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Euclidean triangles have no hot spots

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Abstract

We show that a second Neumann eigenfunction u of a Euclidean triangle has at most one (non-vertex) critical point p, and if p exists, then it is a non-degenerate critical point of Morse index 1. Using this we deduce that (1) the extremal values of u are only achieved at a vertex of the triangle, and (2) a generic acute triangle has exactly one (non-vertex) critical point and that each obtuse triangle has no (non-vertex) critical points. This settles the "hot spots" conjecture for triangles in the plane.

1. Introduction

Let Ω be a domain in Euclidean space with Lipschitz boundary. Let Δ denote the (nonnegative) Euclidean Laplacian, $\Delta f := -\partial_x^2 f - \partial_y^2 f$. The second Neumann eigenvalue, μ_2, is the smallest positive number such that there exists a not identically zero, smooth function $u : \Omega \to \mathbb{R}$ that satisfies

$$\Delta u = \mu_2 \cdot u \quad \text{and} \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} \equiv \ 0, \tag{1}$$

where $\frac{\partial}{\partial n}$ denotes the outward pointing unit normal vector field defined at the smooth points of $\partial\Omega$. A function u that satisfies (1) will be called a *second Neumann eigenfunction for* Ω, or simply a μ_2*-eigenfunction*.

One variant of the "hot spots" conjecture, first proposed by J. Rauch at a conference in 1974,[1] asserts that a second Neumann eigenfunction attains its extrema at the boundary. The main result of this paper, Theorem 1.1, implies the hot spots conjecture for triangles in the plane.

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[1] See [Rau75] for a discussion of hot spots.

THEOREM 1.1. *If u is a second Neumann eigenfunction for a Euclidean triangle T, then u has at most one critical point.*[2] *Moreover, if u has a critical point p, then p lies in a side of T and p is a nondegenerate critical point with Morse index equal to* 1.

In Theorem 13.4, we show that if T is a generic acute triangle, then u has exactly one critical point and that if T is an obtuse triangle, then u has no critical points. Earlier, Bañuelos and Burdzy showed that if T is obtuse, then u has no interior maximum and, in particular, the maximum and minimum values of u are achieved at the acute vertices [BnB99]. We extend the latter statement to all triangles (see Theorem 13.1). Unlike [BnB99], our proof of Theorem 1.1 does not rely on probabilistic techniques.

For a brief history and various formulations of the "hot spots" conjecture, we encourage the reader to consult [BnB99]. We provide some highlights. The first positive result towards this conjecture was due to Kawohl [Kaw85], who showed that the conjecture holds for cylinders in any Euclidean space. Burdzy and Werner in [BW99] (and later Burdzy in [Bur05]) showed that the conjecture fails for domains with two (and one) holes. In the paper [Bur05] Burdzy made two separate ("hot spot") conjectures for "convex" and "simply connected" domains. We believe that the conjecture is true for all convex domains in the plane.

The conjecture has been settled for certain convex domains with symmetry. In 1999, under certain technical assumptions, Bañuelos and Burdzy [BnB99] verified the conjecture for domains with a line of symmetry. A year later Jerison and Nadirashvili [JN00] proved the conjecture for domains with two lines of symmetry. In a different direction, building on the work in [BnB99], Atar and Burdzy [AB04] proved the conjecture for *lip domains* (a domain bounded by the graphs of two Lipschitz functions with Lipschitz constant 1). In 2012, the hot spots conjecture for acute triangles became a "polymath project" [Polymath]. In 2015 Siudeja [Siu15] proved the conjecture for acute triangles with at least one angle less than $\pi/6$ by sharpening the ideas developed by Miyamoto in [Miy09], [Miy13]. Notably, in the same paper, Siudeja proved that the second Neumann eigenvalue of an acute triangle T is simple unless T is an equilateral triangle. An earlier theorem of Atar and Burdzy [AB04] gave that the second Neumann eigenvalue of each obtuse and right triangle is simple.

Our approach to the conjecture differs from most of the previous approaches (but has some features in common with the approach in [JN00]). For each acute or obtuse triangle, T_0, we consider a family of triangles T_t that joins

[2] In this paper, we do not consider a vertex of a triangle to be a possible critical point.

T_0 to a right isosceles triangle T_1. Using the simplicity of μ_2 (due to [AB04], [Miy13] and [Siu15]) we then consider a family of second Neumann eigenfunctions associated to T_t.[3] Because T_1 is the right isosceles triangle, the function u_1 is explicitly known up to a constant, and a straightforward computation shows that u_1 has no critical points (see equation (26)). Therefore, if u_0 were to have a critical point, then it would have to somehow "disappear" as t tends to 1. Each nondengenerate critical point cannot disappear immediately; that is, it is "stable." On the other hand, a degenerate critical point can instantaneously disappear; that is, it could be "unstable." Thus, as t varies from 0 to 1, either a critical point p_t of u_t converges to a vertex, or p_t is degenerate or becomes degenerate and then disappears. Understanding the first case, among the last two possibilities, is more or less straightforward, and we do it by studying the expansion of u_t in terms of Bessel functions near each vertex. Understanding the second case is more complicated. One particular reason for this complication is that disappearance of this type probably does occur for perturbations of general domains.

The study of how eigenvalues and eigenfunctions vary under perturbations of the domain is a classical topic (see, for example, [Kat95]). Jerison and Nadirashvilli [JN00] considered one-parameter families of domains with two axes of symmetry and studied how the nodal lines of the directional derivatives of the associated eigenfunctions varied. In particular, they used the fact that each constant vector field L commutes with the Laplacian, and hence if u is an eigenfunction, then Lu is also an eigenfunction with the same eigenvalue. The eigenfunctions Lu were also used in [Siu15] and implicitly in [BnB99] and [AB04].

In the current paper, we consider the vector field R_p, called the *rotational vector field*, that corresponds to the counter-clockwise rotational flow about a point p. To be precise, if $p = p_1 + ip_2$, then

$$R_p = -(y - p_2) \cdot \partial_x + (x - p_1) \cdot \partial_y.$$

We will call $R_p u$ the *angular derivative of u about p*. Each rotational vector field R_p commutes with the Laplacian, and hence the angular derivative $R_p u$ is an eigenfunction. By studying the nodal sets of $R_v u$ where v is a vertex v of the triangle, one finds that if u has an interior critical point, then u also has a critical point on each side of the triangle (see Lemma 7.1). Moreover, we show that each of these three critical points is stable under perturbation even though the interior critical point might not be stable (see Proposition 10.4). We also use the nodal sets of both $R_p u$ and $L_e u$, where L_e is parallel to the side e of T, to show that, if u has a degenerate critical point, then although this critical

[3]In fact, one can avoid using the simplicity of the second eigenvalue; see Section 12.

point of u might not be stable under perturbation, there are at least two other critical points of u that are stable under perturbation (see Proposition 10.5).

Outline of the paper. In Section 2, we recall Cheng's [Che76] theorem concerning the structure of the nodal set of an eigenfunction on a surface. Using a result of Lojasiewicz [Łoj59], we show that the critical set of each eigenfunction is a disjoint union of isolated points and analytic one-dimensional manifolds. In Section 3, we consider an open domain Ω that contains a polygon P with its vertices removed; see Figure 1. We study the restriction of an eigenfunction φ defined on Ω to the polygon P. In applications, the eigenfunction φ will be the derivative of the extension of a Neumann eigenfunction of P obtained by reflecting about the sides of P. We show that each *maximal subset* of the nodal set of the restriction of φ to P is either an isolated point, an immersed arc, or an immersed loop. At the end of Section 3, we derive two lemmas that involve the comparison of Dirichlet and Neumann conditions. These lemmas will be used to study the structure of nodal sets in later sections.

In Section 4, we consider the Bessel expansion of a Neumann eigenfunction on a sector. Using the radial and angular derivatives of this expansion, we obtain a qualitative description of the critical set of a Neumann eigenfunction on a sector. We use this description in Section 5 to prove that the critical set of a second Neumann eigenfunction u on a convex polygon has finitely many components. There we also (re)prove the fact that the nodal set of a second Neumann eigenfunction u of a polygon P is a simple arc, and we use this fact to obtain information about the first two Bessel coefficients of u at the vertices of P. For example, when P is a triangle, we further deduce that u can vanish at only one vertex of T. In Section 6, we study the nodal set of both the angular derivatives, $R_v u$, about vertices v and the directional derivatives, $L_e u$, parallel to an edge e. We show that each component of each of these nodal sets is a finite tree, and we use this to obtain information about the critical set of u. In Section 7 we specialize to the case of triangles. We show that if u has an interior critical point, then it has at least three more critical points, one critical point per side (Lemma 7.1), and if u has a degenerate critical point on a side e, then u has at least two critical points that lie on sides that are distinct from e (Theorem 7.7). These last two mentioned results are crucial to the proof of Theorem 1.1.

In Section 8, we use topological arguments and results from Section 4, 5, and 7 to understand the structure of a second Neumann eigenfunction on a triangle when it has at most one critical point.

In Section 9, we begin the proof of Theorem 1.1. Given an obtuse or (non-equilateral) acute triangle T_0, we consider a "straight line path" of triangles T_t that joins T_0 to a right isosceles triangle and an associated path $t \mapsto u_t$ of second Neumann eigenfunctions. We suppose t_n converges to t and consider the accumulation points of a sequence p_n where each p_n is a critical point of u_{t_n}. Using the Bessel expansion of u_{t_n}, we find that if each p_n lies in the interior of T_{t_n}, then a vertex is not an accumulation point of p_n. We also show that if each p_n lies in a side e and a vertex v is an accumulation point of p_n, then there does not exist a sequence of critical points q_n lying in a side $e' \neq e$ that has v as an accumulation point.

In Section 10, we address the issue of the stability of critical points. We regard a critical point p of u_t as "stable" if for each neighborhood U of p, the function u_s has a critical point in U for s sufficiently close to t. Non-degenerate critical points are stable but, in general, degenerate critical points are not. Nonetheless, we use the results of Section 7 to show that if p is a degenerate critical point of u_t, then u_t has at least two stable critical points. In Section 11, we use the existence of these two stable critical points to show that if u_0 has an interior critical point, then u_t also has at least two critical points for each $t < 1$ that is near 1. In contrast, the eigenfunction u_1 for the right isosceles triangle has no critical points, and thus, to prove Theorem 1.1 for acute and obtuse triangles, it suffices to show that the number of critical points cannot drop from two to zero in the limit as t tends to "1." This is accomplished by using the results of Section 9 and certain elementary properties of u_1.

To make the exposition of the proof of Theorem 1.1 easier to follow, we use the known simplicity of the second Neumann eigenvalue [BnB99], [AB04], [Miy13], [Siu15]. However, we indicate in Section 12 how to avoid this assumption.

In Section 13, we use results from Sections 8 and 11 to show that u has a critical point if and only if each vertex is an isolated local extremum of u. In particular, if u is associated to an acute triangle, then u has a critical point if and only if u does not vanish at any of the vertices. In the final part of Section 13, we consider the parameter space $\mathcal{T}$ of all labeled triangles up to homothety. The second Neumann eigenspaces of Δ_T define a nontrivial line bundle over $\mathcal{T} - \{T_{\text{eq}}\}$ where T_{eq} is the equilateral triangle. The unit normalized second Neumann eigenfunctions may be naturally regarded as local sections of the bundle. Using analytic perturbation theory and Hartog's separate analyticity theorem we find that each such local section is analytic. In particular, the value of each of these eigenfunctions at each vertex varies analytically, and we conclude that the generic acute triangle has exactly one critical point and that obtuse triangles have no critical points.

Notation and terminology. For notational convenience, we will often regard the Euclidean plane as the complex plane. That is, we will often use $z = x + iy$ to represent a point (x, y) in the plane. In particular, $x = \text{Re}(z)$ and $y = \text{Im}(z)$, and if $z = re^{i\theta}$, then $\theta = \arg(z)$ and $r = |z|$. We will use

A° to denote the interior of a set A. For us, a Laplace eigenfunction φ is a smooth real valued solution to the equation $\Delta\varphi = \lambda \cdot \varphi$ where $\Delta = -(\partial_x^2 + \partial_y^2)$ and $\lambda \in \mathbb{R}$. We will sometimes call such a solution φ a λ-eigenfunction. Let f be a function of t. We denote $f(t) = O(t^s)$ as $t \to 0$ if there exists $C > 0$ such that $|f(t)|/t^s \leq C$ for every $t > 0$ sufficiently small. For a bounded plane domain Ω, we will denote the first Sobolev space by $H^1(\Omega)$.

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2. The nodal set and the critical set of an eigenfunction

Let $\Omega \subset \mathbb{C}$ be an open set, and let $\varphi : \Omega \to \mathbb{R}$ be an eigenfunction of the Laplacian. In this section, we recall some facts about the nodal set $\mathcal{Z}(\varphi) := \varphi^{-1}(0)$ and the set, $\mathrm{crit}(\varphi)$, of critical points of φ. The intersection $\mathcal{Z}(\varphi) \cap \mathrm{crit}(\varphi)$ is the set of *nodal critical points*.

The following is a special case of the stratification of real-analytic sets due to Lojasiewicz [Łoj59]. An elementary proof can be found in the proof of Proposition 5 in [OR09].

LEMMA 2.1. *Let Ω be an open subset of $\mathbb{C}$. If $f : \Omega \to \mathbb{R}$ is a real-analytic function, then each $z \in \mathcal{Z}(f)$ has a neighborhood U such that $U \cap \mathcal{Z}(f)$ is either equal to $\{z\}$ or is homeomorphic to a properly embedded finite graph. Moreover, if $\nabla f(z) \neq 0$, then $U \cap \mathcal{Z}(f)$ is a real-analytic arc.*

Because the Laplacian is a constant coefficient elliptic operator, the eigenfunction φ is real-analytic function. Therefore, it follows from Lemma 2.1 that $\mathcal{Z}(\varphi)$ is a locally finite graph whose vertices are the nodal critical points, and the complement of these vertices is a disjoint union of real-analytic loops and arcs. Cheng observed [Che76] that (in dimension 2) the nodal set has a special structure in a neighborhood of each nodal critical point.

LEMMA 2.2 (Theorem 2.5 in [Che76]). *Let φ be an eigenfunction of the Laplacian on an open set $\Omega \subset \mathbb{C}$. If $p \in \Omega$ is a nodal critical point, then there exist a neighborhood U of p, a positive integer $n \geq 2$, a real number θ, and simple real-analytic arcs $\{\alpha_1, \ldots, \alpha_n\}$, such that*

(1) *$\mathcal{Z}(\varphi) \cap U$ equals $\bigcup_{i=1}^n \alpha_i$,*
(2) *$\bigcap_{i=1}^n \alpha_i = \{p\}$, and*
(3) *for each i, the arc α_i is tangent at p to the line $\left\{z : \arg(z-p) = \frac{i}{n} \cdot \pi + \theta\right\}$.*

In particular, the set of nodal critical points is discrete and the nodal set contains no isolated points.

Remark 2.3. Arcs satisfying condition (3) of Lemma 2.2 are called *equiangular.*

Sketch of proof of Lemma 2.2. Without loss of generality, $p = 0$. The Taylor series of φ about p may be regarded as a sum $\sum_k h_k$ of homogeneous polynomials h_k of degree k in x and y. Because p is a nodal critical point, h_0 and h_1 vanish identically. Since φ is an eigenfunction and Δ maps homogeneous polynomials of degree k to homogeneous polynomials of degree $k-2$, we have $\Delta h_k = \lambda \cdot h_{k-2}$. By a result of [Aro57], there exists a smallest n such that $h_n \neq 0$. Thus, $\Delta h_n = 0$ and h_n is harmonic. By Taylor's theorem, $\varphi = h_n + O(|z|^{n+1})$. The restriction of the harmonic polynomial h_n to the unit circle centered at $p = 0$ is a Laplace eigenfunction with eigenvalue $(\pi n)^2$, and so since h_n is homogeneous, the nodal set of h_n equals the union of lines $\{z : \arg(z) = \frac{i}{n} \cdot \pi + \theta\}$ for some $\theta \in \mathbb{R}$. One obtains the claim by applying the method of [Kuo69]. See Lemma 2.4 in [Che76].[4] □

As a consequence of Lemma 2.2, the nodal set $\mathcal{Z}(\varphi)$ is the union of immersed C^1 loops and properly immersed C^1 arcs.[5] We will call these the *Cheng curves* associated with φ. We emphasize that though a Cheng curve is closed in Ω, it does not have endpoints. A Cheng curve is either the image of a map $c : \mathbb{R}/\mathbb{Z} \to \Omega$, or it is parametrized by a proper map $c : \mathbb{R} \to \Omega$.

We shall be interested in whether certain Cheng arcs cross a line or not. To make this precise, we note the following.

LEMMA 2.4. *Let ℓ be a line in the plane, and let k be a component of $\ell \cap \Omega$. Either the nodal set $\mathcal{Z}(\varphi)$ contains k or the intersection $\mathcal{Z}(\varphi) \cap k$ is discrete. In the latter case, for each Cheng curve $\alpha \subset \mathcal{Z}(\varphi)$ and each $p \in \alpha \cap k$, there exists a local C^1 parametrization $c : (-\epsilon, \epsilon) \to U$ of $\alpha \cap U$ such that $c(0) = p$ and either*

(1) *the sets $c((-\epsilon, 0))$ and $c((0, \epsilon))$ lie in different components of $\Omega \setminus k$, or*
(2) *the sets $c((-\epsilon, 0))$ and $c((0, \epsilon))$ lie in the same component of $\Omega \setminus k$.*

In case (1), we say that the curve α *crosses* the line segment k.

Proof. The restriction of φ to the line segment k is a real-analytic function. Thus this restriction either vanishes identically or at a discrete set of points.

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[4] Lemma 2.2 holds more generally for eigenfunctions of the Laplace-Beltrami operator on a smooth surface [Che76].

[5] By Lemma 2.1, each Cheng curve is real-analytic except perhaps at some nodal critical points.

If the restriction vanishes identically—that is $k \in \mathcal{Z}(\varphi)$—but α does not coincide with k, then each point of intersection in $\alpha \cap k$ belongs to the set of nodal critical points, a discrete set (Lemma 2.2). Moreover, since the nodal set is equiangular at each nodal critical point, and $k \subset \mathcal{Z}(\varphi)$, we see that the C^1 curve α is transverse to k at each intersection point.

If the restriction of φ to k does not vanish identically, then the set $\alpha \cap k$ is discrete. Thus, each $p \in \alpha \cap k$ has a neighborhood U such that $\alpha \cap k \cap U = \{p\}$. Let α' be the component of $\alpha \cap U$ that contains p. Since α is C^1, we may choose a C^1 parametrization $c : (a, b) \to \Omega$ of α so that $c(0) = p$. By choosing $\epsilon > 0$ so that $c((-\epsilon, \epsilon)) \subset \alpha'$, we obtain the desired parametrization. □

Our proof of Lemma 2.4 relied only on the real-analyticity of φ, and not on the fact that each Cheng curve is real-analytic except possibly at nodal critical points.

The set, $\mathrm{crit}(\varphi)$, of critical points has the following description parallel to that of the nodal set $\mathcal{Z}(\varphi)$.

PROPOSITION 2.5. *Let φ be a Laplace eigenfunction on an open set $\Omega \subset \mathbb{R}^2$. Each connected component of* $\mathrm{crit}(\varphi)$ *is either*

(1) *an isolated point*,

(2) *an embedded proper real-analytic arc*, *or*

(3) *a real-analytic curve that is homeomorphic to a circle.*

Proof. The function $f = |\nabla\varphi|^2$ is analytic, and hence by Lemma 2.1 each critical point is either isolated or lies in a component of $\mathrm{crit}(\varphi)$ that is a locally finite graph.

Let A be a component of $\mathrm{crit}(\varphi)$. Suppose that A contains a nodal critical point. Then since A is connected and $\nabla\varphi = 0$ on A, it would follow that $A \subset \mathcal{Z}(\varphi)$ and hence each point of A is a nodal critical point. But the set of nodal critical points is discrete (Lemma 2.2) and so A consists of a single point.

If each $z \in A$ is not a nodal critical point, then for each $z \in A$, we have $\Delta\varphi(z) \neq 0$. Therefore, either $\partial_x^2\varphi(z) \neq 0$ or $\partial_y^2\varphi(z) \neq 0$. Without loss of generality, we may assume that $\partial_x^2\varphi(z) \neq 0$, and hence $\nabla(\partial_x\varphi)(z) \neq 0$. Hence, by applying the analytic implicit function at each point $z \in A$, we find that A is a real-analytic 1-manifold (without boundary). If A is compact, then A is homeomorphic to a circle. Otherwise, there exist a possibly infinite open interval $I \subset \mathbb{R}$ and a real-analytic unit speed parametrization $\gamma : I \to A$. Since A is closed in Ω, the map γ is proper. □

3. **Eigenfunctions on polygons**

In this section, we consider a (compact) polygon P whose interior is contained in an open set Ω so that $\partial P \cap \partial\Omega$ consists of the vertices of P, and we apply the analysis of Section 2 to study the restriction of an eigenfunction

φ on Ω to P.[6] In applications, φ will be the derivative of an extension of a Neumann eigenfunction on P to a domain Ω; see Section 6.

Let ψ denote the restriction of φ to P. The nodal set of ψ consists of the portion of the nodal set of φ that lies in P together with perhaps some vertices of P.

Let α be a Cheng curve of φ.

Definition 3.1. The closure of a component of $\alpha \cap P$ will be called a *maximal subset* of the nodal set of u.

The nodal set of ψ is the union, not necessarily disjoint, of these maximal subsets and perhaps some isolated vertices.

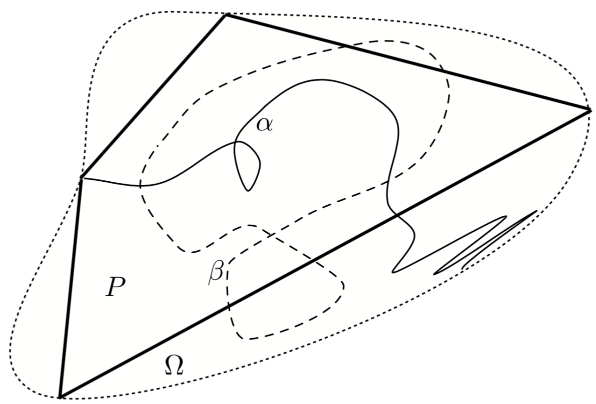

Figure 1. The nodal set of φ on Ω consists of two Cheng curves associated with Ω: the solid arc α and the hashed immersed loop β. The nodal set of the restriction of φ to P consists of three maximal arcs.

Proposition 3.2. *Each maximal subset is either an immersed loop, an isolated point in ∂P, or an immersed arc with endpoints in ∂P.*

If a maximal subset is an immersed arc (resp. loop), then we will call it a *maximal arc* (resp. *loop*). Each intersection among maximal arcs and loops is equiangular (see Remark 2.3).

Proof. Let α be a Cheng curve. Each component of $\alpha \cap P$ is either an arc, a loop, or a point that lies in a side e of P. A loop and a point are both closed and so each defines a maximal subset. Each point component corresponds to alternative (2) in Lemma 2.4 with $k = e$. In particular, each point component is isolated.

[6]In fact, P need not be a polygon. We only need to suppose that a finite subset of the Lipschitz boundary of P intersect Ω.

To finish the proof we need to show that the closure of each arc component β of $\alpha \cap P$ is either a loop or an arc with endpoints lying in ∂P.

Suppose that the Cheng curve α is a loop. Then $\alpha \cap P$ is compact and hence each component is compact. Thus since α is proper, the component β is either an immersed closed arc or an immersed loop, and if $\beta \neq \alpha$, then β is an arc with endpoints in ∂P.

If α is not a loop, then α is parametrized by a proper map $c : \mathbb{R} \to \Omega$. The subarc β is the image of an interval I. Regard the endpoints of I as lying in the extended real numbers. If an endpoint a of I is finite, then $c(a)$ lies in ∂P and $\partial\beta$ contains $c(a)$.

Suppose that ∞ is an endpoint of I. Let U_ϵ be the ϵ-neighborhood of the set of vertices of P. For ϵ sufficiently small, each vertex v belongs to exactly one component of U_ϵ. The relative complement $P - U_\epsilon$ is a compact subset of Ω, and hence $c^{-1}(P - U_\epsilon)$ is compact. In particular, there exists $M > \inf I$ such that if $t > M$, then $c(t) \notin P - U_\epsilon$. Thus, since c is continuous, the arc $c((M, \infty))$ belongs the component of U_ϵ that contains v. Since ϵ is arbitrary, $\lim_{t\to\infty} c(t) = v$.

A similar argument shows that if $-\infty$ is an endpoint of I, then $\lim_{t\to-\infty} c(t)$ exists and equals a vertex of P. If $I = \mathbb{R}$, then both limits lie in the vertex set. If the limits are equal, then the closure of β is an immersed loop, and otherwise the closure is an immersed arc whose endpoints are distinct vertices. In sum, the closure of β is either an immersed loop or an immersed arc with endpoints in ∂P. □

In applications, the eigenfunction φ will arise from "reflecting" an eigenfunction u across the sides of a polygon P on which u satisfies Neumann conditions.[7] More specifically, in Section 6, we will apply these results to a directional or angular derivative of the extension of u to a suitable domain Ω.

Extensions of a Neumann eigenfunction on a polygon can be constructed via reflection. For example, suppose T is a triangle. Given a side e of T, let σ_e denote the reflection about the line containing e. Following [Siu15], we define the *kite* K_e to be the closed set $T \cup \sigma_e(T)$. If φ is an eigenfunction of the Laplacian that satisfies Neumann conditions along e, then φ extends uniquely to a real-analytic Neumann eigenfunction $\widetilde{\varphi}$ on K_e that is reflection invariant: $\widetilde{\varphi} \circ \sigma_e = \widetilde{\varphi}$.

For a general polygon P, there exist disjoint triangles $T_e \subset P$ indexed by the edges e of P so that $\partial T_e \cap \partial P = e$. Moreover, the triangles T_e may be chosen so that $\sigma_e(T_e) \cap \sigma_{e'}(T_{e'}) = \emptyset$ if $e \neq e'$. Thus, we can umabiguously

[7]The analysis applies also to eigenfunctions with Dirichlet or mixed boundary conditions on more general polygons.

extend u to the union of $P \cup \bigcup_e \sigma_e(T_e)$. The interior of this polygon will serve as Ω in applications and will be implicit in what follows. In particular, Proposition 3.2 applies to a Neumann eigenfunction u on any polygon. It also applies to eigenfunctions obtained by differentiating u with respect to a rotational or directional vector field. Indeed, each such derivative of u extends to a derivative of φ.

The following does not depend on the preceding discussion. It should be compared to Lemma 5 in [Siu15].

LEMMA 3.3. *Let ψ be an eigenfunction on a polygon P that satisfies Neumann conditions along a side e. If a piecewise smooth arc α in the nodal set $\mathcal{Z}(\psi)$ has both endpoints in e, then the eigenvalue of ψ is strictly greater than the second Neumann eigenvalue of P.*

Proof. By hypothesis, the arc α and the side e together bound a topological disc D. Define $\widehat{\psi} : P \to \mathbb{R}$ by setting $\widehat{\psi}(z) = \psi(z)$ if $z \in D$ and $\widehat{\psi}(z) = 0$ otherwise. The function $\widehat{\psi}$ lies in $H^1(P)$ and satisfies Neumann conditions along e and Dirichlet conditions along the other sides of P. Hence the eigenvalue, λ, of ψ is larger than the first eigenvalue of the mixed eigenvalue problem on P corresponding to Neumann conditions on e and Dirichlet conditions on the other two sides. In turn, by Theorem 3.1 in [LR17], the first eigenvalue of the mixed problem is greater than the second Neumann eigenvalue of P. □

From the proof one sees that P need not be a polygon. It could be a more general domain whose boundary contains a line segment e. One need only know that ψ satisfies Neumann conditions on e and that there is a topological arc in the nodal set of ψ whose endpoints lie in e.

By the *mean* of $\varphi \in H^1(\Omega)$, we mean the integral of φ over Ω. Let ∂_ν denote the outward normal derivative along the boundary of Ω.

LEMMA 3.4. *Let Ω be a bounded plane domain with Lipschitz boundary, and let $U \subset \Omega$ be an open subset with Lipschitz boundary. Let μ_2 be the second eigenvalue of the Neumann Laplacian on Ω. Let $\varphi \in H^1(\Omega)$ be real-valued with non-zero mean and with support in the closure $\overline{U}$. If the restriction of φ to U satisfies $\Delta\varphi = \mu_2 \cdot \varphi$, then*

$$\int_{\partial\Omega} \varphi \cdot \partial_\nu \varphi \;\geq\; 0.$$

Proof. Since φ satisfies $\Delta\varphi = \mu_2 \cdot \varphi$ and $\mathrm{supp}(\varphi) \subset \overline{U}$, integration by parts gives

$$(2)\qquad \int_\Omega \nabla\varphi \cdot \nabla\varphi \;=\int_U \varphi \cdot \Delta\varphi \;+\int_{\partial U} \varphi \cdot \partial_\nu\varphi \;= \mu_2 \cdot \int_U |\varphi|^2 \;+\int_{\partial\Omega} \varphi \cdot \partial_\nu\varphi.$$

Let v be a second Neumann eigenfunction for Ω. Then by Courant's nodal domain theorem, the complement of the nodal set of v has exactly two components. Let W be one of these components, and define $w := v \cdot \mathbb{1}_W$, where $\mathbb{1}_W$ denotes the characteristic function on the set W. Note that $w \in H^1(\Omega)$ and has nonzero mean over Ω.

Since w satisfies $\Delta w = \mu_2 \cdot w$ on its support W and since w satisfies Neumann conditions on $\partial\Omega$, integration by parts gives

$$\int_\Omega \nabla w \cdot \nabla w \;= \int_W w \cdot \Delta w \;+ \int_{\partial W \cap \partial \Omega} w \cdot \partial_\nu w \;= \mu_2 \cdot \int_\Omega |w|^2. \tag{3}$$

Similarly, we have

$$\int_\Omega \nabla \varphi \cdot \nabla w = \int_W \varphi \cdot \Delta w \;+ \int_{\partial W \cap \partial \Omega} \varphi \cdot \partial_\nu w \;= \mu_2 \cdot \int_\Omega \varphi \cdot w, \tag{4}$$

where we have used the fact that $\partial_\nu w \equiv 0$ along $\partial\Omega \cap \partial W$.

By combining (2), (3) and (4), we find that for each pair a, b of real numbers, we have

$$\int_\Omega |a \cdot \nabla\varphi + b \cdot \nabla w|^2 \;= \mu_2 \cdot \int_\Omega |a \cdot \varphi + b \cdot w|^2 \;+ a^2 \cdot \int_{\partial\Omega} \varphi \cdot \partial_\nu \varphi.$$

On the other hand, since φ and w both have non-zero mean, there exist $a \neq 0$ and $b \neq 0$ so that the mean of $a \cdot \varphi + b \cdot w$ equals zero. Thus, by the variational characterization of the eigenvalue μ_2, we obtain

$$\int_\Omega |a \cdot \nabla\varphi + b \cdot \nabla w|^2 \;\geq\; \mu_2 \cdot \int_\Omega |a \cdot \varphi + b \cdot w|^2.$$

Since $a \neq 0$, the claim follows. □

4. **Neumann eigenfunctions on sectors**

Let $\Omega \subset \mathbb{C}$ be a sector of angle β and radius $\epsilon > 0$, that is,

$$\Omega : = \{z : 0 \leq \arg(z) \leq \beta \text{ and } |z| < \epsilon\}.$$

In this section, u is a (real) eigenfunction of the Laplacian on Ω with eigenvalue $\mu > 0$ that satisfies Neumann boundary conditions along the boundary edges corresponding to $\arg(z) = 0, \beta$ respectively. (We impose no conditions on the circle of radius ϵ.) We will use the expansion of u in Bessel functions near the "vertex" 0 to derive information about both the nodal set and the critical set of u.

Separation of variables leads to the following expansion valid near 0:

$$u\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} c_n \cdot J_{\frac{n\pi}{\beta}}(\sqrt{\mu} \cdot r) \cdot \cos\left(\frac{n\pi\theta}{\beta}\right). \tag{5}$$

Here $c_n \in \mathbb{R}$ and J_ν denotes the Bessel function of the first kind of order ν [Leb72]. The series converges uniformly on compact sets that miss the origin.

The Bessel function J_ν has the expansion [Leb72]

$$J_\nu(r) = r^\nu \cdot \sum_{k=0}^{\infty} \frac{(-1)^k \cdot r^{2k}}{2^{2k} \cdot \Gamma(k+\nu) \cdot \Gamma(k+\nu+1)},$$

where Γ is the Gamma function. In particular, for each $\nu \geq 0$, there exists an entire function g_ν so that $J_\nu(\sqrt{\mu} \cdot r) = r^\nu \cdot g_\nu(r^2)$.[8] Note that none of the Taylor coefficients of g_ν vanish. In particular, neither g_ν nor g'_ν vanishes in a neighborhood of 0 for each $\nu \geq 0$. With this notation, the expansion in (5) takes a more compact form:

$$u\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} c_n \cdot r^{n\cdot\nu} \cdot g_{n\cdot\nu}\left(r^2\right) \cdot \cos\left(n \cdot \nu \cdot \theta\right), \tag{6}$$

where $\nu = \pi/\beta$.

We will be interested in the level set, $u^{-1}(u(0))$, that contains the vertex 0. In particular, if $u(0) = 0$, then $u^{-1}(u(0))$ is the nodal set of u.

Lemma 4.1. *There exists a neighborhood U of 0 such that $u^{-1}(u(0)) \cap U$ either equals $\{0\}$ or equals the union of $m-1$ real-analytic arcs $\alpha_1, \ldots, \alpha_{m-1}$ such that the pairwise intersection of α_j and α_k equals $\{0\}$ for each $j \neq k$.*

Proof. By expanding each g_ν, the expansion in (6) becomes

$$u\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} c_n \cdot a_{n,j} \cdot r^{n\cdot\nu+2j} \cdot \cos\left(n \cdot \nu \cdot \theta\right), \tag{7}$$

where each $a_{n,j}$ is nonzero. We have $u(0) = c_0 \cdot a_0$. Let $\eta = \min\{n\cdot\nu + 2j : (n,j) \neq (0,0),\ c_n \neq 0\}$, and let $A = \{(n,j) : n \cdot \nu + 2j = \eta\}$. Then

$$\frac{u(z) - u(0)}{r^\eta} = \sum_{(n,j)\in A} c_n \cdot a_{n,j} \cdot \cos(n \cdot \nu \cdot \theta) + h(z), \tag{8}$$

where h is a real-analytic function and both $|h(z)|$ and $|\partial_\theta h(z)|$ are of order $O(|z|^\epsilon)$ as $|z|$ tends to zero for some $\epsilon > 0$. The claim follows from the implicit function theorem. □

Remark 4.2. The m in Lemma 4.1 need not be equal to η.

We will require more specialized information about the level sets that contain a vertex of a triangle when the vertex angle is acute or obtuse.

Lemma 4.3. *If the angle $\beta < \pi/2$, then there exists a neighborhood U of 0 such that*

(1) *if $c_0 \neq 0$, then $U \cap u^{-1}(u(0))$ equals $\{0\}$, and*

[8] Note that though g_ν depends on the eigenvalue μ, we will suppress μ from the notation.

(2) *if* $c_0 = 0$ *and* $c_1 \neq 0$, *then* $U \cap u^{-1}(u(0))$ *is a simple arc containing* 0.

If $\pi/2 < \beta < \pi$, *then there exists a neighborhood* U *of* 0 *such that*

(1) *if* $c_1 \neq 0$, *then* $U \cap u^{-1}(u(0))$ *is a simple arc containing* 0, *and*
(2) *if* $c_1 = 0$ *and* $c_0 \neq 0$, *then* $U \cap u^{-1}(u(0)) = \{0\}$

If $c_0 = 0 = c_1$, *then there exists a neighborhood* U *of* 0 *such that* $U \cap u^{-1}(u(0))$ *consists of at least two arcs.*

Proof. Suppose $\beta < \pi/2$. If $c_0 \neq 0$, then η defined in Lemma 4.1 equals 2 and $A = \{(0,1)\}$. In particular, the trigonometric polynomial appearing on the right-hand side of (8) is a constant, and hence $U \cap u^{-1}(u(0)) = \{0\}$. On the other hand, if $c_0 = 0$ and $c_1 \neq 0$, then $\eta = \nu$ and $A = \{(1,0)\}$. In this case, the trigonometric polynomial of the right-hand side of (8) equals $c_1 \cdot a_{1,0} \cdot \cos(\nu \cdot \theta)$, and hence $U \cap u^{-1}(u(0))$ is a simple arc.

Suppose $\pi/2 < \beta < \pi$. If $c_1 \neq 0$, then $\eta = \nu < 2$ and $A = \{(1,0)\}$. Thus, the trigonometric polynomial equals $c_1 \cdot a_{1,0} \cdot \cos(\nu \cdot \theta)$, and $U \cap u^{-1}(u(0))$ is an arc. On the other hand, if $c_1 = 0$ and $c_0 \neq 0$, then the trigonometric polynomial is a constant, and hence $U \cap u^{-1}(u(0)) = \{0\}$.

Finally, if $c_0 = 0 = c_1$, then each term in the trigonometric polynomial in (8) is the product of a constant and $\cos(n \cdot \nu \cdot \theta)$, where $n \geq 2$ and $n \cdot \nu + 2j = \eta$. Such a function has at least two roots. □

PROPOSITION 4.4. *If* β *is not an integer multiple of* $\pi/2$, *then there exists a deleted neighborhood of* $0 \in \Omega$ *that contains no critical points of* u. *If* $\beta = \pi/2$, *then there exists a neighborhood* U *of* 0 *such that* $\mathrm{crit}(u) \cap U$ *is either empty, equals* $\{0\}$, *or equals exactly one edge of the sector.*

Remark 4.5. The conditions on β are necessary. For example, on the square $[0,\pi] \times [0,\pi]$ we have the Neumann eigenfunction $u(z) = \cos(x)$. In this case, the set $\{z : x = 0\}$ lies in the critical set of u.

Proof. The point $z = re^{i\theta}$ is a critical point of u if and only if both the radial derivative $\partial_r u$ and the angular derivative $\partial_\theta u$ vanish at z. Let c_m be the first nonzero coefficient in the Bessel expansion in (6). By differentiating term-by-term, we obtain

$$\partial_\theta u\,(z) \;= -\sum_{n=m}^{\infty} c_n \cdot r^{n\cdot\nu} \cdot g_{n\cdot\nu}(r^2) \cdot n \cdot \nu \cdot \sin\left(n \cdot \nu \cdot \theta\right) \tag{9}$$

and

$$\partial_r u\,(z) = \sum_{n=m}^{\infty} c_n \cdot r^{n\cdot\nu-1} \cdot \left(n \cdot \nu \cdot g_{n\cdot\nu}(r^2) + 2r^2 \cdot g'_{n\cdot\nu}(r^2)\right) \cdot \cos\left(n \cdot \nu \cdot \theta\right). \tag{10}$$

In particular,

$$\partial_\theta u\,(z) = -c_m \cdot r^{m\cdot\nu} \cdot g_{m\cdot\nu}(r^2) \cdot m \cdot \nu \cdot \sin\left(m \cdot \nu \cdot \theta\right) \; + O\left(r^{(m+1)\nu}\right) \tag{11}$$

and

$$\begin{aligned} \partial_r u\,(z) = c_m \cdot r^{m\cdot\nu-1} \cdot \left(m \cdot \nu \cdot g_{m\cdot\nu}(r^2) + 2r^2 \cdot g'_{m\cdot\nu}(r^2)\right) \\ \cdot \cos\left(m \cdot \nu \cdot \theta\right) \; + O\left(r^{(m+1)\nu-1}\right), \end{aligned} \tag{12}$$

where $O\left(r^k\right)$ represents a function defined in a neighborhood of 0 that is bounded by a constant times r^k.

Suppose that $\partial_\theta u(z) = 0$ and $m \geq 1$. Then since $g_\nu(0) \neq 0$, we find from (11) that $|\sin(m \cdot \nu \cdot \theta)| \;\leq\; O\left(r^\nu\right)$. It follows that there exists $k \in \{0, 1, \ldots, m\}$ so that

$$\left|\theta \;-\; \frac{k}{m} \cdot \beta\right| = O(r^\nu). \tag{13}$$

Suppose that $\partial_r u(z) = 0$. If $m \geq 1$, then $m \cdot \nu \neq 0$, and so from (12) we find that $|\cos(m \cdot \nu \cdot \theta)| \leq O(r^\nu)$. It follows that there exists $k \in \{0, \ldots, m-1\}$ so that

$$\left|\theta \;-\; \frac{2k+1}{2m} \cdot \beta\right| = O(r^\nu).$$

Therefore, if $m \geq 1$, there exists $\epsilon > 0$ such that if $0 < |z| < \epsilon$, then $\partial_r u(z)$ and $\partial_\theta u(z)$ cannot both be zero.

If $m = 0$, then the term associated to $c_m \neq 0$ in (10) might not be dominant, and so (12) might not be useful. Which term is dominant depends on the index of the next nonzero coefficient $n_0 := \inf\{n \in \mathbb{Z}^+ : c_n \neq 0\}$.

If $\beta < n_0 \cdot \pi/2$, then the term in (12) associated to c_0 is dominant, and thus $\partial_r u$ does not vanish for small r. If $\beta > n_0 \cdot \pi/2$, then, on the one hand, the term in (12) associated to c_{n_0} is dominant, and we find that there exists $k \in \{0, \ldots, n_0 - 1\}$ so that $|\theta \,-\, (2k+1) \cdot \beta/(2n_0)| = O(r^\epsilon)$ for some $\epsilon > 0$. On the other hand, the leading term in (9) corresponds to n_0 and hence we find that $|\sin(n_0 \cdot \nu \cdot \theta)| \;\leq\; O\left(r^\nu\right)$, and so $|\theta - (k/n_0) \cdot \beta| = O(r^\nu)$ for some integer k. It follows that $\partial_\theta u$ and $\partial_r u$ cannot both vanish near 0.

If $\beta = \pi/2$, then since u satisfies Neumann conditions along the edges, we may use the reflection principle to extend u to a smooth eigenfunction on the disk $|z| < \epsilon$. By Proposition 2.5, if 0 lies in the critical locus of u, then there exists a disk neighborhood U of zero such that $\text{crit}(u) \cap U = \{0\}$ or $\text{crit}(u) \cap U$ is a real-analytic arc α. Because the extended eigenfunction is invariant under reflection across both the real and imaginary axes, the arc α is also invariant under these reflections and hence lies either in the real or imaginary axis. ☐

Remark 4.6. If $\beta = \pi$, then the sector is a half-disk. One can apply the reflection principle to extend u to the disk. Using Proposition 2.5, we find that if 0 is a critical point, then there exists a neighborhood U of 0 such that

$\mathrm{crit}(u) \cap U$ is either $\{0\}$, equals the real-axis, or is an arc that is orthogonal to the real-axis.

Remark 4.7. If $c_1 \neq 0$, then there exists $r_0 > 0$, such that if $0 < z < r_0$ and $0 < \arg(z) < \beta$, then z is not a critical point of u. Indeed, for each n, $\theta \mapsto \sin(n \cdot \nu \cdot \theta)/\sin(\nu \cdot \theta)$ defines an analytic function on $\mathbb{R}$, and hence from (9) we have

$$\partial_\theta u(z) \; = -\sin(\nu \cdot \theta) \cdot r^\nu \left(c_1 \cdot \nu \cdot g_\nu(0) + O\left(r^{\nu'}\right)\right),$$

where $\nu' = \min\{\nu, 2\}$. Thus, if $z = re^{i\theta}$ is a critical point and $0 < \theta < \beta$, then there exists C such that

$$|c_1| \cdot \nu \cdot g_\nu(0) \leq C \cdot r^{\nu'}, \tag{14}$$

and therefore $r \geq (|c_1| \cdot \nu \cdot g_\nu(0)/C)^{1/\nu'}$.

5. A second Neumann eigenfunction of a polygon

In this section, P is a simply connected polygon, and u is a second Neumann eigenfunction for P.

We will use the following well-known fact many times in the sequel. Let Ω be a bounded domain with Lipschitz boundary.

LEMMA 5.1. *Let Ω' be a subset of Ω with piecewise smooth boundary, and suppose that $f \in H^1(\Omega)$ satisfies Dirichlet boundary conditions on Ω', that is $f|_{\partial\Omega'} = 0$. Then the Rayleigh quotient $\mathcal{R}(f)$ is strictly greater than $\mu_2(\Omega)$. In particular, if f itself is a λ-eigenfunction on Ω' with Dirichlet boundary condition, then $\lambda > \mu_2(\Omega)$.*

Proof. By the variational characterization of the first Dirichlet eigenvalue, we have $\mathcal{R}(f) \geq \lambda_1(\Omega')$. By the domain monotonicity of the first Dirichlet eigenvalue, we have $\lambda_1(\Omega') \geq \lambda_1(\Omega)$, and by a result of Pólya [Pól52], we have $\lambda_1(\Omega) > \mu_2(\Omega)$. □

The following fact is also well known.

THEOREM 5.2. *The nodal set of u consists of one simple maximal arc that divides P into two components.*

Proof. By Lemma 2.2, the nodal set $\mathcal{Z}(u)$ is a collection of loops and maximal arcs. Lemma 5.1 implies that there are no loops. By Courant's nodal domain theorem, the complement $T \setminus \mathcal{Z}(u)$ has exactly two components. The claim follows. □

If v is a vertex of the polygon P, then an ϵ-neighborhood of v can be identified with a subset of a sector. For each vertex v, we consider the Bessel expansion of u about v, and we let c_j^v denote the associated j^{th} Bessel coefficient. Note that the sign of each odd coefficient depends on a choice of orientation of the plane. Here we will adopt the standard orientation.

Corollary 5.3. *Let v be a vertex of P. The first two Bessel coefficients, c_0^v and c_1^v, cannot both equal zero.*

Proof. If both c_0^v, c_1^v were both zero, by the last statement of Lemma 4.3, there would exist (at least) two arcs in $\mathcal{Z}(u)$ that emanate from v. These arcs could not be part of a loop by Lemma 5.1, and so they would have to be distinct, but this would contradict Theorem 5.2. □

The following is a consequence of a more general result of [Nad86], but it follows easily from the previous corollary.

Corollary 5.4. *The dimension of the space E of second Neumann eigenfunctions is at most two.*

Proof. Define the linear map $f : E \to \mathbb{R}^2$ by $f(u) = (c_0^v, c_1^v)$, where v is a vertex of P. By Corollary 5.3, the map has no kernel, and so the dimension of E is at most two. □

Theorem 5.2 also implies the following.

Corollary 5.5. *Suppose that P is a triangle. If v and v' are two distinct vertices of P, then $c_0^v = u(v)$ and $c_0^{v'} = u(v')$ cannot both equal zero.*

Corollary 5.5 is not true for general polygons. For example, the function $\cos(\pi x) - \cos(\pi y)$ is a second Neumann eigenfunction of the square $[0, 1] \times [0, 1]$ that vanishes at both $(0, 0)$ and $(1, 1)$.

Proof. Suppose to the contrary that c_0^v and $c_0^{v'}$ are both zero. Then by Corollary 5.3, the coefficients c_1^v and $c_1^{v'}$ are both nonzero. Thus, by Lemma 4.3, there would exist an arc in $\mathcal{Z}(u)$ emanating from v and an arc in $\mathcal{Z}(u)$ emanating from v'. By Theorem 5.2 these arcs would belong to the same maximal arc in $\mathcal{Z}(u)$ that joins v and v'. Because P is a triangle, the endpoints of this arc would lie in the same side of P. This would contradict Lemma 3.3. □

As discussed in Section 3, the eigenfunction u can be extended by reflection to an eigenfunction φ on a domain Ω that contains P with its vertices removed.

Proposition 5.6. *Let P be a convex polygon. If e is a side of P, then either e belongs to the critical set,* $\mathrm{crit}(u)$, *or* $\mathrm{crit}(u) \cap e$ *is finite. Moreover,* $\mathrm{crit}(u)$ *is a finite union of properly embedded real-analytic arcs and isolated points.*

Proof. By Proposition 2.5, each component of the critical locus of φ is a properly embedded, real-analytic arc or loop, or an isolated point. Since P is convex, the angle β at each vertex is less than π. Hence Proposition 4.4 implies that there is a neighborhood U of the vertex set of P such that if a component of $\mathrm{crit}(u)$ intersects U, then that component equals a side of P. It follows that either a side e belongs to $\mathrm{crit}(u)$ or $\mathrm{crit}(u) \cap e$ is finite. Hence, the number of components of $\mathrm{crit}(u)$ is finite.

For each constant vector field L, the critical set of φ is a subset of the nodal set of $L\varphi$. Since L commutes with Δ, the function $L\varphi$ is a Laplace eigenfunction with eigenvalue $\mu_2(P)$. Thus, by applying Lemma 5.1 to Lu on P, we find that the nodal set of Lu does not contain any loops. □

Corollary 5.7. *If P is a triangle, then* $\text{crit}(u)$ *is a finite set.*

Proof. It suffices to show that $\text{crit}(u)$ does not contain any arc. Suppose to the contrary that $\text{crit}(u)$ contains an arc. Because each arc α of $\text{crit}(\varphi)$ is properly embedded, the two endpoints of α lie in ∂T. Because P is a triangle, there exists a vertex v so that the endpoints of α lie in the union of the two sides adjacent to v. (If α coincides with a side of P, then let v be the opposing vertex.) Now consider the rotational derivative $R_v u$ (see (15)). Observe that $R_v u$ vanishes on the sides adjacent to v and along α. In particular, the nodal set of $R_v u$ contains a loop. Since the rotational derivative commutes with the Laplacian, $R_v u$ is a Laplacian eigenfunction with eigenvalue $\mu_2(P)$. Thus, we have a contradiction to Lemma 5.1. □

6. Derivatives and critical points of a second Neumann eigenfunction of a polygon

In this section, u is a second Neumann eigenfunction for a polygon P. If X is a vector field that commutes with the Laplacian, then Xu is a Laplace eigenfunction with eigenvalue $\mu_2(P)$.[9] In this section, we use the results of the preceding sections to analyze the nodal sets of Xu. Note that each vector field that commutes with the Laplacian is either a constant vector field or a rotational vector field.[10]

In the following X will denote a vector field that commutes with the Laplacian. Given a side e of P, let e° denote the interior of e, that is, the side e with its vertices removed.

Lemma 6.1. *If e is a side that is not contained in $\mathcal{Z}(Xu)$, then each point in $e^\circ \cap \mathcal{Z}(Xu)$ is a critical point and, in particular, $e^\circ \cap \mathcal{Z}(Xu)$ is finite.*

Proof. The intersection $e^\circ \cap \mathcal{Z}(Xu)$ is either finite or coincides with e°. Indeed, if $e^\circ \cap \mathcal{Z}(Xu)$ is not discrete, then it follows from Lemma 2.4 that the side e is contained in $\mathcal{Z}(Xu)$. If $e^\circ \cap \mathcal{Z}(Xu)$ is discrete, then the vector field X restricted to e is independent of the constant vector field that is normal to e, and so each point in $e^\circ \cap \mathcal{Z}(Xu)$ is a critical point of u. By Proposition 5.6 the set of critical points in e° is finite, and so the set $e^\circ \cap \mathcal{Z}(Xu)$ is finite. □

[9] The function Xu may not satisfy any reasonable boundary conditions.

[10] These vector fields are the Killing fields for the Riemannian metric tensor associated to the Euclidean metric.

Let $\mathcal{E}(Xu)$ denote the union of the sides of P that lie in $\mathcal{Z}(Xu)$.

A (topological) graph is said to be *finite* if and only it has finitely many vertices and edges. The *degree* of a vertex p of a graph is the number of edges that emanate from p.[11] A simply connected graph is called a *tree*. A tree that contains at least one edge has at least two degree 1 vertices.

PROPOSITION 6.2. *The nodal set of Xu is a finite disjoint union of finite trees. Moreover, each degree 1 vertex of $\mathcal{Z}(Xu)$ lies in ∂P.*

Proof. The nodal set $\mathcal{Z}(Xu)$ is the union of its maximal subsets (see Section 3). By combining Lemma 5.1 and Proposition 3.2, we find that each maximal subset is either an embedded C^1 arc with distinct endpoints in ∂P or an isolated point in a side of P. Lemma 5.1 also implies that any pair of distinct maximal arcs has at most one point of intersection.

Each endpoint q of a maximal arc lies in some side e of P. Lemma 6.1 implies that either q lies in $\mathcal{E}(Xu)$, q is a critical point of u, or q is a vertex of P. Thus, by Proposition 5.6, the subset of $\partial P \setminus \mathcal{E}(Xu)$ consisting of endpoints of maximal arcs is finite. By Lemma 2.2, the number of maximal arcs that meet at a given point is finite, and hence the set of maximal arcs that have at least one endpoint in $\partial P \setminus \mathcal{E}(Xu)$ is finite. Lemma 5.1 implies that at most one maximal arc joins a side $e \subset \mathcal{E}(Xu)$ to a distinct side $e' \subset \mathcal{E}(Xu)$. Thus, the set of maximal arcs that have at least one point on a side in $\mathcal{E}(Xu)$ is finite.

In sum, the collection of maximal arcs is finite and each pair of maximal arcs has at most one intersection point. Hence $\mathcal{Z}(Xu)$ is a finite graph, and thus, by Lemma 5.1, a finite union of finite trees. □

We will be especially interested in degree 1 vertices of $\mathcal{Z}(Xu)$ that are not vertices of P. By Lemma 6.1, such a vertex is a critical point of u. We will show in Section 10 that each such critical point is stable under perturbation.

LEMMA 6.3. *If a component C of $\mathcal{Z}(Xu)$ intersects the interior of P and $\mathcal{E}(Xu)$ is connected, then C has a degree 1 vertex in $\partial P \setminus \mathcal{E}(Xu)$.*

Proof. Let C be a component that intersects the interior of P. By Proposition 6.2, the component C is a tree whose degree 1 vertices lie in ∂P. If C does not contain $\mathcal{E}(Xu)$, then each degree 1 vertex of C lies in $\partial P \setminus \mathcal{E}(Xu)$.

Suppose, on the other hand, that C contains $\mathcal{E}(Xu)$. The closure C' of each connected component of $C \backslash \mathcal{E}(x)$ is a subtree of C. Since $\mathcal{E}(Xu)$ is connected, by Proposition 6.2 at most one of the vertices of this tree lies in $\mathcal{E}(Xu)$. Since C' has at least two degree 1 vertices, some degree 1 vertex of C lies in $\partial P \backslash \mathcal{E}(Xu)$. □

[11] For example, an isolated point in the set $\mathcal{Z}(Xu)$ is regarded as a degree 0 vertex.

Recall that the vector fields X that commute with the Laplacian consist of two types: the constant vector fields and the rotational vectors fields. We will use the term "directional derivative" to describe the result of applying a (real) constant vector field L.

We are particularly interested in the unit vector field, L_e, that is parallel to a side e of a polygon P such that a $\pi/2$ counterclockwise rotation of L_e points into the interior of P. Note that if φ satisfies Neumann conditions along e, then $L_e\varphi$ also satisfies Neumann conditions along e.

We will let $L_e^\perp$ denote the unit vector field that is outward normal to the side e. Note that φ satisfies Neumann conditions if and only if $L_e^\perp\varphi = 0$ along each side e of P.

By "angular derivative" we will mean the result of applying the rotational vector field R_p that corresponds to the counter-clockwise rotational flow about a point p. To be precise, if $p = p_1 + ip_2$, then

$$R_p = -(y - p_2) \cdot \partial_x + (x - p_1) \cdot \partial_y. \tag{15}$$

The vector field R_p commutes with the Laplacian, and so if u is a Laplace eigenfunction, then $R_p u$ is also an eigenfunction with the same eigenvalue.

Lemma 6.4. *If v is a vertex of P, then $\mathcal{E}(R_v u)$ contains the sides adjacent to v.*

Proof. Let e be an edge that is adjacent to v. The vector field R_v restricted to e coincides up to scaling with the vector field $L_e^\perp$ that is orthogonal to e. Since u satisfies Neumann conditions, we have $e \subset \mathcal{Z}(Xu)$. □

Corollary 6.5. *Let v be a vertex of a triangle T. If $\mathcal{Z}(R_v u)$ intersects the interior of T, then $\mathcal{Z}(R_v u)$ has a degree 1 vertex that lies in the side opposite to v.*

Proof. By Lemma 6.4 the two sides adjacent to v are in $\mathcal{E}(R_v u)$. By Proposition 6.2 the third side, the side opposing v, is not in $\mathcal{E}(R_v u)$ and hence the assertion follows from Lemma 6.3. □

Lemma 6.6. *Let e be a side of P that is not contained in* $\mathrm{crit}(u)$. *Then*

(1) *e is not contained in $\mathcal{Z}(L_e u)$,*
(2) *$\mathcal{Z}(L_e u) \cap e^\circ = \mathrm{crit}(u) \cap e^\circ$, and*
(3) *each point in $\mathcal{Z}(L_e u) \cap e$ is an endpoint of a maximal arc of $\mathcal{Z}(L_e u)$ that intersects the interior of P.*

Proof. The first assertion follows from the fact that L_e and $L_e^\perp$ are independent. The second assertion then follows from Lemma 6.1.

The third assertion follows from the fact that $L_e u$—and hence $\mathcal{Z}(L_e u)$—is symmetric with respect to the reflection along e. Indeed, suppose p lies in

$\mathcal{Z}(L_e u) \cap e$. Choose a disc neighborhood D of p, and extend u to an eigenfunction $\widetilde{u}$ in D that is invariant under the reflection σ_e about e (see Section 3). By applying Lemma 2.2, we find a perhaps smaller neighborhood D' of p so that $D' \cap \mathcal{Z}(L_e u)$ is a finite union of embedded arcs that intersect only at p and in an equiangular fashion. Because e is not contained in $\mathcal{Z}(L_e u)$, none of these arcs coincides with e. Since the nodal set is invariant under reflection each arc intersects the interior of P. □

LEMMA 6.7. *Let e be a side of P. For each maximal arc of $\mathcal{Z}(L_e u)$ that intersects the interior of P, there exists a degree 1 vertex of $\mathcal{Z}(L_e u)$ that lies in $\partial P \setminus e$. Distinct maximal arcs with an endpoint lying in e give rise to distinct degree 1 vertices in $\partial P \setminus e$.*

Proof. Because u satisfies Neumann conditions on e, the function $L_e u$ satisfies Neumann conditions along e. By Proposition 6.2, the nodal set $\mathcal{Z}(L_e u)$ is a finite union of finite trees. Each maximal arc of $\mathcal{Z}(L_e u)$ that intersects the interior of P lies in the closure C of a component of $\mathcal{Z}(L_e u) \setminus e$. The tree C has at least two degree 1 vertices. If both vertices were to lie in e, then there would be a topological arc in $\mathcal{Z}(L_e u)$ that joins two points in e. This would contradict Lemma 3.3.

If the closure, C, of a component of $\mathcal{Z}(L_e u) \setminus e$ were to contain two maximal arcs each of which has an endpoint in e, then there would be a topological arc in C with endpoints in e. If the endpoints of this arc were distinct, then this would contradict Lemma 3.3, and if the endpoints were equal, then this would contradict Lemma 5.1. Therefore, each such C contains at most one maximal arc with an endpoint in e, and the components corresponding to two distinct such maximal arcs are distinct. Hence they give rise to two distinct degree 1 vertices that lie in $\partial P \setminus e$. □

LEMMA 6.8. *Let e be a side of P. If a maximal arc of $\mathcal{Z}(L_e u)$ joins the side e to an adjacent side e', then e and e' are not perpendicular.*

Proof. If e' is perpendicular to e, then L_e and $L_{e'}^{\perp}$ differ by a scalar. Thus since u satisfies Neumann conditions along e', e' is contained in $\mathcal{Z}(L_e u)$. If e and e' were adjacent and a maximal arc in $\mathcal{Z}(L_e u)$ were to join e to e', then we would have a contradiction to Lemma 3.3. □

7. Derivatives and critical points of a second Neumann eigenfunction of a triangle

In this section, we use the analysis of the previous sections to show that if P is a triangle and u has either an interior critical point or a degenerate critical point on the boundary, then u has at least three critical points on the boundary.

Lemma 7.1. *Let u be a second Neumann eigenfunction of a triangle T. If u has a critical point that lies in the interior of T, then for each vertex v of T, the nodal set of $R_v u$ has a degree 1 vertex that lies in the interior of the side opposite to v. In particular, if u has a critical point that lies in the interior of T, then u has at least three more critical points each lying in a distinct side of T.*

Proof. If p is a critical point of u, then $R_v u(p) = 0$ for each vertex v of T. Hence $\mathcal{Z}(R_v u)$ intersects the interior of T, and we may apply Corollary 6.5 to obtain a degree 1 vertex p of $\mathcal{Z}(R_v u)$ that lies in the side e that is opposite to v. The side e is not contained in $\mathcal{E}(R_v u)$, and so, by Lemma 6.1, the point p is a critical point of u. □

We will also show that if u has a degenerate critical point on the boundary of a triangle T, then u contains at least three critical points on the boundary.

Lemma 7.2. *Suppose that the side e of T lies in the real axis and that 0 lies in the interior of e. If 0 is a critical point of u, then the Taylor expansion of u has the form*

$$u(z) = a_{00} + a_{20} \cdot x^2 + a_{02} \cdot y^2 + a_{30} \cdot \left(x^3 - 3x \cdot y^2\right) + O(4). \tag{16}$$

Moreover, either $a_{20} \neq 0$ or $a_{02} \neq 0$.

Proof. Let $\widetilde{u}$ be the extension of u to the interior of the kite K_e obtained by reflecting about e. Let h_k be the homogeneous polynomial of degree k in the Taylor expansion of $\widetilde{u}$. Since 0 is a critical point, $h_1 \equiv 0$. We also have $\Delta h_3 = \mu \cdot h_1 \equiv 0$, and hence h_3 is harmonic. Since $\widetilde{u}(x,-y) = \widetilde{u}(x,y)$, we have $h_k(x,-y) = h_k(x,y)$. In particular, $h_2 = a_{20} \cdot x^2 + a_{02} \cdot y^2$. A straightforward argument shows that if a harmonic polynomial of degree 3 is invariant under $(x,y) \mapsto (x,-y)$, then it is a multiple of $x^3 - 3x \cdot y^2$. The first claim follows.

By Theorem 5.2, the nodal set $\mathcal{Z}(u)$ is a simple arc that intersects ∂T at its endpoints. Thus, since $\widetilde{u}$ is obtained from u by reflecting across e, the nodal set of $\widetilde{u}$ is a simple arc or loop containing 0. If both a_{20} and a_{02} were equal zero, then $h_2 \equiv 0$ and so we would have $h_0 \equiv 0$. Thus 0 would be a nodal critical point, and by Lemma 2.2, the valence of the vertex 0 in $\mathcal{Z}(\widetilde{u})$ would be at least 4. Because $\mathcal{Z}(\widetilde{u})$ is invariant under $(x,y) \mapsto (x,-y)$, the nodal set of u would consist of at least two arcs, thus contradicting Theorem 5.2. So either $a_{20} \neq 0$ or $a_{02} \neq 0$. □

In what follows, when considering the Taylor expansion of a critical point p that lies in a side e, we may assume, without loss of generality, that coordinates have been chosen so that $p = 0$, the side e lies in the x-axis, and the triangle T lies in the upper half-plane. In particular, from (16) we find that such a critical point p of u is degenerate if and only if $a_{20} = 0$ or $a_{02} = 0$.

PROPOSITION 7.3. *Suppose that p is a degenerate critical point of u that lies in a side e. If $a_{02} = 0$, then u has at least one critical point in each side of T. Moreover, at least two of these critical points are such that each of these is a degree 1 vertex in the nodal set of a rotational derivative of u.*

Proof. Let $v = x_0 + i \cdot 0$ be an endpoint of e, and let $\widetilde{u}$ be the reflection u across e. Because u satisfies Neumann conditions on e, the function $R_v u$ vanishes on e. From the Taylor expansion (16) we find that

$$R_v \widetilde{u} = -2a_{20} \cdot x \cdot y + 2a_{02} \cdot (x - x_0) \cdot y + O(2).$$

In particular, if $a_{02} = 0$, then p is a nodal critical point of $R_v u$. Thus, by Lemma 2.2, at least two Cheng curves for $R_v \widetilde{u}$ intersect transversely at p. Since u is a Neumann function, one of these curves coincides with the x-axis, and so the other curve must intersect the interior of T. Since T is a triangle, it follows from Corollary 6.5 that u has a critical point on the side opposite to v that is a degree 1 vertex in $\mathcal{Z}(R_v u)$. By applying the argument to both endpoints of e, we obtain the claim. □

PROPOSITION 7.4. *Suppose that p is a degenerate critical point of u in a side e. If $a_{20} = 0$ and $a_{30} = 0$, then u has at least three critical points in the boundary of T. Moreover, at least two of these critical points are degree 1 vertices of the nodal set of $L_e u$.*

Proof. Let $\widetilde{u}$ denote the reflection of u across the edge e which, as before, we may assume lies in the x-axis. From the Taylor expansion (16) we have

$$L_e \widetilde{u} = \partial_x \widetilde{u} = 2a_{20} \cdot x + 3a_{30} \cdot (x^2 - y^2) + O(3). \tag{17}$$

Thus, if $a_{20} = 0$, the function $L_e \widetilde{u}$ has a nodal critical point at $p = 0$. And if, in addition, $a_{30} = 0$, then by Lemma 2.2 there are at least three C^1 arcs in the nodal set of $L_e \widetilde{u}$ that pass through $p = 0$. By Corollary 5.7 and Lemma 6.6, none of these curves can coincide with the side e. Therefore, since $\mathcal{Z}(L_e \widetilde{u})$ is invariant under reflection about e, each Cheng curve that passes through p intersects the interior of T. In other words, at least three maximal arcs of $\mathcal{Z}(L_e u)$ enter the interior of T at p. By Lemma 6.7, the set $\mathcal{Z}(L_e u) \setminus \{p\}$ has three components, and each component has a degree 1 vertex in $\partial T \setminus e$. At most one of these degree 1 vertices lies at the vertex of T opposite to e, and so at least two lie in the interiors of edges. Each of these points is a critical point of u that is distinct from p. □

LEMMA 7.5. *Suppose that p is a degenerate critical point of u that lies in the side e. Suppose that $a_{20} = 0$, $a_{02} \neq 0$, and $a_{30} \neq 0$. Let L denote the directional derivative $c_1 \cdot \partial_x + c_2 \cdot \partial_y$, and suppose that $c_1 \neq 0 \neq c_2$. If $(a_{30}/a_{02}) \cdot$*

$(c_1/c_2) > 0$, *then* p *is a degree* 0 *vertex of* $\mathcal{Z}(Lu)$, *and if* $(a_{30}/a_{02}) \cdot (c_1/c_2) < 0$, *then* p *is a degree* 2 *vertex*[12] *of* $\mathcal{Z}(Lu)$.

Proof. From (16) we find that

$$L\widetilde{u} \ = 2c_2 \cdot a_{02} \cdot y \ + 3a_{30} \left(c_1 \cdot \left(x^2 - y^2\right) \ - 2c_2 \cdot x \cdot y\right) \ + O(3).$$

We have $\partial_y L\widetilde{u}(0,0) \neq 0$, and so by the implicit function theorem there exists a function $f : (-\epsilon, \epsilon) \to \mathbb{R}^2$ such that the zero level set of $L\widetilde{u}$ near $(0,0)$ is the image of $x \mapsto (x, f(x))$. Implicit differentiation gives $f'(0) = 0$ and $f''(0) = -3(a_{30}/a_{02}) \cdot (c_1/c_2)$. It follows that there exists a neighborhood U of p so that if $(a_{30}/a_{02}) \cdot (c_1/c_2) > 0$, then $(U - \{p\}) \cap \mathcal{Z}(L\widetilde{u})$ lies below the x-axis, and, if $(a_{30}/a_{02}) \cdot (c_1/c_2) < 0$, then $(U - \{p\}) \cap \mathcal{Z}(L\widetilde{u})$ lies above the x-axis. The claim follows. □

PROPOSITION 7.6. *Suppose that* p *is a degenerate critical point of* u *in a side* e. *If* $a_{20} = 0$ *and* $a_{30} \neq 0$, *then* u *has at least three critical points in the boundary of* T. *Moreover, there exist two critical points of* u *in the boundary such that each of these critical points is a degree* 1 *vertex of the nodal set of some directional or angular derivative of* u.

We will use the following fact. If G is a graph, then for each component C of G and each vertex v in C, the degree of v in C equals the degree of v in G.

Proof. Let $\widetilde{u}$ be as in the proof of Proposition 7.4. Since $a_{30} \neq 0$, we find from (17) that there are exactly two C^1 arcs in the nodal set of $L_e\widetilde{u}$ that meet transversely at p, and moreover, both curves intersect the interior of T. In particular, if A denotes the component of $\mathcal{Z}(L_e u)$ that contains p, then the vertex p of $\mathcal{Z}(L_e u)$ has degree 2. Since $L_e u$ satisfies Neumann conditions, Lemma 3.3 implies that p is the only vertex of the tree A that lies in e. In particular, A contains at least two degree 1 vertices that lie in $\partial P \setminus e$. If at least two of these degree 1 vertices do not equal v, then we are done. Hence, we may assume that A is a topological arc with one endpoint equal to v and the other endpoint equal to a critical point q that lies in a side e' distinct from e.

If $A \neq \mathcal{Z}(L_e u)$, then $\mathcal{Z}(L_e u)$ has a second component B. By Lemma 6.7, the component B would have a degree 1 vertex, r, in $\partial P \setminus e$ that is distinct from v and q. In particular, the point $r \neq q$ lies in the interior of a side distinct from e. Thus, the proposition is proven in this case.

Therefore, in the only remaining case, the nodal set $\mathcal{Z}(L_u e)$ is a topological arc with a degree 2 vertex at p in e, with one endpoint q in a side e' distinct from e, and with the other endpoint equal to v. See Figure 2. In particular, by Lemma 6.8, the side e is not perpendicular to either e' or the remaining side e''.

[12] One may also regard such a point as a smooth point of $\mathcal{Z}(Lu)$.

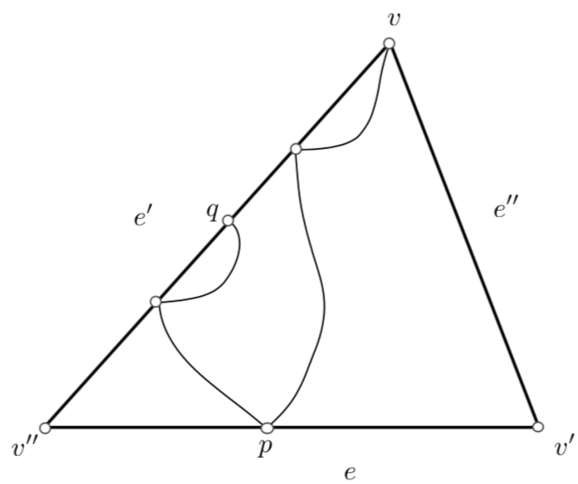

Figure 2. The nodal set of $L_e u$ in the proof of Proposition 7.6.

Consider the nodal set of $L_{e'}u$. Since q is a critical point of u, Lemma 6.6 implies that the component C of $\mathcal{Z}(L_{e'}u)$ that contains q intersects the interior of T. Hence, by Lemma 6.7, C has a degree 1 vertex z that lies in $\partial T \setminus e'$. If z is not equal to either p or the vertex v' opposite e', then we would have another critical point that is a degree 1 vertex of the nodal set of a rotational or directional derivative, and the proof would be complete. So we may assume that $z = p$ or $z = v'$.

Since e' is not perpendicular to e, there exist $c_1 \neq 0 \neq c_2$ so that $L_{e'} = c_1 \cdot \partial_x + c_2 \cdot \partial_y$. By applying Lemma 7.5 with $L = L_{e'}$, we find that z cannot equal p, and thus $z = v'$. Hence there is a (topological) arc in C that joins q to v'. This arc necessarily intersects the (topological) arc in $\mathcal{Z}(L_e u)$ that joins p and v. If the point of intersection lies in the interior, then it is an interior critical point, and then the proposition follows from Lemma 7.1. Thus, for the remainder of the proof we may assume that we are in the case where the point of intersection is p. In particular, by Lemma 7.5, we may assume that the quantity $(a_{30}/a_{02}) \cdot (c_1/c_2)$ is negative.

Next, consider the nodal set of $L_{e'}^{\perp}u = c_2 \cdot \partial_x - c_1 \cdot \partial_y$. The quantity $(a_{30}/a_{02}) \cdot (-c_2/c_1)$ is positive, and therefore Lemma 7.5 implies that the point p is a degree 0 vertex of $\mathcal{Z}(L_{e'}^{\perp}u)$. If $L_{e'}^{\perp}$ has a critical point on e', then this critical point belongs to a component C of $\mathcal{Z}(L_{e'}^{\perp}u)$ that intersects the interior of T. Thus, by Lemma 6.3, C has a degree 1 vertex z' that lies in $\partial P \setminus e'$. Since p is a degree 0 vertex, if $z' \neq v'$, then we have an additional degree 1 vertex, not equal to p, of the nodal set of a rotational or directional derivative and we are done. If, on the other hand, $z' = v'$, then there exists a topological arc α joining e' to v'. Since p is a degree 0 vertex, the arc α intersects $\mathcal{Z}(L_e u)$

in the interior of T. This intersection point would be an interior critical point, and Lemma 7.1 would give the proposition.

Thus, to finish the proof of the proposition it suffices to show that $L_{e'}^{\perp}u$ has a critical point in e'. For the remainder of the proof, we will suppose to the contrary that $L_{e'}^{\perp}u$ does not have a critical point in e' and derive a contradiction.

We claim that the normal derivative of $L_e u$ does not equal zero on the interior of e'. To see this, note that there exist b_1, b_2 so that $L_e = b_1 \cdot L_{e'} + b_2 \cdot L_{e'}^{\perp}$ and $b_2 \neq 0$. Since u satisfies Neumann boundary conditions, the function $L_{e'} L_{e'}^{\perp} u = L_{e'}^{\perp} L_{e'} u$ vanishes on e'. Hence, along e' we have $L_{e'}^{\perp}(L_e u) = b_2 \cdot L_{e'}^{\perp} L_{e'}^{\perp} u$, and so the normal derivative of $L_e u$ along e' equals $b_2 \cdot (L_{e'}^{\perp})^2 u$. Thus, if the normal derivative of $L_e u$ were to vanish at a point z_0 in the interior of e', then we would have $(L_{e'}^{\perp})^2 u(z_0) = 0$. But then z_0 would be a critical point of $L_{e'}^{\perp}$ on e'.

Recall that $\mathcal{Z}(L_e u)$ is a topological arc with endpoints q and v. In particular, $\mathcal{Z}(L_e u) \setminus \{p\}$ has two arc components: a component C_v that contains v and a component C_q that contains q. See Figure 2.

Let K denote the connected component of $T \setminus C_v$ that contains q. Then $\partial K \cap \partial T$ is contained in $e \cup e'$. Let U^+ be the subset of K on which u is positive, and let U^- be the subset of K on which u is negative. We claim that U^+ and U^- are both nonempty. Indeed, if $K = U^{\pm}$, then each point in $C_q \cap K^{\circ}$ would be a nodal critical point of u, and this would violate Lemma 2.2.[13]

By the discussion above, the normal derivative, $\partial_\nu L_e u$, of $L_e u$ is nonzero on the interior of e'. Thus, the function $L_e u \cdot \partial_\nu L_e u$ on e' is either negative on $e' \cap \partial U^+$ or it is negative on $e' \cap \partial U^-$. Let $U = U^+$ if $L_e u \cdot \partial_\nu L_e u$ is negative on e', and let $U = U^-$ otherwise. Because u satisfies Neumann conditions along e, the normal derivative of $L_e u$ along e vanishes identically on e.

Define $\varphi := \mathbb{1}_U \cdot L_e u$. Then the integral of $\varphi \cdot \partial_\nu \varphi$ over ∂T is negative. But φ satisfies the hypotheses of Lemma 3.4 with $T = \Omega$, and so the integral of $\varphi \cdot \partial_\nu \varphi$ over ∂T is nonnegative. This is the desired contradiction. □

The following theorem summarizes Propositions 7.3, 7.4, and 7.6. It plays a prominent role in the proof of Theorem 1.1.

THEOREM 7.7. *If u has a degenerate critical point p that lies in the boundary, then there exist at least three boundary critical points. Moreover, there exist two critical points of u in the boundary such that each of these critical points is a degree 1 vertex of the nodal set of some directional or angular derivative of u.*

[13] In fact, Lemma 2.2 implies the well-known fact that the eigenfunction u takes opposite signs on adjacent nodal domains.

COROLLARY 7.8. *If u has exactly one critical point p, then p is a nondegenerate critical point in ∂T.*

Proof. By Lemma 7.1, the point p lies in ∂T. By Theorem 7.7, the point p is nondegenerate. □

The following lemma will be used in the proof of the obtuse case of Theorem 1.1,

LEMMA 7.9. *If the first Bessel coefficient c_1 at v equals zero, then u has a critical point lying in the interior of the side that is opposite to v.*

Proof. Let m be the smallest positive integer n such that $c_n \neq 0$. By hypothesis, we have $m \geq 2$, and so from (9) we find that

$$R_v u(z) \; = c_m \cdot m \cdot \nu \cdot g_{m \cdot \nu}\left(r^2\right) \cdot r^{m \cdot \nu} \cdot \sin(m \cdot \nu \cdot \theta) \; + O\left(r^{(m+1) \cdot \nu}\right).$$

It follows that the nodal set of $R_v u$ intersects the interior, and hence by Lemma 6.5, the edge opposite to v contains a critical point. □

The following will be used in Section 11 in the proof of the main theorem.

PROPOSITION 7.10. *If a side e of T contains at least two critical points of u, then u has a third critical point that lies in the interior of a side e' of T that is distinct from e.*

Proof. Let p_+ and p_- be critical points of u that lie on e. By Corollary 5.7 the side e is not contained in $\mathrm{crit}(u)$, and hence by Lemma 6.6, there exists a maximal arc $\alpha_\pm$ of $\mathcal{Z}(L_e u)$ that intersects the interior of T and has $p_\pm$ as an endpoint in e. By Lemma 6.7, the other endpoint of $\alpha_\pm$ lies in $\partial T \setminus e$. By Lemma 3.3, the endpoints of α_+ and α_- cannot both be equal to the vertex opposite to e. Thus one of the arcs has an endpoint p in the interior of a side e' that is distinct from e. By Lemma 6.8, the side e is not orthogonal to e', and so the vector fields L_e and $L_{e'}^{\perp}$ are independent. Therefore, p is a critical point of u. □

8. **One nondegenerate critical point on a side of a triangle**

In this section we show that if a second Neumann eigenfunction u has exactly one critical point, then each vertex is an isolated local extremum of u. To prove this we will consider the "double" of the triangle and the extension of u to the double.

The construction of the double goes as follows. Let T' be a second triangle in the plane that is isometric to T but disjoint from T, and let $f : T \to T'$ be an isometry. The *double of* T, $DT := T \bigcup_{f|_{\partial T}} T'$, is the topological space obtained by identifying ∂T to $\partial T'$ via the restriction of f. The space DT is homeomorphic to a two-dimensional sphere. The space DT has three "cone points," Dv_1, Dv_2, Dv_3, each corresponding to a vertex of T. The complement of these cone points has a smooth Riemannian metric whose restriction to T and T' coincides with the Euclidean metric.

Define $Du : DT \to \mathbb{R}$ by setting $Du(z) = u(z)$ if $z \in T$ and $Du(z) = u \circ f^{-1}(z)$ if $z \in T'$. The restriction of Du to $DT \setminus \{Dv_1, Dv_2, Dv_3\}$ is a smooth Laplace eigenfunction.

Let $\chi(X)$ denote the Euler characteristic of a cell complex; that is, $\chi(X)$ is the alternating sum of the number of k-cells. Each surface (resp. graph) is a cell complex, and the Euler characteristic only depends on the topology of the surface (resp. graph).

Recall that the Morse index of a nondegenerate critical point is the sum of the dimensions of the eigenspaces of the Hessian that have have negative eigenvalues. In particular, the Morse index of a nondegenerate critical point of a smooth function defined on a surface equals 1 if and only if the determinant of the Hessian is negative. We will say that a point p is an *isolated local extrema* of u provided the connected component of $u^{-1}(u(p))$ that contains p equals $\{p\}$.

PROPOSITION 8.1. *If u has exactly one critical point p, then each vertex v is an isolated local extremum of u and the critical point is nondegenerate with Morse index* 1. *If u has no critical points, then exactly two of the vertices are isolated local extrema.*

Proof. Consider the level sets of $Du : DT \to \mathbb{R}$. Let A be the union of the level sets that contain a critical point of u or a vertex of T. The complement $S := DT \setminus A$ is foliated by the levels sets of Du that are each homeomorphic to a circle. In particular, the set $DT \setminus A$ is a disjoint union of annuli. Since each annulus in DT is obtained by adding a single 1-cell and a single 2-cell to A, the surface DT is obtained from the graph A by adding the same number of 1-cells and 2-cells. It follows that $2 = \chi(DT) = \chi(A)$.

By Lemma 4.3 and Corollary 5.3, the component of a level set of u that contains a vertex v either equals $\{v\}$ or is a simple arc. Thus, the component of a level set of Du that contains a cone point either consists of the cone point or is a simple loop. Note that each isolated point has Euler characteristic equal to one, and each loop has Euler characteristic equal to zero.

Suppose that u has exactly one critical point, p. By Corollary 7.8, the critical point p is nondegenerate and belongs to a side of T. The level set, Γ, of Du that contains Dp either equals $\{Dp\}$ or is homeomorphic to a figure eight. The Euler characteristic of the figure eight is -1, and so $2 = \chi(A) = \pm 1 + k$, where k is the number of vertices that are isolated extrema of u. Thus, $k = 1$ or $k = 3$. The case $k = 1$ is impossible by Corollary 5.5. Hence Γ is a figure eight, and it follows that the Morse index at p equals 1.

If u has no critical points, then $2 = \chi(A) = k$, and so exactly two vertices are isolated local extrema of u. □

9. The behavior of critical points along a path of triangles

In this section we consider the behavior of second Neumann eigenfunctions associated to a one parameter family of labeled triangles.

Let (v_1, v_2, v_3) be the labeled vertices of a non-equilateral, non-right triangle. Define the "straight line" path[14] that joins this triangle to the right isosceles triangle with vertices $(0, 1, i)$ by

$$(v_1(t), v_2(t), v_3(t)) := (1-t) \cdot (v_1, v_2, v_3) + t \cdot (0, 1, i). \tag{18}$$

Let T_t denote the triangle with vertices $(v_1(t), v_2(t), v_3(t))$. By relabeling the vertices of (v_1, v_2, v_3) if necessary, we may assume that the angle at v_1 is greater than $\pi/3$. If T_0 is acute, then for each $t < 1$, the triangle T_t is acute and not equal to the equilateral triangle. If T_0 is obtuse, then for each $t < 1$, the triangle T_t is obtuse and hence cannot be the equilateral triangle. By the results of [AB04], [Miy13], and [Siu15] the second Neumann eigenvalue of T_t is simple for each $t \in [0, 1]$.[15] Let h_t be the unique real affine homeomorphism that maps the ordered triple $(0, 1, i)$ to $(v_1(t), v_2(t), v_3(t))$. Standard perturbation theory implies that the second Neumann eigenvalue $\mu_2(t)$ of the triangle T_t varies continuously with t, and for each t, there exists a $\mu_2(t)$-eigenfunction, $u_t : T_t \to \mathbb{R}$, such that $t \mapsto u_t \circ h_t$ is continuous.[16]

Let $v = v_i(t)$ be one of the vertices, and consider the "Bessel expansion" about v as in (6):

$$u_t\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} c_n(t) \cdot r^{n \cdot \nu_t} \cdot g^t_{n \cdot \nu_t}\left(r^2\right) \cdot \cos\left(n \cdot \nu_t \cdot \theta\right), \tag{19}$$

where $\nu_t = \pi/\beta_t$ and β_t is the angle at $v_i(t)$. Because the functions $t \mapsto \mu_2(t)$ and $t \mapsto u_t \circ h_t$ are both continuous, each quantity in (19) depends continuously on t.

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[14] Our methods apply provided the path is continuous and, for each $t < 1$, the triangle has no right angle and is not equilateral.

[15] In fact, one can avoid using the simplicity of μ_2 by making some additional arguments (see Section 12). Also, note that when we first considered this question, the foreknowledge that μ_2 is simple made our approach seem more feasible.

[16] The function u_t lies in $C^0(T_t)$, and in the complement of the vertices $T_t \setminus \{v_1(t), v_2(t), v_3(t)\}$ it lies in the Sobolev space H^s for each s, and hence in C^k for each k. Continuity takes place in these spaces.

PROPOSITION 9.1. *Let t_n converge to $t \leq 1$, and for each n, let p_n be a critical point of u_{t_n} that lies in the interior of the triangle T_{t_n}. Then the sequence p_n cannot converge to a vertex $v_i(t)$.*

Proof. Suppose to the contrary that p_n converges to a vertex $v = v_i(t)$. Since the angle β_t at v is less than π, we have $\nu_t > 1$. In particular, $\nu_n := \nu_{t_n}$ is uniformly bounded from below by some $\overline{\nu} > 1$. From the Bessel expansion (19), we find that the radial derivative of u_s satisfies

$$\partial_r u_s(z) \; = 2c_0(s) \cdot r \cdot g_0'(0) \; + c_1(s) \cdot \nu_s \cdot g_{\nu_s}(0) \cdot r^{\nu_s - 1} \cdot \cos(\nu_s \cdot \theta) + O\left(r^{\nu_s^*}\right), \tag{20}$$

where $\nu_s^* = \min\{3, 2\nu_s - 1, \nu_s + 1\}$ and the remainder term depends continuously on s. Since $p_n = r_n \cdot e^{i\theta_n}$ is a critical point, we have $\partial_r u_{t_n}(p_n) = 0$. Thus, from (20) we find that there exist a constant $\nu^* > 1$ and a constant $C > 0$ so that for each n,

$$\left|2c_0(t_n) \cdot g_0'(0) + c_1(t_n) \cdot \nu_n \cdot g_{\nu_n}(0) \cdot r_n^{\nu_n - 2} \cdot \cos(\nu_n \cdot \theta_n)\right| \leq C \cdot (r_n)^{\nu^* - 1}. \tag{21}$$

On the other hand, from (14) we obtain a constant $C' > 0$ such that for each n,

$$|c_1(t_n)| \cdot \nu_n \cdot g_{\nu_n}(0) \leq C' \cdot r_n.$$

By combining this with (21), we obtain

$$|2c_0(t_n) \cdot g_0'(0)| \leq (C + C') \cdot (r_n)^{\nu^* - 1}.$$

Hence $c_0(t) = 0 = c_1(t)$, but this contradicts Corollary 5.3. □

LEMMA 9.2. *Let t_n converge to $t \leq 1$, and suppose that for each n, the points p_n and q_n are critical points of u_{t_n} that lie in the boundary of the triangle. Suppose that p_n converges to a vertex v and q_n converges to a vertex v'. If for each n, the points p_n and q_n lie in distinct sides, then $v \neq v'$.*

Proof. Suppose to the contrary that $v = v'$, and consider the Bessel expansion of u_{t_n} about this vertex. Let β_n be the angle at v, and let $\nu_n = \pi/\beta_n$. By hypothesis, for each n, we have either $\arg(p_n) = 0$ and $\arg(q_n) = \beta_n$ or $\arg(p_n) = \beta_n$ and $\arg(q_n) = 0$. Thus, since p_n and q_n are both critical points, we find from (19) that

$$\begin{aligned} 0 \; &= 2c_0(t_n) \cdot |p_n| \cdot g_0'\left(|p_n|^2\right) \\ &\quad + c_1(t_n) \cdot \nu_n \cdot g_{\nu_n}\left(|p_n|^2\right) \cdot |p_n|^{\nu_n - 1} + O\left(|p_n|^{\nu_n + 1}\right) + O\left(|p_n|^{2\nu_n - 1}\right) \end{aligned} \tag{22}$$

and

$$\begin{aligned} 0 &= 2c_0(t_n) \cdot |q_n| \cdot g_0'\left(|q_n|^2\right) \\ &\quad - c_1(t_n) \cdot \nu_n \cdot g_{\nu_n}\left(|q_n|^2\right) \cdot |q_n|^{\nu_n - 1} + O\left(|q_n|^{\nu_n + 1}\right) + O\left(|q_n|^{2\nu_n - 1}\right). \end{aligned} \tag{23}$$

Divide (22) by $|p_n| \cdot g_0'\left(|p_n|^2\right)$ and (23) by $|q_n| \cdot g_0'\left(|q_n|^2\right)$, and subtract the resulting equations to find that

$$0 = c_1(t_n) \cdot \nu_n \cdot \left(A_n \cdot |p_n|^{\nu_n - 2} + B_n \cdot |q_n|^{\nu_n - 2}\right) + O\left(|p_n|^{\nu_n^*} + |q_n|^{\nu_n^*}\right), \tag{24}$$

where $A_n = g_{\nu_n}(|p_n|)/g_0'(|p_n|^2)$, where $B_n = g_{\nu_n}(|q_n|)/g_0'(|q_n|^2)$, and where $\nu_n^* = \min\{2\nu - 2, \nu_n\}$. Since $A_n \cdot B_n$ converges to $(g_{\nu_n}(0)/g_0'(0))^2$, for sufficiently large n, we have $A_n \cdot B_n > 0$. Also note that $\nu_n^* > \nu_n - 2$. In general, if $a, b > 0$ and $x, y \in \mathbb{R}$, then $(a^x + b^x)/(a^y + b^y) \leq a^{x-y} + b^{x-y}$. Since A_n and B_n are bounded sequences, we may apply this inequality to (24) and find that $c_1(t_n) = O(|p_n|^{\nu_n^* - \nu_n + 2} + |q_n|^{\nu_n^* - \nu_n + 2})$. Therefore, as n tends to infinity, the sequence $c_1(t_n)$ tends to zero.

Divide (22) by $|p_n|^{\nu_n - 1} \cdot g_{\nu_n}\left(|p_n|^2\right)$ and (23) by $|q_n|^{\nu_n - 1} \cdot g_{\nu_n}\left(|q_n|^2\right)$, and add the resulting equations to find that

$$0 = 2c_0(t_n) \cdot \left(A_n^{-1} \cdot |p_n|^{2-\nu_n} + B_n^{-1} \cdot |q_n|^{2-\nu_n}\right) + O\left(|p_n|^{\nu_n'} + |q_n|^{\nu_n'}\right),$$

where $\nu_n' = \min\{2, \nu_n\}$. Because $\beta < \pi$, we have that $\nu_n' > 2 - \nu_n$. Since A_n^{-1} and B_n^{-1} are bounded sequences, it follows that $c_0(t_n) = O(|p_n|^{\nu_n' - 2 + \nu_n} + |q_n|^{\nu_n' - 2 + \nu_n})$. In particular, $c_0(t_n)$ converges to zero.

So we have shown that $c_0(t) = 0 = c_1(t)$, but this contradicts Corollary 5.3. □

LEMMA 9.3. *Let t_n converge to $t \leq 1$. Suppose that for each n, the point p_n is a critical point of u_{t_n} and p_n converges to a vertex v. If the limiting angle β at v is less than $\pi/2$, then $u_t(v) = 0$. If $\pi/2 < \beta < \pi$, then the first Bessel coefficient, c_1, of u_t at v equals 0.*

Proof. By Proposition 9.1 and passing to a subsequence if necessary, we may assume, without loss of generality, that each p_n lies in a side e of T_{t_n}. Without loss of generality, we may assume $v = 0$. Let β_n be the angle at the vertex v of T_{t_n}. For each n, consider the Bessel expansion of u in the sector with vertex v so that e corresponds to $\theta = 0$. From (10) we find that

$$\begin{aligned} 0 = \partial_r u(p_n) &= 2c_0(n) \cdot |p_n| \cdot g_0'(0) \\ &\quad + c_1(n) \cdot \nu \cdot |p_n|^{\nu_n - 1} \cdot g_\nu(0) + O\left(|p_n|^3\right) + O\left(|p_n|^{\nu+1}\right), \end{aligned} \tag{25}$$

where $c_0(n)$ (resp. $c_1(n)$) is the zeroth (resp. first) Bessel coefficient of u_{t_n} at v, where $\nu_n = \pi/\beta_n$. We have $\beta = \lim \beta_n$, and we let $\nu = \lim \nu_n$.

If $\beta < \pi/2$, then there exists $N > 0$ so that if $n > N$, then $\nu_n - 2 \geq (\nu - 2)/2 := \epsilon > 0$. Thus, from (25) we have

$$0 = 2c_0(n) \cdot g_0'(0) + O\left(|p_n|^\epsilon\right) + O\left(|p_n|^2\right),$$

and hence $c_0(n)$ converges to zero. Thus, since u_{t_n} converges to u_t, the zeroth Bessel coefficient of u_t at v equals zero. Hence $u_t(0) = 0$.

If $\pi/2 < \beta < \pi$, then there exists $N > 0$ so that if $n > N$, then $2 - \nu_n \geq (2-\nu_n)/2 := \epsilon > 0$. Thus, from (25) we have

$$0 = c_1(n) \cdot \nu_n \cdot g_{\nu_n}(0) + O\left(|p_n|^{\epsilon}\right) + O\left(|p_n|^2\right),$$

and hence $c_1(n)$ converges to zero. Thus, since u_{t_n} converges to u_t, first Bessel coefficient of u_t at v equals zero. □

10. On the stability of critical points

In this section, we study the behavior of the critical points of u_t as t varies. In particular, we prove some results about the "stability" of critical points of a continuous family of eigenfunctions $s \mapsto \varphi_s$ defined on an open domain $\Omega \subset \mathbb{C}$. We say that a critical point p of φ_t is *stable* if and only if for each neighborhood U of p, there exists $\epsilon > 0$ so that if $|s - t| < \epsilon$, then U contains a critical point of u_s. Our first lemma is more or less standard; it says that nondegenerate critical points are stable.

LEMMA 10.1. *Let $\Omega \subset \mathbb{C}$ be an open set and, for each $s \in (-\delta, \delta)$, let $\varphi_s : \Omega \to \mathbb{R}$ be an eigenfunction of the Laplacian such that $s \mapsto \varphi_s$ is continuous. If $p_0 \in \Omega$ is a nondegenerate critical point of φ_0, then there exist $\epsilon > 0$ and a path $p : (-\epsilon, \epsilon) \to \Omega$ such that $p(0) = p_0$ and $p(s)$ is a nondegenerate critical point of φ_s for each $s \in (-\epsilon, \epsilon)$.*

Proof. The Hessian of φ_0 at p has two nonzero (real) eigenvalues. Thus, there exists a real-affine map $h : \mathbb{C} \to \mathbb{C}$ so that $h(0) = p$ and the Taylor expansion of $\varphi_0 \circ h$ at 0 has the form

$$\varphi_0 \circ h(z) = \varphi_0(p) + x^2 \pm y^2 + O(3),$$

where $O(3)$ is a function such that vanishes to order 3 in x, y. By the continuity of $s \mapsto \varphi_s$, this expansion extends to s near t:

$$\varphi_s \circ h(z) = \varphi_s(p) + a_{10}(s) \cdot x + a_{01}(s) \cdot y + a_{20}(s) x^2 + a_{11}(s) \cdot xy + a_{02}(s) \cdot y^2 + O(3).$$

We have

$$\partial_x^2 \varphi_s \circ h(z) = 2a_{20}(s) + O(1),$$
$$\partial_y^2 \varphi_s \circ h(z) = 2a_{02}(s) + O(1),$$

where $a_{20}(0) = 1$ and $a_{02}(0) = \pm 1$. Since a_{jk} is continuous in s, there exists $\epsilon > 0$ so that if $|s| < \epsilon$, then $a_{20}(s) > 1/2$ and $|a_{02}(s)| > 1/2$. It follows from the implicit function theorem that there exists a neighborhood U of 0 such that for each $|s| < \epsilon$, the intersection $\mathcal{Z}(\partial_x \varphi_s \circ h) \cap U$ (resp. $\mathcal{Z}(\partial_y \varphi_s \circ h) \cap U$) is a real-analytic arc α_s (resp. β_s) that depends continuously on s. The arcs α_0 and β_0 intersect transversely at the origin, and hence there exists a neighborhood $U' \subset U$ of 0 and $\epsilon' > 0$ so that if $|s - t| < \epsilon'$, then α_s and β_s have a unique intersection point $p(s) \in U'$, and moreover, p_s depends continuously on s. In particular, if $|s| < \epsilon'$, then φ_s has no critical points in U' other than $p(s)$. $\square$

Now we apply the above lemma to our path u_t of second Neumann eigenfunctions.

Corollary 10.2. *If $p_t \in T_t$ is a nondegenerate critical point of u_t, then there exists $\epsilon > 0$ and a path $p : (t-\epsilon, t+\epsilon) \to \mathbb{C}$ such that $p(t) = p_t$ and $p(s)$ is a nondegenerate critical point of u_s that lies in T_s. Moreover, if p_t lies in the boundary of T_t, then p_s lies in the boundary of T_s for each $s \in (t-\epsilon, t+\epsilon)$.*

Proof. If p_t lies in the interior of T_t, then there exist $\epsilon' > 0$ and an open ball B about p_t so that if $|s - t| < \epsilon$, then $B \subset T_s^\circ$. By applying Lemma 10.1 to the restriction of $t \mapsto u_t$ to B, we find the desired $\epsilon < \epsilon'$ and path $s \mapsto p(s)$.

If p_t lies in the boundary of T_t, then p_t lies on the interior of a side e_t of T_t. Let K_e^s denote the kite obtained from reflecting T_s across e_s with the reflection σ_e^s. Since u_s is a Neumann eigenfunction, we may extend u_s via reflection to an eigenfunction $\widetilde{u}_s$ defined on K_e^s for each $s \in (t-\epsilon, t+\epsilon)$. Note that there exist $\epsilon' > 0$ and an open ball B about p_t so that if $|s-t| < \epsilon'$, then B lies inside K_e^s.

The point p_t is a nondegenerate critical point of $\widetilde{u}_t$, and hence Lemma 10.1 implies that there exists $0 < \epsilon < \epsilon'$ and a path $p(s) : (t-\epsilon, t+\epsilon) \to B$ so that $p(t) = p_t$ and so that $p(s)$ is a nondegenerate critical point of $\widetilde{u}_s$ for each $s \in (t-\epsilon, t+\epsilon)$.

Suppose that for some s, the critical point $p(s)$ does not lie in the side e_s. Then $\sigma_e^s(p(s))$ would be a distinct critical point of $\widetilde{u}_s$. On the other hand, we have $\sigma_e^t(p(t)) = p(t)$, and so we would find an s' such that $p(s')$ is a degenerate critical point, a contradiction. Therefore, $p(s)$ lies in e_s for each $s \in (t-\epsilon, t+\epsilon)$. $\square$

The following lemma will be used to show that if the nodal set of some directional or angular derivative of u has a degree 1 vertex on the interior of a side, then the corresponding critical point is "stable." Let $H = \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\}$ denote the closed upper half plane.

Lemma 10.3. *Let $\Omega \subset \mathbb{C}$ be an open set that contains 0 and, for each $s \in (-\delta, \delta)$, let $\varphi_s : \Omega \to \mathbb{R}$ be an eigenfunction of the Laplacian such that $s \mapsto \varphi_s$ in continuous. Suppose that the intersection $\mathcal{Z}(\varphi_0) \cap H \cap \Omega$ consists of a simple arc α whose intersection with the real axis is the point 0. Then there exists $\epsilon > 0$ so that if $|s| < \epsilon$, then $\mathcal{Z}(\varphi_s)$ intersects the real axis.*

Proof. It follows from Lemmas 2.2 and 2.4 that there exists a circle $C = \{z \in \mathbb{C} : |z| = r\}$ so that $\mathcal{Z}(\varphi_0) \cap C$ is finite, each intersection is transverse, and $\mathcal{Z}(\varphi_0) \cap C \cap H$ contains exactly one point z_0. Thus, since $s \to \varphi_s$ is continuous, there exists $\epsilon > 0$ so that each intersection point $z_s \in \mathcal{Z}(\varphi_s) \cap C$ depends continuously on $s \in (-\epsilon, \epsilon)$ and each intersection is transverse. Since $s \mapsto z_s$ is continuous, we may assume, by choosing $\epsilon > 0$ smaller if necessary, that if $|s - t| < \epsilon$, then $\mathcal{Z}(\varphi_0) \cap C \cap H$ consists of exactly one point z_s^0.

Let α_s be the (proper) Cheng curve of $\mathcal{Z}(\varphi)$ that contains z_s^0. Since the intersection of α_s and C is transverse, the arc α_s intersects the open disc $\{z \in \mathbb{C} : |z| < r\}$. Since α_s^0 is a proper curve in Ω, the curve α_s intersects C at a point z_s^1 distinct from z_s^0. The point z_s^1 lies outside of H. Therefore, by the intermediate value theorem for example, the Cheng curve α_{z_s} intersects the real axis. □

PROPOSITION 10.4. *If u_t has a critical point that lies in the interior of the triangle T_t, then there exists $\epsilon > 0$ such that if $|t - s| < \epsilon$, then u_s has at least three critical points.*

Proof. Lemma 7.1 implies that for each vertex v of the triangle, the nodal set of $R_v u$ has a degree 1 vertex p in the side e that is opposite to v. For each s, let $\widetilde{u}_s$ denote the extension of u to the kite $K_{e,s}$ obtained by reflecting about the side e. There exists $\epsilon_1 > 0$ and a neighborhood U of p so that if $|s-t| < \epsilon_1$, then U lies in the interior of $K_{e,s}$. For such s, consider the restriction of $\widetilde{u}_s$ to U. Without loss of generality, $p = 0$ and e lies in the real axis, and so we may apply Lemma 10.3 to find $\epsilon > 0$ so that if $|s - t| < \epsilon$, then $\mathcal{Z}(R_v \widetilde{u}_s) \cap e$ contains a point p_s in the interior of e. Since the vectors $L_e^{\perp}(p_s)$ and $R_v(p_s)$ are independent, the point p_s is a critical point of u_s. Thus, for each vertex v of T_s and $s \in (t - \epsilon, t + \epsilon)$, there exists a critical point of u that lies in the interior of the side opposite to v. □

PROPOSITION 10.5. *If u_t has a degenerate critical point p, then there exists $\epsilon > 0$ such that if $|t - s| < \epsilon$, then u_s has at least two critical points.*

Proof. If p lies in the interior of T_s, then this follows from Proposition 10.4. If p lies in a side, then by Theorem 7.7, the nodal set of some directional or angular derivative of u_t has at least two degree 1 vertices that lie in the interior of sides of T. By applying Lemma 10.3 in the same manner as it was applied in the proof of Proposition 10.4, we find that the nodal sets of the perturbed eigenfunctions intersect the relevant sides giving critical points for each s near t. □

11. **The proof of Theorem 1.1**

Let $t \mapsto u_t$ be the one-parameter family of eigenfunctions defined at the beginning of Section 9. For each $t \in [0, 1]$, let $N(t)$ denote the number of critical points of u_t.

LEMMA 11.1. *Suppose that T_0 is an acute triangle. If $N(0) \geq 2$, then $N(t) \geq 2$ for each $t < 1$.*

Proof. It suffices to show that the set $\{t \in [0,1) : N(t) \geq 2\}$ is both open and closed in $[0,1)$.

(Open) Suppose $t < 1$ and $N(t) \geq 2$. If u_t has an interior critical point, then Proposition 10.4 implies that $N(s) \geq 2$ for each s in a neighborhood of t. If u_t has a degenerate critical point on a side, then Proposition 10.5 implies that $N(s) \geq 2$ for each s in a neighborhood of t. If each critical point of u_t is nondegenerate, then it follows from Corollary 10.2 that $N \geq 2$ in a neighborhood of t.

(Closed) Let t_n converge to $t < 1$, and suppose that $N(t_n) \geq 2$ for each n. Let p_n and q_n be distinct critical points of u_{t_n}. By Lemma 7.1, we may assume that p_n and q_n lie in ∂T. Suppose that a subsequence of p_n converges to a vertex v, and a subsequence of q_n converges to a vertex v'. Abusing notation slightly, we denote the subsequences with p_n and q_n. By Proposition 7.10 we may assume that p_n and q_n lie in distinct sides. Lemma 9.2 gives that $v \neq v'$. Since the limiting angles at v and v' are both less than $\pi/2$, Lemma 9.3 implies that $u(v) = 0$ and $u(v') = 0$. This contradicts Corollary 5.5.

Therefore, at most one of the sequences, p_n, q_n, has a vertex as an accumulation point. Suppose, without loss of generality, that p_n has an accumulation point p that is not a vertex. Since u_{t_n} converges to u_t, the point p is a critical point of u_t, and so $N(t) \geq 1$.

If q_n has an accumulation point q that is not a vertex, then q is also a critical point u_t. If $p \neq q$, then $N(t) \geq 2$. If $p = q$, then p is a degenerate critical point, and hence $N(t) \geq 2$ by Theorem 7.7.

If q_n has a subsequence that converges to a vertex v, then it follows from Lemma 9.3 that $u(v) = 0$. Thus, by Lemma 4.3, the vertex v is not an isolated local extremum. Therefore, Proposition 8.1 implies that $N(t) \neq 1$, and so $N(t) \geq 2$. □

Next, we consider the case where T_t is an obtuse triangle for each $t < 1$. Let $v_o(t)$ denote the vertex of T_t whose angle is greater than $\pi/2$. Let $c_1(t)$ denote the first Bessel coefficient of u_t at $v_o(t)$.

LEMMA 11.2. *Suppose that T_0 is an obtuse triangle. If $N(0) \geq 2$, then $N(t) \geq 1$ for each $t < 1$. If $t < 1$ and $N(t) = 1$, then $c_1(t) = 0$.*

Proof. Let $A := \{t \in [0,1) : N(t) \geq 2\}$, and let $B := \{t \in [0,1) : c_1(t) = 0 \text{ and } N(t) = 1\}$. To prove the lemma, it suffices to show that $A \cup B$ is both open and closed in $[0,1)$.

(Open) The set A is open by the same argument given in the proof of Lemma 11.1. If $t \in B$, then u_t has exactly one critical point and it is non-degenerate by Corollary 7.8. Thus, by Corollary 10.2, there exists $\epsilon > 0$ such that if $0 < |s-t| < \epsilon$, then $N(s) \geq 1$. If $|s-t| < \epsilon$ and $c_1(s) = 0$, then $s \in B$. If $|s-t| < \epsilon$ and $c_1(s) \neq 0$, then Lemma 4.3 implies that the obtuse vertex v_o is not an isolated local extrema of u_s, and hence, by Proposition 8.1, we have $N(s) \geq 2$. Thus, $s \in A$.

(Closed) Let $t_n \in A \cup B$ be a sequence that converges to $t < 1$. Up to extracting a subsequence we may assume that either $t_n \in B$ for each n or $t_n \in A$ for each n. In the case that $t_n \in B$ for each n, the continuity of c_1 implies that $c_1(t) = 0$. Lemma 7.9 provides that $N(t) \geq 1$ and hence $t \in A \cup B$.

Suppose now that $t_n \in A$ for each n. Let p_n and q_n be distinct critical points of u_{t_n}. By Lemma 7.1, we may assume that p_n and q_n both lie in ∂T and, by Proposition 7.10, that these two points lie in distinct sides of T. Up to extracting a subsequence we may further assume that both the sequences have accumulation points.

Suppose that neither p_n nor q_n has a vertex as an accumulation point. Let p (resp. q) be an accumulation point of p_n (resp. q_n). Since the points p and q lie on different sides, $p \neq q$, and hence $N(t) \geq 2$ implying that $t \in A$.

Now suppose that a vertex v is an accumulation point of p_n, and suppose that v' is an accumulation point of q_n. We may argue as in the proof of Lemma 11.1 to show that it is not possible that both of the angles at v and v' are less than $\pi/2$. Thus, without loss of generality, the vertex v equals the vertex $v_o(t)$ whose angle is greater than $\pi/2$. In particular, by Lemma 9.3, we have $c_1(t) = 0$, and hence, by Lemma 7.9, we have $N(t) \geq 1$. Thus, $t \in A \cup B$.

Finally, suppose that p_n has an accumulation point that equals a vertex v, whereas q_n has an accumulation point, q, which is not a vertex. The point q is a critical point and so $N(t) \geq 1$. If $v = v_o$, then as before $c_1(t) = 0$, and hence $t \in A \cup B$. If $v \neq v_o$, then by Lemma 9.3, we have $u(v) = 0$. Thus, Lemma 4.3 implies that v is not an isolated local extremum. It follows from Proposition 8.1 that $N(t) \geq 2$, and hence $t \in A$. □

We will show that Lemmas 11.1 and 11.2 imply that if u_0 has at least two critical points, then u_t has at least two critical points for each $t < 1$ and sufficiently close to 1. In contrast, the function u_1 has no critical points.[17] Indeed, each eigenfunction for the right isosceles triangle $(0, 1, i)$ is a multiple of the function

$$u(z) = \cos(\pi x) - \cos(\pi y), \tag{26}$$

where as usual $z = x + iy$.

[17] Recall that a vertex is not, by our consistent definition, a critical point of u_t.

Proof of Theorem 1.1. We assume that u_0 has an interior critical point and derive a contradiction. If u_0 has an interior critical point, then Lemma 7.1 implies that $N(0) \geq 2$. In the acute case, Lemma 11.1 shows that there exists a sequence t_n converging to 1 such that $N(t_n) \geq 2$ for each n. In the obtuse case, observe that the second Neumann eigenvalue of the right isoceles triangle is simple, and so u_1 is a multiple of the function u in (26). This implies that for $t < 1$ and sufficiently close to 1, the eigenvalue μ_t is simple and $c_1(t) \neq 0$. Hence by the last part of Lemma 11.2 there exists a sequence t_n converging to 1 such that $N(t_n) \geq 2$ for each n.

Let p be an accumulation point of a sequence of critical points, p_n, of u_{t_n}. If p is not a vertex of the triangle T_1, then p is a critical point of u_1. But u_1 is a multiple of the function u described in (26), and u has no critical points. Therefore, each accumulation point of $\{p_n\}$ is a vertex. It follows from Proposition 9.1 that there exists $K > 0$ such that if $n > K$, then each critical point of u_{t_n} lies in a side of the triangle.

Since $N(t_n) \geq 2$, Proposition 7.10 implies that that for each $n > K$, there exist distinct sides e and e' and sequences of critical points p_n and p'_n so that for each n, we have p_n in e and p'_n in e'. By passing to a subsequence if necessary we may assume that p_n converges to a vertex v of T_1, and p'_n converges to a vertex v' of T_1. By Lemma 9.2, we have $v \neq v'$. The sets $\{v, v'\}$ and $\{1, i\}$ are both contained in a three element set, and hence we may assume without loss of generality that $v = 1$ or $v = i$. Thus, by Lemma 9.3, the function u_1 vanishes at either 1 or i. But u_1 is a multiple of the function u described in (26), and u does not vanish at 1 or i.

We have thus proven that if T_0 is either an obtuse or a non-equilateral acute triangle, then a second Neumann eigenfunction for T_0 has at most one critical point, and if such a critical point exists, then it lies in ∂T_0. Now we use this to prove the claim for right triangles. (The case of the equilateral triangle can be established by direct computation [Lam52], [Pin80].)

Given a (labeled) right triangle T, let $t \mapsto T_t$ be a path of labeled triangles such that $T_0 = T$, and if $t > 0$, then T_t is an acute triangle (and not equilateral). Let u be an $\mu_2(T_0)$-eigenfunction. For each t, the eigenvalue $\mu_2(T_t)$ is simple [Siu15], and hence standard perturbation theory implies that there exists a continuous path $t \mapsto u_t$ of $\mu_2(T_t)$-eigenfunctions, such that $u_0 = u$. If u were to have an interior critical point, then a variant Proposition 10.4 would imply that u_t has at least three critical points for small t. If u were to have a degenerate critical point that belonged to a side of T, then a variant of Proposition 10.5 would imply that u_t would have at least two critical points for t small. Finally, if u were to have more than one nondegenerate critical point, then Lemma 10.1 would imply that u_t has at least two critical points for t small. Each is a contradiction to the first part of the theorem since T_t is acute for $t > 0$. □

12. Working without the assumption of simplicity

In this section, we indicate the modifications needed to avoid using the simplicity of μ_2 for non-equilateral triangles. Our discussion begins with a standard application of analytic perturbation theory [Kat95].

LEMMA 12.1. *For each $i \in \mathbb{N}$, there exist an analytic path*[18] $t \mapsto \varphi_i(t)$ *and an analytic path $t \mapsto \lambda_i(t)$ so that for each $t \in [0,1]$, the function $\varphi_i(t)$ is a Neumann eigenfunction on T_t with eigenvalue $\lambda_i(t)$ and the collection $\{\varphi_i(t) : i \in \mathbb{N}\}$ is an orthonormal basis*[19] *of $L^2(T_t)$.*

Proof. For each pair of smooth functions $f, g : T_t \to \mathbb{R}$, define

$$q_t(f,g) = \int_{T_t} \nabla f \cdot \nabla g \quad \text{and} \quad n_t(f,g) = \int_{T_t} f \cdot g.$$

Let $\mathcal{D}_t$ be the completion of the smooth functions with respect to the norm $f \mapsto \sqrt{q_t(f,f) + n_t(f,f)}$. This space may be naturally regarded as a dense subspace of $L^2(T_t)$, and the form q_t extends to a closed form with domain $\mathcal{D}_t$. A function $u \in \mathcal{D}_t$ is an eigenfunction of the Neumann Laplacian on T_t with eigenvalue λ if and only if for each $v \in \mathcal{D}_t$, we have $q_t(u,v) = \lambda \cdot n_t(u,v)$.[20] For each labeled triangle T_t, let h_t be the unique real-affine map that sends the ordered triple $(0,1,i)$ to $(v_1(t), v_2(t), v_3(t))$. The map $f \to h_t \circ f =: h_t^*(f)$ sends smooth functions on T_t to smooth functions on T_1, and a straightforward argument shows that h_t^* is a bounded isomorphism from $L^2(T_t)$ to $L^2(T_1)$ that maps $\mathcal{D}_t$ onto $\mathcal{D}_1$. Note that for each $f, g \in L^2(T_1)$, we have $n_t(f \circ h_t^{-1}, g \circ h_t^{-1}) = 2 \cdot a_t \cdot n_1(f,g)$ where a_t is the area of T_t. For each $f, g \in \mathcal{D}_1$, define $\widetilde{q}_t(f,g) := (2a_t)^{-1} \cdot q_t(f \circ h_t^{-1}, g \circ h_t^{-1})$. By tracing through the definitions, one finds that u is a Neumann eigenfunction on T_t with eigenvalue λ if and only if for each $w \in \mathcal{D}_1$, we have

$$\widetilde{q}_t(u \circ h_t, w) = \lambda \cdot n_1(u \circ h_t, w).$$

The family $t \mapsto \widetilde{q}_t$ is an analytic family of type (a) in the sense of Kato (see Theorem 4.2 in Chapter VII of [Kat95]). Moreover, the resolvent of the associated operator is compact, and hence it follows[21] that for each $i \in \mathbb{N}$, there exist an analytic path $t \mapsto \psi_i(t)$ and an analytic path $t \mapsto \lambda_i(t)$ so that for each $t \in [0,1]$, we have $\widetilde{q}_t(\psi_i(t), w) = \lambda_i(t) \cdot n_1(\psi_i(t), w)$ and the collection $\{\psi_i(t) : i \in \mathbb{N}\}$ is an orthonormal basis of $L^2(T_1)$. Set $\varphi_i(t) := (2a_t)^{-\frac{1}{2}} \cdot \psi_i(t) \circ h_t^{-1}$. □

[18] By "analytic" we mean $t \mapsto \varphi_t \circ h_t$ is an analytic path in each Sobolev space on T_1.

[19] By "orthonormal basis," we mean that $\int_{T_t} \varphi_i(t) \cdot \varphi_j(t) = \delta_{ij}$ and the finite linear combinations are dense in $L^2(T_t)$.

[20] Indeed, Neumann conditions are the "natural boundary conditions."

[21] See Chapter VII of [Kat95], especially Remark 4.22.

It is important to note that the analytic eigenvalue "branches" $t \mapsto \lambda_i(t)$ of Lemma 12.1 cannot, in general, be ordered according to the size. Indeed, two eigenvalue branches λ_i and λ_j may "cross" at some $t \in [0,1]$ in the sense that $\lambda_i(s) < \lambda_j(t)$ for $s < t$ and $\lambda_i(s) > \lambda_j(t)$ for $s > t$.

LEMMA 12.2. *There exist a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ and for each $j = 0, \ldots, k-1$, an analytic path $t \mapsto u_t^j$ such that for each $t \in [t_j, t_{j+1}]$, the function u_t^j is a second Neumann eigenfunction of T_t.*

Proof. Let $s \mapsto \varphi_i(s)$ and $s \mapsto \lambda_i(s)$ be the eigenfunction and eigenvalue branches provided by Lemma 12.1. For each t, there exists $i(t) \in \mathbb{N}$ so that $\mu_2(t) = \lambda_{i(t)}$. By Corollary 5.4, the dimension of the space of second Neumann eigenfunctions is at most two. Let A be the set of $t \in [0,1]$ such that $\mu_2(t)$ has multiplicity exactly equal to two. It suffices show that A is discrete. Indeed, then A would be finite, and $i(t)$ would be locally constant on the complement of A. The set A would give the desired partition.

Fix $t \in A$. Since the Neumann spectrum of T_t is discrete, there exists $\epsilon > 0$ so that $\mu_2(t)$ is the only eigenvalue of T_t that lies in $(\mu_2(t) - \epsilon, \mu_2(t) + \epsilon)$. There exist unique integers i and j so that the $\mu_2(t)$-eigenspace of Δ_t is spanned by $\varphi_i(t)$ and $\varphi_j(t)$. We have $\lambda_i(t) = \mu_2(t) = \lambda_j(t)$. By continuity of the eigenvalue branches, there exists $\delta > 0$ so that if $|s - t| < \delta$, then $\lambda_i(s)$ and $\lambda_j(s)$ are the only eigenvalues of T_s that lie in $(\mu_2(t) - \epsilon, \mu_2(t) + \epsilon)$. In particular, we have $\mu_2(s) = \min\{\lambda_i(s), \lambda_j(s)\}$ for $|s - t| < \delta$.

Define $C_t := \{s \in [0,1] : \lambda_i(s) = \lambda_j(s)\}$. Real-analyticity implies that C_t is either discrete or $C_t = [0,1]$. Thus, to finish the proof, it suffices to show that $C_t \neq [0,1]$.

Suppose to the contrary that $C = [0,1]$. Then for $|s - t| < \delta$, we have $\mu_2(s) = \lambda_i(s) = \lambda_j(s)$. Let t^* be the supremum of s so that $\mu_2(s) = \lambda_i(s) = \lambda_j(s)$. The triangle T_1 is right isosceles, the eigenvalue $\mu_2(1)$ is simple, and so $t^* < 1$. The eigenspace associated to $\mu_2(t^*)$ is two-dimensional and is spanned by $\varphi_i(t^*)$ and $\varphi_j(t^*)$. Let s_k be a decreasing sequence that limits to t^*. For each k, let u_k be a $\mu_2(s_k)$-eigenfunction with L^2-norm equal to one. Note that u_k is orthogonal to the space spanned by $\varphi_i(s_k)$ and $\varphi_j(s_k)$. The sequence u_k has a subsequence that limits to a $\mu_2(t^*)$-eigenfunction u. The function u is orthogonal to the span of $\varphi_i(t^*)$ and $\varphi_j(t^*)$. But this contradicts the fact that $\varphi_i(t^*)$ and $\varphi_j(t^*)$ span the $\mu_2(t^*)$-eigenspace. □

Lemma 12.2 allows us to construct a continuous family of second Neumann eigenfunctions to which we can apply our methods. Indeed, let E_t denote the space of second Neumann eigenfunctions, and for each $j = 1, \ldots k-1$, choose a continuous path of eigenfunctions inside E_{t_j} that joins the eigenfunction $u_{t_j}^{j-1}$ to the eigenfunction $u_{t_j}^j$. By concatenating such paths with the paths u_t^j of Lemma 12.2 we obtain a continuous path of second Neumann eigenfunctions that joins u_0 to u_1. The methods of this paper apply to this path, and we obtain Theorem 1.1 without using simplicity.

13. Triangles with no critical point

Let u be a second Neumann eigenfunction eigenvalue $\mu_2(T)$ of a triangle T. From Section 11, it follows that u has at most one critical point. By combining this with Proposition 8.1, we obtain

THEOREM 13.1. *The eigenfunction u has a critical point if and only if each vertex is an isolated local extremum. Moreover, the maximum (resp. minimum) value of u is achieved only at the vertices of the triangle.*

Proof. For the second statement, note that neither the maximum nor the minimum value is achieved at a nondegenerate critical point of Morse index 1. □

COROLLARY 13.2. *If T is an acute triangle, then u has a critical point if and only if u does not vanish at each vertex of T. Moreover, if T is an obtuse triangle, then u has a critical point if and only if u does not vanish at the acute vertices of T and the first Bessel coefficient of u at the obtuse vertex is zero.*

Proof. This follows from Lemma 4.3 and Proposition 8.1. □

PROPOSITION 13.3. *Let T be a nonequilateral isosceles triangle, and let β be the angle of the apex of T. If $\beta > \pi/3$, then u has no critical points. If $\beta < \pi/3$, then u has exactly one critical point.*

Proof. Up to rescaling and rigid motion, each isosceles triangle may be identified with the triangle with vertices $(-t, t, i)$. If $\beta > \pi/3$, then $t > 1/\sqrt{3}$, and the second Neumann eigenvalue $\mu_2(t)$ is simple [Siu15]. Let $t \mapsto u_t$ be an analytic family of second Neumann eigenfunctions associated to the path $t \mapsto (-t, t, i)$. Each triangle is preserved by the reflection $\sigma(x + iy) = -x + iy$ that has fixed point set $x = 0$. Since $\mu_2(T_t)$ is simple, we have either $u_t \circ \sigma = u_t$ or $u_t \circ \sigma = -u_t$ for each t. The triangle with vertices $(-1, 1, i)$ is a right isosceles of triangle, and inspection of (26) shows that $u_1 \circ \sigma = -u_1$. Thus, by continuity, for each $t > 1/\sqrt{3}$, we have $u_t \circ \sigma = -u_t$. In particular, $u_t(i) = 0$ for each $t > 1/\sqrt{3}$. If $t \neq 1$, then Lemma 4.3 implies that the vertex i is not an isolated local extremum, and if $t = 1$, then T_t is a right isoceles triangle with right angle at i, and so i is not an isolated local extremum. Therefore, by Theorem 13.1, the eigenfunction u has no critical points for $t > 1/\sqrt{3}$.

For small t, the triangle may be approximated by a sector with angle $2 \cdot \arctan(t)$ and radius 1. In particular, one can show that for sufficiently small t, the function satisfies $u_t \circ \sigma = u_t$; see, for example, Proposition 2.4

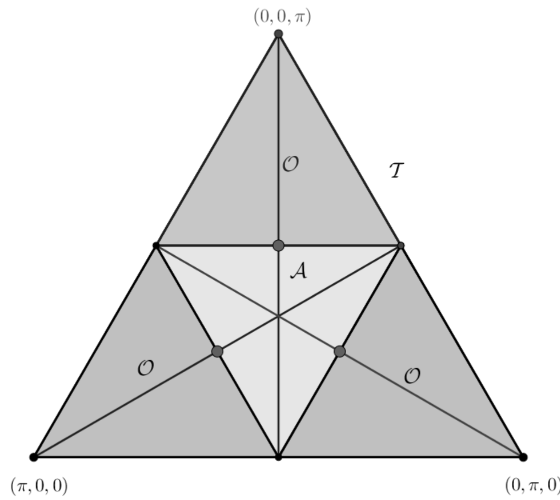

Figure 3. The "Teichmüller space" of labeled triangles. The darker regions correspond to the set $\mathcal{O}$ of obtuse triangles. The lighter region corresponds to the set $\mathcal{A}$ of acute triangles.

in [BnB99]. Therefore, by continuity, $u_t \circ \sigma = u_t$ for each $t < 1/\sqrt{3}$. Thus, $\partial_x u_t(0) = 0$ and 0 is a critical point of u_t for each $t < 1/\sqrt{3}$. □

Let G be the group of linear transformations of the plane generated by isometries and homotheties. If u is a second Neumann eigenfunction for a labeled triangle (v_1, v_2, v_3) and $g \in G$, then $u \circ g$ is a second Neumann eigenfunction for a labeled triangle $(g(v_1), g(v_2), g(v_3))$. In particular, $u \circ g$ has a critical point if and only if u does. Let $\mathcal{T}$ denote the quotient of the set of labeled triangles by G. That is, two labeled triangles (v_1, v_2, v_3) and (v_1', v_2', v_3') define the same point in $\mathcal{T}$ if and only if there exists $g \in G$ so that $(g(v_1), g(v_2), g(v_3)) = (v_1', v_2', v_3')$.

Let $\beta(v_i)$ be the angle at v_i. The map $(v_1, v_2, v_3) \mapsto (\beta(v_1), \beta(v_2), \beta(v_3))$ defines a bijection onto the interior of the convex hull of $(\pi, 0, 0), (0, \pi, 0)$ and $(0, 0, \pi)$ in $\mathbb{R}^3$. See Figure 3.[22] We equip $\mathcal{T}$ with the topology and real-analytic structure of this simplex.

Let $\mathcal{A}$ and $\mathcal{O}$ respectively denote the subspace of $\mathcal{T}$ consisting of (equivalence classes of) acute and obtuse triangles. Let $\mathcal{C}$ denote set of (equivalence

[22]See https://polymathprojects.org/tag/polymath7/ for the same picture together with a description of the triangles that were known to have no hot spots as of 2013.

classes of) triangles T such that each eigenfunction corresponding to the first Neumann eigenvalue of T has a critical point.[23]

THEOREM 13.4. *The set* $\mathcal{C}$ *is open in* $\mathcal{T}$, *the set* $\mathcal{C} \cap \mathcal{A}$ *is dense in* $\mathcal{A}$, *and* $\mathcal{C} \cap \mathcal{O}$ *is empty.*

Proof. Suppose that $T \in \mathcal{C}$. Let u be a second Neumann eigenfunction of T. By Theorem 1.1, the function u has exactly one critical point, p and, by Corollary 7.8, the point p belongs to a side e of T and is nondegenerate. Since p is nondegenerate, by Lemma 10.1 the critical point is stable under a small perturbation of T. In particular, $\mathcal{C}$ is an open subset of $\mathcal{T}$.

Let T^* denote the equilateral triangle. For each $T \neq T^*$, the vector space E_T of second Neumann eigenfunctions of T is a one-dimensional subspace of $L^2(T^*)$ [Siu15], [Miy13], [AB04]. In particular, we have a real line bundle $\mathcal{E}$ over the punctured simplex $\mathcal{T} - \{T^*\}$ such that the fiber over T equals E_T. Let $\mathcal{S} \to \mathcal{T}$ denote the associated "sphere bundle." That is, the fiber of $\mathcal{S}$ over T consists of the two eigenfunctions in E_T whose $L^2(T^*)$-norm equals one.

Let $\mathcal{U}$ be the subset of $\mathcal{T} - \{T^*\}$ obtained by removing the segment $\mathcal{L}$ that joins $T^* = (\pi/3, \pi/3, \pi/3)$ to $(\pi/2, \pi/2, 0)$. The set $\mathcal{U}$ is simply connected, and hence the bundle $\mathcal{S}$ is trivial over $\mathcal{U}$. In particular, there are exactly two sections of $\mathcal{S}$ defined over $\mathcal{U}$. Let $T \to u(T)$ denote one of the sections defined over $\mathcal{U}$. The angles (β_1, β_2) of T at the labeled vertices v_1 and v_2 provide coordinates for $\mathcal{T}$. For each fixed β_2, standard perturbation theory implies that the map $\beta_1 \mapsto u(\beta_1, \beta_2)$ is analytic (away from $\mathcal{L}$), and similarly, for each fixed β_1, the map $\beta_2 \mapsto u(\beta_1, \beta_2)$ is analytic. Therefore, Hartog's separate analyticity theorem implies that $T \to u(T)$ is analytic on $\mathcal{U}$.[24]

In particular, for each i, the value of $u(T)$ at the vertex v_i is a real-analytic function on $\mathcal{U}$. By using (26), we find that $u(\pi/2, \pi/4, \pi/4)(v_2) \neq 0$, $u(\pi/4, \pi/2, \pi/4)(v_3) \neq 0$, and $u(\pi/4, \pi/4, \pi/2)(v_1) \neq 0$. Therefore, for each i, the map $T \mapsto u(T)(v_i)$ is nonzero on a dense subset of $\mathcal{U}$. Corollary 13.2 then implies that the set $\mathcal{C} \cap \mathcal{A}$ is dense in $\mathcal{A}$.

By Corollary 13.2, the set $\mathcal{O} \cap \mathcal{C} \cap \mathcal{U}$ is contained in the set of T such that the first Bessel coefficient $c_1(T)$ of $u(T)$ is zero at the obtuse vertex. The map $T \mapsto c_1(T)$ is a real-analytic function on $\mathcal{U}$. By Proposition 13.3, if T is an obtuse isosceles triangle, then $u(T)$ has no critical point, and hence $c_1(T) \neq 0$. Thus c_1 is nonzero on an open dense subset of $\mathcal{U}$ which, in turn, is open and

[23] Since $\mu_2(T)$ is simple unless T is the equilateral triangle [Siu15], the set $\mathcal{C} \cap \mathcal{A}$ equals the set of triangles such that at least one second Neumann eigenfunction has a critical point.

[24] Similarly, we could trivialize the bundle over the set $\mathcal{U}'$ obtained by removing the segment that joins $(\pi/3, \pi/3, \pi/3)$ to $(0, \pi/2, \pi/2)$. As in the case of $\mathcal{U}$, each section over $\mathcal{U}'$ is also real-analytic. In this way, we see that each local section over $\mathcal{T} - \{T^*\}$ is real-analytic.

dense in $\mathcal{T}$. Hence $\mathcal{O} \cap \mathcal{C}$ is nowhere dense. But from above we have that $\mathcal{C}$ is open and hence $\mathcal{O} \cap \mathcal{C}$ is open. Therefore, $\mathcal{O} \cap \mathcal{C}$ is empty. □

COROLLARY 13.5. *If T is a right triangle, then u does not have a critical point.*

We end the article with the following conjecture.

CONJECTURE 13.6. *If T is not an equilateral triangle, then a second Neumann eigenfunction of T has a critical point if and only if T is an acute triangle that is not isosceles with apex angle greater than $\pi/3$.*

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