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Counterexamples to Hedetniemi's conjecture

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Abstract

The chromatic number of $G \times H$ can be less than the minimum of the chromatic numbers of finite simple graphs G and H.

The *tensor product* $G \times H$ of finite simple graphs G and H has vertex set $V(G) \times V(H)$, and pairs (g, h) and (g', h') are adjacent if and only if $\{g, g'\} \in E(G)$ and $\{h, h'\} \in E(H)$. One can easily see that $\chi(G \times H) \leqslant \chi(G)$ because a proper coloring Ψ of the graph G can be lifted to the coloring $(g, h) \to \Psi(g)$ of $G \times H$. Similarly, a proper coloring of H leads to a proper coloring of $G \times H$ with the same number of colors, so we get

$$\text{(H)} \qquad \chi(G \times H) \leqslant \min\{\chi(G), \chi(H)\}.$$

The classical conjecture of S. T. Hedetniemi [8] posited the equality for all G and H. More than fifty years have passed since the conjecture appeared, and it keeps attracting serious attention of researchers working in graph theory and combinatorics; we mention four exhaustive survey papers [9], [12], [14], [18] for more detailed information on the topic. Here, we briefly recall that Hedetniemi's conjecture was proved in many special cases, including graphs with chromatic number at most four [3], graphs containing large cliques [1], [2], [16], circular graphs and products of cycles [15], and Kneser graphs and hypergraphs [6]. The conjecture gave an impetus to the study of *multiplicative graphs*, which remains remarkably active and important in its own right [5], [13], [17]. A generalization of Hedetniemi's conjecture to fractional chromatic numbers turned out to be true [19], but the version with directed graphs is false [10], as well as the one with infinite chromatic numbers [7], [11]. We show that the inequality (H) can be strict for finite simple graphs.

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A standard tool in the study of Hedetniemi's conjecture is the concept of the *exponential graph* as introduced in [3]. Let c be a positive integer, and let Γ be a finite graph that we allow to contain loops; the graph $\mathcal{E}_c(\Gamma)$ has all mappings $V(\Gamma) \to \{1, \ldots, c\}$ as vertices, and two distinct mappings φ, ψ are adjacent if, and only if, the condition $\varphi(x) \neq \psi(y)$ holds whenever $\{x, y\} \in E(\Gamma)$. The relevance of $\mathcal{E}_c(\Gamma)$ to the problem is easy to see because the graph $\Gamma \times \mathcal{E}_c(\Gamma)$ has the proper c-coloring $(h, \psi) \to \psi(h)$. The idea of our approach lies in the fact that the proper c-colorings of $\mathcal{E}_c(\Gamma)$ become quite well behaved if the graph Γ is fixed and c gets large; let us proceed to technical details and exact statements. A basic result in [3] tells that the constant mappings form a c-clique in $\mathcal{E}_c(\Gamma)$, which means that these mappings get different colors in a proper c-coloring. So a relabeling of colors can turn any proper c-coloring $\Psi : \mathcal{E}_c(\Gamma) \to \{1, \ldots, c\}$ into a *suited* one, in which a color i is assigned to the constant mapping sending every vertex of Γ to i.

Observation 1. *If Ψ is a suited proper c-coloring of $\mathcal{E}_c(\Gamma)$, then $\Psi(\varphi) \in \operatorname{Im} \varphi$.*

Proof. The mapping φ is adjacent to the constant mapping $\{v \to j\}$ for any j not in $\operatorname{Im} \varphi$, so φ cannot get colored with such a j. □

Claim 2. *Consider a graph Γ with $|V(\Gamma)| = n$ and a suited proper c-coloring Ψ of $\mathcal{E}_c(\Gamma)$. Then there is a vertex $v = v(\Psi)$ of Γ such that $\geqslant c - \sqrt[n]{n^3 c^{n-1}}$ color classes $\Psi^{-1}(b)$ are v-robust, which means that, for any $\varphi_b \in \Psi^{-1}(b)$, there is a $w \in N(v)$ satisfying $\varphi_b(w) = b$, where $N(v)$ stands for the closed neighborhood of v in Γ.*

Proof. For any color b and any vertex $u \in V(\Gamma)$, we define $I(u, b)$ as the set of all $\varphi \in \Psi^{-1}(b)$ that satisfy $\varphi(u) = b$. According to Observation 1, every vertex of $\mathcal{E}_c(\Gamma)$ belongs to at least one of the classes $I(u, b)$.

Assume that $I(u, b)$ is a *large* class, that is, it contains more than $n^2 c^{n-2}$ elements, and consider an arbitrary mapping $\varphi_b \in \Psi^{-1}(b)$. If every element ψ_{ub} of $I(u, b)$ admitted a vertex $u' \neq u$ with $\psi_{ub}(u') \in \operatorname{Im} \varphi_b$, then there would be at most n^2 ways to choose u' and $\psi_{ub}(u')$, while the remaining $n-2$ vertices would contribute at most a factor of c^{n-2}. This contradicts the cardinality assumption on $I(u, b)$, so we can actually find a $\psi_{ub} \in I(u, b)$ under which u is an only vertex taking the color b and also $\operatorname{Im} \varphi_b \cap \operatorname{Im} \psi_{ub} = \{b\}$. In other words, the equality $\Psi(\varphi) = b$ cannot hold unless there is a vertex $w \in N(u)$ satisfying $\varphi(w) = b$.

If there is a vertex $v \in V(\Gamma)$ for which $I(v, b)$ is large for at least $c - \sqrt[n]{n^3 c^{n-1}}$ colors b, then we are done. Conversely, we can define more than $n^3 c^{n-1}$ mappings $\varphi : V(\Gamma) \to \{1, \ldots, c\}$ for which the value of φ on a vertex w does not equal those colors b for which $I(w, b)$ is large. None of these mappings belongs to a large class $I(u, b)$, but the non-large classes are too small to cover all of them. □

Now we are ready to proceed with counterexamples. For a simple graph G, we define the graph Γ_G by adding the loops to all the vertices, and the *strong product* $G \boxtimes K_q$ as the graph with vertex set $V(G) \times \{1, \dots, q\}$ and edges between (u, i) and (v, j) when, and only when, $\{u, v\} \in E(G)$ or $(u = v)$ and $(i \neq j)$.

Claim 3. *Let G be a finite simple graph with finite girth $\geqslant 6$. Then, for sufficiently large q, one has $\chi(\mathcal{E}_c(G \boxtimes K_q)) > c$ with $c = \lceil 3.1q \rceil$.*

Proof. The restriction of a suited proper coloring $\Lambda : \mathcal{E}_c(G \boxtimes K_q) \to \{1, \dots, c\}$ to the mappings that are constant on the cliques $\{g\} \times K_q \subset G \boxtimes K_q$ is a proper coloring $\Psi : \mathcal{E}_c(\Gamma_G) \to \{1, \dots, c\}$ up to the identification of every such clique with g. We find a vertex $v = v(\Psi) \in V(G)$ as in Claim 2 and define the clique $\mathcal{M} = \{\mu_{q+1}, \dots, \mu_c\}$ in $\mathcal{E}_c(G \boxtimes K_q)$ by setting, for all $i \in \{1, \dots, q\}$ and $t \in \{q + 1, \dots, c\}$,

(1.1) $\mu_t(g, i) = i$ for all $g \in V(G)$ satisfying $\mathrm{dist}(v, g) \in \{0, 2\}$,

(1.2) $\mu_t(g, i) = q + i$ for all $g \in V(G)$ satisfying $\mathrm{dist}(v, g) = 1$,

(1.3) $\mu_t(g, i) = t$ for all $g \in V(G)$ satisfying $\mathrm{dist}(v, g) \geqslant 3$.

Due to the assumption on the girth of G, no pair of vertices defined in (1.1) and (1.2) can be adjacent in $G \boxtimes K_q$ and monochromatic at the same time; the condition (1.3) uses different colors for different t, and these colors are also different from those of the neighboring vertices dealt with in (1.1). Therefore, $\mathcal{M}$ is indeed a clique and requires $c - q \geqslant 2.1q$ colors. Using the pigeonhole principle, one finds a $\tau \in \{q + 1, \dots, c\}$ such that $\Lambda(\mu_\tau) \notin \{1, \dots, 2q\}$, and due to Observation 1 we have $\tau = \Lambda(\mu_\tau)$. Further, it is only $o(q)$ classes that are not v-robust with respect to Ψ in the terminology of Claim 2, so we can find a v-robust class $\sigma \notin \{1, \dots, 2q, \tau\}$. Finally, we note that the mapping $\nu : G \boxtimes K_q \to \{1, \dots, c\}$ defined as, for all i,

(2.1) $\nu(g, i) = \tau$ for all $g \in V(G)$ in the closed neighborhood $N(v)$,

(2.2) $\nu(g, i) = \sigma$ for all $g \in V(G)$ satisfying $\mathrm{dist}(v, g) \geqslant 2$,

is adjacent to μ_τ in $\mathcal{E}_c(G \boxtimes K_q)$. Since σ is v-robust, we cannot have $\Lambda(\nu) = \sigma$ by Claim 2, but rather we have $\Lambda(\nu) = \tau$ according to Observation 1. So we have $\Lambda(\nu) = \Lambda(\mu_\tau)$, which is a contradiction. □

The classical paper [4] proves the existence of graphs with arbitrarily large girth and fractional chromatic number; so we can find a graph G of girth at least 6 that satisfies $\chi_f(G) > 3.1$. We set $c = \lceil 3.1q \rceil$ and pass to sufficiently large q; we immediately get $\chi(G \boxtimes K_q) \geqslant q \cdot \chi_f(G) > c$ and also $\chi(\mathcal{E}_c(G \boxtimes K_q)) > c$

by Claim 3. The equality $\chi\left((G \boxtimes K_q) \times \mathcal{E}_c(G \boxtimes K_q)\right) = c$ follows by standard theory [3] as the mapping $(u, \varphi) \to \varphi(u)$ is a proper c-coloring of any graph of the form $\Gamma \times \mathcal{E}_c(\Gamma)$.

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