

# Ax-Schanuel for Shimura varieties

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## Abstract

We prove the Ax-Schanuel theorem for a general (pure) Shimura variety. A basic version of the theorem concerns the transcendence of the uniformization map from a bounded Hermitian symmetric space to a Shimura variety. We then prove a version of the theorem with derivatives in the setting of jet spaces, and finally a version in the setting of differential fields.

Our method of proof builds on previous work, combined with a new approach that uses higher-order contact conditions to place varieties yielding intersections of excessive dimension in natural algebraic families.

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## 1. Introduction

Let  $\Omega$  be a bounded Hermitian symmetric domain corresponding to a semisimple arithmetic group  $G$ , and let  $\Gamma \subset G(\mathbb{Z})$  be a finite index subgroup. Then  $X = \Gamma \backslash \Omega$  has the structure of a quasi-projective algebraic variety. A variety  $X$  arising in this way is called a (*connected, pure*) *Shimura variety*. We refer to [8], [9] or [22] for a detailed introduction to Shimura varieties. A Shimura variety  $X$  is endowed with a collection of *weakly special subvarieties*. (There is a smaller collection of *special subvarieties*, where a special subvariety is precisely a weakly special subvariety that contains a *special point*; these play no role in this paper.) For a description of these, see, e.g., [18].

Let  $q : \Omega \rightarrow X$  be the natural projection map, and let  $D \subset \Omega \times X$  be the graph of  $q$ . Recall that  $\Omega$  sits naturally as an open subset in its *compact dual*  $\widehat{\Omega}$ , which has the structure of a projective variety. By an *irreducible algebraic subvariety*  $W \subset \Omega \times X$  we mean a complex-analytically irreducible component of  $\widehat{W} \cap (\Omega \times X)$  for some algebraic subvariety  $\widehat{W} \subset \widehat{\Omega} \times X$ . By an algebraic subvariety of  $\Omega \times X$  we mean a finite union of irreducible algebraic subvarieties of  $\Omega \times X$ . In the sequel,  $\dim U$  denotes the complex dimension of a complex analytic set. Though at some points we will refer implicitly to sets in real Euclidean spaces, any reference to real dimensions will be specifically noted.

Our basic result is the following.

**THEOREM 1.1.** *With notation as above, let  $W \subset \Omega \times X$  be an algebraic subvariety such that  $W = \widehat{W} \cap (\Omega \times X)$  for some irreducible subvariety  $\widehat{W} \subset \widehat{\Omega} \times X$ . Let  $U$  be an irreducible component of  $W \cap D$  whose dimension is larger than expected, that is,*

$$(*) \quad \text{codim } U < \text{codim } W + \text{codim } D,$$

*the codimensions being in  $\Omega \times X$  or, equivalently,*

$$(**) \quad \dim W < \dim U + \dim X.$$

*Then the projection of  $U$  to  $X$  is contained in a proper weakly special subvariety of  $X$ .*

If one takes  $q : \Omega \rightarrow X$  to be the map

$$\exp : \mathbb{C}^n \rightarrow (\mathbb{C}^\times)^n,$$

namely, the Cartesian power of the complex exponential, then the statement is an equivalent form of the Ax-Schanuel theorem of Ax [3]. In this form it is given a new proof in [36]. Note, however, that  $(\mathbb{C}^\times)^n$  is a “mixed” Shimura variety but not a “pure” one, so this case is not formally covered by the above theorem.

One expects equality in (\*) and (\*\*) above, on dimensional considerations, and such a component  $U$  always has dimension *at least*  $\dim W - \dim X$  (see, e.g., Lojasiewicz [19, III.4.6]). Thus the theorem asserts that all components of such intersections that are atypical in dimension are accounted for by weakly special subvarieties.

Since weakly special varieties are “bi-algebraic” in the sense of [18], they do indeed give rise to atypical intersections. For example, in the extreme case that  $W = \Omega_1 \times X_1$ , where  $X_1$  is a weakly special subvariety of  $X$  and  $\Omega_1$  a connected component of its preimage in  $\Omega$ , we get  $\dim W = \dim U + \dim X_1$ .

Upon taking  $W$  to be a product variety in [Theorem 1.1](#), we recover the following 2-sorted version.

**THEOREM 1.2.** *Let  $X, \Omega$  be as in [Theorem 1.1](#). Let  $Y \subset \Omega$  be an irreducible complex-analytic subset. We let  $Y^{\text{WS}} \subset \Omega$  be the smallest weakly special subvariety containing  $Y$ . Then*

$$\dim Y^{\text{zar}} + \dim q(Y)^{\text{zar}} \geq \dim Y + \dim Y^{\text{WS}},$$

where  $Y^{\text{zar}}$  and  $q(Y)^{\text{zar}}$  denote the Zariski closures of  $Y$  and  $q(Y)$  respectively.

As in earlier papers [36], [18], [31], the proof combines arguments from complex geometry (Hwang-To), the geometry/group theory underlying Shimura varieties, o-minimality, and monodromy (Deligne-André). The ingredients from o-minimality include the counting theorem of Pila-Wilkie, and results of Peterzil-Starchenko giving powerful “definable” versions of the classical theorems of Remmert-Stein and Chow.

What these ingredients end up giving is the invariance of  $W$  under a non-trivial arithmetic subgroup  $H \subset G$ . However, in order to use  $H$  to reduce to a smaller bounded Hermitian symmetric domain one needs to know that  $H$  is normal, and this is where one encounters difficulties with previous methods.

The crucial new ingredient in this paper is the observation that one may put  $W$  in a natural *algebraic* family of varieties with similar intersection properties with respect to  $D$ . We then leverage the algebraicity to realize strong monodromy restrictions on the invariance subgroup  $H$  and thereby establish its normality. We establish the algebraicity by way of the results of Peterzil-Starchenko. However, there is also a purely complex analytic approach, which we allude to in [Section 3](#); see the remark at the end of [Section 3](#).

We expect that [Theorem 1.1](#), which is sometimes called the “hyperbolic Ax-Schanuel conjecture” [7], will find applications to the Zilber-Pink conjecture,<sup>1</sup> where it can play a role analogous to that of the “Ax-Lindemann theorem,” which it generalizes, in proving cases of the André-Oort conjecture; see,

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<sup>1</sup>In the Zilber-Pink conjecture, it is really the 2-sorted [Theorem 1.2](#) that is used.

e.g., [30]. One such application has been given by Daw-Ren [7]. An application in a different direction is given in [2]. A generalization of [Theorem 1.1](#) to variations of Hodge structures is given by Bakker-Tsimerman [4].

We will prove a strengthening of [Theorem 1.1](#) involving the uniformizing function together with its derivatives, along the same lines as the result of Pila-Tsimerman [30] for Cartesian powers of the  $j$ -function. For this we first observe the following generalization of a result of Bertrand-Zudilin [5] in the case of the Siegel modular varieties. (But note that their result holds over any algebraically closed subfield of  $\mathbb{C}$ ; we do not have any control over the field of definition of the differential equations.) The following result is established in [Section 9](#) as [Corollary 9.3](#). Let  $N^+ \subset G$  be the unipotent radical of an opposite parabolic subgroup of the complex parabolic subgroup  $B \subset G$  defining the symmetric space  $\widehat{\Omega} (= G(\mathbb{C})/B(\mathbb{C}))$ .

**THEOREM 1.3.** *Let  $z_1, \dots, z_n$  be an  $N^+(\mathbb{C})$ -invariant algebraic coordinate system on  $\Omega$ . Let  $\{\phi_1, \dots, \phi_N\}$  be a  $\mathbb{C}$ -basis of modular functions. Then the field generated by  $\{\phi_i\}$  and their partial derivatives with respect to the  $z_j$  up to order  $k \geq 2$  has transcendence degree over  $\mathbb{C}$  equal to  $\dim G$ . Furthermore, the transcendence degree is the same over  $\mathbb{C}(z_1, \dots, z_n)$ .*

As an example, one may consider the Shimura Variety  $\mathcal{A}_g$ , the moduli space of principally polarized abelian varieties of dimension  $g$ . In this case,  $\Omega = \mathbb{H}_g$  is Siegel upper half space, and  $G = \mathrm{Sp}_{2g}$ . Recall that  $\mathbb{H}_g$  consists of symmetric  $g \times g$  complex matrices whose imaginary parts are positive definite. Let  $z_{ij}$  be the natural matrix co-ordinates on  $\mathbb{H}_g$ , which are an  $N^+(\mathbb{C})$ -invariant set of algebraic co-ordinates for  $N^+$  being the strictly upper triangular elements of  $G$ . Then one may take the  $\phi_i$  to be a basis of Siegel modular functions, and [Theorem 1.3](#) applies to this case, recovering the result of Bertrand-Zudilin [5] (though only over  $\mathbb{C}$ ).

To frame our result we need to study the form of the differential equations satisfied by the uniformization map, for which in [Sections 7](#) and [8](#) we introduce and study the *Schwarzian derivative* for a Hermitian symmetric domain. Differential equations associated with covering maps are studied by Scanlon [34], who shows under quite general assumptions that one gets algebraic differential equations. A key ingredient there, as here, is definability and the results of Peterzil-Starchenko. However, our focus is on getting more specific information (such as [Theorem 1.3](#)) in the special case of Shimura varieties. On this circle of differential ideas, see also Buium [6]. For a description of the Schwarzian for  $\mathrm{PSL}_m$ , see [38].

Our Ax-Schanuel theorem for  $q$  and its derivatives ([Theorem 9.1](#)) is most naturally stated in the setting of *jet spaces*. These are introduced in [Sections 5](#) and [6](#). Here we give the following jet-space-free consequence of [Theorem 9.1](#).

**THEOREM 1.4.** *Let  $V \subset \Omega$  be an irreducible complex analytic variety, not contained in a proper weakly special subvariety. Let  $\{z_i, i = 1, \dots, n\}$  be an algebraic coordinate system on  $\Omega$ . Let  $\{\phi_j^{(\nu)}\}$  consist of a basis  $\phi_1, \dots, \phi_N$  of modular functions, all defined at at least one point of  $V$ , together with their partial derivatives with respect to the  $z_j$  up to order  $k \geq 2$ . Then*

$$\text{tr.deg.}_{\mathbb{C}}(\{z_i\}, \{\phi_j^{(\nu)}\}) \geq \dim G + \dim V,$$

where all functions are considered restricted to  $V$ .

We also give, in [Section 12](#), a version of Ax-Schanuel in the setting of a differential field, and we show that in turn it directly implies the jet version. This depends on the fact that all solutions of the relevant differential system are  $G(\mathbb{C})$ -translates of  $q$ , and this is due to the provenance of the system in properties of the Schwarzian associated with  $G$  and  $\Omega$ .

Let  $(K, \mathcal{D}, C)$  be a differential field with a finite set  $\mathcal{D}$  of commuting derivations and constant field  $C$ . We consider  $K$ -points  $(z, x, y)$  of suitable varieties over  $C$  and establish a differential algebraic condition under which such a point corresponds to a locus  $z$  in  $\widehat{\Omega}$ , whose dimension equals the rank of  $z$ , a corresponding locus  $x$  in  $X$  under some  $G(\mathbb{C})$ -translate  $Q$  of  $q$ , and the restrictions  $y$  of suitable derivatives of  $Q$  to the locus  $x$ .

The precise definition of such a *uniformized locus of rank  $k$*  in  $K$  is given in [Section 12](#), after the differential algebraic condition is established in [Section 11](#). Under suitable identifications, we can also speak of  $x$  being *contained in a proper weakly special subvariety* of  $X$ . With these notions, the differential version of Ax-Schanuel may be stated as follows.

**THEOREM 1.5 (Differential Ax-Schanuel).** *Let  $G, q, (K, \mathcal{D}, C)$  be as above. Let  $(z, x, y)$  be a uniformized locus. Then*

$$\text{tr.deg.}_{\mathbb{C}} C(z, x, y) \geq \text{rank}(z) + \dim G$$

unless  $x$  is contained in a proper weakly special subvariety.

## Part I. Basic Ax-Schanuel

This part is devoted to the proof of [Theorem 1.1](#).

### 2. Preliminaries

We gather some preliminary remarks, definitions, and results.

**2.1. Shimura varieties.** According to the definition given, a Shimura variety  $X$  may not be smooth, and the covering  $q : \Omega \rightarrow X$  may be ramified, if  $\Gamma$  contains elliptic elements. For example,  $j : \mathbb{H} \rightarrow \mathbb{C}$  is ramified at  $\text{SL}_2(\mathbb{Z})i$

and  $\mathrm{SL}_2(\mathbb{Z})\rho$ , where  $\rho = \exp(2\pi i/3)$ , even though in this case the quotient  $\mathbb{C}$  is still smooth.

By passing to a finite index subgroup we may always assume that the uniformization is unramified and the Shimura variety is smooth, and hence a complex manifold. This does not affect the validity of [Theorem 1.1](#). Hence we may and do assume throughout that  $X$  is smooth and that  $q : \Omega \rightarrow X$  is unramified.

**2.2. Definability.** The definition and basic results on o-minimal structures over a real closed field may be found in [\[28\]](#). In this paper, “definable” will mean “definable in the o-minimal structure  $\mathbb{R}_{\mathrm{an},\mathrm{exp}}$ .” However, the results of [Section 2](#) hold more generally.

Let  $\mathcal{F}$  be the classical Siegel domain for the action of  $\Gamma$  on  $\Omega$ . Then the uniformization  $q : \Omega \rightarrow X$  restricted to  $\mathcal{F}$  is definable in the o-minimal structure  $\mathbb{R}_{\mathrm{an},\mathrm{exp}}$ ; see [\[11\]](#) and [\[10\]](#), where the o-minimality of  $\mathbb{R}_{\mathrm{an},\mathrm{exp}}$  is established, building on [\[37\]](#). For a general Shimura variety, this result is due to Klingler-Ullmo-Yafaev [\[18\]](#), generalizing results of Peterzil-Starchenko [\[29\]](#) for moduli spaces of abelian varieties.

We shall need the following results, which can be seen as definable generalizations of GAGA-type theorems.

**THEOREM 2.1** (“Definable Remmert-Stein” [\[28, Th. 5.3\]](#)). *Let  $M$  be a definable complex manifold and  $E$  a definable complex analytic subset of  $M$ . If  $A$  is a definable, complex analytic subset of  $M \setminus E$ , then its closure  $\overline{A}$  is a complex analytic subset of  $M$ .  $\square$*

The following is a slight generalization of a theorem stated by Peterzil-Starchenko [\[28, Th. 4.5\]](#), which may be proved by combining their statement with “Definable Remmert-Stein” above. This strengthening has also been observed by Scanlon [\[34, Th. 2.11\]](#) and, in a slightly less general form, in [\[30\]](#).

**THEOREM 2.2** (“Definable Chow”). *Let  $Y$  be a quasiprojective algebraic variety, and let  $A \subset Y$  be definable, complex analytic, and closed in  $Y$ . Then  $A$  is algebraic.*

*Proof.* We follow the proof in [\[30\]](#). By taking an affine open set in  $Y$ , it suffices to consider the case where  $Y$  is an affine subset of projective space. Then  $Y$  is a definable, complex analytic subset of  $M \setminus E$ , where  $M$  is a projective variety and  $E$  is a closed algebraic subset of  $M$ . Then, by “Definable Remmert-Stein” above, the closure of  $A$  in  $M$  is a definable, complex analytic subset of  $M$ , hence complex analytic in the ambient projective space. Thus  $A$  must be algebraic by Chow’s theorem, or by the Peterzil-Starchenko version [\[28, Th. 4.5\]](#).  $\square$

We say that a set is “constructible complex analytic” if it is in the Boolean algebra generated by closed, complex analytic varieties.

**COROLLARY 2.3.** *Let  $Y$  be a quasiprojective algebraic variety, and let  $A \subset Y$  be definable, constructible complex analytic. Then  $A$  is constructible algebraic.*

*Proof.* Let  $B = \bar{A} \setminus A$ . Then  $B$  is definable, constructible complex analytic, and has  $\dim B < \dim A$ . By induction,  $B$  is constructible algebraic. Since  $\bar{A}$  is closed, it follows from [Theorem 2.2](#) that  $\bar{A}$  is constructible algebraic. Thus  $A$  is constructible algebraic.  $\square$

### 3. Some algebraicity results

We have the uniformization  $q : \Omega \rightarrow X$  in which  $X$  is a quasi-projective variety, and the map  $q$  is complex analytic and surjective. It is further invariant under the action of some discrete group  $\Gamma$  of holomorphic automorphisms of  $\Omega$ , and as noted above the restriction of  $q$  to a suitable fundamental domain  $\mathcal{F}$  for this action is definable.

Suppose that  $V \subset X$  is a relatively closed algebraic subvariety. Then  $q^{-1}(V) \subset \Omega$  is a closed complex analytic set that is  $\Gamma$ -invariant and definable on a fundamental domain  $\mathcal{F}$ . The same statement holds for the uniformization

$$q \times \text{id} : \Omega \times X \rightarrow X \times X$$

and  $V \subset X \times X$ , that is invariant under  $\Gamma \times \{\text{id}\}$ , where now  $q \times \text{id}$  is definable on  $\mathcal{F} \times X$ . We observe that the converse holds.

**THEOREM 3.1.** *Let  $A \subset \Omega \times X$  be a closed, complex analytic set which is  $\Gamma \times \{\text{id}\}$ -invariant, and such that  $A \cap (\mathcal{F} \times X)$  is definable. Then  $(q \times \text{id})(A) \subset X \times X$  is a closed algebraic subset.*

*Proof.* The image  $(q \times \text{id})(A)$  is closed and complex analytic in  $X \times X$ . Since  $(q \times \text{id})(A) = (q \times \text{id})(A \cap (\mathcal{F} \times X))$ , it is also definable, and so it is algebraic by “Definable Chow” ([Theorem 2.2](#)).  $\square$

**COROLLARY 3.2.** *Let  $A \subset \Omega \times X$  be a closed, constructible complex analytic set that is  $\Gamma \times \{\text{id}\}$ -invariant, and such that  $A \cap (\mathcal{F} \times X)$  is definable. Then  $(q \times \text{id})(A) \subset X \times X$  is a constructible algebraic subset.*

*Proof.* It follows from [Corollary 2.3](#) as above.  $\square$

**3.1. Descending Hilbert scheme loci.** For the purposes of proving [Theorem 1.1](#), we need to work with a family of varieties containing  $W$  that is both closed under the action of  $G(\mathbb{C})$  and proper. (The latter assumption is made

for reasons that are somewhat technical, and might be removable with more effort.) A convenient, well-known family to work with is provided by the theory of the Hilbert scheme, and so that is the context in which we work.

Fix a smooth, projective compactification  $\widehat{X}$  of  $X$ . Now we fix some algebraic subvariety  $W \subset \Omega \times X$  such that  $W = \widehat{W} \cap \Omega \times X$  for some irreducible algebraic subvariety  $\widehat{W} \subset \widehat{\Omega} \times X$ , and we let  $U$  be an irreducible component of  $W \cap D$ . We make no assumptions here on the dimension of  $U$ . By the *Hilbert polynomial*  $P = P_W(\nu)$  of  $W$  we mean the Hilbert polynomial of  $\widehat{W}$ .

Let  $M$  be the Hilbert scheme of all subvarieties of  $\widehat{\Omega} \times \widehat{X}$  with Hilbert polynomial  $P$ . Then  $M$  also has the structure of an algebraic variety. Corresponding to  $y \in M$  we have the subvariety  $W_y \subset \Omega \times X$ , and we have the incidence variety (universal family)

$$B = \{(z, x, y) \in \Omega \times X \times M : (z, x) \in W_y\}$$

and the family of the intersections of its fibres over  $M$  with  $D$ , namely,

$$A = \{(z, x, y) \in \Omega \times X \times M : (z, x) \in W_y \cap D\}.$$

Then  $A$  is a closed complex analytic subset of  $\Omega \times X \times M$ . It has natural projection  $\theta : A \rightarrow M$ , with  $(z, x, y) \mapsto y$ . Then, for each natural number  $k$ , the set

$$A(k) = \{(z, x, y) \in \Omega \times X \times M : \dim_{(z,x)} \theta^{-1}\theta(z, x, y) \geq k\},$$

the dimension being the dimension at  $(z, x)$  of the fibre of the projection in  $A$ , is closed and complex analytic; see, e.g., the proof of [27, Lemma 8.2] and references therein.

Now we have the projection  $\psi : \Omega \times X \times M \rightarrow \Omega \times X$ , and we consider

$$Z = Z(k) = \psi(A(k)).$$

Then, as  $M$  is compact,  $\psi$  is proper and so  $Z$  is closed in  $\Omega \times X$ . Note that  $Z$  is  $\Gamma$ -invariant and  $Z \cap (\mathcal{F} \times X)$  is definable.

LEMMA 3.3. *Let  $T = (q \times \text{id})(Z)$ . Then  $T \subset X \times X$  is closed and algebraic.*

*Proof.* Since  $Z$  is  $\Gamma$ -invariant and  $Z \cap (\mathcal{F} \times X)$  is definable, this follows as in Theorem 3.1. □

*Remark.* One may also prove Lemma 3.3 by more geometric methods along the lines of the argument in [24], which uses the method of compactification of complete Kähler manifolds of finite volume of [25] based on  $L^2$ -estimates of  $\bar{\partial}$ .



4. Proof of Theorem 1.1

*Proof.* We argue by induction, in the first instance (upward) on  $\dim \Omega$ . For a given  $\dim \Omega$ , we argue (upward) on  $\dim W - \dim U$ . Finally, we argue by induction (downward) on  $\dim U$ .

We now establish the base cases for the above inductions: First, if  $\dim \Omega = 0$  then the statement is tautologically true as all varieties are dimension 0. Now, if  $\dim U = \dim W$ , then  $W \subset D$ , and the projection  $q(W)$  has the same dimension as  $W$ . It follows that  $W$  is a component of the pre-image  $q^{-1}(q(W))$ , and thus invariant under the monodromy group of  $q(W)$ . Now, assume  $q(W)$  is not contained in a proper weakly special subvariety of  $X$ , and thus by André-Deligne [1, §5, Th. 1] the monodromy group of  $q(W)$  is Zariski dense in  $G$ . It follows that  $W$  is invariant under  $G(\mathbb{R})^+$ , which contradicts the assumption that  $W \subset D$ . Finally, assume  $\dim U = \dim X$ . Then  $W$  has to contain all of  $D$ , and thus  $W$  is invariant under  $G(\mathbb{Z})$ . Since  $W$  is algebraic, it follows that  $W$  is invariant under all of  $G(\mathbb{R})^+$ , and hence that  $W = \Omega \times X$ , which contradicts the assumptions on dimension. This completes the analysis of the base cases.

We carry out the constructions of Section 3.1 with  $k = \dim U$  and keep the notation there. Let  $A(k)' \subset A(k)$  be the irreducible component that contains  $U \times [W]$ , where  $[W]$  is the moduli point of  $W$  in  $M$ . Let  $Z' = \psi(A(k)') \subset Z$  be the corresponding irreducible component of  $Z$ , and let  $V = (q \times \text{id})(Z')$  be the irreducible component of  $T$ , which is therefore algebraic by Lemma 3.3. Now, by assumption,  $V$  contains  $(q \times \text{id})(U)$ , and so it is not contained in any proper weakly special of the diagonal  $\Delta_X$ , and thus its monodromy group is Zariski-dense in  $G$  by André-Deligne [1, §5, Th. 1].

Consider the family  $F_0$  of algebraic varieties corresponding to  $A(k)'$ . Let  $\Gamma_0 \subset \Gamma$  be the subgroup of elements  $\gamma$  such that every member of  $F_0$  is invariant by  $\gamma$ . For any  $\mu \in \Gamma$ , define  $E_\mu \subset F_0$  to be the subset corresponding to algebraic subvarieties invariant under  $\mu$ . Then, for  $\mu \in \Gamma - \Gamma_0$ ,  $E_\mu \subsetneq F_0$  is an algebraic subvariety. Hence, a very general<sup>2</sup> element  $W'$  of  $F_0$  is invariant by exactly the subset  $\Gamma_0$  of  $\Gamma$ . Let  $\Theta$  be the connected component of the Zariski closure of  $\Gamma_0$  in  $G$ .

LEMMA 4.1.  $\Theta$  is a normal subgroup of  $G$ .

*Proof.* Note that there is an action of  $\Gamma$  on  $A(k)$  given by  $\gamma \cdot (z, x, [W]) = (\gamma z, x, [\gamma W])$ , and the map  $A(k) \rightarrow Z$  is equivariant with respect to this action. Since  $A(k) \rightarrow Z$  is proper and the action of  $\Gamma$  is discrete, it follows that  $\Gamma \backslash A(k) \rightarrow \Gamma \backslash Z \cong T$  is a proper map of analytic varieties.

<sup>2</sup>In the sense of being in the complement of countably many proper subvarieties.

Let  $\Gamma \backslash A(k)'$  denote the image of  $A(k)'$  in  $\Gamma \backslash A(k)$ . Then  $\phi : \Gamma \backslash A(k)' \rightarrow V$  is a proper map of analytic varieties, and thus the fibers of  $\phi$  have only finitely many components. Let  $\Gamma_1$  be the image of  $\pi_1(\Gamma \backslash A(k)') \rightarrow \pi_1(V) \rightarrow \Gamma$ . Then  $\pi_1(A(k)')$  has finite index image in the monodromy group of  $V$ , and thus  $\Gamma_1$  is Zariski-dense in  $G$  by André-Deligne [1, §5, Th. 1].

It is clear that  $F_0$  is invariant under  $\Gamma_1$ .

Now, letting  $\text{stab}(W')$  denote the stabilizer of  $W'$ , we have  $\text{stab}(\gamma \cdot W') = \gamma \cdot \text{stab}(W') \cdot \gamma^{-1}$ . It follows that  $\Gamma_0$ , and hence also  $\Theta$  is invariant under conjugation by  $\Gamma_1$ , and hence under its Zariski closure, which is all of  $G$ . This completes the proof.  $\square$

LEMMA 4.2.  $\Theta$  is the identity subgroup.

*Proof.* We argue by contradiction. Without loss of generality we may assume that  $W$  is a very general member of  $F_0$  and hence it is invariant by exactly  $\Theta$ . Since  $\Theta$  is a  $\mathbb{Q}$ -group by construction, it follows that  $G$  is isogenous to  $\Theta \times \Theta'$  and we have a splitting  $\Omega = \Omega_\Theta \times \Omega_{\Theta'}$  of Hermitian symmetric domains. Replacing  $\Gamma$  by a finite index subgroup we also have a splitting  $X \cong X_\Theta \times X_{\Theta'}$ .

Now, as  $W$  is invariant under  $\Theta$ , it is of the form  $\Omega_\Theta \times W_1$  where  $W_1 \subset \Omega_{\Theta'} \times X_{\Theta'} \times X_\Theta$ . Moreover,  $D$  splits as  $D_\Theta \times D_{\Theta'}$ . Let  $U_1$  be the projection of  $U$  to  $W_1$ . Since the map from  $D_\Theta$  to  $X_\Theta$  has discrete pre-images, it follows that  $\dim U = \dim U_1$ .

Now let  $W'$  be the projection of  $W_1$  to  $\Omega_{\Theta'} \times X_{\Theta'}$ . Then letting  $U'$  be a component of  $W' \cap D_{\Theta'}$  we easily see that  $U'$  is the projection of  $U$  to  $\Omega_{\Theta'} \times X_{\Theta'}$ . Then the projection of  $U'$  to  $X_{\Theta'}$  is not contained in a proper weakly special subvariety of  $X_{\Theta'}$ . Now let  $W''$  be the Zariski closure of  $U'$ . It follows by induction on  $\dim \Omega$  that

$$\dim U' + \dim X_{\Theta'} \leq \dim W''.$$

Now for the projection map  $\psi : W_1 \rightarrow W'$ , the generic fiber dimension over  $W'' \subset W'$  is the same as the generic fiber dimension over  $U'$ , and thus

$$\dim U_1 + \dim X_{\Theta'} \leq \dim \psi^{-1}(W'') \leq \dim W_1,$$

from which it follows that

$$\dim U + \dim X \leq \dim W,$$

contradicting the hypothesis  $\dim W < \dim U + \dim X$  as in the statement of [Theorem 1.1](#), as desired.  $\square$

It follows that  $W$  is not invariant by any infinite subgroup of  $\Gamma$ . The following lemma thus reaches a contradiction, and completes the proof.

LEMMA 4.3.  $W$  is invariant by an infinite subgroup of  $\Gamma$ .

*Proof.* As before, let  $\mathcal{F}$  be a fundamental domain for  $\Gamma$  on which the map  $q$  is definable, and consider the definable set

$$I = \{\gamma \in G(\mathbb{R}) \mid \dim_{\mathbb{R}} ((\gamma \cdot W) \cap D \cap (\mathcal{F} \times X)) = \dim_{\mathbb{R}} U\}.$$

Clearly,  $I$  contains  $\gamma \in \Gamma$  whenever  $U$  intersects  $\gamma\mathcal{F} \times X$ .

We claim that the  $\dim_{\mathbb{R}} U$ -dimensional volume of  $((\gamma \cdot W) \cap D \cap (\mathcal{F} \times X))$  is uniformly bounded over  $\gamma \in I$ . To see this, we proceed as in the work of Klingler-Ullmo-Yafaev [18, Lemma 5.8]. They show that using Siegel coordinates one can cover  $\mathcal{F}$  by a finite union of sets  $\Sigma$  that embed into a product of 1-dimensional sets  $\prod_{i=1}^m J_i$ , and that there are (1,1)-volume forms  $\omega_i$  on  $J_i$  such that  $\sum_i \omega_i$  dominates the Kähler form of an invariant hyperbolic metric, and  $\int_{J_i} \omega_i < \infty$ . For a subset  $I \subset \{1, \dots, m\}$  containing  $d = \dim_{\mathbb{R}} U$  elements, let  $J_I = \prod_{i \in I} J_i$ . Thus, it suffices to show that the projections of  $((\gamma \cdot W) \cap D \cap (\mathcal{F} \times X)) \cap \Sigma$  onto  $J_I$  have finite fibers of uniformly bounded cardinality. However, this is an immediate consequence of the definability, which establishes the claim.

By the work of Hwang–To [14, Th. 2], the volume of  $\gamma \cdot W \cap X \cap B(R)$  grows exponentially with  $R$ , where  $B(R)$  is a hyperbolic ball of radius  $R$ . Thus, as in [31], it follows that  $I$  contains at least a polynomial number of integer matrices.<sup>3</sup> It follows by the Pila-Wilkie theorem [32] that  $I$  contains irreducible real algebraic curves  $C$  containing arbitrarily many integer points, in particular, containing at least two integer points.

Write  $W_c := c \cdot W$ . If  $W_c$  is constant in  $c$ , then  $W$  is stable under  $C \cdot C^{-1}$ . Since  $C$  contains at least two integer points, it follows that  $W$  is stabilized by a non-identity integer point, completing the proof that  $W$  is invariant under an infinite group (since  $\Gamma$  is torsion free). So we assume that  $W_c$  varies with  $c \in C$ . Since  $C$  contains an integer point, it follows from our assumption on  $U$  that  $(q \times \text{id})(W_c \cap D)$  is not contained in a proper weakly special subvariety for at least one  $c \in C$ , and thus for all but a countable subset of  $C$  (since there are only countably many families of weakly special subvarieties).

We now have two cases to consider. First, suppose that  $U \subset W_c$  for all  $c \in C$ . Then we may replace  $W$  by  $W_c \cap W_{c'}$  for generic  $c, c' \in C$  and lower  $\dim W$ , contradicting our induction hypothesis on  $\dim W - \dim U$ .

On the other hand, if it is not true that  $U \subset W_c$  for all  $c \in C$ , then  $W_c \cap D$  varies with  $C$ , and so we may set  $W'$  to be the Zariski closure of  $C \cdot W$ . This increases the dimension of  $W$  by 1, but then  $\dim W' \cap D = \dim U + 1$  as well, and thus we again contradict our induction hypothesis, this time on  $\dim U$ . This completes the proof. □

This contradiction completes the proof of [Theorem 1.1](#). □

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<sup>3</sup>Ordered by height, there are at least  $T^\delta$  integer points of height at most  $T$  for some fixed  $\delta > 0$  and arbitrarily large  $T$ .

**Part II. Ax-Schanuel with derivatives**

In this part we establish versions of Ax-Schanuel for  $q$  together with its derivatives. The result is formulated in the setting of jet spaces. For background on jet spaces, see [12, §16.7] or [33] for a more differential-geometric perspective.

**5. Jet Spaces**

5.1. *Definition.* Let  $X$  be a smooth complex algebraic variety and  $k, g \geq 1$  be positive integers. Set

$$\mathbb{D}_k^g = \text{Spec } \mathbb{C}[\varepsilon_1, \dots, \varepsilon_g]/M^{k+1},$$

where  $M$  is the ideal  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_g)$ . We define the *jet space of order  $k$*  to be<sup>4</sup>

$$J_k^g X := \text{Hom}(\mathbb{D}_k^g, X).$$

Note that  $J_0^g X = X$  and that  $J_1^1 X$  is the tangent bundle of  $X$ . It is evident that  $J_k^g$  is a functor and also that there are natural projection maps  $J_k^g \rightarrow J_r^g$  whenever  $k > r$ . As a matter of notation, we simply write  $J_k X$  to denote  $J_k^{\dim X} X$ .

5.2. *Maps between Jet Spaces.* For  $a, b > 0$ , there is a natural map

$$\pi_{a,b}^\# : \mathbb{C}[\varepsilon_1, \dots, \varepsilon_g]/M^{a+b+1} \rightarrow \mathbb{C}[t_1, \dots, t_g]/M^{a+1} \otimes \mathbb{C}[s_1, \dots, s_g]/M^{b+1}$$

defined by  $\pi_{a,b}^\#(\varepsilon_i) = 1 \otimes s_i + t_i \otimes 1$ . This induces a map  $\pi_{a,b} : \mathbb{D}_a^g \times \mathbb{D}_b^g \rightarrow \mathbb{D}_{a+b}^g$ , which is just the truncation of the addition map. Now there are natural identifications

$$J_a^{\dim X} J_b X \cong \text{Hom}(\mathbb{D}_a^g, \text{Hom}(\mathbb{D}_b^g, X)) \cong \text{Hom}(\mathbb{D}_a^g \times \mathbb{D}_b^g, X),$$

and therefore  $\phi$  induces a natural map

$$\pi_{a,b} : J_{a+b} X \rightarrow J_a^{\dim X} J_b X.$$

Since  $\pi_{a+b}^\#$  is injective as a map of rings,  $\pi_{a,b}$  is injective on the level of points.

Moreover, since  $J_{a+b}$  is postcomposition and  $\pi_{a,b}$  is precomposition, it is easy to see that they commute. In other words, for a map  $f : X \rightarrow Y$ , we have

$$(1) \quad \pi_{a,b} \circ J_{a+b} f = J_a^{\dim X} (J_b f) \circ \pi_{a,b}.$$

To see this, consider the following diagram:

$$\mathbb{D}_a^g \times \mathbb{D}_b^g \rightarrow \mathbb{D}_{a+b}^g \rightarrow X \rightarrow Y.$$

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<sup>4</sup> This may also be naturally defined as a scheme representing the functor  $Y \rightarrow \text{Hom}(\mathbb{D}_k^g \times Y, X)$  on the category of  $\mathbb{C}$ -schemes.

### 6. Differential equations

6.1. *The jet space formulation.* Suppose  $\phi : \mathbb{C}^g \rightarrow \mathbb{C}^g$  is a holomorphic function that satisfies an algebraic differential equation of degree  $m$  given by a set of polynomials in the derivatives of the components of  $\phi$ , which we may write as  $\vec{F}(\frac{\partial^{|J|}\phi_i}{\partial z_J})_{|J|<m, j \leq g} = 0$ . We record this geometrically as follows. Consider the natural section  $\text{id}_m : \mathbb{C}^g \rightarrow J_m\mathbb{C}^g$  given by  $\text{id}_m(z) : \mathbb{D}_m \rightarrow \mathbb{C}^g$ , where the latter is given by

$$\text{id}_m(z)(\varepsilon_1, \dots, \varepsilon_g) = (z + \varepsilon_1, \dots, z + \varepsilon_g)$$

and the corresponding map on sheaves is given by

$$\text{id}_m(z)^\#(Z_1, \dots, Z_g) = (z_1 + \varepsilon_1, \dots, z_g + \varepsilon_g).$$

Then, in the natural coordinates for  $J_mX$ , the partial derivatives of  $\phi$  are the coordinates of  $J_m\phi \circ \text{id}_m : \mathbb{C}^g \rightarrow J_k\mathbb{C}^g$ .

We may record our differential equation in the following way. We stipulate that  $J_m\phi \circ \text{id}_m(\mathbb{C}^g) \subset W$  for a specific subvariety  $W \subset J_m\mathbb{C}^g$ . Of course, we may then replace the target  $\mathbb{C}^g$  with any space  $X$  and formulate a differential equation by picking a subvariety  $W \subset J_mX$  without having to pick local coordinates on the image.

6.2. *Differentiating a differential equation.* If  $f(z)$  satisfies  $f'(z) = R(f(z))$ , then it also satisfies  $f''(z) = f'(z)R'(f(z)) = R(f(z))R'(f(z))$ . We explain now how to derive such relations geometrically.

Returning now to the case  $z \in \mathbb{C}^g$ , we first note that  $J_a^g \text{id}_b \circ \text{id}_a(z)$  is the local map  $s \mapsto (t \mapsto z + t + s)$ . Thus, we see that

$$(2) \quad J_a^g \text{id}_b \circ \text{id}_a = \pi_{a,b} \circ \text{id}_{a+b}.$$

Now suppose that  $\phi : \mathbb{C}^g \rightarrow X$  satisfies  $\text{Im} J_b^g \phi \circ \text{id}_b \subset W$ . For any  $a > 0$ , we have

$$\begin{aligned} \pi_{a,b} \circ J_{a+b} \phi \circ \text{id}_{a+b} &= J_a^g(J_b \phi) \circ \pi_{a,b} \circ \text{id}_{a+b} \text{ by (1)} \\ &= J_a^g(J_b \phi) \circ J_a^g \text{id}_b \circ \text{id}_a \text{ by (2)} \\ &\subset J_a W \subset J_a^g J_b X. \end{aligned}$$

Thus, we learn that  $\text{Im} J_{a+b} \phi \circ \text{id}_{a+b}$  is contained in  $\pi_{a,b}^{-1}(J_a W)$ .

6.2.1. *Example.* Suppose that  $\phi'(z) = R(\phi(z))$ . Let  $X = \mathbb{C}$ , so  $J_1X$  can be identified with  $\mathbb{A}_{z,r}^2$  by the maps  $t \mapsto z + rt$ . Let  $W \subset J_1X$  be defined by the relation  $r = R(z)$ , so that  $\text{Im} J_1 \phi \circ \text{id}_1 \subset W$ . Now  $J_1^1 J_1X$  can be defined by  $(z, r, z_1, r_1)$  by

$$s \mapsto (t \mapsto z + rt + sz_1 + str_1).$$

Now an element of  $J_1W$  has the form  $s \mapsto (z + es, R(z) + eR'(z)s)$  and so it maps to

$$s \mapsto (t \mapsto z + es + tR(z) + teR'(z)s).$$

Now the image  $\pi_{1,1}J_2X$  in  $J_1^1J_1X$  consists of those elements that are functions of  $s + t$  in the ring  $\mathbb{C}[s, t]/(s^2, t^2)$ . Thus, we need the  $s$  and  $t$ -coefficients to be the same, so we must have  $e = R(z)$ , and we get the map

$$s + t \rightarrow z + R(z)(s + t) + \frac{1}{2}R(z)R'(z)(s + t)^2.$$

Note that this exactly records the relation  $\phi''(z) = R(\phi(z))R'(\phi(z))$  as desired.

### 7. Schwarzians for Hermitian symmetric spaces

7.1. *Setup.* Let  $\Omega$  be a Hermitian symmetric space of dimension  $n$ . We may write  $\Omega$  as  $G(\mathbb{R})/K$  for a semisimple group  $G$  and maximal compact real subgroup  $K$  inside it. Then  $\Omega$  sits naturally inside the flag manifold  $\widehat{\Omega} := G(\mathbb{C})/B$  for a complex parabolic subgroup  $B$ . Now the Lie algebra of  $G(\mathbb{C})$  decomposes as  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{n}^+$  giving  $\mathfrak{g}$  a Hodge structure of weight 0, where  $\mathfrak{k}$  is the Lie algebra of  $K$  and  $\mathfrak{b} = \mathfrak{n}^- \oplus \mathfrak{k}_{\mathbb{C}}$  is the Lie algebra of  $B$ . Let  $N^-, N^+$  be the corresponding (abelian) unipotent groups. Picking a base-point  $o \in X$ , we may give coordinates on an open subset of  $\widehat{\Omega}$  by identifying it with  $N^+$  by  $\nu \rightarrow \nu \cdot o$ . Fixing an identification  $\mathbb{C}^n \rightarrow N^+$  of vector spaces once and for all, we get sections  $\text{id}_m : N^+ \rightarrow J_m N^+$  that are compatible and invariant under  $N^+$ .

We would like to characterize those functions  $F : \widehat{\Omega} \rightarrow \widehat{\Omega}$  which look like  $F(z) = g \cdot z$  for an element  $g \in G(\mathbb{C})$ . As such, we define the *Schwarzian differential equation* to be

$$W_m := G(\mathbb{C}) \cdot \text{id}_m(N^+) = G(\mathbb{C}) \cdot \text{id}_m(o) \subset J_m \widehat{\Omega}.$$

*Definition.* We define  $J_k^{\text{nd},r}Y \subset J_k^r Y$  to be all those infinitesimal maps which are surjective on tangent spaces. Note that this is only non-empty for  $r \geq \dim Y$ .

7.1.1. *Example.* Consider  $G = \text{PSL}_2, \Omega = \mathbb{H}, o = 0 \in \mathbb{C}$ . Let  $N^-$  be the strictly lower triangular matrices and  $N^+$  the strictly upper triangular matrices. Then  $J_3\mathbb{C}$  can naturally be given coordinates  $(z, a, b, c)$  corresponding to the map  $t \rightarrow z + at + \frac{bt^2}{2} + \frac{ct^3}{6}$ . Note that  $W_3$  is  $G(\mathbb{C})$ -invariant. In particular, since  $W_3$  is  $N^+$ -invariant, it must be cut out by a function of  $a, b, c$ , and it is sufficient to consider it at  $z = 0$ , which is fixed by all of the lower triangular matrices. Acting first by a diagonal element we transform  $(0, a, b, c)$  to  $(0, 1, b/a, c/a)$  via  $z \rightarrow z/a$ . Now acting by a lower triangular matrix  $z \rightarrow \frac{z}{\frac{b}{2a}z+1}$

we get

$$\frac{t + \frac{b}{2a}t^2 + \frac{c}{6a}t^3}{1 + \frac{b}{2a}t + \frac{b^2}{4a^2}t^2 + \frac{bc}{12a^2}t^3} = t + \left( \frac{c}{6a} - \frac{b^2}{4a^2} \right) t^3,$$

which transforms  $(0, 1, b/a, c/a)$  to  $(0, 1, 0, \frac{c}{6a} - \frac{b^2}{4a^2})$ . Thus we recover the classical Schwarzian in this setting.

7.2. *Fixing the order 2 infinitesimal neighborhood.* We have the following lemma.

LEMMA 7.1. *The subgroup of  $B$  that fixes  $\text{id}_1(o)$  is  $N^-$ .*

*Proof.* We are looking for elements that act trivially on the tangent space at  $o$ ,  $T_o\widehat{\Omega} \cong \mathfrak{g}/\mathfrak{b}$ . The action of  $B$  on this space is induced the adjoint action.

We first claim that  $N^-$  acts trivially. To see this, note that the Lie bracket respects the Hodge grading on  $\mathfrak{g}$ , so  $[\mathfrak{n}^-, \mathfrak{n}^-] = 0$  and  $[\mathfrak{n}^-, \mathfrak{k}] \subset \mathfrak{n}^-$ , from which the claim follows.

Since  $B = K_{\mathbb{C}}N^+$ , to finish the proof it suffices to show that no element of  $K_{\mathbb{C}}$  acts trivially. Conjugation by  $K_{\mathbb{C}}$  preserves  $\mathfrak{n}^+$  so if an element  $k \in K_{\mathbb{C}}$  acts trivially on  $T_o\widehat{\Omega}$ , it must also act trivially on  $\mathfrak{n}^+$  and thus commute with  $N^+$ . Since  $N^+o$  is open, and since  $k$  fixes  $o$ , it must be the case that  $k$  acts trivially on all of  $\widehat{\Omega}$  and thus be the identity, as desired.  $\square$

PROPOSITION 7.2. *The group  $G(\mathbb{C})$  acts freely on  $\text{id}_2(o)$ , and thus on all of  $J_k^{\text{nd}}\widehat{\Omega}$  for  $k \geq 2$ .*

*Proof.* By Lemma 7.1 we need only show that no element of  $N^-$  fixes  $\text{id}_2(o)$ . Let  $N \subset N^-$  be the stabilizer of  $\text{id}_2(o)$ , and note that  $N$  is normal in  $B$ . Since  $K_{\mathbb{C}}$  contains a maximal Cartan algebra, the Lie algebra of  $N$  must be a direct sum of root spaces. Assuming  $N$  is not trivial, we let  $\alpha$  be one of those roots and  $N_{\alpha}$  be the corresponding 1-dimensional subgroup. Now there is a map  $\text{SL}_2(\mathbb{R}) \rightarrow G(\mathbb{R})$  that sends  $\text{SO}(2)$  to  $K$  and the roots of  $\text{SL}_2(\mathbb{R})$  (which lie in  $\text{SL}_2(\mathbb{C})$ ) to  $N_{\alpha}$  and its conjugate  $N_{-\alpha}$ . This map induces a map of symmetric spaces  $\mathbb{H} \rightarrow \Omega$  and  $\mathbb{P}^1(\mathbb{C}) \rightarrow \widehat{\Omega}$ . Moreover, this map is evidently holomorphic. We have thus reduced the claim to the case of  $\text{SL}_2(\mathbb{C})$ , where it may be easily checked by hand, since  $\frac{z}{1+az} = z - az^2 + O(z^3)$ .  $\square$

Note that it follows from the proposition that  $W_m$  is closed in  $J_m^{\text{nd}}$  for  $m \geq 2$ .

COROLLARY 7.3. *If  $F : \widehat{\Omega} \rightarrow \widehat{\Omega}$  satisfies  $J_3F \circ \text{id}_3(N^+) \subset W_3$ , then there exists  $g \in G(\mathbb{C})$  such that  $F(z) = gz$ .*

*Proof.* By Proposition 7.2, the variety  $W_3$  projects bijectively onto  $W_2$ . Thus, in local coordinates, we may write equations for all the third order derivatives of  $F$  in terms of its lower order derivatives. Differentiating further

gives us the full power series expansion of  $F$  and thus characterizes it completely. This completes the proof.  $\square$

7.3. *Lowering the order of the differential equation.* In this section we improve over [Corollary 7.3](#) in the following ways.

**THEOREM 7.4.** *Let  $n \geq 2$ ,  $\Omega = \mathbb{B}^n$ ,  $G(\mathbb{C}) = \mathbb{PGL}(n + 1; \mathbb{C})$ . If  $f : (\widehat{\Omega}; x_0) \rightarrow (\widehat{\Omega}; y_0)$  satisfies  $J_2 f \circ \text{id}_2(N^+) \subset W_2$ , then there exists  $g \in G(\mathbb{C})$  such that  $f(z) = gz$  wherever  $f$  is defined.*

**THEOREM 7.5.** *Let  $\Omega$  be an irreducible bounded symmetric domain of rank  $\geq 2$ ,  $G(\mathbb{C}) = \text{Aut}(\widehat{\Omega})$ . If  $f : (\widehat{\Omega}; x) \rightarrow (\widehat{\Omega}; y)$  satisfies  $J_1 f \circ \text{id}_1(N^+) \subset W_1$ , then there exists  $g \in G(\mathbb{C})$  such that  $f(z) = gz$  wherever  $f$  is defined.*

Both [Theorems 7.4](#) and [7.5](#) follow from known results, [Theorem 7.4](#) from a local version (for holomorphic maps) of the Fundamental Theorem of Projective Geometry in the case of the complex field, and [Theorem 7.5](#) from Ochiai's Theorem characterizing automorphisms of  $S$ -structures (Ochiai [[26](#)]) for  $S$  being an irreducible Hermitian symmetric space of the compact type of rank  $\geq 2$ . They can be stated as follows.

**THEOREM A.** *Let  $n \geq 2$ ,  $\Omega = \mathbb{B}^n$ ,  $G(\mathbb{C}) = \mathbb{PGL}(n + 1; \mathbb{C})$ , and let  $\mathbb{B}^n \Subset \mathbb{C}^n \subset \widehat{B}^n = \mathbb{P}^n$  be the standard embeddings. Suppose  $U \subset \mathbb{C}^n$  is a convex open set and  $f : U \xrightarrow{\cong} V$  is a biholomorphism onto an open subset  $V \subset \mathbb{P}^n$  such that, for any nonempty (connected) intersection  $\ell \cap U$  of an affine line  $\ell$  with  $U$ , we have  $f(\ell \cap U) \subset \ell'$  for some affine line  $\ell'$  on  $\mathbb{C}^n$ . Then,  $f$  extends to a biholomorphic automorphism  $F : \mathbb{P}^n \rightarrow \mathbb{P}^n$ , i.e.,  $F \in \mathbb{PGL}(n + 1; \mathbb{C})$ .*

A proof of this is given in Mok [[23](#), §(2.3)]. For the formulation of Ochai's Theorem, recall that an irreducible Hermitian symmetric space of the compact type is given by  $S = G/B$ , where  $G$  is a simple complex Lie group and  $B \subset G$  is some maximal parabolic subgroup. For any point  $x \in S$ , let  $B_x \subset G$  be the isotropy (parabolic) subgroup at  $x$ . Let  $B_x = U \cdot L$  be the Levi decomposition of  $B_x$ , where  $U \subset B_x$  is the unipotent radical and  $L \subset B_x$  is a Levi factor. As is well known, for any  $v \in U$ ,  $dv(x) = \text{id}_{T_x(X)}$ , hence the map  $\Phi(\gamma) = d\gamma(x)$  defines a representation of  $L = B_x/U$  on  $T_x(X)$  that is independent of the choice of the Levi decomposition. We denote by  $\mathcal{W}_x \subset \mathbb{P}T_x(X)$  the highest weight orbit of the action of  $L$  on  $\mathbb{P}T_x(X)$ . Then, Ochiai's Theorem can be formulated as follows.

**THEOREM B (Ochiai [[26](#)]).** *Let  $S$  be an irreducible compact Hermitian symmetric manifold of the compact type and of rank  $\geq 2$ , let  $U, V \subset S$  be connected open subsets, and let  $f : U \xrightarrow{\cong} V$  be a biholomorphism. Suppose that for every  $x \in U$  the differential  $df(x) : T_x(S) \rightarrow T_{f(x)}(S)$  satisfies  $[df(x)](\mathcal{W}_x) = \mathcal{W}_{f(x)}$ . Then there exists  $F \in \text{Aut}(S)$  such that  $F|_U \equiv f$ .*



Ochiai [26] made use of Lie algebra cohomology. Observing that  $\mathcal{W}_x$  agrees with  $\mathcal{C}_x(S)$ , the variety of minimal rational tangents (VMRT) at  $x \in S$  consisting of  $[\alpha] \in \mathbb{P}T_x(X)$  tangent to minimal rational curves passing through  $x \in X$ , Hwang-Mok [13] generalized Theorem B to Fano manifolds of Picard number 1, proving analogously the Cartan-Fubini extension theorem for VMRT-preserving germs of biholomorphisms between Fano manifolds of Picard number 1 under mild geometric conditions. A differential-geometric proof of Theorem B was given in Mok [23, §§(2.2)–(2.4)].

*Deduction of Theorem 7.4 from Theorem A.*

*Proof of Theorem 7.4.* Let  $U \subset \mathbb{C}^n$  be a domain of definition of  $f : (\widehat{\Omega}; x_0) \rightarrow (\widehat{\Omega}; y_0)$ , which we may assume to be convex. By assumption,  $J_2f \circ id_2 \subset W_2$ . In other words, for any  $x \in U$ ,  $J_2f(x) = J_2\gamma_x(x)$  for some  $\gamma_x \in \text{PGL}(n + 1; \mathbb{C})$  such that  $\gamma_x(x) = y := f(x)$ . Let  $\ell \subset \mathbb{P}^n$  be a projective line passing through  $x$ . Then  $\gamma_x(\ell) =: \ell' \subset \mathbb{P}^n$  is a projective line passing through  $y$ , thus from the assumption  $f(\ell)$  is tangent to  $\ell'$  to the order  $\geq 2$ .

On  $\mathbb{P}^n$  there is a canonical projective structure, defined as follows. For each projective line  $\ell$  on  $\mathbb{P}^n$ , denote by  $\widehat{\ell} \subset \mathbb{P}T_{\mathbb{P}^n}$  the tautological lifting of  $\ell$  obtained as the image of the map associating the pair  $(\ell, t)$ ,  $t \in \ell$ , to  $\mathbb{P}T_t(\ell) \in \mathbb{P}T_t(\mathbb{P}^n)$ . Then this gives a 1-dimensional (holomorphic) foliation  $\mathcal{F}$  on the total space  $\mathbb{P}T(\mathbb{P}^n)$  of the projective tangent bundle over  $\mathbb{P}^n$ . We call  $\mathcal{F}$  the canonical projective structure on  $\mathbb{P}^n$ . In the notation of the last paragraph, the germ  $(f(\ell); y)$  is second order tangent to  $(\gamma_x(\ell); y)$ . Lifting to  $\mathbb{P}T_{\mathbb{P}^n}$ , it means that the tautological lifting  $\Lambda$  of the germ of holomorphic curve  $(f(\ell); y)$  is tangent (to the order  $\geq 1$ ) to the lifting  $\widehat{\ell}'$  of the unique projective line  $\ell' = \gamma_x(\ell)$  passing through  $y$  and tangent to  $f(\ell)$  at  $y$ . Let  $\ell_0 \subset \mathbb{P}^n$  be an arbitrary projective line passing through  $x$ . We have proven that for some neighborhood  $\mathcal{V}$  of  $[df](\mathbb{P}T_{x_0}(\ell_0)) \in \mathbb{P}T_{y_0}(\mathbb{P}^n)$ , the tautological liftings of images of projective lines under  $f$  define a holomorphic 1-dimensional foliation  $\mathcal{E} = f_*\mathcal{F}$  on  $\mathcal{V}$  which is tangent to the canonical projective structure  $\mathcal{F}$  at every point  $[\alpha] \in \mathcal{V}$ . But  $\mathcal{F}$  itself is a 1-dimensional foliation, hence  $\mathcal{E}$ , where defined, agrees with  $\mathcal{F}$ . This translates to the statement that  $f(\ell \cap U) \subset \ell'$ , and we may apply Theorem A to deduce that  $f : U \xrightarrow{\cong} V$  extends to a biholomorphic automorphism  $F : \mathbb{P}^n \xrightarrow{\cong} \mathbb{P}^n$ , as desired.  $\square$

*Deduction of Theorem 7.5 from Theorem B.*

*Proof of Theorem 7.5.* In analogy to the proof of Theorem 7.4 we deduce Theorem 7.5 from Theorem B (Ochiai’s Theorem) and the hypothesis  $J_1f \circ id_1 \subset W_1$  that given  $x \in U$  and  $\ell$  a projective line passing through  $x$ ,  $f(\ell \cap U)$  must be tangent to some projective line  $\ell' = \gamma_x(\ell)$  at  $y = f(x)$  for some  $\gamma_x \in G$  such that  $\gamma_x(x) = y$ . Thus  $[df](\mathcal{C}(S)|_U) = \mathcal{C}(S)|_V$ , and by Theorem B  $f$  extends to a biholomorphic automorphism  $F \in \text{Aut}(S)$ .  $\square$

*Theorems 7.4 and 7.5 when  $\Omega$  is reducible.* The analogue of [Theorem 7.4](#) holds for any bounded symmetric domain  $\Omega$  provided that there are no irreducible factors biholomorphic to the disk  $\mathbb{B}^1$ . Likewise, the analogue of [Theorem 7.5](#) holds for any bounded symmetric domain  $\Omega$  provided that each irreducible factor of  $\Omega$  is of rank  $\geq 2$ . The proofs are small variations of the irreducible case. (For  $\Omega = \Omega_1 \times \cdots \times \Omega_m$ , one considers the  $m$  moduli spaces of projective lines  $\mathcal{K}_1, \dots, \mathcal{K}_m$  of the compact duals  $\widehat{\Omega}_1, \dots, \widehat{\Omega}_m$ .)

### 8. Connection formula and automorphic functions

8.1. *Definition of Schwarzian variety.* Notice that since  $N^+$  acts with an open orbit on  $\widehat{\Omega}$ , it follows that

$$G(\mathbb{C}) \backslash J_k \widehat{\Omega} \cong B \backslash (J_k \widehat{\Omega})_o$$

birationally, where  $(J_k \widehat{\Omega})_o$  denotes the fiber above  $o$ . We define this latter variety to be the  $k$ -th Schwarzian variety  $S_k(\widehat{\Omega})$ . Now for any rational map  $F : \widehat{\Omega} \rightarrow \widehat{\Omega}$  we may define the Schwarzian of  $F$  to be

$$S_k(F)(z) = G \cdot J_k F(\text{id}_k(z)) \in S_k(\widehat{\Omega})$$

at any point  $z \in \widehat{\Omega}$  where  $F$  is a morphism. If we restrict to the upper half plane and  $k = 3$ , then  $S_3(\mathbb{P}^1(\mathbb{C}))$  may be identified with  $\mathbb{P}^1(\mathbb{C})$  and  $S_k(F)$  becomes the usual Schwarzian.

We may define a rational map

$$\psi_k : S_k(\widehat{\Omega}) \times (N^+ \backslash J_k(\widehat{\Omega})) \rightarrow S_k(\widehat{\Omega})$$

as follows. An element of  $J_k(\widehat{\Omega})$  corresponds naturally to a point  $z \in \widehat{\Omega}$  and a germ of a map  $F : \widehat{\Omega} \rightarrow \widehat{\Omega}$  sending  $o$  to  $z$ . An element of  $S_k(\widehat{\Omega})$  corresponds to the germ at  $o$  of a map  $H : \widehat{\Omega} \rightarrow \widehat{\Omega}$  sending  $o$  to  $o$  modulo terms of order  $> k$  in the Taylor expansion of  $H$  at  $o$  and up to a left  $B$ -translation. If  $z \in N^+o$ , we let  $\nu_z \in N^+$  be the element sending  $z$  to  $o$ , and we may consider the composition  $H \circ \nu_z \circ F$  as defining an element of  $S_k(\widehat{\Omega})$ . It is clear that this is well defined.

For any positive integer  $d$ , we can naturally extend this to a map

$$\psi_k : S_k(\widehat{\Omega})^d \times (N^+ \backslash J_k(\widehat{\Omega}))^d \rightarrow S_k(\widehat{\Omega})^d.$$

8.2. *Connection formula.* For two maps  $F_1, F_2 : \widehat{\Omega} \rightarrow \widehat{\Omega}$ , it follows that  $J_k F_1 \circ J_k F_2 = J_k(F_1 \circ F_2)$ , from which it follows that

$$\psi_k \left( S_k(F_1)(F_2(z)), J_k F_2(\text{id}_k(z)) \right) = S_k(F_1 \circ F_2)(z).$$

We refer to this equation as the *connection formula*.

8.3. *Schwarzians of automorphic functions.* Let  $\Gamma \subset G(\mathbb{R})$  be a discrete subgroup such that  $X = \Gamma \backslash \Omega$  is a Shimura Variety. Consider a rational embedding<sup>5</sup>  $\phi = (\phi_1, \phi_2) : X \rightarrow \widehat{\Omega}^2$ , and let  $p_i = \phi_i \circ q : \Omega \rightarrow \widehat{\Omega}$ . Set  $p = (p_1, p_2)$ . Writing the connection formula for  $p_i, p_i^{-1}$  gives

$$\psi_k \left( S_k(p_i^{-1}(p_i(z))), J_k p_i(\text{id}_k(z)) \right) = S_k(\text{id}).$$

Now, inverting the action yields the relation

$$(*) \quad S_k(p_i^{-1}(p_i(z))) = \psi_k \left( S_k(\text{id}), J_k p_i(\text{id}_k(z))^{-1} \right).$$

Since (\*) shows that  $(S_k(p_1^{-1}(p_1(z))), S_k(p_2^{-1}(p_2(z))))$  is definable on a fundamental domain, and it is clearly  $\Gamma$  invariant, it must be a (single-valued) algebraic function of  $q(z)$ . Since  $\phi = (\phi_1, \phi_2)$  is a rational embedding, we conclude the following:

PROPOSITION 8.1. *We have that  $(S_k(p_1^{-1}(\phi_1(x))), S_k(p_2^{-1}(\phi_2(x))))$  is an algebraic function  $R_k(x)$  for  $x \in X$ .*

THEOREM 8.2. *Suppose that  $U \subset \widehat{\Omega}$  is a connected open set and  $F : U \rightarrow \phi(X) \subset \widehat{\Omega}^2$  satisfies*

$$\psi_3 \left( R_3(\phi^{-1}(F(z))), J_3 F(\text{id}_3(z)) \right) = (S_3(\text{id}), S_3(\text{id})).$$

*Then there exists  $g \in G(\mathbb{C})$  such that  $F(z) = p(gz)$ .*

*Proof.* Using the connection formula and Proposition 8.1, we see that

$$\begin{aligned} S(p^{-1} \circ F) &= \psi_3 \left( S_3(p^{-1})(F(z)), J_3 F(\text{id}_3(z)) \right) \\ &= \psi_3 \left( R_3(\phi^{-1}(F(z))), J_3 F(\text{id}_3(z)) \right) \\ &= (S_3(\text{id}), S_3(\text{id})). \end{aligned}$$

But now Theorem 7.3 shows that  $(p^{-1} \circ F)(z) = gz$  for some  $g \in G(\mathbb{C})$ , which completes the proof. □

Note that Theorem 7.4 shows that one may use  $R_2(z)$  in Theorem 8.2 in the case where  $\Omega = \mathbb{B}^n$ ,  $n \geq 2$ , and Theorem 7.5 shows that one may use  $R_1(z)$  in the case where  $\Omega$  is irreducible and of rank  $\geq 2$ , with obvious generalizations to the reducible cases (cf. the last paragraph of Section 7).

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<sup>5</sup>Since the complex function field of the  $n$ -dimensional projective variety  $X$  is always isomorphic to that of an irreducible hypersurface in  $\mathbb{P}^{n+1}$  and since  $\widehat{\Omega}$  is rational, such an embedding always exists, and we could deal with only a single function if  $X$  were rational, as is the case of  $X(1)$  and the  $j$ -function.

### 9. Ax-Schanuel with derivatives

#### 9.1. Main theorem with derivatives.

**THEOREM 9.1.** *Let  $k \geq 2$  and  $r \geq \dim X$  be positive integers. With notation as above, let  $W \subset J_k^{\text{nd},r} \Omega \times J_k^{\text{nd},r} X$  be an algebraic subvariety such that  $W = \widehat{W} \cap J_k^{\text{nd},r} \Omega \times J_k^{\text{nd},r} X$  for some irreducible subvariety  $\widehat{W} \subset J_k^{\text{nd},r} \widehat{\Omega} \times J_k^{\text{nd},r} X$ . Let  $U$  be an irreducible component of  $W \cap J_k^{\text{nd},r} D$  whose dimension is larger than expected, that is,*

$$\dim W < \dim U + \dim G.$$

*Then the projection of  $U$  to  $X$  is contained in a proper weakly special subvariety of  $X$ .*

Here  $J_k^{\text{nd},r} \widehat{\Omega} \times J_k^{\text{nd},r} X$  inherits the structure of a quasi-projective manifold resulting from a projective compactification<sup>6</sup> of  $J_k^r Y$  for a quasi-projective manifold  $Y$ .

We first need to state a lemma.

**LEMMA 9.2.** *For  $k \geq 2$ , consider the graph*

$$J_k^{\text{nd},r} D \subset J_k^{\text{nd},r} \Omega \times J_k^{\text{nd},r} X$$

*of the projection map  $J_k^r q : J_k^{\text{nd},r} \Omega \rightarrow J_k^{\text{nd},r} X$ . Then  $G(\mathbb{C}) \cdot J_k^{\text{nd},r} D$  is a closed, algebraic subvariety, and it is the Zariski closure of  $J_k^{\text{nd},r} D$ .*

*Proof.* First, note that  $J_k^{\text{nd},r} D$  is invariant under  $\Gamma$ , and therefore its Zariski closure is invariant under  $G(\mathbb{C})$ . Thus, it is sufficient to show that  $Y := G(\mathbb{C}) \cdot J_k^{\text{nd},r} D \subset J_k^{\text{nd},r} \widehat{\Omega} \times J_k^{\text{nd},r} X$  is a (closed) algebraic subvariety.

We claim that  $Y \subset J_k^{\text{nd},r} \widehat{\Omega} \times J_k^{\text{nd},r} X$  is constructible complex analytic. Consider the holomorphic map  $\Psi_0 : G(\mathbb{C}) \times J_k^{\text{nd},r} \widehat{\Omega} \rightarrow J_k^{\text{nd},r} \widehat{\Omega}$  defined by  $\Psi_0(\gamma, \omega) = \gamma \cdot \omega$ . Since  $G(\mathbb{C})$  acts algebraically on  $\widehat{\Omega}$ , there exist projective compactifications  $G(\mathbb{C})'$  of  $G(\mathbb{C})$  and  $(J_k^{\text{nd},r} \widehat{\Omega})'$  of  $J_k^{\text{nd},r} \widehat{\Omega}$  such that  $\Psi_0$  extends to a rational map  $\Psi'_0 : G(\mathbb{C})' \times (J_k^{\text{nd},r} \widehat{\Omega})' \rightarrow (J_k^{\text{nd},r} \widehat{\Omega})'$ . Write  $\Psi = \Psi_0 \times \text{id}$ , i.e.,

$$\Psi : (G(\mathbb{C}) \times (J_k^{\text{nd},r} \widehat{\Omega})) \times J_k^{\text{nd},r} X \rightarrow J_k^{\text{nd},r} \widehat{\Omega} \times J_k^{\text{nd},r} X$$

is given by  $\Psi(\gamma, \omega, \nu) = (\gamma \cdot \omega, \nu)$ . Define

$$\Psi' := \Psi'_0 \times \text{id} : (G(\mathbb{C})' \times (J_k^{\text{nd},r} \widehat{\Omega})') \times J_k^{\text{nd},r} X \rightarrow (J_k^{\text{nd},r} \widehat{\Omega})' \times J_k^{\text{nd},r} X.$$

Applying the proper mapping theorem to the graph of the restriction of  $\Psi'$  to  $G(\mathbb{C})' \times J_k^{\text{nd},r} D \subset G(\mathbb{C})' \times ((J_k^{\text{nd},r} \widehat{\Omega})' \times J_k^{\text{nd},r} X)$  and noting that  $Y$  is  $\Gamma$ -invariant,

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<sup>6</sup>We postpone to [Section 9.2](#) the construction of a projective compactification of  $J_k^r Y$  as a bundle of weighted projective spaces of a nonsingular projective compactification  $\widehat{Y}$  of  $Y$ .

we deduce that  $\bar{Y} \subset J_k^{\text{nd},r}\widehat{\Omega}' \times J_k^{\text{nd},r}X$  is a complex analytic subvariety, hence  $Y \subset \bar{Y}$  is constructible complex analytic.

Next, we argue that  $Y \subset J_k^{\text{nd},r}\widehat{\Omega} \times J_k^{\text{nd},r}X$  is a closed subset. Observe by Proposition 7.2 that  $G(\mathbb{C})$  acts freely on  $J_k^{\text{nd},r}\widehat{\Omega} \times J_k^{\text{nd},r}X$  so that any  $G$ -invariant constructible complex analytic subset of  $J_k^{\text{nd},r}\widehat{\Omega} \times J_k^{\text{nd},r}X$  is of dimension  $\geq \dim G$ . Now, note that  $Y$  is invariant under the automorphisms  $A$  of the disc  $\mathbb{D}_k^r$  and under the action of  $G(\mathbb{C})$ . Moreover, these actions commute and  $A \times G(\mathbb{C})$  acts transitively on  $J_k^{\text{nd},r}\widehat{\Omega}$ . It follows that if we let  $Z = \bar{Y} - Y$  be the boundary of  $Y$ , then  $Z$  is equidimensional over the first factor  $J_k^{\text{nd},r}\widehat{\Omega}$ . However, the pullback of  $Y$  to  $J_k^{\text{nd},r}\widehat{\Omega} \times J_k^{\text{nd},r}\widehat{\Omega}$  is symmetric, and so  $Z$  is also equidimensional over the second factor  $J_k^{\text{nd},r}X$ . Note that the fibers of  $Y$  over the second factor  $J_k^{\text{nd},r}X$  are of dimension  $\dim G$ . Now since  $\dim Z < \dim Y$ , it follows that the dimension of the fibers of  $Z$  over the second factor  $J_k^{\text{nd},r}X$  are less than  $\dim G$ , which means they are empty since  $Z$  is closed under the action of  $G(\mathbb{C})$ , proving the claim that  $Y \subset J_k^{\text{nd},r}\widehat{\Omega} \times J_k^{\text{nd},r}X$  is closed.

Since  $Y \subset J_k^{\text{nd},r}\widehat{\Omega} \times J_k^{\text{nd},r}X$  is closed and constructible complex analytic, it must be a complex analytic subvariety. On the other hand, since  $Y$  is given by  $G(\mathbb{C})$  acting on the restriction of  $J_k^{\text{nd},r}D$  to  $J_k^{\text{nd},r}\mathcal{F}$ , it follows that it is definable, and by Theorem 2.2,  $Y$  is an algebraic subvariety, as desired.  $\square$

**COROLLARY 9.3.** *Let  $z_1, \dots, z_n$  be an  $N^+$ -invariant algebraic coordinate system on  $\Omega$ . Let  $\{\phi_1, \dots, \phi_N\}$  be a  $\mathbb{C}$ -basis of modular functions. Then the field generated by  $\{\phi_i\}$  and their partial derivatives with respect to the  $z_j$  up to order  $k \geq 2$  has transcendence degree over  $\mathbb{C}$  equal to  $\dim G$ . Furthermore, the transcendence degree is the same over  $\mathbb{C}(z_1, \dots, z_n)$ .*

*Proof.* From Lemma 9.2 and Proposition 7.2 applied to  $r = \dim X = n$ , it follows that

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}(\{z_i\}, \{\phi_j^{(\nu)}\}) = \dim G + n.$$

The algebraic independence will therefore follow as soon as we show that the transcendence degree of  $\mathbb{C}(z_1, \dots, z_n)$  is equal to  $\dim G$ .

To see this, consider

$$V = J_k(\text{id} \times q)(\text{id}_k(\Omega)) \subset J_k^{\text{nd},n}D.$$

We have to show that the Zariski-closure  $W$  of the projection of  $V$  to  $J_k^{\text{nd},n}X$  has dimension  $\dim G$ . Let  $p \in X$  be a point and, without loss of generality, let  $o \in \Omega$  be a pre-image.

Identifying  $N^+ \backslash J_k^{\text{nd},n}(N^+o)$  with  $J_k^{\text{nd},n}(\text{id}_k(o))$  by quotienting out by the action of  $N^+$ , we get a rational map  $\psi : G(\mathbb{C}) \rightarrow J_k^{\text{nd},n}(o)$  by  $\psi(g) = g \cdot \text{id}_k(o)$ . Now, the pre-image of  $p$  in  $W$  contains  $\psi(\Gamma)$  so it must contain  $\psi(G(\mathbb{C}))$ .

It follows that  $W$  contains  $q(G(\mathbb{C}) \cdot \text{id}_k(o))$ , hence by definability it must be equal to its closure (definable since we can restrict to a fundamental domain, and its closure is complex analytic since it is algebraic in  $\widehat{\Omega}$ ). Since  $G(\mathbb{C})$  acts freely on  $J_k^{\text{nd},n}\widehat{\Omega}$ , by Proposition 7.2, it follows that  $\dim G(\mathbb{C}) \cdot \text{id}_k(o) = \dim G$ , as desired.  $\square$

*Remark.* The same argument shows that the projection

$$J_{k+1}q \left( G(\mathbb{C}) \cdot J_{k+1}^{\text{nd},n}(\text{id}_{k+1}(o)) \right) \rightarrow J_kq \left( G(\mathbb{C}) \cdot J_k^{\text{nd},n}(\text{id}_k(o)) \right)$$

is bijective for  $k \geq 2$  and that  $J_kq \left( G(\mathbb{C}) \cdot J_k^{\text{nd},n}(\text{id}_k(o)) \right)$  is the Zariski closure of the graph of the  $\nu$ -th partial derivatives of  $q$  for  $|\nu| \leq k$ . It follows that, for  $k \geq 2$ , the  $k$ -th partial derivatives of  $q$  are rational in the  $\nu$ -th partial derivatives of  $q$  for  $|\nu| \leq 2$ . In other words, the field generated by all the partial derivatives of  $q$  is generated by the partial derivatives of order  $\leq 2$ .

9.2. *Compactifying jet spaces.* We shall require a compactification of jet spaces to discuss Hilbert schemes. Thus, for a complex manifold  $Y$ , we define

$$BJ_k^r Y := \left( \text{Hom}(\mathbb{D}_k^r, Y) \times \mathbb{A}_{\mathbb{C}}^1 \right) / \mathbb{G}_{m,\mathbb{C}},$$

where the action is defined via  $r \cdot (t \mapsto f(t), s) := (t \mapsto f(rt), rs)$ . It is easy to see, by expanding into local coordinates given by Taylor series coefficients, that  $BJ_k^r Y$  is a weighted projective space over  $Y$  and is thus a projective variety if  $Y$  is projective. This then gives a functorial compactification of  $J_k^r Y$  for  $Y$  projective. For  $Y$  quasi-projective, taking  $\widehat{Y} \supset Y$  to be a nonsingular projective compactification,  $BJ_k^r \widehat{Y} \supset J_k^r \widehat{Y} \supset J_k^r Y$  gives a projective compactification of  $J_k^r Y$ .

9.3. *Descending Hilbert scheme loci.* Now we fix some algebraic subvariety  $W \subset J_k^{\text{nd},r}\Omega \times J_k^{\text{nd},r}X$ , with  $\widehat{W} \subset BJ_k^r \widehat{\Omega} \times BJ_k^r \widehat{X}$  its Zariski closure, and  $U$  an irreducible component of  $W \cap J_k^{\text{nd},r}D$ . We make no assumptions here on the dimension of  $U$ .

Let  $M$  be the Hilbert scheme of all subvarieties of  $BJ_k^r \widehat{\Omega} \times BJ_k^r \widehat{X}$  with Hilbert polynomial  $P$ . Then  $M$  also has the structure of an algebraic variety. Corresponding to  $y \in M$  we have the subvariety  $W_y \subset J_k^{\text{nd},r}\Omega \times J_k^{\text{nd},r}X$ , and we have the incidence variety (universal family)

$$B = \{ (z, x, y) \in J_k^{\text{nd},r}\Omega \times J_k^{\text{nd},r}X \times M : (z, x) \in W_y \}$$

and the family of the intersections of its fibres over  $M$  with  $J_k^{\text{nd},r}D$ , namely,

$$A = \{ (z, x, y) \in J_k^{\text{nd},r}\Omega \times J_k^{\text{nd},r}X \times M : (z, x) \in W_y \cap J_k^{\text{nd},r}D \}.$$

Then  $A$  is a closed complex analytic subset of  $J_k^{\text{nd},r}\Omega \times J_k^{\text{nd},r}X \times M$ . It has natural projection  $\theta : A \rightarrow M$ , with  $(z, x, y) \mapsto y$ . Then, for each natural

number  $k$ , the set

$$A(k) = \{(z, x, y) \in J_k^{\text{nd},r}\Omega \times J_k^{\text{nd},r}X \times M : \dim_{(z,x)}\theta^{-1}\theta(z, x, y) \geq k\},$$

the dimension being the dimension at  $(z, x)$  of the fibre of the projection in  $A$ , is closed and complex analytic; see, e.g., the proof of [27, Lemma 8.2] and references therein.

Now we have the projection  $\psi : J_k^{\text{nd},r}\Omega \times J_k^{\text{nd},r}X \times M \rightarrow J_k^{\text{nd},r}\Omega \times J_k^{\text{nd},r}X$ , and we consider

$$Z = Z(k) = \psi(A(k)).$$

Then as  $M$  is compact,  $\psi$  is proper and so  $Z$  is closed in  $J_k^{\text{nd},r}\Omega \times J_k^{\text{nd},r}X$ . Note that  $Z$  is  $\Gamma$ -invariant and  $Z \cap J_k^{\text{nd},r}\mathcal{F} \times J_k^{\text{nd},r}X$  is definable.

LEMMA 9.4. *Let  $T = (q \times \text{id})(Z)$ . Then  $T \subset J_k^{\text{nd},r}X \times J_k^{\text{nd},r}X$  is closed and algebraic.*

*Proof.* This is proven in the same way as Lemma 3.3. □

### 10. Proof of Theorem 9.1

*Proof.* We argue by induction, in the first instance (upward) on  $\dim \Omega$ . For a given  $\dim \Omega$ , we argue (upward) on  $\dim W - \dim U$ . We then argue by induction (downward) on  $\dim U$ , and finally upward on  $r$ . The base cases are established in a way similar to Theorem 1.1.

We carry out the constructions of Section 9.3 with  $k = \dim U$  and keep the notation there. We let  $A(k)' \subset A(k)$  be the irreducible component that contains  $U \times [W]$ ,  $Z' = \psi(A'(k)) \subset Z$  be the corresponding irreducible component of  $Z$ , and  $V = (q \times \text{id})(Z')$  be the irreducible component of  $T$ , which is therefore algebraic by Lemma 3.3. Now, by assumption  $V$  contains  $q(U)$ , and so it is not contained in any proper weakly special of the diagonal  $J_k^{\text{nd},r}\Delta_X$ , and thus its monodromy group is Zariski-dense in  $G$  by [1, §1,Th. 5.1].

Consider the family  $F_0$  of algebraic varieties corresponding to  $A(k)'$ . Let  $\Gamma_0 \subset \Gamma$  be the subgroup of elements  $\gamma$  such that a very general member of  $F_0$  is invariant by  $\gamma$ . Note that a very general element  $W'$  of  $F_0$  is invariant by exactly the subset  $\Gamma_0$  of  $\Gamma$ . Let  $\Theta$  be the connected component of the Zariski closure of  $\Gamma_0$  in  $G(\mathbb{R})$ .

LEMMA 10.1.  *$\Theta$  is a normal subgroup of  $G$ .*

*Proof.* This is proven exactly as in Lemma 4.1. □

LEMMA 10.2.  *$\Theta$  is the identity subgroup.*

*Proof.* We argue by contradiction. Without loss of generality we may assume that  $W$  is a very general member of  $F_0$ , and hence it is invariant by exactly  $\Theta$ . Since  $\Theta$  is a  $\mathbb{Q}$ -group by construction, it follows that  $G$  is

isogenous to  $\Theta \times \Theta'$  and we have a splitting of Hermitian symmetric domains  $\Omega = \Omega_\Theta \times \Omega_{\Theta'}$ . Replacing  $\Gamma$  by a finite index subgroup we also have a splitting  $X \cong X_\Theta \times X_{\Theta'}$ . Moreover,  $D$  splits as  $D_\Theta \times D_{\Theta'}$ .

By our induction on  $\dim W$ , it follows from [Lemma 9.2](#) that  $W \subset G(\mathbb{C}) \cdot J_k^{\text{nd},r} D$ . Now, we let

$$W_1 \subset J_k^{\text{nd},r} \Omega_{\Theta'} \times J_k^{\text{nd},r} X_{\Theta'} \times J_k^{\text{nd},r} X_\Theta$$

be the projection of  $W$ . Let  $U_1$  be the projection of  $U$  to  $W_1$ . Since the map from  $D_\Theta$  to  $X_\Theta$  has discrete pre-images, it follows that  $\dim U = \dim U_1$ .

Now let  $W'$  be the projection of  $W_1$  to  $J_k^{\text{nd},r} \Omega_{\Theta'} \times J_k^{\text{nd},r} X_{\Theta'}$  and  $U'$  be a component of  $W' \cap D_{\Theta'}$ . Since  $W \subset G(\mathbb{C}) \cdot J_k^{\text{nd},r} D$  and  $W$  is closed under  $\Theta(\mathbb{R})^+$ , we see that  $U'$  is the projection of  $U$  to  $J_k^{\text{nd},r} \Omega_{\Theta'} \times J_k^{\text{nd},r} X_{\Theta'}$ . Now let  $W'' \subset W'$  be the Zariski closure of  $U'$ . It follows by the induction hypothesis that

$$\dim U' + \dim \Theta' \leq \dim W''.$$

Now for the projection map  $\psi : W_1 \rightarrow W'$ , the generic fiber dimension of  $W''$  is the same as the generic fiber dimension over  $U'$ , and thus

$$\dim U_1 + \dim \Theta' \leq \dim \psi^{-1}(W'') \leq \dim W_1.$$

Since  $W$  is invariant under  $\Theta(\mathbb{C})$ , it follows that

$$\dim U + \dim G \leq \dim W,$$

contradicting the hypothesis  $\dim W < \dim U + \dim G$  as in the statement of [Theorem 9.1](#), as desired. □

It follows that  $W$  is not invariant by any infinite subgroup of  $\Gamma$ . The following lemma thus gives a contradiction, and completes the proof.

LEMMA 10.3.  *$W$  is invariant by an infinite subgroup of  $\Gamma$ .*

*Proof.* This is proven exactly as [Lemma 4.3](#). □

This contradiction completes the proof of [Theorem 9.1](#). □

As a corollary, we have the following concrete statement (stated as [Theorem 1.4](#)) about transcendence degrees of modular functions and their derivatives on analytic subvarieties.

COROLLARY 10.4. *Let  $V \subset \Omega$  be an irreducible complex analytic variety, not contained in a proper weakly special subvariety. Let  $\{z_i, i = 1, \dots, n\}$  be an algebraic coordinate system on  $\Omega$ . Let  $\{\phi_j^{(\nu)}\}$  consist of a basis  $\phi_1, \dots, \phi_N$  of modular functions, all defined at at least one point of  $V$ , together with their partial derivatives with respect to the  $z_j$  up to order  $k \geq 2$ . Then*

$$\text{tr.deg.}_{\mathbb{C}} \mathbb{C}(\{z_i\}, \{\phi_j^{(\nu)}\}) \geq \dim G + \dim V,$$

where all functions are considered restricted to  $V$ .



*Proof.* Let

$$U = (\text{id}_2 \times J_2q \circ \text{id}_2)(V) \subset J_2\Omega \times J_2X,$$

so that  $U$  is an analytic subvariety of the diagonal  $J_2D$ . Note that  $U$  does not record any of the differential information concerning  $V$ , but instead a coordinate system for  $U$  is given by  $z_1, \dots, z_n, \phi_1, \dots, \phi_N$ . The result now follows immediately from [Theorem 9.1](#) applied to the Zariski closure of  $U$ .  $\square$

**Part III. Ax-Schanuel in a differential field**

In this part we formulate a version of Ax-Schanuel in the setting of a differential field. We further show that the jet version (9.1) may be deduced directly from the differential version (12.1).

**11. Characterizing the uniformization map**

We would like to have a criterion in a differential field to determine when a pair of maps

$$w : \Delta \rightarrow \widehat{\Omega}, \quad u : \Delta \rightarrow X = \Gamma \backslash \Omega,$$

where  $\Delta$  is a disk of given dimension, satisfies  $u = q(gv)$  for some  $g \in G(\mathbb{C})$ . To this end, we need a finer version of the Schwarzian varieties introduced in [Section 8](#) that serves the same purpose for jets of lower rank, i.e., those that are non-surjective on tangent spaces.

LEMMA 11.1. *Let  $k$  be a positive integer. There exists a positive integer  $m = m(G, q, k)$  with the following property. Let  $\Delta^k$  be a  $k$ -dimensional disk, and consider a pair of maps  $(w, u) : \Delta^k \rightarrow \widehat{\Omega} \times \widehat{\Omega}$ . If at every  $t \in \Delta^k$  there exists  $g_t \in G(\mathbb{C})$  such that  $u(t) = g_t w(t)$  up to order  $m$ , then there exists a global  $g \in G(\mathbb{C})$  with  $u(t) = gw(t)$ .*

*Proof.* Let  $d = \dim G$ . For positive integers  $a \geq b$ , we define

$$X_{a,b} = \{(\phi, \psi) \mid \exists \gamma \in G(\mathbb{C}), \gamma \circ \phi = \psi \text{ up to order } b\} \subset J_a^k \widehat{\Omega} \times J_b^k \widehat{\Omega}.$$

We partition  $X_{d,d}$  into subsets  $X_{d,d}^i$  consisting of all pairs  $(\phi, \psi)$  satisfying  $\text{Stab}\phi_i = \text{Stab}\phi_{i+1}$ , where  $\phi_i$  denotes  $\phi$  up to the  $i$ -th order, and  $i$  is the smallest such integer. Note that this is indeed a partition since each time  $\text{Stab}\phi_i \neq \text{Stab}\phi_{i+1}$  the dimension of  $\text{Stab}\phi_{i+1}$  is at least 1 smaller than that of  $\text{Stab}\phi_i$ . Without loss of generality, assume that  $(w, u)$  has image generically in  $X_{d,d}^i$ . Now, for all elements in  $X_{d,d}^i$ , it follows that there exists a universal equation for all  $(i + 1)$ -st partial derivatives of  $\psi$  in terms of the degree  $\leq d$  partial derivatives of  $\phi$  and the degree  $\leq i$  partial derivatives of  $\psi$  at  $o$ . Thus one can solve for  $u$  given  $w$  and all the partial derivatives of degree  $\leq i$  of  $u$  at  $0 \in \Delta^k$ . Clearly  $u(t) = g_o^{-1}w(t)$  is one such solution, and so that must be the only solution. This completes the proof.  $\square$

With  $k, m$  as above, let  $J_m^k D \subset J_m^k \widehat{\Omega} \times J_m^k X$  be the graph of the projection morphism. Now let  $V_m^k = G(\mathbb{C}) \cdot J_m^k D$ , where the group  $G(\mathbb{C})$  acts only on the factor  $J_m^k \widehat{\Omega}$ . Note that  $V_m^k$  is definable, since

$$V_m^k = G(\mathbb{C}) \cdot (J_m^k D|_{J_m^k \mathcal{F} \times J_m^k X}).$$

Also,  $V_m^k$  is the image of  $X_{m,m}$  under the projection map on the second factor, so its closure is analytic and of the same dimension. It follows from Definable Chow that  $V_m^k$  is constructible algebraic.

Now if  $w : \Delta^k \rightarrow \widehat{\Omega}$  and  $g \in G(\mathbb{C})$ , then the image of  $(w, q(gw))$  in  $J_m^k \widehat{\Omega} \times J_m^k X$  is contained in  $V_m^k$ . We show the converse.

**THEOREM 11.2.** *Let  $k$  be a positive integer, and let  $m = m(G, q, k)$  and  $V_m^k$  be as above. Let  $(w, u) : \Delta^k \rightarrow \widehat{\Omega} \times X$ . If the image of  $J^k(w, u)$  is contained in  $V_m^k$ , then there exists a global  $g \in G(\mathbb{C})$  with  $u(t) = q(gw(t))$ .*

*Proof.* Suppose  $J^k(w, u)$  lands in  $V_m^k$ . Let  $\tilde{u}(t)$  be a local lift of  $u(t)$  such that  $u(t) = q(\tilde{u}(t))$ . By [Lemma 11.1](#) it follows that there exists some  $g_0 \in G(\mathbb{C})$  such that  $w(t) = g_0 \tilde{u}(t)$ . Thus,  $u(t) = q(g_0^{-1}w(t))$  as desired.  $\square$

**11.1. Uniformized loci.** Given a map  $w : \Delta^k \rightarrow \widehat{\Omega}$ , an element  $g \in G(\mathbb{C})$ , and an integer  $r \geq 2$ , we get in a natural way a map

$$L(w, g, r) : \Delta^k \rightarrow \widehat{\Omega} \times X \times J_r^n X,$$

where the second map is  $q(gw)$  and the third map is  $J_r^n(q \circ g) \circ \text{id}_r \circ w$ , which records the partial derivatives of  $q_g$ , where  $q_g(z) = q(gz)$ , to order  $|\nu| \leq r$ , restricted to the image of  $w$ . We will call such a map a *uniformized locus*. Note that the second map is repeated in the zero-order terms of the third map, and so it is superfluous in a way, but we find it convenient to keep track of it.

From such a map we obtain an image in the jet spaces (to some order  $m$ )

$$J_m^* L(w, g, r) = \widetilde{v}_k^g : \Delta^k \rightarrow J_m^k \widehat{\Omega} \times J_m^k X \times J_r^n X,$$

where the first map is  $J_m^k(w) \circ \text{id}_m$ , the second map is  $J_m^k(q \circ g \circ w) \circ \text{id}_m$ , and the third map is (again)  $J_r^n(q \circ g) \circ \text{id}_r \circ w$ .

We want differential equations that characterize when a trio of maps

$$(w, v, u) : \Delta^k \rightarrow \widehat{\Omega} \times X \times J_r^n X$$

arises in this way. For  $(w, u)$ , this is dealt with by [Theorem 11.2](#).

Now, we cannot directly talk about the map  $q$ , though we do have access to the map to  $J_m^n X$ . There is a complication when  $w$  has a stabilizer, in that we could replace  $w$  by  $g \circ w$  for any  $g$  in the stabilizer, and in fact by a different, holomorphically varying,  $g$  at every point. This would not affect the map or any of its derivatives, but it would affect the restrictions of the derivatives of  $q \circ g$ . Thus, we will equip the third coordinate with an extra differential equation to insist that the  $g$  “stays constant.” We do this as follows.

We have an algebraic map  $G(\mathbb{C}) \times J_\ell^n \widehat{\Omega} \rightarrow J_\ell^n \widehat{\Omega}$  given by the natural action. Let  $V'_{n,\ell}$  be the image of  $G(\mathbb{C}) \times \text{id}_\ell(\widehat{\Omega})$  restricted to  $J_\ell^n \Omega$ , and  $V_{n,\ell}$  its constructible algebraic (by [Corollary 3.2](#)) image in  $J_\ell^n X$ . Note that for  $\ell \geq 3$ , the variety  $V'_{n,\ell}$  is fibered by varieties  $\Omega$  over  $G(\mathbb{C})/N^+(\mathbb{C})$ .<sup>7</sup> If we descend to  $X$ , we lose the fibration but we retain an algebraic foliation.

To make this precise, let  $W'_{n,\ell}$  be the restriction to  $TV'_{n,\ell}$  of the image of the natural holomorphic map  $G(\mathbb{C}) \times \text{Tid}_\ell(\widehat{\Omega}) \rightarrow TJ_\ell^n \widehat{\Omega}$ . Then the image  $W_{n,\ell}$  of  $W'_{n,\ell}$  in  $TJ_\ell^n X$  determines an integrable algebraic (by [Corollary 3.2](#)) foliation of  $V_{n,\ell}$ .

A map  $L(w, g, r)$  has its image in  $V_{k,m} \times V_{n,r}$ , and the tangent lands in  $W_{n,r}$ . We show that these properties characterize such maps.

**THEOREM 11.3.** *Let  $k$  be a positive integer,  $r \geq 2$ , and  $m$  as above. Consider a map*

$$(w, u, v) : \Delta^k \rightarrow \widehat{\Omega} \times X \times J_r^n X.$$

*If the image of  $(w, u, v)$  lands in  $V_{k,m} \times V_{n,r}$  and the image of  $Tv$  lands in  $W_{n,r}$ , and if  $v$  restricts to  $u$ , then there exists  $g \in G(\mathbb{C})$  such that  $(w, u, v) = L(w, g, r)$ .*

*Proof.* Suppose that the hypotheses are satisfied. By [Theorem 11.2](#) it follows that there exists  $g \in G(\mathbb{C})$  with  $u = q \circ g \circ w$ . So it remains to address the third coordinate. Note that if  $w$  has no stabilizer, the claim follows immediately. As it stands, the third map must be of the form  $t \rightarrow q \circ g(t) \circ \text{id}_k \circ w$  for some function  $g : \Delta^k \rightarrow G(\mathbb{C})$ . However, by assumption,  $Tv$  lands in  $W_{n,r}$ , which is an integrable foliation whose leaves are precisely the set where  $g(t) \in gN$ . Thus we may write  $g(t) = gn(t)$  for a function  $n(t) \in N$ . However, since  $v$  restricts to  $u$ , it follows that  $n(t)$  is the identity function. Thus the function  $g(t) = g$  must be constant and the claim is proved.  $\square$

## 12. Ax-Schanuel in a differential field

12.1. *The setting.* We fix  $G$  and  $q : \Omega \rightarrow X$ . Let  $n = \dim X$ . We take a field of definition  $L_0 \subset \mathbb{C}$ , of finite type, for  $X$  and for the system of differential equations satisfied by  $q$ .

The weakly special subvarieties of  $X$  come in countably many families, and so correspond to points in suitable (possibly constructible) subvarieties of countably many Hilbert schemes. These families are defined over  $\overline{\mathbb{Q}}$  and the collection of families is stable under Galois automorphisms. So we may take them to be (not necessarily irreducible but) defined over  $L_0$ .

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<sup>7</sup>In fact,  $V'_{n,\ell}$  is fibered by  $G(\mathbb{C})/N^+(\mathbb{C})$  over  $\Omega$  and is indeed isomorphic to the product  $\Omega \times G(\mathbb{C})/N^+(\mathbb{C})$ .

We consider a differential field  $(K, \mathcal{D}, C)$ . Here  $\mathcal{D} = \{D_1, \dots, D_d\}$  is a finite set of commuting derivations and  $C$  is the constant field.

It is a deep theorem of Kazhdan [15], [16], [17] (see also Milne [21], [20]) that a conjugate of an arithmetic variety is again arithmetic. It will therefore be important to ensure that our differential fields are identified correctly with the complex object. We therefore assume (initially) that  $C$  contains a subfield  $\Lambda_0$  isomorphic to  $L_0$  under  $\iota_0 : \Lambda_0 \rightarrow L_0$ .

We take further a field of finite type  $L \subset \mathbb{C}$  that is a field of definition for the constructible algebraic varieties in Section 11, and assume that  $C$  contains a field  $\Lambda$  isomorphic (by  $\iota$ ) to  $L$ . Then we can identify the various varieties in Section 11 with their corresponding varieties in  $K$ .

12.2. *Rank and transcendence degree.* Let  $V$  be an algebraic variety <sup>8</sup> over the constant field  $C$ , and let  $p \in V(K)$  be a  $K$ -point. Let  $U \subset V$  be an open affine set defined over  $C$  that contains  $p$ . Let  $R = \mathcal{O}(U)$  be the ring of functions on  $U$ , and let  $S$  be the image of  $R$  in  $K$  induced by evaluation on  $p$ . We define the *transcendence degree* of  $p$  to be the transcendence degree of the fraction field of  $S$  over  $C$ .

We define the *rank* of  $p$  to be the rank of the matrix  $\text{rank}(p) = (D_i s)_{\substack{1 \leq i \leq n \\ s \in S}}$ .

Finally, given two varieties  $V_1, V_2$  and points  $p_1 \in V_1(K), p_2 \in V_2(K)$ , we say that  $p_2$  is a function of  $p_1$  if  $\text{rank}(p_1, p_2) = \text{rank}(p_2)$ .

12.3. *Statement and proof of differential Ax-Schanuel.* We consider  $K$  points  $(z, x, y)$  of  $\widehat{\Omega} \times X \times J_r^n X$  where  $r \geq 2$ . We assume that  $x$  and  $z$  have the same rank and that  $x$  is a function of  $z$ . Set  $k = \text{rank}(x) = \text{rank}(z)$ , and assume  $k \geq 1$ . (For  $k = 0$ , our theorems are true but trivial.)

We consider Seidenberg embeddings of  $K$  into fields of meromorphic functions [35]. If  $(z, x, y)$  satisfies the differential conditions corresponding to the hypotheses of Theorem 11.3 then, under any Seidenberg embedding *over*  $\iota$ , meaning extending  $\iota$ , as may always be assumed (see the version given in Scanlon [34]), we get tuples of regular functions

$$(\bar{z}, \bar{x}, \bar{y}) : \Delta^k \rightarrow \widehat{\Omega} \times X \times J_r X$$

that give a uniformized locus. It is thus a natural abuse of notation to refer to a tuple  $(z, x, y)$  satisfying these conditions as a *uniformized locus* in  $K$ . We will say similarly that  $x$  is *contained in a weakly special subvariety* if it gives a  $K$  point of one of the varieties defining the weakly special families.

We can now state a differential version of Ax-Schanuel, making precise Theorem 1.5.

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<sup>8</sup>Or a scheme of finite type.

**THEOREM 12.1** (Differential Ax-Schanuel). *Fix  $G, q, L$  as above, and let  $(K, \mathcal{D}, C)$  be a differential field with  $\Lambda \subset K, \iota$  as above. Let  $(z, x, y)$  be a uniformized locus. Then*

$$\text{tr.deg.}_C C(z, x, y) \geq \text{rank}(z) + \dim G$$

*unless  $x$  is contained in a proper weakly special subvariety.*

*Proof.* Suppose the transcendence degree  $\text{tr.deg.}_C C(z, x, y)$  is less, so that there is a variety  $W$  defined over  $C$  containing these quantities with  $\dim W < \text{rank}(z) + \dim G$ . Take a suitable finitely generated differential  $K' \subset K$  field containing  $\Lambda$  and all constants appearing in the algebraic dependencies, the  $z$  and the associated  $q$ -quantities, and a field of definition of  $W$ . Take a Seidenberg embedding of  $K'$  over  $\iota$  into a field of meromorphic functions of  $t \in \Delta^k$ , where  $k = \text{rank}(z)$ .

Then  $\bar{x} = q_g(\bar{z})$  for some  $g \in G(\mathbb{C})$  for which the partial derivatives of  $q_g$ , restricted to the image of  $\bar{z}$  as a function of  $t \in \Delta^k$ , agree with the Seidenberg embeddings  $\bar{y}$  of  $y$ . (Note that the tuple  $y$  records the co-ordinates of  $x$ .) Define  $Q = q_g$ , and let  $\Omega_g = g^{-1}\Omega$  be the domain of  $Q$ .

Let  $U'$  be the locus in  $J_2^n \Omega_g \times J_2^n X$  that is the graph under  $J^n Q$  of the locus

$$(\bar{z}, 1_{n \times n}, 0, \dots, 0) \subset J_2^n \Omega_g.$$

By assumption  $(z, x, y)$  is a uniformized locus, and therefore  $U'$  is the locus (i.e. over  $t \in \Delta^k$ ) described by

$$(\bar{z}, 1_{n \times n}, 0, \dots, 0; \bar{x}, \bar{y})$$

and is clearly in  $J_2^{\text{nd},n} \Omega_g \times J_2^{\text{nd},n} X$ . (Recall the assumption that  $q$  is unramified.)

We let  $W'$  be the Zariski closure (over  $\mathbb{C}$ ) of  $U'$  and  $U \subset W' \cap J^n Q$  the component containing  $U'$ . We have  $\dim_{\mathbb{C}} U' = k = \text{rank}(z)$ , while  $W'$  is a subvariety of the image of  $W$  under the Seidenberg embedding, so  $\dim_{\mathbb{C}} W' \leq \dim_C W$ . By assumption, we have  $\text{rank}(z) > \dim_C W - \dim G$ , and therefore we have

$$\dim_{\mathbb{C}} U \geq \dim_{\mathbb{C}} U' = \text{rank}(z) > \dim_C W - \dim G \geq \dim_{\mathbb{C}} W' - \dim G.$$

Hence by [Theorem 9.1](#), for the map  $Q$ , which is just the same statement as for  $q$ , we conclude that  $\bar{x}$  is contained in a proper weakly special subvariety of  $X$ . But then  $x$  also has this property. □

We next show that the jet version of Ax-Schanuel ([Theorem 9.1](#)) may in fact be deduced directly from [Theorem 12.1](#).

**12.4. Direct proof of Theorem 9.1 from Theorem 12.1.** The proof follows the corresponding deduction in [\[30\]](#). We assume [Theorem 12.1](#) holds. If  $A = \{f_1, \dots, f_\ell\}$  is a set of regular functions of  $t \in \Delta^k$ , we set

$$\dim A = \dim(f_1, \dots, f_M) = \dim\{(f_1(t), \dots, f_M(t)) : t \in \Delta^k\},$$

where  $\{(f_1(t), \dots, f_M(t)) : t \in \Delta^k\}$  is the locus parametrized by  $A$ . The transcendence degree  $\text{tr.deg.}_{\mathbb{C}} \mathbb{C}(f_1, \dots, f_M)$  is the dimension of the Zariski closure of this locus, which we denote  $\dim \text{Zcl}(A) = \dim \text{Zcl}(f_1, \dots, f_M)$ .

We consider a locus  $U \subset J_\ell^{\text{nd},n} D$ , of dimension  $k$  say, where  $\ell \geq 2$ , meaning the graph of  $q$  on some locus of non-degenerate jets.

We take  $z = (z_1, \dots, z_n), x = (x_1, \dots, x_N)$  as affine coordinates on  $\Omega$  and an open affine subset of  $X$  containing an open subset of  $U$ . We assume that  $x_1, \dots, x_n$  are algebraically independent on  $X$  and the further variables dependent upon them. We take coordinates

$$(z, r, s, \dots; x, R, S, \dots)$$

in  $J_\ell^n \Omega \times J_\ell^n X$ , where  $r = (r_j^i, i, j = 1, \dots, n)$  with  $r^i$  representing the coordinates of the first derivatives of  $z_i$ , likewise  $R = (R_j^i)$  for  $x_i, i = 1, \dots, N$ ,  $s = (s_{jk}^i)$  and  $S = (S_{jk}^i)$  the second derivatives of  $z_i, x_i$  etc. The non-degeneracy condition means that the matrix  $r$  has rank  $n$ .

Then the action of  $q$  on the jets is given by  $x_i = q_i(z)$  and

$$R_j^i = \sum_p \frac{\partial q_i}{\partial z_p} r_j^p, \quad S_{jh}^i = \sum_p \frac{\partial q_i}{\partial z_p} s_{jh}^p + \sum_m \sum_p \frac{\partial q_i}{\partial z_p \partial z_m} r_h^m r_j^p, \quad \text{etc.}$$

with summations  $p, m = 1, \dots, n$ .

Thus  $U$  is locally parametrized by  $t \in \Delta^k$  in the form

$$(z(t), r(t), s(t), \dots; q(z(t)), R(t), S(t), \dots),$$

where

$$R_j^i(t) = \sum_p \frac{\partial q_i}{\partial z_p} r_j^p(t), \quad S_{jh}^i(t) = \sum_p \frac{\partial q_i}{\partial z_p} s_{jh}^p(t) + \sum_m \sum_p \frac{\partial q_i}{\partial z_p \partial z_m} r_h^m(t) r_j^p(t), \quad \text{etc.},$$

and the derivatives of  $q$  are evaluated at  $z(t)$ .

We must prove that, as functions of  $t$ ,

$$\dim \text{Zcl}(z, r, s, \dots; x, R, S, \dots) \geq \dim(z, r, s, \dots) + \dim G.$$

We claim that

$$\mathbb{C}(z, r, s, \dots; x, R, S, \dots) = \mathbb{C}(z, r, s, \dots; x, q_i^{(\nu)} \circ z(t), |\nu| \leq \ell).$$

Clearly the left-hand side is contained in the right. On the other hand, since  $r$  has full rank, by our non-degeneracy assumption we may find a map  $\phi : \Omega \rightarrow \Delta^k$  such that  $(z, r, s, \dots) \circ \phi$  is the identity, and thus the two fields are equal. Thus we need to prove

$$\dim \text{Zcl}(z, r, s, \dots; x, q_i^{(\nu)} \circ z(t), |\nu| \leq \ell) \geq \dim(z, r, s, \dots) + \dim G.$$

We consider the differential field containing the functions  $z, x, y = q_i^{(\nu)} \circ z$ . The hypotheses of [Theorem 12.1](#) hold; that is,  $(z, x, y)$  is a uniformized locus of rank  $k = \text{rank}(z) = \dim U$ .

If the projection of  $U$  to  $X$  is not contained in a proper weakly special subvariety then we have

$$\dim \text{Zcl}(z, x, y, \dots) \geq \dim(z) + \dim G.$$

The conclusion then follows because, for any sets  $A, B$  of functions,

$$\dim \text{Zcl}(A, B) - \dim \text{Zcl}(A) \geq \dim(A, B) - \dim A.$$

This concludes the proof. □

12.5. *A special case.* We state a special case of [Theorem 12.1](#), which clarifies the relationship between it and the modular and exponential cases. This version views  $q$  as an analogue of  $\exp / j$ . It concerns differential avatars of the Cartesian product map

$$q^\ell : \Omega^\ell \rightarrow X^\ell$$

which are suitably non-degenerate on each factor so that the differential equation can be straightforwardly imposed on the corresponding coordinate functions.

The point is that if  $z, x$  have rank  $n$  and  $(z, x) \in V_m^k$ , then  $y$  is uniquely determined and lies in  $J_r^n(K)$  by solving suitable systems of linear equations in the derivatives of  $z, x$ .

**THEOREM 12.2.** *Suppose  $z = (z^{(1)}, \dots, z^{(\ell)}) \in \widehat{\Omega}^\ell(K)$ , with  $\text{rank}(z^{(k)}) = n$  for each  $k = 1, \dots, \ell$ , and  $x = (x^{(1)}, \dots, x^{(\ell)}) \in X^\ell(K)$ , each of rank  $n$ , such that  $x^{(i)}$  is a function of  $z^{(i)}$  for each  $i$ . Let  $y^{(i)}$  be the partial derivatives of  $x^{(i)}$  with respect to  $z^{(i)}$  up to order  $r \geq 2$ , and put  $y = (y^{(1)}, \dots, y^{(\ell)})$ .*

*Suppose further that  $(z^{(k)}, x^{(k)}) \in V_m^k$  for each  $k = 1, \dots, \ell$ . Then*

$$\text{tr.deg.}_C C(z, x, y) \geq \text{rank}(z) + \ell \dim G$$

*unless  $x$  is contained in a proper weakly special subvariety of  $X$ .*

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