Annals of Mathematics **189** (2019), 863–883
https://doi.org/10.4007/annals.2019.189.3.5

A bound on the cohomology of quasiregularly elliptic manifolds

By Eden Prywes

Abstract

We show that a closed, connected and orientable Riemannian manifold M of dimension d that admits a nonconstant quasiregular mapping from $\mathbb{R}^d$ must have bounded dimension of the cohomology independent of the distortion of the map. The dimension of the degree l de Rham cohomology of M is bounded above by $\binom{d}{l}$. This is a sharp upper bound that proves the Bonk-Heinonen conjecture. A corollary of this theorem answers an open problem posed by Gromov in 1981. He asked whether there exists a d-dimensional, simply connected manifold that does not admit a quasiregular mapping from $\mathbb{R}^d$. Our result gives an affirmative answer to this question.

1. Introduction

Let M be a closed, connected and orientable Riemannian manifold of dimension d. A K-*quasiregular mapping*, $K \geq 1$, is a continuous mapping $f\colon \mathbb{R}^d \to M$ such that $f \in W^{1,d}_{\text{loc}}(\mathbb{R}^d, M)$ and the differential, $Df : T\mathbb{R}^d \to TM$, satisfies

$$\|Df(x)\|^d \leq K J_f(x)$$

for almost every $x \in \mathbb{R}^d$, where $J_f = \det(Df)$. If M admits such a nonconstant quasiregular mapping, then we call M *quasiregularly elliptic*. The main result of this paper is as follows.

Theorem 1.1. *Let M be a closed, connected and orientable Riemannian manifold of dimension d. If M admits a nonconstant quasiregular mapping from $\mathbb{R}^d$, then $\dim H^l(M) \leq \binom{d}{l}$, for $0 \leq l \leq d$, where $H^l(M)$ is the de Rham cohomology of M of degree l.*

Theorem 1.1 is the first result that gives a restriction, independent of the fundamental group of M and the distortion K of the mapping, on quasiregular

Keywords: quasiregular ellipticity, de Rham cohomology
AMS Classification: Primary: 30C65; Secondary: 58A12.

ellipticity of closed manifolds. A K-dependent version of Theorem 1.1 was proved by Bonk and Heinonen [2]. They showed that $\dim H^l(M) \leq C(d,l,K)$ and conjectured that the constant is independent of K. Theorem 1.1 answers this with a sharp bound. The d-dimensional torus, $T^d = S^1 \times \cdots \times S^1$, is quasiregularly elliptic and $\dim H^l(T^d) = \binom{d}{l}$.

This theorem also gives an answer to a longstanding open problem first posed by Gromov in 1981 [9, p. 200]. He asked whether their exists a d-dimensional, simply connected manifold that does not admit a nonconstant quasiregular mapping from $\mathbb{R}^d$. Theorem 1.1 implies the following corollary.

COROLLARY 1.2. *The simply connected manifold* $M = \#^n(S^2 \times S^2)$, *the connected sum of* n *copies of* $S^2 \times S^2$, *is not quasiregularly elliptic for* $n \geq 4$.

Proof. Firstly, the 2-sphere S^2, and hence $S^2 \times S^2$, is simply connected. Furthermore, since the dimension is larger than 2, the connected sum of simply connected manifolds is simply connected. So M is simply connected.

The sphere S^2 satisfies $\dim H^2(S^2) = 1$. By the Künneth formula,

$$\dim H^2(S^2 \times S^2) = 2.$$

By the Mayer-Vietoris Theorem,

$$H^2(\#^n(S^2 \times S^2)) \cong \oplus^n H^2(S^2 \times S^2).$$

Therefore $\dim H^2(M) = 2n > \binom{4}{2}$. So by Theorem 1.1, M is not quasiregularly elliptic. □

Theorem 1.1 is a generalization of a classical theorem for holomorphic functions in dimension 2. Let M be a Riemann surface. By the uniformization theorem, the universal cover of M is either $\widehat{\mathbb{C}}, \mathbb{C}$, or $\mathbb{D}$. If $f\colon \mathbb{C} \to M$ is holomorphic, then f lifts to a holomorphic mapping from $\mathbb{C}$ to the universal cover of M. If the universal covering space is $\mathbb{D}$, then Liouville's theorem states that f is constant. This implies that the only compact Riemann surfaces that admit holomorphic mappings are homeomorphic to $\widehat{\mathbb{C}}$ and $S^1 \times S^1$. Every quasiregular mapping f can be decomposed as $f = g \circ \phi$, where g is holomorphic and $\phi\colon \mathbb{C} \to \mathbb{C}$ is a quasiconformal homeomorphism [16, p. 247]. So in dimension 2, any manifold admitting a quasiregular mapping also admits a holomorphic mapping.

A 1-quasiregular mapping on $\mathbb{C}$ is a holomorphic function. If we study quasiregular ellipticity for $K = 1$ in higher dimensions, then the results are as restrictive as in the $d = 2$ case. If M admits a 1-quasiregular mapping from $\mathbb{R}^d$, then M must be a quotient of the d-dimensional sphere or torus (see [2, Prop. 1.4]). The condition of 1-quasiregularity is consequently too restrictive. When $K \geq 1$, there are several results regarding ellipticity. For manifolds of dimension 3, Theorem 1.1 has been shown by Jormakka in [14]. He proved that if M is quasiregularly elliptic, then M must be a quotient of S^3, T^3, or $S^2 \times S^1$. In dimension 4, in [23], Rickman showed that $(S^2 \times S^2)\#(S^2 \times S^2)$ is quasiregularly elliptic. Theorem 1.1 shows that the connected sum of four copies of $S^2 \times S^2$ is not quasiregularly elliptic.

For $d \geq 4$, there are very few conditions on the topology of M that restrict which manifolds can be quasiregularly elliptic, independent of K. A theorem by Varopoulos gives such a result. It states that the polynomial order of growth of the Cayley graph of the fundamental group of a quasiregularly elliptic manifold is bounded by d (see [24, Th. X.5.1] or [10, Ch. 6]). This result gives a K-independent bound on the size of the fundamental group of the manifold. However, it does not apply to manifolds with small fundamental groups and especially gives no information regarding simply-connected spaces.

A recent theorem due to Kangasniemi [15] gives a K-independent bound on the cohomology for manifolds that admit uniformly quasiregular self-mappings. He proved an analogue to Theorem 1.1 with the added assumption that M admits a non-injective K-quasiregular mapping $f\colon M \to M$ such that the iterates of f are also K-quasiregular. Such a mapping is called uniformly quasiregular. The bound in this theorem is sharp since the torus admits uniformly quasiregular self-mappings.

There are also related results when a quasiregularly elliptic manifold M is open. In dimension 2, one can use the same arguments as in the compact case to deduce that M is homeomorphic to $\mathbb{R}^2$ or $S^1 \times \mathbb{R}$. This result implies Picard's theorem as a corollary. In higher dimensions, Rickman [20] proved what is now known as the Rickman-Picard theorem, showing that a K-quasiregular mapping from $\mathbb{R}^d$ to the d-dimensional sphere S^d can omit at most $C(d, K)$ points. The fact that the constant depends on K is unavoidable as seen in the constructions by Rickman [21] and Drasin and Pankka [6].

We next outline the proof for Theorem 1.1. We argue by contradiction. Let $k > \binom{d}{l}$, and let $\alpha_1, \ldots, \alpha_k$ be representatives of cohomology classes that form a basis in $H^l(M)$. Using Poincaré duality we can choose closed $(n-l)$-forms $\beta_1, \ldots, \beta_k$ on M such that

$$\int_M \alpha_i \wedge \beta_j = \delta_{ij}$$

for $1 \leq i, j \leq k$ and where δ_{ij} is the Kronecker delta. In previous papers on quasiregular ellipticity, p-harmonic forms were used instead of smooth forms arising from Poincaré duality. Our approach allows us to avoid the use of this machinery.

Since we argue by contradiction, there exists a quasiregular mapping $f\colon \mathbb{R}^d \to M$. The pullbacks $\eta_i = f^*\alpha_i$ and $\theta_i = f^*(\beta_i)$ are closed forms on $\mathbb{R}^d$, and they satisfy local L^p-bounds depending on the Jacobian of f. This allows us to use a rescaling procedure to obtain forms on the unit ball in $\mathbb{R}^d$ such that the rescaled forms are pointwise orthogonal almost everywhere.

In the papers by Eremenko and Lewis, [7] and [17], the authors applied a similar rescaling to $\mathcal{A}$-harmonic functions in order to prove the Rickman-Picard theorem for quasiregular mappings. Instead of rescaling functions, we consider rescalings of differential forms. We also note that Kangasniemi [15] rescaled differential forms in the uniformly quasiregular case. The differential forms in his case rescale so that they are orthogonal at every point to each other. The main connection between the techniques used in this paper and the above two results is that in the limit the rescaled objects obey pointwise results. This is the crucial ingredient of the proof.

The rescaling captures how $f\colon \mathbb{R}^d \to M$ behaves on average. Since quasiregular mappings have equidistribution properties similar to holomorphic mappings, f will map a large set evenly over M. This can be measured by the size of the Jacobian of f on a set. We choose a sequence of balls, B_n, so that the integral of the Jacobian of f on B_n tends to infinity. The differential forms, η_i and θ_i, rescaled from B_n to $B(0,1)$, will converge to averages of themselves on M. The limits in this rescaling will be both non-zero and pair to 0 pointwise. On M, we have that

$$\int_M \alpha_i \wedge \beta_j = 0$$

for $i \neq j$. However, the limits of the rescaled forms, $\widetilde{\eta}_i$ and $\widetilde{\theta}_j$, will satisfy

$$\widetilde{\eta}_i \wedge \widetilde{\theta}_j = 0$$

for almost every $x \in B(0,1)$.

Once the differential forms on the unit ball are constructed and we know that they pair pointwise to 0, we see that at most $\binom{d}{l} = \dim(\bigwedge^l \mathbb{R}^d)$ of the forms can be non-zero. This will imply that the sets where at least one of the forms is 0 covers the entire ball, apart from a set of measure 0. However, the size of the rescaled forms is governed by the size of the Jacobian of f. In order to prove this we need to first show that the Jacobian of f satisfies a reverse Hölder inequality. In general, the Jacobian of a quasiregular mapping is in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Bojarski and Iwaniec [1], using a method similar to Gehring's lemma [8], showed that if $f\colon \mathbb{R}^d \to \mathbb{R}^d$, then the Jacobian of f is in $L^{1+\epsilon}_{\mathrm{loc}}(\mathbb{R}^d)$ for a sufficiently small $\epsilon > 0$. In addition, they show that f satisfies a reverse Hölder inequality, i.e.,

$$\left(\frac{1}{|B(x,\frac{r}{2})|}\int_{B(x,\frac{r}{2})} J_f^{(1+\epsilon)}\right)^{1/(1+\epsilon)} \leq C(d,\epsilon,K)\frac{1}{|B(x,r)|}\int_{B(x,r)} J_f,$$

where $x \in \mathbb{R}^d$ and $r > 0$. If $f\colon \mathbb{R}^d \to M$, then the Jacobian of f will be in $L^{1+\epsilon}_{\mathrm{loc}}(\mathbb{R}^d)$, but it will not necessarily satisfy a reverse Hölder inequality. The reverse Hölder inequality only holds when $H^l(M) \neq 0$ for some l, where $1 \leq l \leq d-1$.

An example of a map that does not satisfy a reverse Hölder inequality is $f(z) = e^z : \mathbb{C} \to \widehat{\mathbb{C}}$. The Jacobian of f in this case is $e^{2x}/(1+e^{2x})^2$, where $x = \mathrm{Re}(z)$. If we consider balls of the form $B(0,r)$, then the term on the left-hand side of the inequality will be comparable to $r^{-1/(1+\epsilon)}$ while the term on the right-hand side be comparable to r^{-1}. This is not possible and hence such an inequality cannot be satisfied. Crucially, $H^1(\widehat{\mathbb{C}}) = \{0\}$ and so the inequality is not expected to hold.

Once we know that the Jacobian of f satisfies a reverse Hölder inequality, we prove that the size of the Jacobian governs the size of the rescaled forms, η_i and θ_i, on B_n. In turn, this shows that the integral of the Jacobian of f on B_n will be arbitrarily small as $n \to \infty$. At this point we arrive at a contradiction since the balls were exactly chosen so that the integral of the Jacobian of f is bounded away from 0. Hence the number of forms is bounded by $\binom{d}{l}$. These forms correspond to the dimension of the degree l de Rham cohomology on M, proving Theorem 1.1.

The structure of the paper is as follows. Section 2 gives a brief introduction to differential forms on manifolds and pullbacks of differential forms by quasiregular mappings. We also show the reverse Hölder inequality for the Jacobian of f. For the relationship between quasiregular mappings and differential forms, see [2, §3] and [13]. The use of differential forms in this setting is inspired by the work of Bonk and Heinonen [2], Donaldson and Sullivan [5] and Iwaniec and Martin [13].

In Section 3 we discuss equidistribution properties for f. In Section 4 we define the rescalings of the differential forms and prove certain required convergence results. Section 5 gives the proof of Theorem 1.1. Some of the methods in the proof are influenced by techniques developed by Pankka [19]. For a reference on the facts used for quasiregular mappings, see [2], [5] and [22].

1.1. *Acknowledgments.* The author thanks Mario Bonk for both introducing him to the problem and the many discussions and comments on the paper. The author would also like to thank Pekka Pankka for conversations in Helsinki on this topic. The author was partially supported by NSF grant DMS-1506099.

2. **Exterior algebra and differential forms**

This section gives an introduction to the tools needed to prove Theorem 1.1.

Let $\bigwedge^l(\mathbb{R}^d)$ denote the space of degree l exterior powers of the cotangent bundle of $\mathbb{R}^d$ for $1 \leq l \leq d-1$. Let $D \subset \mathbb{R}^d$ be an open domain. By $C_c^\infty(D)$, we denote the space of smooth functions with compact support in D. We say a differential form α is in $L^p(D)$ whenever the component functions of α are in the usual L^p-space. Similarly, α is in the Sobolev space $W^{1,p}(D)$ whenever the component functions are in the standard Sobolev space, i.e., $\alpha_i \in L^p(D)$

of functions and α_i has weak derivatives in $L^p(D)$ of functions. The pointwise norm of a differential form α will be denoted by $|\alpha|$ and will refer to the pointwise L^2-norm on the component functions of α. The exponents p and q will always denote d/l and $d/(d-l)$ respectively, where l is the degree of the differential form whose norm is being taken. For $x \in \mathbb{R}^d$ and $r > 0$, the set $B(x,r) \subset \mathbb{R}^d$ denotes the ball of radius r, centered at x.

The space M will always be a closed, connected and orientable Riemannian manifold of dimension d. By $\Omega^l(M)$, we mean the space of smooth differential forms on M of degree l. On $\Omega^l(M)$, there exists an inner product induced by the Riemannian metric on M. For $\omega \in \Omega^l(M)$, we denote by $\|\omega\|_\infty$ the L^∞-norm given by this inner product. The de Rham cohomology of M will be denoted by $H^l(M)$.

In the following it suffices to consider l such that $1 \leq l \leq d-1$. This is because $H^d(M) \cong H^0(M) \cong \mathbb{R}$ for the manifolds considered in Theorem 1.1.

In order to select suitable differential forms from the cohomology classes on M, we use Poincaré duality (see [4, p. 44]).

THEOREM 2.1. *Let $k = \dim H^l(M)$. Then there exist closed forms*

$$\alpha_1, \ldots, \alpha_k \in \Omega^l(M)$$

and $\beta_1, \ldots, \beta_k \in \Omega^{d-l}(M)$ such that the cohomology classes $\{[\alpha_i]\}_{i=1}^k$ form a basis for $H^l(M)$ and

$$\int_M \alpha_i \wedge \beta_j = \delta_{ij} \tag{2.1}$$

for $1 \leq i, j \leq k$.

In estimating integrals of differential forms, the following inequality will be useful later on. If $\alpha \in \bigwedge^{l_1}(\mathbb{R}^d)$ and $\beta \in \bigwedge^{l_2}(\mathbb{R}^d)$, then

$$|\alpha \wedge \beta| \leq C(d)|\alpha||\beta|, \tag{2.2}$$

where $C(d)$ only depends on the dimension. To prove this note that the bilinear operator $(\alpha, \beta) \mapsto \alpha \wedge \beta$ is defined on two finite-dimensional vector spaces when x is fixed. Therefore it is bounded and and we arrive at (2.2).

A key tool we use is the pullback of a differential form by a quasiregular map. If $f : \mathbb{R}^d \to M$ is quasiregular and $\omega \in \Omega^l(M)$, then $f^*\omega$ is a well-defined measurable form in $L^p_{\text{loc}}(\mathbb{R}^d)$ and

$$d(f^*\omega) = f^*(d\omega). \tag{2.3}$$

Here, $d(f^*\omega)$ is interpreted in the weak sense. For a thorough discussion of this, see [5, §2].

We also have the following well-known inequality for pullbacks of differential forms by f:

$$|f^*\omega(x)| \leq C(d)\|\omega\|_\infty \|Df(x)\|^l \tag{2.4}$$

for almost every $x \in \mathbb{R}^d$, where $\|Df\|$ is the operator norm for Df and $C(d) > 0$ is a constant that depends only on d.

The inequality is a pointwise estimate. To prove it, without loss of generality, we may assume that $\omega \in \Omega^l(B(0,1))$. For almost every $x \in \mathbb{R}^d$,

$$f^*\omega(x) = \sum_I (\omega_I \circ f(x)) df^I(x),$$

where $I = \{i_1, \dots, i_l\}$ is a multi-index of length l. That is,

$$df^I = df_{i_1} \wedge \cdots \wedge df_{i_l},$$

where f_i is i-th component function of f and we sum over all multi-indices, $1 \le i_1 < \cdots < i_l \le d$. By Hadamard's inequality,

$$|df_{i_1} \wedge \cdots \wedge df_{i_l}| \le |df_{i_1}| \cdots |df_{i_l}| \le \|Df\|^l.$$

Thus,

$$|f^*\omega(x)| \le C(d)\|\omega\|_\infty \|Df(x)\|^l.$$

Bojarski and Iwaniec [1, Th. 5.1] showed that a quasiregular mapping $f\colon \mathbb{R}^d \to \mathbb{R}^d$ has a Jacobian that satisfies a reverse Hölder inequality; that is, there exists $b > 1$ so that if $F, \Omega \subset \mathbb{R}^d$ are sets such that F is compact, Ω is open and $F \subset \Omega$, then

$$\left(\int_F J_f^b\right)^{1/b} \le C(d,b,K)\frac{1}{\operatorname{dist}(F,\partial\Omega)^{d/a}}\int_\Omega J_f, \tag{2.5}$$

where $\frac{1}{a} + \frac{1}{b} = 1$. Crucially, b and $C(d,b,K)$ are independent of f, F and Ω. They prove this by showing a weaker reverse Hölder inequality, where the exponents are 1 and 1/2. They then use Gehring's lemma to upgrade to the above inequality. We would like to have such a statement for $f\colon \mathbb{R}^d \to M$. If $H^l(M) = 0$ for $1 \le l \le d-1$, then the Jacobian of f does not necessarily satisfy a reverse Hölder inequality. An example of such a map is $f(z) = e^z$ as a map from $\mathbb{C} \to \widehat{\mathbb{C}}$, as mentioned in the introduction. Note that, in the setting of Theorem 1.1, there exists an index $1 \le l \le d-1$ such that $H^l(M) \ne 0$, otherwise Theorem 1.1 is trivially true.

Proposition 2.2. *Let M be a closed Riemannian manifold, and let $f\colon \mathbb{R}^d \to M$ be K-quasiregular. If there exists an integer $1 \le l \le d-1$ such that $H^l(M) \ne 0$, then the Jacobian of f satisfies the weak reverse Hölder inequality,*

$$\frac{1}{|\frac{1}{2}B|}\int_{\frac{1}{2}B} J_f \le C(d,K,M)\left(\frac{1}{|B|}\int_B J_f^{d/(d+1)}\right)^{(d+1)/d},$$

where $B \subset \mathbb{R}^d$ is an arbitrary ball.

In order to prove the proposition we will need two lemmas. In general, a top-dimensional product that integrates to the volume of M can be expressed as $\alpha \wedge \beta = V + d\tau$, where $\tau \in \Omega^{d-1}(M)$. In order to prove the revere Hölder inequality of Proposition 2.2, we would like to write V as solely the sum of products of differential forms. The following lemma describes how to absorb the $d\tau$ term into the product term.

LEMMA 2.3. *If there exists a pair of differential forms $\alpha \in \Omega^l(M)$ and $\beta \in \Omega^{n-l}(M)$ that are closed and satisfy*

$$\int_M \alpha \wedge \beta = \int_M V,$$

where V is the volume form on M, then there exists a partition of unity $\{\lambda_\nu\}_{\nu=1}^m$ such that V can be expressed as

$$V = c\sum_{\nu=1}^{m} \lambda_\nu \alpha_\nu \wedge \beta_\nu,$$

where $\alpha_\nu \in \Omega^l(M)$, $\beta_\nu \in \Omega^{d-l}(M)$ and $c \in C^\infty(M)$.

Proof. Without loss of generality, we may assume that the volume of M is 1. Since $\alpha \wedge \beta$ is a top-dimensional form, we have that

$$\alpha \wedge \beta = gV,$$

where V is the volume form on M and $g \in C^\infty(M)$. Let $a \in M$ be a point for which $g(a) > 0$.

Let $x \in M$. By the Isotopy lemma [11, p. 142], there exists an orientation preserving diffeomorphism $\Phi_x\colon M \to M$ such that $\Phi_x(x) = a$. Let U be an open set containing a such that g is positive on U. Then $\{\Phi_x^{-1}(U)\}_{x\in M}$ is an open cover of M and there exists a finite subcover, $U_1, \ldots, U_m$. By the construction, $g \circ \Phi_x$ is positive on U_x. Let Φ_ν be the diffeomorphism corresponding to U_ν, and let $\{\lambda_\nu\}_{\nu=1}^m$ be a partition of unity subordinate to $\{U_\nu\}_{\nu=1}^m$. Define

$$\omega := \sum_{\nu=1}^{m} \lambda_\nu \Phi_\nu^*(\alpha \wedge \beta).$$

From this definition we get that for any $x \in M$,

$$\begin{aligned}\omega|_x &= \sum_{\nu=1}^{m} \lambda_\nu(x)(g \circ \Phi_\nu(x))\Phi_\nu^*(V)|_x \\ &= \sum_{\nu=1}^{m} \lambda_\nu(x)(g \circ \Phi_\nu(x))J_{\Phi_\nu}(x)V|_{\Phi_\nu(x)},\end{aligned}$$

where J_{Φ_ν} is the Jacobian of Φ_ν. The diffeomorphism Φ_ν is orientation preserving, so $J_{\Phi_\nu}(x) > 0$. The functions λ_ν are non-negative and non-zero only on U_ν. On the set U_ν, $g \circ \Phi_\nu(x)$ is also positive. So ω is a positive top-dimensional form and thus $V = c\omega$, where $c\colon M \to (0,\infty)$ is a positive, smooth function on M. □

The following lemma is well known to experts.

LEMMA 2.4. *Let $f\colon \mathbb{R}^d \to M$ be a K-quasiregular mapping, and let $\alpha \in \Omega^l(M)$ and $\beta \in \Omega^{d-l}(M)$ be closed forms. If B is a ball in $\mathbb{R}^d$ such that, in B, $f^*(\alpha\wedge\beta) = g dx^1 \wedge \cdots \wedge dx^d$ for a non-negative $g\colon B \to \mathbb{R}$, then*

$$\frac{1}{|\frac{1}{2}B|}\int_{\frac{1}{2}B} f^*(\alpha\wedge\beta) \le C(d,K)\|\alpha\|_\infty\|\beta\|_\infty\left(\frac{1}{|B|}\int_B J_f^{d/(d+1)}\right)^{(d+1)/d},$$

where $C(d,K)$ depends only on d and K.

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative function that is 1 on $\frac{1}{2}B$ and 0 on the complement of B. Note that we can choose ψ so that $|d\psi| \le r^{-1}$, where r is the radius of B. By the non-negativity of $f^*(\alpha\wedge\beta)$,

$$\int_{\frac{1}{2}B} f^*(\alpha\wedge\beta) \le \int_B \psi f^*(\alpha\wedge\beta).$$

On M, α is closed. By (2.3), $f^*\alpha = du$ on B. We can choose u so that u satisfies a Poincaré-Sobolev inequality. For a precise formulation of this, see [12, Cor. 4.2]. Integration by parts gives that

$$\left|\int_B \psi f^*\alpha\wedge f^*\beta\right| = \left|\int_B d\psi\wedge u\wedge f^*\beta\right|.$$

By (2.2), Hölder's inequality and because $|d\psi| \le r^{-1}$,

$$\left|\int_B d\psi\wedge u\wedge f^*\beta\right| \le \frac{C(d)}{r}\|u\|_{d^2/(l(d+1)-d)}\|f^*\beta\|_{d^2/((d+1)(d-l))}.$$

Since $du = f^*\alpha$, we have, by the Poincaré-Sobolev inequality, that

$$\frac{C(d)}{r}\|u\|_{d^2/(l(d+1)-d)}\|f^*\beta\|_{d^2/((d+1)(d-l))} \le \frac{C(d)}{r}\|f^*\alpha\|_{d^2/(l(d+1))}\|f^*\beta\|_{d^2/((d+1)(d-l))}.$$

We remark that the Poincaré-Sobolev inequality is only valid here because $1 \le l \le d-1$. The forms α and β are smooth on M and therefore are bounded independently of f. So by (2.2),

$$\begin{aligned}\frac{C(d)}{r}&\|f^*\alpha\|_{d^2/(l(d+1))}\|f^*\beta\|_{d^2/((d+1)(d-l))}\\ &\le \frac{C(d,K)}{r}\|\alpha\|_\infty\|\beta\|_\infty\|J_f\|_{d/(d+1)}^{l/d}\|J_f\|_{d/(d+1)}^{(d-l)/d}\\ &= \frac{C(d,K)}{r}\|\alpha\|_\infty\|\beta\|_\infty\left(\int_B J_f^{d/(d+1)}\right)^{(d+1)/d}.\end{aligned}$$

By taking averages, we arrive at the lemma. □

We can now proceed in showing the proposition.

Proof of Proposition 2.2. Since $H^l(M) \neq 0$, there exists a Poincaré pair, α and β, given in Theorem 2.1, with

$$\int_M \alpha \wedge \beta = 1.$$

Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a function such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $\frac{1}{2}B$ and $\psi \equiv 0$ on the complement of B. The Jacobian of f satisfies

$$J_f dx^1 \wedge \cdots \wedge dx^d = f^*V.$$

So, by Lemma 2.3,

$$\begin{aligned}\int_{\frac{1}{2}B} J_f &\leq \int_B \psi J_f \\ &= \int_B \psi f^* V \\ &= \sum_{\nu=1}^m \int_B \psi (c \circ f)(\lambda_\nu \circ f) f^*(\alpha_\nu \wedge \beta_\nu).\end{aligned}$$

We also know that c and λ_ν are positive and bounded above by constants depending only on M. So by Lemma 2.4,

$$\frac{1}{|\frac{1}{2}B|}\int_{\frac{1}{2}B} J_f \leq C(d,K)\sum_{\nu=1}^m \|\alpha_\nu\|_\infty \|\beta_\nu\|_\infty \left(\frac{1}{|B|}\int_B J_f^{d/(d+1)}\right)^{(d+1)/d}.$$

The number m and the L^∞-norms of α_ν and β_ν depend only on M and can be absorbed into the constant. Therefore

$$\frac{1}{|\frac{1}{2}B|}\int_{\frac{1}{2}B} J_f \leq C(d,K,M)\left(\frac{1}{|B|}\int_B J_f^{d/(d+1)}\right)^{(d+1)/d}. \qquad \square$$

Proposition 2.2 and [1, Th. 4.2] together imply the following statement.

PROPOSITION 2.5. *There exists $b > 1$ such that for any ball $B \subset \mathbb{R}^d$,*

$$\left(\frac{1}{|\frac{1}{2}B|}\int_{\frac{1}{2}B} J_f^b\right)^{1/b} \leq C(d,M,K,b)\frac{1}{|B|}\int_B J_f.$$

3. **Equidistribution**

In this section we provide equidistribution results for a quasiregular mapping $f\colon \mathbb{R}^d \to M$. These results will help show that the limits of our rescaled forms, which will be constructed in Section 4, are non-zero. Define

$$A(B) := \int_B J_f$$

for a Borel set $B \subset \mathbb{R}^d$ to be the averaged counting function for f (see [22, Ch. IV]). The following theorem [2, Th. 1.11] shows that $A(B(0,r))$ is unbounded.

THEOREM 3.1. *Let $f\colon \mathbb{R}^d \to M$ be a quasiregular mapping. If $H^l(M) \neq \{0\}$ for some $l \in \{1,\dots,d-1\}$, then there exists a constant $\alpha > 0$ such that*

$$\liminf_{r\to\infty} \frac{A(B(0,r))}{r^\alpha} > 0.$$

In particular, $A(\mathbb{R}^d) = \infty$.

We also record a lemma due to Rickman; for the proof, see [20, Lemma 5.1].

LEMMA 3.2 (Rickman's Hunting Lemma). *Let μ be a Borel measure on $\mathbb{R}^d$ that is absolutely continuous with respect to Lebesgue measure. If $\mu(\mathbb{R}^d) = \infty$, then, for all $M > 0$, there exist a point $a \in \mathbb{R}^d$ and a radius $r > 0$ such that*

$$\mu(B(a,r)) \geq M \quad \text{and} \quad \mu(B(a,r)) \leq D(d)\mu(B(a,r/2)),$$

where $D(d)$ is a constant that depends only on the dimension d.

We remark that this lemma has been used in most proofs of the Rickman-Picard theorem. For a recent paper by Bonk and Poggi-Corradini that proves the Rickman-Picard theorem, see [3].

The next proposition is the key equidistribution result for the quasiregular mapping f. Equidistribution results for quasiregular mappings were first shown by Mattila and Rickman in [18]. The following result also bears some similarity to an equidistribution result due to Pankka [19, Th. 4]. The proof also uses some methods developed there.

PROPOSITION 3.3. *Let $f\colon \mathbb{R}^d \to M$ be a quasiregular mapping, and let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of balls such that*

$$\lim_{n\to\infty} A(B_n) = \infty.$$

Suppose $\psi \in C_c^\infty(B(0,1))$ is a non-negative function that satisfies

$$A(B_n) \leq C(d,K) \int_{B_n} (\psi \circ T_n^{-1})^d J_f, \tag{3.1}$$

where $T_n(x) = a_n + r_n x$, $B_n = B(a_n, r_n)$. If $\omega \in \Omega^d(M)$ satisfies

$$\int_M \omega = \int_M V,$$

where V is the volume form on M, then

$$\lim_{n\to\infty} \left| \frac{1}{\int_{B_n} (\psi \circ T_n^{-1})^d J_f} \int_{B_n} (\psi \circ T_n^{-1})^d f^*\omega - 1 \right| = 0.$$

Proof. Let $\psi_n(x) = \psi \circ T_n^{-1}$. Consider the following difference:

$$\left|\int_{B_n} \psi_n^d \eta_i \wedge \theta_i - \int_{B_n} \psi_n^d J_f\right| = \left|\int_{B_n} \psi_n^d f^*(\omega - V)\right|.$$

Since $\int_M \omega = \int_M V$, the d-form $\omega - V$ integrates to 0 on M. By de Rham's theorem, it is exact and $\omega - V = d\tau$, where $\tau \in \Omega^{d-1}(M)$. We apply integration by parts,

$$\left|\int_{B_n} \psi_n^d f^*(\omega - V)\right| = \left|\int_{B_n} \psi_n^d d(f^*\tau)\right| = \left|d\int_{B_n} \psi_n^{d-1} d\psi_n \wedge f^*\tau\right|.$$

By (2.2) and Hölder's inequality,

$$\left|\int_{B_n} \psi_n^d f^*(\omega - V)\right| \leq C(d)\|d\psi_n\|_{d,B_n}\left(\int_{B_n} \psi_n^d |f^*\tau|^{d/(d-1)}\right)^{(d-1)/d}.$$

By (2.4) and quasiregularity of f,

$$\left|\int_{B_n} \psi_n^d f^*(\omega - V)\right| \leq C(d)K^{(d-1)/d}\|\tau\|_\infty \|d\psi_n\|_{d,B_n}\left(\int_{B_n} \psi_n^d J_f\right)^{(d-1)/d}.$$

Thus,

$$\left|\frac{1}{(\int_{B_n} \psi_n^d J_f)}\int_{B_n} \psi_n^d \omega - 1\right| \leq C(K,d,M)\|d\psi_n\|_{d,B_n}\left(\int_{B_n} \psi_n^d J_f\right)^{-1/d}.$$

Note that

$$\|d\psi_n\|_{d,B_n} = \|d\psi\|_{d,B(0,1)},$$

by the conformal invariance of the d-energy. In other words, the term with ψ_n is independent of n. This and (3.1) give that

$$\left|\frac{1}{(\int_{B_n} \psi_n^d J_f)}\int_{B_n} \psi_n^d \omega - 1\right| \leq C(K,d,M)\|d\psi\|_{d,B(0,1)} A(B_n)^{-1/d} \to 0$$

as $n \to \infty$. □

4. **Rescaling principle**

In this section we construct rescaled forms on $B(0,1)$. By Theorem 2.1, there exist closed differential forms $\alpha_1, \dots, \alpha_k \in \Omega^l(M)$ and $\beta_1, \dots, \beta_k \in \Omega^{d-l}(M)$ such that the cohomology classes $[\alpha_1], \dots, [\alpha_k]$ form a basis for $H^l(M)$. In addition, they satisfy the orthogonality relation

$$\int_M \alpha_i \wedge \beta_j = \delta_{ij}$$

for $1 \leq i, j \leq k$.

By Theorem 3.1 and Lemma 3.2, there exist balls $B_n \subset \mathbb{R}^d$ such that $\lim_{n\to\infty} A(B_n) = \infty$ and

$$A(B_n) \leq D(d)A(\tfrac{1}{2}B_n). \tag{4.1}$$

In the following, $\{B_n\}_{n\in\mathbb{N}}$ will always refer to a sequence of balls satisfying these conditions.

We can now rescale the pullbacks $\eta_i = f^*\alpha_i$ and $\theta_i = f^*\beta_i$ on the sequence of balls, $\{B_n\}_{n\in\mathbb{N}}$. Let $T_n\colon B(0,1) \to B_n = B(a_n, r_n)$ be the map $x \mapsto a_n + r_n x$. The rescaled forms are defined as

$$\eta_i^n := \frac{1}{A(B_n)^{1/p}} T_n^* \eta_i \tag{4.2}$$

and

$$\theta_i^n := \frac{1}{A(B_n)^{1/q}} T_n^* \theta_i. \tag{4.3}$$

Note that by (2.3), η_i^n and θ_i^n are closed. By quasiregularity of f, we have that $f \in W^{1,d}_{\mathrm{loc}}(\mathbb{R}^d, M)$. By (2.4), $\eta_i^n \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ and $\theta_i^n \in L^q_{\mathrm{loc}}(\mathbb{R}^d)$, where $p = d/l$ and $q = d/(d-l)$.

The following lemma provides a convergence result for the sequences of forms $\{\eta_i^n\}_{n\in\mathbb{N}}$ and $\{\theta_i^n\}_{n\in\mathbb{N}}$.

Lemma 4.1. *For each $n \in \mathbb{N}$, there exists a $(d-l-1)$-form $u_i^n \in W^{1,q}(B(0,1))$, where $q = d/(d-l)$, such that*

$$du_i^n = \theta_i^n.$$

Furthermore, we can pass to a subsequence so that the following convergence results hold:

(i) *There exist an l-form $\widetilde{\eta}_i \in L^p(B(0,1))$ and a $(d-l)$-form $\widetilde{\theta}_i \in L^q(B(0,1))$ such that*

$$\lim_{n\to\infty} \eta_i^n = \widetilde{\eta}_i \quad \textit{and} \quad \lim_{n\to\infty} \theta_i^n = \widetilde{\theta}_i,$$

where the convergence of η_i^n is in the weak topology on $L^p(B(0,1))$ and the convergence of θ_i^n is in the weak topology on $L^q(B(0,1))$.

(ii) *There exists a $(d-l-1)$-form $\widetilde{u}_i \in W^{1,q}(B(0,1))$ such that*

$$\lim_{n\to\infty} u_i^n = \widetilde{u}_i$$

in $L^q(B(0,1))$.

(iii) *On $B(0,1)$,*

$$d\widetilde{u}_i = \widetilde{\theta}_i,$$

in the weak sense.

Proof. In this proof we will construct several subsequences of the sequences mentioned in the lemma. It is understood that the subsequences should be taken simultaneously for all the forms mentioned in the lemma.

For the proof of (i), we compute the L^p-norm of η_i^n. Indeed, by equation (4.2),

$$\int_{B(0,1)} |\eta_i^n|^p = \frac{r_n^d}{A(B_n)} \int_{B(0,1)} |\eta_i(a_n + r_n x)|^p dx = \frac{1}{A(B_n)} \int_{B_n} |\eta_i|^p.$$

By quasiregularity of f and (2.4),

$$\frac{1}{A(B_n)} \int_{B_n} |\eta_i|^p \leq KC(d) \frac{\|\alpha_i\|_\infty^p}{A(B_n)} \int_{B_n} J_f \leq KC(d) \|\alpha_i\|_\infty^p.$$

Hence, the L^p-norm of η_i^n is uniformly bounded. By the Banach-Alaoglu theorem, we can pass to a subsequence so that

$$\lim_{n\to\infty} \eta_i^n = \widetilde{\eta}_i$$

weakly in $L^p(B(0,1))$.

The proof for θ_i^n is very similar. By (4.3),

$$\begin{aligned}\int_{B(0,1)} |\theta_i^n|^q &= \frac{r_n^d}{A(B_n)} \int_{B(0,1)} |\theta_i(a_n + r_n x)|^q \\ &= \frac{1}{A(B_n)} \int_{B_n} |\theta_i|^q \\ &\leq KC(d) \frac{\|\beta_i\|_\infty^q}{A(B_n)} \int_{B_n} J_f \\ &\leq KC(d) \|\beta_i\|_\infty^q.\end{aligned}$$

Again, by the Banach-Alaoglu theorem, we can pass to a subsequence so that

$$\lim_{n\to\infty} \theta_i^n = \widetilde{\theta}_i$$

weakly in $L^q(B(0,1))$.

We next prove (ii). By part (i), the L^q-norm of θ_i^n is uniformly bounded. The forms θ_i^n are closed by (2.3). By the Sobolev embedding theorem, there exist $(d-l-1)$-forms $u_i^n \in W^{1,q}(B(0,1))$ such that $du_i^n = \theta_i^n$ and $\|u_i^n\|_{d/(d-l-1)} \leq C\|\theta_i^n\|_q$, where C does not depend on n, u_i^n or θ_i^n; see [12, Cor. 4.2] for the formulation of the Sobolev embedding theorem and the Sobolev-Poincaré inequality for differential forms. Thus there exists a subsequence of u_i^n that converges to $\widetilde{u}_i$ strongly in $L^q(B(0,1))$. We will also denote this subsequence as u_i^n.

Finally, we show (iii). We demonstrate that $d\widetilde{u}_i = \widetilde{\theta}_i$ in the weak sense. By duality, we can consider test forms $\phi \in \Omega^{l+1}(B(0,1))$ with compact support.

We pair $\widetilde{u}_i$ with $d\phi$,

$$\begin{aligned}\int_{\mathbb{R}^d} \widetilde{u}_i \wedge d\phi &= \lim_{n\to\infty} \int_{\mathbb{R}^d} u_i^n \wedge d\phi \\ &= \lim_{n\to\infty} (-1)^{d-l} \int_{\mathbb{R}^d} \theta_i^n \wedge \phi \\ &= (-1)^{d-l} \int_{\mathbb{R}^d} \widetilde{\theta}_i \wedge \phi.\end{aligned}$$

This proves the claims in the lemma. □

The following convergence result is a key tool in proving the main result.

LEMMA 4.2. *Let $\psi \in C_c^\infty(B(0,1))$. Then*

$$\lim_{n\to\infty} \int_{B(0,1)} \psi\eta_i^n \wedge \theta_j^n = \int_{B(0,1)} \psi\widetilde{\eta}_i \wedge \widetilde{\theta}_j$$

for $1 \le i, j \le k$.

Proof. Consider the difference,

$$\begin{aligned}\left|\int_{B(0,1)} \psi\eta_i^n \wedge \theta_j^n - \int_{B(0,1)} \psi\widetilde{\eta}_i \wedge \widetilde{\theta}_j\right| &\le \left|\int_{B(0,1)} \psi\eta_i^n \wedge (\theta_j^n - \widetilde{\theta}_j)\right| \\ &\quad + \left|\int_{B(0,1)} \psi(\eta_i^n - \widetilde{\eta}_i) \wedge \widetilde{\theta}_j\right| \\ &= I + II.\end{aligned}$$

Lemma 4.1 gives that

$$I = \left|\int_{B(0,1)} \psi\eta_i^n \wedge (du_j^n - d\widetilde{u}_j)\right|.$$

By integration by parts and the compactness of the support of ψ,

$$\begin{aligned}\int_{B(0,1)} \psi\eta_i^n \wedge d(u_j^n - \widetilde{u}_j) &= (-1)^{l+1} \int_{B(0,1)} d(\psi\eta_i^n) \wedge (u_j^n - \widetilde{u}_j) \\ &= (-1)^{l+1} \int_{B(0,1)} d\psi \wedge \eta_i^n \wedge (u_j^n - \widetilde{u}_j)\end{aligned}$$

because η_i^n is weakly closed and $\psi(u_j^n - \widetilde{u}_j) \in W^{1,q}(\mathbb{R}^d)$. By (2.2),

$$|d\psi \wedge \eta_i^n \wedge (u_j^n - \widetilde{u}_j)| \le C(d)|d\psi \wedge \eta_i^n||u_j^n - \widetilde{u}_j|,$$

where $C(d)$ only depends on d. By Hölder's inequality,

$$I \le C(d)\|d\psi \wedge \eta_i^n\|_p \|u_j^n - \widetilde{u}_j\|_q.$$

By Lemma 4.1, the term $\|d\psi \wedge \eta_i^n\|_p$ is bounded independently of n and $u_i^n \to \widetilde{u}_i$ in $L^q(B(0,1))$. So $\lim_{n\to\infty} |I| = 0$.

For the term II, we observe first that, by Lemma 4.1, $\eta_i^n \to \widetilde{\eta}_i$ in $L^p(B(0,1))$ in the weak sense. Since $\psi\widetilde{\theta}_j \in L^q(B(0,1))$, it follows that

$$\lim_{n\to\infty} II = \lim_{n\to\infty}\left|\int_{B(0,1)} (\eta_i^n - \widetilde{\eta}_i) \wedge (\psi\widetilde{\theta}_j)\right| = 0. \qquad \square$$

We show that, as a result of the rescaling, condition (2.1) transfers to a pointwise property of the forms $\widetilde{\eta}_i$ and $\widetilde{\theta}_j$.

Lemma 4.3. *For almost every $x \in B(0,1)$,*

$$\widetilde{\eta}_i \wedge \widetilde{\theta}_j(x) = 0 \tag{4.4}$$

when $i \neq j$.

Proof. When $i \neq j$,

$$\int_M \alpha_i \wedge \beta_j = 0,$$

by (2.1). By de Rham's theorem [4, Cor. 5.8], there exists $\tau \in \Omega^{d-1}(M)$ such that $d\tau = \alpha_i \wedge \beta_j$. Let $\psi \in C_c^\infty(B(0,1))$, using integration by parts and the compactness of the support of ψ,

$$\begin{aligned}\int_{B(0,1)} \psi\eta_i^n \wedge \theta_j^n &= \frac{1}{A(B_n)} \int_{B_n} \psi\left(\frac{x-a_n}{r_n}\right) d(f^*\tau)(x) \\ &= \frac{-1}{A(B_n)} \int_{B_n} d\left(\psi\left(\frac{x-a_n}{r_n}\right)\right) \wedge f^*\tau(x).\end{aligned}$$

By (2.2) and Hölder's inequality,

$$\left|\int_{B(0,1)} \psi\eta_i^n \wedge \theta_j^n\right| \leq \frac{C(d)}{A(B_n)} \|d\psi\|_{d,B(0,1)} \left(\int_{B_n} |f^*\tau|^{d/(d-1)}\right)^{(d-1)/d}.$$

By (2.4) and quasiregularity of f,

$$\left|\int_{B(0,1)} \psi\eta_i^n \wedge \theta_j^n\right| \leq C(d)K^{(d-1)/d} \frac{\|d\psi\|_{d,B(0,1)}\|\tau\|_\infty}{A(B_n)} \left(\int_{B_n} J_f\right)^{(d-1)/d} \to 0$$

as $n \to \infty$. By Lemma 4.2,

$$\int_{B(0,1)} \psi\widetilde{\eta}_i \wedge \widetilde{\theta}_j = 0.$$

Since ψ was an arbitrary test function, $\widetilde{\eta}_i \wedge \widetilde{\theta}_j(x) = 0$ for almost every $x \in B(0,1)$. $\square$

We finish this section with a corollary of the lemma.

COROLLARY 4.4. *Suppose $\widetilde{\eta}_i, \widetilde{\theta}_i$ are forms satisfying the conclusion of Lemma 4.4 for each index $i \in \{1, \dots, k\}$. If $k > \binom{d}{l}$, then for almost every $x \in B(0,1)$, there exists an $i \in \{1, \dots, k\}$ such that*

$$\widetilde{\eta}_i \wedge \widetilde{\theta}_i(x) = 0.$$

Proof. Fix $x \in B(0,1)$ such that (4.4) holds for all pairs $i \neq j$. Let $\{\widetilde{\eta}_{i_1}(x), \dots, \widetilde{\eta}_{i_m}(x)\}$ be a basis for $\operatorname{span}(\{\widetilde{\eta}_i(x)\}_{i=1}^k) \subset \bigwedge^l \mathbb{R}^d$. Since the dimension of $\bigwedge^l \mathbb{R}^d$ is $\binom{n}{l}$, we have that $m \leq \binom{n}{l}$. By our assumption $k > \binom{d}{l}$, there exists a form $\widetilde{\eta}_j \notin \{\widetilde{\eta}_{i_1}, \dots, \widetilde{\eta}_{i_m}\}$. It follows that, by (4.4),

$$\widetilde{\eta}_j \wedge \widetilde{\theta}_j(x) = \sum_{a=1}^{m} \lambda_{i_a} \widetilde{\eta}_{i_a} \wedge \widetilde{\theta}_j(x) = 0. \qquad \square$$

5. Proof of Theorem 1.1

In this section we complete the proof of the main result. Recall that $\widetilde{\eta}_i$ and $\widetilde{\theta}_i$, for $i \in \{1, \dots, k\}$, are the rescalings of $f^*\alpha_i$ and $f^*\beta_i$, which were constructed in Lemma 4.1. For each $i = 1, \dots, k$, let

$$D_i = \{x \in B(0,1) : \widetilde{\eta}_i \wedge \widetilde{\theta}_i(x) = 0\}$$

and define $D_i^n = a_n + r_n D_i$.

We will prove Theorem 1.1 by contradiction; assume $k > \binom{d}{l}$. By Corollary 4.4, $|B_n| = |\bigcup_{i=1}^k D_i^n|$ and

$$A(\tfrac{1}{2}B_n) \leq \sum_{i=1}^{k} \int_{D_i^n \cap \frac{1}{2}B_n} J_f.$$

So for each $n \in \mathbb{N}$, there exists by (4.1) an index $1 \leq i_n \leq k$ so that

$$\int_{D_{i_n}^n \cap \frac{1}{2}B_n} J_f \geq \frac{1}{k} A(\tfrac{1}{2}B_n) \geq \frac{A(B_n)}{kD(d)}.$$

We may assume, by passing to a subsequence, that the index i_n is always the same.

LEMMA 5.1. *For all $\epsilon > 0$, there exist a compact set $C_i \subset D_i \cap B(0, \frac{1}{2})$ and an open set U_i containing $D_i \cap B(0, \frac{1}{2})$ such that*

$$\int_{C_i^n} J_f \geq \frac{A(B_n)}{2kD(d)}, \tag{5.1}$$

where $C_i^n = a_n + r_n C_i$ and

$$\int_{U_i} |\widetilde{\eta}_i \wedge \widetilde{\theta}_i| < \epsilon. \tag{5.2}$$

Proof. Fix $\epsilon > 0$. Since $\int_{D_i} |\widetilde{\eta}_i \wedge \widetilde{\theta}_i| = 0$, there exists an open set U_i containing $D_i \cap B(0, \frac{1}{2})$ such that (5.2) is satisfied.

To construct C_i, first note that, for each $\delta > 0$, there exist compact sets $C_i(\delta) \subset D_i \cap B(0, \frac{1}{2})$ satisfying

$$|(D_i \cap B(0, \tfrac{1}{2})) \setminus C_i(\delta)| < \delta.$$

Let $C_i^n(\delta) = a_n + r_n C_i(\delta)$ and $D_i^n = a_n + r_n D_i$. To simplify notation, denote $\frac{1}{2}D_i := D_i \cap B(0, \frac{1}{2})$ and $\frac{1}{2}D_i^n := a_n + r_n \frac{1}{2} D_i$. Then, by Hölder's inequality,

$$\int_{\frac{1}{2}D_i^n \setminus C_i^n(\delta)} J_f \le |\tfrac{1}{2}D_i^n \setminus C_i^n(\delta)|^{1/a} \left(\int_{\frac{1}{2}D_i^n \setminus C_i^n(\delta)} J_f^b \right)^{1/b},$$

where $\frac{1}{a} + \frac{1}{b} = 1$ and $b > 1$ can be chosen arbitrarily close to 1. Continuing the calculation, we get

$$\begin{aligned}\int_{\frac{1}{2}D_i^n \setminus C_i^n(\delta)} J_f &\le r_n^{d/a} |\tfrac{1}{2}D_i \setminus C_i(\delta)|^{1/a} \left(\int_{\frac{1}{2}D_i^n \setminus C_i^n(\delta)} J_f^b \right)^{1/b} \\ &\le r_n^{d/a} |\tfrac{1}{2}D_i \setminus C_i(\delta)|^{1/a} \left(\int_{\frac{1}{2}B_n} J_f^b \right)^{1/b}.\end{aligned}$$

We now use the higher integrability for Jacobians of quasiregular mappings given in Proposition 2.5,

$$\begin{aligned}& r_n^{d/a} |\tfrac{1}{2}D_i \setminus C_i(\delta)|^{1/a} \left(\int_{\frac{1}{2}B_n} J_f^b \right)^{1/b} \\ &\qquad \le C(K, M, d, b) |\tfrac{1}{2}D_i \setminus C_i(\delta)|^{1/a} r_n^{d/a} r_n^{-d/a} \int_{B_n} J_f \\ &\qquad = C(K, M, d, b) |\tfrac{1}{2}D_i \setminus C_i(\delta)|^{1/a} A(B_n).\end{aligned}$$

We now choose $\delta >$ to be so small that $|\frac{1}{2}D_i \setminus C_i(\delta)|^{1/a} < \frac{1}{2C(K,M,d,b)kD(d)}$. This proves the lemma. ∎

We now have all of the ingredients to finish the proof of the main theorem.

Proof of Theorem 1.1. Recall that we proceed by contradiction and assume that $k > \binom{d}{l}$. We may assume that $\mathrm{vol}(M) = 1$. Let C_i, U_i be the sets given in Lemma 5.1. Define C_i^n, U_i^n as in Lemma 5.1. Choose $\widetilde{\psi} \in C_c^\infty(B(0,1))$ so that $0 \le \widetilde{\psi} \le 1$, $\widetilde{\psi} \equiv 1$ on C_i and $\widetilde{\psi} \equiv 0$ on the complement of U_i. Next we define $\psi_n(x) = \widetilde{\psi} \circ T_n^{-1}$. By Proposition 3.3 and Lemma 5.1,

$$\lim_{n \to \infty} \left| \frac{1}{(\int_{B_n} \psi_n^d J_f)} \int_{B_n} \psi_n^d \eta_i \wedge \theta_i - 1 \right| = 0. \tag{5.3}$$

By Lemma 4.2,

$$\lim_{n\to\infty}\left|\frac{1}{A(B_n)}\int_{B_n}\psi^d\eta_i^n\wedge\theta_i^n\right| = \left|\int_{B(0,1)}\widetilde{\psi}^d\widetilde{\eta}_i\wedge\widetilde{\theta}_i\right| \le \int_{B(0,1)}\widetilde{\psi}^d|\widetilde{\eta}_i\wedge\widetilde{\theta}_i|.$$

Since the support of $\widetilde{\psi}$ is contained in U_i,

$$\int_{B(0,1)}\widetilde{\psi}^d|\widetilde{\eta}_i\wedge\widetilde{\theta}_i| \le \int_{U_i}|\widetilde{\eta}_i\wedge\widetilde{\theta}_i| < \epsilon,$$

by (5.2). So, for n sufficiently large, we have that

$$\left|\frac{1}{A(B_n)}\int_{B_n}\psi^d\eta_i\wedge\theta_i\right| \le 2\epsilon. \tag{5.4}$$

Therefore, by (5.1) and (5.4),

$$\frac{1}{(\int_{B_n}\psi^d J_f)}\left|\int_{B_n}\psi^d\eta_i\wedge\theta_i\right| = \frac{A(B_n)}{(\int_{B_n}\psi^d J_f)}\left|\frac{1}{A(B_n)}\int_{B_n}\psi^d\eta_i\wedge\theta_i\right| \le 4kD(d)\epsilon.$$

This bound is independent of n and contradicts (5.3) for small ϵ and large n. Therefore $|\bigcup D_i| \ne |B(0,1)|$ and $k \le \binom{d}{l}$. This proves Theorem 1.1. □

References

[1] B. Bojarski and T. Iwaniec, Analytical foundations of the theory of quasiconformal mappings in $\mathbf{R}^n$, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **8** no. 2 (1983), 257–324. MR 0731786. Zbl 0548.30016. https://doi.org/10.5186/aasfm.1983.0806.

[2] M. Bonk and J. Heinonen, Quasiregular mappings and cohomology, *Acta Math.* **186** no. 2 (2001), 219–238. MR 1846030. Zbl 1088.30011. https://doi.org/10.1007/BF02401840.

[3] M. Bonk and P. Poggi-Corradini, The Rickman-Picard theorem, 2019, to appear in *Ann. Acad. Sci. Fenn. Ser. A I Math.* **44**, no. 2. arXiv 1807.07683.

[4] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, *Grad. Texts in Math.* **82**, Springer-Verlag, New York-Berlin, 1982. MR 0658304. Zbl 0496.55001. https://doi.org/10.1007/978-1-4757-3951-0.

[5] S. K. Donaldson and D. P. Sullivan, Quasiconformal 4-manifolds, *Acta Math.* **163** no. 3-4 (1989), 181–252. MR 1032074. Zbl 0704.57008. https://doi.org/10.1007/BF02392736.

[6] D. Drasin and P. Pankka, Sharpness of Rickman's Picard theorem in all dimensions, *Acta Math.* **214** no. 2 (2015), 209–306. MR 3372169. Zbl 1326.30025. https://doi.org/10.1007/s11511-015-0125-x.

[7] A. Eremenko and J. L. Lewis, Uniform limits of certain A-harmonic functions with applications to quasiregular mappings, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **16** no. 2 (1991), 361–375. MR 1139803. Zbl 0727.35022. https://doi.org/10.5186/aasfm.1991.1609.

[8] F. W. Gehring, The L^p-integrability of the partial derivatives of a quasiconformal mapping, *Acta Math.* **130** (1973), 265–277. MR 0402038. Zbl 0258.30021. https://doi.org/10.1007/BF02392268.

[9] M. Gromov, Hyperbolic manifolds, groups and actions, in *Riemann Surfaces and Related Topics*: *Proceedings of the* 1978 *Stony Brook Conference* (State Univ. New York, Stony Brook, N.Y., 1978), *Ann. of Math. Stud.* **97**, Princeton Univ. Press, Princeton, N.J., 1981, pp. 183–213. MR 0624814. Zbl 0467.53035. https://doi.org/10.1515/9781400881550.

[10] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, English ed., *Modern Birkhäuser Classics*, Birkhäuser Boston, Inc., Boston, MA, 2007, based on the 1981 French original, with appendices by M. Katz, P. Pansu and S. Semmes, translated from the French by Sean Michael Bates. MR 2307192. Zbl 1113.53001. https://doi.org/10.1007/978-0-8176-4583-0.

[11] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974. MR 0348781. Zbl 05788282. https://doi.org/10.1090/chel/370.

[12] T. Iwaniec and A. Lutoborski, Integral estimates for null Lagrangians, *Arch. Rational Mech. Anal.* **125** no. 1 (1993), 25–79. MR 1241286. Zbl 0793.58002. https://doi.org/10.1007/BF00411477.

[13] T. Iwaniec and G. Martin, Quasiregular mappings in even dimensions, *Acta Math.* **170** no. 1 (1993), 29–81. MR 1208562. Zbl 0785.30008. https://doi.org/10.1007/BF02392454.

[14] J. Jormakka, The existence of quasiregular mappings from $\mathbf{R}^3$ to closed orientable 3-manifolds, *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes* no. 69 (1988), 44. MR 0973719. Zbl 0662.57007.

[15] I. Kangasniemi, Sharp cohomological bound for uniformly quasiregularly elliptic manifolds, 2017. arXiv 1711.11410.

[16] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, second ed., *Grundlehren Math. Wissen.* **126**, Springer-Verlag, New York-Heidelberg, 1973, translated from the German by K. W. Lucas. MR 0344463. Zbl 0267.30016.

[17] J. L. Lewis, Picard's theorem and Rickman's theorem by way of Harnack's inequality, *Proc. Amer. Math. Soc.* **122** no. 1 (1994), 199–206. MR 1195483. Zbl 0807.30010. https://doi.org/10.2307/2160861.

[18] P. Mattila and S. Rickman, Averages of the counting function of a quasiregular mapping, *Acta Math.* **143** no. 3-4 (1979), 273–305. MR 0549779. Zbl 0443.30043. https://doi.org/10.1007/BF02392097.

[19] P. Pankka, Mappings of bounded mean distortion and cohomology, *Geom. Funct. Anal.* **20** no. 1 (2010), 229–242. MR 2647140. Zbl 1203.30023. https://doi.org/10.1007/s00039-010-0054-y.

[20] S. Rickman, On the number of omitted values of entire quasiregular mappings, *J. Analyse Math.* **37** (1980), 100–117. MR 0583633. Zbl 0451.30012. https://doi.org/10.1007/BF02797681.

[21] S. Rickman, The analogue of Picard's theorem for quasiregular mappings in dimension three, *Acta Math.* **154** no. 3-4 (1985), 195–242. MR 0781587. Zbl 0617.30024. https://doi.org/10.1007/BF02392472.

[22] S. Rickman, *Quasiregular Mappings*, *Ergeb. Math. Grenzgeb.* **26**, Springer-Verlag, Berlin, 1993. MR 1238941. Zbl 0816.30017. https://doi.org/10.1007/978-3-642-78201-5.

[23] S. Rickman, Simply connected quasiregularly elliptic 4-manifolds, *Ann. Acad. Sci. Fenn. Math.* **31** no. 1 (2006), 97–110. MR 2210111. Zbl 1116.30011.

[24] N. T. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and Geometry on Groups*, *Cambridge Tracts in Math.* **100**, Cambridge University Press, Cambridge, 1992. MR 1218884. Zbl 0813.22003. https://doi.org/10.1017/CBO9780511662485.

(Received: June 21, 2018)
(Revised: February 18, 2019)

University of California, Los Angeles, CA
E-mail: eprywes@math.ucla.edu