int_\Gamma |\nabla_\Gamma f|^2 + f^2 \, d\mu_\Gamma + \gamma \int_\Omega \varepsilon|\nabla w^\perp|^2 + \varepsilon^{-1}(w^\perp)^2 \, d\mu_g \\
&\quad - o(\varepsilon^2) \int_\Gamma |\nabla f|^2 + f^2 \, d\mu_\Gamma - o(1) \int_\Omega \varepsilon|\nabla w^\perp|^2 + \varepsilon^{-1}(w^\perp)^2 \, d\mu_g \\
&\geq \varepsilon\sigma' \int_\Omega w^2 \, d\mu_g.
\end{aligned}$$

The claim follows. ☐

Claim. If $w \in C^\infty(M)$ satisfies $\mathcal{Q}_u(w, w) \leq 0$, then

$$\int_{M\setminus\Omega} w^2 \, d\mu_g \leq C\varepsilon^2 \int_\Omega w^2 \, d\mu_g \tag{5.16}$$

for $C = C(M, g, W, \Sigma, \eta) > 0$ and $\varepsilon > 0$ sufficiently small.

Proof of claim. Using Lemma 5.10 and that $W''(u) \geq \kappa > 0$ on $M \setminus \Omega$ for $\varepsilon > 0$ small, we compute

$$\begin{aligned}
0 \geq \mathcal{Q}_u(w, w) &\geq \mathcal{Q}_u^\Omega(w, w) + \varepsilon^{-1}\kappa \int_{M\setminus\Omega} w^2 d\mu_g \\
&\geq -\varepsilon\sigma \int_\Omega w^2 d\mu_g + \varepsilon^{-1}\kappa \int_{M\setminus\Omega} w^2 d\mu_g.
\end{aligned}$$

Rearranging this completes the proof. ☐

Now, let $w_1, \ldots, w_{I_0} \in C^\infty(M)$ denote an $L^2(M)$-orthonormal set of eigenfunctions for $\mathcal{Q}_u$ with non-positive eigenvalue, and let

$$W_\Omega \triangleq \operatorname{span}\{w_1|_\Omega, \ldots, w_{I_0}|_\Omega\} \subset C^\infty(\Omega),$$
$$W_\Gamma \triangleq \{\Pi_\varepsilon(w) : w \in \operatorname{span}\{w_1, \ldots, w_{I_0}\}\}.$$

We emphasize that

$$\mathcal{Q}_u(w, w) \leq 0 \text{ for all } w \in \operatorname{span}\{w_1, \ldots, w_{I_0}\} \subset C^\infty(M). \tag{5.17}$$

Claim. $\dim W_\Omega = \dim W_\Gamma = I_0$ for $\varepsilon > 0$ sufficiently small.

Proof of claim. To see $\dim W_\Omega = I_0$, it suffices to note that no non-trivial linear combination w of $w_1, \ldots, w_{I_0}$ can vanish on Ω because of (5.16) and (5.17).

Likewise, to see $\dim W_\Gamma = I_0$, it suffices to note that no non-trivial linear combination w of $w_1, \ldots, w_{I_0}$ has $\Pi_\varepsilon(w) = 0$ because of (5.15), (5.16), and (5.17). □

Finally, suppose, for the sake of contradiction, that $I_\Sigma < I_0$. Because $\dim W_\Gamma = I_0 > I_\Sigma$, it must hold that there exists $w \in \operatorname{span}\{w_1, \ldots, w_{I_0}\} \setminus \{0\}$ such that $\mathcal{I}_\Gamma(\Pi_\varepsilon(w)) = 0$. For $\varepsilon > 0$ sufficiently small so that $W''(u) \geq 0$ on $M \setminus \Omega$,

$$\begin{aligned} 0 \geq \mathcal{Q}_u(w, w) &= \mathcal{Q}_u^\Omega(w, w) + \int_{M \setminus \Omega} \varepsilon |\nabla w|^2 + \varepsilon^{-1} W''(u) w^2 \, d\mu_g \\ &\geq \mathcal{Q}_u^\Omega(w, w) \geq \varepsilon \sigma' \int_\Omega w^2 d\mu_g. \end{aligned}$$

We used (5.15) in the last step. Thus, $w \equiv 0$ on Ω, so $w \equiv 0$ on M by (5.16) and (5.17), a contradiction. □

6. Geometric applications

Corollary 6.1 (Multiplicity one, two-sidedness, and index of Allen–Cahn limits for bumpy or positive Ricci curvature metrics). *Let (M^3, g) denote a closed 3-manifold with a bumpy metric (see Definition 1.6) or with positive Ricci curvature. Suppose that $u_i \in C^\infty(M; [-1, 1])$, $\varepsilon_i > 0$, u_i is a critical point of E_{ε_i}, $E_{\varepsilon_i}[u_i] \leq E_0$, $\operatorname{ind}(u_i) \leq I_0$ for all $i = 1, 2, \ldots$ and $\lim_i \varepsilon_i = 0$. Passing to a subsequence, denote by $V \triangleq \lim_i h_0^{-1} V_{\varepsilon_i}[u_i]$ the limit varifold. Then,*

- *the support Σ of V is a smooth, embedded, two-sided, closed minimal surface with $\operatorname{ind}(\Sigma) \leq I_0$;*
- *the limiting varifold V is equal to the varifold associated to Σ with multiplicity one;*
- *for $\beta \in (0, 1)$ fixed, the level sets $u_i^{-1}(t)$, $|t| < 1 - \beta$, converge in $C^{2,\theta}$ with multiplicity one to Σ;*

- *for i sufficiently large*, $\mathrm{nul}(\Sigma) + \mathrm{ind}(\Sigma) \geq \mathrm{nul}(u_i) + \mathrm{ind}(u_i)$.

Proof. By Theorem 4.1, any component of Σ that does not satisfy the conclusion at hand must admit a two-sided double cover with a positive Jacobi field. This cannot happen if g is bumpy (irrespective of the sign of the Jacobi field). Similarly, because a positive Jacobi field implies that the two-sided double cover is stable, this cannot occur for positive Ricci curvature. The $C^{2,\theta}$ convergence follows from Lemma 5.4. The index upper bounds for Σ follow from [Hie18] (also from [Gas17]). Finally, the index lower bounds follow from Theorem 5.11. □

Finally, we note that Corollary 6.1 proves Yau's conjecture for bumpy metrics (or those with positive Ricci curvature) on a 3-manifold. In fact, we establish the following strengthened version of Yau's conjecture, which describes certain geometric properties of the minimal surfaces. That a generic Riemannian manifold contains an embedded two-sided minimal surface of each positive Morse index was conjectured by Marques and Neves (cf. [Nev14, p. 17], [MN16b, Conj. 6.2]).

Corollary 6.2 (Yau's conjecture for bumpy metrics and geometric properties of the minimal surfaces). *Let (M^3, g) be a closed 3-manifold with a bumpy metric. There is $C = C(M, g, W) > 0$ and a smooth embedded closed minimal surface Σ_p for each positive integer p so that*

- *each component of Σ_p is two-sided*;
- *the area of Σ_p satisfies* $C^{-1}p^{\frac{1}{3}} \leq \mathrm{area}_g(\Sigma_p) \leq Cp^{\frac{1}{3}}$;
- *the index of Σ_p is satisfies* $\mathrm{ind}(\Sigma_p) = p$; *and*
- *the genus of Σ_p satisfies* $\mathrm{genus}(\Sigma_p) \geq \frac{p}{6} - Cp^{\frac{1}{3}}$.

In particular, thanks to the index estimate, all of the Σ_p are geometrically distinct.

Proof. Gaspar–Guaraco set up a min-max procedure for the Allen–Cahn energy functional and showed [GG18, Ths. 3, 4] that there is $C = C(M, g, W)$ > 1 so that for each integer $p > 0$, there exists $\varepsilon_0(p) > 0$ so that for $\varepsilon \in (0, \varepsilon_0)$, there exists $u_{p,\varepsilon}$, a critical point of E_ε with

$$C^{-1}p^{\frac{1}{3}} \leq E_\varepsilon[u_{p,\varepsilon}] \leq Cp^{\frac{1}{3}}, \ \mathrm{ind}(u_{p,\varepsilon}) \leq p, \ \mathrm{nul}(u_{p,\varepsilon}) + \mathrm{ind}(u_{p,\varepsilon}) \geq p;$$

see [GG18, Th. 3.3(2)]. Now, the first three bullet points follow from Corollary 6.1 applied to an arbitrary sequence $(u_{p,\varepsilon_i}, \varepsilon_i)$ with $\varepsilon_i \to 0$.

The genus bounds follow from an estimate of Ejiri–Micallef [EM08, Th. 4.3] who prove that there is a constant $C = C(M, g)$ so that writing $\Sigma_p = \cup_{m=1}^N \Sigma_p^{(m)}$, where $\Sigma_p^{(m)}$ are connected and $N = |\pi_0(\Sigma_p)|$ is the number of connected components of Σ_p, we have

$$\operatorname{ind}(\Sigma_p^{(m)}) \leq C \operatorname{area}(\Sigma_p^{(m)}) + r(\operatorname{genus}(\Sigma_p^{(m)})),\ m = 1, \ldots, N,$$

where $r(g)$ is the dimension of the space of conformal structures on a genus g surface, i.e.,

$$r(g) = \begin{cases} 0 & g = 0, \\ 2 & g = 1, \\ 6(g-1) & g > 1. \end{cases}$$

Thus, we find that

$$p = \sum_{m=1}^{N} \operatorname{ind}(\Sigma_p^{(m)}) \leq C \operatorname{area}_g(\Sigma_p) + \sum_{m=1}^{N} r(\operatorname{genus}(\Sigma_p^{(m)})).$$

Using $r(g) \leq 6g$ and $\operatorname{area}(\Sigma_p) \leq Cp^{\frac{1}{3}}$ (for some $C = C(M, g)$ as explained above), we find that,

$$\frac{p}{6} - Cp^{\frac{1}{3}} \leq \sum_{m=1}^{M} \operatorname{genus}(\Sigma_p^{(m)}) = \operatorname{genus}(\Sigma_p)$$

for $C = C(M, g)$. This proves the fourth bullet point, completing the proof. □

Remark 6.3. We note that for (M^3, g) with positive Ricci curvature, the same conclusion as in Corollary 6.2 holds, except the third bullet point is replaced by $\operatorname{ind}(\Sigma_p) + \operatorname{nul}(\Sigma_p) = p$. The genus bound still holds by the same result of Ejiri–Micallef [EM08, Th. 4.3].

When (M^3, g) does not have positive Ricci, Σ_p might have several components. We can use the discrepancy between the linear index growth and sublinear area growth to prove that at least one of the components has large index and genus. (Note that this discrepancy has been leveraged in a rather different manner by Marques–Neves [MN17] in their proof of Yau's conjecture in positive Ricci curvature.)

Corollary 6.4 (Connected components of the p-width having large index and genus). *Let* (M^3, g) *denote a closed* 3*-manifold with a bumpy metric. There is* $C = C(M, g, W) > 0$ *so that some connected component* Σ_p' *of the minimal surface* Σ_p *discussed in Corollary* 6.2 *has* $\operatorname{genus}(\Sigma_p') \geq C^{-1} \operatorname{ind}(\Sigma_p') \geq C^{-1} p^{\frac{2}{3}}$.

Proof. Write the surfaces Σ_p obtained in Corollary 6.2 above as a union of their connected components, i.e., $\Sigma_p = \cup_{m=1}^{N} \Sigma_p^{(m)}$. By the monotonicity formula, there is $c = c(M, g) > 0$ so that any closed minimal surface Σ' in M has $\operatorname{area}_g(\Sigma') \geq c$. Thus,

$$Nc \leq \sum_{m=1}^{N} \operatorname{area}_g(\Sigma_p^{(m)}) = \operatorname{area}_g(\Sigma_p) \leq Cp^{\frac{1}{3}}. \tag{6.1}$$

Because $p = \sum_{m=1}^{N} \operatorname{ind}(\Sigma_p^{(m)})$, (6.1) implies that $\operatorname{ind}(\Sigma_p^m) \geq Cp^{\frac{2}{3}}$ for some $m \in \{1, \ldots, N\}$. For this particular m, $\operatorname{area}_g(\Sigma_p^{(m)}) \leq Cp^{\frac{1}{3}}$ and the estimate of Ejiri–Micallef [EM08] used above implies that

$$\operatorname{genus}(\Sigma_p^{(m)}) \geq C^{-1} \operatorname{ind}(\Sigma_p^{(m)}) \geq C^{-1} p^{\frac{2}{3}}.$$

This completes the proof. □

7. Barriers with Dirichlet data

7.1. *Setup.* The heteroclinic solution from Section 1.3 lifts trivially to a solution of the Allen–Cahn PDE, (1.1), on $\mathbf{R}^n$, for any $n \geq 1$; indeed, one may just take $u(x^1, \ldots, x^n) \triangleq \mathbb{H}_\varepsilon(x^n)$. Notice that this solution is "centered" on the $\{x^n = 0\}$ hyperplane. One may just as easily center it on any hyperplane in $\mathbf{R}^n$ by a suitable translation and rotation.

The question of centering approximate heteroclinic solutions on arbitrary minimal $\Sigma^{n-1} \subset (M^n, g)$ has been well-studied in the compact setting; see, e.g., [PR03] for the boundary-less case and the geometrically natural case of Neumann conditions at the boundary when ∂M, $\partial\Sigma \neq \emptyset$, or see [Pac12] for a more general survey with a faster construction than [PR03], albeit only presented in the boundary-less case.

In this section we establish a corresponding existence theorem similar in spirit to those in [PR03], [Pac12], except we prescribe Dirichlet data. This theorem provides the barriers that were a crucial ingredient in the final "sliding" argument of Section 4.

The setup is as follows. Define $C^{k,\alpha}_\varepsilon$, $\alpha \in (0,1)$, $\varepsilon > 0$, to be the standard Hölder space after rescaling by ε, i.e., whose Banach norm is

$$\|v\|_{C^{k,\alpha}_\varepsilon} \triangleq \sum_{j=0}^{k} \varepsilon^j \|\nabla^j v\|_{L^\infty} + \varepsilon^{k+\alpha} [\nabla^k v]_\alpha. \tag{7.1}$$

Various choices of domain and metric will be specified below. See Remarks 7.1 and 7.12.

Next, suppose that D^{n-1} is an $(n-1)$-dimensional manifold with nonempty boundary, over which we take a topological cylinder $\Omega \triangleq D \times [-1, 1]$, whose coordinates we label $X = (y, z) \in D \times [-1, 1]$. Let g be a smooth metric on Ω, given in (y, z) coordinates (Fermi coordinates) by

$$g = g_z + dz^2.$$

We require that

$$\Sigma \triangleq D \times \{0\} \subset (\Omega, g) \text{ is a minimal surface} \tag{7.2}$$

whose second fundamental form is uniformly bounded in $C^{0,\theta}$ for some $\theta \in (0,1)$ that will be eventually chosen to be near 1 (see Theorem 7.4):

$$|\,\mathrm{I\!I}_\Sigma\,| + [\mathrm{I\!I}_\Sigma]_\theta \leq \eta, \tag{7.3}$$

and also[10] in $C^{1,\theta}_\varepsilon$:

$$\varepsilon|\nabla_\Sigma \mathrm{I\!I}_\Sigma\,| + \varepsilon^{1+\theta}[\nabla_\Sigma \mathrm{I\!I}_\Sigma]_\theta \leq \eta, \tag{7.4}$$

with $\eta > 0$ small. We furthermore assume that there are $C^{2,\theta}$-coordinate charts on Σ so that the induced metric g_0 is $C^{0,\theta}$ and $C^{1,\theta}_\varepsilon$-close to the Euclidean metric in the sense that

$$|(g_0)_{ij} - \delta_{ij}| + [(g_0)_{ij}]_\theta \leq \eta, \tag{7.5}$$

$$\varepsilon|\partial_k (g_0)_{ij}| + \varepsilon^{1+\theta}[\partial_k (g_0)_{ij}]_\theta \leq \eta, \tag{7.6}$$

where i, j, k run through the coordinates $(y^1, \ldots, y^{n-1})$ on Σ in the given coordinate chart.

Note that (7.3) implies that Fermi coordinates (y, z) with respect to Σ are a diffeomorphism which is $C^{1,\theta}$-close to the identity so, in particular, together with (7.5), it follows that the metric g is $C^{0,\theta}$-close to being Euclidean in Fermi coordinates

$$|g_{\kappa\lambda} - \delta_{\kappa\lambda}| + [g_{\kappa\lambda}]_\theta \leq \eta' \tag{7.7}$$

for small $\eta' = \eta'(\eta, n) > 0$. Here, κ, λ run through all n Fermi coordinates $(y^1, \ldots, y^{n-1}, z)$.

Likewise, (7.4) and (7.6) imply that Fermi coordinates are $C^{2,\theta}_\varepsilon$-close to the identity and

$$\varepsilon|\partial_\mu g_{\kappa\lambda}| + \varepsilon^{1+\theta}[\partial_\mu g_{\kappa\lambda}]_\theta \leq \eta'. \tag{7.8}$$

Here, κ, λ, μ run through all n Fermi coordinates.

We also require that Σ carries no non-trivial Jacobi fields with Dirichlet boundary conditions in the following quantitative sense:

$$\int_\Sigma (J_\Sigma f)^2 \, d\mu_{g_0} \geq \eta \int_\Sigma f^2 \, d\mu_{g_0} \text{ for every } f \in C^\infty_c(\Sigma \setminus \partial\Sigma), \tag{7.9}$$

where

$$J_\Sigma f \triangleq -\Delta_{g_0} f - (|\,\mathrm{I\!I}_\Sigma\,|^2 + \mathrm{Ric}_g(\partial_z, \partial_z)|_\Sigma) f \tag{7.10}$$

[10] It is crucial for Section 4 that we only work with the weaker bounds on derivatives of $\mathrm{I\!I}$ given in (7.3) and (7.4), which are precisely the types of estimates we derived in Section 3.

denotes the Jacobi operator on Σ. (Note that our sign convention for the Jacobi operator differs from the one in [Pac12].)

Let us also fix $\delta_* \in (0,1)$ and define cutoff functions $\chi_j : \mathbf{R} \to [0,1]$, with $\chi_j' \geq 0$ on $[0,\infty)$, so that

$$\chi_j(t) = \begin{cases} 1 & |t| \leq \varepsilon^{\delta_*}\left(1 - \frac{2j-1}{100}\right), \\ 0 & |t| \geq \varepsilon^{\delta_*}\left(1 - \frac{2j-2}{100}\right), \end{cases} \tag{7.11}$$

as well as $\|\chi_j\|_{C^3_{\varepsilon^{\delta_*}}(\mathbf{R})} \leq 200$. We further require that the χ_j be even functions.

For $\varepsilon > 0$, set

$$\widetilde{\mathbb{H}}_\varepsilon(t) \triangleq \chi_1(t)\mathbb{H}_\varepsilon(t) \pm (1 - \chi_1(t)), \tag{7.12}$$

where the $\pm$ corresponds to $t > 0$, $t < 0$, respectively, and $\mathbb{H}_\varepsilon$ is as in (1.8). This is a truncation of the one-dimensional solution, $\mathbb{H}_\varepsilon$, which coincides with $\mathbb{H}_\varepsilon$ near Σ and with ± 1 away from Σ.

The functions χ_j, $\mathbb{H}_\varepsilon$, $\widetilde{\mathbb{H}}_\varepsilon$ lift trivially to $\Sigma \times \mathbf{R}$. We also set

$$\Omega_j \triangleq \{(y,z) \in \Sigma \times \mathbf{R} : z \in \operatorname{spt} \chi_j\}.$$

Using the Fermi coordinates (y,z), χ_j, $\mathbb{H}_\varepsilon$, $\widetilde{\mathbb{H}}_\varepsilon$ also give functions on Ω that depend only on z. By (7.6) and (7.11), these functions are uniformly $C^{2,\theta}_\varepsilon$ in $\Sigma \times \mathbf{R}$ with respect to the product metric $g_0 + dz^2$ and also in Ω with respect to the metric g. Likewise, by (7.3) and (7.4), the slab Ω_j can also be viewed as a subset of (Ω, g) whose boundary is $C^{1,\theta}$ and $C^{2,\theta}_\varepsilon$-close to being totally geodesic.

Remark 7.1. By (7.6) and (7.8), there exists a constant $C = C(\eta)$ such that

$$C^{-1}\|f\|_{C^{k,\alpha}_\varepsilon(\Omega)} \leq \|f\|_{C^{k,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \leq C\|f\|_{C^{k,\alpha}_\varepsilon(\Omega)},\ k = 0,1,2,\ \alpha \in (0,\theta]$$

for any function $f : \Omega \to \mathbf{R}$ with support in the interior of Ω. The norms above are taken with respect to the product metric $g_0 + dz^2$ on $\Sigma \times \mathbf{R}$ and the metric g on Ω.

Remark 7.2. We cannot reuse the truncation from Section 2, because we now need a truncation that trivializes outside a polynomial window instead of a logarithmic window.

For subsets $S \subset \Sigma$, let us define

$$\Pi_\varepsilon : L^2(S \times \mathbf{R}) \to L^2(S),\ \Pi_\varepsilon^\perp : L^2(S \times \mathbf{R}) \to L^2(S \times \mathbf{R})$$

to be given by

$$\Pi_\varepsilon(f)(y) \triangleq \varepsilon^{-1} h_0^{-1} \int_{-\infty}^{\infty} f(y,z) \cdot \mathbb{H}'(\varepsilon^{-1} z)\, dz, \tag{7.13}$$

$$\Pi_\varepsilon^\perp(f)(y,z) \triangleq f(y,z) - \Pi_\varepsilon(f)(y)\mathbb{H}'(\varepsilon^{-1}z). \tag{7.14}$$

We note two things:

(1) S does not appear in the projection notation, but it will clear from the context when it is relevant.

(2) Our normalization is such that $\Pi_\varepsilon(\{z \mapsto \mathbb{H}'(\varepsilon^{-1}z)\}) = \varepsilon \Pi_\varepsilon(\mathbb{H}'_\varepsilon) = 1$.

From this point forward we also consider another Hölder exponent, $\alpha \in (0,1)$, which is such that

$$\alpha \leq \theta$$

(with θ as in (7.3)–(7.6)). The exponent α will be eventually taken to be near 0 (see Theorem 7.4).

We point out the following trivial lemma:

Lemma 7.3. *Both Π_ε and $\Pi_\varepsilon^\perp$ lift to linear maps*

$$\Pi_\varepsilon : C^{0,\alpha}_\varepsilon(S \times \mathbf{R}) \to C^{0,\alpha}_\varepsilon(S),\ \Pi_\varepsilon^\perp : C^{0,\alpha}_\varepsilon(S \times \mathbf{R}) \to C^{0,\alpha}_\varepsilon(S \times \mathbf{R}).$$

The $C^{0,\alpha}_\varepsilon(S \times \mathbf{R})$ norm is taken with respect to the product metric $g_0 + dz^2$. Viewed as linear maps over these Hölder spaces, we have $\sup_{\varepsilon > 0}\big(\|\Pi_\varepsilon\| + \|\Pi_\varepsilon^\perp\|\big) < \infty$.

For $\zeta \in C^{2,\alpha}(\Sigma)$, we define D_ζ to be the map

$$D_\zeta(y,t) \triangleq (y, t - \chi_2(t)\zeta(y)). \tag{7.15}$$

Finally, we introduce the modified Hölder norm:

$$\|v\|_{\widetilde{C}^{k,\alpha}_\varepsilon(\Omega)} \triangleq \varepsilon^{-2}\|\chi_5 v\|_{C^{k,\alpha}_\varepsilon(\Omega)} + \|v\|_{C^{k,\alpha}_\varepsilon(\Omega)}. \tag{7.16}$$

Recall that $\|\cdot\|_{C^{k,\alpha}_\varepsilon}$ is as in (7.1). As with Remark 7.1, the $C^{k,\alpha}_\varepsilon(\Omega)$ norm is taken with respect to g.

The main result of this section is

Theorem 7.4. *If $\alpha \leq \alpha_0$, $\varepsilon \leq \varepsilon_0$ and we are given boundary data*

(1) $\widehat{v}^\flat \in \widetilde{C}^{2,\alpha}_\varepsilon(\partial\Omega)$, $\|\widehat{v}^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\partial\Omega)} \leq \mu\varepsilon^2$, $\widehat{v}^\flat = 0$ *on* $\{\chi_4 = 1\} \cap \partial\Omega$,

(2) $\widehat{v}^\sharp \in C^{2,\alpha}_\varepsilon(\partial\Sigma \times \mathbf{R})$, $\|\widehat{v}^\sharp\|_{C^{2,\alpha}_\varepsilon(\partial\Sigma\times\mathbf{R})} \leq \mu\varepsilon^2$, $\Pi_\varepsilon(\widehat{v}^\sharp) \equiv 0$ *on* $\partial\Sigma$,

(3) $\widehat{\zeta} \in C^{2,\alpha}(\partial\Sigma)$, $\varepsilon^{2\alpha}\|\widehat{\zeta}\|_{C^{2,\alpha}(\partial\Sigma)} \leq \mu\varepsilon^2$,

and a metric g for which (7.2)–(7.6) hold with $\theta \geq \theta_0 \geq \alpha_0$, then there exist

(1) $v^\flat \in \widetilde{C}^{2,\alpha}(\Omega)$, $v^\flat|_{\partial\Omega} = \widehat{v}^\flat$, $\|v^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} \leq C\varepsilon^2$,

(2) $v^\sharp \in C^{2,\alpha}(\Sigma \times \mathbf{R})$, $v^\sharp|_{\partial\Sigma\times\mathbf{R}} = \widehat{v}^\sharp$, $\Pi_\varepsilon v^\sharp \equiv 0$, $\|v^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \leq C\varepsilon^2$,

(3) $\zeta \in C^{2,\alpha}(\Sigma)$, $\zeta|_{\partial\Gamma} = \widehat{\zeta}$, $\varepsilon^{2\alpha}\|\zeta\|_{C^{2,\alpha}(\Sigma)} \leq C\varepsilon^2$,

so that $\mathfrak{u} = (\widetilde{\mathbb{H}}_\varepsilon + \chi_4 v^\sharp + v^\flat) \circ D_\zeta$ *satisfies*

$$\varepsilon^2 \Delta_g \mathfrak{u} = W'(\mathfrak{u}) \text{ on } \Omega. \tag{7.17}$$

The solution map $(\widehat{v}^\flat, \widehat{v}^\sharp, \widehat{\zeta}, g) \mapsto (v^\flat, v^\sharp, \zeta)$ *is Lipschitz continuous, with Lipschitz constant* L, *as a map*

$$\widetilde{C}^{2,\alpha}_\varepsilon(\partial\Omega) \times C^{2,\alpha}_\varepsilon(\partial\Sigma \times \mathbf{R}) \times C^{2,\alpha}(\partial\Sigma) \\ \times \mathrm{Met}_{\varepsilon,\eta}(\Omega) \to \widetilde{C}^{2,\alpha}_\varepsilon(\Omega) \times C^{2,\alpha}_\varepsilon(\Sigma \times \mathbf{R}) \times C^{2,\alpha}(\Sigma),$$

where $\mathrm{Met}_{\varepsilon,\eta}(\Omega)$ *denotes the set of metrics satisfying* (7.7)–(7.8) *with the obvious topology. The spaces* $\widetilde{C}^{2,\alpha}_\varepsilon(\Omega) \times C^{2,\alpha}_\varepsilon(\Sigma \times \mathbf{R}) \times C^{2,\alpha}(\Sigma)$, $\widetilde{C}^{2,\alpha}(\partial\Omega) \times C^{2,\alpha}_\varepsilon(\partial\Sigma \times \mathbf{R}) \times C^{2,\alpha}(\partial\Sigma)$ *are topologized using the norms in* (7.86) *and* (7.87), *respectively. Here,* $\varepsilon_0 = \varepsilon_0(n, \eta, W, \delta_*, \mu, \alpha)$, $\alpha_0 = \alpha_0(n, \eta, W, \delta_*, \mu)$, $\theta_0 = \theta_0(\delta_*)$, $C = C(n, \eta, W, \delta_*, \mu, \alpha)$, $L = L(n, \eta, W, \delta_*, \mu, \alpha, \theta)$.

This follows along the lines of [Pac12, §3], provided one makes the necessary modifications to account for (possibly non-zero, but small) Dirichlet data as well as the important fact that our Fermi coordinate regularity is constrained by the weaker assumptions (7.3) and (7.4). This lower regularity situation makes certain aspects of Theorem 7.4 delicate, so we describe the proof in detail below.

7.2. *Linear scheme.* In this section we generalize linear estimates found in [Pac12, §3] to allow Dirichlet boundary conditions, possibly with non-zero data. The operators we will study are

$$L_* \triangleq \Delta_{\mathbf{R}^n} + \partial_z^2 - W''(\mathbb{H}) \text{ on } \mathbf{R}^n_+ \times \mathbf{R}, \tag{7.18}$$

$$L_\varepsilon \triangleq \varepsilon^2(\Delta_{g_0} + \partial_z^2) - W''(\mathbb{H}_\varepsilon) \text{ on } \Sigma \times \mathbf{R}, \tag{7.19}$$

$$\mathcal{L}_\varepsilon \triangleq \varepsilon^2 \Delta_g - W''(\pm 1) \text{ on } \Omega. \tag{7.20}$$

Lemma 7.5 (cf. [Pac12, Lemma 3.7]). *Assume that* $w \in L^\infty(\mathbf{R}^n_+ \times \mathbf{R})$ *satisfies* $L_* w = 0$ *and* $w \equiv 0$ *on* $\partial\mathbf{R}^n_+ \times \mathbf{R}$. *Then* $w \equiv 0$.

Proof. The result follows from [Pac12, Lemma 3.7] after an odd reflection of w across $\partial\mathbf{R}^n_+$. □

The next results that need to be adapted pertain to L_ε and functions $\varphi \in L^\infty(\Sigma \times \mathbf{R})$ satisfying $\Pi_\varepsilon(\varphi) \equiv 0$ on Σ, where Π_ε is as in (7.13).

Lemma 7.6 (cf. [Pac12, Prop. 3.1]). *If* $\varepsilon \le \varepsilon_0$, $w \in C^{2,\alpha}_\varepsilon(\Sigma \times \mathbf{R})$, *and* $\Pi_\varepsilon(w) \equiv 0$ *on* Σ, *then*

$$\|w\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \le C(\|L_\varepsilon w\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|w|_{\partial\Sigma\times\mathbf{R}}\|_{C^{2,\alpha}_\varepsilon(\partial\Sigma\times\mathbf{R})}).$$

Here, $\varepsilon_0 = \varepsilon_0(n, \eta, W)$, $C = C(n, \eta, W, \alpha)$.

Proof. This follows from the $C^{1,\alpha}_\varepsilon$ control of g_0 by way of (7.6), [Pac12, Prop. 3.1], Lemma 7.5, and boundary Schauder estimates (e.g., [Sim97, Th. 5]). $\square$

LEMMA 7.7 (cf. [Pac12, Prop. 3.2]). *There exists $\varepsilon_0 > 0$ depending on n, $\eta > 0$, W, such that for all $\varepsilon \in (0, \varepsilon_0)$, all $f \in C^{0,\alpha}_\varepsilon(\Sigma \times \mathbf{R})$ with $\Pi_\varepsilon(f) \equiv 0$ on Σ, and all $\hat{f} \in C^{2,\alpha}_\varepsilon(\partial\Sigma \times \mathbf{R})$ with $\Pi_\varepsilon(\hat{f}) \equiv 0$ on $\partial\Sigma$, there exists a unique function $w \in C^{2,\alpha}_\varepsilon(\Sigma \times \mathbf{R})$, also with $\Pi_\varepsilon(w) \equiv 0$ on Σ, such that*

$$L_\varepsilon w = f \text{ in } \Sigma \times \mathbf{R},\ w = \hat{f} \text{ on } \partial\Sigma \times \mathbf{R}.$$

Proof. When $\hat{f} \equiv 0$ this follows from the functional analytic argument already found in [Pac12, Prop. 3.2] applied, instead, to $W^{1,2}_0(\Sigma \times \mathbf{R})$.

When $\hat{f} \not\equiv 0$, this follows by extending $\hat{f}$ to $C^{2,\alpha}(\Sigma \times \mathbf{R})$, $\Pi_\varepsilon(\hat{f}) \equiv 0$, and applying the previous existence result with zero boundary data to solve $L_\varepsilon w = f - L_\varepsilon \hat{f}$. $\square$

Finally, [Pac12] deals with $\mathcal{L}_\varepsilon$.

LEMMA 7.8 (cf. [Pac12, Prop. 3.3]). *If $\varepsilon \in (0, 1)$, then*

$$\|w\|_{C^{2,\alpha}_\varepsilon(\Omega)} \leq C(\|\mathcal{L}_\varepsilon w\|_{C^{0,\alpha}_\varepsilon(\Omega)} + \|w|_{\partial\Omega}\|_{C^{2,\alpha}_\varepsilon(\partial\Omega)}).$$

Here, $C = C(n, \eta, W, \alpha)$.

Proof. The interior estimate follows from interior Schauder theory, since g is $C^{1,\alpha}_\varepsilon$ by (7.8). The boundary estimate on the regular portion of $\partial\Omega$ follows from boundary Schauder theory, because $\partial\Omega$ is $C^{2,\alpha}_\varepsilon$ at those points by (7.4). Finally, the estimate at the corners of $\partial\Omega$ follows from the boundary theory as well. This is because we can carry out odd reflections across $D \times \{\pm 1\}$ since the angles at the corners are all $\pi/2$. $\square$

We also derive an *improved* estimate for functions satisfying $\mathcal{L}_\varepsilon w = 0$ on a strip of height $O(\varepsilon^{\delta_*})$, and $w = 0$ on its lateral boundary. Recall the definition of the norm $\widetilde{C}^{2,\alpha}_\varepsilon$ in (7.16).

LEMMA 7.9 (cf. [Pac12, (3.26)]). *If $\varepsilon \leq \varepsilon_0$, $w \in C^{2,\alpha}_\varepsilon(\Omega)$, and*

$$\mathcal{L}_\varepsilon w = 0 \text{ on } \Omega_4 \text{ and } w = 0 \text{ on } \partial\Omega_4 \cap \partial\Omega,$$

then

$$\|w\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} \leq C(\|\mathcal{L}_\varepsilon w\|_{C^{0,\alpha}_\varepsilon(\Omega)} + \|w|_{\partial\Omega}\|_{C^{2,\alpha}_\varepsilon(\partial\Omega)}).$$

Here, $\varepsilon_0 = \varepsilon_0(n, \eta, W, \delta_*)$, $C = C(n, \eta, W, \delta_*, \alpha)$.

Proof. Considering Lemma 7.8, it suffices to check that

$$\|\chi_5 w\|_{C^{2,\alpha}_\varepsilon(\Omega)} \leq C\varepsilon^2(\|\mathcal{L}_\varepsilon w\|_{C^{0,\alpha}_\varepsilon(\Omega)} + \|w|_{\partial\Omega}\|_{C^{2,\alpha}_\varepsilon(\partial\Omega)}). \tag{7.21}$$

Since $\mathcal{L}_\varepsilon = 0$ on Ω_4, $w = 0$ on $\partial\Omega_4 \cap \partial\Omega$, and $\delta_* \in (0,1)$, Schauder's *interior* estimates estimates on $\partial\Omega_5 \setminus \partial\Omega$, Schauder's *boundary* estimates near $\partial\Omega_5 \cap \partial\Omega$, (7.4), and (7.8), imply that

$$\|w\|_{C^{2,\alpha}_\varepsilon(\Omega_5)} \leq C\|w\|_{L^\infty(\{\chi_4=1\})}.$$

In particular, given the decay of the first and second derivatives of χ_j from (7.11) and $\delta_* \in (0,1)$, (7.21) will follow as long as

$$\|w\|_{L^\infty(\{\chi_4=1\})} \leq C\varepsilon^2\|w\|_{L^\infty(\Omega)}. \tag{7.22}$$

We use the same barrier argument as in [Pac12, Rem. 3.2], paying closer attention to the boundary and to the regularity. Define

$$\varphi_{z_0}(z) \triangleq \cosh(\gamma\varepsilon^{-1}(z - z_0))$$

with $|z_0| \leq \varepsilon^{\delta_*}$ and $\gamma \in (0, (W''(\pm 1))^{\frac{1}{2}})$. If H_z denotes the mean curvature of of a z-level set in Fermi coordinates, then

$$\varepsilon^2\Delta_g\varphi_{z_0}(z) = \gamma^2\varphi_{z_0}(z) + H_z\gamma\varepsilon\sinh(\gamma\varepsilon^{-1}(z - z_0)) \leq (\gamma^2 + \gamma\varepsilon|H_z|)\varphi_{z_0}(z).$$

It follows from (7.3) as well as (A.2) and (A.3) that $|H_z|$ is uniformly bounded. In particular, for sufficiently small ε, depending on γ, η, n, we have

$$\varepsilon^2\Delta_g\varphi_{z_0}(z) \leq W''(\pm 1)\varphi_{z_0}(z),$$

so φ_{z_0} is a barrier, as it was in [Pac12]. It therefore follows from the maximum principle applied to $w - t\varphi_{z_0}$ that, for $(y, z_0) \in \Omega_4$,

$$|w(y, z_0)| \leq \left(\inf_{\Omega\setminus\Omega_4} \varphi_{z_0}\right)^{-1} \max_{\partial\Omega_4} |w|,$$

which is trivially bounded by $c\varepsilon^2\|w\|_{L^\infty(\Omega)}$ whenever $(y, z_0) \in \{\chi_4 = 1\}$, and $\varepsilon > 0$ is small. This implies (7.22) and, in turn, (7.21). □

7.3. *Nonlinear scheme.* We consider the following non-linear functionals, originally defined in [Pac12, §3]:

$$\mathscr{E}_\varepsilon(\zeta) \triangleq \varepsilon^2\Delta_g(\widetilde{\mathbb{H}}_\varepsilon \circ D_\zeta) \circ D_\zeta^{-1} - W'(\widetilde{\mathbb{H}}_\varepsilon), \tag{7.23}$$

$$Q_\varepsilon(v) \triangleq W'(\widetilde{\mathbb{H}}_\varepsilon + v) - W'(\widetilde{\mathbb{H}}_\varepsilon) - W''(\widetilde{\mathbb{H}}_\varepsilon)v, \tag{7.24}$$

(7.25)

$$\begin{aligned} M_\varepsilon(v^\flat, v^\sharp, \zeta) \triangleq \chi_3\Big[&L_\varepsilon v^\sharp - \varepsilon^2\Delta_g(v^\sharp \circ D_\zeta) \circ D_\zeta^{-1} + W''(\mathbb{H}_\varepsilon)v^\sharp \\ &- \varepsilon^2(\Delta_g(v^\flat \circ D_\zeta) \circ D_\zeta^{-1} - \Delta_g v^\flat) - \mathscr{E}_\varepsilon(\zeta) + \varepsilon^2(J_\Sigma\zeta)\partial_z\mathbb{H}_\varepsilon \\ &- Q_\varepsilon(\chi_4 v^\sharp + v^\flat) + (W''(\mathbb{H}_\varepsilon) - W''(\pm 1))v^\flat\Big], \end{aligned}$$

(7.26)

$$N_\varepsilon(v^\flat, v^\sharp, \zeta) \triangleq (\chi_4 - 1)\Big[\varepsilon^2(\Delta_g(v^\flat \circ D_\zeta) \circ D_\zeta^{-1} - \Delta_g v^\flat) + (W''(\widetilde{\mathbb{H}}_\varepsilon) - W''(\pm 1))v^\flat - \mathscr{E}_\varepsilon(\zeta) - Q_\varepsilon(\chi_4 v^\sharp + v^\flat)\Big] - \varepsilon^2(\Delta_g((\chi_4 v^\sharp) \circ D_\zeta) - \chi_4 \Delta_g(v^\sharp \circ D_\zeta)) \circ D_\zeta^{-1}.$$

These functionals allow us to pose (7.17) as a fixed point problem:

$$\mathcal{L}_\varepsilon v^\flat = N_\varepsilon(v^\flat, v^\sharp, \zeta), \tag{7.27}$$

$$L_\varepsilon v^\sharp = \Pi_\varepsilon^\perp M_\varepsilon(v^\flat, v^\sharp, \zeta), \tag{7.28}$$

$$J_\Sigma \zeta = \varepsilon^{-1} \Pi_\varepsilon M_\varepsilon(v^\flat, v^\sharp, \zeta); \tag{7.29}$$

cf. [Pac12, (3.31), (3.32), (3.33)]. We impose, as does [Pac12, §3], the additional constraint

$$\Pi_\varepsilon v^\sharp \equiv 0 \text{ on } \Sigma.$$

LEMMA 7.10 (cf. [Pac12, Lemma 3.8]). *The following estimate holds*:

$$\|N_\varepsilon(0,0,0)\|_{C^{0,\alpha}_\varepsilon(\Omega)} + \|\Pi_\varepsilon^\perp M_\varepsilon(0,0,0)\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \varepsilon^{-1}\|\Pi_\varepsilon M_\varepsilon(0,0,0)\|_{C^{0,\alpha}(\Sigma)} \leq c_0 \varepsilon^2.$$

Here, $\varepsilon \in (0, \frac{1}{2})$, $c_0 = c_0(n, \eta, W, \delta_*, \alpha)$.

Proof. Note that

$$M_\varepsilon(0,0,0) = -\chi_3 \mathscr{E}_\varepsilon(0), \; N_\varepsilon(0,0,0) = (1-\chi_4)\mathscr{E}_\varepsilon(0).$$

Straightforward computation shows $\mathscr{E}_\varepsilon(0) = \varepsilon^2 \Delta_g \widetilde{\mathbb{H}}_\varepsilon - W'(\widetilde{\mathbb{H}}_\varepsilon)$. From (7.12),

$$\widetilde{\mathbb{H}}_\varepsilon - \mathbb{H}_\varepsilon = (1-\chi_1)(\pm 1 - \mathbb{H}_\varepsilon), \tag{7.30}$$

($\pm$ depends on $z > 0$ or $z < 0$), a quantity that decays exponentially to all orders with $\varepsilon \to 0$. Since $\mathbb{H}_\varepsilon$ does too on $\operatorname{spt}(1-\chi_4)$, we in fact get

$$\|N_\varepsilon(0,0,0)\|_{C^{0,\alpha}_\varepsilon(\Omega)} \leq C_m \varepsilon^m$$

for all $m \in \mathbf{N}$. (Taking $m = 2$ will suffice.)

To estimate $M_\varepsilon(0,0,0)$, we proceed to further rewrite

$$\begin{aligned}\mathscr{E}_\varepsilon(0) &= \varepsilon^2 \Delta_g \widetilde{\mathbb{H}}_\varepsilon - W'(\widetilde{\mathbb{H}}_\varepsilon) \\ &= \varepsilon^2 \Delta_g \mathbb{H}_\varepsilon - W'(\mathbb{H}_\varepsilon) + \varepsilon^2 \Delta_g(\widetilde{\mathbb{H}}_\varepsilon - \mathbb{H}_\varepsilon) - (W'(\widetilde{\mathbb{H}}_\varepsilon) - W'(\mathbb{H}_\varepsilon)) \\ &= \varepsilon^2 H_z \partial_z \mathbb{H}_\varepsilon - (\varepsilon^2 \Delta_g - W''(\widetilde{\mathbb{H}}_\varepsilon) - Q_\varepsilon)(\mathbb{H}_\varepsilon - \widetilde{\mathbb{H}}_\varepsilon).\end{aligned}$$

Note that

$$\widetilde{\mathbb{H}}_\varepsilon \equiv 1 \text{ on } \Omega \setminus \Omega_1 \implies \mathscr{E}_\varepsilon(0) \equiv 0 \text{ on } \Omega \setminus \Omega_1.$$

If $\chi : \Omega \to [0,1]$ is the cutoff function $\chi(z) = \chi_1(z/2)$, then note that $\chi \equiv 1$ on $\operatorname{spt} \mathscr{E}_\varepsilon(0)$ so that

$$\mathscr{E}_\varepsilon(0) = \chi \cdot \varepsilon^2 H_z \partial_z \mathbb{H}_\varepsilon - \chi \cdot (\varepsilon^2 \Delta_g - W''(\widetilde{\mathbb{H}}_\varepsilon) - Q_\varepsilon)(\mathbb{H}_\varepsilon - \widetilde{\mathbb{H}}_\varepsilon).$$

It follows from (7.8), (7.11), and (7.30) that

$$\|\chi \cdot (\varepsilon^2 \Delta_g - W''(\widetilde{\mathbb{H}}_\varepsilon) - Q_\varepsilon)(\mathbb{H}_\varepsilon - \widetilde{\mathbb{H}}_\varepsilon)\|_{C^{0,\alpha}_\varepsilon(\Sigma \times \mathbf{R})} \leq C_m \varepsilon^m \tag{7.31}$$

for $m \in \mathbf{N}$. (Taking $m = 4$ will suffice.)

Recalling (A.3),

$$\partial_z H_z = -|\,\mathrm{I\!I}_z\,|^2 + \operatorname{Ric}_g(\partial_z, \partial_z)|_{D \times \{z\}},\ z \in [-1, 1]. \tag{7.32}$$

Certainly, since $\alpha \leq \theta$, this already implies

$$\sup_{|z| \leq 1} \|y \mapsto \partial_z H_z\|_{C^{0,\alpha}(\Sigma)} \leq C.$$

Combining (7.32) with (7.3), $\alpha \leq \theta$, (A.1), and (A.2), we even find that

$$\sup_{|z| \leq 1} \|y \mapsto \partial_z^2 H_z(y, z)\|_{C^{0,\alpha}(\Sigma)} \leq C. \tag{7.33}$$

In particular, (7.2), (7.33) and Taylor's theorem imply

$$H_z = -(|\,\mathrm{I\!I}_0\,|^2 + \operatorname{Ric}_g(\partial_z, \partial_z)|_\Sigma) z + \mathcal{R}(y, z) z^2, \tag{7.34}$$

where

$$\sup_{|z| \leq 1} \|y \mapsto \mathcal{R}(y, z)\|_{C^{0,\alpha}(\Sigma)} \leq C. \tag{7.35}$$

From the trivial estimate $|z| \partial_z \mathbb{H}_\varepsilon \leq C$, (7.11), and (7.34), we find that

$$\|\chi \cdot \varepsilon^2 H_z \partial_z \mathbb{H}_\varepsilon\|_{C^{0,\alpha}_\varepsilon(\Sigma \times \mathbf{R})} \leq C\varepsilon^2. \tag{7.36}$$

Put together, (7.31), (7.36), and Lemma 7.3 imply

$$\|\Pi_\varepsilon^\perp M_\varepsilon(0,0,0)\|_{C^{0,\alpha}_\varepsilon(\Sigma \times \mathbf{R})} \leq C\varepsilon^2.$$

Finally, by (7.34),

$$\Pi_\varepsilon(\chi \cdot \varepsilon^2 H_z \partial_z \mathbb{H}_\varepsilon) = h_0^{-1} \int_{-\infty}^{\infty} \chi(z)(\partial_z H_z(y, 0) \cdot z + \mathcal{R}(y, z) z^2)(\mathbb{H}'(\varepsilon^{-1} z))^2 \, dz.$$

Recalling that from parity (since $\chi(z)$ is even)

$$\int_{-\infty}^{\infty} \chi(z) z (\mathbb{H}'(\varepsilon^{-1} z))^2 \, dz = 0,$$

it follows that

$$\Pi_\varepsilon(\chi \cdot \varepsilon^2 H_z \partial_z \mathbb{H}_\varepsilon) = h_0^{-1} \int_{-\infty}^{\infty} \chi(z) \mathcal{R}(y, z) z^2 (\mathbb{H}'(\varepsilon^{-1} z))^2 \, dz,$$

at which point we can directly estimate using (1.6), (1.8), and (7.35) and get

$$\|\Pi_\varepsilon(\chi \cdot \varepsilon^2 H_z \partial_z \mathbb{H}_\varepsilon)\|_{C^{0,\alpha}(\Sigma)} \leq C\varepsilon^3.$$

Together with (7.31) (with $m = 4$), this implies

$$\|\Pi_\varepsilon M_\varepsilon(0,0,0)\|_{C^{0,\alpha}(\Sigma)} \leq C\varepsilon^3.$$

This completes the proof. □

LEMMA 7.11 (cf. [Pac12, Lemma 3.9]). *For* $\alpha \leq \alpha_0$, $\varepsilon \leq \varepsilon_0$,

$$\begin{aligned}(7.37)\quad & \|N_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - N_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1)\|_{C^{0,\alpha}_\varepsilon(\Omega)} \\ & \leq c_1\varepsilon^\delta\Big(\|v_2^\flat - v_1^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)} + \|v_2^\sharp - v_1^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}\Big),\end{aligned}$$

$$\begin{aligned}(7.38)\quad & \|\Pi_\varepsilon^\perp(M_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - M_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1))\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \\ & \leq c_1\varepsilon^\delta\Big(\|v_2^\flat - v_1^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} + \|v_2^\sharp - v_1^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}\Big),\end{aligned}$$

$$\begin{aligned}(7.39)\quad & \|\Pi_\varepsilon(M_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - M_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1))\|_{C^{0,\alpha}(\Sigma)} \\ & \leq c_1\varepsilon^{1+\delta}\|v_2^\flat - v_1^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} \\ & \quad + c_1\varepsilon^{1-\alpha}\|v_2^\sharp - v_1^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + c_1\varepsilon^{1+\delta}\|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)},\end{aligned}$$

provided (7.3)–(7.6) *hold with* $\theta \geq \theta_0 \geq \alpha_0$, *and*

$$\sum_{j=1,2} \|v_j^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} + \|v_j^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \varepsilon^{2\alpha}\|\zeta_j\|_{C^{2,\alpha}(\Sigma)} \leq C'\varepsilon^2.$$

Here, $\varepsilon_0 = \varepsilon_0(n,\eta,W,\delta_*)$, $\delta = \delta(\delta_*)$, $\theta_0 = \theta_0(\delta_*)$, $\alpha_0 = \alpha_0(\delta_*)$, *and* $c_1 = c_1(n,\eta,W,\delta_*,C',\alpha)$.

Remark 7.12. We emphasize that three different norms are used:

(1) On $v^\flat$, we use the *modified* weighted Hölder norm

$$\|w\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} = \|w\|_{C^{2,\alpha}_\varepsilon(\Omega)} + \varepsilon^{-2}\|\chi_5 w\|_{C^{2,\alpha}_\varepsilon(\Omega)}.$$

Here, the Hölder norms are measured with respect to the metric g.

(2) On $v^\sharp$, we use the standard weighted Hölder norm $C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})$. Here, the Hölder norms are measured with respect to the product metric $g_0 + dz^2$.

(3) On ζ, we use the *unweighted* Hölder norm $C^{2,\alpha}(\Sigma)$, which strictly dominates $C^{2,\alpha}_\varepsilon(\Sigma)$:

$$\|\zeta\|_{C^{2,\alpha}_\varepsilon(\Sigma)} \leq \|\zeta\|_{C^{2,\alpha}(\Sigma)}.$$

Here, the Hölder norms are measured with respect to the metric g_0 induced on Σ.

Proof of Lemma 7.11. In what follows we may assume that $\alpha_0 \leq \frac{1}{4}$. Note, from (7.11) and (7.26), that

$$N_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1) \equiv N_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) \equiv 0 \text{ on } \{\chi_4 = 1\}.$$

Therefore, since $\delta_* \in (0,1)$,

$$\begin{aligned}
&\|N_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - N_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1)\|_{C_\varepsilon^{0,\alpha}(\Omega)} \\
&\quad = \|N_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - N_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1)\|_{C_\varepsilon^{0,\alpha}(\{\chi_4 \neq 1\})} \\
&\quad \leq \|N_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - N_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1)\|_{C_\varepsilon^{0,\alpha}(\Omega \setminus \Omega_5)}.
\end{aligned}$$

We will estimate this by pairing up the terms, making sure to use use the fact that our Hölder norm is taken over $\Omega \setminus \Omega_5$ instead of over Ω, in order to gain a factor of ε^δ for some $\delta > 0$ that depends on δ_*.

In all that follows, we will repeatedly (and implicitly) use that our Fermi coordinates (and thus also D_ζ, D_ζ^{-1}) are $C_\varepsilon^{2,\alpha}$ close to the identity, and that our metric g in Fermi coordinates is $C_\varepsilon^{1,\alpha}$ close to Euclidean.

We start by estimating

$$\|\varepsilon^2(\Delta_g(v_2^\flat \circ D_{\zeta_2}) \circ D_{\zeta_2}^{-1} - \Delta_g v_2^\flat) - \varepsilon^2(\Delta_g(v_1^\flat \circ D_{\zeta_1}) \circ D_{\zeta_1}^{-1} - \Delta_g v_1^\flat)\|_{C_\varepsilon^{0,\alpha}(\Omega)}.$$

(We can deduce a good estimate on all of Ω, not just on $\Omega \setminus \Omega_5$.) By working in Fermi coordinates in scale $O(\varepsilon)$, we see that

$$\mathcal{F}_1(v, \zeta) \triangleq \varepsilon^2 \Delta_g(v \circ D_\zeta) \circ D_\zeta^{-1} \tag{7.40}$$

is a *smooth* non-linear Banach space functional $\mathcal{F}_1 : C_\varepsilon^{2,\alpha}(\Omega) \times C_\varepsilon^{2,\alpha}(\Sigma) \to C_\varepsilon^{0,\alpha}(\Omega)$ and is *linear* in v. In particular,

$$\begin{aligned}
&\varepsilon^2\big[(\Delta_g(v_2^\flat \circ D_{\zeta_2}) \circ D_{\zeta_2}^{-1} - \Delta_g v_2^\flat) - (\Delta_g(v_1^\flat \circ D_{\zeta_1}) \circ D_{\zeta_1} - \Delta_g v_1^\flat)\big] \\
&\quad = (\mathcal{F}_1(v_2^\flat, \zeta_2) - \mathcal{F}_1(v_1^\flat, \zeta_1)) - (\mathcal{F}_1(v_2^\flat, 0) - \mathcal{F}_1(v_1^\flat, 0)) \\
&\quad = \int_0^1 \langle D_v \mathcal{F}_1(v_1^\flat + t(v_2^\flat - v_1^\flat), \zeta_1 + t(\zeta_2 - \zeta_1)), v_2^\flat - v_1^\flat \rangle \\
&\qquad + \langle D_\zeta \mathcal{F}_1(v_1^\flat + t(v_2^\flat - v_1^\flat), \zeta_1 + t(\zeta_2 - \zeta_1)), \zeta_2 - \zeta_1 \rangle \, dt \\
&\qquad - \int_0^1 \langle D_v \mathcal{F}_1(v_1^\flat + t(v_2^\flat - v_1^\flat), 0), v_2^\flat - v_1^\flat \rangle dt \\
&\quad = \int_0^1 \int_0^1 \langle D_\zeta D_v \mathcal{F}_1(v_1^\flat + t(v_2^\flat - v_1^\flat), s\zeta_1 + st(\zeta_2 - \zeta_1)), \\
&\qquad\qquad (\zeta_1 + t(\zeta_2 - \zeta_1)) \otimes (v_2^\flat - v_1^\flat) \rangle \, ds \, dt \\
&\quad + \int_0^1 \langle D_\zeta \mathcal{F}_1(v_1^\flat + t(v_2^\flat - v_1^\flat), \zeta_1 + t(\zeta_2 - \zeta_1)), \zeta_2 - \zeta_1 \rangle \, dt.
\end{aligned}$$

Seeing as to how $\|v_j^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)} \leq C'\varepsilon^2$, $\|\zeta_j\|_{C^{2,\alpha}(\Sigma)} \leq C'\varepsilon^{2-2\alpha}$, and using the linearity in v of $\mathcal{F}_1$ (and thus of $D_\zeta\mathcal{F}_1$), we can directly estimate the following:

(7.41)
$$\begin{aligned}
&\|\varepsilon^2((\Delta_g(v_2^\flat \circ D_{\zeta_2}) \circ D_{\zeta_2}^{-1} - \Delta_g v_2^\flat) - (\Delta_g(v_1^\flat \circ D_{\zeta_1}) \circ D_{\zeta_1} - \Delta_g v_1^\flat))\|_{C^{0,\alpha}_\varepsilon(\Omega)}\\
&\quad \leq C(\|\zeta_1\|_{C^{2,\alpha}_\varepsilon(\Sigma)} + \|\zeta_2\|_{C^{2,\alpha}_\varepsilon(\Sigma)})\|v_2^\flat - v_1^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)}\\
&\quad + C(\|v_1^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)} + \|v_2^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)})\|\zeta_2 - \zeta_1\|_{C^{2,\alpha}_\varepsilon(\Sigma)}\\
&\quad \leq C(\|\zeta_1\|_{C^{2,\alpha}(\Sigma)} + \|\zeta_2\|_{C^{2,\alpha}(\Sigma)})\|v_2^\flat - v_1^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)}\\
&\quad + C(\|v_1^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)} + \|v_2^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)})\|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}\\
&\quad \leq C\varepsilon^{2-2\alpha}\|v_2^\flat - v_1^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)}\\
&\quad + C\varepsilon^2\|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}.
\end{aligned}$$

This estimate is of the desired form.

Next, we estimate
$$\|(W''(\widetilde{\mathbb{H}}_\varepsilon) - W''(\pm 1))(v_2^\flat - v_1^\flat)\|_{C^{0,\alpha}_\varepsilon(\Omega\setminus\Omega_5)}.$$

The desired estimate is a simple consequence of Remark 7.1 and how, on $\Omega\setminus\Omega_5$, we have

$$\|W''(\widetilde{\mathbb{H}}_\varepsilon) - W''(\pm 1)\|_{C^{0,\alpha}_\varepsilon(\Omega\setminus\Omega_5)} \leq C_m\varepsilon^m \tag{7.42}$$

for all $m \in \mathbf{N}$; thus, any $\delta > 0$ will do.

Next, we estimate
$$\|\mathscr{E}_\varepsilon(\zeta_2) - \mathscr{E}_\varepsilon(\zeta_1)\|_{C^{0,\alpha}_\varepsilon(\Omega\setminus\Omega_5)}.$$

We have
$$\begin{aligned}
\mathscr{E}_\varepsilon(\zeta_2) - \mathscr{E}_\varepsilon(\zeta_1) &= \varepsilon^2(\Delta_g(\widetilde{\mathbb{H}}_\varepsilon \circ D_{\zeta_2}) \circ D_{\zeta_2}^{-1} - \Delta_g(\widetilde{\mathbb{H}}_\varepsilon \circ D_{\zeta_1}) \circ D_{\zeta_1}^{-1})\\
&= \mathcal{F}_1'(\widetilde{\mathbb{H}}_\varepsilon, \zeta_2) - \mathcal{F}_1'(\widetilde{\mathbb{H}}_\varepsilon, \zeta_1),
\end{aligned}$$

where $\mathcal{F}_1' : C^{2,\alpha}_\varepsilon(\Omega\setminus\Omega_5) \times C^{2,\alpha}_\varepsilon(\Sigma) \to C^{0,\alpha}_\varepsilon(\Omega\setminus\Omega_5)$ is the restriction of $\mathcal{F}_1$ from (7.40). Arguing as before, we get

$$\begin{aligned}
\|\mathscr{E}_\varepsilon(\zeta_2) - \mathscr{E}_\varepsilon(\zeta_1)\|_{C^{0,\alpha}_\varepsilon(\Omega\setminus\Omega_5)} &\leq C\|\widetilde{\mathbb{H}}_\varepsilon\|_{C^{2,\alpha}_\varepsilon(\Omega\setminus\Omega_5)}\|\zeta_2 - \zeta_1\|_{C^{2,\alpha}_\varepsilon(\Sigma)}\\
&\leq C_m\varepsilon^m\|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}
\end{aligned} \tag{7.43}$$

for all $m \in \mathbf{N}$, which implies what we want for any $\delta > 0$.

Next, we estimate
$$\|Q_\varepsilon(\chi_4 v_2^\sharp + v_2^\flat) - Q_\varepsilon(\chi_4 v_1^\sharp + v_1^\flat)\|_{C^{0,\alpha}_\varepsilon(\Omega)}.$$

Note that

$$Q_\varepsilon(\chi_4 v_2^\sharp + v_2^\flat) - Q_\varepsilon(\chi_4 v_1^\sharp + v_1^\flat) = W'(\widetilde{\mathbb{H}}_\varepsilon + \chi_4 v_2^\sharp + v_2^\flat) \\ -W'(\widetilde{\mathbb{H}}_\varepsilon + \chi_4 v_1^\sharp + v_1^\flat) - W''(\widetilde{\mathbb{H}}_\varepsilon)(\chi_4(v_2^\sharp - v_1^\sharp) + (v_2^\flat - v_1^\flat)).$$

Define

$$\mathcal{F}_2(v) \triangleq W'(\widetilde{\mathbb{H}}_\varepsilon + v), \tag{7.44}$$

viewed as a *smooth* non-linear Banach space functional $\mathcal{F}_2 : C^{0,\alpha}_\varepsilon(\Omega) \to C^{0,\alpha}_\varepsilon(\Omega)$. Note that

$$\langle D_v \mathcal{F}_2(v), w\rangle = W''(\widetilde{\mathbb{H}}_\varepsilon + v)w, \ \langle D_v D_v \mathcal{F}_2(v), w \otimes w'\rangle = W''(\widetilde{\mathbb{H}}_\varepsilon + v)ww'$$

for w, $w' \in C^{0,\alpha}_\varepsilon(\Omega)$. In particular, the expression we are trying to bound equals

$$\begin{aligned}
&\mathcal{F}_2(\chi_4 v_2^\sharp + v_2^\flat) - \mathcal{F}_2(\chi_4 v_1^\sharp + v_1^\flat) - \langle D_v\mathcal{F}_2(0), \chi_4(v_2^\sharp - v_1^\sharp) + v_2^\flat - v_1^\flat\rangle \\
&= \int_0^1 \langle D_v \mathcal{F}_2(\chi_4 v_1^\sharp + v_1^\flat + t(\chi_4(v_2^\sharp - v_1^\sharp) + v_2^\flat - v_1^\flat)), \chi_4(v_2^\sharp - v_1^\sharp) + v_2^\flat - v_1^\flat\rangle \, dt \\
&\quad - \langle D_v\mathcal{F}_2(0), \chi_4(v_2^\sharp - v_1^\sharp) + v_2^\flat - v_1^\flat\rangle \\
&= \int_0^1\int_0^1 \langle D_v D_v \mathcal{F}_2(s(\chi_4 v_1^\sharp + v_1^\flat + t(\chi_4(v_2^\sharp - v_1^\sharp) + v_2^\flat - v_1^\flat)), \\
&\qquad (\chi_4 v_1^\sharp + v_1^\flat + t(\chi_4(v_2^\sharp - v_1^\sharp) + v_2^\flat - v_1^\flat)) \otimes (\chi_4(v_2^\sharp - v_1^\sharp) + v_2^\flat - v_1^\flat))\rangle \, ds\, dt.
\end{aligned}$$

Recalling Remark 7.1, (7.11), and $\delta_* \in (0,1)$, we can estimate

$$\begin{aligned}
&\|Q_\varepsilon(\chi_4 v_2^\sharp + v_2^\flat) - Q_\varepsilon(\chi_4 v_1^\sharp + v_1^\flat)\|_{C^{0,\alpha}_\varepsilon(\Omega)} \\
&\quad \leq C(\|v_1^\sharp\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|v_1^\flat\|_{C^{0,\alpha}_\varepsilon(\Omega)} + \|v_2^\sharp\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|v_2^\flat\|_{C^{0,\alpha}_\varepsilon(\Omega)}) \\
&\qquad \cdot (\|v_2^\sharp - v_1^\sharp\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|v_2^\flat - v_1^\flat\|_{C^{0,\alpha}_\varepsilon(\Omega)}).
\end{aligned}$$

This gives

$$\begin{aligned}
&\|Q_\varepsilon(\chi_4 v_2^\sharp + v_2^\flat) - Q_\varepsilon(\chi_4 v_1^\sharp + v_1^\flat)\|_{C^{0,\alpha}_\varepsilon(\Omega)} \\
&\quad \leq C\varepsilon^2(\|v_2^\sharp - v_1^\sharp\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|v_2^\flat - v_1^\flat\|_{C^{0,\alpha}_\varepsilon(\Omega)}),
\end{aligned} \tag{7.45}$$

using $\|v_j^\sharp\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})}, \|v_j^\flat\|_{C^{0,\alpha}_\varepsilon(\Omega)} \leq C'\varepsilon^2$.

Next, we consider

$$\|\varepsilon^2((\Delta_g((\chi_4 v_2^\sharp)\circ D_{\zeta_2}) - \chi_4\Delta_g(v_2^\sharp \circ D_{\zeta_2}))\circ D_{\zeta_2}^{-1}) \\ -(\Delta_g((\chi_4 v_1^\sharp)\circ D_{\zeta_1}) - \chi_4\Delta_g(v_1^\sharp \circ D_{\zeta_1}))\circ D_{\zeta_1}^{-1})\|_{C^{0,\alpha}_\varepsilon(\Omega)}.$$

Define

$$\mathcal{F}_3(v,\zeta) \triangleq \varepsilon^2(\Delta_g((\chi_4 v)\circ D_\zeta) - \chi_4\Delta_g(v\circ D_\zeta))\circ D_\zeta^{-1},$$

which once again is viewed as a map $\mathcal{F}_3 : C^{2,\alpha}_\varepsilon(\Omega) \times C^{2,\alpha}_\varepsilon(\Sigma) \to C^{0,\alpha}_\varepsilon(\Omega)$, is a smooth non-linear Banach space functional. We can then write

$$\begin{aligned}
&\varepsilon^2((\Delta_g((\chi_4 v_2^\sharp)\circ D_{\zeta_2}) - \chi_4\Delta_g(v_2^\sharp \circ D_{\zeta_2}))\circ D_{\zeta_2}^{-1}) \\
&\qquad - (\Delta_g((\chi_4 v_1^\sharp)\circ D_{\zeta_1}) - \chi_4\Delta_g(v_1^\sharp \circ D_{\zeta_1}))\circ D_{\zeta_1}^{-1}) \\
&\quad = \mathcal{F}_3(v_2^\sharp, \zeta_2) - \mathcal{F}_3(v_1^\sharp, \zeta_1) \\
&\quad = \int_0^1 \langle D_v \mathcal{F}_3(v_1^\sharp + t(v_2^\sharp - v_1^\sharp), \zeta_1 + t(\zeta_2 - \zeta_1)), v_2^\sharp - v_1^\sharp\rangle \\
&\qquad + \langle D_\zeta \mathcal{F}_3(v_1^\sharp + t(v_2^\sharp - v_1^\sharp), \zeta_1 + t(\zeta_2 - \zeta_1)), \zeta_2 - \zeta_1\rangle \, dt.
\end{aligned}$$

The second term can be estimated by using the linearity in v of $\mathcal{F}_3$ (and thus of $D_\zeta \mathcal{F}_3$) and Remark 7.1 to give

$$\begin{aligned}
&\|\langle D_\zeta \mathcal{F}_3(v_1^\sharp + t(v_2^\sharp - v_1^\sharp), \zeta_1 + t(\zeta_2 - \zeta_1)), \zeta_2 - \zeta_1\rangle\|_{C^{0,\alpha}_\varepsilon(\Omega)} \\
&\quad \le C(\|v_1^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|v_2^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})})\|\zeta_2 - \zeta_1\|_{C^{2,\alpha}_\varepsilon(\Sigma)} \\
&\quad \le C\varepsilon^2 \|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)},
\end{aligned}$$

which is of the desired form with $\delta = 2$.

The first term instead requires that we use the product rule on $\mathcal{F}_3$ to recast it as

$$\mathcal{F}_3(v,\zeta) = \varepsilon^2(2\langle \nabla_g(\chi_4 \circ D_\zeta), \nabla_g(v\circ D_\zeta)\rangle + (\Delta_g(\chi_4\circ D_\zeta))(v\circ D_\zeta))\circ D_\zeta^{-1},$$

which can, in turn, be differentiated in v to give

$$\langle D_v\mathcal{F}_3(v,\zeta), w\rangle = \varepsilon^2(2\langle \nabla_g(\chi_4 \circ D_\zeta), \nabla_g(w\circ D_\zeta)\rangle + (\Delta_g(\chi_4\circ D_\zeta))(w\circ D_\zeta))\circ D_\zeta^{-1}.$$

At this point, we note that there are no zero-order χ_4's remaining, so we use Remark 7.1, (7.11), and $\delta_* \in (0,1)$ to get

$$\begin{aligned}
&\|\langle D_v \mathcal{F}_3(v_1^\sharp + t(v_2^\sharp - v_1^\sharp), \zeta_1 + t(\zeta_2 - \zeta_1)), v_2^\sharp - v_1^\sharp\rangle\|_{C^{0,\alpha}_\varepsilon(\Omega)} \\
&\quad \le C\varepsilon^{1-\delta_*}\|v_2^\sharp - v_1^\sharp\|_{C^{1,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \le C\varepsilon^{1-\delta_*}\|v_2^\sharp - v_1^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})}.
\end{aligned}$$

Summarizing, we have shown that

$$\begin{aligned}
&\|\varepsilon^2((\Delta_g((\chi_4 v_2^\sharp)\circ D_{\zeta_2}) - \chi_4\Delta_g(v_2^\sharp \circ D_{\zeta_2}))\circ D_{\zeta_2}^{-1}) \\
&\qquad - (\Delta_g((\chi_4 v_1^\sharp)\circ D_{\zeta_1}) - \chi_4\Delta_g(v_1^\sharp \circ D_{\zeta_1}))\circ D_{\zeta_1}^{-1})\|_{C^{0,\alpha}_\varepsilon(\Omega)} \\
&\quad \le C\varepsilon^{1-\delta_*}(\|v_2^\sharp - v_1^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}).
\end{aligned} \tag{7.46}$$

The contraction estimate on N_ε, (7.37), now follows from (7.41), (7.42), (7.43), (7.45), and (7.46).

We move on to the contraction estimates on M_ε, (7.38) and (7.39). Before we derive those two precise estimates, we investigate several of the easier terms in $M_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - M_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1)$.

We note, right away, that we have already shown in (7.41) that

$$\|\varepsilon^2(\Delta_g(v_2^\flat \circ D_{\zeta_2}) \circ D_{\zeta_2}^{-1} - \Delta_g v_2^\flat) - (\Delta_g(v_1^\flat \circ D_{\zeta_1}) \circ D_{\zeta_1}^{-1} - \Delta_g v_1^\flat)\|_{C^{0,\alpha}_\varepsilon(\Omega_3)} \leq C\varepsilon^{2-2\alpha}\|v_2^\flat - v_1^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)} + C\varepsilon^2\|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}.$$

In particular, Remark 7.1, Lemma 7.3, and $\|\cdot\|_{C^{0,\alpha}(\Sigma)} \leq \varepsilon^{-\alpha}\|\cdot\|_{C^{0,\alpha}_\varepsilon(\Sigma)}$ imply

(7.47)
$$\begin{aligned}
&\varepsilon^\alpha\Big\|\Pi_\varepsilon\Big[\varepsilon^2(\Delta_g(v_2^\flat \circ D_{\zeta_2}) \circ D_{\zeta_2}^{-1} - \Delta_g v_2^\flat) - (\Delta_g(v_1^\flat \circ D_{\zeta_1}) \circ D_{\zeta_1}^{-1} - \Delta_g v_1^\flat)\Big]\Big\|_{C^{0,\alpha}(\Sigma)} \\
&\quad + \Big\|\Pi_\varepsilon^\perp\Big[\varepsilon^2(\Delta_g(v_2^\flat \circ D_{\zeta_2}) \circ D_{\zeta_2}^{-1} - \Delta_g v_2^\flat) \\
&\qquad - (\Delta_g(v_1^\flat \circ D_{\zeta_1}) \circ D_{\zeta_1}^{-1} - \Delta_g v_1^\flat)\Big]\Big\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \\
&\leq C\varepsilon^{2-2\alpha}\|v_2^\flat - v_1^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)} + C\varepsilon^2\|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}.
\end{aligned}$$

Next, from Remark 7.1, (7.45), we conclude

$$\begin{aligned}
&\varepsilon^\alpha\Big\|\Pi_\varepsilon\big[Q_\varepsilon(\chi_4 v_2^\sharp + v_2^\flat) - Q_\varepsilon(\chi_4 v_1^\sharp + v_1^\flat)\big]\Big\|_{C^{0,\alpha}(\Sigma)} \\
&\quad + \Big\|\Pi_\varepsilon^\perp\big[Q_\varepsilon(\chi_4 v_2^\sharp + v_2^\flat) - Q_\varepsilon(\chi_4 v_1^\sharp + v_1^\flat)\big]\Big\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \\
&\leq C\varepsilon^2(\|v_2^\sharp - v_1^\sharp\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|v_2^\flat - v_1^\flat\|_{C^{0,\alpha}_\varepsilon(\Omega)}).
\end{aligned} \tag{7.48}$$

Next, we estimate

$$\|(W''(\mathbb{H}_\varepsilon) - W''(\pm 1))(v_2^\flat - v_1^\flat)\|_{C^{0,\alpha}_\varepsilon(\Omega_3)}.$$

This is the only time we will use $\|\cdot\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)}$ for the purposes of (7.38). We have

$$\begin{aligned}
&\|(W''(\mathbb{H}_\varepsilon) - W''(\pm 1))(v_2^\flat - v_1^\flat)\|_{C^{0,\alpha}_\varepsilon(\Omega_3)} \\
&\quad \leq \|(W''(\mathbb{H}_\varepsilon) - W''(\pm 1))\chi_5(v_2^\flat - v_1^\flat)\|_{C^{0,\alpha}_\varepsilon(\Omega)} \\
&\qquad + \|(W''(\mathbb{H}_\varepsilon) - W''(\pm 1))(1-\chi_5)(v_2^\flat - v_1^\flat)\|_{C^{0,\alpha}_\varepsilon(\Omega)} \\
&\quad \leq \varepsilon^2\|v_2^\flat - v_1^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} + \|(W''(\mathbb{H}_\varepsilon) - W''(\pm 1))(1-\chi_5)(v_2^\flat - v_1^\flat)\|_{C^{0,\alpha}_\varepsilon(\Omega\setminus\Omega_5)}.
\end{aligned}$$

Recalling $\|W''(\mathbb{H}_\varepsilon) - W''(\pm 1)\|_{C^{0,\alpha}_\varepsilon(\Omega\setminus\Omega_5)} \leq C_m\varepsilon^m$ for all $m \in \mathbf{N}$, e.g., as in (7.42), we deduce

$$\|(W''(\mathbb{H}_\varepsilon) - W''(\pm 1))(v_2^\flat - v_1^\flat)\|_{C^{0,\alpha}_\varepsilon(\Omega_3)} \leq C\varepsilon^2\|v_2^\flat - v_1^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)},$$

so, combined with Remark 7.1, Lemma 7.3, (7.11), and $\delta_* \in (0,1)$, this gives

$$\begin{aligned}
&\varepsilon^\alpha\|\Pi_\varepsilon\big[\chi_3(W''(\mathbb{H}_\varepsilon) - W''(\pm 1))(v_2^\flat - v_1^\flat)\big]\|_{C^{0,\alpha}(\Sigma)} \\
&\quad + \|\Pi_\varepsilon^\perp\big[\chi_3(W''(\mathbb{H}_\varepsilon) - W''(\pm 1))(v_2^\flat - v_1^\flat)\big]\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \\
&\leq C\varepsilon^2\|v_2^\flat - v_1^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)}.
\end{aligned} \tag{7.49}$$

We now proceed to the more involved contraction estimates pertaining to M_ε. We will estimate

$$\begin{aligned}\|(L_\varepsilon v_2^\sharp - \varepsilon^2 \Delta_g (v_2^\sharp \circ D_{\zeta_2}) \circ D_{\zeta_2}^{-1} + W''(\mathbb{H}_\varepsilon) v_2^\sharp) \\ - (L_\varepsilon v_1^\sharp - \varepsilon^2 \Delta_g (v_1^\sharp \circ D_{\zeta_1}) \circ D_{\zeta_1}^{-1} + W''(\mathbb{H}_\varepsilon) v_1^\sharp)\|_{C^{0,\alpha}_\varepsilon(\Omega_3)}.\end{aligned} \tag{7.50}$$

Note that, by repeating the argument carried out to obtain (7.41), except with $v_j^\sharp$ in place of $v_j^\flat$, and also using Remark 7.1, we get

$$\begin{aligned}&\|\varepsilon^2((\Delta_g(v_2^\sharp \circ D_{\zeta_2}) \circ D_{\zeta_2}^{-1} - \Delta_g v_2^\sharp) - (\Delta_g(v_1^\sharp \circ D_{\zeta_1}) \circ D_{\zeta_1}^{-1} - \Delta_g v_1^\sharp))\|_{C^{0,\alpha}_\varepsilon(\Omega_3)} \\ &\quad \leq C\varepsilon^{2-2\alpha} \|v_2^\sharp - v_1^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma \times \mathbf{R})} + C\varepsilon^2 \|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}.\end{aligned} \tag{7.51}$$

In view of Remark 7.1 and Lemma 7.3, this allows us to estimate

$$\begin{aligned}(L_\varepsilon v_2^\sharp &- \varepsilon^2 \Delta_g v_2^\sharp + W''(\mathbb{H}_\varepsilon) v_2^\sharp) - (L_\varepsilon v_1^\sharp - \varepsilon^2 \Delta_g v_1^\sharp + W''(\mathbb{H}_\varepsilon) v_1^\sharp) \\ &= L_\varepsilon (v_2^\sharp - v_1^\sharp) - \varepsilon^2 \Delta_g (v_2^\sharp - v_1^\sharp) + W''(\mathbb{H}_\varepsilon)(v_2^\sharp - v_1^\sharp) \\ &= \varepsilon^2 (\Delta_{g_0} + \partial_z^2 - \Delta_g)(v_2^\sharp - v_1^\sharp)\end{aligned}$$

instead of (7.50) in both (7.38) and (7.39). Let us denote

$$\mathcal{F}_4(v) \triangleq \varepsilon^2 (\Delta_g - \Delta_{g_0} - \partial_z^2) v,$$

which is evidently a linear functional $\mathcal{F}_4 : C^{2,\alpha}_\varepsilon(\Omega_3) \to C^{0,\alpha}_\varepsilon(\Omega_3)$. Because $\Delta_g = \Delta_{g_z} + \partial_z^2 + H_z \partial_z$ in Fermi coordinates, we can rewrite

$$\mathcal{F}_4(v) = \varepsilon^2 (\Delta_{g_z} - \Delta_{g_0}) v + \varepsilon^2 H_z \partial_z v.$$

We now make use of (A.7) to write

$$\mathcal{F}_4(v) = \Big[- \varepsilon^2 \int_0^z (2 \langle \mathrm{I\!I}_t, \nabla^2_{g_t} v \rangle_{g_t} + \langle \nabla_{g_t} H_t, \nabla_{g_t} v \rangle_{g_t})\, dt \Big] + \varepsilon^2 H_z \partial_z v.$$

First, let us derive C^0 bounds. Let $(y,z) \in \Omega_3$. It follows from (7.3), (A.1), (A.2), and (A.6) that

$$\left| 2\varepsilon^2 \int_0^z \langle \mathrm{I\!I}_t, \nabla^2_{g_t} v \rangle_{g_t}\, dt \right| \leq C|z| \|v\|_{C^2_\varepsilon(\Omega_3)}. \tag{7.52}$$

It follows from (7.4), (A.1), (A.2), (A.4), and (A.11) that

$$\left| \varepsilon^2 \int_0^z \langle \nabla_{g_t} H_t, \nabla_{g_t} v \rangle_{g_t}\, dt \right| \leq C|z| \|v\|_{C^1_\varepsilon(\Omega_3)}. \tag{7.53}$$

It follows from (7.2), (7.3), and (A.2) that

$$|\varepsilon^2 H_z \partial_z v| \leq C\varepsilon |z| \|v\|_{C^1_\varepsilon(\Omega_3)}. \tag{7.54}$$

Altogether, (7.52)–(7.54), show

$$|\mathcal{F}_4(v)| \leq C|z| \|v\|_{C^{2,\alpha}_\varepsilon(\Omega_3)} \text{ on } \Omega_3. \tag{7.55}$$

Next, let us derive Hölder bounds. For fixed $z \in \Omega_3$, an analogous argument gives

$$\varepsilon^\alpha [y \mapsto \mathcal{F}_4(v)(y,z)]_\alpha \le C|z| \|v\|_{C^{2,\alpha}_\varepsilon(\Omega_3)}. \tag{7.56}$$

Now fix y. By (7.3), (A.1), (A.2), and (A.6), we have the *Lipschitz* bound

$$\left| \frac{\partial}{\partial z} \left(2\varepsilon^2 \int_0^z \langle \mathrm{I\!I}_t, \nabla^2_{g_t} v \rangle_{g_t} \, dt \right) \right| \le C \|v\|_{C^2_\varepsilon(\Omega_3)}.$$

In view of the a priori height bound $|z| \le \varepsilon^{\delta_*}$, this trivially implies the Hölder bound

$$\varepsilon^\alpha \left[z \mapsto \varepsilon^2 \int_0^z \langle \mathrm{I\!I}_t, \nabla^2_{g_t} v \rangle_{g_t} \, dt \right]_\alpha \le C\varepsilon^\alpha \varepsilon^{\delta_*(1-\alpha)} \|v\|_{C^2_\varepsilon(\Omega_3)}. \tag{7.57}$$

By (7.4), (A.1), (A.2), (A.4), and (A.11), we have another Lipschitz bound,

$$\left| \frac{\partial}{\partial z} \left(\varepsilon^2 \int_0^z \langle \nabla_{g_t} H_t, \nabla_{g_t} v \rangle_{g_t} \, dt \right) \right| \le C \|v\|_{C^1_\varepsilon(\Omega_3)}, \tag{7.58}$$

which, again by $|z| \le \varepsilon^{\delta_*}$, implies

$$\varepsilon^\alpha \left[z \mapsto \varepsilon^2 \int_0^z \langle \nabla_{g_t} H_t, \nabla_{g_t} v \rangle_{g_t} \, dt \right]_\alpha \le C\varepsilon^\alpha \varepsilon^{\delta_*(1-\alpha)} \|v\|_{C^1_\varepsilon(\Omega_3)}. \tag{7.59}$$

Finally, from (A.3) we have the Lipschitz bound

$$\left| \frac{\partial}{\partial z} (\varepsilon^2 H_z \partial_z v) \right| \le C \|v\|_{C^2_\varepsilon(\Omega_3)},$$

which, again by $|z| \le \varepsilon^{\delta_*}$, improves to

$$\varepsilon^\alpha [z \mapsto \varepsilon^2 H_z \partial_z v]_\alpha \le C\varepsilon^\alpha \varepsilon^{\delta_*(1-\alpha)} \|v\|_{C^2_\varepsilon(\Omega_3)}. \tag{7.60}$$

Altogether, (7.57)–(7.60) imply

$$\varepsilon^\alpha [z \mapsto \mathcal{F}_4(v)(y,z)]_\alpha \le C\varepsilon^{\delta_* + \alpha(1-\delta_*)} \|v\|_{C^2_\varepsilon(\Omega_3)}. \tag{7.61}$$

Together, (7.55), (7.56), and (7.61) imply

$$\|\mathcal{F}_4(v)\|_{C^{0,\alpha}_\varepsilon(\Omega_3)} \le C\varepsilon^{\delta_*} \|v\|_{C^{2,\alpha}_\varepsilon(\Omega_3)}. \tag{7.62}$$

Together with Remark 7.1 and Lemma 7.3, this gives

$$\|\Pi^\perp_\varepsilon \mathcal{F}_4(v)\|_{C^{0,\alpha}_\varepsilon(\Omega_3)} \le C\varepsilon^{\delta_*} \|v\|_{C^{2,\alpha}_\varepsilon(\Omega_3)}. \tag{7.63}$$

It remains to estimate $\Pi_\varepsilon \mathcal{F}_4(v)$. Note that the obvious inequality (which follows from (1.6) and (1.8))

$$\int_{-\infty}^\infty |z| \partial_z \mathbb{H}_\varepsilon(z) \, dz = \varepsilon \int_{-\infty}^\infty |t| \mathbb{H}'(t) \, dt \le C\varepsilon$$

combined with (7.55) and (7.56) readily implies

$$\|\Pi_\varepsilon \mathcal{F}_4(v)\|_{C^{0,\alpha}_\varepsilon(\Sigma)} \le C\varepsilon \|v\|_{C^{2,\alpha}_\varepsilon(\Omega_3)} \implies \|\Pi_\varepsilon \mathcal{F}_4(v)\|_{C^{0,\alpha}(\Sigma)} \le C\varepsilon^{1-\alpha} \|v\|_{C^{2,\alpha}_\varepsilon(\Omega_3)}. \tag{7.64}$$

This completes our study of $\mathcal{F}_4$, as we have the desired estimates in view of Remark 7.1.

We proceed to the final contraction estimate pertaining to M_ε, which involves Π_ε, $\Pi_\varepsilon^\perp$ of

$$\chi_3(\mathscr{E}_\varepsilon(\zeta_2) - \mathscr{E}_\varepsilon(\zeta_1) - \varepsilon^2 J_\Sigma(\zeta_2 - \zeta_2)\partial_z \mathbb{H}_\varepsilon).$$

By (7.11), $\delta_* \in (0,1)$, and Lemma 7.3, we may just estimate $\mathscr{E}_\varepsilon(\zeta_2) - \mathscr{E}_\varepsilon(\zeta_1) - \varepsilon^2 J_\Sigma(\zeta_2 - \zeta_2)\partial_z \mathbb{H}_\varepsilon$ on Ω_3.

Fix $(y,z) \in \Omega_3$. Recall the definition of D_ζ in (7.15) and the estimate

$$\operatorname{dist}_g(\Omega_3, \{\chi_2 \neq 1\}) = O(\varepsilon^{\delta_*}) \gg \|\zeta_2\|_{C^0(\Sigma)} + \|\zeta_1\|_{C^0(\Sigma)} \tag{7.65}$$

that follows from the a priori bound on ζ_1, ζ_2. Also recall that $\mathbb{H}_\varepsilon \equiv \widetilde{\mathbb{H}}_\varepsilon$ on Ω_3. Then, in Fermi coordinates (y,z), we have

$$\begin{aligned}
&\mathscr{E}_\varepsilon(\zeta_2)(y,z) - \mathscr{E}_\varepsilon(\zeta_1)(y,z) - \varepsilon^2 J_\Sigma(\zeta_2 - \zeta_1)(y) \cdot \partial_z \mathbb{H}_\varepsilon(z) \\
&\quad = \varepsilon^2 \Delta_g(\mathbb{H}_\varepsilon \circ D_{\zeta_2}) \circ D_{\zeta_2}^{-1}(y,z) - \varepsilon^2 \Delta_g(\mathbb{H}_\varepsilon \circ D_{\zeta_1}) \circ D_{\zeta_1}^{-1}(y,z) \\
&\qquad - \varepsilon^2 J_\Sigma(\zeta_2 - \zeta_1)(y) \cdot \partial_z \mathbb{H}_\varepsilon(z) \\
&\quad = \varepsilon^2 \Big[\partial_z^2 \mathbb{H}_\varepsilon(z) \big(|\nabla_{g_{z+\zeta_2(y)}} \zeta_2(y)|^2 - |\nabla_{g_{z+\zeta_1(y)}} \zeta_1(y)|^2 \big) \\
&\qquad - \partial_z \mathbb{H}_\varepsilon(z) \big((\Delta_{g_{z+\zeta_2(y)}} \zeta_2(y) - H_{z+\zeta_2(y)}(y)) - (\Delta_{g_{z+\zeta_1(y)}} \zeta_1(y) \\
&\qquad\qquad - H_{z+\zeta_1(y)}(y)) + J_\Sigma(\zeta_2 - \zeta_1)(y) \big) \Big].
\end{aligned} \tag{7.66}$$

For $\zeta \in C^{1,\alpha}(\Sigma)$, denote

$$\mathcal{F}_5(\zeta)(y,z) \triangleq |\nabla_{g_{z+\zeta(y)}} \zeta(y)|^2 = g^{ij}_{z+\zeta(y)} \zeta_i(y) \zeta_j(y)$$

to be the smooth non-linear functional $\mathcal{F}_5 : C^{1,\alpha}(\Sigma) \to C^{0,\alpha}(\Omega_3)$. By virtue of (A.1), we know that

$$\langle D_\zeta \mathcal{F}_5(\zeta), w \rangle (y,z) = -2\, \mathrm{I\!I}^{ij}_{z+\zeta(y)}\, \zeta_i(y)\zeta_j(y) w(y) + 2 g^{ij}_{z+\zeta(y)} w_i(y) \zeta_j(y). \tag{7.67}$$

By the fundamental theorem of calculus,

$$\mathcal{F}_5(\zeta_2) - \mathcal{F}_5(\zeta_1) = \int_0^1 \langle D_\zeta \mathcal{F}_5(\zeta_1 + t(\zeta_2 - \zeta_1)), \zeta_2 - \zeta_1 \rangle \, dt,$$

so together with (7.3), the a priori estimates on ζ_1, ζ_2, (7.67), (A.1), and (A.2),

$$\|\mathcal{F}_5(\zeta_2) - \mathcal{F}_5(\zeta_1)\|_{C^{0,\alpha}(\Omega_3)} \leq C\varepsilon^{2-2\alpha} \|\zeta_2 - \zeta_1\|_{C^{1,\alpha}(\Sigma)}.$$

Alongside (1.7), Remark 7.1, Lemma 7.3, (7.11), and $\delta_* \in (0,1)$, this implies

$$\begin{aligned}
&\varepsilon^\alpha \left\| \Pi_\varepsilon \big(\chi_3 \varepsilon^2 (\partial_z^2 \mathbb{H}_\varepsilon)(\mathcal{F}_5(\zeta_2) - \mathcal{F}_5(\zeta_1)) \big) \right\|_{C^{0,\alpha}(\Sigma)} \\
&\qquad + \left\| \Pi_\varepsilon^\perp \big(\chi_3 \varepsilon^2 (\partial_z^2 \mathbb{H}_\varepsilon)(\mathcal{F}_5(\zeta_2) - \mathcal{F}_5(\zeta_1)) \big) \right\|_{C^{0,\alpha}_\varepsilon(\Sigma \times \mathbf{R})} \\
&\quad \leq C\varepsilon^{2-2\alpha} \|\zeta_2 - \zeta_1\|_{C^{1,\alpha}(\Sigma)}.
\end{aligned} \tag{7.68}$$

Finally, let us denote

$$\mathcal{F}_6(\zeta)(y,z) \triangleq \varepsilon\Big(\Delta_{z+\zeta(y)}\zeta(y) - H_{z+\zeta(y)}(y) + J_\Sigma\zeta(y)\Big)$$

to be the smooth non-linear Banach space functional $\mathcal{F}_6 : C^{2,\alpha}(\Sigma) \to C^{0,\alpha}_\varepsilon(\Omega_3)$. By (A.3) and (A.7),

$$\begin{aligned}
&\langle D_\zeta \mathcal{F}_6(\zeta), w\rangle \\
&\quad = \varepsilon\Big(\Delta_{z+\zeta} w + \big(-2\langle \mathrm{II}_{z+\zeta}, \nabla^2_{g_{z+\zeta}}\zeta\rangle_{g_{z+\zeta}} - \langle \nabla_{g_{z+\zeta}} H_{z+\zeta}, \nabla_{g_{z+\zeta}}\zeta\rangle_{g_{z+\zeta}}\big)w \\
&\qquad + (|\,\mathrm{II}_{z+\zeta}\,|^2 + \mathrm{Ric}_g(\partial_z,\partial_z)|_{D\times\{z+\zeta\}})w + J_\Sigma w\Big) \\
&\quad = \varepsilon\Big(\big(-2\langle \mathrm{II}_{z+\zeta}, \nabla^2_{g_{z+\zeta}}\zeta\rangle_{g_{z+\zeta}} - \langle \nabla_{g_{z+\zeta}} H_{z+\zeta}, \nabla_{g_{z+\zeta}}\zeta\rangle_{g_{z+\zeta}}\big)w \\
&\qquad - \int_0^{z+\zeta} (2\langle \mathrm{II}_t, \nabla^2_{g_t} w\rangle_{g_t} + \langle \nabla_{g_t} H_t, \nabla_{g_t} w\rangle_{g_t})\,dt \\
&\qquad + \Big(\int_0^{z+\zeta} \tfrac{\partial}{\partial t}(|\,\mathrm{II}_t\,|^2 + \mathrm{Ric}_g(\partial_z,\partial_z)|_{D\times\{t\}})\,dt\Big)w\Big).
\end{aligned}$$

By the fundamental theorem of calculus,

$$\mathcal{F}_6(\zeta_2) - \mathcal{F}_6(\zeta_1) = \int_0^1 \langle D_\zeta \mathcal{F}_6(\zeta_1 + t(\zeta_2-\zeta_1)), \zeta_2-\zeta_1\rangle\,dt.$$

We now estimate $\langle D_\zeta \mathcal{F}_6(\zeta), w\rangle$ for $\zeta = \zeta_1 + t(\zeta_2-\zeta_1)$ and $w = \zeta_2 - \zeta_1$. We will make repeated use of (7.3), (7.4), (A.1), (A.2), (A.4), (A.5), (A.6), $\|\zeta\|_{C^{2,\alpha}(\Sigma)} \le C'\varepsilon^{2-2\alpha}$, and $\|\cdot\|_{C^{0,\alpha}_\varepsilon(\Sigma)} \le \|\cdot\|_{C^{2,\alpha}(\Sigma)}$. First,

$$\begin{aligned}
&\Big\|\varepsilon\big(2\langle \mathrm{II}_{z+\zeta}, \nabla^2_{g_{z+\zeta}}\zeta\rangle_{g_{z+\zeta}} + \langle \nabla_{g_{z+\zeta}} H_{z+\zeta}, \nabla_{g_{z+\zeta}}\zeta\rangle_{g_{z+\zeta}}\big)w\Big\|_{C^{0,\alpha}_\varepsilon(\Omega_3)} \\
&\qquad\qquad \le C\varepsilon^{2-2\alpha}\|\zeta_2-\zeta_1\|_{C^{2,\alpha}(\Sigma)}.
\end{aligned}\tag{7.69}$$

Additionally using the $O(\varepsilon^{\delta_*})$ height bound on Ω_3, we also have

$$\Big\|\varepsilon\int_0^{z+\zeta}\langle \mathrm{II}_{z+\zeta}, \nabla^2_{g_{z+\zeta}} w\rangle_{g_{z+\zeta}}\Big\|_{C^{0,\alpha}_\varepsilon(\Omega_3)} \le C\varepsilon^{1+\delta_*}\|\zeta_2-\zeta_1\|_{C^{2,\alpha}(\Sigma)}.\tag{7.70}$$

Likewise,

$$\begin{aligned}
&\Big\|\varepsilon\Big(\int_0^{z+\zeta}\tfrac{\partial}{\partial t}(|\,\mathrm{II}_t\,|^2 + \mathrm{Ric}_g(\partial_z,\partial_z)|_{D\times\{t\}})\,dt\Big)w\Big\|_{C^{0,\alpha}_\varepsilon(\Omega_3)} \\
&\qquad\qquad \le C\varepsilon^{1+\delta_*}\|\zeta_2-\zeta_1\|_{C^{0,\alpha}(\Sigma)}.
\end{aligned}\tag{7.71}$$

It remains to estimate

$$\Big\|\varepsilon\int_0^{z+\zeta}\langle \nabla_{g_t} H_t, \nabla_{g_t} w\rangle_{g_t}\,dt\Big\|_{C^{0,\alpha}_\varepsilon(\Omega_3)}.$$

Now is the only place in the proof where we need to distinguish the Hölder exponents $\alpha \leq \theta$, taking the prior to be small and the latter to be large. From (7.3), (7.4), (A.1), (A.2), (A.4) and the interpolation of (unweighted) Hölder spaces $C^{1,\theta} \hookrightarrow C^{1,\alpha} \hookrightarrow C^{0,\theta}$ (Lemma E.1), we have

$$\|\nabla_{g_z} H_z\|_{C^{0,\alpha}(\Omega_3)} \leq C\|H_z\|^{\theta-\alpha}_{C^{0,\theta}(\Omega)}\|H_z\|^{1+\alpha-\theta}_{C^{1,\theta}(\Omega)} \leq C\varepsilon^{-2(1+\alpha-\theta)} \leq C\varepsilon^{-\frac{1}{2}\delta_*},$$

as long as α_0, θ_0 are chosen sufficiently close to 0 and to 1, respectively, depending on δ_*. It is now easy to see, as before, that

$$\left\|\varepsilon \int_0^{z+\zeta} \langle \nabla_{g_t} H_t, \nabla_{g_t} w\rangle_{g_t}\, dt\right\|_{C^{0,\alpha}_\varepsilon(\Omega_3)} \leq C\varepsilon^{1+\frac{1}{2}\delta_*}\|\zeta_2 - \zeta_1\|_{C^{1,\alpha}(\Sigma)}. \tag{7.72}$$

Altogether, (7.69), (7.70), (7.71), and (7.72) imply

$$\|\mathcal{F}_6(\zeta_2) - \mathcal{F}_6(\zeta_1)\|_{C^{0,\alpha}_\varepsilon(\Omega_3)} \leq C\varepsilon^{1+\frac{1}{2}\delta_*}\|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}.$$

Alongside (1.6), Remark 7.1, Lemma 7.3, (7.11), and $\delta_* \in (0,1)$, this implies

$$\begin{aligned}
&\varepsilon^\alpha \left\|\Pi_\varepsilon\big(\chi_3\varepsilon(\partial_z\mathbb{H}_\varepsilon)(\mathcal{F}_6(\zeta_2) - \mathcal{F}_6(\zeta_1))\big)\right\|_{C^{0,\alpha}(\Sigma)} \\
&\qquad + \left\|\Pi_\varepsilon^\perp\big(\chi_3(\varepsilon\partial_z\mathbb{H}_\varepsilon)(\mathcal{F}_6(\zeta_2) - \mathcal{F}_6(\zeta_1))\big)\right\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \\
&\quad \leq C(\varepsilon^{2-2\alpha} + \varepsilon^{1+\frac{1}{2}\delta_*})\|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}.
\end{aligned} \tag{7.73}$$

Together, (7.47), (7.48), (7.49), (7.51), (7.63), (7.66), (7.68), and (7.73) imply (7.38) for α_0, θ_0 depending on δ_*.

Likewise, (7.47), (7.48), (7.49), (7.51), (7.64), (7.66), (7.68), (7.73) imply (7.39) for α_0, θ_0 depending on δ_*. □

Proof of Theorem 7.4. As was already pointed out, we can rewrite (7.17) as the non-linear fixed point problem (7.27)–(7.29). We will take α, θ, δ as in Lemma 7.11, and $M \geq 1$.

Consider g as in Section 7, and also define

$$\begin{aligned}
\mathcal{U}(\varepsilon; M) \triangleq \Big\{&(v^\flat, v^\sharp, \zeta) \in \widetilde{C}^{2,\alpha}_\varepsilon(\Omega) \times C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R}) \times C^{2,\alpha}(\Sigma) : \\
&\|v^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} + \|v^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \varepsilon^{2\alpha}\|\zeta\|_{C^{2,\alpha}(\Sigma)} \leq M\varepsilon^2\Big\}
\end{aligned} \tag{7.74}$$

and

$$\begin{aligned}
\mathcal{B}(\varepsilon; \mu) \triangleq \Big\{&(\widehat{v}^\flat, \widehat{v}^\sharp, \widehat{\zeta}) \in C^{2,\alpha}_\varepsilon(\partial\Omega) \times C^{2,\alpha}_\varepsilon(\partial\Sigma\times\mathbf{R}) \times C^{2,\alpha}(\partial\Sigma) : \\
&\widehat{v}^\flat \equiv 0 \text{ on } \{\chi_4 = 1\},\ \Pi_\varepsilon(\widehat{v}^\sharp) \equiv 0 \text{ on } \partial\Sigma, \\
&\|\widehat{v}^\flat\|_{C^{2,\alpha}_\varepsilon(\partial\Omega)} + \|\widehat{v}^\sharp\|_{C^{2,\alpha}_\varepsilon(\partial\Sigma\times\mathbf{R})} + \|\widehat{\zeta}\|_{C^{2,\alpha}(\partial\Sigma)} \leq \mu\varepsilon^2\Big\}.
\end{aligned} \tag{7.75}$$

Lemmas 7.10 and 7.11 guarantee that for every $(v^\flat, v^\sharp, \zeta) \in \mathcal{U}(\varepsilon; M)$,

$$\|N_\varepsilon(v^\flat, v^\sharp, \zeta)\|_{C^{0,\alpha}_\varepsilon(\Omega)} \leq c_1' \varepsilon^{2+\delta-2\alpha} + c_0\varepsilon^2, \tag{7.76}$$

$$\|\Pi_\varepsilon^\perp M_\varepsilon(v^\flat, v^\sharp, \zeta)\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \leq c_1' \varepsilon^{2+\delta-2\alpha} + c_0\varepsilon^2, \tag{7.77}$$

$$\|\varepsilon^{-1}\Pi_\varepsilon M_\varepsilon(v^\flat, v^\sharp, \zeta)\|_{C^{0,\alpha}(\Sigma)} \leq c_1' \varepsilon^{2+\delta-2\alpha} + c_1'\varepsilon^{2-\alpha} + c_0\varepsilon^2, \tag{7.78}$$

with c_0 as in Lemma 7.10, and with $c_1' = M \cdot c_1$, $\varepsilon \leq \varepsilon_0$ as in Lemma 7.11.

Let

$$\Phi : \mathcal{U}(\varepsilon; M) \times \mathcal{B}(\varepsilon; \mu) \times \mathrm{Met}_{\varepsilon,\eta}(\Omega) \to \widetilde{C}^{2,\alpha}_\varepsilon(\Omega) \times C^{2,\alpha}_\varepsilon(\Sigma \times \mathbf{R}) \times C^{2,\alpha}(\Sigma)$$

be the solution map $\Phi : (v^\flat, v^\sharp, \zeta, \widehat{v}^\flat, \widehat{v}^\sharp, \widehat{\zeta}, g) \mapsto (V^\flat, V^\sharp, Z)$ for the *linear* system

$$\mathcal{L}_\varepsilon V^\flat = N_\varepsilon(v^\flat, v^\sharp, \zeta) \text{ on } \Omega,\ V^\flat|_{\partial\Omega} = \widehat{v}^\flat, \tag{7.79}$$

$$L_\varepsilon V^\sharp = \Pi_\varepsilon^\perp M_\varepsilon(v^\flat, v^\sharp, \zeta) \text{ on } \Sigma \times \mathbf{R},\ V^\sharp|_{\partial\Sigma\times\mathbf{R}} = \widehat{v}^\sharp, \tag{7.80}$$

$$J_\Sigma Z = \varepsilon^{-1}\Pi_\varepsilon M_\varepsilon(v^\flat, v^\sharp, \zeta) \text{ on } \Sigma,\ Z|_{\partial\Sigma} = \widehat{\zeta}. \tag{7.81}$$

The existence of $V^\flat$ follows from Fredholm theory. In fact, together with Lemma 7.9 and (7.76), we have

$$\begin{aligned} \|V^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} &\leq C(\|N_\varepsilon(v^\flat, v^\sharp, \zeta)\|_{C^{0,\alpha}_\varepsilon(\Omega)} + \|\widehat{v}^\flat\|_{C^{2,\alpha}_\varepsilon(\Omega)}) \\ &\leq Cc_1'\varepsilon^{2+\delta-2\alpha} + C(c_0+\mu)\varepsilon^2. \end{aligned} \tag{7.82}$$

The existence of $V^\sharp$ follows from Lemma 7.6. In fact, together with Lemma 7.7 and (7.77), we have

$$\begin{aligned} \|V^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} &\leq C(\|\Pi_\varepsilon^\perp M_\varepsilon(v^\flat, v^\sharp, \zeta)\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|\widehat{v}^\sharp\|_{C^{2,\alpha}_\varepsilon(\partial\Sigma\times\mathbf{R})} \\ &\leq Cc_1'\varepsilon^{2+\delta-2\alpha} + C(c_0+\mu)\varepsilon^2. \end{aligned} \tag{7.83}$$

Finally, the existence of Z follows from Fredholm theory and (7.9). In fact, by Schauder theory on the elliptic operator J_Σ on Σ, and (7.78), we find that

$$\begin{aligned} \|Z\|_{C^{2,\alpha}(\Sigma)} &\leq C(\|\varepsilon^{-1}\Pi_\varepsilon M_\varepsilon(v^\flat, v^\sharp, \zeta)\|_{C^{0,\alpha}(\Sigma)} + \|\widehat{\zeta}\|_{C^{2,\alpha}(\partial\Sigma)}) \\ &\leq Cc_1'\varepsilon^{2+\delta-2\alpha} + Cc_1'\varepsilon^{2-\alpha} + Cc_0\varepsilon^2 + C\mu\varepsilon^{2-2\alpha}, \\ \implies \varepsilon^{2\alpha}\|Z\|_{C^{2,\alpha}(\Sigma)} &\leq Cc_1'\varepsilon^{2+\delta} + Cc_1'\varepsilon^{2+\alpha} + Cc_0\varepsilon^{2+2\alpha} + C\mu\varepsilon^2. \end{aligned} \tag{7.84}$$

We emphasize that the constant C in (7.82), (7.83), and (7.84) depends only on n, $\eta > 0$, and W.

The expressions in (7.82), (7.83), and (7.84) can all be made to be $\leq \frac{1}{3}M\varepsilon^2$ as follows:

(1) Choose M large, depending on c_0, C, μ, so that $C(c_0 + \mu) \leq \frac{1}{6}M$.

(2) Then, choose $\varepsilon \le \varepsilon_0$ small depending on C, c_1', M, so that

$$Cc_1'\varepsilon^\alpha \ll 1; \tag{7.85}$$

note that, since $M \ge 1$, the left-hand side is also $\le \frac{1}{12}M$.

(3) Using $\alpha \in (0, \frac{\delta}{3})$ we find that $\varepsilon^{\delta-2\alpha} \le \varepsilon^\alpha$, so $Cc_1'\varepsilon^\delta \le Cc_1'\varepsilon^{\delta-2\alpha} \le \frac{1}{12}M$.

Thus, for such a choice of $M = M(n, \eta, W, \delta_*, \mu)$, $\varepsilon \le \varepsilon_0' = \varepsilon_0'(n, \eta, W, \delta_*, \mu, \alpha)$, we have

$$\Phi\big(\mathcal{U}(\varepsilon; M) \times \mathcal{B}(\varepsilon; \mu) \times \mathrm{Met}_{\varepsilon,\eta}(\Omega)\big) \subset \mathcal{U}(\varepsilon; M).$$

We show that $\Phi(\cdot, \cdot, \cdot, \widehat{v}^\flat, \widehat{v}^\sharp, \widehat{\zeta}, g)$ is a *contraction* with respect to the norm

$$\|(v^\flat, v^\sharp, \zeta)\|_{\mathcal{U}} \triangleq \|v^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} + \|v^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \varepsilon^{2\alpha}\|\zeta\|_{C^{2,\alpha}(\Sigma)}, \tag{7.86}$$

uniformly with respect to $\widehat{v}^\flat$, $\widehat{v}^\sharp, \widehat{\zeta}, g$. Let us also define

$$\|(\widehat{v}^\flat, \widehat{v}^\sharp, \widehat{\zeta})\|_{\mathcal{B}} \triangleq \|\widehat{v}^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\partial\Omega)} + \|\widehat{v}^\sharp\|_{C^{2,\alpha}_\varepsilon(\partial\Sigma\times\mathbf{R})} + \|\widehat{\zeta}\|_{C^{2,\alpha}(\partial\Sigma)}. \tag{7.87}$$

Let us set

$$(V_1^\flat, V_1^\sharp, Z_1) \triangleq \Phi(v_1^\flat, v_1^\sharp, \zeta_1, \widehat{v}^\flat, \widehat{v}^\sharp, \widehat{\zeta}, g),$$
$$(V_2^\flat, V_2^\sharp, Z_2) \triangleq \Phi(v_2^\flat, v_2^\sharp, \zeta_2, \widehat{v}^\flat, \widehat{v}^\sharp, \widehat{\zeta}, g).$$

By Lemmas 7.9 and 7.11,

$$\begin{aligned} &\|V_2^\flat - V_1^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} \\ &\le C\|\mathcal{L}_\varepsilon V_2^\flat - \mathcal{L}_\varepsilon V_1^\flat\|_{C^{0,\alpha}_\varepsilon(\Omega)} \\ &= C\|N_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - N_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1)\|_{C^{0,\alpha}_\varepsilon(\Omega)} \\ &\le Cc_1'\varepsilon^\delta\Big(\|v_2^\flat - v_1^\flat\|_{\widetilde{C}^{2,\alpha}(\Omega)} + \|v_2^\sharp - v_1^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}\Big). \end{aligned} \tag{7.88}$$

By Lemmas 7.7 and 7.11,

$$\begin{aligned} &\|V_2^\sharp - V_1^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \\ &\le C\|L_\varepsilon V_2^\sharp - L_\varepsilon V_1^\sharp\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \\ &= C\|\Pi_\varepsilon^\perp M_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - \Pi_\varepsilon^\perp M_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1)\|_{C^{0,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} \\ &\le Cc_1'\varepsilon^\delta\Big(\|v_2^\flat - v_1^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} + \|v_2^\sharp - v_1^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})} + \|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}\Big). \end{aligned} \tag{7.89}$$

Finally, by Lemma 7.11, (7.9), and Schauder theory,

(7.90)
$$\begin{aligned} &\|Z_2 - Z_1\|_{C^{2,\alpha}(\Sigma)} \\ &\le C\|J_\Sigma Z_2 - J_\Sigma Z_1\|_{C^{0,\alpha}(\Sigma)} \\ &= C\|\varepsilon^{-1}\Pi_\varepsilon M_\varepsilon(v_2^\flat, v_2^\sharp, \zeta_2) - \varepsilon^{-1}\Pi_\varepsilon M_\varepsilon(v_1^\flat, v_1^\sharp, \zeta_1)\|_{C^{0,\alpha}(\Sigma)} \\ &\le Cc_1'\Big[\varepsilon^\delta\big(\|v_2^\flat - v_1^\flat\|_{\widetilde{C}^{2,\alpha}_\varepsilon(\Omega)} + \|\zeta_2 - \zeta_1\|_{C^{2,\alpha}(\Sigma)}\big) + \varepsilon^{-\alpha}\|v_2^\sharp - v_1^\sharp\|_{C^{2,\alpha}_\varepsilon(\Sigma\times\mathbf{R})}\Big]. \end{aligned}$$

Adding (7.88), (7.89), and $\varepsilon^{2\alpha}$ times (7.90), using $\alpha < \frac{1}{3}\delta$ and the $\|\cdot\|_{\mathcal{U}}$ norm on $\mathcal{U}(\varepsilon; M)$,

$$\|(V_2^\flat, V_2^\sharp, Z_2) - (V_1^\flat, V_1^\sharp, Z_1)\|_{\mathcal{U}} \leq Cc_1'\varepsilon^\alpha \|(v_2^\flat, v_2^\sharp, \zeta_2) - (v_1^\flat, v_1^\sharp, \zeta_1)\|_{\mathcal{U}}. \tag{7.91}$$

This implies that $\Phi(\cdot, \cdot, \cdot, \widehat{v}^\flat, \widehat{v}^\sharp, \widehat{\zeta}, g)$ is uniformly Lipschitz, with Lipschitz constant $\leq Cc_1'\varepsilon^\alpha$, and by (7.85) we conclude that it is , in fact, a contraction mapping. This readily implies the existence of a fixed point $(v^\flat, v^\sharp, \zeta)$, which therefore satisfies (7.17).

We finally move to prove the continuity of the solution map

$$\mathcal{S} : \mathcal{B}(\varepsilon; \mu) \times \mathrm{Met}_{\varepsilon,\eta}(\Omega) \to \mathcal{U}(\varepsilon; M).$$

For $(\widehat{v}_1^\flat, \widehat{v}_1^\sharp, \widehat{\zeta}_1, g_1)$, $(\widehat{v}_2^\flat, \widehat{v}_2^\sharp, \widehat{\zeta}_2, g_2) \in \mathcal{B}(\varepsilon; \mu) \times \mathrm{Met}_{\varepsilon,\eta}(\Omega)$, we have, by the fixed point property,

$$\begin{aligned}
&\mathcal{S}(\widehat{v}_2^\flat, \widehat{v}_2^\sharp, \widehat{\zeta}_2, g_2) - \mathcal{S}(\widehat{v}_1^\flat, \widehat{v}_1^\sharp, \widehat{\zeta}_1, g_1) \\
&\quad = \Big(\Phi(\mathcal{S}(\widehat{v}_2^\flat, \widehat{v}_2^\sharp, \widehat{\zeta}_2, g_2), \widehat{v}_2^\flat, \widehat{v}_2^\sharp, \widehat{\zeta}_2, g_2) - \Phi(\mathcal{S}(\widehat{v}_2^\flat, \widehat{v}_2^\sharp, \widehat{\zeta}_2, g_2), \widehat{v}_1^\flat, \widehat{v}_1^\sharp, \widehat{\zeta}_1, g_1)\Big) \\
&\qquad - \Big(\Phi(\mathcal{S}(\widehat{v}_1^\flat, \widehat{v}_1^\sharp, \widehat{\zeta}_1, g_1), \widehat{v}_1^\flat, \widehat{v}_1^\sharp, \widehat{\zeta}_1, g_1) - \Phi(\mathcal{S}(\widehat{v}_2^\flat, \widehat{v}_2^\sharp, \widehat{\zeta}_2, g_2), \widehat{v}_1^\flat, \widehat{v}_1^\sharp, \widehat{\zeta}_1, g_1)\Big).
\end{aligned}$$

The last parenthesis will be bounded using the contraction mapping property (7.91) on $(\widehat{v}_1^\flat, \widehat{v}_1^\sharp, \widehat{\zeta}_1, g_1)$. The second-to-last parenthesis will be bounded by varying the four slots of $\Phi(\mathcal{S}(\widehat{v}_2^\flat, \widehat{v}_2^\sharp, \widehat{\zeta}_2, g_2), \cdot, \cdot, \cdot, \cdot)$ using the fundamental theorem of calculus. The $\widehat{v}^\flat$, $\widehat{v}^\sharp$, $\widehat{\zeta}$ derivatives of $\Phi(\mathcal{S}(\widehat{v}_2^\flat, \widehat{v}_2^\sharp, \widehat{\zeta}_2, g_2), \cdot, \cdot, \cdot, \cdot)$ can be controlled using Lemmas 7.9 and 7.6, and Schauder theory on J_Σ, respectively. Likewise, it is not hard to see that for $g \in \mathrm{Met}_{\varepsilon,\eta}(\Omega)$, the map

$$g \mapsto \Phi(v^\flat, v^\sharp, \zeta, \widehat{v}^\flat, \widehat{v}^\sharp, \widehat{\zeta}, g)$$

is uniformly Lipschitz with respect to $(v^\flat, v^\sharp, \zeta, \widehat{v}^\flat, \widehat{v}^\sharp, \widehat{\zeta}) \in \mathcal{U}(\varepsilon; M) \times \mathcal{B}(\varepsilon; \mu)$. Altogether, we have

$$\begin{aligned}
&\|\mathcal{S}(\widehat{v}_2^\flat, \widehat{v}_2^\sharp, \widehat{\zeta}_2, g_2) - \mathcal{S}(\widehat{v}_1^\flat, \widehat{v}_1^\sharp, \widehat{\zeta}_1, g_1))\|_{\mathcal{U}} \\
&\quad \leq c\Big(\|(\widehat{v}_2^\flat, \widehat{v}_2^\sharp, \widehat{\zeta}_2) - (\widehat{v}_1^\flat, \widehat{v}_1^\sharp, \widehat{\zeta}_1)\|_{\mathcal{B}} + d(g_2, g_1)\Big) \\
&\qquad + Cc_1'\varepsilon^\alpha \|\mathcal{S}(\widehat{v}_2^\flat, \widehat{v}_2^\sharp, \widehat{\zeta}_2, g_2) - \mathcal{S}(\widehat{v}_1^\flat, \widehat{v}_1^\sharp, \widehat{\zeta}_1, g_1))\|_{\mathcal{U}},
\end{aligned}$$

and the result follows by rearranging. □

Appendix A. Mean curvature of normal graphs

The purpose of this appendix is to record a form of the second variation of mean curvature that is convenient for our paper, since this computation is not easily found in the literature.

We consider Fermi coordinates (y, z) near a hypersurface $\Sigma \subset M$ satisfying the conditions of Section 2.1, where the normal graph of a function $f : \Sigma \to \mathbf{R}$

will eventually look like

$$G[f] \triangleq \{(y, f(y)) : y \in \Sigma\}.$$

Before discussing the geometry of the graph over Σ, let us first discuss the geometry of the distance level sets $\{z = \text{const}\}$ relative to Σ.

We will denote the restriction of the metric to the parallel hypersurface $\{(y, z) : y \in \Sigma\}$ by g_z, i.e., $g_z = Z_\Sigma(\cdot, z)^* g$, and the corresponding upward pointing unit normal, area form, second fundamental form, mean curvature, divergence, gradient, Hessian, and Laplacian by ∂_z, $d\mu_{g_z}$, $\mathrm{I\!I}_z$, H_z, div_{g_z}, ∇_{g_z}, $\nabla^2_{g_z}$, Δ_{g_z}. We recall that the ∂_z (Lie) derivative of g_z is known to be

$$\mathscr{L}_{\partial_z} g_z = 2\,\mathrm{I\!I}_z, \tag{A.1}$$

and also the corresponding derivative of the second fundamental form $\mathrm{I\!I}_z$ is

$$\mathscr{L}_{\partial_z} \mathrm{I\!I}_z = \mathrm{I\!I}_z^2 - \mathrm{Rm}_g(\cdot, \partial_z, \partial_z, \cdot), \tag{A.2}$$

where $\mathrm{I\!I}_z^2$ denotes a single trace of $\mathrm{I\!I}_z \otimes \mathrm{I\!I}_z$, and our Riemann curvature convention is such that $(\mathrm{Rm}_g)_{ijji}$ (suitably normalized) denotes a sectional curvature. From (A.2) we recover the well-known Jacobi equation

$$\partial_z H_z = \partial_z(g_z^{ij}\,\mathrm{I\!I}^z_{ij}) = -g_z^{ik} g_z^{j\ell} (\mathscr{L}_{\partial_z} g_z)_{k\ell}\,\mathrm{I\!I}^z_{ij} + g_z^{ij} \mathscr{L}_{\partial_z}\mathrm{I\!I}^z_{ij} = -|\,\mathrm{I\!I}_z\,|^2 - \mathrm{Ric}_g(\partial_z, \partial_z). \tag{A.3}$$

From (A.1) we also find the evolution of the gradient operator:

$$\mathscr{L}_{\partial_z} \nabla_{g_z} f = -2\,\mathrm{I\!I}_z(\nabla_{g_z} f, \cdot),\ f \in C^\infty(\Sigma). \tag{A.4}$$

Next, we seek the evolution of the divergence operator on 1-forms. To find it, we first need to find the evolution of the Christoffel symbols. Recall that the Christoffel symbols do not transform like tensors but that their difference does. In particular, $\partial_z \Gamma$ is a vector-valued 2-tensor given by (using the Codazzi equation to get the second form)

$$\begin{aligned}(\partial_z\Gamma)(\mathbf{X}, \mathbf{Y}) &= [\nabla^{g_z}_{\mathbf{X}}\,\mathrm{I\!I}(\cdot, \mathbf{Y}) + \nabla^{g_z}_{\mathbf{Y}}\,\mathrm{I\!I}(\mathbf{X}, \cdot) - \nabla^{g_z}_{\cdot}\,\mathrm{I\!I}(\mathbf{X}, \mathbf{Y})]^\sharp \\ &= [\nabla^{g_z}_{\cdot}\,\mathrm{I\!I}(\mathbf{X}, \mathbf{Y}) + \mathrm{Rm}_g(\partial_z, \mathbf{X}, \mathbf{Y}, \cdot) + \mathrm{Rm}_g(\partial_z, \mathbf{Y}, \mathbf{X}, \cdot)]^\sharp,\end{aligned} \tag{A.5}$$

where the indices are raised with the $\sharp$ operator using g_z. From this we find the evolution of the Hessians of scalar fields,

$$\mathscr{L}_{\partial_z} \nabla^2_{g_z} f = -\nabla^{g_z}_{\nabla_{g_z} f}\,\mathrm{I\!I}_z, \tag{A.6}$$

and their Laplacians

$$\mathscr{L}_z \Delta_{g_z} f = -2\langle \mathrm{I\!I}_z, \nabla^2_{g_z} f\rangle_{g_z} - \langle \nabla_{g_z} H_z, \nabla_{g_z} f\rangle_{g_z}. \tag{A.7}$$

Likewise, the evolution of the divergence of 1-forms is

$$\mathscr{L}_z \operatorname{div}_{g_z} \omega = -2\langle \mathrm{I\!I}_z, \nabla_{g_z}\omega\rangle_{g_z} - \langle \nabla_{g_z} H + \mathrm{Ric}_g(\partial_z, \cdot), \omega\rangle_{g_z}, \qquad \omega \in \Omega^1(\Sigma). \tag{A.8}$$

Next, we seek to calculate the evolution of $\nabla_{g_z} \mathrm{I\!I}_z$. To do so, we pick coordinates so that the vectors ∂_{y_i} are parallel (with respect to ∇_{g_z}) at the base point where we are computing the derivative. Then

$$\begin{aligned}
\partial_z(\partial_{y_i} \mathrm{I\!I}^z_{jk}) &= \partial_{y_i}(\partial_z \mathrm{I\!I}^z_{jk}) \\
&= \partial_{y_i}(\mathscr{L}_{\partial_z} \mathrm{I\!I}^z_{jk}) \\
&= \partial_{y_i}((\mathrm{I\!I}^2_z)_{jk} - \mathrm{Rm}_g(\partial_{y_j}, \nu_z, \nu_z, \partial_{y_k})) \\
&= \partial_{y_i}(g_z^{\ell m} \mathrm{I\!I}^z_{j\ell} \mathrm{I\!I}^z_{km} - \mathrm{Rm}_g(\partial_{y_j}, \partial_z, \partial_z, \partial_{y_k})) \\
&= -g_z^{\ell m} \partial_{y_i} \mathrm{I\!I}^z_{j\ell} \mathrm{I\!I}^z_{km} - g_z^{\ell m} \mathrm{I\!I}^z_{j\ell} \partial_{y_i} \mathrm{I\!I}^z_{km} - \partial_{y_i}(\mathrm{Rm}_g(\partial_{y_j}, \partial_z, \partial_z, \partial_{y_k})) \\
&= -g_z^{\ell m}(\nabla^{g_z}_i \mathrm{I\!I}^z_{j\ell}) \mathrm{I\!I}^z_{km} - g_z^{\ell m}(\nabla^{g_z}_i \mathrm{I\!I}^z_{km}) \mathrm{I\!I}^z_{j\ell} - \nabla^g_{\partial_{y_i}} \mathrm{Rm}_g(\partial_{y_j}, \partial_z, \partial_z, \partial_{y_k}) \\
&\quad + 2\,\mathrm{Rm}_g(\partial_{y_j}, g_z^{\ell m} \mathrm{I\!I}^z_{i\ell} \partial_{y_m}, \partial_z, \partial_{y_k});
\end{aligned}$$

i.e.,

(A.9)

$$\begin{aligned}
\partial_z(\partial_{y_i} \mathrm{I\!I}^z_{jk}) = {} & -g_z^{\ell m}(\nabla^{g_z}_i \mathrm{I\!I}^z_{j\ell}) \mathrm{I\!I}^z_{km} - g_z^{\ell m}(\nabla^{g_z}_i \mathrm{I\!I}^z_{km}) \mathrm{I\!I}^z_{j\ell} \\
& - \nabla^g_{\partial_{y_i}} \mathrm{Rm}_g(\partial_{y_j}, \partial_z, \partial_z, \partial_{y_k}) + 2 g_z^{\ell m} \mathrm{I\!I}^z_{i\ell} \mathrm{Rm}_g(\partial_{y_j}, \partial_{y_m}, \partial_z, \partial_{y_k}).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathscr{L}_{\partial_z}(\nabla^{g_z}_{\partial_{y_i}} \partial_{y_j}) &= \mathscr{L}_{\partial_z}(\nabla^g_{\partial_{y_i}} \partial_{y_j} + \mathrm{I\!I}^z_{ij} \partial_z) \\
&= \mathscr{L}_{\partial_z} \nabla^g_{\partial_{y_i}} \partial_{y_j} + (\mathscr{L}_{\partial_z} \mathrm{I\!I}_z)_{ij} \partial_z \\
&= \nabla^g_{\partial_z} \nabla^g_{\partial_{y_i}} \partial_{y_j} + (\mathrm{I\!I}^2_z)_{ij} \partial_z - \mathrm{Rm}_g(\partial_{y_i}, \partial_z, \partial_z, \partial_{y_j}) \partial_z \\
&= \nabla^g_{\partial_z} \nabla^g_{\partial_{y_i}} \partial_{y_j} + (\mathrm{I\!I}^2_z)_{ij} \partial_z - \mathrm{Rm}_g(\partial_z, \partial_{y_i}, \partial_{y_j}, \partial_z) \partial_z \\
&= \nabla^g_{\partial_z} \nabla^g_{\partial_{y_i}} \partial_{y_j} + (\mathrm{I\!I}^2_z)_{ij} \partial_z - (\nabla^g_{\partial_z} \nabla^g_{\partial_{y_i}} \partial_{y_j} - \nabla^g_{\partial_{y_i}} \nabla^g_{\partial_z} \partial_{y_j}) \\
&= \nabla^g_{\partial_{y_i}} \nabla^g_{\partial_z} \partial_{y_j} + (\mathrm{I\!I}^2_z)_{ij} \partial_z = \nabla^g_{\partial_{y_i}} \nabla^g_{\partial_{y_j}} \partial_z + (\mathrm{I\!I}^2_z)_{ij} \partial_z.
\end{aligned}$$

Recall that $\mathscr{L}_{\partial_z}(\nabla^{g_z}_{\partial_{y_i}} \partial_{y_j})$ is tangential to $\{z = \text{const}\}$, and so is $\nabla^g_{\partial_{y_j}} \partial_z = g_z^{k\ell} \mathrm{I\!I}^z_{jk} \partial_{y_\ell}$. By projecting onto $\{z = \text{const}\}$, the expression above reduces to

$$\mathscr{L}_{\partial_z}(\nabla^{g_z}_{\partial_{y_i}} \partial_{y_j}) = \nabla^{g_z}_{\partial_{y_i}}(g_z^{k\ell} \mathrm{I\!I}^z_{jk} \partial_{y_\ell}) = g_z^{k\ell}(\nabla^{g_z}_i \mathrm{I\!I}^z_{jk}) \partial_{y_\ell}. \tag{A.10}$$

Combining with (A.9), we deduce that

$$\begin{aligned}
\partial_z(\nabla_i^{g_z} \mathrm{I\!I}_{jk}^z) &= \partial_z(\partial_{y_i} \mathrm{I\!I}_{jk}^z) - \mathrm{I\!I}_z(\nabla_{\partial_{y_i}}^{g_z} \partial_{y_j}, \partial_{y_k}) - \mathrm{I\!I}_z(\partial_{y_j}, \nabla_{\partial_{y_i}}^{g_z} \partial_{y_k})) \\
&= \partial_z(\partial_{y_i} \mathrm{I\!I}_{jk}^z) - \mathrm{I\!I}_z(\mathscr{L}_{\partial_z}(\nabla_{\partial_{y_i}}^{g_z} \partial_{y_j}), \partial_{y_k}) - \mathrm{I\!I}_z(\partial_{y_j}, \mathscr{L}_{\partial_z}(\nabla_{\partial_{y_i}}^{g_z} \partial_{y_k})) \\
&= -g_z^{\ell m}(\nabla_i^{g_z} \mathrm{I\!I}_{j\ell}^z) \mathrm{I\!I}_{km}^z - g_z^{\ell m}(\nabla_i^{g_z} \mathrm{I\!I}_{km}^z) \mathrm{I\!I}_{j\ell}^z \\
&\quad - \nabla_{\partial_{y_i}}^g \mathrm{Rm}_g(\partial_{y_j}, \partial_z, \partial_z, \partial_{y_k}) \\
&\quad + 2g_z^{\ell m} \mathrm{I\!I}_{i\ell}^z \mathrm{Rm}_g(\partial_{y_j}, \partial_{y_m}, \partial_z, \partial_{y_k}) - \mathrm{I\!I}_z(g_z^{m\ell}(\nabla_i^{g_z} \mathrm{I\!I}_{jm}^z)\partial_{y_\ell}, \partial_{y_k}) \\
&\quad - \mathrm{I\!I}_z(\partial_{y_j}, g_z^{m\ell}(\nabla_i^{g_z} \mathrm{I\!I}_{km}^z)\partial_{y_\ell}) \\
&= -g_z^{\ell m}(\nabla_i^{g_z} \mathrm{I\!I}_{j\ell}^z) \mathrm{I\!I}_{km}^z - g_z^{\ell m}(\nabla_i^{g_z} \mathrm{I\!I}_{km}^z) \mathrm{I\!I}_{j\ell}^z \\
&\quad - \nabla_{\partial_{y_i}}^g \mathrm{Rm}_g(\partial_{y_j}, \partial_z, \partial_z, \partial_{y_k}) \\
&\quad + 2g_z^{\ell m} \mathrm{I\!I}_{i\ell}^z \mathrm{Rm}_g(\partial_{y_j}, \partial_{y_m}, \partial_z, \partial_{y_k}) \\
&\quad - g_z^{m\ell} \mathrm{I\!I}_{\ell k}^z \nabla_i^{g_z} \mathrm{I\!I}_{jm}^z - g_z^{m\ell} \mathrm{I\!I}_{j\ell}^z \nabla_i^{g_z} \mathrm{I\!I}_{km}^z \\
&= -2g_z^{\ell m}(\nabla_i^{g_z} \mathrm{I\!I}_{j\ell}^z) \mathrm{I\!I}_{km}^z - 2g_z^{\ell m}(\nabla_i^{g_z} \mathrm{I\!I}_{km}^z) \mathrm{I\!I}_{j\ell}^z \\
&\quad - \nabla_{\partial_{y_i}}^g \mathrm{Rm}_g(\partial_{y_j}, \partial_z, \partial_z, \partial_{y_k}) + 2g_z^{\ell m} \mathrm{I\!I}_{i\ell}^z \mathrm{Rm}_g(\partial_{y_j}, \partial_{y_m}, \partial_z, \partial_{y_k}).
\end{aligned}$$

In particular,

$$\mathscr{L}_{\partial_z} \nabla_{g_z} \mathrm{I\!I}_z = \nabla_{g_z} \mathrm{I\!I}_z * \mathrm{I\!I}_z + \nabla_g \mathrm{Rm}_g + \mathrm{I\!I}_z * \mathrm{Rm}_g \tag{A.11}$$

as a symmetric 2-tensor on the $\{z = \text{const}\}$ level sets.

We now proceed to use these evolution equations to understand the second variation of the mean curvature of a graph in Fermi coordinates. These computations are motivated by the ones in [PS].

Continuing to work in Fermi coordinates (y, z) relative to Σ, we write

$$G[f] \triangleq \{(y, f(y)) : y \in \Sigma\}.$$

Note that the induced metric on $G[f]$ is

$$g|_{G[f]} = g_{f(y)} + df(y)^2, \ y \in \Sigma.$$

The induced area form on $G[f]$ is therefore

$$d\mu_{G[f]}(y) = (1 + g_{f(y)}^{ij} f_i(y) f_j(y))^{1/2} \, d\mu_{g_{f(y)}}(y), \ y \in \Sigma.$$

Thus,

$$\operatorname{area}(G[f]) = \int_\Sigma (1 + g_f^{ij} f_i f_j)^{1/2} \, d\mu_{g_{f(\cdot)}}. \tag{A.12}$$

We now consider the variation $f + t\varphi$, $\varphi \in C^2_c(\Sigma \setminus \partial\Sigma)$. We have (we are using integration by parts in the second step)

$$\begin{aligned}
&\left[\frac{d}{dt}\operatorname{area}(G[f+t\varphi])\right]_{t=0}\\
&\quad = \int_\Sigma \frac{g_f^{ij} f_i \varphi_j}{(1+g_f^{ij} f_i f_j)^{1/2}}\, d\mu_{g_f} - \int_\Sigma \frac{\mathrm{I\!I}_f^{ij}\, f_i f_j \varphi}{(1+g_f^{ij} f_j f_j)^{1/2}}\, d\mu_{g_f}\\
&\qquad + \int_\Sigma (1+g_f^{ij} f_i f_j)^{1/2} H_f \varphi \, d\mu_{g_f}\\
&\quad = -\int_\Sigma \operatorname{div}_{g_f}\left(\frac{\nabla_{g_f} f}{(1+g_f^{ij} f_i f_j)^{1/2}}\right)\varphi\, d\mu_{g_f} - \int_\Sigma \frac{\mathrm{I\!I}_f^{ij}\, f_i f_j \varphi}{(1+g_f^{ij} f_j f_j)^{1/2}}\, d\mu_{g_f}\\
&\qquad + \int_\Sigma (1+g_f^{ij} f_i f_j)^{1/2} H_f \varphi \, d\mu_{g_f}.
\end{aligned}$$

Note that, if ν denotes the normal to $G[f]$, then

$$g(\nu, \partial_z) = (1+g_f^{ij} f_i f_j)^{-1/2} \implies d\mu_{g_f} = g(\nu,\partial_z) d\mu_{G[f]},$$

and, therefore,

$$\begin{aligned}
&\left[\frac{d}{dt}\operatorname{area}(G[f+t\varphi])\right]_{t=0}\\
&\quad = \int_\Sigma \Big[-\operatorname{div}_{g_f}\left(\frac{\nabla_{g_f} f}{(1+g_f^{ij} f_i f_j)^{1/2}}\right) - \frac{\mathrm{I\!I}_f^{ij}\, f_i f_j}{(1+g_f^{ij} f_i f_j)^{1/2}}\\
&\qquad + (1+g_f^{ij} f_i f_j)^{1/2} H_f \Big] g(\nu,\partial_z)\varphi\, d\mu_{G[f]}.
\end{aligned}$$

On the other hand, by definition,

$$\left[\frac{d}{dt}\operatorname{area}(G[f+t\varphi])\right]_{t=0} = \int_\Sigma H_{G[f]} g(\nu,\partial_z)\varphi\, d\mu_{G[f]},$$

so we conclude

(A.13)

$$H[f] = -\operatorname{div}_{g_f}\left(\frac{\nabla_{g_f} f}{(1+g_f^{ij} f_i f_j)^{1/2}}\right) - \frac{\mathrm{I\!I}_f^{ij}\, f_i f_j}{(1+g_f^{ij} f_i f_i)^{1/2}} + (1+g_f^{ij} f_i f_j)^{1/2} H_f.$$

We now claim that the quantity

(A.14)

$$\widetilde{\mathcal{Q}} f \triangleq H[f] - H_0 + \frac{\sqrt{g_0}}{\sqrt{g_f}} \operatorname{div}_{g_0}\Big(\frac{\sqrt{g_f}}{\sqrt{g_0}} (1+g_f^{ij} f_i f_j)^{-1/2} \nabla_{g_f} f\Big) + (|\,\mathrm{I\!I}_0\,|^2 + \operatorname{Ric}_g(\partial_z,\partial_z)) f$$

is a quadratic error term in the Taylor expansion of $H[f]$ with respect to $\{z = 0\}$:

Lemma A.1. *We have the pointwise estimate*

$$|\widetilde{\mathcal{Q}} f| \leq c(|f|^2 + |\partial f|^2),$$

where $c = c(n, \Lambda) > 0$ *and* $\Lambda = \Lambda(f, y) > 0$ *is such that*

$$\sup_{|z| \leq |f(y)|} |\mathrm{I\!I}_z(y)| \leq \Lambda.$$

Proof. First, note that

$$\operatorname{div}_{g_f}\left(\frac{\nabla_{g_f} f}{(1 + g_f^{ij} f_i f_j)^{1/2}}\right) = \frac{\sqrt{g_0}}{\sqrt{g_f}} \operatorname{div}_{g_0}\left(\frac{\sqrt{g_f}}{\sqrt{g_0}} \frac{\nabla_{g_f} f}{(1 + g_f^{ij} f_i f_j)^{1/2}}\right),$$

which means that

$$\widetilde{\mathcal{Q}} f = -\frac{\mathrm{I\!I}_f^{ij} f_i f_j}{(1 + g_f^{ij} f_i f_j)^{1/2}} + (1 + g_f^{ij} f_i f_j)^{1/2} H_f - H_0 + (|\mathrm{I\!I}_0|^2 + \operatorname{Ric}_g(\partial_z, \partial_z)) f.$$

The result follows by adding and subtracting H_f,

$$\frac{|\mathrm{I\!I}_f^{ij} f_i f_j|}{(1 + g_f^{ij} f_i f_j)^{1/2}} + |(1 + g_f^{ij} f_i f_j)^{1/2} H_f - H_f| \leq c |\partial f|^2,$$

and

$$|H_f - H_0 + (|\mathrm{I\!I}_0|^2 + \operatorname{Ric}_g(\partial_z, \partial_z)) f| \leq c |f|^2.$$

We have used (A.3) in the last estimate. □

Appendix B. Some results of Wang–Wei

For completeness, we recall several results proven in [WW19a] by Wang–Wei. We will assume (2.3)–(2.11) and will use the notation $\overline{\mathbb{H}} \triangleq \overline{\mathbb{H}}^{3|\log \varepsilon|}$ and $\overline{\mathbb{H}}_{\varepsilon,\ell}$ from Section 2.1 throughout this appendix. We emphasize (see Remark 2.1) that we are working at the original scale, rather than the ε-scale as in [WW19a], so these expressions have changed relative to [WW19a] by appropriate factors of ε.

Lemma B.1 ([WW19a, Lemma 8.3]). *For* $m \neq \ell \in \{1, \ldots, Q\}$, *consider* $X \in Z_\ell(\Gamma_\ell(\frac{3}{2}) \times [-1, 1])$ *with* $|d_m(X)|, |d_\ell(X)| \leq K\varepsilon |\log \varepsilon|$. *Then,*

$$d_{\Gamma_m}(\Pi_m \circ \Pi_\ell(X), \Pi_m(X)) \leq C(K) \varepsilon^{\frac{3}{2}} |\log \varepsilon|^{\frac{3}{2}},$$
$$|d_m(\Pi_\ell(X)) + d_\ell(\Pi_m(X))| \leq C(K) \varepsilon^{\frac{3}{2}} |\log \varepsilon|^{\frac{3}{2}},$$
$$|d_m(X) - d_\ell(X) + d_m(\Pi_\ell(X))| \leq C(K) \varepsilon^{\frac{3}{2}} |\log \varepsilon|^{\frac{3}{2}},$$
$$|d_\ell(X) - d_m(X) - d_\ell(\Pi_m(X))| \leq C(K) \varepsilon^{\frac{3}{2}} |\log \varepsilon|^{\frac{3}{2}},$$
$$1 - \nabla d_\ell(X) \cdot \nabla d_m(X) \leq C(K) \varepsilon^{\frac{1}{2}} |\log \varepsilon|^{\frac{3}{2}}.$$

Recall the definition of ϕ in (2.17). Wang–Wei compute [WW19a, (9.4)] in Fermi coordinates with respect to Γ_ℓ that

$$\begin{aligned}
&\varepsilon^2(\Delta_{\Gamma_{\ell,z}}\phi + H_{\Gamma_{\ell,z}}\partial_z\phi + \partial_z^2\phi) \\
&= W''(U[\mathbf{h}])\phi + \mathcal{R}(\phi) + \left(W'(U[\mathbf{h}]) - \sum_{m=1}^{Q} W'(\overline{\mathbb{H}}_{\varepsilon,m})\right) \\
&\quad + \varepsilon^2(\Delta_{\Gamma_{\ell,z}}h_\ell - H_{\Gamma_{\ell,z}})\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell} - \varepsilon^2|\nabla_{\Gamma_{\ell,z}}h_\ell|^2\partial_z^2\overline{\mathbb{H}}_{\varepsilon,\ell} \\
&\quad + \sum_{m\neq\ell}\left(\varepsilon\mathcal{R}_{m,1}((Z_{\Gamma_m})_*\partial_z)\overline{\mathbb{H}}_{\varepsilon,m} - \varepsilon^2\mathcal{R}_{m,2}((Z_{\Gamma_m})_*\partial_z)^2\overline{\mathbb{H}}_{\varepsilon,m}\right) \\
&\quad - \sum_{m=1}^{Q}\overline{\xi}((-1)^{m-1}\varepsilon^{-1}(d_m - h_m\circ\Pi_m)).
\end{aligned} \tag{B.1}$$

Note the slight differences in signs relative to [WW19a, (9.4)], which arise from our different sign convention on the mean curvature and our choice to avoid introducing extraneous notation for $(Z_{\Gamma_m})_*\partial_z$ derivatives of $\overline{\mathbb{H}}_{\varepsilon,m}$ that introduce factors of $(-1)^m$ for $m=\ell$ or $m\neq\ell$ (cf. g'_α, g''_α in [WW19a, §9]). Above, we have written

$$\begin{aligned}
\mathcal{R}(\phi) &\triangleq W'(U[\mathbf{h}]+\phi) - W'(U[\mathbf{h}]) - W''(U[\mathbf{h}])\phi = O(\phi^2), \\
((Z_{\Gamma_m})^*\mathcal{R}_{m,1})(y',z') &\triangleq \varepsilon(\Delta_{\Gamma_{m,z'}}h_m(y') - H_{\Gamma_{m,z'}}(y')), \\
((Z_{\Gamma_m})^*\mathcal{R}_{m,2})(y',z') &\triangleq |\nabla_{\Gamma_{m,z'}}h_m(y')|^2
\end{aligned}$$

as well as (cf. (2.15))

$$\overline{\xi}(t) \triangleq \overline{\mathbb{H}}''(t) - W'(\overline{\mathbb{H}}(t)). \tag{B.2}$$

It is useful to remember that the terms involving $\mathcal{R}_{m,1}$, $\mathcal{R}_{m,2}$ in (B.1) vanish when $d_m > 6\varepsilon|\log\varepsilon|$.

Lemma B.2 (cf. [WW19a, Lemma A.1]). *For $\kappa > 0$, we have*

$$\int_{-\infty}^{\infty}(W''(\mathbb{H}(t)) - 2)\mathbb{H}'(t-T)\mathbb{H}'(t)\,dt = -4\sqrt{2}(A_0)^2e^{-\sqrt{2}T} + O(e^{-2(1-\kappa)\sqrt{2}T})$$

as $T\to\infty$.

Proof. Let us denote the left-hand side as $I(T)$. Recall that $W(\pm 1) = 2$. We can rewrite

$$\begin{aligned} I(T) = \sqrt{2}A_0 e^{-\sqrt{2}T} & \int_{-\infty}^{\infty} (W''(\mathbb{H}(t)) - 2) e^{\sqrt{2}t} \mathbb{H}'(t)\, dt \\ & + \int_{-\infty}^{\infty} (W''(\mathbb{H}(t)) - 2)(\mathbb{H}'(t-T) - \sqrt{2}A_0 e^{\sqrt{2}(t-T)}) \mathbb{H}'(t)\, dt \\ = {} & \sqrt{2}A_0 e^{-\sqrt{2}T} \int_{-\infty}^{\infty} (W''(\mathbb{H}(t)) - 2) e^{\sqrt{2}t} \mathbb{H}'(t)\, dt \\ & + O(1) \int_{-\infty}^{\infty} \left| \mathbb{H}'(t-T) - \sqrt{2}A_0 e^{\sqrt{2}(t-T)} \right| \mathbb{H}'(t)^2\, dt, \end{aligned}$$

where we have used (1.5) and (1.6) in the last step.

We can directly evaluate the first integral by writing $W''(\mathbb{H}(t))\mathbb{H}'(t) = \mathbb{H}'''(t)$ and integrating by parts:

$$\begin{aligned} \int_{-L}^{L} W''(\mathbb{H}(t)) e^{\sqrt{2}t} \mathbb{H}'(t) dt &= \int_{-L}^{L} \mathbb{H}'''(t) e^{\sqrt{2}t} dt \\ &= \mathbb{H}''(L) e^{\sqrt{2}L} - \mathbb{H}''(-L) e^{-\sqrt{2}L} - \sqrt{2} \int_{-L}^{L} \mathbb{H}''(t) e^{\sqrt{2}t} dt \\ &= \mathbb{H}''(L) e^{\sqrt{2}L} - \mathbb{H}''(-L) e^{-\sqrt{2}L} - \sqrt{2}\mathbb{H}'(L) e^{\sqrt{2}L} \\ &\quad + \sqrt{2}\mathbb{H}'(-L) e^{-\sqrt{2}L} + 2 \int_{-L}^{L} \mathbb{H}'(t) e^{\sqrt{2}t} dt. \end{aligned}$$

Recalling (1.6) and (1.7), sending $L \to \infty$ gives

$$\int_{-\infty}^{\infty} \left(W''(\mathbb{H}(t)) - 2 \right) e^{\sqrt{2}t} \mathbb{H}'(t)\, dt = -4A_0.$$

Plugging this into the expression for $I(T)$, we have

(B.3)
$$I(T) = -4\sqrt{2}(A_0)^2 e^{-\sqrt{2}T} + O(1) \int_{-\infty}^{\infty} (\mathbb{H}'(t-T) - \sqrt{2}A_0 e^{\sqrt{2}(t-T)}) \mathbb{H}'(t)^2\, dt.$$

It remains to show that the last integral is $O(e^{-2(1-\kappa)\sqrt{2}T})$. Let $\alpha \in (0,1)$ be fixed. When $t \in (-\infty, \alpha T)$,

$$\mathbb{H}'(t-T) - \sqrt{2}A_0 e^{\sqrt{2}(t-T)} = O(e^{2\sqrt{2}(t-T)}),$$

by (1.6), so

$$\int_{-\infty}^{\alpha T} (\mathbb{H}'(t-T) - \sqrt{2}A_0 e^{\sqrt{2}(t-T)}) \mathbb{H}'(t)^2\, dt = O(T e^{-2\sqrt{2}T}). \tag{B.4}$$

To bound the integral over $[\alpha T, \infty)$, it suffices to observe the following bound on its dominant term:

$$\begin{aligned}\text{(B.5)} \quad & \int_{\alpha T}^{\infty} e^{\sqrt{2}(t-T)} \mathbb{H}'(t)^2 \, dt = O(1) \int_{\alpha T}^{\infty} e^{-\sqrt{2}(t+T)} \, dt = O(e^{-\sqrt{2}(1+\alpha)T}) \\ & \implies \int_{\alpha T}^{\infty} (\mathbb{H}'(t-T) - \sqrt{2} A_0 e^{\sqrt{2}(t-T)}) \mathbb{H}'(t)^2 \, dt = O(e^{-\sqrt{2}(1+\alpha)T}).\end{aligned}$$

The result follows by plugging (B.4) and (B.5) into (B.3). □

Appendix C. Proof of Lemma 2.8

We follow the proof of [WW19a, Lemma 9.6], using Lemma 2.7 to gain improved estimates on the error terms. We continue to use the notation of Appendix B.

Fix $\ell \in \{1, \ldots, Q\}$, $y \in \Gamma_\ell(\frac{8}{10})$. In what follows we work in Fermi coordinates with respect to Γ_ℓ. Because $u(y) = 0$, we have

$$\text{(C.1)} \qquad \phi(y,0) = -\overline{\mathbb{H}}((-1)^{\ell-1}\varepsilon^{-1} h_\ell(y)) - \sum_{m \neq \ell} \left(\overline{\mathbb{H}}_{\varepsilon,m}(y,0) \pm (-1)^{m-1} \right),$$

where the "±" is a "−" for $m < \ell$ and "+" for $m > \ell$. This implies the first inequality immediately, using the fact that $|\mathbb{H}(\varepsilon^{-1} h_\ell(y))| \simeq \varepsilon^{-1}|h_\ell(y)|$ by a Taylor expansion.

Differentiating (C.1) once with respect to y, we find that (recalling (2.13))

$$\begin{aligned}\varepsilon \nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y,0) &= -(-1)^{\ell-1} \overline{\mathbb{H}}'((-1)^{\ell-1}\varepsilon^{-1} h_\ell(y)) \nabla_{\Gamma_\ell} h_\ell(y) \\ &\quad - \varepsilon \sum_{m \neq \ell} \partial_z ((Z_{\Gamma_m})^* \overline{\mathbb{H}}_{\varepsilon,m})(y,0) (\nabla_{\Gamma_\ell} d_m(y,0) \\ &\quad - \nabla_{\Gamma_\ell}(h_m \circ \Pi_m)(y,0)).\end{aligned}$$

Define

$$\mathcal{I}_\ell \triangleq \{ m \in \{1, \ldots, Q\} \ : \ m \neq \ell, \ d_m(y,0) \leq K\varepsilon |\log \varepsilon| \}$$

for $K > 6$ fixed. Then, the exponential decay of $\overline{\mathbb{H}}'$ (and definition of $\overline{\mathbb{H}}$) gives

$$\begin{aligned}& |\nabla_{\Gamma_\ell} h_\ell(y)| \\ & \leq c \Bigg(\varepsilon |\nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y)| \\ & \qquad + \sup_{m \in \mathcal{I}_\ell(y)} (|\nabla_{\Gamma_\ell} d_m(y,0)| + |\nabla_{\Gamma_\ell}(h_m \circ \Pi_m)(y,0)|) \exp(-\sqrt{2}\varepsilon^{-1} D_\ell(y)) \Bigg) \\ & \leq c \left(\varepsilon |\nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y)| + \varepsilon^{\kappa} \exp(-\sqrt{2}\varepsilon^{-1} D_\ell(y)) \right).\end{aligned}$$

We have used Lemma B.1 to bound the first term in the supremum and the bounds from Lemmas 2.3 and 2.7 to bound the second term. (Note that in the proof of Lemma 2.3, the second term was simply bounded by $o(1)$ since at that point Lemma 2.7 was not available.)

Differentiating (C.1) again, we find

$$\begin{aligned}
&\varepsilon^2\nabla^2_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y,0)\\
=&-\varepsilon(-1)^{\ell-1}\overline{\mathbb{H}}'((-1)^{\ell-1}\varepsilon^{-1}h_\ell(y))\nabla^2_{\Gamma_\ell}h_\ell(y)\\
&-\overline{\mathbb{H}}''((-1)^{\ell-1}\varepsilon^{-1}h_\ell(y))\nabla_{\Gamma_\ell}h_\ell(y)\otimes\nabla_{\Gamma_\ell}h_\ell(y)\\
&-\varepsilon^2\sum_{m\neq\ell}\partial_z^2\left((Z_{\Gamma_m})^*\overline{\mathbb{H}}_{\varepsilon,m}\right)(y,0)(\nabla_{\Gamma_\ell}d_m(y,0)-\nabla_{\Gamma_\ell}(h_m\circ\Pi_m)(y,0))\\
&\otimes(\nabla_{\Gamma_\ell}d_m(y,0)-\nabla_{\Gamma_\ell}(h_m\circ\Pi_m)(y,0))\\
&-\varepsilon^2\sum_{m\neq\ell}\partial_z((Z_{\Gamma_m})^*\overline{\mathbb{H}}_{\varepsilon,m})(y,0)(\nabla^2_{\Gamma_\ell}d_m(y,0)-\nabla^2_{\Gamma_\ell}(h_m\circ\Pi_m)(y,0)).
\end{aligned}$$

Because Γ_ℓ, Γ_m have bounded second fundamental form by (2.6), (A.2) shows that

$$|\nabla^2_{\Gamma_\ell}d_m(y,0)|\le c,\ m\in\mathcal{I}_\ell.$$

Thus, we find that, as claimed,

$$\varepsilon|\nabla^2_{\Gamma_\ell}h(y)|\le c\left(\varepsilon^2|\nabla^2_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y)|+\varepsilon^2|\nabla_{\Gamma_\ell}(\phi|_{\Gamma_\ell})(y)|^2+\varepsilon^\kappa\exp(-\sqrt{2}\varepsilon^{-1}D_\ell(y))\right).$$

The Hölder estimate follows similarly, with one important change: we do not know (at this point) that $[\mathrm{I\!I}_{\Gamma_\ell}]_\theta$ is uniformly bounded, and thus cannot conclude that $[\nabla^2_{\Gamma_\ell}d_m(y,0)]_\theta\le c$. Instead we use (2.6) and (2.7) in conjunction with (A.2) and (A.11) to conclude that

$$\varepsilon^\theta[\nabla^2 d_m(y,0)]_\theta\le c,\ m\in\mathcal{I}_\ell.$$

This, combined with the factor of ε in front of the last line suffices to complete the Hölder estimate.

Appendix D. Proof of (3.2)

We follow [WW19a, §19], except we keep track of how the error terms improve upon strengthened sheet separation estimates, as well as keeping track of the constant in front of the main term on the right-hand side of the stability inequality. We assume that $\ell\in\{2,\dots,Q-1\}$; i.e., there are sheets above and below Γ_ℓ. (When $\ell=1$ or Q, the argument is similar.) Similarly, we can assume that

$$(-1)^{\ell-1}=1. \tag{D.1}$$

Here, and throughout this appendix, we will write $\mathcal{E}_\zeta$ for any term which is bounded as follows:

$$|\mathcal{E}_\zeta| \le c'\varepsilon^2 + c' \sum_{m=1}^{Q} \sup\Big\{ \exp(-\sqrt{2}(1+\kappa)\varepsilon^{-1}D_m(y')) : \\ y' \in \Gamma_m \cap \Pi_\ell^{-1}(B^{n-1}_{2K\varepsilon|\log\varepsilon|}(\operatorname{spt}\zeta))\Big\} \tag{D.2}$$

for some $\kappa > 0$ fixed throughout sufficiently small. We emphasize that the constant c' is uniform in ε sufficiently small. Here, ζ is just the test function from the statement of (3.2).

We emphasize that Lemma 2.7 holds, so by (2.20) and Lemma 2.3,

$$\sum_{m=1}^{Q} \|\phi\|_{C^{2,\theta}_\varepsilon(\mathcal{M}_m(r))} + \varepsilon\|\Delta_{\Gamma_m} h_m - H_{\Gamma_m}\|_{C^{0,\theta}_\varepsilon(\Gamma_m(r))} \\ + \varepsilon^{-1}\|h_m\|_{C^{2,\theta}_\varepsilon(\Gamma_m(r))} \le c'\varepsilon^2 + c'\sum_{m=1}^{Q} A_m(r + K\varepsilon|\log\varepsilon|) \le c'\varepsilon, \tag{D.3}$$

and the improved estimate on the tangential derivatives of ϕ from (2.21), which we will write as

$$\varepsilon\|(Z_{\Gamma_\ell})_*\partial_{y_i}\phi\|_{C^{1,\theta}_\varepsilon(\mathcal{M}_\ell(r))} \le c'\varepsilon^2 + c'\sum_{m=1}^{Q} A_m(r + 2K\varepsilon|\log\varepsilon|)^{1+\kappa} \\ + c'\varepsilon^\kappa \sum_{m=1}^{Q} A_m(r + 2K\varepsilon|\log\varepsilon|) \le c'\varepsilon^{1+\kappa}. \tag{D.4}$$

In fact, we will often use the localized version of (D.3) and (D.4) on $\mathcal{M}_\ell(1) \cap \Pi_\ell^{-1}(\operatorname{spt}\zeta)$:

$$\|\phi\|_{C^{2,\theta}_\epsilon} + \varepsilon\|\Delta_{\Gamma_\ell} h_\ell - H_{\Gamma_\ell}\|_{C^{0,\theta}_\varepsilon} + \varepsilon^{-1}\|h_\ell\|_{C^{2,\theta}_\varepsilon} \\ \le c'\varepsilon^2 + c'\sum_{m=1}^{Q} \sup\left\{\exp(-\sqrt{2}\varepsilon^{-1}D_m(\cdot))\right\} \tag{D.5}$$

and

$$\varepsilon\|(Z_{\Gamma_\ell})_*\partial_{y_i}\phi\|_{C^{1,\theta}_\varepsilon} \le O(|\mathcal{E}_\zeta|); \tag{D.6}$$

the Hölder norms are taken over $\mathcal{M}_\ell(1)\cap\Pi_\ell^{-1}(\operatorname{spt}\zeta)$, $\Gamma_\ell(1)\cap\Pi_\ell^{-1}(\operatorname{spt}\zeta)$, $\Gamma_\ell(1)\cap \Pi_\ell^{-1}(\operatorname{spt}\zeta)$ for (D.5) and over $\mathcal{M}_\ell(1)\cap\Pi_\ell^{-1}(\operatorname{spt}\zeta)$ for (D.6), and the sup is over $\mathcal{M}_m(1) \cap \Pi_\ell^{-1}(B^{n-1}_{2K\varepsilon|\log\varepsilon|}(\operatorname{spt}\zeta))$. Note how (D.5) and (D.6) imply (D.3) and (D.4) by Lemma 2.7.

We will write $\overline{\mathbb{H}}$ for $\overline{\mathbb{H}}^{3|\log\varepsilon|}$ throughout this appendix, where $\overline{\mathbb{H}}^{3|\log\varepsilon|}$ is as in (2.14) with $\Lambda = 3|\log\varepsilon|$. We also recall the definition of $\overline{\xi}$ from (B.2). We then define the following functions by their expression in Γ_m Fermi coordinates:

$$((Z_{\Gamma_m})^*\overline{\mathbb{H}}_{\varepsilon,m})(y,z) \triangleq \overline{\mathbb{H}}((-1)^{m-1}\varepsilon^{-1}(z - h_m(y))),$$
$$((Z_{\Gamma_m})^*\overline{\xi}_{\varepsilon,m})(y,z) \triangleq \overline{\xi}((-1)^{m-1}\varepsilon^{-1}(z - h_m(y))).$$

Recall that $Z_{\Gamma_m}(y,z)$ is the point (y,z) in Fermi coordinates over Γ_m (see the definition after (2.11)), and that $g = dz^2 + g_z$ in Fermi coordinates. Recall also (B.2) and (2.15).

Choose functions $\rho_\ell^\pm(y) = \frac{1}{2} f_{\ell,\ell\pm 1}(y)$, where we recall $\Gamma_{\ell\pm 1}$ is the normal graph of $f_{\ell,\ell\pm 1}$ over Γ_ℓ. Note that $\rho_\ell^\pm$ is thus uniformly bounded in $C^1(\Gamma_\ell(\frac{8}{10}))$ by (2.6)–(2.7). We consider a vertical cutoff function $\chi(y,z)$ defined by

$$\chi(y,z) \triangleq \tilde\chi\left(\varepsilon^{-1}L^{-1}(z-\rho_\ell^+(y))\right)\tilde\chi\left(\varepsilon^{-1}L^{-1}(\rho_\ell^-(y)-z)\right),$$

where $\tilde\chi$ is a smooth function with $\tilde\chi(t) = 1$ for $t \in (-\infty,-1)$ and $\operatorname{spt}\tilde\chi \subset (-\infty,0)$. We will fix $L > 0$ sufficiently large (independent of $\varepsilon > 0$ small) below. Note that

$$\varepsilon L|\nabla\chi| \le c \tag{D.7}$$

and for fixed y,

$$\operatorname{spt}|\partial_z\chi(y,\cdot)| \subset [\rho_\ell^-(y), \rho_\ell^-(y)+\varepsilon L] \cup [\rho_\ell^+(y)-\varepsilon L, \rho_\ell^+(y)]. \tag{D.8}$$

We will frequently use the observation that on $\operatorname{spt}|\nabla\chi| \cap \{\pm z > 0\}$,

(D.9)
$$\begin{aligned}\varepsilon^k|\partial_z^k\overline{\mathbb{H}}_{\varepsilon,\ell}(y,z)| &\le c'\exp(-\sqrt{2}\varepsilon^{-1}\rho_\ell^\pm(y))\\ &\le c'\varepsilon^2 + c'\sup\left\{\exp(-\tfrac{1}{2}\sqrt{2}\varepsilon^{-1}D_\ell(y')) : y' \in B^{n-1}_{\varepsilon|\log\varepsilon|}(y)\right\}\end{aligned}$$

for integers $k \ge 1$ (we used Lemma B.1 in the last step), as well as the fact that on $\operatorname{spt}\chi$, we have

$$|\partial_z d\mu_{g_z}| + |\partial_z^2 d\mu_{g_z}| = O(1)d\mu_{g_z}, \tag{D.10}$$

which follows from (2.6), (A.1), (A.2), and (A.3). Moreover, we note for future reference that the following expression holds on $\operatorname{spt}\chi$,

$$\begin{aligned}u = \overline{\mathbb{H}}_{\varepsilon,\ell} + \phi + \sum_{m<\ell}\left(\overline{\mathbb{H}}_{\varepsilon,m} - (-1)^{m-1}\right)\\ + \sum_{m>\ell}\left(\overline{\mathbb{H}}_{\varepsilon,m} + (-1)^{m-1}\right) = \overline{\mathbb{H}}_{\varepsilon,\ell} + \phi + \sum_{m\ne\ell} O(\varepsilon(\partial_z\overline{\mathbb{H}}_{\varepsilon,m})),\end{aligned}$$

so

$$W''(u) = W''(\overline{\mathbb{H}}_{\varepsilon,\ell}) + O(\phi) + \sum_{m\ne\ell} O(\varepsilon(\partial_z\overline{\mathbb{H}}_{\varepsilon,m})). \tag{D.11}$$

Let us set $\varphi(y,z) \triangleq \zeta(y)\chi(y,z)(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell}(y,z))$. Because u is stable,

$$\int_{C_{8/10}(0)} (\varepsilon|\nabla\varphi|^2 + \varepsilon^{-1}W''(u)\varphi^2)\,d\mu_g \ge 0.$$

We will write this integral in Fermi coordinates over Γ_ℓ and expand using the choice of φ. Note that

$$|\nabla\varphi|^2 = (\partial_z\varphi)^2 + |\nabla_{\Gamma_{\ell,z}}\varphi|^2.$$

We begin with the contribution of the vertical derivative, $\partial_z\varphi = \zeta(\partial_z\chi)(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell}) + \zeta\chi(\partial_z^2\overline{\mathbb{H}}_{\varepsilon,\ell})$, to stability:

$$\begin{aligned}
\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon(\partial_z\varphi)^2\,d\mu_{g_z}\,dz &= \varepsilon\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2(\partial_z^2\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,d\mu_{g_z}\,dz \\
&\quad + \varepsilon\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2(\partial_z\chi)^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,d\mu_{g_z}\,dz \\
&\quad + 2\varepsilon\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2(\partial_z\chi)\chi(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})(\partial_z^2\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz \\
&= -\varepsilon^{-1}\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2 W''(\overline{\mathbb{H}}_{\varepsilon,\ell})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,d\mu_{g_z}\,dz \\
&\quad + \varepsilon\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2(\partial_z\chi)^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,d\mu_{g_z}\,dz \\
&\quad - \varepsilon^{-1}\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2(\partial_z\overline{\xi}_{\varepsilon,\ell})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz \\
&\quad - \varepsilon\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})(\partial_z^2\overline{\mathbb{H}}_{\varepsilon,\ell})\,(\partial_z d\mu_{g_z})\,dz \\
&= -\varepsilon^{-1}\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2 W''(\overline{\mathbb{H}}_{\varepsilon,\ell})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,d\mu_{g_z}\,dz \\
&\quad + \varepsilon\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2(\partial_z\chi)^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,d\mu_{g_z}\,dz \\
&\quad - \varepsilon^{-1}\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2(\partial_z\overline{\xi}_{\varepsilon,\ell})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz \\
&\quad + \varepsilon\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi(\partial_z\chi)(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,(\partial_z d\mu_{g_z})\,dz \\
&\quad + \tfrac{1}{2}\varepsilon\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,(\partial_z^2 d\mu_{g_z})\,dz,
\end{aligned}$$

where we integrated by parts on the final term after the first equality and the second equality. Using (2.15), (D.7), (D.8), (D.9), and (D.10), we find that

$$\begin{aligned}
&\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon(\partial_z\varphi)^2\,d\mu_{g_z}\,dz \\
&\quad = -\varepsilon^{-1}\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2 W''(\overline{\mathbb{H}}_{\varepsilon,\ell})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,d\mu_{g_z}\,dz \\
&\qquad + O(\varepsilon^{-2}L^{-1} + \varepsilon^{-1})\int_{\Gamma_\ell} \zeta^2\left(\exp(-2\sqrt{2}\varepsilon^{-1}\rho_\ell^+) + \exp(-2\sqrt{2}\varepsilon^{-1}\rho_\ell^-)\right)\,d\mu_{\Gamma_\ell} \\
&\qquad + O(1)\int_{\Gamma_\ell} \zeta^2\,d\mu_{\Gamma_\ell}.
\end{aligned}$$

We now turn to the second term. We use Cauchy–Schwartz to estimate the mixed terms with a factor of $L^{-1/2}$ and $L^{1/2}$ respectively, in the first inequality below:

$$\begin{aligned}
&\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon|\nabla_{\Gamma_{\ell,z}}\varphi|^2\,d\mu_{g_z}\,dz\\
&\leq (1+O(L^{-\frac{1}{2}}))\cdot\varepsilon\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}}\left(|\nabla_{\Gamma_{\ell,z}}\zeta|^2\chi^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\right)d\mu_{g_z}\,dz\\
&\quad+O(L^{\frac{1}{2}})\cdot\varepsilon\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}}\left(\zeta^2|\nabla\chi|^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2+\zeta^2\chi^2(\partial_z^2\overline{\mathbb{H}}_{\varepsilon,\ell})^2|\nabla_{\Gamma_{\ell,z}}h_\ell|^2\right)d\mu_{g_z}\,dz\\
&=(1+O(L^{-\frac{1}{2}}))\cdot h_0\int_{\Gamma_\ell}|\nabla_{\Gamma_\ell}\zeta|^2\,d\mu_{\Gamma_\ell}\\
&\quad+O(\varepsilon^{-2}L^{-\frac{1}{2}})\int_{\Gamma_\ell}\zeta^2\left(\exp(-2\sqrt{2}\varepsilon^{-1}\rho_\ell^+)+\exp(-2\sqrt{2}\varepsilon^{-1}\rho_\ell^-)\right)d\mu_{\Gamma_\ell}\\
&\quad+O(L^{\frac{1}{2}})\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell}.
\end{aligned}$$

We have used (A.1), (A.2), (A.3), (D.7), (D.8), (D.9), and (D.10).

Putting these two computations together and multiplying by ε^2, the stability condition becomes

$$\begin{aligned}
&(1+O(L^{-\frac{1}{2}}))\cdot\varepsilon^2 h_0\int_{\Gamma_\ell}|\nabla_{\Gamma_\ell}\zeta|^2\,d\mu_\Gamma\\
&\geq\varepsilon\cdot\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}}\zeta^2\chi^2(W''(\overline{\mathbb{H}}_{\varepsilon,\ell})-W''(u))(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,d\mu_{g_z}\,dz\\
&\quad+O(L^{-\frac{1}{2}}+\varepsilon)\int_{\Gamma_\ell}\zeta^2\left(\exp(-2\sqrt{2}\varepsilon^{-1}\rho_\ell^+)+\exp(-2\sqrt{2}\varepsilon^{-1}\rho_\ell^-)\right)d\mu_{\Gamma_\ell}\\
&\quad+O(L^{\frac{1}{2}}\varepsilon^2)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell}.
\end{aligned}\tag{D.12}$$

The first term of the right-hand side represents the interaction between the sheets and requires further consideration. To this end, we rewrite (B.1) slightly, using the definition of $\mathcal{R}(\phi)$:

$$\begin{aligned}
&\varepsilon^2(\Delta_{\Gamma_{\ell,z}}\phi + H_{\Gamma_{\ell,z}}\partial_z\phi + \partial_z^2\phi)\\
&\quad = W'(u) - \sum_{m=1}^{Q} W'(\overline{\mathbb{H}}_{\varepsilon,m})\\
&\qquad + \varepsilon^2(\Delta_{\Gamma_{\ell,z}}h_\ell - H_{\Gamma_{\ell,z}})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell}) - \varepsilon^2|\nabla_{\Gamma_{\ell,z}}h_\ell|^2(\partial_z^2\overline{\mathbb{H}}_{\varepsilon,\ell})\\
&\qquad + \sum_{m\neq\ell}\big(\varepsilon\mathcal{R}_{m,1}((Z_{\Gamma_m})_*\partial_z)\overline{\mathbb{H}}_{\varepsilon,m}) - \varepsilon^2\mathcal{R}_{m,2}((Z_{\Gamma_m})_*\partial_z)^2\overline{\mathbb{H}}_{\varepsilon,m})\big) - \sum_{m=1}^{Q}\overline{\xi}_{\varepsilon,m}.
\end{aligned}$$

We then differentiate this with respect to z to obtain

$$\begin{aligned}
&\varepsilon^2(\partial_z\Delta_{\Gamma_{\ell,z}}\phi + \partial_z(H_{\Gamma_{\ell,z}}\partial_z\phi) + \partial_z^3\phi)\\
&= W''(u)(\partial_z\phi) + \big(W''(u) - W''(\overline{\mathbb{H}}_{\varepsilon,\ell})\big)(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\\
&\quad + \sum_{m\neq\ell}\big(W''(u) - W''(\overline{\mathbb{H}}_{\varepsilon,m})\big)\partial_z\overline{\mathbb{H}}_{\varepsilon,m}\\
&\quad + \varepsilon^2\partial_z\left((\Delta_{\Gamma_{\ell,z}}h_\ell - H_{\Gamma_{\ell,z}})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\right) - \varepsilon^2\partial_z\left(|\nabla_{\Gamma_{\ell,z}}h_\ell|^2(\partial_z^2\overline{\mathbb{H}}_{\varepsilon,\ell})\right)\\
&\quad + \sum_{m\neq\ell}\left(\varepsilon\partial_z\big(\mathcal{R}_{m,1}((Z_{\Gamma_m})_*\partial_z)\overline{\mathbb{H}}_{\varepsilon,m}\big) - \varepsilon^2\partial_z\big(\mathcal{R}_{m,2}((Z_{\Gamma_m})_*\partial_z)^2\overline{\mathbb{H}}_{\varepsilon,m})\big)\right)\\
&\quad - \sum_{m=1}^{Q}\partial_z\overline{\xi}_{\varepsilon,m}.
\end{aligned}\tag{D.13}$$

We multiply this by $\zeta(y)^2\chi(y,z)^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell}(y,z))$, integrate in (y,z), and estimate each term. The first term on the left-hand side of (D.13) yields

$$\begin{aligned}
&\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}}\varepsilon^2(\partial_z\Delta_{\Gamma_{\ell,z}}\phi)\zeta^2\chi^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz\\
&\qquad = -\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}}\varepsilon^2(\Delta_{\Gamma_{\ell,z}}\phi)\partial_z(\zeta^2\chi^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z})\,dz\\
&\qquad = O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell}.
\end{aligned}\tag{D.14}$$

Here, we have bounded $\varepsilon^2\Delta_{\Gamma_{\ell,z}}\phi$ by (D.6) and the remaining terms using (2.6), (A.1), (A.3), (D.7), (D.8), (D.9), and (D.10). Continuing on, the second term on the left-hand side of (D.13) can be estimated similarly as

$$\begin{aligned}
&\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}}\varepsilon^2\partial_z(H_{\Gamma_{\ell,z}}\partial_z\phi)\zeta^2\chi^2\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell}\,d\mu_{g_z}\,dz\\
&\qquad = -\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}}\varepsilon^2H_{\Gamma_{\ell,z}}\partial_z\phi\,\partial_z(\zeta^2\chi^2\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell}\,d\mu_{g_z})\,dz\\
&\qquad = O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2 d\mu_{\Gamma_\ell}.
\end{aligned}$$

We now consider the third term on the left-hand side of (D.13). It is *not* an error term, but instead will cancel (up to error terms) with the first term on the right-hand side:

(D.15)

$$
\begin{aligned}
&\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon^2(\partial_z^3\phi)\zeta^2\chi^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz\\
&= -\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon^2(\partial_z^2\phi)\zeta^2\chi^2(\partial_z^2\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz\\
&\quad -\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon^2(\partial_z^2\phi)\zeta^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\partial_z(\chi^2\,d\mu_{g_z})\,dz\\
&= \int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon^2(\partial_z\phi)\zeta^2\chi^2(\partial_z^3\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz\\
&\quad +\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon^2(\partial_z\phi)\zeta^2(\partial_z^2\overline{\mathbb{H}}_{\varepsilon,\ell})\partial_z(\chi^2\,d\mu_{g_z})\,dz\\
&\quad -\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon^2(\partial_z^2\phi)\zeta^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\partial_z(\chi^2\,d\mu_{g_z})\,dz\\
&= \int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} (\partial_z\phi)\zeta^2\chi^2 W''(\overline{\mathbb{H}}_{\varepsilon,\ell})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz + O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell}\\
&= \int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} (\partial_z\phi)\zeta^2\chi^2 W''(u)(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz\\
&\quad +\int_{-\eta}^{\eta}\int_{\Gamma_{\ell}} (\partial_z\phi)\zeta^2\chi^2 (W''(\overline{\mathbb{H}}_{\varepsilon,\ell})-W''(u))(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz\\
&\quad + O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell}.
\end{aligned}
$$

We have used (D.5), (D.7), (D.8), (D.9), (D.10), (A.1), and (A.3). The first term of the final expression above cancels with the first term on the right-hand side of (D.13). We now study the second term of (D.15). Using (D.11), the second term of the right-hand side of (D.15) can be rewritten as

(D.16)

$$
\begin{aligned}
&\int_{-\eta}^{\eta}\int_{\Gamma_{\ell_z}} (\partial_z\phi)\zeta^2\chi^2 (W''(\overline{\mathbb{H}}_{\varepsilon,\ell})-W''(u))(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz\\
&= O(\varepsilon)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell} + O(1)\sum_{m\neq\ell}\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}}\chi^2\zeta^2\cdot\varepsilon|\partial_z\overline{\mathbb{H}}_{\varepsilon,m}|(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz\\
&= O(\varepsilon)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell} + O\left(\varepsilon^{-1}\exp(-\sqrt{2}\varepsilon^{-1}D_\ell(\cdot)\right)\sum_{m\neq\ell}\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}}\chi^2\zeta^2\,d\mu_{g_z}\,dz\\
&= O\left(\varepsilon + \sup_{\operatorname{spt}\zeta}\left[\varepsilon^{-1}D_\ell(\cdot)\exp(-\sqrt{2}\varepsilon^{-1}D_\ell(\cdot))\right]\right)\int_{\Gamma_\ell}\zeta^2 d\mu_{\Gamma_\ell}\\
&= O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2 d\mu_{\Gamma_\ell}.
\end{aligned}
$$

Note that we estimated $|(\partial_z\overline{\mathbb{H}}_{\varepsilon,m})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})|$ using (1.6) and Lemma B.1:

$$\varepsilon^2|(\partial_z\overline{\mathbb{H}}_{\varepsilon,m})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})|(y,z) \le c'\varepsilon^2 + c'\exp\left(-\sqrt{2}\varepsilon^{-1}D_\ell(y)\right),\ m \neq \ell. \tag{D.17}$$

We continue estimating terms on the right-hand side in (D.13).

We have just seen that the first term on the right-hand side will cancel with a term of (D.15). The second term of (D.13) is the term we are interested in estimating. We now consider the third term of (D.13). For $m \neq \ell$, we note that on $\operatorname{spt}\chi$,

$$W''(\overline{\mathbb{H}}_{\varepsilon,m}) = W''(\pm 1) + O(\varepsilon(\partial_z\overline{\mathbb{H}}_{\varepsilon,m})).$$

Thus, combined with (D.11), we find that on $\operatorname{spt}\chi$,

$$W''(u) - W''(\overline{\mathbb{H}}_{\varepsilon,m}) = W''(\overline{\mathbb{H}}_{\varepsilon,\ell}) - W''(\pm 1) + O(\varepsilon) + \sum_{m'\neq\ell} O(\varepsilon(\partial_z\overline{\mathbb{H}}_{\varepsilon,m'})).$$

Hence, using Lemma B.1, (D.3), and bounding $|(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})(\partial_z\overline{\mathbb{H}}_{\varepsilon,m})(\partial_z\overline{\mathbb{H}}_{\varepsilon,m'})|$ as in (D.17),

$$\begin{aligned}
&\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \left(W''(u) - W''(\overline{\mathbb{H}}_{\varepsilon,m})\right) \\
&\qquad \times (\partial_z\overline{\mathbb{H}}_{\varepsilon,m})(\partial_z d_m + O(|\nabla_{\Gamma_m} h_m|))\zeta^2\chi^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz \\
&\quad = \int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2\left(W''(\overline{\mathbb{H}}_{\varepsilon,\ell}) - W''(\pm 1)\right)(\partial_z\overline{\mathbb{H}}_{\varepsilon,m})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz \\
&\qquad + O\Bigg(\sup_{\operatorname{spt}\zeta}\Big[\varepsilon^{-1}D_\ell(\cdot)\exp(-\sqrt{2}\varepsilon^{-1}D_\ell(\cdot)) \\
&\qquad\qquad + \varepsilon^{-2}D_\ell(\cdot)\exp(-\tfrac{3}{2}\sqrt{2}\varepsilon^{-1}D_\ell(\cdot))\Big]\Bigg)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell} \\
&\quad = \int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2\left(W''(\overline{\mathbb{H}}_{\varepsilon,\ell}) - W''(\pm 1))\right)(\partial_z\overline{\mathbb{H}}_{\varepsilon,m})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz \\
&\qquad + O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell}.
\end{aligned} \tag{D.18}$$

We now turn to the next term of (D.13). Note that on $\operatorname{spt}\chi$,

$$|\Delta_{\Gamma_{\ell,z}} h_\ell| + |\partial_z\Delta_{\Gamma_{\ell,z}} h_\ell| + |H_{\Gamma_{\ell,z}}| + |\partial_z H_{\Gamma_{\ell,z}}| \le c'$$

by (D.3), (2.6), (A.3), (A.4), and (A.7). Thus,

$$\begin{aligned}
&\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon^2\partial_z\left((\Delta_{\Gamma_{\ell,z}} h_\ell - H_{\Gamma_{\ell,z}})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\right)\zeta^2\chi^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz \\
&\qquad\qquad = O(\varepsilon)\int_{\Gamma_\ell}\zeta(y)^2 d\mu_\Gamma.
\end{aligned} \tag{D.19}$$

The next term of (D.13) is estimated similarly.

The term of (D.13) involving $\mathcal{R}_{m,1}$ is estimated by an integration by parts as follows. First, recall the definition of $\mathcal{R}_{m,1}$ from Appendix B and note that (D.5) implies that

$$|\mathcal{R}_{m,1}| \leq c'\varepsilon^2 + c' \sum_{m=1}^{Q} \sup\left\{\exp(-\sqrt{2}\varepsilon^{-1} D_m(\cdot))\right\},$$

with the sup taken as in (D.5). This bound thus implies

$$\begin{aligned}
&\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon\partial_z\left(\mathcal{R}_{m,1}((Z_{\Gamma_m})_*\partial_z)\overline{\mathbb{H}}_{\varepsilon,m}\right)\zeta^2\chi^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz \\
&= -\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \varepsilon\mathcal{R}_{m,1}\big(((Z_{\Gamma_m})_*\partial_z)\overline{\mathbb{H}}_{\varepsilon,m}\big)\zeta^2\partial_z\left(\chi^2(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\right)dz \\
&= O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2\,d\mu_\Gamma,
\end{aligned} \tag{D.20}$$

where we additionally used Lemma B.1, (D.7), (D.8), (D.10), and (D.17). The terms in (D.13) involving $\mathcal{R}_{m,2}$, $\overline{\xi}_{\varepsilon,m}$, are estimated similarly.

Plugging (D.14), (D.15), (D.15), (D.18), (D.19), and (D.20) into (D.13), we find that

$$\begin{aligned}
&\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2(W''(u)-W''(\overline{\mathbb{H}}_{\varepsilon,\ell}))(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,d\mu_{g_z}\,dz \\
&\quad= \sum_{m\neq\ell}\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2\left(W''(\pm1)-W''(\overline{\mathbb{H}}_{\varepsilon,\ell})\right)(\partial_z\overline{\mathbb{H}}_{\varepsilon,m})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz \\
&\qquad+ O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell}.
\end{aligned}$$

Observe that for $m \notin \{\ell-1,\ell+1\}$,

$$\varepsilon^2|(\partial_z\overline{\mathbb{H}}_{\varepsilon,m})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})|(y,z) = O(|\mathcal{E}_\zeta|^{\frac{2}{1+\kappa}})$$

on $\operatorname{spt}\chi$. Thus, we can write (because $(-1)^{\ell\pm1} = -1$ by (D.1))

$$\begin{aligned}
&\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2(W''(u)-W''(\overline{\mathbb{H}}_{\varepsilon,\ell}))(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})^2\,d\mu_{g_z}\,dz \\
&= \sum_{m\in\{\ell\pm1\}}\int_{-\eta}^{\eta}\int_{\Gamma_{\ell,z}} \zeta^2\chi^2\left(W''(\overline{\mathbb{H}}_{\varepsilon,\ell})-W''(\pm1))\right)(\partial_z\overline{\mathbb{H}}_{\varepsilon,m})(\partial_z\overline{\mathbb{H}}_{\varepsilon,\ell})\,d\mu_{g_z}\,dz \\
&\quad+ O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell} \\
&= \varepsilon^{-2}\int_{\Gamma_\ell}\zeta^2\left(\int_{-\eta}^{\eta}\big(W''(\overline{\mathbb{H}}(\varepsilon^{-1}t))-W''(\pm1)\big)\,\overline{\mathbb{H}}'(-\varepsilon^{-1}(d_{\ell+1}(y)-t))\overline{\mathbb{H}}'(\varepsilon^{-1}t)\,dt\right)d\mu_{\Gamma_\ell} \\
&\quad+ \varepsilon^{-2}\int_{\Gamma_\ell}\zeta^2\left(\int_{-\eta}^{\eta}\big(W''(\overline{\mathbb{H}}(\varepsilon^{-1}t))-W''(\pm1)\big)\,\overline{\mathbb{H}}'(\varepsilon^{-1}(t+d_{\ell-1}(y)))\overline{\mathbb{H}}'(\varepsilon^{-1}t)\,dt\right)d\mu_{\Gamma_\ell}
\end{aligned}$$

$$
\begin{aligned}
&+ O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell}\\
&= \varepsilon^{-1}\int_{\Gamma_\ell}\zeta^2\Big(\int_{-\infty}^{\infty}(W''(\mathbb{H}(t))-2)\,\mathbb{H}'(t-\varepsilon^{-1}|d_{\ell+1}(y)|)\mathbb{H}'(t)dt\Big)\,d\mu_{\Gamma_\ell}\\
&\quad+\varepsilon^{-1}\int_{\Gamma_\ell}\zeta^2\Big(\int_{-\infty}^{\infty}(W''(\mathbb{H}(t))-2)\,\mathbb{H}'(t-\varepsilon^{-1}|d_{\ell-1}(y))|)\mathbb{H}'(t)dt\Big)\,d\mu_{\Gamma_\ell}\\
&\quad+O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell}\\
&= -4\sqrt{2}(A_0)^2\cdot\varepsilon^{-1}\int_{\Gamma_\ell}\zeta^2\Big(\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+1}|)+\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-1}|)\Big)\,d\mu_{\Gamma_\ell}\\
&\quad+O(\varepsilon^{-1})\int_{\Gamma_\ell}\zeta^2\Big(\exp(-2(1-\kappa)\sqrt{2}\varepsilon^{-1}|d_{\ell+1}|)+\exp(-2(1-\kappa)\sqrt{2}\varepsilon^{-1}|d_{\ell-1}|)\Big)\,d\mu_{\Gamma_\ell}\\
&\quad+O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell}\\
&= -(4\sqrt{2}(A_0)^2+o(1))\cdot\varepsilon^{-1}\int_{\Gamma_\ell}\Big(\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell+1}|)+\exp(-\sqrt{2}\varepsilon^{-1}|d_{\ell-1}|)\Big)\,\zeta^2\,d\mu_{\Gamma_\ell}\\
&\quad+O(\varepsilon^{-1}|\mathcal{E}_\zeta|)\int_{\Gamma_\ell}\zeta^2\,d\mu_{\Gamma_\ell}.
\end{aligned}
$$

In the last two equalities we used Lemma B.2. Together with (D.12) and Lemma B.1, we get (3.2).

Appendix E. An interpolation lemma

We record a proof of the following interpolation inequality:

Lemma E.1. *For $0<\alpha<\theta<1$ and $f:\mathbf{R}^n\to\mathbf{R}$, we have*

$$\|\nabla f\|_{C^{0,\alpha}(\mathbf{R}^n)}\le C\|f\|^{\theta-\alpha}_{C^{0,\theta}(\mathbf{R}^n)}\|\nabla f\|^{1+\alpha-\theta}_{C^{0,\theta}(\mathbf{R}^n)},$$

with $C=C(n)$.

Proof. We assume $\nabla f\not\equiv 0$. Fix $\mathbf{x}\in\mathbf{R}^n$ with $\nabla f(\mathbf{x})\neq 0$, and set $\mathbf{e}:=\nabla f(\mathbf{x})/|\nabla f(\mathbf{x})|$. For $t>0$:

$$
\begin{aligned}
f(\mathbf{x}+t\mathbf{e})-f(\mathbf{x}) &= \int_0^1 \nabla f(\mathbf{x}+st\mathbf{e})\cdot t\mathbf{e}\,ds\\
&= \int_0^1(\nabla f(\mathbf{x}+st\mathbf{e})-\nabla f(\mathbf{x}))\cdot t\mathbf{e}\,ds+\nabla f(\mathbf{x})\cdot t\mathbf{e}\\
&= \int_0^1(\nabla f(\mathbf{x}+st\mathbf{e})-\nabla f(\mathbf{x}))\cdot t\mathbf{e}\,ds+t|\nabla f(\mathbf{x})|.
\end{aligned}
$$

Rearranging, and using the Hölder estimate on $f(\mathbf{x}+t\mathbf{e})-f(\mathbf{x})$ and $\nabla f(\mathbf{x}+st\mathbf{e}) - \nabla f(\mathbf{x})$ we deduce

$$t|\nabla f(\mathbf{x})|\le [f]_\theta t^\theta+[\nabla f]_\theta t^{1+\theta}.$$

Dividing through by t and optimizing in t (using calculus) and using the fact that x was arbitrary,

$$\|\nabla f\|_{C^0(\mathbf{R}^n)} \leq 2[f]_\theta^\theta [\nabla f]_\theta^{1-\theta}. \tag{E.1}$$

By the trivial $C^{0,\theta} \hookrightarrow C^{0,\alpha} \hookrightarrow C^0$ interpolation on ∇f and the previous estimate we conclude that

$$\begin{aligned}[\nabla f]_\alpha &\leq C\|\nabla f\|_{C^0(\mathbf{R}^n)}^{\frac{\theta-\alpha}{\theta}}[\nabla f]_\theta^{\frac{\alpha}{\theta}} \\ &\leq 2C[f]_\theta^{\theta-\alpha}[\nabla f]_\theta^{\frac{(1-\theta)(\theta-\alpha)}{\theta}+\frac{\alpha}{\theta}} = 2C[f]_\theta^{\theta-\alpha}[\nabla f]_\theta^{1+\alpha-\theta}.\end{aligned} \tag{E.2}$$

Together, (E.1) and (E.2) give the required estimate when we replace the seminorms by norms. □

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