

On the dimension of Bernoulli convolutions for all transcendental parameters

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Dedicated to the memory of Jean Bourgain

Abstract

The Bernoulli convolution ν_λ with parameter $\lambda \in (0, 1)$ is the probability measure supported on \mathbf{R} that is the law of the random variable $\sum \pm \lambda^n$, where the \pm are independent fair coin-tosses. We prove that $\dim \nu_\lambda = 1$ for all transcendental $\lambda \in (1/2, 1)$.

Introduction

Fix a number $\lambda \in (0, 1)$ and let X_0, X_1, \dots be a sequence of independent random variables, which take the values ± 1 with equal probability. The Bernoulli convolution ν_λ with parameter λ is the probability measure on \mathbf{R} that is the law of the random variable of

$$\sum_{n=0}^{\infty} X_n \lambda^n.$$

Bernoulli convolutions are one of the most studied examples of self-similar measures, and they are objects of great interest in fractal geometry. The following two basic questions about them are still open.

QUESTION 1. *For which values of λ is ν_λ absolutely continuous with respect to the Lebesgue measure?*

QUESTION 2. *For which values of λ does $\dim \nu_\lambda = 1$ hold?*

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The latter question requires some explanation. Feng and Hu [8] proved that Bernoulli convolutions are exact dimensional, that is to say, there is a number, which we denote by $\dim \nu_\lambda$, such that

$$\lim_{r \rightarrow 0} \frac{\log \nu_\lambda([x-r, x+r])}{\log r} = \dim \nu_\lambda$$

for ν_λ -almost every x . It is worth noting that $\dim \nu_\lambda$ is also the infimum of the Hausdorff dimensions of all Borel subsets of \mathbf{R} that have positive ν_λ -measure; see [7, Prop. 10.2] for a proof of this fact, which holds for all exact dimensional measures.

For $\lambda < 1/2$, the Bernoulli convolution ν_λ is the Cantor-Lebesgue measure on a Cantor set; it is singular and has dimension $\log 2 / \log \lambda^{-1}$. For $\lambda > 1/2$, the above questions are still not completely understood. The study of [Question 1](#) goes back to Erdős [5], who exhibited the first and only known examples of parameters $\lambda \in (1/2, 1)$ that make the Bernoulli convolution singular. These are the inverses of Pisot numbers. Erdős [6] also proved that there is a number $a < 1$ such that ν_λ is absolutely continuous for almost all $\lambda \in (a, 1)$. Solomyak proved in the landmark paper [20] that one can take $a = 1/2$ in the above statement.

In this paper, we make progress on the second question. The above mentioned result of Solomyak implies that the set

$$E := \{\lambda \in (1/2, 1) : \dim \nu_\lambda < 1\}$$

is of 0 Lebesgue measure. Hochman proved in another landmark paper [12] that the set E is of 0 packing and Hausdorff dimension. We take this a step further and show that E is countable.

THEOREM 3. *We have $\dim \nu_\lambda = 1$ for all transcendental $\lambda \in (1/2, 1)$.*

The dimension of Bernoulli convolutions for algebraic parameters is not fully understood. The only known examples of parameters $\lambda \in (1/2, 1)$ with $\dim(\nu_\lambda) < 1$ are the inverses of Pisot numbers; this fact can be traced back to Garsia [10] in some form. Hochman [12] expressed the dimension for algebraic parameters in terms of the so-called Garsia entropy of λ , a quantity that has been studied recently in [4], [1]. We will briefly recall these results in [Section 1.4](#).

There is a folklore conjecture predicting that the dimension of self-similar measures equal their similarity dimension unless exact overlaps occur. See [12, §§1.1 and 1.2], where this conjecture is discussed. (See also [16, Question 2.6] for a closely related question about self-similar sets due to Simon.) [Theorem 3](#) together with the results of Hochman [12, Th. 1.5] for algebraic parameters establish this conjecture for Bernoulli convolutions.

It is not our aim to give a thorough discussion of the rich literature on Bernoulli convolutions. Instead, we refer the interested readers to the surveys [14], [21], [22], [11]. The reader interested in [Question 1](#) is also recommended to consult the recent papers [18], [19], [23].

Hochman [12] proved [Theorem 3](#) conditionally on the hypothesis that there is a number $C > 0$ such that any two algebraic numbers ξ_1 and ξ_2 that are roots of (not necessarily the same) polynomials of degree at most n with coefficients $-1, 0$ and 1 satisfy the separation condition $|\xi_1 - \xi_2| > C^{-n}$. This hypothesis is very plausible, because there are at most $n3^{n+1}$ such algebraic numbers; however, we do not know how to prove it.

On the other hand, we are able to prove the following weaker property. Let $\xi \in (1/2, 1/2^{2/3})$ be an algebraic number of degree at most d with Mahler measure at most M . (See [Section 1.1](#) for the definition.) Then there are numbers C_1, C_2, C_3 depending only on M such that the following holds. Let $n > C_1 d \log d$ be an integer, and let P be a polynomial of degree at most n with coefficients $-1, 0$ and 1 . Then $|P(\lambda)| > C_2^{-n}$ for all λ that satisfy $C_3^{-n} \leq |\lambda - \xi| \leq C_3^{-n+1}$.

The proof of [Theorem 3](#) relies heavily on the work of several mathematicians. The statement in the previous paragraph is proved using an estimate on the values polynomials with small coefficients evaluated on algebraic numbers, which was first used in the context of Bernoulli convolutions by Garsia [9], together with the transversality property of these polynomials proved by Solomyak [20]. (See also Peres and Solomyak [15].) The algebraic number ξ in the statement is found using a characterization of parameters with $\dim \nu_\lambda < 1$ by Breuillard and Varjú [3]. The Mahler measure of ξ is estimated using another paper [4] of the same authors. Then the conclusion of the above statement is plugged into a result of Hochman [12] to prove $\dim \nu_\lambda = 1$.

The paper is organized as follows. In [Section 1](#), we recall some facts from the above mentioned five papers. We then prove [Theorem 3](#) in [Section 2](#) in just a few strokes.

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1. Preliminaries

We recall some facts from the literature in this section, which will be used in the proof of [Theorem 3](#).

1.1. *Estimates for the values of polynomials.* Let λ be an algebraic number with minimal polynomial

$$a_d x^d + \cdots + a_1 x + a_0 = a_d (x - \lambda_1) \cdots (x - \lambda_d).$$

The Mahler measure of λ is defined as

$$M(\lambda) = |a_d| \prod_{j: |\lambda_j| > 1} |\lambda_j|.$$

For more on the basic properties of this quantity, we refer to [2, Ch. 1].

The utility of the following simple lemma (or a variant of it, rather — see [9, Lemma 1.51]) in the study of Bernoulli convolutions was first pointed out by Garsia.

THEOREM 4. *Let P be a polynomial of degree at most n with coefficients -1 , 0 and 1 , and let λ be an algebraic number of degree d . Suppose that $P(\lambda) \neq 0$. Then*

$$|P(\lambda)| \geq M(\lambda)^{-n} (n+1)^{1-d}.$$

We adapt the proof from [9, Lemma 1.51].

Proof. Let a_d and $\lambda = \lambda_1, \dots, \lambda_d$ be as above. Then

$$a_d^n \prod_{j=1}^d P(\lambda_j) \in \mathbf{Z}.$$

Indeed, it is clearly in \mathbf{Q} , since it is invariant under all automorphisms of $\overline{\mathbf{Q}}$. To show that it is an integer, it is enough to show that

$$\left| a_d^n \prod_{j=1}^d P(\lambda_j) \right|_v \leq 1$$

for all finite places v of $K(\lambda_1, \dots, \lambda_d)$.⁽¹⁾ Using that $|\cdot|_v$ is an ultrametric, we have

$$\left| a_d^n \prod_{j=1}^d P(\lambda_j) \right|_v \leq |a_d|_v^n \prod_{j=1}^d \max(|\lambda_j|_v^n, 1).$$

Applying Gauss's lemma (see, e.g., [2, Lemma 1.6.3]) for the product of polynomials $\prod_j (x - \lambda_j)$, we get

$$\prod_{j=1}^d \max(|\lambda_j|_v, 1) = |a_d|_v^{-1},$$

which combined with our previous inequality gives the claim.

⁽¹⁾ Indeed, if a number $a \in \mathbf{Q}$ is not an integer, then the denominator of a is divisible by a prime p , hence $|a|_p > 1$, and we also have $|a|_v > 1$ for all places v of $K(\lambda_1, \dots, \lambda_d)$ that lie above p . For basic properties of absolute values, we refer to [2, §§1.2, 1.3].

For each j , we can write

$$|P(\lambda_j)| \leq (n + 1) \max(1, |\lambda_j|)^n.$$

Using

$$\left| a_d^n \prod_{j=1}^d P(\lambda_j) \right| \geq 1,$$

we get

$$\begin{aligned} |P(\lambda_1)| &\geq a_d^{-n} \prod_{j=2}^d |P(\lambda_j)|^{-1} \\ &\geq (n + 1)^{1-d} a_d^{-n} \prod_{j=2}^d \max(1, |\lambda_j|)^{-n} \\ &\geq (n + 1)^{1-d} M(\lambda)^{-n}. \end{aligned} \quad \square$$

1.2. *Transversality.* It is clear that a polynomial with coefficients $-1, 0$ and 1 cannot have a root in the interval $(0, 1/2)$. It is natural to expect that there is a larger interval $(0, a)$ with $a > 1/2$, where such a polynomial may have at most one root. This is indeed true and was established by Solomyak [20] building on ideas from [17]. In fact, a slightly stronger property called transversality also holds, which we recall now.

Let $A \subset \mathbf{Z}_{>0}$. We write \mathcal{P}_A for the set of power series of the form

$$1 + \sum_{n \in A} a_n x^n,$$

where $a_n \in \{-1, 0, 1\}$. Let $x_0, \delta > 0$ be numbers. We say that the interval $[0, x_0]$ is an interval of δ -transversality for \mathcal{P}_A if for all $x \in [0, x_0]$ and $f \in \mathcal{P}_A$, the inequality $f(x) < \delta$ implies $f'(x) < -\delta$.

We note a consequence of this definition. If $f(x_1) < \delta$ for some $x_1 \in [0, x_0]$, then $f(x) < \delta - \delta(x - x_1)$ for all $x \in [x_1, x_0]$. Indeed, if this is not true, by continuity, there is a smallest $x \in [x_1, x_0]$ with $f(x) \geq \delta - \delta(x - x_1)$. Clearly $f(t) < \delta$ for all $t \in [x_1, x]$, which implies $f'(t) < -\delta$ by the definition of transversality, which contradicts the mean value theorem. This means, in particular, that a function $f \in \mathcal{P}_A$ may have at most one zero in an interval of δ -transversality.

There are three sets, which we will use in the role of A in this paper, and now we introduce short notation for them. We write $\mathcal{P} := \mathcal{P}_{\mathbf{Z}_{>0}}$ and

$$\mathcal{P}_i := \mathcal{P}_{\{n \in \mathbf{Z}_{>0} : 3 \nmid n - i\}}$$

for $i = 1, 2$.

We recall the following result from [20], [15].

THEOREM 5. *There is an absolute constant $\delta > 0$ such that $[0, 2^{-2/3}]$ is an interval of δ -transversality for \mathcal{P} and $[0, 2^{-1/2}]$ is an interval of δ -transversality for \mathcal{P}_1 and \mathcal{P}_2 .*

See the proof of Theorem 1 at the end of Section 3 in [15], where these statements are deduced from the lemma at the beginning of Section 3 in [15].

1.3. *Parameters with dimension drop I.* Hochman proved that parameters $\lambda \in [1/2, 1)$ with $\dim \nu_\lambda < 1$ can be approximated by algebraic numbers with high precision.

THEOREM 6 ([12, Th. 1.9]). *Suppose that $\dim \nu_\lambda < 1$ for some $\lambda \in [1/2, 1)$. Then for every $\theta \in (0, 1)$ and for all large enough n (depending on λ and θ), there is a polynomial $P \neq 0$ of degree at most n with coefficients $-1, 0$ and 1 such that $|P(\lambda)| < \theta^n$.*

Since $[0, 2^{-1/2}]$ is not an interval of δ -transversality for \mathcal{P} for any $\delta > 0$, we will need to borrow a trick from [15] and consider trimmed Bernoulli convolutions. For a parameter $\lambda \in (0, 1)$, we denote by $\tilde{\nu}_\lambda$ the law of the random variable $\sum_{n:3|n-2} X_n \lambda^n$, where X_n is a sequence of independent random variables taking the values ± 1 with equal probability. These trimmed Bernoulli convolutions satisfy an analogue of Theorem 6 as follows. We note that $\tilde{\nu}_\lambda$ are exact dimensional by [8], so we can talk about their dimensions in the same way as in the case of Bernoulli convolutions. We write \mathcal{Q} for the set of polynomials whose coefficients are $-1, 0$ and 1 and the coefficient of x^n is always 0 when $3 \nmid n - 2$.

THEOREM 7. *Suppose that $\dim \tilde{\nu}_\lambda < 1$ for some $\lambda \in [2^{-2/3}, 1)$. Then for every $\theta \in (0, 1)$ and for all large enough n (depending on λ and θ), there is a polynomial $P \neq 0 \in \mathcal{Q}$ of degree at most n such that $|P(\lambda)| < \theta^n$.*

This theorem can be deduced from [12, Th. 1.7] in exactly the same way as [12, Th. 1.9] (which we recalled above in Theorem 6).

1.4. *Entropy and Mahler measure.* Theorem 6 implies that $\dim \nu_\lambda = 1$ for all algebraic parameters $\lambda \in (1/2, 1)$ that are not roots of polynomials with coefficients $-1, 0$ and 1 . Moreover, Hochman [12] has the following even more precise result about algebraic parameters.

THEOREM 8. *Let $\lambda \in [1/2, 1)$ be algebraic. Then*

$$\dim \nu_\lambda = \min \left(1, \frac{h_\lambda}{\log \lambda^{-1}} \right),$$

where

$$h_\lambda = \lim_{N \rightarrow \infty} \frac{1}{N} H \left(\sum_{n=0}^{N-1} X_n \lambda^n \right)$$

and $H(\cdot)$ denotes the Shannon entropy of a discrete random variable.

See [4, §3.4], where this is formally deduced from the main result of Hochman [12].

Theorem 8 reduces Question 2 for algebraic parameters to determining when $h_\lambda \geq \log \lambda^{-1}$ holds. The quantity h_λ (also called Garsia entropy) received considerable attention recently. In particular, it was proved by Breuillard and Varjú [4] that

$$c \min(\log 2, \log M_\lambda) \leq h_\lambda \leq \min(\log 2, \log M_\lambda)$$

for an absolute constant $c > 0$. According to (not rigorous) numerical calculations reported in [4], the constant c can be taken as 0.44. A recent paper of Akiyama, Feng, Kempton and Persson [1] gives an algorithm that allows one to approximate h_λ with arbitrary precision with a finite computation. This means that for all algebraic parameters that satisfy $h_\lambda > \log \lambda^{-1}$, $\dim \nu_\lambda = 1$ can, in principle, be proved by finite computation.

The results in [4] also allow us to deduce the following.

THEOREM 9. *For any $h \in (0, \log 2)$, there is a number $C(h)$ such that $h_\lambda \geq h$ for all algebraic numbers λ with $M(\lambda) \geq C(h)$.*

Remark 10. This theorem is of independent interest. Following the proof below, one may compute an explicit function $C(h)$ with which the theorem holds and use it to give new examples of algebraic parameters that make the Bernoulli convolution have dimension 1.

Proof. Let X_0 be a random variable taking the values ± 1 with equal probability, and let G be an independent (from X_0) standard Gaussian random variable. For $a \in \mathbf{R}_{\geq 1}$, we define

$$\Phi(a) = \sup_{t>0} (H(X_0ta + G) - H(X_0t + G)),$$

where $H(\cdot)$ is now the differential entropy of an absolutely continuous random variable.

By [4, Prop. 13], we have $h_\lambda \geq \Phi(M(\lambda))$ for all algebraic numbers λ . It is easy to see that

$$\lim_{a \rightarrow \infty} H(X_0a^{-1/2} + G) = H(G)$$

and

$$\lim_{a \rightarrow \infty} H(X_0a^{1/2} + G) = \log 2 + H(G).$$

Plugging in $t = a^{-1/2}$ to the definition of Φ , we see that

$$\lim_{a \rightarrow \infty} \Phi(a) = \log 2.$$

(Here we also used the fact that $\Phi(a) \leq \log 2$, which can be seen for example from $h_\lambda \leq \log 2$.) This proves the claim. \square

1.5. *Parameters with dimension drop II.* Breuillard and Varjú also gave approximations for parameters $\lambda \in [1/2, 1)$ with $\dim \nu_\lambda < 1$ by algebraic numbers. The following is a simplified version of [3, Th. 1].

THEOREM 11. *Suppose that $\dim \nu_\lambda < 1$ for some $\lambda \in [1/2, 1)$. Then there are arbitrarily large integers d such that there is an algebraic number $\eta = \eta(d)$ with $\deg \eta \leq d$, $\dim \nu_\eta < 1$ and*

$$|\lambda - \eta| \leq \exp(-d^2).$$

There are a number of differences compared with [Theorem 6](#). The approximation is not claimed in [Theorem 11](#) on all sufficiently large scales, but only on a (possibly very sparse) sequence of scales. However, the approximating parameter satisfies $\dim \nu_\eta < 1$ and a better estimate for the distance to λ . Both of these features and the fact that the approximation is available at all (sufficiently large) scales in [Theorem 6](#) are critically important for the success of our proof of [Theorem 3](#).

2. Proof of Theorem 3

We begin by formalizing the statement about lower bounds on the values of polynomials that we made after [Theorem 3](#). Recall the notation \mathcal{Q} from [Section 1.3](#).

PROPOSITION 12. *Let $\xi \in (1/2, 2^{-1/2})$ be an algebraic number of degree at most d of Mahler measure at most M . Let $n > 10d \log d$ be an integer. Let $P \neq 0 \in \mathcal{Q}$ be a polynomial of degree at most n . Then $|P(\lambda)| > (20M)^{-n}$ for all λ satisfying $(5M)^{-n} \leq |\lambda - \xi| \leq (5M)^{-n+1}$ provided d is larger than an absolute constant. If $\xi \in (1/2, 2^{-2/3}]$, then the claim also holds for all polynomials $P \neq 0$ of degree at most n with coefficients $-1, 0$ and 1 .*

Proof. Suppose to the contrary that $|P(\lambda)| \leq (20M)^{-n}$, where λ is a number in the range specified by the proposition and $P \neq 0$ is a polynomial of degree at most n with coefficients in $-1, 0$ and 1 and $P \in \mathcal{Q}$ if $\xi > 2^{-2/3}$.

If $P(0) \neq 1$, we replace P by $\pm P/x^k$ for a suitable k such that $P(0) = 1$ holds. For this new P , we have $P \in \mathcal{P}$ and, moreover, $P \in \mathcal{P}_1 \cup \mathcal{P}_2$ if $\xi > 2^{-2/3}$, and we still have $|P(\lambda)| \leq (10M)^{-n}$. (Recall the definitions of \mathcal{P} , \mathcal{P}_1 and \mathcal{P}_2 from [Section 1.2](#).)

Let $\delta > 0$ be a number such that $(0, 2^{-2/3})$ is an interval of δ -transversality for \mathcal{P} and $(0, 2^{-1/2})$ is an interval of δ -transversality for \mathcal{P}_1 and \mathcal{P}_2 . Such a number exists by [Theorem 5](#).

We first show that $P(\xi) \neq 0$. If this is not the case, then $P(\xi), P(\lambda) < \delta$ (provided n is large enough), so we have $P'(t) < -\delta$ for all t between ξ and λ . Since $|\xi - \lambda| \geq (5M)^{-n}$, we have $|P(\xi) - P(\lambda)| > \delta(5M)^{-n}$ by the mean value theorem, a contradiction.

Now we can apply [Theorem 4](#) and conclude that

$$|P(\xi)| \geq (n + 1)^{1-d} M(\xi)^{-n} \geq (2M)^{-n}$$

since $n > 10d \log d$. Since $|P(\lambda)| \leq (10M)^{-n}$ and $|\lambda - \xi| \leq (5M)^{-n+1}$, we get a contradiction with $|P'(t)| \leq n^2$, which holds for all $t \in [0, 1]$. \square

Remark 13. After circulating a previous version of this paper, Vesselin Dimitrov pointed out to me that a variant of [Proposition 12](#) follows from a result of Mignotte [[13](#)], which gives an estimate for the distance between algebraic numbers if one of the degrees is much larger than the other. In this version, one may relax the condition $\xi \in (1/2, 2^{-2/3})$ at the expense of requiring $n > d(\log d)^2$. With this approach, one may avoid using the trimmed version of Bernoulli convolutions $\tilde{\nu}_\lambda$ later in the proof.

We turn to the proof of [Theorem 3](#). Let $\lambda \in (1/2, 1)$ be a transcendental number, and assume to the contrary that $\dim \nu_\lambda < 1$. We show that $\dim \nu_{\lambda^k} \leq \dim \nu_\lambda$ for any $k \in \mathbf{Z}_{>0}$. Note that $\nu_\lambda = \nu_{\lambda^k} * \mu$, where μ is the law of the random variable $\sum_{n:k \nmid n} \pm \lambda^n$. Now let E be a Borel set with $\nu_\lambda(E) > 0$. Then for t belonging to a set of positive μ -measure, we have $\nu_{\lambda^k}(E - t) > 0$, hence $\dim_H(E) = \dim_H(E - t) \geq \dim \nu_{\lambda^k}$. This proves $\dim \nu_\lambda \geq \dim \nu_{\lambda^k}$. Therefore, we can assume $\lambda < 2^{-1/2}$, for otherwise we can replace λ by λ^k for some positive integer k so that $\lambda^k \in (1/2, 2^{-1/2})$.

By [Theorem 11](#), there is an arbitrarily large integer d , such that there is an algebraic number ξ of degree at most d with $\dim \nu_\xi < 1$ and $|\xi - \lambda| \leq \exp(-d^2)$. By [Theorems 8 and 9](#), there is a number M depending only on λ (but crucially not on d) such that $M(\xi) \leq M$. Choose an integer n such that

$$(5M)^{-n} \leq |\lambda - \xi| \leq (5M)^{-n+1}.$$

If $\xi \leq 2^{-2/3}$, we apply [Theorem 6](#) with $\theta = (20M)^{-1}$ to find a polynomial $P \neq 0$ of degree at most n with coefficients $-1, 0$ and 1 such that $|P(\lambda)| < (20M)^{-n}$. Crucially, [Theorem 6](#) holds at every sufficiently large scale n . Now, we have a contradiction with [Proposition 12](#).

If $\xi > 2^{-2/3}$, then we note that $\dim \tilde{\nu}_\lambda \leq \dim \nu_\lambda < 1$, for ν_λ is the convolution of $\tilde{\nu}_\lambda$ with another measure and convolution may only increase the dimension of measures by the above argument. Hence, we can apply [Theorem 7](#) and find that there is a polynomial $P \neq 0 \in \mathcal{Q}$ of degree at most n such that $|P(\lambda)| < (20M)^{-n}$, and we reach a contradiction with [Proposition 12](#) again.

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