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Solenoidal attractors with bounded combinatorics are shy

By Daniel Smania

Dedicated to the memory of Welington de Melo (1946–2016)

Abstract

We show that in a generic finite-dimensional real-analytic family of real-analytic multimodal maps, the subset of parameters on which the corresponding map has a solenoidal attractor with bounded combinatorics is a set with zero Lebesgue measure.

Contents

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1. **Introduction**

A multimodal map $f: I \to I$ is a smooth map defined in an interval I, with a finite number of critical points c_i, all of them local maximum or local minimum, and such that $f(\partial I) \subset \partial I$. We are going to assume f is real-analytic.

For *unimodal* maps with a *quadratic* critical point, the understanding of the *typical* behavior is very satisfactory. Lyubich [25] and Graczyk and Światek [17] proved the density of hyperbolic parameters in the quadratic family. But this was not enough to understand the typical behavior at almost every parameter of the quadratic family. Indeed earlier Jakobson [19] proved that in the complement of the hyperbolic parameters there is a subset of parameters with positive measure for which the dynamics admit an absolutely continuous invariant probability. (The map is stochastic.) Finally Lyubich [28] proved that for almost every parameter in the quadratic family, the map is either regular (a hyperbolic map) or stochastic. Avila, Lyubich and de Melo [3] generalized this result for a non-degenerate real-analytic family of quadratic real-analytic unimodal maps, and Avila and Moreira [5] improved this, proving that in a non-degenerate family the map is either regular or Collet-Eckmann at almost every parameter. There are similar results for real-analytic unimodal maps with higher order by Clark [11]. See also Bruin, Shen and van Strien [10], Avila, Lyubich and Shen [4] and Shen [38] for related results.

Similar studies for multimodal maps (or even unimodal maps with higher order) pose new difficulties. New phenomena appear, as non-renormalizable maps without decay of geometry. (See Bruin, Keller, Nowicki and van Strien [8], Keller and Nowicki [21].) Decay of geometry was an essential tool in the study of unimodal quadratic maps. This was a major difficulty in the study of the so-called Fibonacci renormalization for unimodal maps with higher order in Smania [42] and the proof of the density of hyperbolicity for polynomials in Kozlovski, Shen, van Strien [23], [22]. Moreover, the lack of decay of geometry allows additional metric behaviors, as the existence of wild attractors. See Milnor [33], Bruin, Keller, Nowicki and van Strien [8] and Bruin, Keller and St. Pierre [9].

Another issue is that for families of polynomials with more than one critical point (as in the cubic family), the parameter space has dimension larger than one. This implies that the parapuzzle approach as used in the unimodal case (see Lyubich [27], Avila, Lyubich and de Melo [3]) does not seem to be easily adaptable here, since the fact that holomorphic maps with one-variable are conformal was used in a crucial way.

So as a consequence there are a lot of unanswered questions concerning the typical behavior in the *measure-theoretical* sense in families of polynomials and/or multimodal maps.

One of them is how often maps with *solenoidal attractors* appear in these families. We say that a set $\Lambda \subset I$ is a *solenoidal attractor* of a multimodal map f if there exists an increasing sequence of positive integers n_k, $k \in \mathbb{N}$, and a family of closed intervals $I^k_j \subset I$, $k \in \mathbb{N}$ and $0 \leq j < n_k$, such that

(A) for each k, the intervals in the family $\{I^k_j\}_{j<n_k}$ has pairwise disjoint interior;

(B) we have $f(I^k_j) \subset I^k_{j+1 \bmod n_k}$;

(C) for every k,
$$\{c_i\}_i \cap \cup_{j<n_k} I^k_j \neq \emptyset.$$
and
$$\cup_{j<n_{k+1}} I^{k+1}_j \subset \cup_{j<n_k} I^k_j;$$

(D) we have
$$\Lambda = \cap_k \cup_{j<n_k} I^k_j.$$

See Blokh and Lyubich [6], [7] for more information on attractors for multimodal maps. The solenoidal attractor Λ has *bounded combinatorics* if
$$\sup_k \frac{n_{k+1}}{n_k} < \infty.$$

One important step in previous results about the typical behavior in families of unimodal maps is to prove that at a typical parameter the map *does not* have solenoidal attractors. This was done in the quadratic family by Lyubich [27] and for no degenerate families of unimodal maps by Avila, Lyubich and de Melo [3]. An important tool in many of these results on unimodal maps is the fact that the topological classes of unimodal maps extend to an analytic, codimension one lamination (except a few combinatorial types). This implies that the holonomy of this lamination is quite regular. Our goal is to prove that

Theorem A. *On a generic real-analytic finite-dimensional family of real-analytic multimodal maps with quadratic critical points and negative Schwarzian derivative, the set of parameters whose corresponding maps have a solenoidal attractor with bounded combinatorics has zero Lebesgue measure.*

The precise statement is given in Theorem 7. We also have an analogous result for families with finite smoothness and continuous families. The method used in the unimodal case in Avila, Lyubich and de Melo [3] no longer works in the multimodal case, once the lamination of topological classes has higher codimension, so we are going to use a quite different approach. If a map f has a solenoidal attractor with bounded combinatorics, one can find an induced map F of f that is a composition of unimodal maps, and it is infinitely renormalizable as defined in [40]. In particular, the iterations of the *renormalization operator* $\mathcal{R}$ *for multimodal maps* are well defined for F. Using the universality property proved in [40] one can prove that F belongs to the stable lamination of the omega-limit set Ω of $\mathcal{R}$. The renormalization operator is a real-analytic, compact and non-linear operator acting on a Banach space of real-analytic multimodal maps.

Our main technical result is that

THEOREM B. *Consider the renormalization operator $\mathcal{R}$ acting on real-analytic multimodal maps that are renormalizable with combinatorics bounded by some $p > 0$. Then the omega-limit set Ω of $\mathcal{R}$ is a hyperbolic set.*

The precise statement is given in Section 5. Lyubich [26] proved the hyperbolicity of the omega-limit set in the unimodal case using the so-called Small Orbits Theorem. We use a different approach, reducing the study of the hyperbolicity of Ω to the study of the existence and regularity of solutions for a certain linear cohomological equation. This new method allows us to deal only with real-analytic maps and its complex analytic extensions.

The relationship between renormalization and cohomological equations appears in many contexts, as for instance in the study of rigidity of circle diffeomorphisms and generalized interval exchange transformations. Closer to our setting we have the introduction by Lyubich [26] of the concept of horizontal direction in the study of the renormalization operator for unimodal maps and the study of the hyperbolicity of the fixed point of the action of a pseudo-Anosov map on certain character variety by Kapovich [20].

The final ingredient is a very recent result on partially hyperbolic invariant sets on Banach spaces [43]. The result we use is, roughly speaking, the following (see [43, Th. 1]). Suppose that a "regular" real-analytic operator $\mathcal{R}$ has a hyperbolic set Ω, and its stable lamination $W^s(\Omega)$ satisfies the "Transversal Empty Interior property": every regular manifold M that is transversal to $W^s(\Omega)$ intersects $W^s(\Omega)$ in a subset of empty interior (in the topology of M). Then a generic real-analytic finite-dimensional family intersects $W^s(\Omega)$ on a subset with zero Lebesgue measure. This will give us our main result. The Transversal Empty Interior property for the renormalization operator (see Corollary 9.1) is closely related with the fact that maps F that are infinitely renormalizable with bounded combinatorics can be approximated by hyperbolic maps.

Some of the most classical families of 1-dimensional dynamical systems are families of polynomials. The *cubic family* is the two parameter family

$$f_{a,b}(z) = z^3 - 3a^2z + b.$$

The critical points of $f_{a,b}$ are $a, -a$. We also have

THEOREM C. *The set of of parameters $(a, b) \in \mathbb{R}^2$ such that $f_{a,b}$ is infinitely renormalizable with bounded combinatorics has zero 2-dimensional Lebesgue measure.*

The study of the renormalization operator has a long history. It was first discovered in the unimodal case by Feigenbaum [14], [15] and Coullet and Tresser [45]. They conjectured that the period-doubling renormalization operator has a unique fixed point in a space of quadratic unimodal maps, that this fixed point is hyperbolic and its codimension one stable manifold contains all Feigenbaum maps. Such conjectures could explain certain intriguing universal features of the bifurcation diagram of families of unimodal maps. The existence and hyperbolicity of such a fixed point was proven by Lanford [24]. Such conjectures were later extended for arbitrary bounded combinatorial types, when the fixed point needs to be replaced by an omega-limit set that is hyperbolic. (See Derrida, Gervois and Pomeau [12] and Gol′berg, Sinaĭ and Khanin [16].) Sullivan [44] proved that the orbit by the renormalization operator of a map that is infinitely renormalizable with bounded combinatorics converges to the orbit of a map on the omega limit set and such orbit is determined by the combinatorics of the map. Sullivan's result, in particular, implies the uniqueness of the fixed point of the period-doubling renormalization operator and implies that it attracts all Feigenbaum maps. McMullen [31] proved that the rate of convergence is indeed exponential. Finally Lyubich [26] proved that the omega-limit set of the renormalization operator for unimodal maps is hyperbolic. In particular, Lyubich found a suitable space where the renormalization operator is a complex-analytic non-linear operator. See also de Faria, de Melo and Pinto [13] for the proof of the conjectures in the C^r case.

The renormalization operator for bimodal maps was first considered in MacKay and van Zeijts [29] and Hu [18]. The general multimodal case, with a precise combinatorial description, was described in [39], as well the so-called real and complex a priori bounds for bounded combinatorics. In [41] the phase space universality in the bounded combinatorics case was proved.

It is natural to ask if results such as Theorems A, B and C hold for the full renormalization operator, that is, considering *unbounded* combinatorial types as well. We believe that recent results by Avila and Lyubich [2] on the contraction of the renormalization operator in the hybrid class of infinitely renormalizable unimodal maps with unbounded combinatorics can be carried out for multimodal maps. So the main difficulty seems to be to understand the dynamics of the renormalization operator in the directions transversal to the horizontal spaces, as in the proof of Theorem B. New difficulties arise in the unbounded case, once the omega-limit set of the renormalization operator has not a simple structure anymore. However we are confident that a version of the Key Lemma (Theorem 4) can be obtained in this setting and it will be useful to understand the dynamics of the renormalization operator and the generic behavior in families of multimodal maps.

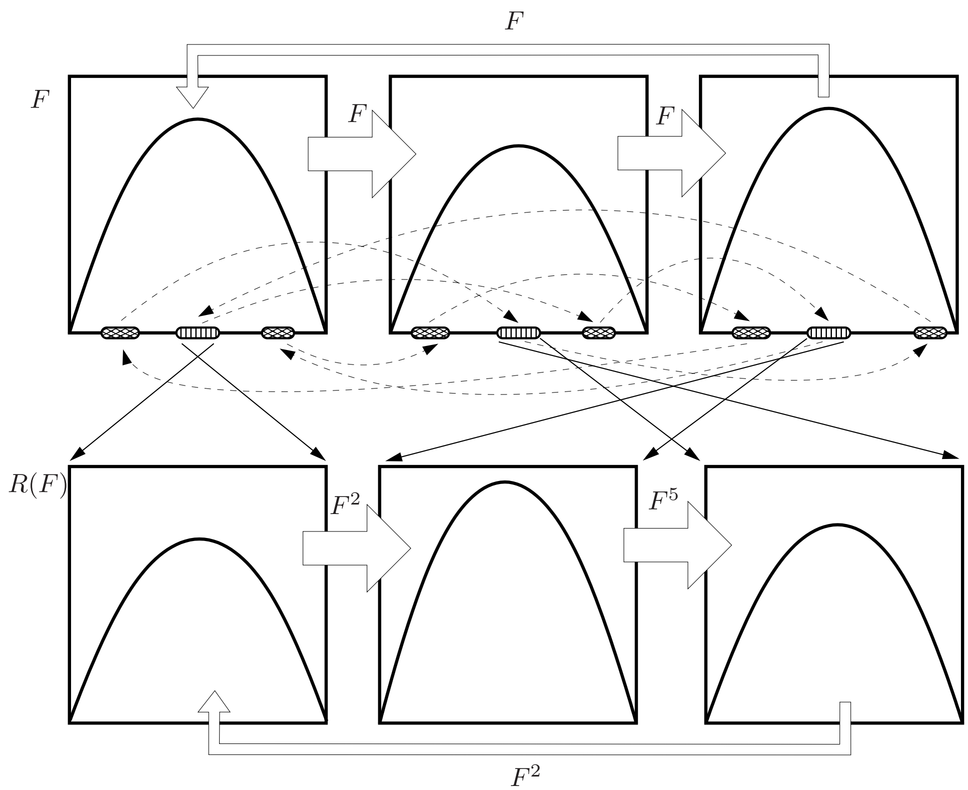

Figure 1. Renormalization of an extended map of type 3.

2. Renormalization of extended maps

To study the renormalisation of multimodal maps, it is more convenient to decompose the dynamics of f in its unimodal parts. Let $I_i = [-1, a_i]$, with $a_i > 0$, be intervals, and let

$$f_i \colon I_i \to I_{i+1 \bmod n} \tag{1}$$

be C^1 maps such that c_i is its unique critical point, which is a maximum and $f_i(\partial I_i) \subset \partial I_{i+1 \bmod n}$. An extended map (see Figure 1) F is defined by a finite sequence $(f_1, \dots, f_n)$ of maps and is the map defined on $I_F^n = \{(x, i) \colon x \in I_i, 1 \leq i \leq n\}$ as

$$F(x, i) = (f_i(x), i + 1 \bmod n). \tag{2}$$

We say that f is a multimodal map of type n if it can be written as a composition of n unimodal maps — to be more precise, if there exist maps $f_1, \dots, f_n$ as above satisfying

(i) $f = f_n \circ \dots \circ f_1$;
(ii) $f_i(c_i) \geq c_{i+1 \bmod n}$.

The n-uple $(f_1, ..., f_n)$ is a decomposition of f. In this paper, we will assume that the unimodal maps are analytic and the critical points of f_i are quadratic. Clearly f has many decompositions.

In [39], we proved that deep renormalizations of infinitely renormalizable multimodal maps are multimodal maps of type n.

2.1. *Renormalization of extended maps.* We say that J is a k-periodic interval, $k \geq 2$, of the extended map F if

- $(c_1, 1) \in J$ ((c_i, i) are the critical points of F);
- $\{J, F(J), \dots, F^{k-1}(J)\}$ is a collection of intervals with disjoint interiors;
- the union of intervals in the above family contains $\{(c_i, i)\}$;
- $F^k(J) \subset J$.

We will call k the period of J. If F has a k-periodic interval, for some k, we say that F is renormalizable.

Suppose that there exists a k-periodic interval for F. Let $P \subset I_1 \times \{1\}$ be the maximal interval that is a k-periodic interval for F. Then $F^k(\partial P) \subset \partial P$. We say that P is a restrictive interval for F of period k. Note that if P and $\tilde{P}$ are, respectively, restrictive intervals for F of period k and $\tilde{k}$, $k < \tilde{k}$, then $\tilde{P} \subset P$. Let P be a restrictive interval, and let $0 = \ell_1 < \cdots < \ell_n$ be the iterations such that $(c_i, i) \in F^{\ell_j}(P)$ for some i. Let P_j be the symmetrization of $F^{\ell_j}(P)$ in relation to (c_i, i). Observe that P_j contains a periodic point in its boundary. If $(c_i, i) \in P_j$, let

$$A_{P_j} \colon \mathbb{C} \times \{i\} \to \mathbb{C} \times \{j\}$$

be the affine map that maps (c_i, i) to $(0, j)$ and this periodic point to -1. Let $[-1, b_j] \times \{j\} = A_{P_j}(P_j)$. Then

$$g_j \colon [-1, b_j] \times \{j\} \to [-1, b_{j+1}] \times \{j+1\}$$

defined by $g_j = A_{P_{j+1}} \circ F^{\ell_{j+1} - \ell_j} \circ A_{P_j}^{-1}$ is a unimodal map. The extended map $G(x, j) = g_j(x, j)$ is called a renormalization of the extended map F. An extended map may have many renormalizations, but at most one with a given period. The renormalization with minimal period k is called the first renormalization of F, and it is denoted $R(F)$.

Following the notation in [40], the primitive marked combinational data (primitive m.c.d) associated with the first renormalization of F is $\sigma = \langle A, \prec, A^c \rangle$, where

- $A = \{1, 2, \dots, k\}$;
- the relation $\prec$ is a partial order on A defined in the following way: $i \prec \ell$ if $F^i J$ and $F^\ell J$ belongs to the same interval in I_F^n and $F^i J$ is on the left side of $F^\ell J$;
- the set A^c is a subset of A and $i \in A^c$ if $F^i J$ intersects $\{(c_i, i)\}$.

The extended map $R(F)$ can be renormalizable again, and so on. If this process can be continued indefinitely, we say that F is infinitely renormalizable. If F is infinitely renormalizable, then all of its renormalizations can be obtained iterating the operator R. Denote by P_0^k the restrictive interval associated to the k-th renormalization $R^k(F)$. If $q \in C(F) := \{(c_i, i)\}$, denote by the corresponding capital letter Q_0^k the symmetrization of the interval $F^\ell(P_0^k)$ that contains q. We reserve the letter p for $(c_1, 1)$. The critical point r for F will be the successor of the critical point q at level k if $r \in F^\ell(Q_0^k)$ for the minimal ℓ so that $F^\ell(Q_0^k)$ contains a critical point. Define $n_r^k = \ell$. Then, for any $r \in C(F)$, $k \in \mathbb{N}$ and $i < n_r^k$, there exists an interval R_{-i}^k so that

- F^i is monotone in R_{-i}^k;
- $F^i(R_{-i}^k) = R_0^k$;
- the interval $F^{n_r^k - i}(Q_0^k)$ is contained in R_{-i}^k.

For details, see [39].

Denote by N_k the period of the restrictive interval P_0^k. We say that F has C-bounded combinatorics if $N_{k+1}/N_k \leq C$ for every k.

For $(x, i), (y, j) \in I_F^n$, we say that $(x, i) < (y, j)$ if $i = j$ and $x < y$. The intervals of I_F^n are the sets $J \times \{i\}$ for some interval $J \subset I_i$ and $1 \leq i \leq n$. If c_i is the critical point of f_i, denote $C(F) = \{(i, c_i)\}_i$.

Let F and G be two infinitely renormalizable extended maps. We say that F and G have same combinatorics if $F^i(c_k) < F^j(c_\ell)$ if and only if $G^i(c_k) < G^j(c_\ell)$ for any i,$j \geq 0$ and k and $\ell < n$.

Let σ_i be the primitive m.c.d. of the (first) renormalization of $R^i(F)$ and $\tilde{\sigma}_i$ be the primitive m.c.d. of the (first) renomalization of $R^i(G)$. It turns out that F and G have the same combinatorics if and only if $\sigma_i = \tilde{\sigma}_i$ for every i. So we say that F has combinatorics $(\sigma_1, \sigma_2, \sigma_3, \dots)$. Moreover, let $\mathcal{C}_{p,n}$ be the set of all primitive m.c.d. that appears as the first renormalization of an extended map with n intervals. It has period either smaller or equal to p. By Corollary 2.3 in [40], for every given sequence $\sigma_i \in \cup_p \mathcal{C}_{p,n}$, $i \geq 1$, there exists a real-analytic extended map (with, say, quadratic critical points) whose i-th renormalization has primitive m.c.d. σ_i.

2.2. *Polynomial-like extended maps.* Denote $\mathbb{C}_n = \{(x, i)\colon x \in \mathbb{C}, 1 \leq i \leq n\}$. (In other words, $\mathbb{C}_n$ is a disjoint union of n copies of $\mathbb{C}$.) Given an open set $O \subset \mathbb{C}_n$, denote

$$O_i = O \cap (\mathbb{C} \times \{i\}).$$

A polynomial-like extended map is a map $F\colon U \to V$, where

- U and V are open sets of $\mathbb{C}_n$, where $\overline{U} \subset V$.
- for each i, $F(U_i) = V_{i+1 \text{ mod } n}$. Moreover, $F\colon U_i \to V_{i+1 \,\mathrm{mod}\, n}$ is a proper map with a unique critical point.
- For each i, we have that U_i and V_i are simply connected domains.

We define

$$\operatorname{mod}(V \setminus U) = \min_i \operatorname{mod}(V_i \setminus U_i).$$

The filled-in Julia set $K(F)$ of a polynomial-like extended map F is defined as

$$K(F) = \cap_{i \geq 0} F^{-i}(V).$$

Note that $K(F)$ is connected if and only if all the critical points of F belong to $K(F)$.

A real-analytic extended map $F\colon I_F^n \to I_F^n$ has a polynomial-like extension if there is a polynomial-like extend map $\tilde{F}\colon U \to V$ such that $I \times \{i\} \subset U_i$ and $F = \tilde{F}$ on I_F^n.

2.3. *Polynomial-like renormalization of real-analytic extended maps.* Let $U \subset \mathbb{C}_n$ be an open set. Given an analytic function $F\colon U \to \mathbb{C}_n$, define the open set

$$\mathcal{D}_U^n(F) := \bigcap_{i=0}^{n-1} F^{-i} U.$$

In other words, $\mathcal{D}_U^n(F)$ is the domain contained in $\mathbb{C}_n$, where F^n is defined.

Let $F\colon U \to V$ be a complex-analytic extension of an extended map. Note that F does not need to be a polynomial-like extension. Suppose that the extended map F is r-times renormalizable, and let P_j and ℓ_j be the intervals and integers associated with the r-th renormalization, as defined in Section 2.1. Define $n_k = \ell_{k+1 \bmod n} - \ell_k$.

Suppose we can find a sequence i_k, $k = 1, \ldots, n$, with $i_1 = 1$, simply connected domains $\hat{U}_k$ and $\hat{V}_k \subset \mathbb{C} \times \{i_k\}$, such that

(1) we have $\{i_k\}_k = \{1, 2, \ldots, n\}$;
(2) we have $P_k \subset \hat{U}_k$ and $\overline{\hat{U}_k} \subset \hat{V}_k$;
(3) the domains $\hat{U}_k$ and $\hat{V}_k$ satisfy $\overline{\hat{U}_k} \subset \mathcal{D}_U^{n_k}(F)$, $F^{n_k}\hat{U}_k = \hat{V}_{k+1 \bmod n}$;
(4) the map
$$F^{n_k}\colon \hat{U}_k \to \hat{V}_{k+1 \bmod n}$$
is a proper map for which $(0, i_k)$ is the unique critical point.

Let $A_j\colon \mathbb{C} \to \mathbb{C}$ be the affine maps defined in Section 2.1. Define $A\colon \mathbb{C}_n \to \mathbb{C}_n$ as $A(x, i) = (A_i(x), i)$. Then we can define

$$g_i\colon A(\hat{U}_j) \to A(\hat{V}_{j+1})$$

as $g_j = A \circ F^{n_k} \circ A^{-1}$. If

$$\hat{U} = \bigcup_i A(\hat{U}_i) \times \{i\},\ \hat{V} = \bigcup_i A(\hat{V}_i) \times \{i\},$$

then the map

$$R^r(F)\colon \hat{U} \to \hat{V}$$

defined by $R^r(F)(x,i) = (g_i(x), i+1)$ is a polynomial-like extension of the real renormalization $R^r(F)$ and it is called a polynomial-like r-th renormalization of F. Note that $\hat{U}_k$ and $\hat{V}_k$ are not uniquely defined, however any polynomial-like extension of $R^r(F)$ does coincide on $I^n_{R^r(F)}$.

LEMMA 2.1. *Let $x_1, \ldots, x_n \in \mathbb{C}$ and $U_i \subset \mathbb{C}$ be open sets such that $x_i \in U_{i_k}$, and let*

$$f_{i,k}\colon U_{i,k} \to \mathbb{C},\ \ k = 1, 2,$$

be holomorphic functions such that

(i) *We have $f_{i,k}(x_i) = x_{i+1 \bmod n}$.*
(ii) *We have $|\lambda_1 \lambda_2 \cdots \lambda_n| > 1$, where $\lambda_i = f'_{i,1}(x_i) = f'_{i,2}(x_i)$.*
(iii) *Let*
$$g_{i,k} = f_{(i+n-1)\, mod\, n,\, k} \circ \cdots \circ f_{(i+1)\, mod\, n,\, k} \circ f_{i,k}.$$
There is $q \geq 1$ such that $g^q_{i,1} = g^q_{i,2}$ for every i.

Then $f_{i,1}(x) = f_{i,2}(x)$ for every i and for every x close to x_i.

Proof. Due to (iii) we have that x_i is a repelling fixed point of $g_{i,k}$ and its multiplier is $\lambda_1 \lambda_2 \cdots \lambda_n$. By the Kœnigs linearization theorem (see, for instance, Milnor [34, Th. 8.2]) there is a *unique* germ of holomorphic function h_i at x_i such that $h'_i(x_i) = 1$ and

$$h_i \circ g_{i,1}(x) = g_{i,2} \circ h_i(x) \tag{3}$$

for x close to x_i. Note that the uniqueness of h_i and (ii) implies

$$h_{i+1} \circ f_{i,1}(x) = f_{i,2} \circ h_i(x) \tag{4}$$

for x close to x_i. Note also that

$$h_i \circ g^q_{i,1}(x) = g^q_{i,2} \circ h_i(x). \tag{5}$$

But since $g^q_{i,1} = g^q_{i,2}$, this implies (due the uniqueness of the solution of the Schröder's equation in the Kœnigs linearization theorem) that $h_i(x) = x$, so by (4) we have $f_{i,1} = f_{i,2}$ for every i. □

PROPOSITION 2.2 (Injectivity of Renormalization). *Let F_1, F_2 be real-analytic extended maps of type n with polynomial-like extensions of type n. Suppose that F_k, $k = 1, 2$ are renormalizable and $R(F_k)$, $k = 1, 2$, also have polynomial-like extensions of type n. Additionally, assume that*

$$r \in \overline{\{F^\ell_k(s),\ \ell \geq 0\}},\ \textit{every}\ r, s \in C(F_k),\ k = 1, 2. \tag{6}$$

Then $R(F_1) = R(F_2)$ implies $F_1 = F_2$.

Proof. We use an argument similar to de Melo and van Strien [32, Ch. VI, Prop. 1.1]. To simplify the notation we assume that $F_k(x,i) = F_k(-x,i)$. Let p_k be the period of the first renormalization of F_k and $q_k = p_k/n$. Using the notation of Section 2.1, we have $c_i = 0$, $a_i = 1$ and $b_i = 1$ for every $i \leq n$. Let $P_{1,k}$ be the interval of the first renormalization of F_k such that $(0, 1) \in P_{1,k}$. Let $0 = \ell_{1,k} < \cdots < \ell_{n,k}$ be the iterations such that $(0, i_{j,k}) \in F_k^{\ell_{j,k}}(P_{1,k})$ for some $i_{j,k}$. Let $P_{j,k} = [-\beta_{j,k}, \beta_{j,k}]$ be the symmetrization of $F_k^{\ell_{j,k}}(P_{j,k})$, where $\beta_{j,k}$ is periodic. Let

$$Y_{j,k} = \left[-\frac{1}{\beta_{j,k}}, \frac{1}{\beta_{j,k}}\right].$$

The map

$$g_{j,k} \colon Y_{j,k} \to Y_{j,k}$$

defined by

$$g_{j,k}(x) = \frac{-1}{\beta_{j,k}} \pi_1(F_k^n(-\beta_{j,k}x, i_{j_k}))$$

is a multimodal map with a polynomial-like extension of degree 2^n and the real trace of its filled-in Julia set is $[-1/\beta_{j,k}, 1/\beta_{j,k}]$. Then $R(F_1) = R(F_2)$ implies that $g_{j,1}^{q_1} = g_{j,2}^{q_2}$ on $[-1, 1]$ and for every j. Moreover, $g_{j,k}^{q_k}$ has a polynomial-like extension of degree 2^n whose real trace of its filled-in Julia set is $[-1, 1]$. Note that if $|\beta_{j,1}| < |\beta_{j,2}|$, then $Y_{j,2}$ is invariant by $g_{j,1}^{q_1}$, which implies that $Y_{j,2}$ would be a restricted interval of $g_{j,1}$, which is not possible since $g_{j,1}^{q_1}$ on $[-1, 1]$ is the first renormalization of $g_{j,1}$. So

$$\beta_{j,1} = \beta_{j,2} \text{ and } Y_{j,1} = Y_{j,2} \text{ for every } j. \tag{7}$$

Counting the number of restricted intervals associated to $[-\beta_{j,k}, \beta_{j,k}]$ in $Y_{2,k}$ we obtain $p_1 = p_2$.

Note that $\ell_{1,k}$ is the number of critical values of $F_k^{\ell_{j,k}}$ in $[-1, 1] \times \{1\}$, which is equal to the number of critical values of $R(F_k)$ in $Y_{1,k} \times \{1\}$. Since $R(F_1) = R(F_2)$ and $Y_{1,2} = Y_{1,2}$, we conclude that $\ell_{1,1} = \ell_{1,2}$ and $i_{2,k} = 1 + \ell_{1,1}$ for $k = 1, 2$. Suppose by induction that $i_{j,1} = i_{j,2}$. Then $\ell_{j,k}$ is the number of critical values of $R(F_k)$ in $Y_{j,k} \times \{j\}$, so $\ell_{j,1} = \ell_{j,2}$ and consequently $i_{j+1,k} = i_{j+1,1} + \ell_{j,1}$. So

$$i_{j,1} = i_{j,2} \text{ and } \ell_{j,1} = \ell_{j,2} \text{ for every } j. \tag{8}$$

Finally, due to (7), (8) and $R(F_1) = R(F_2)$ we have that

$$F_1^{\ell_{j+1,1} - \ell_{j,1}} = F_2^{\ell_{j+1,2} - \ell_{j,2}} \tag{9}$$

in a neighborhood of the point $(-1, i_{j,1})$ and $F_1^{\ell_{j+1,1} - \ell_{j,1}}(-1, i_{j,1}) = (-1, i_{j+1,1})$ for every j.

Let $\lambda_{i,k} = DF_k(-1, i) > 0$. We claim that $\lambda_{i,1} = \lambda_{i,2}$ for every i. Indeed, let $p = p_1 = p_2$, $q = q_1 = q_2$ and $\ell_i = \ell_{i,1} = \ell_{i,2}$. So $g_{j,1}^q = g_{j,2}^q$, and (7) implies that $F_1^{qn} = F_2^{qn}$ in a neighborhood of $[-1, 1] \times \{1, \ldots, n\}$. In particular,

$$(\lambda_{1,1} \cdots \lambda_{n,1})^q = (\lambda_{1,2} \cdots \lambda_{n,2})^q,$$

so

$$\lambda_{1,1}\cdots\lambda_{n,1} = \lambda_{1,2}\cdots\lambda_{n,2}. \tag{10}$$

There is exactly one $j \le n$ such that $(0,1) \in F_1^{\ell_j}(P_{1,1})$, that is the unique i_1 satisfying $\ell_{i_1} = w_1 n + 1$, for some $w_1 \in \mathbb{N}$. In particular

$$DF_k^{\ell_{i_1}}(-1,0) = (\lambda_{1,k}\cdots\lambda_{n,k})^{w_1}\lambda_{1,k}.$$

Due to (9) we have $DF_1^{\ell_{i_1}}(-1,0) = DF_2^{\ell_{i_1}}(-1,0)$, so it follows from (10) that $\lambda_{1,1} = \lambda_{1,2}$. Suppose by induction that $\lambda_{j,1} = \lambda_{j,2}$ for $j < j_0 < n$. Then there is a unique $\ell_{i_{j_0}}$ such that $\ell_{i_{j_0}} = w_{j_0} n + j_0$ for some $w_{j_0} \in \mathbb{N}$ and consequently

$$DF_k^{\ell_{i_{j_0}}}(-1,0) = (\lambda_{1,k}\cdots\lambda_{n,k})^{w_{j_0}}\lambda_{1,k}\lambda_{2,k}\cdots\lambda_{j_0-1,k}\lambda_{j_0,k}.$$

It follows from the induction assumption, $DF_1^{\ell_{i_{j_0}}}(-1,0) = DF_2^{\ell_{i_{j_0}}}(-1,0)$ and (10) that $\lambda_{j_0,1} = \lambda_{j_0,2}$. So $\lambda_{j,1} = \lambda_{j,2}$ for every $j \le n-q$. We conclude that $\lambda_{n,1} = \lambda_{n,2}$ due (10). This concludes the proof of the claim.

Define $f_{i,k}(x) = \pi_1(F_k(x,i))$ and $x_i = -1$. By Lemma 2.1 we have that $f_{i,1}(x) = f_{i,2}(x)$ for every i and x close to -1, so $F_1 = F_2$. □

2.4. *Complex bounds and rigidity of real-analytic, infinitely renormalizable extended maps with bounded combinatorics.* Here we summarize the results in [40].

Theorem 1. *Let $\sigma = (\sigma_i)_{i\in\mathbb{Z}} \in \mathcal{C}_{p,n}^{\mathbb{Z}}$. Then there exists a unique sequence of real-analytic maps $F_{\sigma,i}$, $i \in \mathbb{Z}$, satisfying the following conditions:*

(1) *the map $F_{\sigma,i}$ is renormalizable and $R(F_{\sigma,i}) = F_{\sigma,i+1}$;*
(2) *the first renormalization of $F_{\sigma,i}$ has combinatorics σ_i;*
(3) *there exist polynomial-like extensions $F_{\sigma,i}\colon U_\sigma^i \to V_\sigma^i$, where*

$$\inf_i \operatorname{mod}(V_\sigma^i \setminus U_\sigma^i) > 0.$$

If $U \subset \mathbb{C}$ is a bounded open set such that $0 \in U$, denote by $\mathcal{B}(U)$ the Banach space of all holomorphic functions $g\colon U \to \mathbb{C}$ that has a continuous extension to $\overline{U}$ and a critical point at 0, with the sup norm. If $-1 \in \overline{U}$, let $\mathcal{B}_{\mathrm{nor}}(U)$ be the affine subspace of maps $g \in \mathcal{B}(U)$ such that $g(-1) = -1$.

In an analogous way, let $U \subset \mathbb{C}_n$ be a bounded open set such that

$$U_i = (\mathbb{C} \times \{i\}) \cap U \neq \emptyset$$

and $(0,i) \in U$ for every i. Consider the set $\mathcal{B}(U)$ of all holomorphic functions $G\colon U \to \mathbb{C}_n$ with the following properties:

(1) G has a continuous extension to $\overline{U}$;
(2) G has critical points at $(0,i)$ for every i;
(3) $G(U_i) \subset \mathbb{C} \times \{i+1 \bmod n\}$.

Theorem 2 (Complex Bounds [39], [40]). *There exists* $\epsilon_0 > 0$ *with the following property. If* F *is a real-analytic extended map that is infinitely renormalizable with combinatorics in* $\mathcal{C}^{\mathbb{N}}_{p,n}$ *and with a complex analytic (but not necessarily polynomial-like) extension* $F \in \mathcal{B}(U)$*, then there exist a neighborhood* $V_F \subset \mathcal{B}(U)$ *of* F *and* k_0 *with the following property. For every* $k \geq k_0$ *and every real-analytic and infinitely renormalizable* $G \in V_F$ *with combinatorics in* $\mathcal{C}^{\mathbb{N}}_{p,n}$*, the map* G *has a polynomial-like* k*-th renormalization*

$$R^k(G)\colon U^k \to V^k$$

such that

$$\mathrm{mod}(V^k \setminus U^k) > \epsilon_0.$$

If $(-1, i) \in \overline{U}$ for every i, we can also consider the subset $\mathcal{B}_{\mathrm{nor}}(U)$ of all maps $G \in \mathcal{B}(U)$ such that $G(-1, i) = (-1, i + 1 \bmod n)$ for every i.

Denote $\pi(x, i) = x$. We identify $\mathcal{B}(U)$ with the Banach space

$$\mathcal{B}(\pi(U_1)) \times \mathcal{B}(\pi(U_2)) \times \cdots \times \mathcal{B}(\pi(U_n)) \tag{11}$$

in the following way. For each $G \in \mathcal{B}(U)$, there is a unique decomposition

$$(g_1, \ldots, g_n) \in \mathcal{B}(\pi(U_1)) \times \mathcal{B}(\pi(U_2)) \times \cdots \times \mathcal{B}(\pi(U_n)), \tag{12}$$

where g_i is defined by $g_i(x) = \pi \circ G(x, i)$. For each n-uple as in (12), we can associate $G \in \mathcal{B}(U)$ defined by $G(x, i) = (g_i(x), i + 1 \bmod n)$. With this identification, $\mathcal{B}_{\mathrm{nor}}(U)$ turns out to be an affine subspace of $\mathcal{B}(U)$. So given $F \in \mathcal{B}_{\mathrm{nor}}(U)$ we can consider the tangent space of $\mathcal{B}_{\mathrm{nor}}(U)$ at F, denoted by $T_F\mathcal{B}_{\mathrm{nor}}(U)$. Using the identification (11), then $T_F\mathcal{B}_{\mathrm{nor}}(U)$ is the subspace of

$$(v_1, \ldots, v_n) \in \mathcal{B}(\pi(U_1)) \times \mathcal{B}(\pi(U_2)) \times \cdots \times \mathcal{B}(\pi(U_n))$$

such that $v_i(-1) = 0$ for $i = 1, \ldots, n$. In particular, $T_F\mathcal{B}_{\mathrm{nor}}(U)$ does not depend on F, so sometimes we will write $T\mathcal{B}_{\mathrm{nor}}(U)$.

Given $\delta > 0$ and $\theta > 0$, let $D_{\delta,\theta}$ be the set

$$\{x \in \mathbb{C} : \ \mathrm{dist}(x, [-1, 1]) < \delta \text{ and } |\mathrm{Im}(x)| < \theta(\mathrm{Re}(x) + 1)\} \times \{1, \ldots, n\}.$$

Define

$$\Omega_{p,n} = \{F_{\sigma,0}\}_{\sigma \in \mathcal{C}^{\mathbb{Z}}_{p,n}}.$$

Indeed, due Theorem 1 we have that

$$\Omega_{p,n} = \{F_{\sigma,i}\}_{\sigma \in \mathcal{C}^{\mathbb{Z}}_{p,n}}$$

for every i. Using Theorems 2 and 1, one can show that there exists ϵ_0 such that for every $F_{\sigma,0} \in \Omega_{p,n}$, there exists a polynomial-like extension $F_{\sigma_0}\colon U^0_\sigma \to V^0_\sigma$ such that $\mathrm{mod}(V^0_\sigma \setminus U^0_\sigma) > \epsilon_0$. There exists δ_0 such that for every simply connected domains $Q \supset W \supset [-1, 1]$ such that $\mathrm{mod}(Q \setminus W) \geq \epsilon_0/2$, we have

$$\{x \in \mathbb{C} : \ \mathrm{dist}(x, [-1, 1]) \leq \delta_0\} \subset Q. \tag{13}$$

In particular,

$$\overline{D_{\delta_0,\theta}} \subset U^0_\sigma$$

for every $\theta > 0$ and for every $\sigma \in \mathcal{C}_{p,n}$. In particular, $\Omega_{p,n} \subset \mathcal{B}_{\text{nor}}(D_{\delta_0,\theta})$.

Consider the shift operator on $\mathcal{C}^{\mathbb{Z}}_{p,n}$; that is, if $\sigma = (\sigma_i)_{i\in\mathbb{Z}}$, then $S(\sigma) = \sigma'$, where $\sigma'_i = \sigma_{i+1}$.

Corollary 2.3. *The set $\Omega_{p,n} \subset \mathcal{B}_{\text{nor}}(D_{\delta_0,\theta})$ is a Cantor set. Indeed the map $H\colon \mathcal{C}^{\mathbb{Z}}_{p,n} \to \Omega_{p,n}$ given by*

$$H(\sigma) = F_{\sigma,0}$$

is a homeomorphism. Moreover, $R(F_{\sigma,0}) = F_{S(\sigma),0}$.

Proof. The map H is continuous and onto due to [40, §7.1]. The injectivity of H follows from Proposition 2.2. □

3. Complexification of the renormalization operator $\mathcal{R}$

Given $\theta_0 > 0$, by Theorem 2, for each $F \in \Omega_{p,n}$, there exist a neighborhood $V_F \subset \mathcal{B}_{\text{nor}}(D_{\delta_0,\theta_0})$ of F and k_F such that for every real map $G \in V_F$ that is infinitely renormalizable with combinatorics in $\mathcal{C}_{p,n}$ and for every $k \geq k_F$, we have a polynomial-like k-th renormalization $R^k(G)\colon \hat{U} \to \hat{V}$ with $\text{mod}(\hat{V} \setminus \hat{U}) > \epsilon_0$. In particular, $R^k(G) \in \mathcal{B}_{\text{nor}}(D_{\delta_0,\theta_0})$. Since $\Omega_{p,n}$ is a compact set, choose a finite sub-cover $\{V_{F_i}\}_{i\leq\ell}$ of $\Omega_{p,n}$. Let $k_0 = \max_{i\leq\ell} k_{F_i}$ and $\mathcal{V} = \cup_{i\leq\ell} V_{F_i}$.

Let H be the homeomorphism defined in Corollary 2.3. For every $\gamma = (\gamma_1, \ldots, \gamma_{k_0}) \in \mathcal{C}^{k_0}_{p,n}$, define the compact set

$$\Omega_{p,n}(\gamma) = H(\{\sigma \in \mathcal{C}^{\mathbb{Z}}_{p,n}\colon\ \sigma_i = \gamma_i \text{ for } 1 \leq i \leq k_0\}).$$

We have

$$d_1 = \inf\{\text{dist}_{\mathcal{B}_{\text{nor}}(D_{\delta_0,\theta_0})}(G_1, G_2)\colon G_1 \in \Omega_{p,n}(\hat{\gamma}), G_2 \in \Omega_{p,n}(\tilde{\gamma}),\ \hat{\gamma} \neq \tilde{\gamma}\} > 0.$$

Given $F \in \Omega_{p,n}$, consider the intervals $P_{F,j}$, $j = 1, \ldots, n$, integers n_j, corresponding the restrictive intervals of the k_0-th renormalization of F, as in Section 2.3. Each interval $P_{F,j}$ contains a unique repelling periodic point $(\beta_{F,j}, i_j)$ in its boundary. These repelling periodic points have a complex analytic continuation $(\beta_{G,j}, i_j)$ for every G in a connected neighborhood $\tilde{W}_F$ of F in $\mathcal{B}_{\text{nor}}(D_{\delta_0,\theta_0})$ that is also a repelling periodic point for G. Note that for a real map G, the point $\beta_{G,j}$ is real and we can assume that it has the same combinatorics as $\beta_{F,j}$. We can also assume that $\tilde{W}_F \subset \mathcal{V}$ and that the diameter of $\tilde{W}_F$ is smaller than $d_1/2$.

Let $d_2 < d_1$ be a Lebesgue number of the cover $\{\tilde{W}_F\}_{F\in\Omega_{p,n}}$ of $\Omega_{p,n}$. For every $F \in \Omega_{p,n}$, choose a connected neighborhood $W_F \subset \tilde{W}_F$ of F so that

$$\text{diam}_{\mathcal{B}_{\text{nor}}(D_{\delta_0,\theta_0})} W_F < d_2/4.$$

Let $F_1, F_2 \in \Omega_{p,n}$. Consider the complex analytic continuations $(\beta^1_{G,j}, i^1_j)$, $(\beta^2_{G,j}, i^2_j)$ of $(\beta_{F_1,j}, i^1_j)$ and $(\beta_{F_2,j}, i^2_j)$ defined for every $G \in W_{F_1}$ and $G \in W_{F_2}$ respectively. Suppose that $W_{F_1} \cap W_{F_2} \neq \emptyset$. We claim that $i^1_j = i^2_j$ and $\beta^1_{G,j} = \beta^2_{G,j}$ for every $G \in W_{F_1} \cap W_{F_2}$ and j. Since the diameter of $W_{F_1} \cup W_{F_2}$ is smaller than d_2, we have that $W_{F_1} \cup W_{F_2} \subset \tilde{W}_{F_3}$ for some $F_3 \in \Omega_{p,n}$. Note that the distance between two maps in $\{F_1, F_2, F_3\}$ is smaller than d_1. In particular, the combinatorics of their k_0-th renormalizations are the same, so $i^1_j = i^2_j = i^3_j$ for every j. Consider the complex analytic continuation $(\beta^3_{G,j}, i_j)$ of $(\beta_{F_3,j}, i_j)$ defined for $G \in \tilde{W}_{F_3}$. Then $(\beta^3_{F_1,j}, i_j)$ and $(\beta_{F_1,j}, i_j)$ are repelling periodic points with the same combinatorics. Since F_1 has negative Schwarzian derivative, the minimal principle implies that $\beta^3_{F_1,j} = \beta_{F_1,j}$. In an analogous way, $\beta^3_{F_2,j} = \beta_{F_2,j}$. The uniqueness of the analytic continuation of a repelling periodic point implies that $\beta^3_{G,j} = \beta^1_{G,j}$ for $G \in W_{F_1}$ and $\beta^3_{G,j} = \beta^2_{G,j}$ for $G \in W_{F_2}$. This concludes the proof of the claim.

In particular, the function

$$G \mapsto (\beta_{G,j}, i_j)$$

is well defined and complex analytic in $\mathcal{W} = \cup_{F \in \Omega_{p,n}} W_F$. There is a small abuse of notation here since i_j depends on G, but it is a locally constant function.

Fix $G \in W_F$. Let $A_{G,j} \colon \mathbb{C} \times \{i_j\} \to \mathbb{C} \times \{j\}$ be the affine transformation that maps $(\beta_{G,j}, i_j)$ to $(-1, j)$ and $(0, i_j)$ to $(0, j)$, and $A_G \colon \mathbb{C}_n \to \mathbb{C}_n$ as $A_G(x, i) = (A_{G,i}(x), i)$. Let $D^{F,j}$ be the set

$$\overline{A^{-1}_{F,j}(\{z \in \mathbb{C} : \ \mathrm{dist}(z, [-1,1]) < \delta_0 \text{ and } \ |\mathrm{Im}(z)| < \theta_0(\mathrm{Re}(z) + 1)\} \times \{j\})}.$$

Since $\mathrm{mod}(\hat{V} \setminus \hat{U}) > \epsilon_0$, we have that

$$D^{F,j} \subset \hat{U}_j \subset \mathcal{D}^{n_j}_{D_{\delta_0,\theta_0}}(F).$$

Moreover, due the complex bounds, reducing θ_0 and δ_0 we can assume that the interior of the sets in the family

$$\{F^m(D^{F,j})\}_{m < n_j}$$

are pairwise disjoint, and the intersection of the closure of every two of those sets is contained in

$$\{F^m(\beta_{F,j}, i_j)\}_{m < n_j}.$$

Let $G \in W_F$, and define the set $D^{G,j}$ as

$$\overline{A^{-1}_{G,j}(\{z \in \mathbb{C} : \ \mathrm{dist}(z, [-1,1]) < \delta_0 \text{ and } |\mathrm{Im}(z)| < \theta_0(\mathrm{Re}(z) + 1)\} \times \{j\})}.$$

Reducing the neighborhood W_F of F and θ_0, we can assume that

$$D^{G,j} \subset \mathcal{D}^{n_j}_{D_{\delta_0,\theta_0}}(G)$$

for every $G \in W_F$. Furthermore, the interior of the sets in the family

$$\{G^m(D^{G,j})\}_{m<n_j}$$

are pairwise disjoint, and the intersection of the closure of every two of those sets is contained in

$$\{G^m(\beta_{G,j}, i_j)\}_{m<n_j}.$$

Define the complexification of the renormalization operator

$$\mathcal{R}\colon \mathcal{W} \to \mathcal{B}_{\rm nor}(D_{\delta_0,\theta_0})$$

as

$$\mathcal{R}(G)(x,j) = A_{G,j+1} \circ G^{n_j} \circ A^{-1}_{G,j}(x,j) \tag{14}$$

if $G \in W_F$. The operator $\mathcal{R}$ is a compact complex analytic map. From now on denote $U = D_{\delta_0,\theta_0}$.

Remark 3.1. Let $\tilde{U}$ be a little larger complex open domain that contains $\overline{U}$. Consider the complex analytic transformation

$$\tilde{\mathcal{R}}\colon \mathcal{W} \to \mathcal{B}_{\rm nor}(\tilde{U})$$

defined exactly as in (14). Let

$$i\colon \mathcal{B}_{\rm nor}(\tilde{U}) \to \mathcal{B}_{\rm nor}(U)$$

be the compact linear inclusion between these spaces. Then $\mathcal{R} = i \circ \tilde{\mathcal{R}}$, so the complexification of the renormalization operator is a strongly compact operator as defined in [43].

Let $v \in T_G\mathcal{B}_{\rm nor}(U)$. If $z \in U$ and $G^j(z) \in U$ for every $j < i$, then $(G+tv)^i$ is defined in a neighborhood of z and we can define

$$a_i(z) = \frac{\partial}{\partial t}(G+tv)^i|_{t=0}(z) = \sum_{j=0}^{i-1} DG^{i-j-1}(G^{j+1}(z))v(G^j(z)). \tag{15}$$

Let $F \in \Omega_{p,n}$ and $G \in W_F$. For each $v \in T_G\mathcal{B}_{\rm nor}(U)$ and $z \in U_j$, we have

$$\begin{aligned}(D\mathcal{R}_G \cdot v)(x,j) &= \frac{\partial}{\partial t} A_{G+tv,j+1} \circ (G+tv)^{n_j} \circ A^{-1}_{G+tv,j}(x,j)|_{t=0} \\ &= -\frac{\partial_G \beta_{G,j+1} \cdot v}{\beta_{G,j+1}} \cdot A_{G,j+1} \circ G^{n_j} \circ A^{-1}_{G,j}(x,j) \\ &\quad - \frac{1}{\beta_{G,j+1}}\Big(a_{n_j} \circ A^{-1}_{G,j}(x,j) + (\partial_x G^{n_j}) \\ &\qquad \circ A^{-1}_{G,j}(x,j) \cdot (-\partial_G \beta_{G,j} \cdot v\ x, j)\Big).\end{aligned}$$

THEOREM 3. *Let $F \in \mathcal{W}$. Then $D_F\mathcal{R}(T_F\mathcal{B}_{\rm nor}(U))$ is dense in $T_{\mathcal{R}F}\mathcal{B}_{\rm nor}(U)$.*

Proof. The proof is quite similar to the proof of the analogous statement in [3]. Let $w \in T_{\mathcal{R}F}\mathcal{B}_{\text{nor}}(U)$. Then $w(-1,k) = 0$ and $w'(0,k) = 0$ for every k. We are going to define a function

$$\hat{v}\colon \cup_j \cup_{m<n_j} G^m(D^{G,j}) \to \mathbb{C}$$

in the following way. Define the function $\hat{v}$ as 0 on

$$\cup_j \cup_{0<m<n_j} G^m(D^{G,j})$$

and

$$\hat{v}(z) = [DG^{n_j-1}(G(z))]^{-1} \cdot w \circ A_{G,j}(z)$$

for $z \in D^{G,j}$. Also define $\hat{v}(-1,k) = 0$ for every k. Then $\hat{v}$ is well defined, it is continuous on

$$\Lambda = \cup_j \cup_{m<n_j} G^m(D^{G,j}) \cup \{(-1,k)\}_k,$$

and it is complex analytic in the interior of Λ.

Moreover, $\hat{v}$ vanishes on the orbit of the periodic points $\{\beta_{G,j}\}_j$. Since $\mathbb{C} \times \{i\} \setminus \Lambda$ is a connected set, by Mergelyan's Theorem, for each given $\epsilon > 0$ and i, we can find a polynomial q_i such that $|\hat{v}(z) - q_i(z)| < \epsilon$ for $z \in \Lambda_i = \Lambda \cap \mathbb{C} \times \{i\}$. Define

$$\hat{q}_i(z) = q_i(z) - q_i'(0,i)z - q_i(-1,i) - q_i'(0,i).$$

Note that $\hat{q}_i'(0,i) = 0$ and $\hat{q}_i(-1,i) = 0$. Define $q(x,i) = \hat{q}_i(x)$. We have that $q \in T_G\mathcal{B}_{\text{nor}}(U)$ and

$$|D_G\mathcal{R} \cdot q - w|_{\mathcal{B}(U)} \to_{\epsilon\to 0} 0. \qquad \square$$

4. Action of $D\mathcal{R}$ on horizontal directions

4.1. *Horizontal direction.* Let $F\colon I_F^n \to I_F^n$ be a real-analytic extended map that is either infinitely renormalizable with bounded combinatorics in $\mathcal{C}_{p,n}$ or whose critical points belongs to the same periodic orbit. A continuous function

$$v\colon I_F^n \to T\mathbb{C}_n$$

is a *horizontal direction* of F if

(1) for each $x \in I_F^n$, we have $v(x) \in T_{F(x)}\mathbb{C}_n$;
(2) the function v is real-analytic in the interior of I_F^n;
(3) there is a quasiconformal vector field

$$\alpha\colon W \to T\mathbb{C}_n,$$

defined in a complex neighborhood W of the post critical set of F, such that for every x in the post critical set,

$$v(x) = \alpha(F(x)) - DF(x) \cdot \alpha(x); \tag{16}$$

(4) we have $\alpha(c) = 0$ for every critical point c of F.

Denote by E^h_F the set of $v \in T_F\mathcal{B}_{\mathrm{nor}}(U)$ such that v is horizontal. Of course E^h_F is a linear subspace of $T_F\mathcal{B}_{\mathrm{nor}}(U)$.

PROPOSITION 4.1 (Infinitesimal pullback argument; Avila, Lyubich and de Melo [3]). *Let $F \in \Omega_{n,p}$. Let*

$$F\colon W \to V$$

be a polynomial-like extension of F and $v \in \mathcal{B}(W) \cap T_F\mathcal{B}_{\mathrm{nor}}(U)$ such that there exists a quasiconformal vector field α, defined in a neighborhood of the post critical set of F, such that

$$v(x) = \alpha \circ F(x) - DF(x)\cdot \alpha(x) \tag{17}$$

for every $x \in P(F)$. In particular, $v \in E^h_F$. Reducing a little bit the domain W, there exists a quasiconformal vector field extension $\alpha\colon W \to \mathbb{C}$ such that (17) holds for every $x \in W$.

PROPOSITION 4.2 (Invariance). *Let $F \in \Omega_{n,p}$. Then*

$$D_F\mathcal{R}(E^h_F) \subset E^h_{\mathcal{R}F} \tag{18}$$

and

$$(D_F\mathcal{R})^{-1}(E^h_{\mathcal{R}F}) \subset E^h_F. \tag{19}$$

Proof. The proof of (18) is quite similar to the proof of a similar statement in [41]. Indeed, consider a_i as in (15). Note that

$$a_i(z) = v(F^{i-1}) + DF(F^{i-1}(z))a_{i-1}(z).$$

By an inductive argument one can show that

$$a_i = \alpha \circ F^i - DF^i \cdot \alpha$$

on $P(F)$. Denote

$$\alpha(\beta_{F,j+1}) = \partial_F\beta_{F,j+1}\cdot v.$$

Then if $z = (x,j) \in P(F)$, we have

$$\begin{aligned}
&(D\mathcal{R}_F \cdot v)(x,j)\\
&= -\frac{\partial_F\beta_{F,j+1}\cdot v}{\beta_{F,j+1}} \cdot A_{F,j+1}\circ F^{n_j}\circ A^{-1}_{F,j}(x,j)\\
&\quad - \frac{1}{\beta_{F,j+1}}\big(a_{n_j}\circ A^{-1}_{F,j}(x,j) + (DF^{n_j})\circ A^{-1}_{F,j}(x,j)\cdot(-\partial_F\beta_{F,j}\cdot v\ x,j)\big)\\
&= -\frac{\alpha(\beta_{F,j+1})}{\beta_{F,j+1}}\cdot(\mathcal{R}F)(z)\\
&\quad - \frac{1}{\beta_{F,j+1}}\alpha\circ A^{-1}_{F,j+1}\circ A_{F,j+1}\circ F^{n_j}\circ A^{-1}_{F,j}(x,j)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta_{F,j}}{\beta_{F,j+1}} DF^{n_j} \circ A_{F,j}^{-1}(x,j) \cdot \frac{1}{\beta_{F,j}} \alpha \circ A_{F,j}^{-1}(x,j) \\
&+ D_z(\mathcal{R}F) \cdot (\frac{\alpha(\beta_{F,j})}{\beta_{F,j}}\ x, j) \\
= &-\frac{\alpha(\beta_{F,j+1})}{\beta_{F,j+1}} \cdot (\mathcal{R}F)(z) \\
&- \frac{1}{\beta_{F,j+1}} \alpha \circ A_{F,j+1}^{-1} \circ (\mathcal{R}F)(z) + D_z(\mathcal{R}F) \cdot \frac{1}{\beta_{F,j}} \alpha \circ A_{F,j}^{-1}(x,j) \\
&+ D_z(\mathcal{R}F) \cdot (\frac{\alpha(\beta_{F,j})}{\beta_{F,j}}\ x, j).
\end{aligned}
$$

Define the vector field $r(\alpha)$ as

$$
r(\alpha)(x,j) = -\frac{1}{\beta_{F,j}} \alpha \circ A_{F,j}^{-1}(x,j) - (\frac{\alpha(\beta_{F,j})}{\beta_{F,j}} \cdot x, j) \tag{20}
$$

for $z = (x,j) \in U_j$. Then

$$
(D\mathcal{R}_F \cdot v)(z) = r(\alpha) \circ (\mathcal{R}F)(z) - D_z(\mathcal{R}F) \cdot r(\alpha)(z) \tag{21}
$$

for z in the postcritical set of $\mathcal{R}F$. Note that $r(\alpha)$ is a quasiconformal vector field in a neighborhood of the post critical set of $\mathcal{R}F$. So $D\mathcal{R}_F \cdot v \in E^h_{\mathcal{R}F}$.

Now suppose that $v \in (D_F\mathcal{R})^{-1}(E^h_{\mathcal{R}F})$. Then $D\mathcal{R}_F \cdot v \in E^h_{\mathcal{R}F}$, so there exists a quasiconformal vector field $\gamma\colon \mathbb{C}_n \to \mathbb{C}$ such that

$$
(D\mathcal{R}_F \cdot v)(z) = \gamma \circ (\mathcal{R}F)(z) - D_z(\mathcal{R}F) \cdot \gamma(z) \tag{22}
$$

for every z in a neighborhood of the post critical set of $\mathcal{R}F$. Define

$$
\delta_j = \partial_t \beta_{F+tv,j}\big|_{t=0} = \partial_F \beta_{F,j} \cdot v.
$$

Define α in $A_{F,j}(U)$ as

$$
\alpha(z) = \beta_{F,j}\gamma \circ A_{F,j}^{-1}(z) + \delta_j A_{F,j}^{-1}(z).
$$

Let z be a point very close to the post critical set of F. Then

$$
\{k \geq 0 \text{ such that } \ F^k(z) \in \cup_{j \leq n} A_{F,j}(U)\} \neq \emptyset.
$$

Let $k(z)$ be a minimal element of the above set. Note that $z \mapsto k(z)$ is locally constant. We define $\alpha(z)$ for z close to the post critical set of F by induction of $k(z)$. We already defined $\alpha(z)$ when $k(z) = 0$. If $k(z) > 0$, then $k(F(z)) = k(z) - 1$ and we define

$$
\alpha(z) = \frac{v(z) + \alpha(F(z))}{DF(z)}.
$$

One can check that α is a quasiconformal vector field and

$$
v = \alpha(F(z)) - DF(z)\alpha(z)
$$

in a neighborhood of the post critical set of F, so $v \in E^h_F$. $\square$

PROPOSITION 4.3. *Let $F \in \Omega_{n,p}$. Then $D\mathcal{R}_F$ is injective.*

Proof. Let v be such that $D\mathcal{R}_F \cdot v = 0$. By Proposition 4.2 we have that $v \in E^h_F$. By Proposition 4.1 there is a quasiconformal vector field α, defined in a neighborhood of the Julia set of F, satisfying (17) for every x on its Julia set. Let $r(\alpha)$ be the quasiconformal vector field defined by (20). Then by (21) we have that $r(\alpha)$ satisfies

$$0 = r(\alpha) \circ (\mathcal{R}F)(z) - D_z(\mathcal{R}F) \cdot r(\alpha)(z)$$

for every z in the Julia set of $\mathcal{R}F$. We can easily conclude that $r(\alpha)(z) = 0$ at every repelling periodic point z of $\mathcal{R}F$ and consequently at every point of its Julia set. By (20) we have that α is zero at every point of the small Julia sets of F corresponding to this renormalization, and by (17) we have that v vanishes in these small Julia sets as well. So $v = 0$ everywhere. □

PROPOSITION 4.4 (Closedness). *Let $F_k \in \Omega_{n,p}$ and $v_k \in E^h_{F_k} \subset T\mathcal{B}_{\mathrm{nor}}(U)$ be sequences such that (F_k, v_k) converges to $(F, v) \in \mathcal{B}_{\mathrm{nor}}(U) \times T\mathcal{B}_{\mathrm{nor}}(U)$. Then $F \in \Omega_{p,n}$ and $v \in E^h_F$. In particular, E^h_F is Banach subspace of $T_F\mathcal{B}_{\mathrm{nor}}(U)$.*

Proof. Due to the definition of the operator $\mathcal{R}$, the map $\mathcal{R}F$ has a polynomial-like extension

$$\mathcal{R}F \colon W \to V,$$

with $\overline{U} \subset W$. Reducing V a little bit, we can assume ∂V is a finite union of analytic curves and that for k large enough, the map

$$\mathcal{R}F_k \colon W_k \to V,$$

where $W_k \subset \mathbb{C}_n$, $\overline{U} \subset W_k$, is the set whose connected components are the connected components of

$$(\mathcal{R}F_k)^{-1}V$$

that intersect $\{(0, j)\}_j$, is a polynomial-like extension of $\mathcal{R}F_k$. Since $v_k \in E^h_{F_k}$, we have that $D_{F_k}\mathcal{R} \cdot v_k \in E^h_{\mathcal{R}F_k}$, so there exists a quasiconformal vector field $\tilde{\gamma}^k$ such that

$$(D\mathcal{R}_{F_k} \cdot v_k)(z) = \tilde{\gamma}^k \circ (\mathcal{R}F_k)(z) - D_z(\mathcal{R}F_k) \cdot \tilde{\gamma}^k(z)$$

holds for z in a neighborhood of the post critical set of $\mathcal{R}F_k$.

Now we use the infinitesimal pullback argument in Avila, Lyubich and de Melo [3]. For each k, there exist $C > 0$ and a quasiconformal vector field $\gamma^k_0 \colon \mathbb{C}_n \to \mathbb{C}$ with the following properties:

(1) The vector field γ^k_0 vanishes outside V. Moreover, $\gamma^k_0(-1) = \gamma^k_0(0) = 0$.
(2) It satisfies

$$(D\mathcal{R}_{F_k} \cdot v_k)(z) = \gamma^k_0 \circ (\mathcal{R}F_k)(z) - D_z(\mathcal{R}F_k) \cdot \gamma^k_0(z)$$

for every $z \in \partial W_k$.

(3) The vector field γ_0^k is C^∞ in a neighborhood of

$$\overline{V \setminus W_k}$$

and

$$|\overline{\partial}\gamma_0^k| \leq C$$

on this set.

(4) We have $\gamma_0^k = \tilde{\gamma}^k$ in a neighborhood of the post critical set of $\mathcal{R}F_k$.

Define by induction γ_j^k as 0 outside V and

$$\gamma_{j+1}^k(z) = \frac{\gamma_j^k \circ (\mathcal{R}F_k)(z) - (D\mathcal{R}_{F_k} \cdot v_k)(z)}{D_z(\mathcal{R}F_k)}$$

on $V \setminus \{(0,m)\}_m$, and $\gamma_{j+1}^k(0,m) = 0$.

Using the McMullen compactness criterion for quasiconformal vectors fields [31, Cor. A.11], one can prove that for each k, the sequence

$$\hat{\gamma}_j^k = \frac{1}{j}\sum_{t=0}^{j-1} \gamma_t^k$$

has a convergent subsequence, uniform on compact subsets of $\mathbb{C}_n$. Moreover, such limits are quasiconformal vectors fields. Let γ_∞^k be one of theses limits. Since the filled-in Julia sets of the polynomial-like extensions of $\mathcal{R}F_k$ do not support invariant line fields [40], we conclude that $|\overline{\partial}\gamma_\infty^k| \leq C$ on $\mathbb{C}_n$. Note that

$$(D\mathcal{R}_{F_k} \cdot v_k)(z) = \gamma_\infty^k \circ (\mathcal{R}F_k)(z) - D_z(\mathcal{R}F_k) \cdot \gamma_\infty^k(z), \ z \in \overline{U}. \tag{23}$$

By the compactness criterion for quasiconformal vectors fields in McMullen [31] we can consider a convergent subsequence $\gamma^{k_t}\infty \to_t \gamma$, where γ is a quasiconformal vector field on $\mathbb{C}_n$ and the convergence is uniform on compact subsets of $\mathbb{C}_n$. By (23) we have

$$(D\mathcal{R}_F \cdot v)(z) = \gamma \circ (\mathcal{R}F)(z) - D_z(\mathcal{R}F) \cdot \gamma(z), \ z \in \overline{U}, \tag{24}$$

so $D\mathcal{R}_F \cdot v \in E^h_{\mathcal{R}F}$, and so by (19) we have $v \in E^h_F$. □

Proposition 4.5 (Contraction on the horizontal directions). *There exist K and $\theta_1 > 1$ such that for every $F \in \Omega_{n,p}$ and $v \in E^h_F$, we have*

$$|D_F\mathcal{R}^i \cdot v|_{T\mathcal{B}_{\mathrm{nor}}(U)} \leq K{\theta_1}^{-i}|v|_{T\mathcal{B}_{\mathrm{nor}}(U)}.$$

We do not provide a proof for Proposition 4.5 since it can be proven in exactly the same way as is done in the unimodal setting. One can use the argument by Lyubich [26, Th. 6.3] using the Schwarz's lemma and the rigidity of McMullen's towers [31]. An infinitesimal argument using the rigidity of McMullen's towers and the compactness of the renormalization operator is given in [41, Prop. 3.9] (in the case of the fixed point of the period doubling renormalization) can be also applied here. We also cite the new methods by Avila and Lyubich [2] to prove the contraction in the horizontal directions in the case of unimodal unbounded combinatorics.

PROPOSITION 4.6 (Contraction on the hybrid classes). *There exists* $\lambda_1 \in (0,1)$ *with the following property. Let* F *be a real-analytic polynomial-like map of type* n *that is infinitely renormalizable with combinatorics bounded by* p. *Then there exist* $G \in \Omega_{n,p}$, $k_0 = k_0(F)$ *and* $C = C(F)$ *such that* $\mathcal{R}^k F \in \mathcal{B}(U)$ *for every* $k \geq k_0$ *and*

$$|\mathcal{R}^k F - \mathcal{R}^k G|_{\mathcal{B}_{\mathrm{nor}}(U)} \leq C\lambda_1^k \text{ for } k \geq k_0.$$

Proof. One can prove this in a quite similar way to the proof of the main result in [40]. An alternative proof is obtained using Proposition 4.5 and the same argument as in the proof of Theorem 1 in [41]. □

Next we show that every map in $\Omega_{p,n}$ can be approximated the hyperbolic polynomial-like maps of type n.

PROPOSITION 4.7. *Let* $G \in \mathcal{W}$ *be such that there exist domains* $\hat{U}$ *and* $\hat{V}$, *whose boundaries are analytic Jordan curves, such that* $\mathrm{mod}\, \hat{V} \setminus \hat{U} > \epsilon_0/2$ *and*

$$G\colon \hat{U} \to \hat{V}$$

is a real polynomial-like map of type n *that is infinitely renormalizable with combinatorics bounded by* p. *Then there exist polynomial-like maps of type* n

$$G_i\colon \hat{U}^i \to \hat{V}^i$$

such that

(A) *we have* $\mathrm{mod}\, \hat{V}^i \setminus \hat{U}^i \geq \epsilon_0/2$ *and* $G_i \in \mathcal{B}_{\mathrm{nor}}(U)$;
(B) *all critical points of* G_i *belong to the same periodic orbit*;
(C) *we have*

$$\lim_i |G_i - G|_{\mathcal{B}_{\mathrm{nor}}(U)} = 0.$$

Proof. We use the notation introduced in [40]. Let $\sigma = (\sigma_1, \sigma_2, \dots)$ be the combinatorics of G. By Proposition 2.2 in [40], there exists a sequence of polynomial P_i of type n with combinatorics $\sigma_i \star \cdots \star \sigma_1$. By Corollary 2.3 in [40] any accumulation point of this sequence is a polynomial P of type n that is infinitely renormalizable with combinatorics σ. By the proof of Theorem 2 in [40] there is only one polynomial of type n with combinatorics σ, so the sequence P_i indeed converges to P. Indeed there are now far more general rigidity results for polynomials. See Kozlovski, Shen and van Strien [23], [22].

Since P_i is a convergent sequence of polynomials of type n with connected Julia sets, it is possible to choose domains $\hat{U}^i$ and $\hat{V}^i$ such that

- $\inf_i \mathrm{mod}\, \hat{V}^i \setminus \hat{U}^i > 0$;

- $P_i\colon \hat{U}^i \to \hat{V}^i$ is a polynomial-like map of type n.

Furthermore, for some $K > 0$, there are K-quasiconformal maps

$$\phi_i\colon \mathbb{C}_n \to \mathbb{C}_n$$

such that

- $\phi_i(\hat{U}^i) = \hat{U}$ and $\phi_i(\hat{V}^i) = \hat{V}$;
- $\phi_i(\overline{z}) = \overline{\phi_i(z)}$;
- $P_i\colon \hat{U}^i \to \hat{V}^i$ is a polynomial-like map of type n;
- $G \circ \phi_i = \phi_i \circ P_i$ on $\partial\hat{U}^i$;
- the sequence ϕ_i converges to a K-quasiconformal map ϕ;
- if $\hat{U}^\infty = \phi^{-1}(\hat{U})$ and $\hat{V}^\infty = \phi^{-1}(\hat{V})$, then $P\colon \hat{U}^\infty \to \hat{V}^\infty$ is a polynomial-like map of type n.

Let μ_i be the Beltrami field that coincides with $\mu_i = \overline{\partial}\phi_i/\partial\phi_i$ on $\mathbb{C}_n \setminus \hat{U}^i$ that is invariant under P_i, and $\mu_i = 0$ on $K(P_i)$. Let $\psi_i\colon \mathbb{C}_n \to \mathbb{C}_n$ be the unique quasiconformal map such that $\psi_i(-1, j) = (-1, j)$ and $\psi_i(0, j) = (0, j)$ for every j, and $\mu_i = \overline{\partial}\psi_i/\partial\psi_i$ on $\mathbb{C}_n$. Define

$$G_i = \psi_i \circ P_i \circ \psi_i^{-1}.$$

Then

$$G_i\colon \psi_i(\hat{U}^i) \to \psi_i(\hat{V}^i)$$

is a polynomial-like map of type n. Note that

$$\inf_i \operatorname{mod} \psi_i(\hat{V}^i) \setminus \psi_i(\hat{U}^i) > 0.$$

Every subsequence of G_i has a convergent subsequence. Let F be one these accumulation points. We claim that $F = G$. Note that every accumulation point is of the form $F = \psi \circ P \circ \psi^{-1}$, where ψ is a K-quasiconformal map that is an accumulation point of the sequence ψ_i. We can assume, without loss of generality, that ϕ_i converges to a K-quasiconformal map ϕ.

Notice that

$$\phi_i \circ \psi_i^{-1} \circ G_i \circ \psi_i \circ \phi_i^{-1}(z) = \phi_i \circ P_i \circ \phi_i^{-1}(z) = G(z)$$

for $z \in \phi_i(\partial\hat{U}^i) = \partial\hat{U}$. Taking the limit on i we obtain

$$\phi \circ \psi^{-1} \circ F \circ \psi \circ \phi^{-1}(z) = G(z)$$

for $z \in \partial\hat{U}$. Moreover, since $\psi_i \circ \phi_i^{-1}$ is conformal in $\mathbb{C}_n \setminus \hat{U}$, we conclude that $\psi \circ \phi^{-1}$ is conformal in $\mathbb{C}_n \setminus \hat{U}$. Since F and G are both infinitely renormalizable with the same combinatorics, one can use the Sullivan's pullback argument to conclude that there is quasiconformal conjugacy H between $F\colon \hat{U}^\infty \to \hat{V}^\infty$ and $G\colon \hat{U} \to \hat{V}$ such that H is conformal in $\mathbb{C}_n \setminus K(F)$. Since there are not invariant line fields supported of $K(F)$, we conclude that H is conformal on $\mathbb{C}_n$, so H is affine on each connected component of $\mathbb{C}_n$. Since $H(-1, j) = (-1, j)$

and $H(0,j)=(0,j)$ for every j, we conclude that H is the identity. So $F=G$ and G_i converges to G. Items (A), (B) and (C) of Proposition 4.7 follow easily from this. □

4.2. *Vertical directions, codimension of E^h and vector bundles.* Let $f\colon V_1 \to V_2$ be a polynomial-like map. Let $\mathbb{B}_f$ be the vector space of the germs of holomorphic functions defined in a neighborhood of $K(f)$. We say that $v \in \mathbb{B}_f$ is a *vertical vector* if there exists a holomorphic vector field α defined on $\overline{\mathbb{C}} \setminus K(f)$ such that

$$v(x) = \alpha \circ f(x) - Df(x)\alpha(x) \tag{25}$$

for every x close to $K(f)$ and in the domain of α, and additionally

$$\lim_{z\to 0} z^2\alpha(1/z) = 0. \tag{26}$$

We have an analogous definition for polynomial-like extended maps of type n. Denote the set of vertical directions of f as $\hat{E}^v_f$. Recall that $v \in \mathbb{B}_f$ is a *horizontal vector* ($v \in \hat{E}^h_f$) if there is quasiconformal vector field on $\mathbb{C}$ such that (25) holds in a neighborhood of $K(f)$ and $\overline{\partial}\alpha = 0$ on $K(f)$. Lyubich [26] proved that

$$\mathbb{B}_f = \hat{E}^h_f + \hat{E}^v_f. \tag{27}$$

The same statement holds for polynomial-like extended maps of type n. Note that if $F \in \mathcal{W}$ has an extension that is a real polynomial-like extended map of type n and F is either infinitely renormalizable with bounded combinatorics in $\mathcal{C}_{p,n}$ or whose critical points belongs to the same periodic orbit, then $E^h_F = \hat{E}^h_F \cap T\mathcal{B}_{\mathrm{nor}}(U)$. Here E^h_F is as defined in Section 4.1.

Due to the infinitesimal pullback argument, if f does not have invariant line fields on its Julia set $J(f)$, then $\hat{E}^h_f \cap \hat{E}^v_f$ is exactly the space of vectors v such that there exists a vector field $\alpha(z) = az+b$ on $\overline{\mathbb{C}}$ that satisfies (25) in a neighborhood of $K(f)$. In particular, if $F \in \Omega_{n,p}$, then we have

$$T\mathcal{B}_{\mathrm{nor}}(U) = E^h_F \oplus E^v_F.$$

PROPOSITION 4.8. *If $f\colon V_1 \to V_2$ is a polynomial-like of degree d generated by the restriction of a polynomial of degree d, then $\hat{E}^v_f$ is exactly the linear space of polynomials of degree d. If f is a polynomial-like extended map of type n such that on each $\mathbb{C}\times\{i\}$ the map f coincides with a quadratic polynomial then $\hat{E}^v_f$ is the space of vectors that coincides with quadratic polynomials on each $\mathbb{C}\times\{i\}$.*

Proof. Suppose that f is a polynomial of degree d, and let $v \in \hat{E}^v_f$. Then the right-hand side of (25) implies that v extends to an entire holomorphic function. Of course (26) implies that

$$|\alpha(y)| \leq C|y|$$

for some C, provided $y \in \mathbb{C}$ has large modulus. Since Df is a polynomial of degree $d-1$, it follows from (25) that

$$|v(x)| \leq \tilde{C}|x|^d,$$

for some $\tilde{C}$, provided $|x|$ is large. So v is a polynomial whose degree is at most d. On the other hand, if v is a polynomial of degree at most d, we have that $f_t = f + tv$ is a polynomial of degree d for every small t. Every f_t has the very same external class (see Lyubich [26]). This implies that $v = \partial_t f_t|_{t=0} \in \hat{E}^v_f$. The proof in the case of a polynomial-like extended map of type n is analogous. □

The following is similar to Lyubich [26, Lemma 4.10], but for the sake of completeness we provide details.

Proposition 4.9. *Let $F\colon W \to V$ be a polynomial-like map of type n with connected Julia set satisfying*

(i) $0 < \epsilon_0 < \mathrm{mod}(V \setminus W) < \epsilon_1$;
(ii) $\mathrm{diam}\, K(F) \cap (\mathbb{C} \times \{i\}) \geq 1$ *for every* i;
(iii) $\mathrm{diam}\, V \cap (\mathbb{C} \times \{i\}) \leq C_1$ *for every* i.

Let $v \in E^v_F \cap \mathcal{B}(W)$. Consider the holomorphic vector field

$$\alpha\colon \overline{\mathbb{C}}_n \setminus K(F) \to \mathbb{C}$$

such that $\lim_{z\to 0} z^2\alpha(1/z) = 0$ and

$$v(z) = \alpha(F(z)) - DF(z)\alpha(z)$$

for every $z \in W \setminus K(F)$. Then there is $C_2 > 0$, which depends only on ϵ_0, ϵ_1 and C_1, such that

$$|\alpha|_{\mathrm{sph}\ \overline{\mathbb{C}}_n \setminus F^{-1}W} \leq C_2 |v|_{\mathcal{B}(W)}.$$

Here $|\cdot|_{\mathrm{sph}\ Q}$ denotes the sup norm on Q considering the spherical metric on each component of $\mathbb{C}_n$.

Proof. Define $K_i(F) = K(F) \cap (\mathbb{C} \times \{i\})$, $1 \leq 1 \leq n$. Let

$$\phi_i\colon (\overline{\mathbb{C}} \times \{i\}) \setminus K_i(F) \to \mathbb{D} \times \{i\}$$

be conformal maps such that $\phi_i(\infty, i) = (0, i)$. Define the conformal maps

$$\phi\colon \overline{\mathbb{C}}_n \setminus K(F) \to \mathbb{D} \times \{1, \ldots, n\}$$

as $\phi(z, i) = \phi_i(z, i)$. Define

$$\tilde{W} = \phi(W \setminus K(F)), \tilde{V} = \phi(V \setminus K(F))$$

and

$$\tilde{F}\colon \tilde{W} \to \tilde{V}$$

as

$$\tilde{F}(z,i) = \phi \circ F \circ \phi^{-1}(z,i).$$

Let $\psi(x,i) = (z/|z|^2, i)$, $\hat{W} = \overline{\tilde{W} \cup \psi(\tilde{W})}$ and $\hat{V} = \overline{\tilde{V} \cup \psi(\tilde{V})}$. Then $\tilde{F}$ has a analytic extension to a covering

$$\hat{F}\colon \hat{W} \to \hat{V}$$

satisfying $\hat{F}(\mathbb{S} \times \{1,\dots,n\}) = \mathbb{S} \times \{1,\dots,n\}$. Note that each connected component of

$$\hat{V} \setminus \hat{W}$$

is an annulus with modulus larger than ϵ_0. This implies that there is k (which depends only on ϵ_0, ϵ_1, C_1 and n) such that

$$|D\hat{F}^{kn}(z,i)| \geq 2$$

for every $(z,i) \in \hat{F}^{-(kn+1)}\hat{W}$. Note that there is $C_3 > 1$, which depends only on ϵ_0, ϵ_1, C_1, such that

- $|D\phi(z,i)| \in [1/C_3, C_3]$ for every $(z,i) \in W \setminus F^{-(kn+1)}W$;
- $|DF(z,i)| \in [1/C_3, C_3]$ for every $(z,i) \in F^{-1}W \setminus F^{-(kn+1)}W$;
- $|D\hat{F}(z,i)| \in [1/C_3, C_3]$ for every $(z,i) \in \hat{F}^{-1}\hat{W} \setminus \hat{F}^{-(kn+1)}\hat{W}$;
- we have $1/C_3 \leq |z| \leq C_3$ for every $z \in \partial F^{-1}W$.

Define

$$\hat{\alpha}\colon \mathbb{D} \times \{1,\dots,n\} \to \mathbb{C}$$

as

$$\hat{\alpha}(z,i) = D\phi(\phi^{-1}(z,i)\alpha(\phi^{-1}(z,i))$$

and

$$\hat{v}(z,i) = D\phi(F(\phi^{-1}(z,i)))v(\phi^{-1}(z,i)).$$

Then $\hat{\alpha}(0) = 0$,

$$\hat{v} = \hat{\alpha} \circ \hat{F} - D\hat{F} \cdot \hat{\alpha},$$

and consequently

$$\sum_{j=0}^{nk} D\hat{F}^{nk-j}(\hat{F}^{j+1}(z,i))\hat{v}(\hat{F}^{j}(z,i)) = \hat{\alpha} \circ \hat{F}^{nk}(z,i) - D\hat{F}^{nk}(z,i) \cdot \hat{\alpha}(z,i)$$

for $(z,i) \in \partial \hat{F}^{-(kn+1)}\hat{W}$. Let (z_0, i_0) be such that

$$|\hat{\alpha}(z_0,i_0)| = \max_{(z,i) \in \partial \hat{F}^{-(kn+1)}\tilde{W}} |\hat{\alpha}(z,i)|.$$

Then

$$|\sum_{j=0}^{nk} D\hat{F}^{nk-j}(\hat{F}^{j+1}(z,i))\hat{v}(\hat{F}^{j}(z_0,i_0))|$$
$$\geq 2 \max_{(z,i) \in \partial \hat{F}^{-(kn+1)}\tilde{W}} |\hat{\alpha}(z,i)| - \max_{(z,i) \in \partial \hat{F}^{-1}\tilde{W}} |\hat{\alpha}(z,i)|.$$

Since $\hat{\alpha}$ is holomorphic, the maximum principle implies

$$\max_{(z,i)\in\partial\hat{F}^{-(kn+1)}\tilde{W}} |\hat{\alpha}(z,i)| \geq \max_{(z,i)\in\partial\hat{F}^{-1}\tilde{W}} |\hat{\alpha}(z,i)|,$$

so

$$\max_{(z,i)\in\partial\hat{F}^{-(kn+1)}\tilde{W}} |\hat{\alpha}(z,i)| \leq C_4 \sup_{(z,i)\in\hat{F}^{-1}\tilde{W}\setminus\hat{F}^{-(kn+1)}\tilde{W}} |\hat{v}(z,i)|.$$

Here $C_4 = nkC_3^{nk}$. Consequently,

$$\begin{aligned}
\max_{(z,i)\in\partial F^{-1}W} |\alpha(z,i)| &\leq C_3 \max_{(z,i)\in\partial\hat{F}^{-1}\tilde{W}} |\hat{\alpha}(z,i)| \\
&\leq C_3 \max_{(z,i)\in\partial\hat{F}^{-(kn+1)}\tilde{W}} |\hat{\alpha}(z,i)| \\
&\leq C_3C_4 \max_{(z,i)\in\hat{F}^{-1}\tilde{W}\setminus\hat{F}^{-(kn+1)}\tilde{W}} |\hat{v}(z,i)| \\
&\leq C_3^2C_4 \max_{(z,i)\in F^{-1}W\setminus F^{-(kn+1)}W} |v(z,i)| \\
&\leq C_3^2C_4 \max_{(z,i)\in F^{-1}W} |v(z,i)|.
\end{aligned}$$

Note that

$$\begin{aligned}
\sup_{(z,i)\in\overline{\mathbb{C}}_n\setminus F^{-1}W} |\alpha(z,i)|_{sph\ \overline{\mathbb{C}}} &= \sup_{(z,i)\in\overline{\mathbb{C}}_n\setminus F^{-1}W} \frac{2|\alpha(z,i)|}{1+|z|^2} \\
&\leq \sup_{(z,i)\in\overline{\mathbb{C}}_n\setminus F^{-1}W} \left|\frac{2\alpha(z,i)}{z^2}\right| \\
&\leq \max_{(z,i)\in\partial F^{-1}W} \left|\frac{2\alpha(z,i)}{z^2}\right| \\
&\leq C_3^2 \max_{(z,i)\in\partial F^{-1}W} |\alpha(z,i)|.
\end{aligned}$$

□

PROPOSITION 4.10 (Codimension of E^h_G). *For every $G \in \Omega_{n,p}$, the codimension of E^h_G is n.*

Proof. By Proposition 4.7, for every $G \in \Omega_{n,p}$, one can find a polynomial-like extension $G\colon \tilde{U} \to \tilde{V}$ of type n and a sequence of polynomial-like maps $G_i\colon \hat{U}^i \to \hat{V}^i$ of type n whose periodic points belongs to the same critical orbit and such that U is compactly contained in $\hat{U}^i$ and $\hat{U}^i$ is compactly contained in $\tilde{U}$. Denote $E^j_{G_i}(\tilde{U}) = \hat{E}^j_{G_i} \cap T\mathcal{B}_{\mathrm{nor}}(\tilde{U})$, where $j \in \{v, h\}$. We claim that $\operatorname{codim} E^h_{G_i}(\tilde{U}) = n$.

Indeed for each $q \in C(G_i)$, let $m^i_q > 0$ be such that $G_i^{m^i_q}(q) \in C(G_i)$ and $G_i^k(q) \notin C(G_i)$ for every $0 < k < m^i_q$. Given $v \in T\mathcal{B}_{\mathrm{nor}}(U)$, let $v_{q,k} = v(G_i^k(q))$.

Then there is a unique solution $\{\alpha_{q,k}\}_{q\in C(G_i),1\leq k\leq m^i_q}$ for the homogeneous system of linear equations

$$\begin{aligned} v_{q,k} &= \alpha_{q,k+1} - DG_i(G_i^k(q))\cdot \alpha_{q,k},\\ 1\leq k &< m^i_q, \quad q\in C(G_i),\\ v_{q,0} &= \alpha_{q,1}, \quad q\in C(G_i). \end{aligned}$$

In particular,

$$(v_{q,k})_{q\in C(G_i),0\leq k< m^i_q} \mapsto (\alpha_{q,k})_{q\in C(G_i),1\leq k\leq m^i_q}$$

is a linear bijection. Note that $v\in E^h_{G_i}$ if and only if $\alpha_{q,m^i_q}=0$ for every $q\in C(G_i)$, that is, if and only if v belongs to the kernel of the linear map

$$v\mapsto (\alpha_{q,m^i_q})_{q\in C(G_i)}.$$

Since

$$v\mapsto (v_{q,k})_{q\in C(G_i),0\leq k< m^i_q}$$

and

$$(v_{q,k})_{q\in C(G_i),0\leq k< m^i_q} \mapsto (\alpha_{q,m^i_q})_{q\in C(G_i)}$$

are onto continuous linear maps, it follows that $\operatorname{codim} E^h_{G_i}(\tilde{U}) = n$. (Recall that G_i has n critical points.) So $\dim E^v_{G_i}(\tilde{U}) = n$. Since $\tilde{U}$ is compatible with G, we have that [42, Prop. 10.4] implies $\dim E^v_G(\tilde{U}) = \operatorname{codim} E^h_G(\tilde{U}) = n$.

Unfortunately U is not compatible with G (the repelling fixed point -1 of G does not belong to the interior of U), so we need to be a little more careful to conclude that $\operatorname{codim} E^h_G = n$. Consider the natural affine inclusion

$$\pi\colon \mathcal{B}_{\mathrm{nor}}(\tilde{U}) \to \mathcal{B}_{\mathrm{nor}}(U).$$

Then $D\pi$ is a continuous, injective map, it has dense image, and moreover

$$(D\pi)^{-1}E^h_G = E^h_G(\tilde{U}).$$

Note that if $\operatorname{codim} E^h_G \geq k$, there is a bounded linear onto map $\psi\colon T_G\mathcal{B}_{\mathrm{nor}}(U) \to \mathbb{R}^k$ such that $E^h_G \subset \operatorname{Ker}\,\psi$. Since $D\pi$ have dense image, we have that $\psi\circ D\pi$ is also a bounded linear onto map such that $E^h_G(\tilde{U}) \subset \operatorname{Ker}\,\psi\circ D\pi$, so $\operatorname{codim} E^h_G(\tilde{U}) \geq \operatorname{codim} \operatorname{Ker}\,\psi\circ D\pi = k$. Thus $\operatorname{codim} E^h_G \leq \operatorname{codim} E^h_G(\tilde{U}) = n$.

On the other hand, since π is injective we have that $\dim \pi(E^v_G(\tilde{U})) = n$. If $v\in \pi(E^v_G(\tilde{U}))\cap E^h_G$, then $v\in E^v_G(\tilde{U})\cap E^h_G(\tilde{U}) = \{0\}$. So $\operatorname{codim} E^h_G \geq n$. □

The following lemma is elementary. We included it here for the sake of completeness.

Lemma 4.11. *Let $(\mathcal{B}_i, |\cdot|_i)$, $i=1,2$, be Banach spaces. Let Ω be a compact metric space, and suppose that for every $f\in\Omega$, we associate vector subspaces $E^v_f \subset \mathcal{B}_2 \subset \mathcal{B}_1$, $E^h_{f,i}\subset \mathcal{B}_i$ satisfying*

(A) *for every* $f \in \Omega$, *we have* $\mathcal{B}_2 = E^v_f \oplus E^h_{f,2}$ *and* $E^v_f \cap E^h_{f,1} = \{0\}$;

(B) *the set*

$$\{(f, v) \colon\; f \in \Omega,\; v \in E^h_{f,i}\}$$

is a closed subset of $\Omega \times \mathcal{B}_i$;

(C) *we have that*

$$\{(f, v) \colon\; f \in \Omega,\; v \in E^v_f,\; |v|_i \leq 1\}$$

is a compact subset of $\Omega \times \mathcal{B}_i$, $i = 1, 2$;

(D) *there exists* $n \in \mathbb{N}$ *such that* $\dim E^v_f = n$ *for every* $f \in \Omega$;

(E) *the inclusion* $\imath \colon \mathcal{B}_2 \to \mathcal{B}_1$ *is a compact linear operator and*

$$\imath^{-1}(E^h_{f,1}) = E^h_{f,2}.$$

Then

(I) *The set*

$$E^v = \{(f, v) \colon\; f \in \Omega,\; v \in E^v_f\},$$

with the topology induced by $\Omega \times \mathcal{B}_2$, *is a topological vector bundle (with the obvious linear structure on the fibers* E^v_f*) with fibers of dimension* n.

(II) *Let* $\sim$ *be the equivalent relation on* $\Omega \times \mathcal{B}_2$ *defined by* $(f, v) \sim (g, w)$ *if and only if* $f = g$ *and* $v - w \in E^h_f$. *Then the quotient topological space*

$$E = \{(f, [v]) \text{ such that } f \in \Omega \text{ and } [v] \in \mathcal{B}/E^h_{f,2}\}$$

is a topological vector bundle with fibers of dimension n.

(III) *Define*

$$|(f, [v])| = \mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f,2}) = \inf\{|v - w|_2 \colon\; w \in E^h_{f,2}\}.$$

Then

$$(f, [v]) \in E \mapsto |(f, [v])|_2$$

is continuous.

Proof of (I). All limits in the proof of (I) and (II) are in the topology of $(\mathcal{B}_2, |\cdot|_2)$. Let $u \in \mathcal{B}_2$. Then for every $g \in \Omega$, there are unique vectors $v^{g,u} \in E^v_g$ and $w^{g,u} \in E^h_{g,2}$ such that

$$u = v^{g,u} + w^{g,u}.$$

First note that

$$\sup\,\{|v^{g,u}|_2, g \in \Omega,\; |u|_2 \leq 1\} \cup \{|w^{g,u}|_2, g \in \Omega,\; |u|_2 \leq 1\} < \infty. \tag{28}$$

Otherwise there is a sequence $g_i \in \Omega$, $|u_i| \leq 1$ such that

$$r_i = \max\{|v^{g_i,u_i}|_2, |w^{g_i,u_i}|_2\} \to_i \infty.$$

Since Ω is compact, without loss of generality we can assume that

$$\lim_i g_i = g \in \Omega,$$

and (C) implies that we can assume

$$\lim_i \frac{v^{g_i,u_i}}{r_i} = v \in E^v_g.$$

Consequently

$$\lim_i \frac{w^{g_i,u_i}}{r_i} = -v. \tag{29}$$

On the other hand, by (29) and (B), we have $v \in E^h_g$, so by (A) we conclude $v = 0$. This is a contradiction with the definition of r_i. So (28) holds. We claim that the map

$$S\colon \Omega \times \mathcal{B}_2 \to \Omega \times \mathcal{B}_2$$

defined by

$$S(g,u) = (g, v^{g,u}) \in \mathcal{B}_2$$

is a continuous linear map. Indeed suppose $\lim_i g_i = g$ and $\lim_i u_i = u$. By assumption (C) and (28), taking a subsequence we may assume that

$$\lim_i v^{g_i,u_i} = v \in E^v_g$$

and consequently by (B),

$$\lim_i w^{g_i,u_i} = \lim_i u_i - \lim_i v^{g_i,u_i} = u - v \in E^h_{g,2}.$$

By (A) we conclude that $v = v^{g,u}$ and $u - v = w^{g,u}$. This proves the claim. Let $f \in \Omega$, and choose a basis $v_1, \dots, v_n \in E^v_f$. Let $v^g_i = S(g, v_i) \in E^v_g$ and $w^g_i = v_i - S(g, v_i) \in E^h_{g,2}$, with $g \in \Omega$. Of course

$$v_i = v^g_i + w^g_i.$$

So for every i, we have that $g \mapsto v^g_i$ is continuous and, moreover, $v^f_i = v_i$ and $w^f_i = 0$. In particular, there exists an open neighborhood O_1 of f in Ω such that $\{v^g_i\}_i$ is a basis of E^v_g for every $g \in O$.

Reducing the neighborhood O, we may assume that $E^h_{g,2} \cap E^v_f = \{0\}$ for every $g \in O$. Otherwise there would exist a sequence $g_i \to_i f$ and $w_i \in E^h_{g_i,2} \cap E^v_f$ satisfying $|w_i| = 1$. By (D) we may assume $\lim_i w_i = w \in E^v_f$. By (B) we have $w \in E^h_{f,2}$, which contradicts (A).

Define the map

$$H\colon O \times E^v_f \to \{(g,v)\colon\ g \in O,\ v \in E^v_g\}$$

as

$$H\Big(g, \sum_i c_i v_i\Big) = \Big(g, \sum_i c_i v^g_i\Big).$$

We have that H is a continuous and bijective map. Note that

$$H^{-1}(g,u) = (g,v),$$

where v is the only vector in $E^h_{f,2}$ such that $v - u \in E^h_{g,2}$.

We claim that H^{-1} is continuous. Indeed, suppose that $\lim_i(g_i, u_i) = (g, u)$, with $g, g_i \in O$ and $u, u_i \in E^v_{g_i}$. If $H^{-1}(g_i, u_i) = (g_i, v_i)$, then $w_i = v_i - u_i \in E^h_{g_i,2}$ and $v_i \in E^v_f$. Let $r_i = \max\{|v_i|_2, |w_i|_2\}$. We claim that

$$\sup_i r_i < \infty. \tag{30}$$

Otherwise, without loss of generality we may assume $\lim_i r_i = \infty$. By (D) we may assume that $\lim_i v_i/r_i = \tilde{v} \in E^v_f$ and consequently $\lim_i w_i/r_i = \tilde{v}$. By (B) we have $\tilde{v} \in E^h_{g,2}$. So $\tilde{v} = 0$, in contradiction with the definition of r_i. This proves (30). In particular, without loss of generality we can assume $\lim_i v_i = \hat{v} \in E^v_f$, and consequently by (B) we have $\lim_i w_i = \hat{w} = \hat{v} - u \in E^h_{f,2}$. In particular, $H^{-1}(g, u) = (g, \hat{v})$. Thus H^{-1} is continuous. So

$$\hat{H} \colon O \times \mathbb{R}^n \to \{(g, v)\colon \ g \in O,\ v \in E^v_g\}$$

defined by

$$\hat{H}(g, (c_i)_i) = \Big(g, \sum_i c_i v^g_i\Big)$$

is a local trivialization of the vector bundle E^v in the open set $\{(g, v)\colon \ g \in O$, $v \in E^v_g\}$. □

Proof of (II). Let $\pi \colon \Omega \times \mathcal{B}_2 \to E$ be a natural projection

$$(f, v) \mapsto (f, [v]_f),$$

where $[v]_f$ represents the equivalent class of v in $\mathcal{B}/E^h_{f,2}$. We will define a homeomorphism

$$T \colon E^v \to E$$

that is a vector bundle homeomorphism. Indeed let T be the restriction of π to E^v. Of course T is a continuous map that preserves the linear structure in the fibers. It is also a bijection, since $T(f, v) = T(g, w)$ implies $f = g$, with $v, w \in E^v_f$ and $v - w \in E^h_{f,2}$, so by (A) we have $v = w$. Note that $T^{-1}(f, [u]_f) = (f, v)$, where v is the unique vector that satisfies $v \in E^v_f$ and $u - v \in E^h_{f,2}$. Note that $T^{-1}(f, [u]_f) = S(f, u)$, where S was defined in the proof of (I). Consequently T^{-1} is continuous, since S descends to the quotient space E as T^{-1}. □

Proof of (III). We claim that

$$(f, v) \in \Omega \times \mathcal{B}_2 \to \operatorname{dist}_{\mathcal{B}_2}(v, E^h_f) \tag{31}$$

is continuous. Indeed suppose that $\lim_k(f_k, v_k) = (f, v)$. We have

$$|\operatorname{dist}_{\mathcal{B}_2}(v_k, E^h_{f_k,2}) - \operatorname{dist}_{\mathcal{B}_2}(v, E^h_{f_k,2})| \le \operatorname{dist}_{\mathcal{B}_2}(v_k - v, E^h_{f_k,2}) \le |v - v_k|_2 \to_k 0;$$

in particular,

$$\lim_k \operatorname{dist}_{\mathcal{B}_2}(v_k, E^h_{f_k,2}) - \operatorname{dist}_{\mathcal{B}_2}(v, E^h_{f_k,2}) = 0,$$

so to prove the claim it is enough to show that

$$\lim_k \mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f_k,2}) = \mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f,2}). \tag{32}$$

Fix $\epsilon > 0$, and let $w \in E^h_{f,2}$ be such that

$$|v - w|_2 < \mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f,2}) + \epsilon.$$

Then $\lim_k S(f_k, w) = 0$, where S is as defined in the proof of (I). In particular, $w_k = w - S(f_k, w) \in E^h_{f_k,2}$, and for large k, we have $|v - w_k| < \mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f,2}) + 2\epsilon$. Since $\epsilon > 0$ is arbitrary,

$$\limsup_k \mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f_k,2}) \leq \mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f,2}).$$

On the other hand, if

$$\liminf_k \mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f_k,2}) \leq \mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f,2}) - 2\epsilon,$$

then we can assume (taking a subsequence) that there is $w_k \in E^h_{f_k,2}$ such that

$$|v - w_k|_2 \leq \mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f,2}) - \epsilon.$$

Let $u_k = S(f, w_k) \in E^v_{f,2}$. Note that $\sup_k |w_k|_2 < \infty$, which implies $\sup_k |u_k|_2 < \infty$, and consequently $y_k = w_k - u_k \in E^h_{f,2}$ satisfies

$$\sup_k |y_k|_2 < \infty.$$

By (B) and E. we can find $y \in E^h_{f,1}$ and a subsequence of y_k such that y_k converges to y in $\mathcal{B}_1$. Taking a subsequence we can assume $\lim_k u_k = u \in E^v_f$ (in the topologies of $\mathcal{B}_i$, $i = 1, 2$). So w_k converges to $u + y$ in $\mathcal{B}_1$. By (A), (B) and (E) we have $u = 0$. We conclude that

$$\mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f,2}) \leq \liminf_k |v - y_k|_2 \leq \mathrm{dist}_{\mathcal{B}_2}(v, E^h_{f,2}) - \epsilon,$$

which is a contradiction. So (32) holds. This proves the claim. Since $(f, v) \sim (g, \tilde{v})$ implies $\mathrm{dist}_{\mathcal{B}_2}(v, E^h_f) = \mathrm{dist}_{\mathcal{B}_2}(\tilde{v}, E^h_g)$, the function (31) descends to the quotient topological space E as a continuous function. □

Proposition 4.4 implies that

$$\{(F, v),\ F \in \Omega_{n,p} \text{ and } v \in E^h_F\}$$

is a closed subset of $\Omega_{n,p} \times T\mathcal{B}_{\mathrm{nor}}(U)$. We also have

Lemma 4.12. *The set*

$$\mathcal{E} = \{(F, v),\ F \in \Omega_{n,p} \textit{ and } v \in E^v_F,\ |v|_{T\mathcal{B}_{\mathrm{nor}}(U)} \leq 1\} \tag{33}$$

is a compact subset of $\Omega_{n,p} \times T\mathcal{B}_{\mathrm{nor}}(U)$.

Proof. Let (F_k, v_k) be a sequence in the set $\mathcal{E}$. Due to the complex bounds there exist domains $W_k, V_k \in \mathbb{C}_n$ such that $\overline{W_k} \subset V_k$ and

$$F_k \colon W_k \to V_k$$

are polynomial-like maps of type n satisfying

$$\mathrm{mod}(V_k \setminus W_k) \geq \epsilon_0.$$

Using the same argument as McMullen [30, Th. 5.8] there is a polynomial-like map of type n $F \colon W \to V$ with $\mathrm{mod}(V \setminus W) \geq \epsilon_0$ such that $\lim_k F_k = F$ in the topology defined by McMullen. In particular, $\lim_k W_k = W$ and $\lim_k V_k = V$ in the Carathéodory topology and F_k converges to F uniformly on compact subsets of W. Consequently $\lim_k F_k = F$ in $\mathcal{B}_{\mathrm{nor}}(U)$, and we can find $\tilde{V}$ compactly contained in V such that

$$F_k \colon F_k^{-1}\tilde{V} \to \tilde{V}$$

are polynomial-like maps of type n satisfying

$$\mathrm{mod}(\tilde{V} \setminus F_k^{-1}\tilde{V}) \geq \epsilon_0/2.$$

Let $\tilde{W} = F^{-1}\tilde{V}$. Since $v_k \in E^v_{F_k}$, there exist holomorphic vectors fields

$$\alpha_k \colon \overline{\mathbb{C}}_n \setminus K(F_k) \to \mathbb{C}$$

such that $\alpha_k(\infty) = 0$ and

$$v_k(z) = \alpha_k(F_k(z)) - DF_k(z)\alpha_k(z)$$

for every $z \in W_k \setminus K(F_k)$.

Finally, for every large $j > 0$, it is possible to find a domain V^j such that

$$K(F) \subset V^j \subset \overline{\{z \in \mathbb{C}_n \colon \ \mathrm{dist}(z, K(F)) < 1/j\}} \subset \tilde{V}$$

and

$$F \colon F^{-1}V^j \to V^j$$

is a polynomial-like map of type n. Consequently there is $k_0 = k_0(j)$ such that for every $k \geq k_0$, we have that

$$F_k \colon F_k^{-1}V^j \to V^j$$

is a polynomial-like map of type n. Note that $F_k^{-1}V^j \subset \tilde{W}$ for large k. Due to Proposition 4.9 we have that

$$|\alpha_k|_{\mathrm{sph}\ \overline{\mathbb{C}}_n \setminus F_k^{-2}V^j} \leq C_j |v_k|_{\mathcal{B}(V^j)} \leq C_j |v_k|_{\mathcal{B}(\tilde{W})}$$

for large k. We claim that

$$\sup_k |v_k|_{\mathcal{B}(\tilde{W})} < \infty. \tag{34}$$

Indeed, otherwise we may assume that $r_k = |v_k|_{\mathcal{B}(\tilde{W})}$ diverges to infinity. Then $\hat{\alpha}_k = \alpha_k / r_k$ satisfies

$$|\hat{\alpha}_k|_{\mathrm{sph}\ \overline{\mathbb{C}}_n \setminus F_k^{-2}V^j} \leq C_j.$$

This implies that a subsequence of $\hat{\alpha}_k$ converges uniformly on compact subsets of $\mathbb{C}_n \setminus K(F)$ to a holomorphic vector field $\hat{\alpha}\colon \overline{\mathbb{C}}_n \setminus K(F) \to \mathbb{C}$ and the corresponding subsequence of $\hat{v}_k = v_k/r_k$ converges uniformly on compact subsets of $W \setminus K(F)$ to

$$\hat{v} = \hat{\alpha} \circ F - DF \cdot \hat{\alpha}.$$

By the maximum principle we have that $\hat{v}_k$ is a uniform Cauchy sequence on compact subsets of W, so $\hat{v}$ extends to a holomorphic function on W and $\lim_k \hat{v}_k = \hat{v}$ uniformly on compact subsets of W. Since $|\hat{v}_k|_{\mathcal{B}(\tilde{W})} = 1$ for every k, we have $|\hat{v}|_{\mathcal{B}(\tilde{W})} = 1$. On the other hand,

$$|\hat{v}|_{\mathcal{B}(U)} = \lim_k |\hat{v}_k|_{\mathcal{B}(U)} = \lim_k 1/r_k = 0,$$

so $\hat{v} = 0$ everywhere, in contradiction with $|\hat{v}|_{\mathcal{B}(\tilde{W})} = 1$. This proves the claim. Now we can use the same argument as in the previous paragraph to conclude that there is a subsequence of α_k that converges uniformly on compact subsets of $\mathbb{C}_n \setminus K(F)$ to a holomorphic vector field $\alpha\colon \overline{\mathbb{C}}_n \setminus K(F) \to \mathbb{C}$. The corresponding subsequence of v_k converges in $\mathcal{B}(\tilde{W})$ (and, in particular, in $\mathcal{B}(U)$) to a vector v, which satisfies

$$v = \alpha \circ F - Df \cdot \alpha$$

on $W \setminus K(F)$. This concludes the proof. ☐

As an immediate consequence of Lemmas 4.11 and 4.12, we have

Proposition 4.13. *The quotient topological space*

$$E = \{(F, [v]) \text{ such that } F \in \Omega_{n,p} \text{ and } [v] \in T\mathcal{B}_{\mathrm{nor}}(U)/E^h_F\}$$

is a topological vector bundle with fibers of dimension n. Moreover,

$$(F, [v]) \mapsto |(F, [v])| = \mathrm{dist}_{T\mathcal{B}_{\mathrm{nor}}(U)}(v, E^h_F) \tag{35}$$

is a continuous function.

Proof. Let $\hat{U}$ be a symmetric domain with respect to the real trace of $\mathbb{C}_n$, which is compactly contained in the interior of U, such that the $\hat{U} \cap \mathbb{R}$ is an interval that contains in its interior the convex closure of the postcritical set of every $F \in \Omega_{n,p}$. Let $\mathcal{B}_2 = T\mathcal{B}_{\mathrm{nor}}(U)$, $\mathcal{B}_1 = \mathcal{B}(\hat{U})$, $E^h_{F,2} = E^h_F$ and $E^h_{F,1}$ be the set of horizontal vectors of $\mathcal{B}(\hat{U})$. Apply Lemma 4.11. ☐

5. Hyperbolicity of the ω-limit set $\Omega_{n,p}$ of $\mathcal{R}$

Given $F \in \Omega_{n,p}$, denote

$$\mathfrak{B}_+(F) = \{v \in T_F\mathcal{B}_{\mathrm{nor}}(U) \text{ such that } \sup_{i \in \mathbb{N}} |D\mathcal{R}^i_f \cdot v_i| < \infty\}.$$

Recall that we choose $U = D_{\delta_0,\theta_0}$. The goal of this section is to prove

Proposition 5.1. *Suppose that for every $F \in \Omega_{n,p}$, we have*

$$\mathfrak{B}_+(F) \subset E^h_F. \tag{36}$$

Then $\Omega_{n,p}$ is a hyperbolic set. Moreover, its stable direction is exactly E^h.

Proposition 5.1 reduces the study of the hyperbolicity of $\Omega_{n,p}$ to the study of the existence and regularity of the solutions α of the cohomological equation (16). So to show that $\Omega_{n,p}$ is a hyperbolic set it remains to prove

Theorem 4 (Key Lemma). *If $F \in \Omega_{n,p}$, then*

$$\mathfrak{B}_+(F) \subset E^h_F. \tag{37}$$

We will prove Theorem 4 in Section 8. As an immediate consequence of Proposition 5.1 and Theorem 4 we have

Theorem 5 (Theorem B). *$\Omega_{n,p}$ is a hyperbolic set. Moreover, its stable direction is exactly E^h.*

5.1. *A criterion for hyperbolicity of cocycles.* Let E be a topological vector bundle with base Ω, fiber $\mathbb{R}^n$ and projection $p\colon E \to \Omega$. We denote elements of E by (x, v), where $x \in \Omega$ and $v \in p^{-1}(x)$. We also assume that Ω is compact. Additionally, assume that $|\cdot|$ is a continuous function

$$(x, v) \in E \mapsto |(x, v)| \in \mathbb{R}$$

so that $|\cdot|$ is a norm on each fiber $p^{-1}(x)$. We will abuse the notation writing $|v|$ instead of (x, v).

Let $L\colon E \to E$ be a fiber-preserving homeomorphism that is linear on the fibers. The map L is called a linear cocycle on E. Define

$$\mathfrak{B}_+ = \{(x, v) \in E \text{ such that } \sup_{i\in\mathbb{N}} |v_i| < \infty, \text{ where } L^i(x, v) = (x_i, v_i)\},$$
$$\mathfrak{B} = \{(x, v) \in E \text{ such that } \sup_{i\in\mathbb{Z}} |v_i| < \infty, \text{ where } L^i(x, v) = (x_i, v_i)\},$$
$$\mathcal{S} = \{(x, v) \in E \text{ such that } \lim_{i\to+\infty} |v_i| = 0, \text{ where } L^i(x, v) = (x_i, v_i)\},$$
$$\mathcal{U} = \{(x, v) \in E \text{ such that } \lim_{i\to-\infty} |v_i| = 0, \text{ where } L^i(x, v) = (x_i, v_i)\},$$

and the zero section

$$E_0 = \{(x, 0) \in E\}.$$

We say that the cocycle L is uniformly expanding if there exist $K > 0$ and $\theta > 1$ such that for every $(x, v) \in E$, we have

$$|v_i| \geq K\theta^i |v| \tag{38}$$

for every $i \geq 0$, where $L^i(x, v) = (x_n, v_n)$.

Proposition 5.2. *The cocycle $L\colon E \to E$ is uniformly expanding if and only if*

$$\mathfrak{B}_+ = E_0. \tag{39}$$

Proof. Of course if $L\colon E \to E$ is uniformly expanding, then (39) holds. To prove the reverse implication, note that (39) implies $\mathcal{S} = \mathfrak{B} = E_0$. By Theorem 2 in Sacker and Sell [35] (see also Section 7 there) we have that $L\colon E \to E$ is uniformly expanding. □

One can also prove Proposition 5.2 applying Sacker and Sell's results in [36, Lemma 9 and Th. 2]. We just refer to that because the proof of these results in [36] seems to be more elementary than the proof of Theorem 2 in [35].

5.2. *Unstable invariant cones.* Let $F \in \Omega_{n,p}$, and denote

$$\mathfrak{B}_+(F) = \{v \in T_F\mathcal{B}_{\text{nor}}(U) \text{ such that } \sup_{i\in\mathbb{N}} |D\mathcal{R}^i_f \cdot v_i| < \infty\}.$$

Consider

$$E = \{(F, [v]) \text{ such that } F \in \Omega_{n,p} \text{ and } [v] \in T_F\mathcal{B}_{\text{nor}}(U)/E^h_F\}.$$

Recall that due to Proposition 4.13 we have that E is a topological vector bundle and

$$\dim T_F\mathcal{B}_{\text{nor}}(U)/E^h_F = n.$$

By Proposition 4.2 the linear transformation

$$D\mathcal{R}_F\colon T_F\mathcal{B}_{\text{nor}}(U) \to T_{\mathcal{R}F}\mathcal{B}_{\text{nor}}(U) \tag{40}$$

induces a bounded linear transformation

$$L_F\colon T_F\mathcal{B}_{\text{nor}}/E^h_F \to T_{\mathcal{R}F}\mathcal{B}_{\text{nor}}(U)/E^h_{\mathcal{R}F}.$$

Lemma 5.3. *The map*

$$L(F, v) = (\mathcal{R}F, L_F \cdot v)$$

is a vector bundle isomorphism in the vector bundle E, that is, it is a homeomorphism of E onto itself that preserves the linear structure on the fibers.

Proof. Let $\pi\colon \Omega_{n,p} \times T\mathcal{B}_{\text{nor}}(U) \to E$ be a natural projection

$$(F, v) \mapsto (F, [v]_F),$$

where $[v]_F$ represents the equivalent class of v in $T\mathcal{B}_{\text{nor}}(U)/E^h_F$. Of course

$$\tilde{L}\colon \Omega_{n,p} \times T\mathcal{B}_{\text{nor}}(U) \to E$$

defined by $\tilde{L}(F, v) = \pi(\mathcal{R}F, D\mathcal{R}_F \cdot v)$ is continuous. Then by Proposition 4.2 the map $\tilde{L}$ descends to the topological quotient space E as the continuous map L.

By Theorem 3 we have that the linear transformation (40) has dense image in $T_{\mathcal{R}F}\mathcal{B}_{\text{nor}}(U)$. This implies that for every F, the linear map L_F is invertible.

By Corollary 2.3 we have that $\mathcal{R}\colon \Omega_{n,p} \to \Omega_{n,p}$ is a homeomorphism. We conclude that L is invertible. It remains to show that its inverse is continuous. Since

$$E^1 = \{(F, [v]_F)\colon F \in \Omega_{n,p} \text{ and } |[v]_F| = 1\}$$

is compact, L is invertible, and the function

$$\psi\colon E^1 \to \mathbb{R}^+$$

defined by $\psi(F, [v]_F) = |L(F, [v]_F)|$ is continuous, we have that

$$C = \min_{(F,[v]_F)\in E^1} \psi(F, [v]_F) > 0. \tag{41}$$

So suppose $\lim_k (F_k, [v_k]_{F_k}) = (F, [v]_F)$ and

$$L^{-1}(F_k, [v_k]_{F_k}) = (\mathcal{R}^{-1}F_k, [w_k]_{\mathcal{R}^{-1}F_k}).$$

Then $\lim_k \mathcal{R}^{-1}F_k = \mathcal{R}^{-1}F$, and by (41) we have $\sup_k |[w_k]_{\mathcal{R}^{-1}F_k}| \leq \hat{C}$ for some constant $\hat{C}$. Taking a subsequence we can assume that $\lim_k [w_k]_{\mathcal{R}^{-1}F_k} = [\tilde{w}]_{\mathcal{R}^{-1}F}$. Since L is continuous,

$$(F, [v]_F) = \lim_k L(\mathcal{R}^{-1}F_k, [w_k]_{\mathcal{R}^{-1}F_k}) = (F, L_F[\tilde{w}]_{\mathcal{R}^{-1}F}).$$

From the injectivity of L we conclude that $[\tilde{w}]_{\mathcal{R}^{-1}F} = L_F^{-1}[v]_F$. So L^{-1} is continuous. □

PROPOSITION 5.4. *Suppose that for every $F \in \Omega_{n,p}$, we have*

$$\mathfrak{B}_+(F) \subset E^h_F. \tag{42}$$

Then the cocycle L is uniformly expanding; that is, there are $C > 0$ and $\theta_2 > 1$ such that for every $v \in \mathcal{B}_{\mathrm{nor}}(U)$ and $F \in \Omega_{n,p}$, we have

$$d(D\mathcal{R}^i_F \cdot v, E^h_{\mathcal{R}^iF}) \geq C{\theta_2}^i d(v, E^h_F). \tag{43}$$

Proof. Indeed, suppose that $[v] \in T_F\mathcal{B}_{\mathrm{nor}}/E^h_F$ satisfies

$$\sup_i |L^i_F \cdot [v]| < \infty.$$

By Proposition 5.2, it is enough to show that $[v] = 0$; that is, $v \in E^h_F$. First note that $D\mathcal{R}^i_F \cdot v = u_i + w_i$, where $\sup_i |u_i| = C < \infty$ and $w_i \in E^h_{\mathcal{R}^iF}$. Note that

$$D\mathcal{R}_{\mathcal{R}^iF} \cdot (u_i + w_i) = u_{i+1} + w_{i+1},$$

so

$$w_{i+1} = D\mathcal{R}_{\mathcal{R}^iF} \cdot u_i - u_{i+1} + D\mathcal{R}_{\mathcal{R}^iF} \cdot w_i.$$

Thus

$$w_{i+j} = D\mathcal{R}^j_{\mathcal{R}^iF} \cdot u_i - u_{i+j} + D\mathcal{R}^j_{\mathcal{R}^iF} \cdot w_i;$$

in particular,

$$|w_{i+j}| \leq C(1 + K\theta^{-j}) + K\theta^{-j}|w_i|,$$

where K and $\theta > 1$ are as in Proposition 4.5. This implies that $\sup_i |w_i| < \infty$ and consequently

$$\sup_i |D\mathcal{R}_F^i \cdot v| < \infty.$$

By (37) we have that $v \in E_F^h$. So L is uniformly expanding. □

Let $C > 0$ and $\theta > 1$ be as in (43). Choose $j_0 > 0$ such that

$$C\theta^{j_0} > 1.$$

If $\epsilon > 0$ is small enough, we have that

$$\tilde{\theta} = Ce^{-\epsilon}\theta^{j_0} > 1.$$

Denote

$$\tilde{C} = \sup_{F \in \Omega_{n,p}} |D\mathcal{R}_F^{j_0}|.$$

Define the cone $C_F^u(K)$ as the set of all $v \in T\mathcal{B}_{\rm nor}(U)$ that can be written as $v = u + w$, where

(A) $|u| \leq e^{\epsilon} d(v, E_F^h)$;
(B) $w \in E_F^h$;
(C) $|w| \leq K|u|$.

Note that

$$\bigcup_{K>0} C_F^u(K) = \big(T\mathcal{B}_{\rm nor}(U) \setminus E_F^h\big) \cup \{0\}. \tag{44}$$

Our goal is to show that if (43) holds, then there is $K > 0$ such that

$$F \in \Omega_{n,p} \mapsto C_F^u(K)$$

is a field of unstable $\mathcal{R}$-invariant cones on $\Omega_{n,p}$.

PROPOSITION 5.5. *Assume that (43) holds. Then for $\epsilon > 0$ small enough, the following holds. Let $F \in \Omega_{n,p}$. If $v_0 = u_0 + w_0$, with*

$$0 < |u_0| \leq e^{\epsilon} d(v_0, E_F^h) \text{ and } w_0 \in E_F^h,$$

then for every $w_1 \in E_{\mathcal{R}^{j_0}F}^h$ and u_1 satisfying $D\mathcal{R}_F^{j_0} \cdot v_0 = u_1 + w_1$, we have

$$\frac{|w_1|}{|u_1|} \leq \tilde{C}\tilde{\theta}^{-1} + 1 + \tilde{\theta}^{-2}\frac{|w_0|}{|u_0|}. \tag{45}$$

Proof. Let $v_1 = D\mathcal{R}_F^{j_0} \cdot v_0$. In particular,

$$\frac{|u_1|}{|u_0|} \geq e^{-\epsilon}\frac{d(D\mathcal{R}_F^{j_0} \cdot v_0, E_{\mathcal{R}^{j_0}F}^h)}{d(v_0, E_F^h)} \geq Ce^{-\epsilon}\theta^{j_0} = \tilde{\theta} > 1.$$

Then

$$D\mathcal{R}_F^{j_0} v_0 = D\mathcal{R}_F^{j_0} u_0 + D\mathcal{R}_F^{j_0} w_0 = u_1 + w_1.$$

So

$$D\mathcal{R}_F^{j_0}\frac{u_0}{|u_1|}+D\mathcal{R}_F^{j_0}\frac{w_0}{|u_1|}=\frac{u_1}{|u_1|}+\frac{w_1}{|u_1|}$$

and

$$\begin{aligned}\frac{|w_1|}{|u_1|}&\le |D\mathcal{R}_F^{j_0}|\frac{|u_0|}{|u_1|}+1+\left|D\mathcal{R}_F^{j_0}\frac{w_0}{|u_1|}\right|\\&\le \tilde{C}\tilde{\theta}^{-1}+1+C\theta^{-j_0}\frac{|w_0|}{|u_1|}\\&\le \tilde{C}\tilde{\theta}^{-1}+1+\tilde{\theta}^{-1}C\theta^{-j_0}\frac{|w_0|}{|u_0|}\\&\le \tilde{C}\tilde{\theta}^{-1}+1+\tilde{\theta}^{-2}\frac{|w_0|}{|u_0|}.\end{aligned}\tag{46}$$

□

Corollary 5.6 (Invariant Cones). *Assume* (43) *holds. If* $v_0\in C_F^u(K_0)$, *then* $v_1=D\mathcal{R}_F^{j_0}\cdot v_0\in C_{\mathcal{R}F}^u(K_1)$, *where*

$$K_1=\tilde{C}\tilde{\theta}^{-1}+1+\tilde{\theta}^{-2}K_0.$$

Proof. We can assume that $v_0\neq 0$. Since $v_0\in C_F^u(K_0)$, there exist $w_0\in E_F^h$ and u_0 such that $v_0=u_0+w_0$ and

$$0<|u_0|\le e^{\epsilon}d(v_0,E_F^h)\text{ and }|w_0|\le K_0|u_0|.$$

Moreover, there exist $w_1\in E^h_{\mathcal{R}^{j_0}F}$ and u_1 satisfying $v_1=u_1+w_1$, with

$$|u_1|\le e^{\epsilon}d(v_1,E^h_{\mathcal{R}^{j_0}F}).$$

By Proposition 5.5 we have that $|w_1|\le K_1|u_1|$, so $v_1\in C^u_{\mathcal{R}^{j_0}F}(K_1)$. □

To simplify the notation, we will replace the operator $\mathcal{R}$ by its iteration $\mathcal{R}^{j_0}$. The following two corollaries are an immediate consequence of Corollary 5.6.

Corollary 5.7 (Forward Invariant Cones). *Assume that* (43) *holds. If*

$$K\ge\frac{\tilde{C}\tilde{\theta}^{-1}+1}{1-\tilde{\theta}^{-2}},$$

then for every $F\in\Omega_{n,p}$,

$$D\mathcal{R}_F C_F^u(K)\subset C^u_{\mathcal{R}F}(K).\tag{47}$$

Corollary 5.8 (Absorbing Cones). *Assume that* (43) *holds. For each*

$$K_0>\frac{\tilde{C}\tilde{\theta}^{-1}+1}{1-\tilde{\theta}^{-2}},$$

the following holds: for every $K>0$, *there exists* i *such that for all* $F\in\Omega_{n,p}$,

$$D\mathcal{R}^i_F C_F^u(K)\subset C^u_{\mathcal{R}^iF}(K_0).\tag{48}$$

Corollary 5.9 (Unstable Cones). *Assume that* (43) *holds. For each* $K_0 > 0$, *there exists* $C > 0$ *such that for all* $F \in \Omega_{n,p}$, $v \in C^u_F(K_0)$ *and* $i \geq 0$

$$|D\mathcal{R}^i_F v| \geq C\theta^i |v|. \tag{49}$$

Proof. If $v \in C^u_F(K_0)$, then $v = u + w$, with $w \in E^h_F$,

$$|u| \leq e^{\epsilon} d(v, E^h_F)$$

and

$$|w| \leq K_0 |u|.$$

So

$$|v| \leq |u| + |w| \leq (1 + K_0)e^{\epsilon} d(v, E^h_F).$$

By (43) we have

$$|D\mathcal{R}^i_F \cdot v| \geq d(D\mathcal{R}^i_F \cdot v, E^h_{\mathcal{R}^i F}) \geq C\theta^i d(v, E^h_F) \geq \frac{C}{(1 + K_0)e^{\epsilon}} \theta^i |v|. \qquad \square$$

Now fix

$$K_0 > \frac{\tilde{C}\tilde{\theta}^{-1} + 1}{1 - \tilde{\theta}^{-2}}.$$

Choose $i > 0$ such that

$$\theta_1 = \frac{C}{(1 + K_0)e^{\epsilon}} \theta^i > 1.$$

Replace (once again) the operator $\mathcal{R}$ by its iteration $\mathcal{R}^i$.

Corollary 5.10 (Unstable Invariant Cones near $\Omega_{n,p}$). *Assume that* (43) *holds. For each*

$$K_0 > \frac{\tilde{C}\tilde{\theta}^{-1} + 1}{1 - \tilde{\theta}^{-2}} \tag{50}$$

and $\hat{\theta} \in (1, \theta_1)$, *there exists* $\delta > 0$ *such that if*

$$\mathrm{dist}(F, G_0) < \delta$$

for some $G_0 \in \Omega_{n,p}$, *then*

$$D\mathcal{R}_F C^u_{G_0}(K_0) \subset C^u_{\mathcal{R}G_0}(K_0)$$

and

$$|D\mathcal{R}_F \cdot v| \geq \hat{\theta} |v| \tag{51}$$

for every $v \in C^u_{G_0}(K_0)$.

Proof. Define $K_1 = \tilde{C}\tilde{\theta}^{-1} + 1 + \tilde{\theta}^{-2}K_0$. Then

$$\frac{\tilde{C}\tilde{\theta}^{-1}+1}{1-\tilde{\theta}^{-2}} < K_1 < K_0.$$

Choose $\gamma \in (0, \epsilon)$ small enough such that

$$K_1 e^{\gamma} < K_0.$$

Let $v \in C^u_{G_0}(K_0)$. Then there exist $w_0 \in E^h_{G_0}$ and u_0 such that $v = u_0 + w_0$ and

$$|u_0| \leq e^{\epsilon} d(v_0, E^h_{G_0}) \text{ and } |w_0| \leq K_0|u_0|.$$

Moreover, there exist $w_1 \in E^h_{\mathcal{R}G_0}$ and u_1 satisfying

$$|u_1| \leq e^{\gamma/3} d(D\mathcal{R}_{G_0} \cdot v, E^h_{\mathcal{R}G_0})$$

and $D\mathcal{R}_{G_0} \cdot v = u_1 + w_1$. By Proposition 5.5,

$$|w_1| \leq K_1|u_1|.$$

Then

$$D\mathcal{R}_F \cdot v = u_1 + (D\mathcal{R}_{G_0} - D\mathcal{R}_F) \cdot v + w_1. \tag{52}$$

Note that

$$\begin{aligned} d((D\mathcal{R}_{G_0} - D\mathcal{R}_F)\cdot v, E^h_{\mathcal{R}G_0}) &\leq |(D\mathcal{R}_{G_0} - D\mathcal{R}_F)\cdot v| \\ &\leq |D\mathcal{R}_{G_0} - D\mathcal{R}_F| e^{\epsilon}(1+K_0) d(v, E^h_{G_0}) \\ &\leq |D\mathcal{R}_{G_0} - D\mathcal{R}_F| \frac{e^{\epsilon}(1+K_0)}{\theta} d(D\mathcal{R}_{G_0}\cdot v, E^h_{\mathcal{R}G_0}). \end{aligned}$$

Let $\delta_1 > 0$ be such that $|F - G_0| < \delta_1$ implies

$$\begin{aligned} 1 - |D\mathcal{R}_{G_0} - D\mathcal{R}_F| \frac{e^{\epsilon}(1+K_0)}{\theta} &\geq e^{-\gamma/3}, \\ e^{\gamma/3} + |D\mathcal{R}_{G_0} - D\mathcal{R}_F| \frac{e^{\epsilon}(1+K_0)}{\theta} &\leq e^{2\gamma/3}, \end{aligned}$$

and

$$\tilde{\theta} = \theta_1 - |D\mathcal{R}_{G_0} - D\mathcal{R}_F| > \hat{\theta} > 1.$$

Then

$$\begin{aligned} d(D\mathcal{R}_F \cdot v, E^h_{\mathcal{R}G_0}) &\geq d(D\mathcal{R}_{G_0}\cdot v, E^h_{\mathcal{R}G_0}) - d((D\mathcal{R}_{G_0} - D\mathcal{R}_F)\cdot v, E^h_{\mathcal{R}G_0}) \\ &\geq e^{-\gamma/3} d(D\mathcal{R}_{G_0}\cdot v, E^h_{\mathcal{R}G_0}), \end{aligned} \tag{53}$$

so

(54)
$$\begin{aligned}|u_1 + (D\mathcal{R}_{G_0} - D\mathcal{R}_F)\cdot v| &\leq e^{\gamma/3} d(D\mathcal{R}_{G_0}\cdot v, E^h_{\mathcal{R}G_0}) + |(D\mathcal{R}_{G_0} - D\mathcal{R}_F)\cdot v| \\ &\leq e^{2\gamma/3} d(D\mathcal{R}_{G_0}\cdot v, E^h_{\mathcal{R}G_0}) \\ &\leq e^{\gamma} d(D\mathcal{R}_{F}\cdot v, E^h_{\mathcal{R}G_0}) \\ &\leq e^{\epsilon} d(D\mathcal{R}_{F}\cdot v, E^h_{\mathcal{R}G_0})\end{aligned}$$

and, moreover,

$$\begin{aligned}|u_1 + (D\mathcal{R}_{G_0} - D\mathcal{R}_F)\cdot v| &\geq |u_1| - |(D\mathcal{R}_{G_0} - D\mathcal{R}_F)\cdot v| \\ &\geq d(D\mathcal{R}_{G_0}\cdot v, E^h_{\mathcal{R}G_0}) - |(D\mathcal{R}_{G_0} - D\mathcal{R}_F)\cdot v| \\ &\geq e^{-\gamma/3} d(D\mathcal{R}_{G_0}\cdot v, E^h_{\mathcal{R}G_0}).\end{aligned} \tag{55}$$

Finally

$$\begin{aligned}|w_1| &\leq K_1 |u_1| \\ &\leq K_1 e^{\gamma/3} d(D\mathcal{R}_{G_0}\cdot v, E^h_{\mathcal{R}G_0}) \\ &\leq K_1 e^{2\gamma/3} |u_1 + (D\mathcal{R}_{G_0} - D\mathcal{R}_F)\cdot v|. \\ &\leq K_0 |u_1 + (D\mathcal{R}_{G_0} - D\mathcal{R}_F)\cdot v|.\end{aligned} \tag{56}$$

So (52), (54) and (56) imply that $D\mathcal{R}_F\cdot v \in C^u_{\mathcal{R}G_0}(K_0)$. Furthermore,

$$\begin{aligned}|D\mathcal{R}_F\cdot v| &\geq |D\mathcal{R}_G\cdot v| - |(D\mathcal{R}_{G_0} - D\mathcal{R}_F)\cdot v| \\ &\geq \theta_1|v| - |D\mathcal{R}_{G_0} - D\mathcal{R}_F||v| \\ &\geq \hat{\theta}|v|.\end{aligned} \tag{57}$$

□

Proof of Proposition 5.1. Let K_0 be as in Corollary 5.10. We claim that every cone $C^u_F(K_0)$, with $F \in \Omega_{n,p}$, contains a subspace S_F of dimension n. Indeed, since E^h_G, $G \in \Omega_{n,p}$, has finite codimension n, there is a subspace $E_G \subset T\mathcal{B}_{\text{nor}}(U)$ of dimension n such that

$$E_G \oplus E^h_G = T\mathcal{B}_{\text{nor}}(U).$$

Since $\Omega_{n,p}$ is a Cantor set and $G \mapsto E^h_G$ is a continuous distribution, it is easy to see that we can find a finite covering $\{O_j\}_j$ by compact subsets of $\Omega_{n,p}$ and subspaces E_j with dimension n such that

$$E_j \oplus E^h_G = T\mathcal{B}_{\text{nor}}(U)$$

for every $G \in O_j$. By (44), Proposition 5.5 and Corollary 5.8 there exists i_0 such that

$$D\mathcal{R}^{i_0}_G E_j \subset C^u_{\mathcal{R}^{i_0}G}(K_0)$$

for every $G \in O_j$. Moreover, $\mathcal{R}$ is invertible on $\Omega_{n,p}$, so we can choose G such that $\mathcal{R}^{i_0}G = F$. Since $D\mathcal{R}_G$ is injective for every $G \in \Omega_{n.p}$, we conclude that

$S_F = D\mathcal{R}_G^{i_0} E_j$ is a subspace of dimension n in $C_F^u(K_0)$. This concludes the proof of the claim. Note that

$$S_F \oplus E_F^h = T\mathcal{B}_{\text{nor}}(U).$$

Let $G \in \Omega_{n,p}$. Since $\mathcal{R}$ is invertible on $\Omega_{n,p}$, there is a unique sequence $G_i \in \Omega_{n,p}$ such that $\mathcal{R}G_{i+1} = G_i$ and $G_0 = G$. Let S_i' be an arbitrary subspace of dimension n contained in $C_{G_i}^u(K_0)$. Then $D\mathcal{R}_{G_i}^i(S_i')$ is a subspace of dimension n contained in $C_G^u(K_0)$. Since

$$C_G^u(K_0) \cap E_G^h = \{0\}, \tag{58}$$

we have that there is a linear function

$$\mathcal{H}_i \colon S_G \to E_G^h$$

such that $\{v + \mathcal{H}_i(v),\ v \in S_G\} = D\mathcal{R}_{G_i}^i(S_i')$. Since $v + \mathcal{H}_i(v) \in C_G^u(K_0)$, we have $v + \mathcal{H}_i(v) = u_1 + w_1$, with $w_1 \in E_G^h$,

$$|u_1| \leq e^{\epsilon} d(v + \mathcal{H}_i(v), E_G^h) = e^{\epsilon} d(v, E_G^h) < e^{\epsilon}|v|,$$

and $|w_1| \leq K_0|u_1|$. In particular,

$$|\mathcal{H}_i(v)| = |u_1 + w_1 - v| \leq (1 + e^{\epsilon}(1 + K_0))|v|.$$

So the family of functions $\mathcal{H}_i$ is an equicontinuous family of functions. By the strong compactness of the operator $\mathcal{R}$ and Arzela-Ascoli Theorem there exists a subsequence that converges to a bounded linear function $\mathcal{H}^G \colon S_G \to E_G^h$ satisfying

$$\{v + \mathcal{H}^G(v),\ v \in S_G\} \subset \bigcap_{i=0}^{\infty} \overline{D\mathcal{R}_{G_i}^i C_{G_i}^u(K_0)} \subset C_G^u(K_0).$$

Due to (58) and the contraction in the horizontal directions we have that there is only one possible accumulation point $\mathcal{H}^G$ for sequences as $(\mathcal{H}_i)_i$. Let

$$E_G^u = \{v + \mathcal{H}^G(v),\ v \in S_G\}.$$

Then we can easily conclude that $D\mathcal{R}_G(E_G^u) = E_{\mathcal{R}G}^u$. Then $G \mapsto E_G^u$ is the unstable direction of $\mathcal{R}$. □

6. **Induced expanding maps**

Let $F \in \Omega_{n,p}$. We are going to define a real induced map $G_{\mathbb{R}} \colon D \to \mathbb{R}$ for F whose domain D is the union of intervals R_{-i}^k, $k \geq 0$ and $0 < i < n_r^k$, $r \in C(F)$, satisfying

$$R_{-i}^k \subset \bigcup_{q \in C(F)} Q_0^{k-1}.$$

If $R^k_{-i} \subset Q^{k-1}_0$ and $s \in C(F)$ is the successor of q at the $k-1$ level, we define $G_{\mathbb{R}}(x) = F^{n^{k-1}_s}(x)$ for every $x \in R^k_{-i}$. Note that

$$G_{\mathbb{R}} \colon R^k_{-i} \to R^k_{-i+n^{k-1}_s}$$

is a diffeomorphism. Due to the real bounds, there exists ϵ_0 such that

$$0 < \epsilon_0 < \inf_{r \in C(F), k} \frac{2\mathrm{dist}(P(F), \partial R^k_0)}{|R^k_0|}. \tag{59}$$

We will now define a complex-analytic extension G of $G_{\mathbb{R}}$ that is a conformal iterated dynamical system. Suppose that $R^k_{-i} \subset Q^{k-1}_0$. Let $\mathscr{R}^k_0$ be the ball $B(r, d^k_r)$, where

$$d^k_r = \Big(1 - \frac{\epsilon_0}{16}\Big) \frac{|R^k_0|}{2}. \tag{60}$$

Since F belongs to the Epstein class, there exists a simply connected domain $\mathscr{R}^k_{-i}$ such that $\mathscr{R}^k_{-i} \cap \mathbb{R} \subset R^k_{-i}$ and $\mathscr{R}^k_{-i}$ is contained in the ball whose diameter is $R^k_{-i} \cap \mathbb{R}$, and moreover

$$F^i \colon \mathscr{R}^k_{-i} \to \mathscr{R}^k_0$$

is univalent. Due to the real bounds, we can reduce ϵ_0 if necessary in such way that

$$\mathscr{R}^k_{-i} \subset \mathscr{Q}^{k-1}_0$$

for every $R^k_{-i} \subset Q^{k-1}_0$ and

$$\inf_k \inf_{\substack{\{q,r\} \subset C(F) \\ R^k_{-i} \subset Q^{k-1}_0}} \mathrm{dist}(\mathscr{R}^k_{-i}, \partial \mathscr{Q}^{k-1}_0) > 0.$$

We define G on

$$\mathscr{D} = \bigcup_k \bigcup_{r,q \in C(F)} \bigcup_{R^k_{-i} \subset Q^{k-1}_0} \mathscr{R}^k_{-i}$$

as $G(z) = F^{n^{k-1}_s}(z)$ for every $z \in \mathscr{R}^k_{-i}$, where $R^k_{-i} \subset Q^{k-1}_0$ and $s \in C(F)$ is the successor of q at level $k-1$. Note that $P(F) \setminus C(F) \subset \mathscr{D}$ and

$$G(P(F) \setminus C(F)) = P(F).$$

Lemma 6.1 (Markovian property of the induced map). *Let $r_j \in C(F)$, $m_i \in \mathbb{N}$ and $0 \leq i_j < n^{m_i}_{r_j}$, $j \leq \ell$ be such that*

(A) *Either we have that $m_{j+1} = m_j$, $i_{j+1} < i_j$, $r_{j+1} = r_j = r$ for some $r \in C(F)$ and, moreover,*

$$R^{m_j}_{-i_j}, R^{m_j}_{-i_{j+1}} \subset \bigcup_{q \in C(F)} Q^{m_j - 1}_0$$

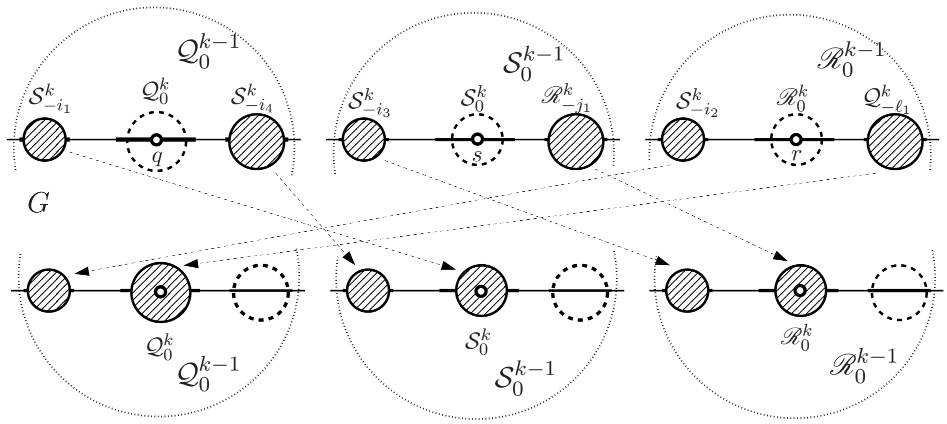

Figure 2. The action of G on the domains $\mathscr{R}_{-i}^k \subset \cup_{q\in C(F)} \mathcal{Q}_0^{k-1}$ if the combinatorics of the k-th renormalization is the same as the combinatorics of the renormalization $R(F)$ in Figure 1. At level $k-1$ we have that s is the successor of q, r is the successor of s and q is the successor of r. At level k we have that r is the successor of q, s is the successor of r and q is the successor of s. Moreover, $\ell_1, j_1 > 0$ and $0 < i_1 < i_2 < i_3 < i_4$.

and

$$R_{-i}^{m_j} \notin \bigcup_{q\in C(F)} Q_0^{m_j-1}$$

for every i satisfying $i_{j+1} < i < i_j$. In particular,

$$G_{\mathbb{R}}\colon R_{-i_j}^{m_j} \mapsto R_{-i_{j+1}}^{m_{j+1}}$$

is a diffeomorphism.

(B) *Or*

$$m_{j+1} > m_j,$$

with

$$R_{-i_j}^{m_j} \subset \bigcup_{q\in C(F)} Q_0^{m_j-1},$$

and

$$R_{-i}^{m_j} \notin \bigcup_{q\in C(F)} Q_0^{m_j-1}$$

for every i satisfying $0 < i < i_j$. Here $r = r_j$. In particular,

$$G\colon R_{-i_j}^{m_j} \mapsto R_0^{m_j}$$

is a diffeomorphism. Moreover, $i_{j+1} > 0$ and

$$S_{-i_{j+1}}^{m_{j+1}} \subset R_0^{m_{j+1}-1},$$

where $s = r_{j+1}$. In particular,

$$G_{\mathbb{R}}\colon R^{m_j}_{-i_j} \mapsto R^{m_j}_0$$

is a diffeomorphism and

$$S^{m_{j+1}}_{-i_{j+1}} \subset R^{m_j}_0 = G_{\mathbb{R}}(R^{m_j}_{-i_j}).$$

Then there exists a unique interval W such that

$$G^{\ell}_{\mathbb{R}}\colon W \to R^{m_\ell}_{-i_\ell}$$

is a diffeomorphism and W is the set of points z such that for every $j \leq \ell$, we have $G^j_{\mathbb{R}}(z) \in Q^{m_j}_{-i_j}$, where $q = r_j$. Moreover, $W = R^{m_\ell}_{-i}$ for some i, where $r = r_\ell$.

Proof. If $\ell = 0$, there is nothing to prove, since $W = R^{m_0}_{-i_0}$, with $r = r_0$. Suppose by induction on ℓ that Lemma 6.1 holds for ℓ. Let $r_j \in C(F)$, $m_i \in \mathbb{N}$ and $0 < i_j < n^{m_i}_{r_j}$, $j \leq \ell + 1$ be as in the statement of the lemma. By the induction assumption there exists b such that

$$G^{\ell}_{\mathbb{R}}\colon R^{m_{\ell+1}}_{-b} \to R^{m_{\ell+1}}_{-i_{\ell+1}}$$

is a diffeomorphism and for every $j \leq \ell$, we have $G^j_{\mathbb{R}} R^{m_{\ell+1}}_{-b} \subset Q^{m_{j+1}}_{-i_{j+1}}$, where $q = r_{j+1}$. In particular, $R^{m_{\ell+1}}_{-b} \subset S^{m_1}_{-i_1}$, with $s = r_1$. There are two cases. If $m_0 = m_1$, then $S^{m_0}_{-i_0} = S^{m_1}_{-i_0}$, $i_0 > i_1$,

$$G_{\mathbb{R}}\colon S^{m_1}_{-i_0} \to S^{m_1}_{-i_1}$$

is a diffeomorphism, and $W = R^{m_\ell}_{-(b+i_0-i_1)}$ is the unique interval $W \subset S^{m_1}_{-i_0}$ such that $G_{\mathbb{R}}(W) = R^{m_\ell}_{-b}$. If $m_1 > m_0$, then $S^{m_1}_{-i_1} \subset Q^{m_0}_0$, with $q = m_0$, and

$$G\colon Q^{m_0}_{-i_0} \to Q^{m_0}_0$$

is a diffeomorphism. Then $W = R^{m_\ell}_{-(b+i_0)}$ is the unique interval $W \subset Q^{m_0}_{-i_0}$ such that $G(W) = R^{m_\ell}_{-b}$. □

The next proposition says that the postcritical set $P(F)$ of F is the maximal invariant set of the induced map G.

Proposition 6.2. *Given $(z, j) \in \mathscr{D}$, we have that (z, j) belongs to $\mathscr{D} \setminus P(F)$ if and only if there exists $k_0 \geq 0$ such that $G^k(z, j) \in \mathscr{D}$ for every $k < k_0$ and $G^{k_0}(z, j) \notin \mathscr{D}$.*

Proof. Let $r_j \in C(F)$, $m_i \in \mathbb{N}$ and $0 < i_j < n^{m_i}_{r_j}$ such that either

$$m_{j+1} > m_j$$

and

$$R^{m_{j+1}}_{-i_{j+1}} \subset Q^{m_j}_0,$$

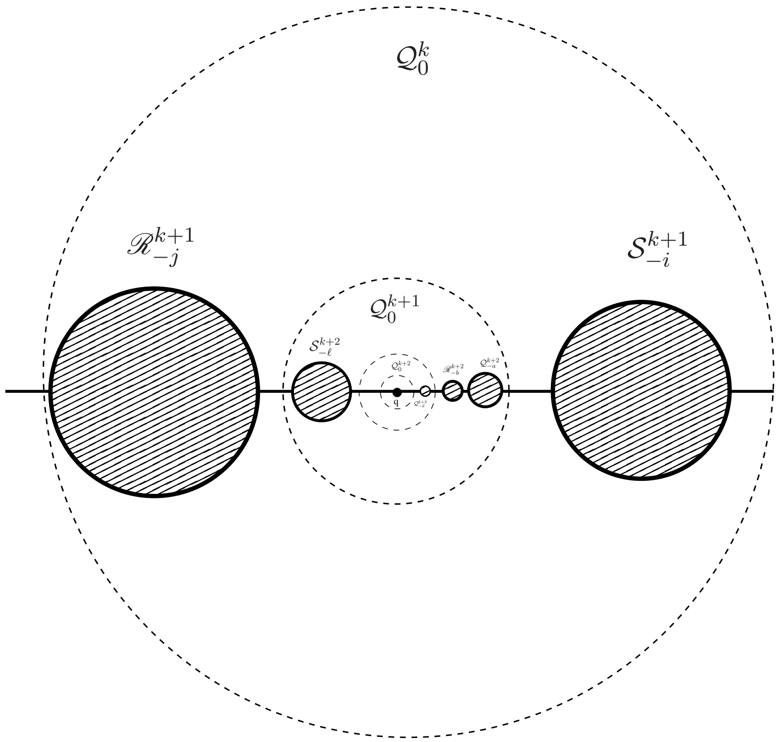

Figure 3. The domain $\mathscr{D}$ of G close to a critical point q (the point in the center); it contains infinitely many topological disks accumulating on the critical point q.

where $r = r_{j+1}$ and $q = r_j$, or $m_{j+1} = m_j$ and $i_{j+1} = i_j - 1$. We claim that there exists a unique $z \in \mathbb{C}_n$ such that

$$G^j(z) \in \mathscr{R}^{m_j}_{-i_j} \tag{61}$$

for every $j \geq 0$. Indeed, for each ℓ, let D_ℓ be the set of points such that (61) holds for every $j \leq \ell$. Of course $D_{\ell+1} \subset D_\ell$. If $m_{\ell+1} = m_\ell$, then $D_{\ell+1} = D_\ell$. If $m_{\ell+1} > m_\ell$, then

$$G^{\ell+1} \colon D_\ell \to \mathcal{Q}_0^{m_\ell},$$

with $q = r_\ell$ and

$$G^{\ell+1} \colon D_{\ell+1} \to \mathcal{R}^{m_{\ell+1}}_{-i_{\ell+1}},$$

with $r = r_{\ell+1}$, are univalent (and onto). By the definition of G,

$$M = \inf_m \inf_{\mathcal{R}^{m+1}_{-i} \subset \mathcal{Q}_0^m} \operatorname{mod}(\mathcal{Q}_0^m \setminus \mathcal{R}^{m+1}_{-i}) > 0,$$

so

$$\operatorname{mod}(D_\ell \setminus D_{\ell+1}) = \operatorname{mod}(\mathcal{Q}_0^{m_\ell} \setminus \mathcal{R}^{m_{\ell+1}}_{-i_{\ell+1}}) \geq M;$$

in particular, since there exist infinitely many ℓ such that $m_{\ell+1} > m_\ell$, we have that

$$\lim_{\ell} \operatorname{diam}(D_\ell) = 0$$

and

$$\cap_\ell D_\ell = \{z_0\}$$

for some $z_0 \in \mathscr{D}$. This completes the proof of the claim. Note that since D_ℓ are symmetric with respect to the $\mathbb{R}$, we have that $z_0 \in \mathbb{R}$ and by (61),

$$G^j(z_0) \in \mathscr{R}^{m_j}_{-i_j} \cap \mathbb{R} \subset R^{m_j}_{-i_j}, \tag{62}$$

where $r = r_j$ for every j. By Lemma 6.1 there exists b_j such that

$$z_0 \in R^{m_j}_{-b_j},$$

where $r = r_j$ for every $j \geq 0$. In particular, $z_0 \in P(F)$. □

7. Induced problem

Let a_i be as in (15). Define the function

$$V \colon \mathscr{D} \to \mathbb{C}$$

as

$$V(z) = a_i(z) = \frac{\partial}{\partial t}(F_t)^{n_s^{k-1}}(z)\big|_{t=0}$$

for every $z \in \mathscr{R}^k_{-i} \subset \mathscr{D}$, provided $R^k_{-i} \subset Q^{k-1}_0$ and s is the successor of q at the $k-1$ level. Here $F_t = F + tv$.

Lemma 7.1. *Suppose that α is a continuous vector field in a neighborhood of $P(F)$, such that*

(A1) *$\alpha(c) = 0$ for every $c \in C(F)$; and*
(A2) *for every $z \in P(F) \setminus C(F)$, we have*

$$V(z) = \alpha \circ G(z) - DG(z) \cdot \alpha(z). \tag{63}$$

Let $x \in P(F)$ and $\ell > 0$ such that $G^j(x) \notin C(F)$ for every $0 \leq j < \ell$. Then for each $j < \ell$, there is $r, q \in C(F)$, with $b_j > 0$ such that $G^j(x) \in R^{k_j}_{-b_j} \subset Q^{k_j-1}_0$. Define $i_j = n_s^{k_j-1}$, where $s \in C(F)$ is the successor of q at level $k-1$. Note that the critical points q, s, r may depend on x and j. Let

$$m = \sum_{j=0}^{\ell-1} i_j.$$

Then

(B1) *For every $z \in \mathbb{C}$ close enough to x, we have that $G^\ell(z)$ is well defined and $G^\ell(z) = F^m(z)$.*

(B2) *For every $z \in P(F)$ close enough to x, we have that $\alpha(G^\ell(z))$ is well defined and*

$$\alpha(z) = \frac{\alpha(G^\ell(z))}{DG^\ell(z)} - \sum_{a=1}^{m} \frac{v(F^{a-1}(z))}{DF^a(z)}.$$

(B3) *In particular, for every $z \in P(F)$ close enough to x such that $G^\ell(x) \in C(F)$, we have*

$$\alpha(x) = -\sum_{a=1}^{m} \frac{v(F^{a-1}(x))}{DF^a(x)}.$$

Proof. We let to the reader to show that $G^\ell(z) = F^m(z)$ for z close enough to x. By (63) we get

$$\alpha(x) = \frac{\alpha(G^\ell(z))}{DG^\ell(z)} - \sum_{j=1}^{\ell} \frac{V(G^{j-1}(x))}{DG^j(z)}.$$

So

$$\sum_{j=1}^{\ell} \frac{V(G^{j-1}(x))}{DG^j(z)} = \sum_{j=1}^{\ell} \frac{(\partial_t (F_t)^{i_j})(F^{i_0+i_1+\cdots+i_{j-1}}(z))}{DF^{i_0+i_1+\cdots+i_j}(z)}$$

$$= \sum_{j=1}^{\ell} \sum_{b=0}^{i_j-1} \frac{DF^{i_j-b-1}(F^{b+1}(F^{i_0+i_1+\cdots+i_{j-1}}(z)))v(F^b(F^{i_0+i_1+\cdots+i_{j-1}}(z)))}{DF^{i_0+i_1+\cdots+i_j}(z)}$$

$$= \sum_{j=1}^{\ell} \sum_{b=0}^{i_j-1} \frac{DF^{i_j-b-1}(F^{b+1}(F^{i_0+i_1+\cdots+i_{j-1}}(z)))v(F^{i_0+i_1+\cdots+i_{j-1}+b}(z))}{DF^{i_j-b-1}(F^{b+1}(F^{i_0+i_1+\cdots+i_{j-1}}(z)))DF^{i_0+i_1+\cdots+i_{j-1}+b+1}(z)}$$

$$= \sum_{j=1}^{\ell} \sum_{b=0}^{i_j-1} \frac{v(F^{i_0+i_1+\cdots+i_{j-1}+b}(z))}{DF^{i_0+i_1+\cdots+i_{j-1}+b+1}(z)} = \sum_{a=1}^{m} \frac{v(F^{a-1}(z))}{DF^a(z)}. \qquad \square$$

Proposition 7.2. *Suppose that α is a continuous vector field in a neighborhood of $P(F)$, such that $\alpha(r) = 0$ for every $r \in C(F)$ and*

$$V(z) = \alpha \circ G(z) - DG(z) \cdot \alpha(z) \tag{64}$$

for every $z \in P(F) \setminus C(F)$. Then

$$v(z) = \alpha \circ F(z) - DF(z) \cdot \alpha(z) \tag{65}$$

for every $z \in P(F)$.

Proof. For each R^k_{-m}, with $r \in C(F)$ and $0 < m < n^k_r$, there exists a unique $r^k_{-m} \in R^k_{-m}$ such that

$$F^m(r^k_{-m}) = r.$$

The set

$$\Gamma = \{r^k_{-m}\}_{r\in C(F), 0<m<n^k_r} \subset P(F)$$

is dense on $P(F)$ and

$$F(\Gamma) = \Gamma \cup C(F).$$

We claim that (65) holds for every $z \in \Gamma$. Indeed, given $r^k_{-m} \in \Gamma$, there exists $\ell > 0$ such that $G^\ell(r^k_{-m}) = 0$. Moreover, by Lemma 7.1, we have that $G^\ell = F^m$ in a neighborhood of r^k_{-m} and

$$\alpha(r^k_{-m}) = -\sum_{a=1}^{m} \frac{v(F^{a-1}(r^k_{-m}))}{DF^a(r^k_{-m})}. \tag{66}$$

Suppose that $m = 1$. Then $\ell = 1$, $G = F$ in a neighborhood of r^k_{-1}, $V(r^k_{-1}) = v(r^k_{-1})$ and $G(r^k_{-1}) \in C(F)$. By (64) it follows that

$$v(r^k_{-1}) = \alpha \circ G(r^k_{-1}) - DG(r^k_{-1}) \cdot \alpha(r^k_{-1}) = \alpha \circ F(r^k_{-1}) - DF(r^k_{-1}) \cdot \alpha(r^k_{-1}).$$

If $m > 1$, then $F(r^k_{-m}) = r^k_{-(m-1)} \in \Gamma$, so by (66) we have

$$\begin{aligned}
&\alpha(F(r^k_{-m})) - DF(r^k_{-m})\alpha(r^k_{-m}) \\
&= -\sum_{a=1}^{m-1} \frac{v(F^{a-1}(r^k_{-(m-1)}))}{DF^a(r^k_{-(m-1)})} + DF(r^k_{-m}) \sum_{b=1}^{m} \frac{v(F^{b-1}(r^k_{-m}))}{DF^b(r^k_{-m})} \\
&= -\sum_{a=1}^{m-1} \frac{v(F^{a-1}(r^k_{-(m-1)}))}{DF^a(r^k_{-(m-1)})} + \sum_{b=1}^{m} \frac{v(F^{b-1}(r^k_{-m}))}{DF^{b-1}(r^k_{-(m-1)})} \\
&= -\sum_{a=1}^{m-1} \frac{v(F^{a-1}(r^k_{-(m-1)}))}{DF^a(r^k_{-(m-1)})} + v(r^k_{-m}) + \sum_{b=2}^{m} \frac{v(F^{b-2}(r^k_{-(m-1)}))}{DF^{b-1}(r^k_{-(m-1)})} \\
&= -\sum_{a=1}^{m-1} \frac{v(F^{a-1}(r^k_{-(m-1)}))}{DF^a(r^k_{-(m-1)})} + v(r^k_{-m}) + \sum_{a=1}^{m-1} \frac{v(F^{a-1}(r^k_{-(m-1)}))}{DF^a(r^k_{-(m-1)})} \\
&= v(r^k_{-m}).
\end{aligned}$$

Thus (65) holds for every $z \in \Gamma$. Since v, α and F are continuous in a neighborhood of $P(F)$ and Γ is dense in $P(F)$, it follows that (65) holds for every $z \in P(F)$. □

Corollary 7.3 (Induced problem). *Let $F \in \Omega_{n,p}$ and $v \in \mathfrak{B}_+(F)$. If there exists a quasiconformal vector field α, defined in a neighborhood of $P(F)$, such that $\alpha(r) = 0$ for every $r \in C(F)$ and*

$$V(z) = \alpha \circ G(z) - DG(z) \cdot \alpha(z) \tag{67}$$

for every $z \in P(F) \setminus C(F)$, then $v \in E^h(F)$.

Proof. This is an immediate consequence of Propositions 4.1 and 7.2. □

8. **Solving the induced problem**

We are going to change the notation slightly. Let P_i^k, $i = 1, \ldots, n$ be the set of restrictive intervals for F associated to the renomalization $\mathcal{R}^k F$; that is,

(A) these intervals are pairwise disjoint, $C(F) \cap P_i^k \neq \emptyset$ and

$$C(F) \subset \cup_i P_i^k;$$

(B) there are integers n_i^k such that

$$F^{n_i^k}: P_i^k \to P_{i+1 \bmod n}^k$$

is an unimodal map;

(C) we have that $P_i^k = [\delta_i^k, b_i^k]$, where δ_i^k is a $\mu^{(k)}$-periodic repelling fixed point, with

$$\mu^{(k)} = \sum_i n_i^k,$$

and $F^{n_i^k}(\delta_i^k) = F^{n_i^k}(b_i^k)$;

(D) the renormalization associated to the restrictive interval P_1^k is $\mathcal{R}^k F$.

If t is small enough, then $F_t = F + tv$ is close to F and δ_i^k has an analytic continuation $\delta_i^k(t)$. Denote by $\partial_t \delta_i^k$ the derivative of this continuation with respect to t at $t = 0$. Consider $\{q_i^k\} = C(F) \cap P_i^k$. Then q_{i+1}^k is the successor of q_i^k at level k. Let $j_i^k \in \{1, \ldots, n\}$ be such that $q_i^k \in I_{j_i^k}$, and let

$$A_i^k : \mathbb{C} \times \{j_i^k\} \to \mathbb{C} \times \{i\}$$

be the only affine transformation such that $A_i^k(\delta_i^k) = (-1, i)$ and $A_i^k(q_i^k) = (0, i)$.

Lemma 8.1. *We have $V = V_1 + V_2$, where*

$$V_1(z, j_i^k) = -\delta_{i+1}^k v^k \circ A_i^k(z, j_i^k), \tag{68}$$

and

$$V_2(z, j_i^k) = \frac{\partial_t \delta_{i+1}^k}{\delta_{i+1}^k} F^{n_i^k}(z, j_i^k) - DF^{n_i^k}(z, j_i^k) \cdot \frac{\partial_t \delta_i^k}{\delta_i^k}(z, j_i^k) \tag{69}$$

for every $(z, j_i^k) \in \mathscr{R}_{-\ell}^{k+1}$, with $R_{-\ell}^{k+1} \subset Q_0^k$, $q = q_i^k$.

Proof. Note that

$$\begin{aligned} v^k(x,i) &= \partial_t(\mathcal{R}^k(F_t))\big|_{t=0}(x,i) = (D\mathcal{R}_F^k \cdot v)(x,i) \\ &= -\frac{\partial_t \delta_{i+1}^k}{\delta_{i+1}^k} \cdot A_{i+1}^k \circ F^{n_i^k} \circ (A_i^k)^{-1}(x,i) \\ &\quad - \frac{1}{\delta_{i+1}^k}\big(a_{n_i^k} \circ (A_i^k)^{-1}(x,i) + (DF^{n_i^k}) \circ (A_i^k)^{-1}(x,i) \cdot (-\partial_t \delta_i^k\ x, i)\big). \end{aligned}$$

So if $(x,i)=A^k_i(z,j^k_i)$, with $(z,j^k_i)\in \mathscr{R}^{k+1}_{-\ell}$, where $R^{k+1}_{-\ell}\subset Q^k_0$, then we have

$$V(z,j_i)=a_{n^k_i}(z,j^k_i)=-\delta^k_{i+1}v^k\circ A^k_i(z,j^k_i) + \frac{\partial_t\delta^k_{i+1}}{\delta^k_{i+1}}F^{n^k_i}(z,j^k_i)-DF^{n^k_i}(z,j^k_i)\cdot\frac{\partial_t\delta^k_i}{\delta^k_i}(z,j^k_i). \qquad \square$$

LEMMA 8.2. *Let $v\in\mathfrak{B}_+(F)$. There exists $C_3>0$ such that for every k and every i_k,i_{k+1} such that $q^k_{i_k}=q^{k+1}_{i_{k+1}}$, we have*

$$\Big|\frac{\partial_t\delta^{k+1}_{i_{k+1}}}{\delta^{k+1}_{i_{k+1}}}-\frac{\partial_t\delta^k_{i_k}}{\delta^k_{i_k}}\Big|\leq C_3. \tag{70}$$

Proof. Let $q=q^k_{i_k}=q^{k+1}_{i_{k+1}}$ and i_j be such that $q^j_{i_j}=q$. Note that $\beta^{k+1}_q=A^k_i(\delta^{k+1}_{i_{k+1}})$ is a periodic point for $\mathcal{R}^kF$ with period $y^k=\mu^{(k+1)}/\mu^{(k)}$. Indeed,

$$\beta^{k+1}_q=\left(\frac{\delta^{k+1}_{i_{k+1}}}{\delta^k_{i_k}},i_k\right).$$

If t is small enough, then there is an analytic continuation $\beta^{k+1}_q(t)$ for β^{k+1}_q, that is a periodic point for $\mathcal{R}^kF_t$. Since

$$\partial_t(\mathcal{R}^k(F_t))\big|_{t=0}(x,i)=(D\mathcal{R}^k_F\cdot v)(x,i)=v^k(x,i),$$

by the Implicit Function Theorem we have that

$$\partial_t(\mathcal{R}^k(F_t))^{y^k}\big|_{t=0}(\beta^{k+1}_q)+D(\mathcal{R}^kF)^{y^k}(\beta^{k+1}_q)\partial_t\beta^{k+1}_q=\partial_t\beta^{k+1}_q.$$

So

$$\begin{aligned}\partial_t\beta^{k+1}_q&=\frac{1}{1-D(\mathcal{R}^kF)^{y^k}(\beta^{k+1}_q)}\\&\quad\times\sum_{j=0}^{y^k-1}D(\mathcal{R}^kF)^{y^k-j-1}((\mathcal{R}^kF)^{j+1}(\beta^{k+1}_q))v^k((\mathcal{R}^kF)^j(\beta^{k+1}_q))\\&=\sum_{\ell=1}^{\infty}\frac{-1}{D(\mathcal{R}^kF)^{y^k\ell}(\beta^{k+1}_q)}\\&\quad\times\sum_{j=0}^{y^k-1}D(\mathcal{R}^kF)^{y^k-j-1}((\mathcal{R}^kF)^{j+1}(\beta^{k+1}_q))v^k((\mathcal{R}^kF)^j(\beta^{k+1}_q))\\&=\sum_{\ell=1}^{\infty}\sum_{j=0}^{y^k-1}-\frac{v^k((\mathcal{R}^kF)^{y^k(\ell-1)+j}(\beta^{k+1}_q))}{D(\mathcal{R}^kF)^{y^k(\ell-1)+j+1}(\beta^{k+1}_q)}\\&=\sum_{a=0}^{\infty}-\frac{v^k((\mathcal{R}^kF)^a(\beta^{k+1}_q))}{D(\mathcal{R}^kF)^{a+1}(\beta^{k+1}_q)}.\end{aligned}$$

Due to the real bounds and $v \in \mathfrak{B}_+(F)$ there exist C_1, C_2 such that

$$\sup_k \{|D(\mathcal{R}^k F)|_{\mathcal{B}(V)},\ |\beta_q^k|,\ \frac{1}{|\beta_q^k|},\ |v^k|_{\mathcal{B}(V)}\} \le C_1$$

and

$$1 < C_2 < \inf_{k,q} |D(\mathcal{R}^k F)^{y^k}(\beta_q^{k+1})|.$$

So

$$\sup_{k,q} |\frac{\partial_t \beta_q^{k+1}}{\beta_q^{k+1}}| = C_3 < \infty. \tag{71}$$

If i_j satisfies $q_{i_j}^j = q$, then

$$\delta_{i_{k+1}}^{k+1}(t) = \prod_{j=0}^{k} \frac{\delta_{i_{j+1}}^{j+1}(t)}{\delta_{i_j}^j(t)} = \prod_{j=1}^{k+1} \beta_q^j(t).$$

If we derive with respect to t at $t=0$, we obtain

$$\partial_t \delta_{i_{k+1}}^{k+1} = \sum_{j=1}^{k+1} \partial_t \beta_q^j \prod_{\ell \ne j} \beta_q^\ell.$$

We conclude that

$$\frac{\partial_t \delta_{i_{k+1}}^{k+1}}{\delta_{i_{k+1}}^{k+1}} = \sum_{j=1}^{k+1} \frac{\partial_t \beta_q^j}{\beta_q^j} \tag{72}$$

and

$$|\frac{\partial_t \delta_{i_{k+1}}^{k+1}}{\delta_{i_{k+1}}^{k+1}} - \frac{\partial_t \delta_{i_k}^k}{\delta_{i_k}^k}| = |\frac{\partial_t \beta_q^{k+1}}{\beta_q^{k+1}}| \le C_3. \tag{73}$$

□

LEMMA 8.3. *For every $(z, j_i^k) \in \mathscr{R}_{-\ell}^{k+1}$, with $R_{-\ell}^{k+1} \subset Q_0^k$, $q = q_i^k$, we have*

$$|V_1(z, j_i^k)| \le |\delta_{i+1}^k| \sup_k |v^k| \tag{74}$$

and

$$|V_2(z, j_i^k)| \le C_3 k |\delta_{i+1}^k| + C_3 k |\delta_i^k| \sup |DG|. \tag{75}$$

In particular,

$$\lim_{w \to C(F)} V(w) = 0.$$

Proof. The proof follows from Lemmas 8.1 and 8.2. □

LEMMA 8.4. *Let $\psi\colon \mathbb{R}_+^\star \to \mathbb{R}$ be a C^∞ function. Define $\gamma\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ as*

$$\gamma(z) = \psi(|z|)z.$$

Then

$$|\overline{\partial}\gamma(z)| = \frac{|zD\psi(|z|)|}{2}.$$

If $|D\psi(t)| \leq Ct^{-1}$, *then* γ *can be extended as a quasiconformal vector field on* $\mathbb{C}$ *such that* $\gamma(0) = 0$.

Proof. If $z = x + iy$, $x, y \in \mathbb{R}$, then

$$\psi(|z|) = \psi(\sqrt{x^2 + y^2}).$$

In particular,

$$\overline{\partial}(\psi(|z|)) = \frac{2xD\psi(\sqrt{x^2+y^2})}{4\sqrt{x^2+y^2}} + i\frac{2yD\psi(\sqrt{x^2+y^2})}{4\sqrt{x^2+y^2}}$$

and

$$|\overline{\partial}(\psi(|z|))| = \frac{|D\psi(|z|)|}{2};$$

that is,

$$|\overline{\partial}(\psi(|z|)z)| = \frac{|zD\psi(|z|)|}{2}.$$

Let $\epsilon \in (0, 1)$. Note that

$$|\psi(1) - \psi(\epsilon)| = |\int_\epsilon^1 D\psi(t)\ dt| \leq \int_\epsilon^1 \frac{1}{t}\ dt = -\ln\epsilon.$$

$$|\psi(\epsilon)| \leq -\ln\epsilon + |\psi(1)|.$$

In particular,

$$\lim_{z\to 0} z\psi(|z|) = 0, \tag{76}$$

so defining $\gamma(0) = 0$ we obtain a continuous extension to $\mathbb{C}$ of γ. To show that γ is a quasiconformal vector field, note that

$$|\overline{\partial}\gamma(z)| = \frac{|zD\psi(|z|)}{2} \leq \frac{C}{2}$$

for every $z \in \mathbb{C} \setminus \{0\}$. By [1, Lemma 3, p. 53] there exists a quasiconformal vector field $\tilde{\gamma}$ on $\mathbb{C}$ such that its distributional derivative belongs to $L^2(\mathbb{C})$, it satisfies $\overline{\partial}\tilde{\gamma}(z) = 0$ if $|z| \geq 1$ and

$$\overline{\partial}\tilde{\gamma}(z) = \overline{\partial}\gamma(z)$$

for almost every z satisfying $|z| < 1$. So $\gamma - \tilde{\gamma}$ is continuous on $\{z \in \mathbb{C}\colon\ |z| < 1\}$ and (by Weyl's Lemma) holomorphic on $\{z \in \mathbb{C}\colon\ 0 < |z| < 1\}$. By Riemann's Theorem on removable singularities we have that 0 is a removable singularity, so $\gamma - \tilde{\gamma}$ is holomorphic on $|z| < 1$, and therefore γ is quasiconformal on $\mathbb{C}$. □

An illustrative example of an application of Lemma 8.4 is obtained considering $\psi(x) = \log(x)$.

PROPOSITION 8.5. *Let $v \in \mathfrak{B}_+(F)$. There exists a quasiconformal vector field $\alpha_2 \colon \mathbb{C}_n \to \mathbb{C}$ such that*

$$V_2(z,j) = \alpha_2 \circ G(z,j) - DG(z,j) \cdot \alpha_2(z,j) \tag{77}$$

for every $(z,j) \in \partial\mathscr{D}$ and, moreover, $\alpha_2(0,j) = 0$ for every j.

Proof. Due to the real bounds we have that

$$\inf_{q \in C(F)} \inf_k \min_{R^{k+1}_{-\ell} \subset Q^k_0} \frac{\operatorname{dist}(R^{k+1}_{-\ell}, \partial Q^k_0)}{|Q^k_0|} = \epsilon_1 > 0. \tag{78}$$

Without loss of generality we can choose ϵ_0 in (59) satisfying $\epsilon_0 < \epsilon_1/4$. Let $\phi \colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that

(i) $\phi(x) \in [0,1]$ for every $x \in \mathbb{R}$;
(ii) if $|x| < 1 - \epsilon_1/4$, then $\phi(x) = 1$;
(iii) if $|x| > 1 - \epsilon_1/8$, then $\phi(x) = 0$.

Given $q \in C(F)$, let i_j be such that $q^j_{i_j} = q$ and $\delta^j_{i_j}$, and let β^j_q, $j \in \mathbb{N}$, be as in the proof of Lemma 8.2. For every $x \in \mathbb{R}^\star$, define

$$\psi_q(x) = \sum_{j=1}^{\infty} \frac{\partial_t \beta^j_q}{\beta^j_q} \cdot \phi\left(\frac{x}{\delta^j_{i_j}}\right).$$

The function ψ_q is well defined in $\mathbb{R} \setminus \{0\}$; it is C^∞ on $\mathbb{R} \setminus \{0\}$ and if

$$(1 - \epsilon_1/8)|\delta^{k+1}_{i_{k+1}}| \le |x| \le (1 - \epsilon_1/2)|\delta^k_{i_k}|, \tag{79}$$

then

$$\psi_q(x) = \sum_{j=1}^{k} \frac{\partial_t \beta^j_q}{\beta^j_q} \cdot \phi\left(\frac{x}{\delta^j_{i_j}}\right) = \sum_{j=1}^{k} \frac{\partial_t \beta^j_q}{\beta^j_q} = \frac{\partial_t \delta^k_{i_k}}{\delta^k_{i_k}}. \tag{80}$$

Moreover, notice that if

$$|\delta^{k+1}_{i_{k+1}}| \le |x| \le |\delta^k_{i_k}|, \tag{81}$$

then

$$\psi_q(x) = \frac{\partial_t \beta^k_q}{\beta^k_q} \cdot \phi\left(\frac{x}{\delta^k_{i_k}}\right) + \sum_{j=1}^{k-1} \frac{\partial_t \beta^j_q}{\beta^j_q},$$

so

$$D\psi_q(x) = \frac{\partial_t \beta^k_q}{\delta^k_{i_k} \beta^k_q} \cdot D\phi\left(\frac{x}{\delta^k_{i_k}}\right),$$

and by (71) and (81),

$$|D\psi_q(x)| \le \frac{1}{|\delta^k_{i_k}|} C_3 \max|D\phi| \le \frac{C_3 \max|D\phi|}{|x|}. \tag{82}$$

If $q = (0,j)$, define

$$\alpha_2(z,j) = \psi_q(|z|)z. \tag{83}$$

By (82) and Lemma 8.4 we conclude that α_2 is a quasiconformal vector field with $\alpha_2(0,j)=0$ for every j.

It remains to show that α_2 satisfies (77). Indeed, let $(z,j)\in\partial\mathscr{R}^{k+1}_{-\ell}$, with $R^{k+1}_{-\ell}\subset Q^k_0$ and $\ell>0$. Since $\mathscr{R}^{k+1}_{-\ell}$ belongs to a disc with diameter given by the interval $R^{k+1}_{-\ell}$ and it does not intercept $\mathcal{Q}^{k+1}_0$, we have

$$(1-\epsilon_1/64)|\delta^{k+1}_{i_{k+1}}|\leq d^{k+1}_q\leq |z|\leq (1-\epsilon_1)|\delta^k_{i_k}| \tag{84}$$

for every $(z,j)\in\partial\mathscr{R}^{k+1}_{-\ell}$. Here d^{k+1}_q is as defined in (60). By (80) we have that

$$\alpha_2(z,j)=\frac{\partial_t\delta^k_{i_k}}{\delta^k_{i_k}}z$$

for every

$$(z,j)\in\partial\mathscr{R}^{k+1}_{-\ell}.$$

Indeed, let $(z,\tilde{j})\in\mathscr{R}^{k+1}_{-\ell}$, with $R^{k+1}_{-\ell}\subset S^k_0$, with $\ell>0$ and such that q is the successor of s at level k. Then

$$G(\mathscr{R}^{k+1}_{-\ell})\subset\mathcal{Q}^k_0.$$

If $G(\mathscr{R}^{k+1}_{-\ell})=\mathscr{R}^{k+1}_{-a}$ for some $a>0$, then the points in this image also satisfy (84). Otherwise $r=q$ and $G(\mathscr{R}^{k+1}_{-\ell})=\mathcal{Q}^{k+1}_0$, so if $(w,j)\in G(\partial\mathscr{R}^{k+1}_{-\ell})$, then

$$(1-\epsilon_1/64)|\delta^{k+1}_{i_{k+1}}|\leq |d^{k+1}_q|=|w|\leq|\delta^{k+1}_{i_{k+1}}|\leq(1-\epsilon_1)|\delta^k_{i_k}|. \tag{85}$$

By (80) we have that

$$\alpha_2(w,j)=\frac{\partial_t\delta^k_{i_k}}{\delta^k_{i_k}}w$$

for every

$$(w,j)\in G(\partial\mathscr{R}^{k+1}_{-\ell}).$$

Since these estimates hold for every $q\in C(F)$, we have that for every $(z,j)\in\partial\mathscr{R}^{k+1}_{-\ell}$, with $R^{k+1}_{-\ell}\subset Q^k_0$ and $\ell>0$,

(86)
$$\begin{aligned}\alpha_2\circ G(z,j)-DG(z,j)\cdot\alpha_2(z,j)&=\frac{\partial_t\delta^k_{i_k+1}}{\delta^k_{i_k+1}}G(z,j)-DG(z,j)\frac{\partial_t\delta^k_{i_k}}{\delta^k_{i_k}}(z,j)\\&=\frac{\partial_t\delta^k_{i_k+1}}{\delta^k_{i_k+1}}F^{n^k_{i_k}}(z,j)-DF^{n^k_{i_k}}(z,j)\frac{\partial_t\delta^k_{i_k}}{\delta^k_{i_k}}(z,j)\\&=V_2(z,j).\end{aligned}$$
□

Proposition 8.6. *Let $v\in\mathfrak{B}_+(F)$. There exists a quasiconformal vector field $\alpha_1\colon\mathbb{C}_n\to\mathbb{C}$ such that*

$$V_1(z,j)=\alpha_1\circ G(z,j)-DG(z,j)\cdot\alpha_1(z,j) \tag{87}$$

for every $(z,j)\in\partial\mathscr{D}$ and, moreover, $\alpha_1(q)=0$ for every $q\in C(F)$.

Proof. Let $\epsilon_1 > 0$ be as in (78). Let $\phi\colon \mathbb{C} \to \mathbb{R}$ be a C^∞ function such that

(i) $\phi(x) \in [0,1]$ for every $x \in \mathbb{C}$;
(ii) if $|x| = 1 - \epsilon_0/16$, then $\phi(x) = 1$;
(iii) if either $|x| > 1 - \epsilon_0/32$ or $|x| < 1 - \epsilon_1/2$, then $\phi(x) = 0$.

Given $r \in C(F)$, let

$$i_0 = 0 < i_1 < \cdots < i_{\ell^k_r} < n^k_r$$

be the sequence of integers $i < n^k_r$ such that

$$R^k_i \subset \bigcup_{q \in C(F)} Q^{k-1}_0.$$

Denote by δ^k_r the periodic point in the boundary of R^k_0. There is an onto univalent extension

$$G^j\colon D^k_{r,-j} \to B(r, |\delta^k_r|),$$

where $D^k_{r,-j} \cap \mathbb{R} = R^k_{-i_j}$. Note that $\mathscr{R}^k_{-i_j} \subset D^k_{r,-j}$ and

$$G^j(\mathscr{R}^k_{-i_j}) = B(r, (1-\epsilon_0/16)|\delta^k_r|) = \mathscr{R}^k_0. \tag{88}$$

Let $q_j \in C(F)$ such that q_j and $R^k_{-i_j}$ belong to the same connected component of I^n_F. Define a function

$$\psi^k_{r,-j}\colon \mathbb{C}_n \to \mathbb{C}$$

in the following way. Let

$$\psi^k_{r,0}(z,a) = 0$$

for every $(z,a) \in \mathbb{C}_n$, and by induction on j define

$$\psi^k_{r,-(j+1)}(z,a) = \phi\Big(\frac{G^{j+1}(z,a)}{\delta^k_r}\Big)\frac{\psi^k_{r,-j}(G(z,a)) - \delta^{k-1}_{q_j} v^{k-1} \circ A^{k-1}_{q_{j+1}}(z,a)}{DG(z,a)} \tag{89}$$

for every $(z,a) \in D^k_{r,-(j+1)}$. Note that $\psi^k_{r,-(j+1)}(z,a) = 0$ for (z,a) in a neighborhood of $\partial D^k_{r,-(j+1)}$, so we can extend $\psi^k_{r,-(j+1)}$ to a C^∞ function on $\mathbb{C}_n$ defining $\psi^k_{r,-(j+1)}(z,a) = 0$ for $(z,a) \notin D^k_{r,-(j+1)}$.

Finally note that by (88),

$$\phi\Big(\frac{G^{j+1}(z,a)}{\delta^k_r}\Big) = 1$$

for every $(z,a) \in \partial \mathscr{R}^k_{-i_{j+1}}$, so by (89),

(90)
$$-V_1(z,a) = \delta^{k-1}_{q_j} v^{k-1} \circ A^{k-1}_{q_{j+1}}(z,a) = \psi^k_{r,-j}(G(z,a)) - DG(z,a)\psi^k_{r,-(j+1)}(z,a)$$

for every $(z,a) \in \partial \mathscr{R}^k_{-i_j}$. Note that given $(z,a) \in \mathbb{C}_n \setminus C(F)$, there exists an open neighborhood of (z,a) that intersects only one of the supports of a function in the family

$$\mathcal{F} = \{\psi^k_{r,-j}\}_{k,\ r\in C(F), j\leq \ell^k_r}.$$

In particular, the function

$$\alpha_1(z,a) = -\sum_k \sum_{r\in C(F)} \sum_{j=0}^{\ell^k_r} \psi^k_{r,-j}(z,a)$$

is well defined and it is C^∞ on $\mathbb{C}_n \setminus C(F)$. Moreover, given $\mathscr{R}^k_{-i_j}$, the function $\psi^k_{r,-j}$ is the unique function in this family whose support intersects $\partial \mathscr{R}^k_{-i_j}$. By (90), this implies that

$$V_1(z,a) = \alpha_1 \circ G(z,a)) - DG(z,a)\alpha_1(z,a)$$

for every $(z,a) \in \partial \mathscr{D}$. It is easy to prove by induction that

$$|\psi^k_{r,-j}(z,a)| \leq \sup_k |v^{k-1}| \sum_{\ell=0}^{j-1} \frac{|\delta^{k-1}_{q_{j-1-\ell}}|}{|DG|^{\ell+1}(z,a)}; \tag{91}$$

in particular,

$$|\psi^k_{r,-j}| \leq \max_{q\in C(F)} |\delta^{k-1}_q| \sup_k |v^{k-1}| \sum_{j=0}^{\sup_k \frac{p_k}{p_{k-1}}} \frac{1}{(\inf |DG|)^j} \tag{92}$$

due to the real bounds

$$|\psi^k_{r,-j}| \leq C\theta^{-k}. \tag{93}$$

If $(z,a) \in \mathcal{Q}^{k-1}_0 \setminus \{q\}$ for some $q \in C(F)$, then either $\alpha_1(z,a) = 0$ or (z,a) belongs to the support of a unique function $\psi^b_{r,-j}$, with $b \geq k$. We conclude that

$$\lim_{(z,a)\to q} \alpha_1(z,a) = 0, \tag{94}$$

so we can extend α_1 as a continuous function on $\mathbb{C}_n$. Note that

$$\begin{aligned} &\overline{\partial}\psi^k_{r,-(j+1)}(z,a) \\ &= \overline{\partial}\phi\Big(\frac{G^{j+1}(z,a)}{\delta^k_r}\Big)\frac{\overline{DG^{j+1}}(z,a)}{\overline{\delta^k_r}}\frac{\psi^k_{r,-j}(G(z,a)) - \delta^{k-1}_{q_j} v^{k-1} \circ A^{k-1}_{q_{j+1}}(z,a)}{DG(z,a)} \\ &\quad + \phi\Big(\frac{G^{j+1}(z,a)}{\delta^k_r}\Big)\overline{\partial}\psi^k_{r,-j}(G(z,a))\frac{\overline{DG}(z,a)}{DG(z,a)}. \end{aligned} \tag{95}$$

By the real bounds there exists $C > 0$ such that

$$\frac{|DG^{j-\ell}|(G^{\ell+1}(z,a))}{|\delta_r^k|} \le \frac{C}{|R^k_{-i_{j-\ell}}|}$$

for every $r \in C(F)$, K, j and $(z,a) \in D^k_{r,-(j+1)}$. In particular,

$$\begin{aligned}
(96)\qquad & \frac{|DG^{j+1}|(z,a)}{|\delta_r^k|}\frac{|\psi^k_{r,-j}(G(z,a)) - \delta^{k-1}_{q_j} v^{k-1}\circ A^{k-1}_{q_{j+1}}(z,a)|}{|DG(z,a)|} \\
&\le \frac{|DG^{j+1}|(z,a)}{|\delta_r^k|}\Big(\sum_{\ell=0}^{j-1}\frac{|\delta^{k-1}_{q_{j-1-\ell}}|}{|DG|^{\ell+2}(z,a)} + \frac{|\delta^{k-1}_{q_j}|}{|DG|(z,a)}\Big) \\
&\le \frac{|DG^{j+1}|(z,a)}{|\delta_r^k|}\sum_{\ell=0}^{j}\frac{|\delta^{k-1}_{q_{j-\ell}}|}{|DG|^{\ell+1}(z,a)} \\
&\le \sum_{\ell=0}^{j}\frac{|DG^{j-\ell}|(G^{\ell+1}(z,a))|\delta^{k-1}_{q_{j-\ell}}|}{|\delta_r^k|} \le C\sum_{\ell=0}^{j}\frac{|\delta^{k-1}_{q_{j-\ell}}|}{|R^k_{-i_{j-\ell}}|} \\
&\le C\sup_k \frac{\mu^{(k)}}{\mu^{(k-1)}}.
\end{aligned}$$

The last inequality follows from the fact that $R^k_{-i_{j-\ell}} \subset Q_0^{k-1}$, with $q = q_{j-\ell}$, and $2\delta^{k-1}_{q_{j-\ell}} = |Q_0^{k-1}|$. By the real bounds there exists $C > 0$ such that

$$\frac{|Q_0^{k-1}|}{|R^k_{-i}|} \le C$$

for every $R^k_{-i} \subset Q_0^{k-1}$. So by (96),

$$\sup|\overline{\partial}\psi^k_{r,-(j+1)}| \le C\sup|\overline{\partial}\phi|\sup_k\frac{\mu^{(k)}}{\mu^{(k-1)}} + \sup|\overline{\partial}\psi^k_{r,-j}|. \qquad (97)$$

Since $\psi_0^k = 0$, we obtain

$$\sup|\overline{\partial}\psi^k_{r,-j}| \le C\sup_k\frac{\mu^{(k)}}{\mu^{(k-1)}}$$

Since for every $(z,a) \in \mathbb{C}_n \setminus C(F)$, there exists an open neighborhood of (z,a) that intersects only one of the supports of the functions in the family $\mathcal{F}$, we conclude that α_1 is a quasiconformal vector field on $\mathbb{C}_n \setminus C(F)$. Since α_1 is continuous at $C(F)$, we can use the argument in the end of the proof of Lemma 8.4 to conclude that α_1 is a quasiconformal vector field on $\mathbb{C}_n$. □

Proof of Theorem 4. Let $v \in \mathfrak{B}_+(F)$. Let α_1 and α_2 be as in Proposition 8.5 and 8.6. Define $\alpha^0 = \alpha_1 + \alpha_2$. Then by Lemma 8.1,

$$V(z,j) = \alpha^0 \circ G(z,j) - DG(z,j)\cdot\alpha^0(z,j) \qquad (98)$$

for every $(z, j) \in \partial\mathscr{D} \setminus C(F)$ and, moreover, $\alpha^0(q) = 0$ for every $q \in C(F)$. Now we will use an argument similar to the infinitesimal pullback argument for polynomial-like maps [3]. We define by induction on m a sequence of quasiconformal vector fields

$$\alpha^m \colon \mathbb{C}_n \to \mathbb{C}$$

such that $\alpha^m(q) = 0$ for every $q \in C(F)$,

$$V(z, j) = \alpha^m \circ G(z, j) - DG(z, j) \cdot \alpha^m(z, j) \tag{99}$$

for every $(z, j) \in \partial\mathscr{D}$ and, moreover,

$$V(z, j) = \alpha^{m-1} \circ G(z, j) - DG(z, j) \cdot \alpha^m(z, j) \tag{100}$$

for every $(z, j) \in \mathscr{D}$, $m \geq 1$ and

$$\sup_{\mathbb{C}_n} |\overline{\partial}\alpha^{m+1}| \leq \sup_{\mathbb{C}_n} |\overline{\partial}\alpha^m|.$$

Indeed, suppose by induction we have defined α^k. Define $\alpha^{k+1}(z, j) = \alpha^k(z, j)$ for every $(z, j) \notin \mathscr{D}$ and

$$\alpha^{k+1}(z, j) = \frac{\alpha^k(G(z, j)) - V(z, j)}{DG(z, j)} \tag{101}$$

for every $(z, j) \in \mathscr{D}$. Note that

$$|\overline{\partial}\alpha^{k+1}(z, j)| = |\overline{\partial}\alpha^k(z, j)|$$

for every $(z, j) \notin \overline{\mathscr{D}}$ and

$$|\overline{\partial}\alpha^{k+1}(z, j)| = |\overline{\partial}\alpha^k(G(z, j))|$$

for $(z, j) \in \mathscr{D}$. So α^{k+1} is a quasiconformal vector field in $\mathbb{C}_n \setminus \partial\mathscr{D}$. Moreover, (100) holds for $m = k+1$ and due to (99) with $m = k$, we have that $\alpha^{k+1} = \alpha^k$ on $\partial\mathscr{D} \setminus C(F)$, so α^{k+1} is continuous at points in

$$\partial\mathscr{D} \setminus C(F).$$

If $(z, j) \in \partial\mathscr{D} \setminus C(F)$ then $(z, j) \in \partial\mathscr{R}^n_{-i}$, for some $r \in C(F)$. In particular, there exists a neighborhood W of (z, j) such that α^{k+1} is continuous on W and a quasiconformal vector field on $W \setminus \partial\mathscr{R}^n_{-i}$. Since $\partial\mathscr{R}^n_{-i}$ is an analytic curve, we conclude that α^{k+1} is a quasiconformal vector field on W. So α^{k+1} is a quasiconformal vector field on $\mathbb{C}_n \setminus C(F)$. Finally, notice that α^{k+1} is continuous at points in $C(F)$. Indeed, suppose that

$$(z_\ell, i) \to_\ell (0, i).$$

If $(z_\ell, i) \notin \mathscr{D}$ for every ℓ, then

$$\lim_{\ell \to \infty} \alpha^{k+1}(z_\ell, i) = \lim_{\ell \to \infty} \alpha^k(z_\ell, i) = 0.$$

If $(z_\ell, i) \in \mathscr{D}$ for every ℓ, then

$$\lim_{\ell\to\infty} \alpha^{k+1}(z_\ell, i) = \lim_{\ell\to\infty} \frac{\alpha^k(G(z_\ell, i)) - V(z_\ell, i)}{DG(z_\ell, i)}. \tag{102}$$

Since the accumulation points of the sequence $G(z_\ell, i)$ belong to $C(F)$, we have

$$\lim_{\ell\to\infty} \alpha^k(G(z_\ell, i)) = 0.$$

By Lemma 8.3 it follows that

$$\lim_{\ell\to\infty} V(z_\ell, i) = 0.$$

Since by the complex bounds we have that

$$\inf_{(z,j)\in\mathscr{D}} |DG(z, j)| > 0,$$

we conclude by (102) that

$$\lim_{\ell\to\infty} \alpha^{k+1}(z_\ell, i) = 0,$$

so α^{k+1} is continuous at points in $C(F)$. By the same argument at the end of the proof of Lemma 8.4, we conclude that α^{k+1} is a quasiconformal vector field on $\mathbb{C}_n$ and

$$\sup_{\mathbb{C}_n} |\overline{\partial}\alpha^{k+1}| \leq \sup_{\mathbb{C}_n} |\overline{\partial}\alpha^k|. \tag{103}$$

Given a point $(z, j) \in \mathbb{C}_n$ such that there exists $k_0 \geq 0$ such that $G^k(z, j) \in \mathscr{D}$ for every $k < k_0$ and $G^{k_0}(z, j) \notin \mathscr{D}$, we claim that $\alpha^k(z, j) = \alpha^{k_0}(z, j)$ for every $k \geq k_0$. Note that $\alpha^k(z, j) = \alpha^0(z, j)$ for every $(z, j) \notin \mathscr{D}$, so the claim holds for $k_0 = 0$. Suppose by induction on k_0 that the claim hold for k_0. If $G^i(z, j) \in \mathscr{D}$ for $i \leq k_0$ and $G^{k_0+1}(z, j) \notin \mathscr{D}$, then $G^i(G(z, j)) \in \mathscr{D}$ for $i < k_0$ and $G^{k_0}(G(z, j)) \notin \mathscr{D}$. Thus by the induction assumption, $\alpha^k(G(z, j)) = \alpha^{k_0}(G(z, j))$ for every $k \geq k_0$. Since $G(z, j) \in \mathscr{D}$, we have by (101) that

$$\alpha^{k+1}(z, j) = \frac{\alpha^k(G(z, j)) - V(z, j)}{DG(z, j)} = \frac{\alpha^{k_0}(G(z, j)) - V(z, j)}{DG(z, j)} = \alpha^{k_0+1}(z, j)$$

for every $k \geq k_0$, which proves the claim.

By Proposition 6.2 the sequence α^k converges almost everywhere. By (103) and McMullen [31] we have that every subsequence α^k has a subsequence that converges uniformly to some quasiconformal vector field. So α^k converges uniformly to a quasiconformal vector field α. Taking $m \to \infty$ in (100), we conclude that

$$V(z, j) = \alpha \circ G(z, j) - DG(z, j) \cdot \alpha(z, j) \tag{104}$$

for every $(z, j) \in \mathscr{D}$ and, in particular, for every $(z, j) \in P(F) \setminus C(F)$. Since $\alpha(q) = 0$ for every $q \in C(F)$, by Corollary 7.3 we conclude that $v \in E^h(F)$. $\square$

9. **Transversal families have hyperbolic parameters**

From now on we will consider only *real maps*. In particular, $\mathcal{B}^{\mathbb{R}}_{\mathrm{nor}}(U)$ will now denote the real Banach space of all $F \in \mathcal{B}_{\mathrm{nor}}(U)$ that are real on the real line. Since $\Omega_{n,p}$ is a hyperbolic set, we can define the stable

$$G \mapsto E^h_G$$

and unstable

$$G \mapsto E^u_G$$

subspace distributions defined for $G \in \Omega_{n,p}$, and the corresponding projections on the spaces

$$\pi^h_G \colon T\mathcal{B}^{\mathbb{R}}_{\mathrm{nor}}(U) \to E^h_G$$

and

$$\pi^u_G \colon T\mathcal{B}^{\mathbb{R}}_{\mathrm{nor}}(U) \to E^u_G.$$

Recall that $U = D_{\delta_0,\theta_0}$. As defined in [43] we also have the adapted norms $|\cdot|_{G,0}$ that satisfy

$$|v|_{G,0} = |\pi^h_G(v)|_{G,0} + |\pi^u_G(v)|_{G,0},$$

and the family of cones $C^u_\epsilon(G)$, with $\epsilon > 0$, for which $v \in C^u_\epsilon(G)$ if and only if

$$|\pi^h_G(v)|_{G,0} \leq \epsilon |\pi^u_G(v)|_{G,0}.$$

Those cones are unstable and forward-invariant for the action of $\mathcal{R}$ on $\Omega_{n,p}$ provided ϵ is small enough. In particular, if ϵ is small, there is $\theta > 1$ such that for every $F \in \mathcal{B}^{\mathbb{R}}_{\mathrm{nor}}(U)$ close enough to some $G \in \Omega_{n,p}$ and $v \in C^u_{2\epsilon}(G)$,

$$|D\mathcal{R}_F \cdot v|_{\mathcal{R}G,0} \geq \theta |v|_{G,0}.$$

Moreover, there is $\epsilon' \in (0,\epsilon)$ such that

$$|\pi^h_{\mathcal{R}G}(D\mathcal{R}_F \cdot v)|_{\mathcal{R}G,0} \leq 2\epsilon' |\pi^u_{\mathcal{R}G}(D\mathcal{R}_F \cdot v)|_{\mathcal{R}G,0}.$$

Furthermore, there is $\lambda \in (0,1)$ such that for every $G \in \Omega_{n,p}$, $v \in E^h_G$ and $k \in \mathbb{N}$,

$$|D\mathcal{R}^k_G \cdot v|_{\mathcal{R}^k G,0} \leq \lambda^k |v|_{G,0}.$$

Define the δ-shadow of G as

$$W^s_\delta(G) = \{F \in \mathcal{W} \colon \ \mathrm{dist}_{\mathcal{B}^{\mathbb{R}}_{\mathrm{nor}}(U)}(\mathcal{R}^k F, \mathcal{R}^k G) \leq \delta \text{ for } \textit{every } k \geq 0\}$$

and the δ-shadow of $\Omega_{p,n}$ as

$$W^s_\delta(\Omega_{p,n}) = \cup_{G \in \Omega_{p,n}} W^s_\delta(G).$$

We also define

$$\mathbb{B}^u_G(v_0,\delta) = \{v \in E^u_G \cap \mathcal{B}^{\mathbb{R}}(U) \colon \ |v - v_0|_{G,0} \leq \delta\},$$

where $v_0 \in E^u_G \cap \mathcal{B}^{\mathbb{R}}(U)$, and

$$E^h_G + G = \{v + G \colon \ v \in E^h_G\}.$$

Let $\delta_3 > 0$. (We will use this notation to follow [43].) Define $\mathcal{T}_0^1(G, \delta, \epsilon)$, with $\delta \in (0, \delta_3)$, as the set of C^1 functions

$$\mathcal{H}\colon \mathbb{B}^u_G(v_0, \delta) \to E^h_G + G,$$

with $v_0 \in E^u_G \cap \mathcal{B}^{\mathbb{R}}(U)$, such that

$$|D\mathcal{H}|_{(G,0),(G,0)} \le \epsilon$$

and

$$F_0 = v_0 + \mathcal{H}(v_0) \in W^s_{\delta_3}(G).$$

We will call F_0 the *base point* of $\mathcal{H}$. In particular, $w + D_x\mathcal{H} \cdot w \in C^u(G)$ for every $x \in \mathbb{B}^u_G(v_0, \delta)$, $w \in E^u(G)$, and $G \in \Omega_{p,n}$. Denote

$$\hat{\mathcal{H}} = \{v + \mathcal{H}(v) \colon\ v \in \mathbb{B}^u_G(v_0, \delta)\}.$$

The Transversal Empty Interior assumption for the renormalization operator is the main result of this section.

Corollary 9.1 (Transversal Empty Interior Assumption). *For every small $\epsilon > 0$, we can choose δ_3 small enough such that the following holds. For every $G \in \Omega_{p,n}$ and for every $C^{1+\mathrm{Lip}}$ function $\mathcal{H} \in \mathcal{T}_0^1(G, \delta', \epsilon)$, with $\delta' < \delta_3$, we have that $\hat{\mathcal{H}} \cap W^s_{\delta_3}(\Omega)$ has empty interior in $\hat{\mathcal{H}}$.*

This property is closely related with the fact that maps F that are infinitely renormalizable with bounded combinatorics can be approximated by hyperbolic maps.

We now introduce notation from [43]. Let $C^{\omega_{\mathbb{R}}}([-1,1]^j, \mathcal{B}^{\mathbb{R}}_{\mathrm{nor}}(U))$ denote the space of functions

$$\gamma\colon (-1,1)^j \to \mathcal{B}^{\mathbb{R}}_{\mathrm{nor}}(U)$$

that can be extended to a complex analytic function

$$\gamma\colon \mathbb{D}^j \to \mathcal{B}_{\mathrm{nor}}(U).$$

Moreover, there is a continuous extension of γ to $\overline{\mathbb{D}}^j$. Endowed with the sup norm on $\overline{\mathbb{D}}^j$, the space $C^{\omega_{\mathbb{R}}}([-1,1]^j, \mathcal{B}^{\mathbb{R}}_{\mathrm{nor}}(U))$ is a real Banach space.

Endow $T_{\mathbb{C}} = \overline{\mathbb{D}}^{\mathbb{N}}$ with the product topology. Let $\Gamma^\omega(\mathcal{B}_{\mathrm{nor}}(U))$ be the set of continuous functions $\gamma\colon T_{\mathbb{C}} \mapsto \mathcal{B}_{\mathrm{nor}}(U)$ that are holomorphic when we fix all but a finite number of entries of $\lambda \in T_{\mathbb{C}}$ and $|\lambda_i| < 1$ for every i. Endowing $\Gamma^\omega(\mathcal{B}_{\mathrm{nor}}(U))$ with the sup norm we obtain a complex Banach space.

Note that since U is symmetric with respect to the real line, that is, $(z, i) \in U$ if and only if $(\overline{z}, i) \in U$, there is a complex conjugation on the complex Banach space $\mathcal{B}(U)$ defined by $\overline{f}(z) = \overline{f(\overline{z})}$ for $f \in \mathcal{B}(U)$. Define $\Gamma^{\omega_{\mathbb{R}}}(\mathcal{B}^{\mathbb{R}}_{\mathrm{nor}}(U))$ as the *real* Banach space that consists of the restrictions to $T = [-1,1]^{\mathbb{N}}$ of functions $\gamma \in \Gamma^\omega(\mathcal{B}_{\mathrm{nor}}(U))$ satisfying $\gamma(\overline{\lambda}) = \overline{\gamma(\lambda)}$.

We say that a set $\Theta \subset \mathcal{B}_{\rm nor}(U)$ is a $\Gamma^{\omega_{\mathbb{R}}}(\mathcal{B}^{\mathbb{R}}_{\rm nor}(U))$-*null set* if there exists a residual subset $\mathcal{F} \subset \Gamma^{\omega_{\mathbb{R}}}(\mathcal{B}^{\mathbb{R}}_{\rm nor}(U))$ such that

$$m(\lambda \in [-1,1]^{\mathbb{N}} \colon \gamma(\lambda) \in \Theta) = 0$$

for every $\gamma \in \mathcal{F}$. Here m is the product measure obtained considering the normalized Lebesgue measure on each copy of $[-1,1]$.

The Transversal Empty Interior property allows us to apply [43, Th. 1] to the renormalization operator. Indeed we already verified that

- $\mathcal{R} \colon \mathcal{W}^{\mathbb{R}} \to \mathcal{B}^{\mathbb{R}}_{\rm nor}(U)$ is a real-analytic map. Here $\mathcal{W}^{\mathbb{R}} = \mathcal{W} \cap \mathcal{B}^{\mathbb{R}}_{\rm nor}(U)$.
- The map $\mathcal{R}$ is a strongly compact operator (Remark 3.1),
- $\Omega_{n,p}$ is a hyperbolic set (Theorem 5),
- For every $F \in \mathcal{R}^{-i} W^s_\delta(\Omega_{n,p})$, with $i \in \mathbb{N}$ and $\delta > 0$, we have that
$$D_F \mathcal{R}^i (T_F \mathcal{B}^{\mathbb{R}}_{\rm nor}(U))$$
is dense in $T_{\mathcal{R}^i F} \mathcal{B}^{\mathbb{R}}_{\rm nor}(U)$. This is an easy consequence of Theorem 3.

Thus [43, Th. 1] in our setting boils down to

Theorem 6 ([43, Th. 1]). *Suppose that the renormalization operator $\mathcal{R}$ satisfies additionally*

(A) *There exists $\delta_3 > 0$ such that $W^s_{\delta_3}(\Omega_{n,p})$ satisfies the Transversal Empty Interior assumption.*

Then $W^s(\Omega_{n,p})$ is a $\Gamma^{\omega_{\mathbb{R}}}(\mathcal{B}^{\mathbb{R}}_{\rm nor}(U))$-null set. Indeed for every j there exists a residual set of real-analytic maps $\gamma \in C^{\omega_{\mathbb{R}}}([-1,1]^j, \mathcal{B}^{\mathbb{R}}_{\rm nor}(U))$ such that

$$m(t \in [-1,1]^j \colon\ \gamma(t) \in W^s(\Omega_{n,p})) = 0.$$

Here m is the Lebesgue measure on $[-1,1]^j$.

In particular, this implies that a generic real-analytic finite-dimensional family in $\mathcal{W}^{\mathbb{R}}$ intersects $W^s(\Omega_{n,p})$ on a subset with zero Lebesgue measure. So we have a version of Theorem A for real-analytic families of extended maps that *belong* to $\mathcal{W}^{\mathbb{R}}$. Indeed the full-blown version of Theorem A is proven in Section 10.

Proposition 9.2. *For every $\epsilon > 0$ small enough, there is $\gamma > 0$ with the following property. Suppose that $F \in \mathcal{B}^{\mathbb{R}}_{\rm nor}(U)$, that F has a polynomial-like extension $F \colon \hat{V}^0 \to \hat{V}^1$, with $\overline{U} \subset \hat{V}^0$, and that*

$$\mathrm{dist}_{\mathcal{B}^{\mathbb{R}}_{\rm nor}(U)}(F,G) < \gamma$$

for some $G \in \Omega_{p,n}$. If

$$v \in \hat{E}^h_F \cap C^u_{2\epsilon}(G)$$

and $v \in \mathcal{B}^{\mathbb{R}}(\hat{V}^0) \cap \mathcal{B}^{\mathbb{R}}(U)$, then $v = 0$.

Proof. Suppose by contradiction that there exist sequences $G_i \in \Omega_{p,n}$, $F_i \in \mathcal{B}^{\mathbb{R}}_{\text{nor}}(U)$ and $v_i \in \mathcal{B}^{\mathbb{R}}(\hat{V}_i^0)$ such that

- we have
$$\text{dist}_{\mathcal{B}^{\mathbb{R}}_{\text{nor}}(U)}(F_i, G_i) \to_i 0;$$
- the maps F_i have a polynomial-like of type n extension $F_i\colon \hat{V}_i^0 \to \hat{V}_i^1$ and $\overline{U} \subset \hat{V}_i^0$;
- the vectors satisfy $v_i \in \hat{E}^h_{F_i} \cap C^u_{2\epsilon}(G_i)$, $|v_i|_{\mathcal{B}^{\mathbb{R}}(U)} \neq 0$ and $v_i \in \mathcal{B}^{\mathbb{R}}(\hat{V}_i^0)$.

In particular, for large i the critical points of F_i belong to $K(F_i)$ and F_i is renormalizable. Without loss of generality we can assume that $|v_i|_{\mathcal{B}(U)} = 1$ for every i. Since $v_i \in \hat{E}^h(F_i)$ and F_i are very close to $\Omega_{n,p}$, we have that $\mathcal{R}F_i$ has a polynomial-like extension of type n,

$$\mathcal{R}F_i\colon V_i^0 \to V_i^1,$$

with $\text{mod}\, V_i^1 \setminus V_i^0 > \epsilon_0$. Moreover, $D_{F_i}\mathcal{R} \cdot v_i \in \hat{E}^h_{\mathcal{R}F_i} \cap \mathcal{B}^{\mathbb{R}}(V_i^0)$, and there is $C > 0$ such that

$$|D_{F_i}\mathcal{R} \cdot v_i|_{\mathcal{B}^{\mathbb{R}}(V_i^0)} \leq C$$

for every large i. Note that $\mathcal{R}F_i\colon V_i^0 \to V_i^1$ is real on the real line and consequently it is hybrid conjugate with a real polynomial of type n. (See the Straightening lemma in [40, Prop. 4.1].) It follows from Shen [37] that $\mathcal{R}F_i$ does not have invariant line fields on its Julia set. So one can use the infinitesimal pullback argument to conclude that there exist quasiconformal vector fields $\alpha_i\colon \mathbb{C}_n \to \mathbb{C}$ with $\sup_i |\overline{\partial}\alpha_i|_\infty < \infty$ such that

$$D_{F_i}\mathcal{R} \cdot v_i = \alpha_i \circ \mathcal{R}F_i - D(\mathcal{R}F_i) \cdot \alpha_i \tag{105}$$

on a domain a little bit smaller than V_i^0 and, in particular, on $U = D_{\delta_0,\theta_0}$. By a compactness argument [31], the sequence α_i has a convergent subsequence that converges to a quasiconformal vector field α. Let

$$D_\delta = \{x \in \mathbb{C}\colon\ \text{dist}(x, [-1,1]) < \delta\} \times \{1, \dots, n\}.$$

Note that by (13) there exists $\delta' > \delta_0$ such that $\overline{U} \subset D_{\delta'} \subset V_i^0$ for every i. Since

$$|D_{F_i}\mathcal{R} \cdot v_i|_{\mathcal{B}(D_{\delta'})} \leq C,$$

there exists a subsequence of $D_{F_i}\mathcal{R} \cdot v_i$ that converges to some v on $\mathcal{B}^{\mathbb{R}}(U)$ satisfying $|v|_{\mathcal{B}(U)} \leq C$. Since $\Omega_{p,n}$ is compact, without loss of generality we can assume that $\mathcal{R}G_i$ (and so $\mathcal{R}F_i$) converges to some $G \in \Omega_{p,n}$. By (105) we obtain

$$v = \alpha \circ G - DG \cdot \alpha$$

on the pos-critical set of G. Since there are no invariant line fields supported in the Julia set of G, by the infinitesimal pullback argument we conclude that $v \in E^h_G$. In particular,

$$|D\mathcal{R}^k_G \cdot v|_{\mathcal{R}^k(G),0} \leq \lambda^k |v|_{G,0} \leq C_1 \lambda^k. \tag{106}$$

On the other hand, since $v_i \in C^u_{2\epsilon}(G_i)$ and ϵ is small, we have

$$|D_{G_i}\mathcal{R} \cdot v_i|_{\mathcal{R}(G_i),0} \geq \theta |v_i|_{G_i,0} \geq C_2 \theta |v_i|_{\mathcal{B}^{\mathbb{R}}(U)} = C_2\theta > 0.$$

The compactness of $\Omega_{p,n}$ gives $\lim_i D_{G_i}\mathcal{R} \cdot v_i = v$ and consequently for every $k \geq 1$,

$$\lim_i D\mathcal{R}^k_{G_i} \cdot v_i = D\mathcal{R}^{k-1}_G \cdot v.$$

We have that $v_i \in C^u_{2\epsilon}(G_i)$, so for $k \geq 1$,

$$|D\mathcal{R}^k_{G_i} \cdot v_i|_{\mathcal{R}^k(G_i),0} \geq \theta^{k-1} |D\mathcal{R}_{G_i} \cdot v_i|_{\mathcal{R}G_i,0} \geq C_2 \theta^k.$$

Taking the limit on i we obtain

$$|D\mathcal{R}^k_G \cdot v|_{\mathcal{R}^k(G),0} \geq C_2 \theta^k. \tag{107}$$

Since $\lambda < 1 < \theta$, we conclude that (106) and (107) give us a contradiction. □

PROPOSITION 9.3. *For $\epsilon > 0$ small, we can choose δ_3 small enough such that for every $\delta' \in (0, \delta_3)$, the following holds. Let $\mathcal{H}$ be a $C^{1+\mathrm{Lip}}$ function*

$$\mathcal{H}\colon \mathbb{B}^u_G(u_0, \delta') \to E^h_G + G$$

such that $\mathcal{H} \in \mathcal{T}^1_0(G, \delta', 2\epsilon)$, where $G \in \Omega_{p,n}$. Then there exists $w \in \mathbb{B}^u_G(u_0, \delta')$ such that $w + \mathcal{H}(w)$ is a map whose critical points belong to the same periodic orbit.

Proof. Define

$$\tilde{\mathcal{H}}\colon \mathbb{B}^u_G(u_0, \delta') \times \{v \in E^h(G)\colon\ |v| \leq \delta'\} \to \mathcal{B}^{\mathbb{R}}_{\mathrm{nor}}(U)$$

as $\tilde{\mathcal{H}}(u, v) = u + \mathcal{H}(u) + v$. Let $F = u_0 + \mathcal{H}(u_0)$. Note that $\tilde{\mathcal{H}}$ is a homeomorphism on its image, which is an open neighborhood of F. Define $\tilde{\mathcal{H}}_1(u, v) = \mathcal{R}\tilde{\mathcal{H}}(u, v)$. If δ_3 is small enough, there is a smooth family of domains $\hat{U}_{(u,v)}$ such that for every (u, v) in the domain of $\tilde{\mathcal{H}}_1$, we have that

$$\tilde{\mathcal{H}}_1(u, v)\colon \hat{U}_{(u,v)} \to \hat{V}$$

is a polynomial-like map of type n such that

$$\mathrm{mod}\, \hat{V} \setminus \hat{U}_{(u,v)} > \epsilon_0.$$

Reducing $\hat{V}$ a little bit, for every

$$(w, z) \in E^u(G) \times E^h(G)$$

we have

$$D_u\tilde{\mathcal{H}}_1(u, v) \cdot w + D_v\tilde{\mathcal{H}}_1(u, v) \cdot z \in \mathcal{B}(\hat{U}_{(u,v)}).$$

Moreover, if $w \in E^u(G) \setminus \{0\}$, we have

$$D_u \tilde{\mathcal{H}}_1(u,v) \cdot w \in C^u_{2\epsilon}(\mathcal{R}G),$$

so by Proposition 9.2,

$$D_u \tilde{\mathcal{H}}_1(u,v) \cdot w \notin E^h_{\tilde{\mathcal{H}}_1(u,v)}. \tag{108}$$

Furthermore, for every $(w,z) \in E^u(G) \times E^h(G)$ such that

$$D_u \tilde{\mathcal{H}}_1(u,v) \cdot w + D_v \tilde{\mathcal{H}}_1(u,v) \cdot z \in E^h_{\tilde{\mathcal{H}}_1(u,v)},$$

we have

$$D_u \tilde{\mathcal{H}}_1(u,v) \cdot w + D_v \tilde{\mathcal{H}}_1(u,v) \cdot z \notin C^u_{2\epsilon}(\mathcal{R}G). \tag{109}$$

The image of $\tilde{\mathcal{H}}$ is an open neighborhood of $u_0 + \mathcal{H}(u_0) \in W^s_{\delta_3}(G)$, with $G \in \Omega_{n,p}$. In particular, by Proposition 4.7 there exists (u_1, v_1) such that $\tilde{\mathcal{H}}(u_1, v_1)$ is a map whose critical points belong to the same periodic orbit, and consequently $\tilde{\mathcal{H}}_1(u_1, v_1) = \mathcal{R}\tilde{\mathcal{H}}(u_1, v_1)$ is also a map whose critical points belong to the same periodic orbit. Furthermore, one can choose (u_1, v_1) arbitrarily close to $(u_0, 0)$. If $v_1 = 0$, choose $w = u_1$; we have finished the proof in this case. Otherwise $v_1 \neq 0$, and we consider the $C^{1+\mathrm{Lip}}$ smooth map

$$(u,t,x) \in E^u(G) \times \mathbb{R} \times U \mapsto f_{(u,t)}(x) := \tilde{\mathcal{H}}_1(u, tv_1)(x).$$

The critical points of $f_{(u_1,1)}$ belong to the same periodic orbit, so there are natural numbers i_k, $k = 1, \dots, n$ and we can index the critical points

$$\mathrm{Crit} = \{(0,j)\}_{0 \leq j \leq n-1} = \{(0,j_k)\}_{0 \leq k \leq n-1}$$

in such way that for every $k \leq n-1$,

$$f^{i_k}_{(u_0,1)}(0,j_k) = (0, j_{k+1 \bmod n}) \text{ and } f^{i}_{(u_0,1)}(0,j_k) \notin \text{ Crit for } i < i_k.$$

We claim that there is a function $t \mapsto u(t)$ defined for every $t \in [0,1]$ such that

$$f^{i_k}_{(u(t),t)}(0,j_k) = (0, j_{k+1 \bmod n}) \text{ and } f^{i}_{(u(t),t)}(0,j_k) \notin \text{ Crit for } i < i_k \tag{110}$$

for every $k \leq n-1$. Indeed, let Y be the set of $q \in [0,1]$ such that there exists a continuous function u defined on $[q,1]$ such that (110) holds for every $t \in [q,1]$ and $u(1) = u_1$. Note that $1 \in Y$. We need to show that $0 \in Y$. It is enough to show that Y is a open and closed subset of $[0,1]$. Indeed, suppose that (u_2, t_2) satisfies

$$f^{i_k}_{(u_2,t_2)}(0,j_k) = (0, j_{k+1 \bmod n}) \text{ and } f^{i}_{(u_2,t_2)}(0,j_k) \notin \text{ Crit for } i < i_k \tag{111}$$

for every $k \leq n-1$. Note that the linear map

$$w \mapsto (D_u f^{i_k}_{(u_2,t_2)}(0,j_k) \cdot w)_{0 \leq k \leq n-1}$$

is invertible; otherwise $w \in E^u(G) \setminus \{0\}$ would exists such that

$$D_u f^{i_k}_{(u_2,t_2)}(0,j_k) \cdot w = 0$$

for every k. Thus using the infinitesimal pullback argument one can conclude that

$$D_u\tilde{\mathcal{H}}_1(u_2, t_2v_1)\cdot w \in E^h_{\tilde{\mathcal{H}}_1(u_2,t_2v_1)},$$

which contradicts (108). So by the Implicit Function Theorem there exists an open interval O with $t_2 \in O$ such that there is a unique continuous function u defined on O such that (110) holds for every $t \in O$ and $u(t_2) = u_2$. We conclude that Y is an open set and that for each $q \in Y$, there exists an *unique* continuous function U defined on $[q, 1]$ and satisfying and (110) and $u(1) = u_1$. To show that Y is closed, suppose that $q_n \in Y$ is a decreasing sequence converging to some $q \in [0, 1]$. Then there exists a unique continuous function u defined in $(q, 1]$ such that $u(1) = u_1$ and (110) holds. We claim that u is a Lipschitz function on $(q, 1]$, so we can extend it to a continuous function u defined in $[q, 1]$. Indeed, note that

$$\partial_t f_{(u(t),t)} \in E^h_{f_{(u(t),t)}},$$

so by (109),

$$\partial_t f_{(u(t),t)} \notin C^u_{2\epsilon}(\mathcal{R}G). \tag{112}$$

Moreover,

$$\partial_t f_{(u(t),t)} = D\mathcal{R}_{u(t)+\mathcal{H}(u(t))+tv_1}\cdot(u'(t) + D_u\mathcal{H}_{u(t)}\cdot u'(t) + v_1). \tag{113}$$

Let

$$y = D\mathcal{R}_{u(t)+\mathcal{H}(u(t))+tv_1}\cdot(u'(t) + D_u\mathcal{H}_{u(t)}\cdot u'(t)).$$

Note that

$$|y|_{\mathcal{R}G,0} \geq \lambda\frac{1-2\epsilon}{1+2\epsilon}|u'(t)|_{G,0}.$$

Suppose that $|u'(t)|_{G,0} \geq L|v_1|_{G,0}$. If δ' is small enough, then there is $C > 0$ such that

$$|\pi^u_{\mathcal{R}G,0}(\partial_t f_{(u(t),t)})|_{\mathcal{R}G,0} \geq \left(1 - \frac{C}{L\lambda}\frac{1+2\epsilon}{1-2\epsilon}\right)|y|_{\mathcal{R}G,0} \tag{114}$$

and

$$|\pi^h_{\mathcal{R}G,0}(\partial_t f_{(u(t),t)})|_{\mathcal{R}G,0} \leq \left(2\epsilon' + \frac{C}{L\lambda}\frac{1+2\epsilon}{1-2\epsilon}\right)|y|_{\mathcal{R}G,0}. \tag{115}$$

If L is large enough, then

$$2\epsilon' + \frac{C}{L\lambda}\frac{1+2\epsilon}{1-2\epsilon} \leq 2\epsilon\left(1 - \frac{C}{L\lambda}\frac{1+2\epsilon}{1-2\epsilon}\right),$$

which implies that $\partial_t f_{(u(t),t)} \in C^u_{2\epsilon}(\mathcal{R}G)$. This contradicts (112). In particular, there is L satisfying $|u'(t)|_{G,0} \leq L|v_1|_{G,0}$ for every $t \in (q, 1]$, and consequently u is a Lipchitz function. So we can extend u to a continuous map to $[q, 1]$. It is easy to see that (110) also holds for $t = q$. We conclude that Y is closed. Since Y is an open, closed, non-empty subset of $[0, 1]$, we conclude that $Y = [0, 1]$ so, in particular, $0 \in Y$. Therefore there exists w such that

$f_{(w,0)} = \tilde{\mathcal{H}}_1(w,0) = \mathcal{R}(w+\mathcal{H}(w))$ is a map whose critical points belong to the same periodic orbit, and consequently $w+\mathcal{H}(w)$ has the same property. □

Proof of Corollary 9.1. Let ϵ be small. It is easy to see that if $\delta_3 > 0$ is small enough, then for every $G \in \Omega_{p,n}$ and for every $C^{1+\mathrm{Lip}}$ function $\mathcal{H} \in \mathcal{T}_0^1(G, \delta', \epsilon)$, with $\delta' < \delta_3$ and for every

$$F \in \hat{\mathcal{H}} \cap W^s_{\delta_3}(\Omega_{n,p}),$$

there is $G_F \in \Omega_{n,p}$ such that $F \in W^s_{\delta_3}(G_F)$ and $\delta'' > 0$ such that

$$\{w+\mathcal{H}(w)\colon\ w \in \mathbb{B}^u(v_0, \delta')\} \\ \cap \{u+v+G_F\colon\ u \in \mathbb{B}^u_{G_F}(\pi^u_{G_F}(F-G_F), \delta''),\ v \in E^h_{G_F}\}$$

is the graph $\hat{\mathcal{H}}_F$ of a $C^{1+\mathrm{Lip}}$ function in $\mathcal{T}_0^1(G_F, \delta'', 2\epsilon)$. By Proposition 9.3 there is $w \in \mathbb{B}^u_{G_F}(\pi^u_{G_F}(F-G_F), \delta'')$ such that $w+\mathcal{H}_F(w)$ is a map such that all its critical points belong to the same periodic orbit. In particular, every map close enough to $w+\mathcal{H}_F(w)$ is a hyperbolic map with an attracting periodic orbit that attracts all its critical points. In particular, we can find hyperbolic maps in $\hat{\mathcal{H}}$ arbitrarily close to F. Note that hyperbolic maps do not belong to $W^s_{\delta_3}(\Omega_{n,p})$, since every map in $W^s_{\delta_3}(\Omega_{n,p})$ is infinitely renormalizable. □

10. Families of multimodal maps

In the beginning of Section 9 we saw that a version of Theorem A for real-analytic families of extended maps that *belong* to $\mathcal{W}^{\mathbb{R}}$ can be obtained from the hyperbolicity of $\Omega_{n,p}$, the Empty Interior Transversality property and [43, Th. 1]. This is not enough to our purposes once Theorem A deals with real-analytic families of multimodal maps. Indeed a multimodal map with more than a critical point is *not* an extended map.

To prove Theorem A we will need a classic tool, *inducing*. We will associate to each real-analytic multimodal map f that is close enough to an infinitely renormalizable multimodal map with bounded combinatorics a renormalization F of f that is an extended map in $\mathcal{W}^{\mathbb{R}}$. Indeed a renormalizable multimodal map can be renormalizable in many ways (it can have distinct cycles of restrictive intervals with disjoint orbits) and many times (it can have deeper and deeper renormalizations), so we need to *mark* f with a restrictive interval P in such way to make this association

$$(f, P) \mapsto \mathcal{I}(f, P) = F$$

well behaved. Indeed we are going to see that $\mathcal{I}$ can be defined in such way that it is a real-analytic map defined in an open set of a real Banach space with image in $\mathcal{W}^{\mathbb{R}}$. The derivative $D_{(f,P)}\mathcal{I}$ of this map has dense image at every infinitely renormalizable marked multimodal maps (f, P), which allows us to use Proposition 8.1 of [43] to conclude that $\mathcal{I}^{-1}W^s_{\delta}(\Omega_{n,p})$ intersects a generic real-analytic family of multimodal maps on a set of parameters with zero Lebesgue measure. This is the main argument of the proof of Theorem A. We provide the complete proof below.

Let $V \subset \mathbb{C}$ be a connected open set, symmetric with respect to the real line ($z \in V$ implies $\overline{z} \in V$) such that $[-1,1] \subset V$. In this section we will denote by $\mathcal{B}_{\mathbb{C}}$ the affine subspace of of functions$f \in \mathcal{B}(V)$ that satisfy $f(-1) = f(-1) = -1$. Denote by $\mathcal{B}_{\mathbb{R}}$ the real Banach space of all functions $f \in \mathcal{B}_{\mathbb{C}}$ that are real on the $V \cap \mathbb{R}$.

Given $m \in \mathbb{N}$, let $\Gamma^{\omega_{\mathbb{R}}}_m(\mathcal{B}_{\mathbb{R}})$ be the set of all continuous functions

$$\gamma\colon \overline{\mathbb{D}}^m \to \mathcal{B}_{\mathbb{C}}$$

that are complex analytic on $\mathbb{D}^m$ and such that $\gamma(\lambda) \in \mathcal{B}_{\mathbb{R}}$ for every $\lambda \in [-1,1]^m$. We can endow $\Gamma^{\omega_{\mathbb{R}}}_m(\mathcal{B}_{\mathbb{R}})$ with the sup norm.

Let $\Gamma \subset \mathcal{B}_{\mathbb{R}}$ be the open subset of multimodal maps $f\colon [-1,1] \to [-1,1]$, where -1 is a repelling fixed point, $f'(1) \neq 0$, with quadratic critical points, negative Schwarzian derivative and $f(-1,1) \subset (-1,1)$.

Denote by $\Gamma_n \subset \Gamma^{\omega_{\mathbb{R}}}_n(\mathcal{B}_{\mathbb{R}})$ the subset of all families γ such that $\gamma(\lambda) \in \Gamma$ for every $\lambda \in [-1,1]^n$. Note that Γ_n is an open subset of $\Gamma^{\omega_{\mathbb{R}}}_n(\mathcal{B}_{\mathbb{R}})$.

10.1. *Generic families.* Our main result for generic families is

THEOREM 7 (Theorem A). *For every γ in a generic subset of Γ_m, the set Λ of parameters λ such that $\gamma(\lambda)$ has (at least) one solenoidal attractor with bounded combinatorics on $(-1,1)$ has zero Lebesgue measure.*

Proof. We divide the proof in several steps.

Step I: *Marking restrictive intervals.* It turns out that a multimodal map may have many disjoint cycles of restrictive intervals. To deal with that we need to "mark" one of those restrictive intervals. To this end fix $j \in \mathbb{N}^*$ and $q, n \in \mathbb{N} \setminus \{0,1\}$. Let $\mathcal{O}_{j,q,n}$ be the set of all pairs (f_0, P_0) such that

(A) The map $f_0 \in \Gamma$ has j critical points in $[-1,1]$.
(B) P_0 is a restrictive interval of f_0 such that each $f_0^i(P_0)$ has at most one critical point for every i, $\cup_i f_0^i(P_0)$ contains n critical points, and P_0 has a repelling periodic point in its boundary, with period $q' < q$. In particular, $f_0^{q'}(\partial P_0) \subset \partial P_0$.
(C) The f_0-forward orbit of any critical point on the orbit of such restrictive interval P_0 *does not* fall in the orbit of such periodic point.

Note that the image $\pi_1(\mathcal{O}_{j,q,n})$ of the projection onto the first coordinate in $\mathcal{O}_{j,q,n}$ is an open subset of Γ. Of course the countable family

$$\{\pi_1(\mathcal{O}_{j,q,n})\}_{j,q,n}$$

covers all infinitely renormalizable multimodal maps.

Fix $(f_0, P_0) \in \mathcal{O}_{j,q,n}$. By the implicit function theorem the repelling periodic point of f_0 in the boundary of P_0 has an analytic continuation that is also repelling, and it defines a restrictive interval P_g for each map g in an open connected neighborhood $\mathcal{V}_0$ of f_0 on Γ and such restrictive interval also satisfies properties (A), (B) and (C). In particular, the family $\mathcal{F}$ of pairs $(\mathcal{V}, P)$ where

(1) $\mathcal{V}$ is an open and connected subset of Γ, with $f_0 \in \mathcal{V}$;
(2) the real-analytic function

$$g \in \mathcal{V} \mapsto P(g)$$

associate with each map $g \in \mathcal{V}$ a restrictive interval $P(g)$ of g satisfying $(g, P(g)) \in \mathcal{O}_{j,q,n}$ and moreover $P(f_0) = P_0$

is non-empty. Consequently, by Zorn's Lemma, $\mathcal{F}$ has a maximal element with respect to the order $(\mathcal{V}_1, P_1) < (\mathcal{V}_2, P_2)$ if and only if $\mathcal{V}_1 \subset \mathcal{V}_2$ and $P_2(g) = P_1(g)$ for every $g \in \mathcal{V}_1$. We claim that such a maximal element is unique.

We claim that if $(\mathcal{V}_0, P_0), (\mathcal{V}_1, P_1) \in \mathcal{F}$, then $P_0 = P_1$ on $\mathcal{V}_0 \cap \mathcal{V}_1$. Indeed since $f \in \mathcal{V}_i$, $i = 0, 1$, always has j critical points (moving continuously with respect to f, since they are quadratic) and a point $b_{f,i} \in \partial P_i(f)$ is a repelling periodic point of f that is analytic continuation of $b_{f_0,0} = b_{f_0,1}$, it follows that all those periodic points have exactly the same combinatorics with respect to the symbolic dynamics defined by partition induced by the critical points. In particular, if $f \in \mathcal{V}_0 \cap \mathcal{V}_1$, then $b_{f,0}, b_{f,1}$ are repelling periodic points of f with the same combinatorics. Since f has negative Schwarzian derivative, the minimal principle implies that $b_{f,0} = b_{f,1}$. This proves the claim.

In particular, the maximal element of $\mathcal{F}$, denoted by $(\mathcal{V}_{f_0,P_0}, P_{f_0,P_0})$, can be described by

$$\mathcal{V}_{f_0,P_0} = \cup_{(\mathcal{V},P)\in\mathcal{F}} \mathcal{V}$$

and $P_{f_0,P_0}(f) = P(f)$ for every $f \in \mathcal{V}$ satisfying $(\mathcal{V}, P) \in \mathcal{F}$. Note that

$$\mathcal{G} = \{\mathcal{V}_{f_0,P_0} \colon (f_0, P_0) \in \mathcal{O}_{j,q,n}\}$$

is a partition of $\mathcal{O}_{j,q,n}$. We claim that such a partition has a countable number of elements. Indeed, suppose that

$$\{(f_\lambda, P_\lambda)\}_{\lambda\in\Lambda}$$

is an uncountable family such that

$$\mathcal{V}_{f_\lambda,P_\lambda} \neq \mathcal{V}_{f_\mu,P_\mu}$$

for every $\lambda \neq \mu$. Choose a complex neighborhood W of $[-1, 1]$ such that $\overline{W} \subset U$. Then there exists a sequence $\lambda_k \in \Lambda$, $k \in \mathbb{N}$ such that

(P1) There are n and $q' < q$ such that $(f_{\lambda_k}, P_{\lambda_k})$ satisfies the conditions (A) and (B) for every k.

(P2) $\lim_k (f_{\lambda_k}, P_{\lambda_k}) = (f_\infty, P_\infty)$ on $\mathcal{B}_{\mathbb{R}}(W)$, where (f_∞, P_∞) is a multimodal map with j quadratic critical points, negative Schwarzian derivative and $f_\infty(-1,1) \subset (-1,1)$, and that also satisfies (A), (B) and (C) for the very same n and q' as in (P1).

(P3) There is $\theta > 1$ such that if b_{λ_k} is the repelling periodic point in the boundary of P_{λ_k}, then $|Df^{q'}_{\lambda_k}(b_{\lambda_k})| > \theta$ for every k.

(P4) If $k \neq k'$, then $\lambda_k \neq \lambda_{k'}$.

By the implicit function theorem there is a ball Y of $\mathcal{B}_{\mathbb{R}}(W)$ around f_∞ and a real-analytic function P defined in Y such that for every $f \in Y$, we have that $P(f)$ is a restrictive interval for f satisfying A and B, and additionally $P(f_{\lambda_k}) = P_{\lambda_k}$ for every large k. In particular, choose k_0, k_1 large enough and a small connected open subset $\tilde{W} \subset \Gamma$ around the segment $\{t f_{\lambda_{k_0}} + (1-t) f_{\lambda_{k_1}},\ t \in [0,1]\}$. Then the function P is defined in $\tilde{W}$, which implies that

$$\mathcal{V}_{f_{\lambda_{k_0}}, P_{\lambda_{k_0}}} = \mathcal{V}_{f_{\lambda_{k_1}}, P_{\lambda_{k_1}}},$$

which is a contradiction. This completes the proof of our claim.

Step II: *Replacing multimodal maps by extended maps of type n.* A real-analytic multimodal map does not have the nice structure of a multimodal map of type n. Fix some open set $\mathcal{V}_{f_0,P_0} \subset \mathcal{O}_{j,q,n}$. We will replace every $g \in \pi_1(\mathcal{V}_{f_0,P_0})$ by an induced map that is an extended map of type n. Denote by I_g the extended map of type n that is the renormalization of g associated with the restrictive interval $P_{f_0,P_0}(g)$. Of course I_g is a real-analytic extended map with negative Schwarzian derivative and quadratic critical points.

Step III: *Compexification.* Fix $p \geq 2$. Let $\mathcal{W} \subset \mathcal{B}_{\mathrm{nor}}(U)$ be the domain of the complexification of the p-bounded renormalization operator $\mathcal{R}$ as defined Section 3. Note that if I_g is infinitely renormalizable with p-bounded combinatorics, we *do not* necessarily have that $I_g \in \mathcal{W}$. It may be the case that I_g is not defined on the domain U, for instance. However by the beau complex bounds given by Proposition 2 and the universality result in Proposition 4.6 there is k such that $R^{k'}(I_g) \in \mathcal{W}$ for every $k' \geq k$. Again by the beau complex bounds and Proposition 4.6 we can find open subsets $\mathcal{V}^k_{f_0,P_0} \subset \mathcal{V}_{f_0,P_0}$ such that

(E1) for every $g \in \mathcal{V}^k_{f_0,P_0}$, we have that $\mathcal{I}_k(g) = R^k(I_g)$ is well defined and it belongs to $\mathcal{W}$;

(E2) for every $g \in \mathcal{V}^k_{f_0,P_0}$ that is infinitely renormalizable with p-bounded combinatorics, we have $R^{k'}(I_g) \in \mathcal{W}$ for every $k' \geq k$;

(E3) we have that

$$\cup_k \mathcal{V}^k_{f_0,P_0}$$

contains all infinitely maps in $\mathcal{V}_{f_0,P_0}$ that are renormalizable with p-bounded combinatorics.

The operator $\mathcal{I}_k$ has a complexification. (It can the proven using *exactly* the same argument as in the complexification of the renormalization operator in Section 3.) From now on we restrict $\mathcal{I}_k$ to real maps. Note that the image of the operator $D_g\mathcal{I}_k$ is dense in $T\mathcal{B}^{\mathbb{R}}_{\text{nor}}(U)$. (Again, the argument is the same as with the renormalization operator in Theorem 3.)

Step IV: *Applying the hyperbolicity of* $\Omega_{n,p}$. Due to Theorem 5 we have that $\Omega_{n,p}$ is a hyperbolic invariant set of $\mathcal{R}$. Moreover, Corollary 9.1 says that $W^s(\Omega_{n,p})$ has transversal empty interior. Theorem 3 tells us that

$$D_F\mathcal{R}(T_F\mathcal{B}^{\mathbb{R}}_{\text{nor}}(U))$$

is dense in $T_{\mathcal{R}F}\mathcal{B}^{\mathbb{R}}_{\text{nor}}(U)$ for every $F \in \mathcal{W}^{\mathbb{R}}$. So we conclude that $\mathcal{R}$ (restricted to real maps) satisfies the assumptions of Theorem 1 in [43] (taking $k = \omega_{\mathbb{R}}$ there). Now we can apply Proposition 8.1 of [43] taking $\mathcal{M} = \mathcal{I}_k$ to conclude that for a generic $\gamma \in \Gamma^{\omega_{\mathbb{R}}}_n(\mathcal{B}_{\mathbb{R}})$, the set of parameters $\lambda \in [-1,1]^n$ where $\mathcal{I}_k(\gamma(\lambda))$ is infinitely renormalizable with p-bounded combinatorics has zero Lebesgue measure. Since there is just a countable number of choices for k, elements of $\mathcal{G}$, q, p and j, we concluded the proof. □

Indeed Proposition 8.1 of [43] implies an analogous result for finitely differentiable families of maps in Γ. We refer the reader to [43] for additional statements and definitions for this setting.

10.2. *Transversal families of polynomial-like maps.* Recall the definition of $\hat{E}^h_f$ and $\hat{E}^v_f$ in Section 4.2.

THEOREM 8 (Transversal families). *Let Λ be an open subset of $\mathbb{C}^d$. Let*

$$\lambda \in \Lambda \mapsto f_\lambda \colon V^1_\lambda \to V^2_\lambda$$

be a complex analytic family of polynomial-like maps such that for every $\lambda \in \Lambda \cap \mathbb{R}^d$, we have that V^1_λ, V^2_λ are symmetric with respect with $\mathbb{R}$, its real trace is an interval, $f_\lambda(x) \in \mathbb{R}$ for every $x \in \mathbb{R}$, f_λ has negative Schwarzian derivative and just quadratic critical points on the real line. Suppose that for every $\lambda_0 \in \mathbb{R}^d$ such that f_{λ_0} is infinitely renormalizable with bounded combinatorics, we have the following:

> TRANSVERSALITY ASSUMPTION. *Every holomorphic vector in a neighborhood of $K(f_{\lambda_0})$ can be written as a sum of a vector in $\hat{E}^h_{f_{\lambda_0}}$ and a vector in $D_\lambda f_\lambda|_{\lambda=\lambda_0}(\mathbb{R}^d)$.*

Then the set of parameters $\lambda \in \Lambda \cap \mathbb{R}^d$ where f_λ is infinitely renormalizable with bounded combinatorics has zero d-dimensional Lebesgue measure.

Proof. We can find a countable family of domains $U_i \subset \mathbb{C}$, symmetric with respect to $\mathbb{R}$, and open subsets $\Lambda_i \subset \Lambda \cap \mathbb{R}^d$ such that $\cup_i \Lambda_i = \Lambda \cap \mathbb{R}^d$,

$f_\lambda \in \mathcal{B}(U_i)$, for $\lambda \in \Lambda_i$, with $K(f_\lambda) \cap \mathbb{R} \subset U_i \cap \mathbb{R}$, and

$$\lambda \in \Lambda_i \mapsto f_\lambda \in \mathcal{B}(U_i)$$

is a real-analytic family. It is enough to prove the conclusion of Theorem 8 for each one of those families. So fix i. The proof proceeds as the proof of Theorem 7. We can define the sets $\mathcal{O}_{j,q,n}$ replacing the pairs (f, P) by pairs of the form (λ, P), where $\lambda \in \Lambda_i$ and P is a restrictive interval of f_λ. In a similar way we can define $(\mathcal{V}_{\lambda_0,P_{\lambda_0}}, P_{\lambda_0,P_{\lambda_0}})$, the sets $\mathcal{V}^k_{\lambda_0,P_{\lambda_0}} \subset \Lambda_i$ and the real-analytic parametrized families

$$\lambda \in \mathcal{V}^k_{\lambda_0,P_{\lambda_0}} \to \mathcal{I}_k(f_\lambda) \in \mathcal{W}.$$

Suppose that f_λ is infinitely renormalizable with p-bounded combinatorics. Then

$$D_{f_\lambda}\mathcal{I}_k(E^h_{f_\lambda}) \subset E^h_{\mathcal{I}_k(f_\lambda)}.$$

This follows exactly as the proof of Proposition 4.2. On the other hand, we know (see the proof of Theorem 7) that $\operatorname{Im} D_{f_\lambda}\mathcal{I}_k$ is dense in $T_{\mathcal{I}_k(f_\lambda)}\mathcal{B}^{\mathbb{R}}_{\mathrm{nor}}(U)$. By the Transversality assumption this implies that there is a subspace $S_\lambda \subset \mathbb{R}^d$, with $\dim S_\lambda = n$, such that

$$D_{f_\lambda}\mathcal{I}_k \cdot D_\lambda f_\lambda(S_\lambda) \pitchfork E^h_{\mathcal{I}_k(f_\lambda)}.$$

Suppose that λ_0 is such that f_{λ_0} is infinitely renormalizable with p-bounded combinatorics. Let $v_1, \dots, v_n, v_{n+1}, \dots, v_d$ be a basis of $\mathbb{R}^d$ such that $v_1, \dots, v_n$ is a basis for S_{λ_0}. Then for every $\gamma = (\gamma_1, \dots, \gamma_{d-n}) \in \mathbb{R}^{n-d}$ that is small enough, we have that the family

$$\theta = (\theta_1, \dots, \theta_n) \mapsto g_\theta = f_{\lambda_0 + \sum_{i=1}^n \theta_i v_i + \sum_{i=n+1}^d \gamma_{i-n} v_i},$$

where θ is also small, satisfies

$$D_{g_\theta}\mathcal{I}_k \cdot D_\theta g_\theta(\mathbb{R}^n) \pitchfork E^h_{\mathcal{I}_k(g_\theta)}.$$

So by [43, Cor. 10.2] we have that for every small γ, the set of small parameters θ such that g_θ is infinitely renormalizable with bounded combinatorics has zero n-dimensional Lebesgue measure. By the Fubini's Theorem it follows that in a small neighborhood the parameter λ_0 the set of parameters λ such that f_λ is infinitely renormalizable with p-bounded combinatorics has zero d-dimensional Lebesgue measure. This completes the proof. □

Proof of Theorem C. Let $f_{\lambda_1,\lambda_2}(z) = z^3 - 3\lambda_1^2 z + \lambda_2$. Note that if $\lambda_1 = 0$, then f_{λ_1,λ_2} is not infinitely renormalizable, so we assume that $\lambda_1 \neq 0$. Let $\lambda_0 = (a, b)$, $a \neq 0$. Then

$$\partial_\lambda f_\lambda|_{\lambda=\lambda_0}(\mathbb{R}^2) = \{cz + d,\ c, d \in \mathbb{R}\}.$$

By Proposition 4.8 we have that $\hat{E}^v_{f_\lambda}$ is the space of cubic polynomials, so $\dim \hat{E}^v_{f_\lambda} = 4$ and

$$\partial_\lambda f_\lambda|_{\lambda=\lambda_0}(\mathbb{R}^2) \subset \hat{E}^v_{f_\lambda}.$$

We claim that

$$\{2z^3 - b, -3z^2 - 3a^2 - 1\} \subset \hat{E}^h_{f_\lambda} \cap \hat{E}^v_{f_\lambda}. \tag{116}$$

Indeed, let $H_t(z) = z/t$. Then

$$2z^3 - b = \partial_t(H_t \circ f_{a,b} \circ H_t^{-1}(z))|_{t=1},$$

so $2z^3 - b = \alpha_1(f_{a,b}(z)) - Df_{a,b}(z)\alpha_1(z)$, where $\alpha_1(z) = \partial_t H_t(z)|_{t=1} = -z$. Let $S_t(z) = z - t$. Then

$$3z^2 - 3a^2 - 1 = \partial_t(S_t \circ f_{a,b} \circ S_t^{-1}(z))|_{t=0},$$

so $3z^2 - 3a^2 - 1 = \alpha_2(f_{a,b}(z)) - Df_{a,b}(z)\alpha_2(z)$, where $\alpha_2(z) = \partial_t S_t(z)|_{t=0} = -1$. Since $\{1, z, 2z^3 - b, 3z^2 - 3a^2 - 1\}$ is a basis of $\hat{E}^v_{f_\lambda}$, by (27) and (116) we have that every holomorphic vector in a neighborhood of $K(f_{a,b})$ can be written as a sum of a vector in $\hat{E}^h_{f_{a,b}}$ and a vector in $\partial_\lambda f_\lambda|_{\lambda=\lambda_0}(\mathbb{R}^2)$. Now we apply Theorem 8. □

10.3. *Compositions of quadratic maps.* We can say something about a specific family of extended maps of type n. This family was introduced in [40]. Let $\lambda = (\lambda_i)_{i \le n}$, with $\lambda_i \in [0, 1]$, and define

$$F_\lambda \colon \mathbb{C} \times \{i\}_{i \le n} \to \mathbb{C} \times \{i\}_{i \le n}$$

as $F_\lambda(z, i) = (-2\lambda_i z^2 + 2\lambda_i - 1, i + 1 \bmod n)$.

It follows from the study in [40] that each possible combinatorial type of an infinitely renormalizable extended map of type n with combinatorics bounded by p can be realized by a unique parameter in $[0, 1]^n$ and the set of such parameters $\Lambda_{p,n} \subset [0, 1]^n$ is a Cantor set [40, Th. 2]. The following result answers a conjecture in [40].

THEOREM 9. *We have that $m(\Lambda_{p,n}) = 0$, where m is the n-dimensional Lebesgue measure.*

Proof. Due to Corollary 10.2 in [43], it is enough to show that this family is transversal to the horizontal distribution $F \to E^h_F$. We will give a proof similar to the proof of the transversality of the quadratic family by Lyubich [26]. Indeed, suppose by contradiction that there exists $\lambda_0 \in \Lambda_{p,n}$ and $w \in \mathbb{R}^n \setminus \{0\}$ such that

$$v = \partial_\lambda F_\lambda|_{\lambda=\lambda_0} \cdot w \in E^h_{F_{\lambda_0}}.$$

So there is a quasiconformal vector field α_0, defined in a neighborhood of the postcritical set $P(F_{\lambda_0})$ satisfying (17) on $P(F_{\lambda_0})$. Since this is a family of polynomials, the conformal dynamics outside the Julia set of F_{λ_0} is always the same, so v is also a *vertical* direction; that is, there exists a conformal vector field α_1 defined outside the Julia set such that (17) holds outside the Julia set. Using the infinitesimal pullback argument we can find a quasiconformal vector field solution α that satisfies (17) everywhere. Moreover, it is conformal outside the Julia set. Since F_{λ_0} does not support invariant line fields on its Julia set, we conclude that α is conformal everywhere. Indeed it is equal to zero, since it is zero at three points of $\overline{\mathbb{C}} \times \{i\}$ for each $i \le n$. So $v = 0$, which implies $w = 0$. □

Remark 10.1. Note that $\Lambda_{p,n}$ only includes the parameters where each renormalization involves all n critical points; that is, each cycle of intervals covers all critical points. If we consider infinitely renormalizable maps where fewer points are involved, then the set of parameters it is not a Cantor set anymore. However it is likely that this larger subset of parameters also has zero Lebesgue measure.

References

[1] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, second ed., *Univ. Lecture Ser.* **38**, Amer. Math. Soc., Providence, RI, 2006, with supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. MR 2241787. Zbl 1103.30001. https://doi.org/10.1090/ulect/038.

[2] A. Avila and M. Lyubich, The full renormalization horseshoe for unimodal maps of higher degree: exponential contraction along hybrid classes, *Publ. Math. Inst. Hautes Études Sci.* no. 114 (2011), 171–223. MR 2854860. Zbl 1286.37047. https://doi.org/10.1007/s10240-011-0034-2.

[3] A. Avila, M. Lyubich, and W. de Melo, Regular or stochastic dynamics in real analytic families of unimodal maps, *Invent. Math.* **154** no. 3 (2003), 451–550. MR 2018784. Zbl 1050.37018. https://doi.org/10.1007/s00222-003-0307-6.

[4] A. Avila, M. Lyubich, and W. Shen, Parapuzzle of the Multibrot set and typical dynamics of unimodal maps, *J. Eur. Math. Soc.* (*JEMS*) **13** no. 1 (2011), 27–56. MR 2735075. Zbl 1213.37076. https://doi.org/10.4171/JEMS/243.

[5] A. Avila and C. G. Moreira, Phase-parameter relation and sharp statistical properties for general families of unimodal maps, in *Geometry and dDynamics*, *Contemp. Math.* **389**, Amer. Math. Soc., Providence, RI, 2005, pp. 1–42. MR 2181956. Zbl 1145.37022. https://doi.org/10.1090/conm/389/07270.

[6] A. M. Blokh and M. Y. Lyubich, Attractors of the transformations of the interval, *Funktsional. Anal. i Prilozhen.* **21** no. 2 (1987), 70–71. MR 0902297. Zbl 0653.58022. Available at http://mi.mathnet.ru/eng/faa/v21/i2/p70.

[7] A. M. Blokh and M. Y. Lyubich, Decomposition of one-dimensional dynamical systems into ergodic components. The case of a negative Schwarzian derivative, *Algebra i Analiz* **1** no. 1 (1989), 128–145. MR 1015337. Zbl 0718.58024. Available at http://mi.mathnet.ru/eng/aa/v1/i1/p128.

[8] H. Bruin, G. Keller, T. Nowicki, and S. van Strien, Wild Cantor attractors exist, *Ann. of Math.* (2) **143** no. 1 (1996), 97–130. MR 1370759. Zbl 0848.58016. https://doi.org/10.2307/2118654.

[9] H. Bruin, G. Keller, and M. St. Pierre, Adding machines and wild attractors, *Ergodic Theory Dynam. Systems* **17** no. 6 (1997), 1267–1287. MR 1488317. Zbl 0898.58012. https://doi.org/10.1017/S0143385797086392.

[10] H. Bruin, W. Shen, and S. van Strien, Existence of unique SRB-measures is typical for real unicritical polynomial families, *Ann. Sci. École Norm. Sup.* (4) **39** no. 3 (2006), 381–414. MR 2265674. Zbl 1112.37018. https://doi.org/10.1016/j.ansens.2006.02.001.

[11] T. Clark, Regular or stochastic dynamics in families of higher-degree unimodal maps, *Ergodic Theory Dynam. Systems* **34** no. 5 (2014), 1538–1566. MR 3255432. Zbl 1322.37019. https://doi.org/10.1017/etds.2013.10.

[12] B. Derrida, A. Gervois, and Y. Pomeau, Universal metric properties of bifurcations of endomorphisms, *J. Phys. A* **12** no. 3 (1979), 269–296. MR 0524170. Zbl 0416.28011. https://doi.org/10.1088/0305-4470/12/3/004.

[13] E. de Faria, W. de Melo, and A. Pinto, Global hyperbolicity of renormalization for C^r unimodal mappings, *Ann. of Math.* (2) **164** no. 3 (2006), 731–824. MR 2259245. Zbl 1129.37021. https://doi.org/10.4007/annals.2006.164.731.

[14] M. J. Feigenbaum, Quantitative universality for a class of nonlinear transformations, *J. Statist. Phys.* **19** no. 1 (1978), 25–52. MR 0501179. Zbl 0509.58037. https://doi.org/10.1007/BF01020332.

[15] M. J. Feigenbaum, The universal metric properties of nonlinear transformations, *J. Statist. Phys.* **21** no. 6 (1979), 669–706. MR 0555919. Zbl 0515.58028. https://doi.org/10.1007/BF01107909.

[16] A. I. Gol′berg, Ya. G. Sinaĭ, and K. M. Khanin, Universal properties of sequences of period-tripling bifurcations, *Uspekhi Mat. Nauk* **38** no. 1(229) (1983), 159–160. MR 0693727. Zbl 0534.30032. https://doi.org/10.1070/RM1983v038n01ABEH003398.

[17] J. Graczyk and G. Światek, Generic hyperbolicity in the logistic family, *Ann. of Math.* (2) **146** no. 1 (1997), 1–52. MR 1469316. Zbl 0936.37015. https://doi.org/10.2307/2951831.

[18] J. Hu, *Renormalization, Rigidity, and Universality in Bifurcation Theory*, ProQuest LLC, Ann Arbor, MI, 1995, thesis (Ph.D.)–City Univ. of New York. MR 2693444. Available at https://search.proquest.com/docview/304170974?accountid=13314.

[19] M. V. Jakobson, Absolutely continuous invariant measures for one-parameter families of one-dimensional maps, *Comm. Math. Phys.* **81** no. 1 (1981), 39–88. MR 0630331. Zbl 0497.58017. https://doi.org/10.1007/BF01941800.

[20] M. Kapovich, On the dynamics of pseudo-Anosov homeomorphisms on representation varieties of surface groups, *Ann. Acad. Sci. Fenn. Math.* **23** no. 1 (1998), 83–100. MR 1601847. Zbl 0892.58058. Available at https://www.acadsci.fi/mathematica/Vol23/kapovich.html.

[21] G. Keller and T. Nowicki, Fibonacci maps re(al)visited, *Ergodic Theory Dynam. Systems* **15** no. 1 (1995), 99–120. MR 1314971. Zbl 0853.58072. https://doi.org/10.1017/S0143385700008269.

[22] O. Kozlovski, W. Shen, and S. van Strien, Density of hyperbolicity in dimension one, *Ann. of Math.* (2) **166** no. 1 (2007), 145–182. MR 2342693. Zbl 1138.37013. https://doi.org/10.4007/annals.2007.166.145.

[23] O. Kozlovski, W. Shen, and S. van Strien, Rigidity for real polynomials, *Ann. of Math.* (2) **165** no. 3 (2007), 749–841. MR 2335796. Zbl 1129.37020. https://doi.org/10.4007/annals.2007.165.749.

[24] O. E. Lanford, III, A computer-assisted proof of the Feigenbaum conjectures, *Bull. Amer. Math. Soc.* (*N.S.*) **6** no. 3 (1982), 427–434. MR 0648529. Zbl 0487.58017. https://doi.org/10.1090/S0273-0979-1982-15008-X.

[25] M. Lyubich, Dynamics of quadratic polynomials. I, II, *Acta Math.* **178** no. 2 (1997), 185–297. MR 1459261. Zbl 0908.58053. https://doi.org/10.1007/BF02392694.

[26] M. Lyubich, Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture, *Ann. of Math.* (2) **149** no. 2 (1999), 319–420. MR 1689333. Zbl 0945.37012. https://doi.org/10.2307/120968.

[27] M. Lyubich, Dynamics of quadratic polynomials. III. Parapuzzle and SBR measures, in *Géométrie Complexe et Systèmes Dynamiques* (Orsay, 1995), *Astérisque* **261**, Soc. Math. France, Paris, 2000, pp. xii–xiii, 173–200. MR 1755441. Zbl 1044.37038.

[28] M. Lyubich, Almost every real quadratic map is either regular or stochastic, *Ann. of Math.* (2) **156** no. 1 (2002), 1–78. MR 1935840. Zbl 1160.37356. https://doi.org/10.2307/3597183.

[29] R. S. MacKay and J. B. J. van Zeijts, Period doubling for bimodal maps: a horseshoe for a renormalisation operator, *Nonlinearity* **1** no. 1 (1988), 253–277. MR 0928956. Zbl 0648.58028.

[30] C. T. McMullen, *Complex Dynamics and Renormalization*, *Ann. of Math. Stud.* **135**, Princeton Univ. Press, Princeton, NJ, 1994. MR 1312365. Zbl 0822.30002. https://doi.org/10.1515/9781400882557.

[31] C. T. McMullen, *Renormalization and* 3*-Manifolds which Fiber over the Circle*, *Ann. of Math. Stud.* **142**, Princeton Univ. Press, Princeton, NJ, 1996. MR 1401347. Zbl 0860.58002. https://doi.org/10.1515/9781400865178.

[32] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, *Ergeb. Math. Grenzgeb.* **25**, Springer-Verlag, Berlin, 1993. MR 1239171. Zbl 0791.58003. https://doi.org/10.1007/978-3-642-78043-1.

[33] J. Milnor, On the concept of attractor, *Comm. Math. Phys.* **99** no. 2 (1985), 177–195. MR 0790735. Zbl 0595.58028. https://doi.org/10.1007/BF01212280.

[34] J. Milnor, *Dynamics in One Complex Variable*, third ed., *Ann. of Math. Stud.* **160**, Princeton Univ. Press, Princeton, NJ, 2006. MR 2193309. Zbl 1085.30002. https://doi.org/10.2307/j.ctt7rnxn.

[35] R. J. Sacker and G. R. Sell, Existence of dichotomies and invariant splittings for linear differential systems. I, *J. Differential Equations* **15** (1974), 429–458. MR 0341458. Zbl 0294.58008. https://doi.org/10.1016/0022-0396(74)90067-9.

[36] R. J. Sacker and G. R. Sell, Existence of dichotomies and invariant splittings for linear differential systems. II, *J. Differential Equations* **22** no. 2 (1976), 478–496. MR 0440620. Zbl 0339.58013. https://doi.org/10.1016/0022-0396(76)90042-5.

[37] W. Shen, On the measurable dynamics of real rational functions, *Ergodic Theory Dynam. Systems* **23** no. 3 (2003), 957–983. MR 1992673. Zbl 1059.37034. https://doi.org/10.1017/S0143385702001311.

[38] W. Shen, Decay of geometry for unimodal maps: an elementary proof, *Ann. of Math.* (2) **163** no. 2 (2006), 383–404. MR 2199221. Zbl 1097.37032. https://doi.org/10.4007/annals.2006.163.383.

[39] D. Smania, Complex bounds for multimodal maps: bounded combinatorics, *Nonlinearity* **14** no. 5 (2001), 1311–1330. MR 1862823. Zbl 1067.37053. https://doi.org/10.1088/0951-7715/14/5/320.

[40] D. Smania, Phase space universality for multimodal maps, *Bull. Braz. Math. Soc.* (*N.S.*) **36** no. 2 (2005), 225–274. MR 2152018. Zbl 1100.37024. https://doi.org/10.1007/s00574-005-0038-y.

[41] D. Smania, On the hyperbolicity of the period-doubling fixed point, *Trans. Amer. Math. Soc.* **358** no. 4 (2006), 1827–1846. MR 2186998. Zbl 1080.37053. https://doi.org/10.1090/S0002-9947-05-03803-1.

[42] D. Smania, Puzzle geometry and rigidity: the Fibonacci cycle is hyperbolic, *J. Amer. Math. Soc.* **20** no. 3 (2007), 629–673. MR 2291915. Zbl 1222.37042. https://doi.org/10.1090/S0894-0347-07-00550-4.

[43] D. Smania, Shy shadows of infinite-dimensional partially hyperbolic invariant sets, *Ergodic Theory Dynam. Systems* **39** no. 5 (2019), 1361–1400. MR 3928622. https://doi.org/10.1017/etds.2017.65.

[44] D. Sullivan, Bounds, quadratic differentials, and renormalization conjectures, in *American Mathematical Society Centennial Publications, Vol. II* (Providence, RI, 1988), Amer. Math. Soc., Providence, RI, 1992, pp. 417–466. MR 1184622. Zbl 0936.37016.

[45] C. Tresser and P. Coullet, Itérations d'endomorphismes et groupe de renormalisation, *C. R. Acad. Sci. Paris Sér. A-B* **287** no. 7 (1978), A577–A580. MR 0512110. Zbl 0402.54046.

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