Annals of Mathematics **190** (2019), 931–947
https://doi.org/10.4007/annals.2019.190.3.5

Knot Floer homology obstructs ribbon concordance

By Ian Zemke

Abstract

We prove that the map on knot Floer homology induced by a ribbon concordance is injective. As a consequence, we prove that the Seifert genus is monotonic under ribbon concordance. Generalizing theorems of Gabai and Scharlemann, we also prove that the Seifert genus is super-additive under band connected sums of arbitrarily many knots. Our results give evidence for a conjecture of Gordon that ribbon concordance is a partial order on the set of knots.

1. Introduction

If K_0 and K_1 are knots in S^3, a *concordance* from K_0 to K_1 is a smoothly embedded annulus in $[0,1]\times S^3$ with boundary $-\{0\}\times K_0\cup\{1\}\times K_1$. A *ribbon concordance* is a concordance C with only index 0 and 1 critical points. A *slice knot* is one that is concordant to the unknot (or equivalently, one that bounds a smoothly embedded disk in B^4). A *ribbon knot* is one that admits a ribbon concordance from the unknot to K.

A major open problem in low-dimensional topology is the *slice-ribbon conjecture*, which asks whether every slice knot is ribbon. In this paper, we discuss the related problem of determining when two concordant knots are ribbon concordant.

Some classical results about ribbon concordances are due to Gordon [Gor81]. Suppose C is a ribbon concordance from K_0 to K_1. Write $\pi_1(K_i)$ for the fundamental group of the complement of K_i in S^3, and $\pi_1(C)$ for the fundamental group of the complement of C in $[0,1]\times S^3$. Gordon [Gor81, Lemma 3.1] proved that

$$\pi_1(K_0)\to\pi_1(C) \text{ is injective and } \pi_1(K_1)\to\pi_1(C) \text{ is surjective.}$$

Keywords: concordance, ribbon concordance, knot Floer homology, Seifert genus
AMS Classification: Primary: 57M25, 57M27, 57Q60, 57R58.
This research was supported by NSF grant DMS-1703685.

Note that in Gordon's terminology, such a concordance goes "from" K_1 "to" K_0, though this is the opposite of the cobordism orientation, which is more convenient for our present paper.

In contrast to the slice-ribbon conjecture, it is well known that there are knots that are concordant, but not ribbon concordant. For example, if T_r and T_l denote the right- and left-handed trefoils and F_8 denotes the figure eight knot, then $K_0 := T_r \# T_l$ and $K_1 := F_8 \# F_8$ are concordant. However since both are fibered and have the same genus, a result of Gordon [Gor81, Lemma 3.4] implies that if K_0 and K_1 were ribbon concordant, then they would be isotopic.

In this paper, we show that knot Floer homology gives an obstruction to ribbon concordance.

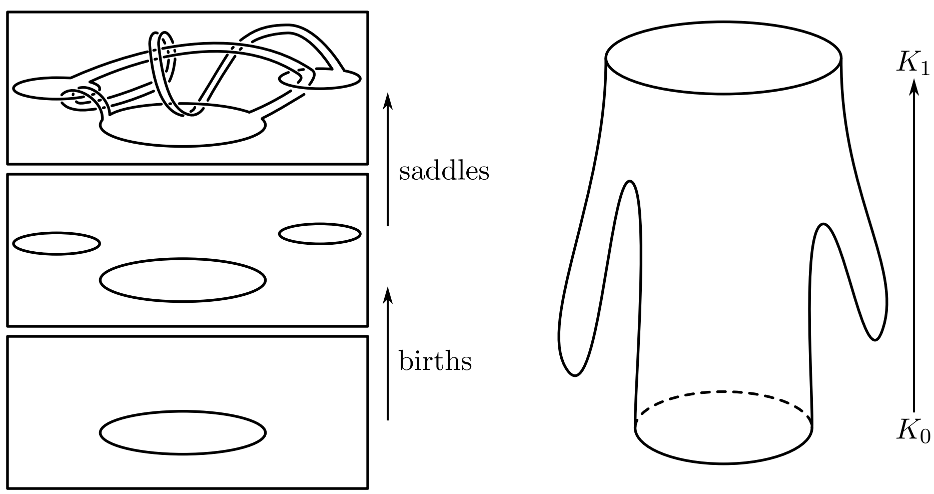

Figure 1. *A ribbon concordance from* K_0 *to* K_1.

1.1. *Knot Floer homology and ribbon concordances.* If $K \subseteq S^3$ is a knot, there is a bigraded $\mathbb{F}_2$ vector space

$$\widehat{HFK}(K) = \bigoplus_{i,j \in \mathbb{Z}} \widehat{HFK}_i(K, j), \tag{1}$$

constructed independently by Ozsváth and Szabó [OS04b], and Rasmussen [Ras03]. The subscript i in equation (1) denotes the Maslov grading, and j denotes the Alexander grading.

If C is a concordance from K_0 to K_1, Juhász and Marengon [JM16] construct a grading preserving cobordism map

$$F_C \colon \widehat{HFK}(K_0) \to \widehat{HFK}(K_1),$$

which is well defined up to two graded automorphisms of knot Floer homology. The ambiguity corresponds to a choice of decoration on C; see Section 2 for further details.

The concordance maps are based on a more general construction of cobordism maps on link Floer homology due to Juhász [Juh16]. We will also make use of an alternate description given by the author [Zem19b], which extends to the minus and infinity flavors of link Floer homology.

Our main theorem is the following:

THEOREM 1.1. *If C is a ribbon concordance from K_0 to K_1, then the map*

$$F_C\colon \widehat{HFK}(K_0) \to \widehat{HFK}(K_1)$$

is an injection.

Our argument is easy to summarize. Let C' denote the concordance from K_1 to K_0 obtained by turning C upside down and reversing its orientation. We will show that

$$F_{C'} \circ F_C = \mathrm{id}_{\widehat{HFK}(K_0)},$$

which immediately implies Theorem 1.1.

We will in fact show that a version of Theorem 1.1 holds for the full knot Floer complex, $\mathit{CFK}^\infty(K)$, which contains more information than $\widehat{HFK}(K)$; see Theorem 1.7 and Section 4.

An immediate corollary of Theorem 1.1 is the following:

THEOREM 1.2. *If there is a ribbon concordance from K_0 to K_1, then for each i and j,*

$$\operatorname{rank}_{\mathbb{F}_2} \widehat{HFK}_i(K_0, j) \le \operatorname{rank}_{\mathbb{F}_2} \widehat{HFK}_i(K_1, j).$$

Gordon made the following conjecture:

CONJECTURE 1.3 ([Gor81]). *Ribbon concordance is a partial ordering, i.e., if there is a ribbon concordance from K_0 to K_1, and also a ribbon concordance from K_1 to K_0, then $K_0 = K_1$.*

Our Theorem 1.1 gives the following immediate corollary, which supports Gordon's conjecture:

THEOREM 1.4. *If there is a ribbon concordance from K_0 to K_1, and also a ribbon concordance from K_1 to K_0, then*

$$\widehat{HFK}(K_0) \cong \widehat{HFK}(K_1),$$

as bigraded vector spaces over $\mathbb{F}_2$.

A caveat to Theorem 1.4 is that although $\widehat{HFK}$ detects the unknot [OS04a], as well as trefoils and the figure-eight knot [Ghi08], [Ni07], there are infinite families of non-isotopic knots that have the same knot Floer homology [HW18, Th. 1].

1.2. *Monotonicity of the Seifert genus.* If K is a knot, let $d(K)$ denote the degree of the Alexander polynomial of K. Gordon [Gor81, Lemma 3.4] showed that if there is a ribbon concordance from K_0 to K_1, then

$$d(K_0) \leq d(K_1). \tag{2}$$

Ozsváth and Szabó [OS04a, Th. 1.2] proved that knot Floer homology detects the Seifert genus:

$$g_3(K) = \max\left\{i : \widehat{HFK}(K, i) \neq \{0\}\right\}. \tag{3}$$

Juhász gave an alternate argument using surface decompositions and sutured manifolds [Juh08, Th. 1.5].

Analogous to Gordon's result in equation (2), an immediate consequence of Theorem 1.1 and equation (3) is the following:

THEOREM 1.5. *If there is a ribbon concordance from K_0 to K_1, then*

$$g_3(K_0) \leq g_3(K_1).$$

1.3. *Seifert genus of band connected sums.* If $K_1, \dots, K_n$ are knots in S^3 that are unlinked from each other, a *band connected sum* of $K_1, \dots, K_n$ is a knot L obtained by connecting $K_1, \dots, K_n$ together with $n-1$ bands. The ordinary connected sum is an example of a band connected sum, but in general, band connected sums will be more complicated.

Gabai [Gab87] proved that if L is a band connected sum of K_1 and K_2, then

$$g_3(L) \geq g_3(K_1) + g_3(K_2). \tag{4}$$

Gabai also proved that if equality holds in equation (4), then $L = K_1 \# K_2$. Scharlemann [Sch85] independently proved that if the the band connected sum of two unknots is an unknot, then the band is a trivial band.

Note that the band connected sum of three or more knots is not in general an iterated band connected sum of pairs of knots. Gabai's proof does not obviously extend to the case of three or more summands. We prove the following:

THEOREM 1.6. *If a knot L is a band connected sum of knots $K_1, \dots, K_n$, then*

$$g_3(L) \geq g_3(K_1) + \cdots + g_3(K_n). \tag{5}$$

Proof. Miyazaki [Miy98] gave an elegant manipulation that shows that if L is a band connected sum of $K_1, \dots, K_n$, then there is a ribbon concordance from $K_1 \# \cdots \# K_n$ to L. Hence, equation (5) follows immediately from our Theorem 1.5, as well as the additivity of the Seifert genus under connected sum. □

Some comments are in order. Miyazaki [Miy18] recently proved superadditivity of the Seifert genus under the assumption that L is fibered, in fact showing that $K_1, \dots, K_n$ must all be fibered as well. Miyazaki combines results of Gordon [Gor81], Silver [Sil92] and Kochloukova [Koc06] to show that if equality holds in equation (5) and L is fibered, then $L = K_1 \# \cdots \# K_n$.

1.4. *Extension to the full knot Floer complex.* Ozsváth and Szabó [OS04b] defined a more general version of knot Floer homology, called the *full knot Floer complex*, denoted $\mathit{CFK}^\infty(K)$. The object $\mathit{CFK}^\infty(K)$ is a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex over the ring $\mathbb{F}_2[U, U^{-1}]$.

The present author gave a functorial construction of cobordism maps for the full knot Floer complex [Zem19b]. As a generalization to Theorem 1.1, we will show the following:

THEOREM 1.7. *If C is a ribbon concordance from K_0 to K_1, and C' is the concordance from K_1 to K_0 obtained by turning around and reversing the orientation of C, then*

$$F_{C'} \circ F_C \simeq \mathrm{id}_{\mathit{CFK}^\infty(K_0)},$$

where $\simeq$ means filtered, $\mathbb{F}_2[U, U^{-1}]$-equivariantly chain homotopic.

1.5. *Further commentary.* A recent paper of Miyazaki [Miy18] points out that work of Silver [Sil92] and Kochloukova [Koc06] together imply that if there is a ribbon concordance from K_0 to K_1, and K_1 is fibered, then K_0 is also fibered. Silver reduced the problem to a conjecture of Rapaport [Str75] about knot-like groups, which Kochloukova proved. In particular, if there is a ribbon concordance from K_0 to K_1 and K_1 is fibered, and further K_0 and K_1 have the same Seifert genus, then [Gor81, Lemma 3.4] implies they must be isotopic. Note that our Theorem 1.1 gives an alternate proof of this latter fact that avoids Kochloukova's result, by using Ni's theorem that knot Floer homology detects fibered knots [Ni07] together with [Gor81, Lemma 3.4].

Finally, we remark that a major open problem in symplectic topology is determining whether every Lagrangian concordance between Legendrian knots in S^3 is *decomposable*; see [Cha12, Def. 1.4], [EHK16, §6]. Decomposable Lagrangian cobordisms are products of elementary cobordisms corresponding to Legendrian Reidemeister moves, saddles and births. In particular, decomposable Lagrangians are ribbon. One strategy for proving that a given Lagrangian concordance is not decomposable might be to show that it is not even ribbon via our Theorem 1.2 (or more ambitiously Theorem 1.1, if one could explicitly compute the map). Unfortunately the only candidates the author is aware of [CNS16, §2.2] are satellites of decomposable Legendrian concordances, and hence are ribbon.

1.6. *Acknowledgments.* I would like to thank Jen Hom, Tye Lidman, Jeffrey Meier and Maggie Miller for helpful correspondences. I would also like to thank David Gabai for pointing out Scharlemann's work. This problem was posed by Sucharit Sarkar at the CMO-BIRS Conference *Thirty Years of Floer*

Theory for 3-*Manifolds* in Oaxaca, Mexico; see Problem 26 of the problem list https://www.birs.ca/cmo-workshops/2017/17w5011/report17w5011.pdf.

2. Background on knot and link Floer homology

Knot Floer homology is an invariant of knots discovered independently by Ozsváth and Szabó [OS04b], and Rasmussen [Ras03]. Ozsváth and Szabó [OS08] constructed a generalization, called link Floer homology, associated to links in 3-manifolds. In this section, we present background material about knot and link Floer homology.

Definition 2.1. A *multi-based link* $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ in a 3-manifold Y is an oriented link $L \subseteq Y$, together with two disjoint and finite collections of basepoints $\mathbf{w}, \mathbf{z} \subseteq L$ such that the following hold:

(1) Each component of L has at least two basepoints.
(2) The basepoints alternate between $\mathbf{w}$ and $\mathbf{z}$, as one traverses L.

To a multi-based link $\mathbb{L}$ in Y, the link Floer homology group

$$\widehat{HFL}(Y, \mathbb{L})$$

is a vector space over $\mathbb{F}_2$. If $\mathbb{K} = (K, w, z)$ is a doubly based knot in S^3, the group $\widehat{HFL}(S^3, \mathbb{K})$ coincides with the knot Floer homology group $\widehat{HFK}(K)$. The group $\widehat{HFL}(Y, \mathbb{L})$ decomposes along Spin^c structures as

$$\widehat{HFL}(Y, \mathbb{L}) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} \widehat{HFL}(Y, \mathbb{L}, \mathfrak{s}),$$

as we outline below.

We briefly describe the construction of link Floer homology. One starts with a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for $\mathbb{L}$; see [OS08, §3.5] for the definition of a Heegaard diagram of a multi-based link. Write

$$\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_n\} \qquad \text{and} \qquad \boldsymbol{\beta} = \{\beta_1, \dots, \beta_n\},$$

where $n = g(\Sigma) + |\mathbf{w}| - 1 = g(\Sigma) + |\mathbf{z}| - 1$, and consider the two half-dimensional tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_n \qquad \text{and} \qquad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_n$$

inside of the symmetric product $\text{Sym}^n(\Sigma)$.

There is a map $\mathfrak{s}_{\mathbf{w}} \colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \text{Spin}^c(Y)$ defined by Ozsváth and Szabó [OS04c, §2.6]. As a module over $\mathbb{F}_2$, the chain complex $\widehat{CFL}(Y, \mathbb{L}, \mathfrak{s})$ is freely generated by the intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ that satisfy $\mathfrak{s}_{\mathbf{w}}(\mathbf{x}) = \mathfrak{s}$. The differential ∂ on $\widehat{CFL}(Y, \mathbb{L}, \mathfrak{s})$ is defined by counting holomorphic disks in

$\mathrm{Sym}^n(\Sigma)$ with zero multiplicity on $\mathbf{w}$ and $\mathbf{z}$:

$$\partial \mathbf{x} = \sum_{\mathbf{y}\in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi\in \pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1 \\ n_{\mathbf{w}}(\phi)=n_{\mathbf{z}}(\phi)=0}} \#(\mathcal{M}(\phi)/\mathbb{R})\cdot \mathbf{y}. \tag{6}$$

The definition of link Floer homology can be extended to disconnected manifolds via a tensor product, as long as each component of the 3-manifold contains a component of the link. By convention, we set

$$\widehat{HFL}(\emptyset) := \mathbb{F}_2.$$

Functorial cobordism maps for the hat flavor of link Floer homology were constructed by Juhász [Juh16]. Juhász's construction made use of the contact gluing map defined by Honda, Kazez and Matić [HKM08]. The present author [Zem19b] gave an alternate construction of link cobordism maps in terms of elementary cobordisms. The construction is independent of the contact-geometric construction of Honda, Kazez and Matić. In a joint work with Juhász, the author showed that the two constructions yield the same cobordism maps [JZ18, Th. 1.4].

Juhász's link Floer TQFT uses the following notion of decorated link cobordism between two multi-based links:

Definition 2.2. Let Y_0 and Y_1 be 3-manifolds containing multi-based links $\mathbb{L}_0 = (L_0, \mathbf{w}_0, \mathbf{z}_0)$ and $\mathbb{L}_1 = (L_1, \mathbf{w}_1, \mathbf{z}_1)$, respectively. A *decorated link cobordism* from $(Y_0, \mathbb{L}_0)$ to $(Y_1, \mathbb{L}_1)$ is a pair $(W, \mathcal{F}) = (W, (\Sigma, \mathcal{A}))$, satisfying the following:

(1) W is an oriented cobordism from Y_0 to Y_1.
(2) Σ is a properly embedded, oriented surface in W with $\partial\Sigma = -L_0 \cup L_1$.
(3) $\mathcal{A}$ is a properly embedded 1-manifold in Σ that divides Σ into two subsurfaces $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$ that meet along $\mathcal{A}$, such that $\mathbf{w}_0, \mathbf{w}_1 \subseteq \Sigma_{\mathbf{w}}$ and $\mathbf{z}_0, \mathbf{z}_1 \subseteq \Sigma_{\mathbf{z}}$.

Using the constructions from [Juh16] and [Zem19b], if $\mathfrak{s} \in \mathrm{Spin}^c(W)$, there is a functorial cobordism map

$$F_{W,\mathcal{F},\mathfrak{s}} \colon \widehat{HFL}(Y_0, \mathbb{L}_0, \mathfrak{s}|_{Y_0}) \to \widehat{HFL}(Y_1, \mathbb{L}_1, \mathfrak{s}|_{Y_1}).$$

When $\mathrm{Spin}^c(W)$ contains only one element, $\mathfrak{s}$, we write simply

$$F_{W,\mathcal{F}} := F_{W,\mathcal{F},\mathfrak{s}}.$$

To a concordance C from K_0 to K_1, we decorate K_0 and K_1 each with a pair of basepoints, and we obtain a decorated link cobordism $([0,1]\times S^3, \mathcal{C})$ by decorating C with two parallel dividing arcs, both going from K_0 to K_1. This configuration is studied in [JM16]. The choice of such dividing arcs is not canonical, since we can always apply a Dehn twist along a homotopically nontrivial curve in C. Hence, if C is an undecorated concordance, the induced cobordism map is only well defined up to the automorphisms of knot Floer homology induced by the diffeomorphisms that twist K_0 or K_1 in one full twist. Note that composition with a grading preserving automorphism does not affect the statement of Theorem 1.1. The basepoint moving automorphism map has been studied by Sarkar [Sar15] and by the author [Zem17].

An important property of the link Floer TQFT is that cobordisms with non-connected ends are allowed. This fact will be important in our proof of Theorem 1.1. The cobordism map associated to the disjoint union of two link cobordisms is the tensor product of the two link cobordism maps.

Next, we discuss gradings. If $\mathbb{L}$ is a null-homologous link in Y (i.e., the total class of $\mathbb{L}$ vanishes in $H_1(Y;\mathbb{Z})$) and $\mathfrak{s}$ is a torsion Spin^c structure on Y, Ozsváth and Szabó construct two gradings on link Floer homology: the Maslov and Alexander gradings. In the framework of our TQFT, it is convenient to repackage these two gradings into three gradings, $\mathrm{gr}_{\mathbf{w}}$, $\mathrm{gr}_{\mathbf{z}}$ and A, which satisfy the linear dependency

$$A = \frac{1}{2}(\mathrm{gr}_{\mathbf{w}} - \mathrm{gr}_{\mathbf{z}}).$$

The Maslov grading described by Ozsváth and Szabó is equal to $\mathrm{gr}_{\mathbf{w}}$, in our notation.

The cobordism maps are graded, and the author [Zem19a, Th. 1.4] showed that if $\mathfrak{s}|_{Y_0}$ and $\mathfrak{s}|_{Y_1}$ are torsion, and $\mathbb{L}_0$ and $\mathbb{L}_1$ are null-homologous, then

$$\mathrm{gr}_{\mathbf{w}}(F_{W,\mathcal{F},\mathfrak{s}}(\mathbf{x})) - \mathrm{gr}_{\mathbf{w}}(\mathbf{x}) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4} + \widetilde{\chi}(\Sigma_{\mathbf{w}}) \tag{7}$$

and

$$\mathrm{gr}_{\mathbf{z}}(F_{W,\mathcal{F},\mathfrak{s}}(\mathbf{x})) - \mathrm{gr}_{\mathbf{z}}(\mathbf{x}) = \frac{c_1(\mathfrak{s} - PD[\Sigma])^2 - 2\chi(W) - 3\sigma(W)}{4} + \widetilde{\chi}(\Sigma_{\mathbf{z}}), \tag{8}$$

where

$$\widetilde{\chi}(\Sigma_{\mathbf{w}}) := \chi(\Sigma_{\mathbf{w}}) - \frac{1}{2}(|\mathbf{w}_0| + |\mathbf{w}_1|),$$

and $\widetilde{\chi}(\Sigma_{\mathbf{z}})$ is defined similarly. Special cases of the above grading formulas were independently proven by Juhász and Marengon [JM18], when $W = [0,1] \times S^3$.

A final property that we will need concerns the behavior of the cobordism maps for 2-knots in S^4:

Lemma 2.3. *Suppose $(S^4, \mathcal{S})\colon \emptyset \to \emptyset$ is a decorated link cobordism such that $\mathcal{S}$ is a smooth 2-knot decorated with a single dividing curve. The induced map*

$$F_{S^4,\mathcal{S}}\colon \mathbb{F}_2 \to \mathbb{F}_2$$

is an isomorphism.

Lemma 2.3 follows from [JM16, Th. 1.2]. Alternatively, we can view it as a consequence of a more general formula for the behavior of the cobordism maps applied to closed surfaces, due to the author; see Lemma 4.1, below.

3. Proof of Theorem 1.1

Having reviewed the necessary background, we now prove our main result:

Proof of Theorem 1.1. Suppose C is a ribbon concordance from K_0 to K_1. Let $\mathcal{C}$ denote C, decorated with two parallel dividing arcs running from K_0 to K_1.

Consider the concordance C' from K_1 to K_0 obtained by turning around and reversing the orientation of C. Let $\mathcal{C}'$ denote the concordance C' with the decorations induced by $\mathcal{C}$. Write $C' \circ C$ for the concordance from K_0 to itself obtained by concatenating C and C', and write $\mathcal{C}' \circ \mathcal{C}$ for $C' \circ C$ decorated with the arcs from $\mathcal{C}$ and $\mathcal{C}'$.

We claim that

$$F_{[0,1]\times S^3,\mathcal{C}'} \circ F_{[0,1]\times S^3,\mathcal{C}} = F_{[0,1]\times S^3,\mathcal{C}'\circ\mathcal{C}} = \mathrm{id}_{\widehat{HFK}(K_0)}. \tag{9}$$

Note that equation (9) immediately implies Theorem 1.1. The first equality in equation (9) follows from the composition law for link cobordisms, so it remains to prove the second.

The concordance $C' \circ C$ will not in general be isotopic to the product

$$[0,1] \times K_0 \subseteq [0,1] \times S^3.$$

Nonetheless, the link Floer TQFT cannot tell the difference, as we now precisely describe.

Pick a movie presentation of C, with the following form:

(M-1) n births, adding unknots $U_1, \dots, U_n$;

(M-2) n saddles, for bands $B_1, \dots, B_n$, such that B_i connects U_i to K_0;

(M-3) an isotopy taking the band surgered knot $(K_0 \cup U_1 \cup \cdots \cup U_n)(B_1, \dots, B_n)$ to K_1.

Such a movie can be obtained by taking the concordance C (which by assumption has only index 0 and 1 critical points) and moving the index 0 critical points below the index 1 critical points. A-priori the bands induced by the index 1 critical points may not have one end on K_0 and one end on one of $U_1, \dots, U_n$. However, after a sequence of band slides, it is easy to arrange for this configuration.

The concordance $C' \circ C$ can be given a movie by concatenating the above movie with its reverse. In this movie for $C' \circ C$, we run the isotopy from (M-3) forward in the C-portion of the movie and then immediately run it backwards in the C'-portion. Consequently, we can delete the two adjacent levels corresponding to isotopy in the movie for $C' \circ C$ and obtain the following movie for $C' \circ C$:

(M'-1) n births, adding $U_1, \dots, U_n$;
(M'-2) n saddles, for the bands $B_1, \dots, B_n$;
(M'-3) n saddles, for bands $B'_1, \dots, B'_n$ obtained by reversing $B_1, \dots, B_n$;
(M'-4) n deaths, deleting $U_1, \dots, U_n$.

If we were to omit all $2n$ bands, the births and deaths determine 2-spheres, $S_1, \dots, S_n$. For each i, the band B_i, together with its reverse B'_i, determines a tube (i.e., an annulus), for which we write T_i. Consequently, we can view the concordance $C' \circ C$ as being obtained by taking the identity concordance $[0,1] \times K_0$, and tubing in the spheres $S_1, \dots, S_n$ using the tubes $T_1, \dots, T_n$. Although the 2-spheres S_i are individually unknotted, the tubes T_i may link the spheres S_i in a complicated manner.

We view the tubes as the boundaries of 3-dimensional 1-handles $h_1, \dots, h_n$ embedded in $[0,1] \times S^3$ that join the surface $[0,1] \times K_0$ to the spheres $S_1, \dots, S_n$. The 1-handle h_i intersects S_i in a disk D_i and intersects $[0,1] \times K_0$ in a disk D_i^0. Define $D'_i := S_i \setminus \operatorname{int}(D_i)$.

The concordance $C' \circ C$ is equal to the union

$$C' \circ C = \left([0,1] \times K_0 \setminus \left(D_1^0 \cup \cdots \cup D_n^0 \right) \right) \cup (T_1 \cup \cdots \cup T_n) \cup (D'_1 \cup \cdots \cup D'_n) . \tag{10}$$

If we replace the expression $D'_1 \cup \cdots \cup D'_n$ appearing in equation (10) with $D_1 \cup \cdots \cup D_n$, we obtain a surface that is isotopic to the identity concordance $[0,1] \times K_0$; see Figure 2.

We now claim that replacing D'_i with D_i does not change the cobordism map. Let $N(S_i)$ denote a regular neighborhood of S_i, and let Y_i denote the boundary of $N(S_i)$. Note that

$$N(S_i) \cong D^2 \times S^2 \qquad \text{and} \qquad Y_i \cong S^1 \times S^2.$$

The concordance $C' \circ C$ intersects Y_i in an unknot O_i (equal to the intersection of T_i with Y_i). We isotope the dividing set on $\mathcal{C}' \circ \mathcal{C}$ so that it intersects O_i in two points and intersects D'_i in a single arc. Let $\mathcal{D}'_i$ denote D'_i decorated with this dividing arc, and let $\mathcal{D}_i$ denote D_i decorated with a single dividing arc. Let $\mathbb{O}_i$ denote the unknot $O_i \subseteq Y_i$, decorated with two basepoints that are compatible with the dividing arcs.

We now claim that

$$F_{N(S_i),\mathcal{D}'_i,\mathfrak{t}_0} = F_{N(S_i),\mathcal{D}_i,\mathfrak{t}_0}, \tag{11}$$

as maps from $\widehat{HFL}(\emptyset) \cong \mathbb{F}_2$ to $\widehat{HFL}(Y_i, \mathbb{O}_i, \mathfrak{s}_0)$, where $\mathfrak{t}_0 \in \mathrm{Spin}^c(N(S_i))$ denotes the Spin^c structure that evaluates trivially on S_i, and $\mathfrak{s}_0$ denotes its restriction to Y_i.

We note that the link Floer TQFT allows cobordisms that have disconnected or empty incoming and outgoing ends. Furthermore, the cobordism map for a disjoint union of link cobordisms is the tensor product of the link cobordism maps for each connected component. Hence, equation (11), together with the composition law for link cobordisms [Zem19b, Th. B], implies the main claim.

We prove equation (11) in the subsequent Lemma 3.1. □

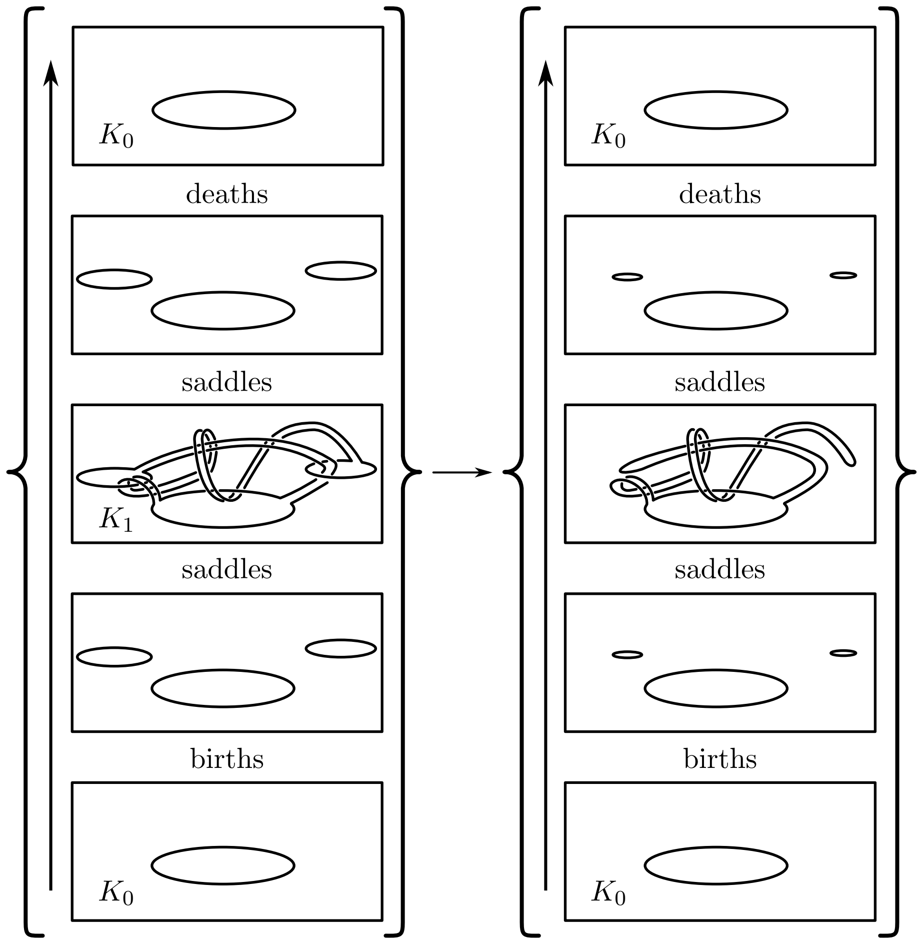

Figure 2. *The modification of $C' \circ C$ from the proof of Theorem 1.1*. On the left is a movie for $C' \circ C$. On the right is a movie for the concordance obtained by replacing the disks D_i' with D_i. The concordance on the right is isotopic to $[0, 1] \times K_0$.

Lemma 3.1. *Suppose D and D' are two smooth, properly embedded disks in $W := D^2 \times S^2$ that intersect $S^1 \times S^2$ in an unknot O. Let $\mathbb{O}$ denote O decorated with two basepoints, and let $\mathcal{D}$ and $\mathcal{D}'$ denote D and D' decorated with a single dividing arc, compatibly with the basepoints of $\mathbb{O}$. Let $\mathfrak{t}_0$ denote the unique* Spin^c *structure on W whose Chern class evaluates trivially on $\{0\} \times S^2$, and let $\mathfrak{s}_0$ denote its restriction to $S^1 \times S^2$. Then*

$$F_{W,\mathcal{D},\mathfrak{t}_0} = F_{W,\mathcal{D}',\mathfrak{t}_0}, \tag{12}$$

as maps from $\mathbb{F}_2 \cong \widehat{HFK}(\emptyset)$ to $\widehat{HFK}(S^1 \times S^2, \mathbb{O}, \mathfrak{s}_0)$.

Proof. A Heegaard diagram for $(S^1 \times S^2, \mathbb{O})$ is shown in Figure 3.

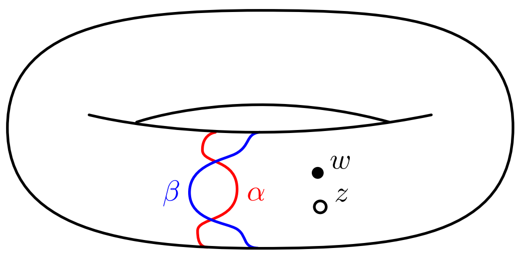

Figure 3. *A diagram for the unknot $\mathbb{O}$ in $S^1 \times S^2$.* The two intersection points both represent the torsion Spin^c structure.

Since $\mathbb{O}$ is a doubly based unknot, the gradings $\mathrm{gr}_{\mathbf{w}}$ and $\mathrm{gr}_{\mathbf{z}}$ coincide on the generators of $\widehat{HFK}(S^1 \times S^2, \mathbb{O}, \mathfrak{s}_0)$. Indeed both $\mathrm{gr}_{\mathbf{w}}$ and $\mathrm{gr}_{\mathbf{z}}$ are defined on intersection points using the same formula, since the basepoints are immediately adjacent. We refer to the common grading on $\widehat{HFK}(S^1 \times S^2, \mathbb{O}, \mathfrak{s}_0)$ simply as the Maslov grading. Note that

$$\widehat{HFK}(S^1 \times S^2, \mathbb{O}, \mathfrak{s}_0) \cong (\mathbb{F}_2)_{-\frac{1}{2}} \oplus (\mathbb{F}_2)_{\frac{1}{2}},$$

where $(\mathbb{F}_2)_p$ denotes a rank 1 summand of $\mathbb{F}_2$, concentrated in Maslov grading $p \in \mathbb{Q}$.

We make two claims:

(c-1) Both $F_{W,\mathcal{D},\mathfrak{t}_0}(1)$ and $F_{W,\mathcal{D}',\mathfrak{t}_0}(1)$ have Maslov grading $-\frac{1}{2}$.
(c-2) $F_{W,\mathcal{D},\mathfrak{t}_0}(1)$ and $F_{W,\mathcal{D}',\mathfrak{t}_0}(1)$ are both non-zero.

Claim (c-1) follows from the grading change formulas in equations (7) and (8). (Both formulas give the same answer.)

Claim (c-2) is proven as follows: Let $(W_0, \mathcal{D}_0) \colon (S^1 \times S^2, \mathbb{O}) \to \emptyset$ denote a decorated link cobordism, where W_0 is a 3-handle cobordism followed by a 4-handle, and $\mathcal{D}_0$ is a smooth disk, decorated with a single dividing arc. Note that $W_0 \cup W$ is diffeomorphic to S^4. Write $\mathcal{S}$ and $\mathcal{S}'$ for the 2-spheres $\mathcal{D}_0 \cup \mathcal{D}$ and $\mathcal{D}_0 \cup \mathcal{D}'$, respectively. By the composition law,

$$F_{S^4,\mathcal{S}} = F_{W_0,\mathcal{D}_0} \circ F_{W,\mathcal{D},\mathfrak{t}_0}. \tag{13}$$

However $F_{S^4,\mathcal{S}}\colon \mathbb{F}_2 \to \mathbb{F}_2$ is an isomorphism by Lemma 2.3. Consequently $F_{W,\mathcal{D},\mathfrak{t}_0}(1)$ must be non-zero in light of equation (13). The same argument shows $F_{W,\mathcal{D}',\mathfrak{t}_0}(1)$ is non-zero, so Claim (c-2) holds.

Noting that $\widehat{HFK}(S^1\times S^2,\mathbb{O},\mathfrak{s}_0)$ has rank 1 in Maslov grading $-\frac{1}{2}$, Claims (c-1) and (c-2) imply that $F_{W,\mathcal{D},\mathfrak{t}_0} = F_{W,\mathcal{D}',\mathfrak{t}_0}$, completing the proof. □

4. Extension to the full knot Floer complex

In this section, we prove Theorem 1.7. The argument is only notationally harder than the one we gave for Theorem 1.1. We first recall the definition of the relevant version of the full knot Floer complex and the cobordism maps from [Zem19b], and we state some basic properties.

If $\mathbb{L}$ is a multi-based link in Y, we let $\mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s})$ denote the module that is freely generated over the two variable polynomial ring $\mathbb{F}_2[u,v]$ by intersection points $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ with $\mathfrak{s}_{\mathbf{w}}(\mathbf{x})=\mathfrak{s}$. Adapting equation (6), we equip $\mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s})$ with the differential

$$\partial\mathbf{x} = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R})\cdot u^{n_{\mathbf{w}}(\phi)}v^{n_{\mathbf{z}}(\phi)}\cdot\mathbf{y},$$

where $n_{\mathbf{w}}(\phi)$ and $n_{\mathbf{z}}(\phi)$ denote the total multiplicity of the class ϕ on the basepoints $\mathbf{w}$ and $\mathbf{z}$.

Note that $\widehat{CFL}(Y,\mathbb{L},\mathfrak{s})$ is obtained from $\mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s})$ by setting $u=v=0$. A complex $\mathcal{CFL}^\infty(Y,\mathbb{L},\mathfrak{s})$ is defined by formally inverting u and v in the module $\mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s})$.

The gradings $\mathrm{gr}_{\mathbf{w}}$, $\mathrm{gr}_{\mathbf{z}}$ and A described in Section 2 all have incarnations on the minus and infinity flavors. On intersection points, their definitions coincide with the gradings on $\widehat{CFL}(Y,\mathbb{L},\mathfrak{s})$. They are extended to $\mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s})$ by defining u to have $(\mathrm{gr}_{\mathbf{w}},\mathrm{gr}_{\mathbf{z}})$-bigrading $(-2,0)$ and defining v to have $(\mathrm{gr}_{\mathbf{w}},\mathrm{gr}_{\mathbf{z}})$-bigrading $(0,-2)$. The Alexander grading satisfies $A=\frac{1}{2}(\mathrm{gr}_{\mathbf{w}}-\mathrm{gr}_{\mathbf{z}})$.

Let $\mathbb{K}$ denote a knot K in S^3, decorated with two basepoints. The full knot Floer complex $CFK^\infty(K)$, as defined in [OS04b], is equal to the subcomplex of $\mathcal{CFL}^\infty(S^3,\mathbb{K})$ in Alexander grading zero. The actions of u and v are not individually well defined on $CFK^\infty(K)$, since they have non-zero Alexander grading. Nonetheless, the group $CFK^\infty(K)$ can be given an action of $\mathbb{F}_2[U,U^{-1}]$, by having U act by the product uv.

The module $\mathcal{CFL}^\infty(S^3,\mathbb{K})$ has a natural $\mathbb{Z}\oplus\mathbb{Z}$-filtration $\mathcal{G}_{(n,m)}(\mathbb{K})\subseteq\mathcal{CFL}^\infty(S^3,\mathbb{K})$, generated by monomials $u^iv^j\cdot\mathbf{x}$ with $i\ge n$ and $j\ge m$. The $\mathbb{Z}\oplus\mathbb{Z}$-filtration on $CFK^\infty(K)$ is then given as the intersection of $\mathcal{G}_{(n,m)}(\mathbb{K})$ with $CFK^\infty(K)$, or equivalently, the homogeneous subset of $\mathcal{G}_{(n,m)}(\mathbb{K})$ in Alexander grading zero.

To a decorated link cobordism $(W,\mathcal{F})\colon (Y_0,\mathbb{L}_0)\to(Y_1,\mathbb{L}_1)$, equipped with a Spin^c structure $\mathfrak{s}\in \mathrm{Spin}^c(W)$, the author [Zem19b] constructs functorial cobordism maps

$$F_{W,\mathcal{F},\mathfrak{s}}\colon \mathcal{CFL}^-(Y_0,\mathbb{L}_0,\mathfrak{s}|_{Y_0})\to \mathcal{CFL}^-(Y_1,\mathbb{L}_1,\mathfrak{s}|_{Y_1}).$$

If $\mathbb{L}_0$ and $\mathbb{L}_1$ are both null-homologous and $\mathfrak{s}|_{Y_0}$ and $\mathfrak{s}|_{Y_1}$ are torsion (so $\mathrm{gr}_{\mathbf{w}}$ and $\mathrm{gr}_{\mathbf{z}}$ are defined), then the grading formulas from equations (7) and (8) hold [Zem19a, Th. 1.4].

Next, we recall the form of the maps for closed surfaces in S^4, as computed by the author [Zem19a, Th. 1.8]:

Lemma 4.1. *Suppose that $\mathcal{S}=(\Sigma,\mathcal{A})$ is a closed, decorated surface in S^4 such that $\Sigma\setminus\mathcal{A}$ has two connected components, $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$. Then the cobordism map*

$$F_{S^4,\mathcal{S}}\colon \mathbb{F}_2[u,v]\to\mathbb{F}_2[u,v]$$

is equal to the map

$$1\mapsto u^{g(\Sigma_{\mathbf{w}})}v^{g(\Sigma_{\mathbf{z}})}.$$

In particular, when $\mathcal{S}$ is a 2-knot, the cobordism map is the identity.

We are now equipped to prove Theorem 1.7:

Proof of Theorem 1.7. Suppose C is a ribbon concordance from K_0 to K_1, and let $\mathcal{C}$ and $\mathcal{C}'$ be decorations of C and C', as in the proof of Theorem 1.1. The proof of the present theorem amounts to showing that

$$F_{[0,1]\times S^3,\mathcal{C}'}\circ F_{[0,1]\times S^3,\mathcal{C}}\simeq F_{[0,1]\times S^3,\mathcal{C}'\circ\mathcal{C}}\simeq \mathrm{id}_{\mathcal{CFL}^-(K_0)}. \tag{14}$$

The proof of equation (14) follows the proof of Theorem 1.1 verbatim until Lemma 3.1. We claim that Lemma 3.1 holds with $\widehat{CFL}(S^1\times S^2,\mathbb{O},\mathfrak{s}_0)$ replaced by $\mathcal{CFL}^-(S^1\times S^2,\mathbb{O},\mathfrak{s}_0)$.

Note that $\mathrm{gr}_{\mathbf{w}}$ and $\mathrm{gr}_{\mathbf{z}}$ do not coincide on $\mathcal{CFL}^-(S^1\times S^2,\mathbb{O},\mathfrak{s}_0)$, since u and v have non-zero $(\mathrm{gr}_{\mathbf{w}},\mathrm{gr}_{\mathbf{z}})$-bigrading. We claim that the following analogs of Claims (c-1) and (c-2) hold:

(c'-1) Both $F_{W,\mathcal{D},\mathfrak{t}_0}(1)$ and $F_{W,\mathcal{D}',\mathfrak{t}_0}(1)$ have $(\mathrm{gr}_{\mathbf{w}},\mathrm{gr}_{\mathbf{z}})$-bigrading $(-\frac{1}{2},-\frac{1}{2})$ in $\mathcal{CFL}^-(S^1\times S^2,\mathbb{O},\mathfrak{s}_0)$.

(c'-2) $F_{W,\mathcal{D},\mathfrak{t}_0}(1)$ and $F_{W,\mathcal{D}',\mathfrak{t}_0}(1)$ are both non-zero.

Claim (c'-1) follows from the grading formulas in equations (7) and (8), as before.

Claim (c'-2) is proven similarly to Claim (c-2), except using Lemma 4.1 for the maps induced by 2-knots.

Finally, we note that

$$\mathcal{CFL}^-(S^1\times S^2,\mathbb{O},\mathfrak{s}_0)\cong\Big((\mathbb{F}_2)_{(-\frac{1}{2},-\frac{1}{2})}\oplus(\mathbb{F}_2)_{(\frac{1}{2},\frac{1}{2})}\Big)\otimes_{\mathbb{F}_2}\mathbb{F}_2[u,v],$$

with vanishing differential. (See Figure 3 for a diagram.) In the above equation, $(\mathbb{F}_2)_{(p,q)}$ denotes a summand of $\mathbb{F}_2$ concentrated in $(\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$-bigrading (p, q). The homogeneous subset in $(\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$-bigrading $(-\frac{1}{2}, -\frac{1}{2})$ has rank 1, so Claims (c'-1) and (c'-2) force

$$F_{W,\mathcal{D},\mathfrak{t}_0}(1) = F_{W,\mathcal{D}',\mathfrak{t}_0}(1)$$

as elements of $\mathcal{CFL}^-(S^1 \times S^2, \mathbb{O}, \mathfrak{s}_0)$. Using the composition law, the proof is complete.

□

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(Received: February 17, 2019)
(Revised: July 17, 2019)

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