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Sharp L^2 estimates of the Schrödinger maximal function in higher dimensions

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Dedicated to the memory of Jean Bourgain

Abstract

We show that, for $n \geq 3$, $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ holds almost everywhere for all $f \in H^s(\mathbb{R}^n)$ provided that $s > \frac{n}{2(n+1)}$. Due to a counterexample by Bourgain, up to the endpoint, this result is sharp and fully resolves a problem raised by Carleson. Our main theorem is a fractal L^2 restriction estimate, which also gives improved results on the size of the divergence set of the Schrödinger solutions, the Falconer distance set problem and the spherical average Fourier decay rates of fractal measures. The key ingredients of the proof include multilinear Kakeya estimates, decoupling and induction on scales.

1. Introduction

The solution to the free Schrödinger equation

$$\begin{cases} iu_t - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & x \in \mathbb{R}^n \end{cases} \tag{1.1}$$

is given by

$$e^{it\Delta} f(x) = (2\pi)^{-n} \int e^{i\left(x\cdot\xi + t|\xi|^2\right)} \widehat{f}(\xi)\, d\xi.$$

In [8], Carleson proposed the problem of identifying the optimal s for which $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ almost everywhere whenever $f \in H^s(\mathbb{R}^n)$, and he proved convergence for $s \geq \frac{1}{4}$ when $n = 1$. Dahlberg and Kenig [10] then showed that this result is sharp. The higher dimensional case has since been studied by several authors [7], [9], [29], [31], [2], [27], [30], [22], [3], [23], [11], [4], [25], [12], [13]. In particular, almost everywhere convergence holds if $s > \frac{1}{2} - \frac{1}{4n}$ when $n \geq 2$ ($n = 2$ due to Lee [22] and $n \geq 2$ due to Bourgain [3]).

Keywords: Fourier restriction, weighted restriction, Schrödinger equation, Schrödinger maximal function, decoupling, refined Strichartz

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Recently Bourgain [4] gave counterexamples showing that convergence can fail if $s < \frac{n}{2(n+1)}$. Since then, Guth, Li and the first author [12] improved the sufficient condition when $n = 2$ to the almost sharp range $s > \frac{1}{3}$. In higher dimensions ($n \geq 3$), Guth, Li and the authors [13] proved the convergence for $s > \frac{n+1}{2(n+2)}$.

In this article, we establish the following theorem, which is sharp up to the endpoint.

THEOREM 1.1. *Let* $n \geq 3$. *For every* $f \in H^s(\mathbb{R}^n)$ *with* $s > \frac{n}{2(n+1)}$, $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$ *almost everywhere.*

We use $B^m(x, r)$ to represent a ball centered at x with radius r in $\mathbb{R}^m$. By a standard smooth approximation argument, Theorem 1.1 is a consequence of the following estimate of the Schrödinger maximal function.

THEOREM 1.2. *Let* $n \geq 3$. *For any* $s > \frac{n}{2(n+1)}$, *the following bound holds. For any function* $f \in H^s(\mathbb{R}^n)$,

$$\left\| \sup_{0<t\leq 1} |e^{it\Delta} f| \right\|_{L^2(B^n(0,1))} \leq C_s \|f\|_{H^s(\mathbb{R}^n)}. \tag{1.2}$$

Via a localization argument, Littlewood-Paley decomposition and parabolic rescaling, Theorem 1.2 is reduced to the following theorem, which we will prove in this paper.

THEOREM 1.3. *Let* $n \geq 3$. *For any* $\varepsilon > 0$, *there exists a constant* C_ε *such that*

$$\left\| \sup_{0<t\leq R} |e^{it\Delta} f| \right\|_{L^2(B^n(0,R))} \leq C_\varepsilon R^{\frac{n}{2(n+1)}+\varepsilon} \|f\|_2 \tag{1.3}$$

holds for all $R \geq 1$ *and all* f *with* $\mathrm{supp}\widehat{f} \subset A(1) = \{\xi \in \mathbb{R}^n : |\xi| \sim 1\}$.

Remark 1.4. When $n = 1, 2$, our proof of Theorem 1.3 remains valid and recovers the almost sharp results of the pointwise convergence problem. However, the sharp L^2 estimates of the Schrödinger maximal function are not as strong as the previous sharp L^p estimates in the cases $n = 1, 2$:

$$\left\| \sup_{t>0} |e^{it\Delta} f| \right\|_{L^4(\mathbb{R})} \leq C \|f\|_{H^{1/4}(\mathbb{R})} \quad \text{[21, Kenig-Ponce-Vega]} \tag{1.4}$$

and

$$\left\| \sup_{0<t\leq 1} |e^{it\Delta} f| \right\|_{L^3(\mathbb{R}^2)} \leq C_s \|f\|_{H^s(\mathbb{R}^2)} \quad \forall s > \frac{1}{3} \quad \text{[12, D.-Guth-Li].}\ ^{1} \tag{1.5}$$

[1] The global L^3 estimate (1.5) follows easily from the local L^3 estimate in [12], via a localization argument using wave packet decomposition.

Testing with the standard examples used in restriction theory seems to suggest that the following estimate holds for all $n \geq 1$:

$$\left\| \sup_{0<t\leq 1} |e^{it\Delta} f| \right\|_{L^{\frac{2(n+1)}{n}}(\mathbb{R}^n)} \leq C \|f\|_{H^{\frac{n}{2(n+1)}}(\mathbb{R}^n)}. \tag{1.6}$$

From (1.4) and (1.5) we see that (1.6) is true for $n = 1$ and is true up to the endpoint for $n = 2$. However, the estimate (1.6) fails in higher dimensions. In a recent work of Kim, Wang and the authors [15], by looking at Bourgain's counterexample [4] in every intermediate dimension, we showed that the following local estimate,

$$\left\| \sup_{0<t\leq 1} |e^{it\Delta} f| \right\|_{L^p(B^n(0,1))} \leq C_s \|f\|_{H^s(\mathbb{R}^n)} \quad \forall s > \frac{n}{2(n+1)}, \tag{1.7}$$

fails if $p > p_0 := 2 + \frac{4}{(n-1)(n+2)}$. Note that $\frac{2(n+1)}{n} > p_0$ when $n \geq 3$ and henceforth (1.6) fails. To our best knowledge, the following two problems are still open when $n \geq 3$: determine the optimal $p = p(n)$ for which we can have (1.7), and identify the optimal $s = s(n,p)$ for which (1.7) with $p > 2$ fixed holds.

Remark 1.5. In our proof of (1.3), no typical L^2 arguments such as Plancherel and TT^* are invoked to take advantage of the particular use of the L^2 norm on the left-hand side of (1.3). In fact, the L^2 norm will be converted to L^p norm (see Proposition 3.1), where $p = \frac{2(n+1)}{n-1}$ is the sharp exponent for the l^2 decoupling theorem in dimension n. The L^2 is used on the left-hand side of (1.3) mostly because the numerology adds up favorably for that space.

By lattice L-cubes we mean cubes of the form $l + [0, L]^n$ with $l \in (L\mathbb{Z})^n$. Our main result is the following fractal L^2 restriction estimate, from which Theorem 1.3 follows.

THEOREM 1.6. *Let $n \geq 1$. For any $\varepsilon > 0$, there exists a constant C_ε such that the following holds for all $R \geq 1$ and all f with* $\operatorname{supp}\widehat{f} \subset B^n(0,1)$. *Suppose that $X = \bigcup_k B_k$ is a union of lattice unit cubes in $B^{n+1}(0,R)$ and each lattice $R^{1/2}$-cube intersecting X contains $\sim \lambda$ many unit cubes in X. Let $1 \leq \alpha \leq n+1$ and γ be given by*

$$\gamma := \max_{\substack{B^{n+1}(x',r) \subset B^{n+1}(0,R) \\ x' \in \mathbb{R}^{n+1}, r \geq 1}} \frac{\#\{B_k : B_k \subset B(x', r)\}}{r^\alpha}. \tag{1.8}$$

Then

$$\|e^{it\Delta} f\|_{L^2(X)} \leq C_\varepsilon \gamma^{\frac{2}{(n+1)(n+2)}} \lambda^{\frac{n}{(n+1)(n+2)}} R^{\frac{\alpha}{(n+1)(n+2)}+\varepsilon} \|f\|_2. \tag{1.9}$$

Note that in Theorem 1.6, $\lambda \le \gamma R^{\alpha/2}$. As a direct result of Theorem 1.6, there holds a slightly weaker fractal L^2 restriction estimate. It has a relatively simpler statement:

Corollary 1.7. *Let $n \ge 1$. For any $\varepsilon > 0$, there exists a constant C_ε such that the following holds for all $R \ge 1$ and all f with* $\operatorname{supp}\widehat{f} \subset B^n(0,1)$. *Suppose that $X = \bigcup_k B_k$ is a union of lattice unit cubes in $B^{n+1}(0,R)$. Let $1 \le \alpha \le n+1$ and γ be given by*

$$\gamma := \max_{\substack{B^{n+1}(x',r)\subset B^{n+1}(0,R) \\ x'\in\mathbb{R}^{n+1}, r\ge 1}} \frac{\#\{B_k : B_k \subset B(x',r)\}}{r^\alpha}. \tag{1.10}$$

Then

$$\|e^{it\Delta} f\|_{L^2(X)} \le C_\varepsilon \gamma^{\frac{1}{n+1}} R^{\frac{\alpha}{2(n+1)}+\varepsilon} \|f\|_2. \tag{1.11}$$

We will see that Corollary 1.7 is sufficient to derive the sharp L^2 estimate of the Schrödinger maximal function (Theorem 1.3) and all other applications in Section 2. This corollary can also be proved directly by a slightly simpler argument. The case $n = 1$ of Corollary 1.7 can be recovered using the ingredients in Wolff's paper [32]. See Section 3.3 for a discussion.

Nevertheless, Theorem 1.6 has two advantages compared to Corollary 1.7. Firstly, it gives us a better L^2 restriction estimate if the set X of unit cubes is fairly sparse at the scale $R^{1/2}$. Secondly, it tells us some geometric information about a set X of unit cubes when $\|e^{it\Delta} f\|_{L^2(X)}$ is comparable to $\|e^{it\Delta} f\|_{L^2(B(0,R))}$. For example, taking $\alpha = n+1$ (hence $\gamma \lesssim 1$), we have

Corollary 1.8. *Let $n \ge 1$. Suppose that $X = \bigcup_k B_k$ is a union of lattice unit cubes in $B^{n+1}(0,R)$ and each lattice $R^{1/2}$-cube intersecting X contains $\sim \lambda$ many unit cubes in X. Suppose there is a function f with* $\operatorname{supp}\widehat{f} \subset B^n(0,1)$ *and $\|f\|_2 \ne 0$ such that $\|e^{it\Delta} f\|_{L^2(X)} \gtrsim R^{1/2}\|f\|_2$. Then $\lambda \gtrapprox R^{\frac{n+1}{2}}$.*

As a remark, the scale $R^{1/2}$ in Corollary 1.8 is the largest one can have. Indeed, with the assumption of the corollary, the unit cubes in X do not have to almost fill R^β-cubes completely for $\beta > 1/2$. One can see this from the Knapp example where we only have one wave packet.

To prove our main result, Theorem 1.6, we will use a broad-narrow analysis, which has similar spirit as the techniques in the work of Bourgain-Guth [6], Bourgain [3], Bourgain-Demeter [5] and Guth [19].

In the broad case, we can exploit the transversality and apply the multilinear refined Strichartz estimate, which is a result obtained by Guth, Li and the authors in [13]. (See [12], [14], [13] for applications of the refined Strichartz estimate.) In the narrow case, we use the l^2 decoupling theorem of Bourgain-Demeter [5] in a lower dimension and perform induction on scales. The way we do induction has its roots in the proof of the linear refined Strichartz estimate, due to Guth, Li and the first author (essentially proved in [12] — see [13] for the statement in the general setting).

Our method is related to Bourgain's in [3], where he has a similar broad-narrow analysis. (Here we have the size of the small ball being K^2 instead of K as in [3] for a technical issue similar to what one has in [5], [19].) He applied multilinear restriction to control the broad part in the sharp range $s > \frac{n}{2(n+1)}$ (except the endpoint). He speculated from this that the above range of s might be sharp. (See the end of the introduction in [4].) In [3] the narrow part was handled following the general approach from [6], which gives non-sharp estimates. Historically, one could view the present non-endpoint solution to Carleson's problem as building on [3], providing a subtler way of handling the narrow part and proving Corollary 1.7. For the stronger Theorem 1.6 and Corollary 1.8, one needs a different ingredient, namely, the multilinear refined Strichartz in [13], to handle the broad part.

In Section 2 we show how Corollary 1.7 and Theorem 1.3 follow from Theorem 1.6, and we also present applications of Theorem 1.6 to other problems, bounding the size of the divergence set of the Schrödinger solutions (Theorem 2.4), the Falconer distance set problem (Theorems 2.6 and 2.7) and the spherical average Fourier decay rates of fractal measures (Theorem 2.8). We prove Theorem 1.6 in Section 3.

Notation. We write

- $A \lesssim B$ if $A \le CB$ for some absolute constant C, $A \sim B$ if $A \lesssim B$ and $B \lesssim A$;
- $A \ll B$ if A is much less than B;
- $A \lessapprox B$ if $A \le C_\varepsilon R^\varepsilon B$ for any $\varepsilon > 0, R > 1$.

Sometimes we also write

- $A \lesssim B$ if $A \le C_\varepsilon B$ for some constant C_ε depending on ε

(when the dependence on ε is unimportant).

By an r-ball (cube) we mean a ball (cube) of radius (side length) r. An $r \times \cdots \times r \times L$-tube (box) means a tube (box) with radius (short sides length) r and length L. For a set $\mathcal{S}$, $\#\mathcal{S}$ denotes its cardinality.

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2. Applications of Theorem 1.6

2.1. *Sharp L^2 estimates of the Schrödinger maximal function.* In this subsection, we show how Corollary 1.7 and Theorem 1.3 follow from Theorem 1.6, via the dyadic pigeonholing argument and the locally constant property.

Proof of (Theorem 1.6 $\Longrightarrow$ Corollary 1.7). Given $X=\bigcup_k B_k$, a union of lattice unit cubes in $B^{n+1}(0,R)$ satisfying the assumptions of Corollary 1.7, we sort the lattice $R^{1/2}$-cubes in $\mathbb{R}^{n+1}$ intersecting X by the number λ of unit cubes B_k contained in it. Since $1\le\lambda\le R^{O(1)}$, there are only $O(\log R)$ choices for the dyadic number λ. So we can choose a dyadic number λ and a subset $\mathcal{B}_\lambda$ of $\{B_k\}$ such that for each unit cube B in $\mathcal{B}_\lambda$, the lattice $R^{1/2}$-cube containing it contains $\sim\lambda$ many unit cubes from $\mathcal{B}_\lambda$ and

$$\|e^{it\Delta}f\|_{L^2(X)}\lessapprox\|e^{it\Delta}f\|_{L^2(\bigcup_{B\in\mathcal{B}_\lambda}B)}.$$

By applying Theorem 1.6 to $\|e^{it\Delta}f\|_{L^2(\bigcup_{B\in\mathcal{B}_\lambda}B)}$, we get

$$\|e^{it\Delta}f\|_{L^2(X)}\lessapprox\gamma^{\frac{2}{(n+1)(n+2)}}\lambda^{\frac{n}{(n+1)(n+2)}}R^{\frac{\alpha}{(n+1)(n+2)}}\|f\|_2,$$

and (1.11) follows from the fact that $\lambda\le\gamma R^{\alpha/2}$. □

Proof of the case $\alpha=n$ of Corollary 1.7 $\Longrightarrow$ Theorem 1.3. We will show that

$$\left\|\sup_{0<t<R}|e^{it\Delta}f|\right\|_{L^2(B^n(0,R))}\lessapprox R^{\frac{n}{2(n+1)}}\|f\|_2 \tag{2.1}$$

holds for all $R\ge1$ and all f with Fourier support in $B^n(0,1)$.

By viewing $|e^{it\Delta}f(x)|$ essentially as constant on unit balls,[2] we can find a set X described as follows: X is a union of unit balls in $B^n(0,R)\times[0,R]$ satisfying the property that each vertical thin tube of dimensions $1\times\cdots\times1\times R$ contains exactly one unit ball in X, and

$$\left\|\sup_{0<t<R}|e^{it\Delta}f|\right\|_{L^2(B^n(0,R))}\lessapprox\|e^{it\Delta}f\|_{L^2(X)}. \tag{2.2}$$

[2] We refer the readers to [6, §§2–5] for the standard formalism of this locally constant property.

The desired estimate (2.1) follows by applying Corollary 1.7 to $\|e^{it\Delta}f\|_{L^2(X)}$ with $\alpha = n$ and $\gamma \lesssim 1$. □

2.2. *Other applications.* By formalizing the locally constant property, from Corollary 1.7 we derive some weighted L^2 estimates, Theorems 2.2 and 2.3, which in turn have applications to several problems described below.

Definition 2.1. Let $\alpha \in (0, d]$. We say that μ is an *α-dimensional measure* in $\mathbb{R}^d$ if it is a probability measure supported in the unit ball $B^d(0,1)$ and satisfies that

$$\mu(B(x,r)) \le C_\mu r^\alpha \quad \forall r > 0, \quad \forall x \in \mathbb{R}^d. \tag{2.3}$$

Denote $d\mu_R(\cdot) := R^\alpha d\mu(\frac{\cdot}{R})$.

THEOREM 2.2. *Let $n \ge 1, \alpha \in (0, n]$ and μ be an α-dimensional measure in $\mathbb{R}^n$. Then*

$$\left\| \sup_{0<t<R} |e^{it\Delta} f| \right\|_{L^2(B^n(0,R); d\mu_R(x))} \lessapprox R^{\frac{\alpha}{2(n+1)}} \|f\|_2, \tag{2.4}$$

whenever $R \ge 1$ and f has Fourier support in $B^n(0,1)$.

THEOREM 2.3. *Let $n \ge 1, \alpha \in (0, n+1]$ and μ be an α-dimensional measure in $\mathbb{R}^{n+1}$. Then*

$$\left\| e^{it\Delta} f \right\|_{L^2(B^{n+1}(0,R); d\mu_R(x,t))} \lessapprox R^{\frac{\alpha}{2(n+1)}} \|f\|_2, \tag{2.5}$$

whenever $R \ge 1$ and f has Fourier support in $B^n(0,1)$.

We defer the proof of these weighted L^2 estimates to the end of this subsection. Let us first see their applications. We omit history and various previous results on the following three problems and refer the readers to [13], [14], [24] and the references therein.

(I) *Hausdorff dimension of the divergence set of the Schrödinger solutions.* A natural refinement of Carleson's problem was initiated by Sjögren and Sjölin [28]: Determine the size of the divergence set; in particular, consider

$$\alpha_n(s) := \sup_{f \in H^s(\mathbb{R}^n)} \dim \left\{ x \in \mathbb{R}^n : \lim_{t \to 0} e^{it\Delta} f(x) \ne f(x) \right\},$$

where dim stands for the Hausdorff dimension.

The following theorem is a direct result of Theorem 2.2 (cf. [13], [24]). When $n = 2$, it recovers the corresponding result derived from the sharp L^3 estimate of the Schrödinger maximal function in Du-Guth-Li [12]. When $n \ge 3$, it improves the previous best known result in Du-Guth-Li-Zhang [13].

THEOREM 2.4. *Let $n \geq 2$. Then*

$$\alpha_n(s) \leq n + 1 - \frac{2(n+1)s}{n}, \quad \frac{n}{2(n+1)} < s < \frac{n}{4}. \tag{2.6}$$

(II) *Falconer distance set problem.* Let $E \subset \mathbb{R}^d$ be a compact subset. Its distance set $\Delta(E)$ is defined by

$$\Delta(E) := \{|x - y| : x, y \in E\}.$$

CONJECTURE 2.5 (Falconer [16]). *Let $d \geq 2$, and let $E \subset \mathbb{R}^d$ be a compact set. Then*

$$\dim(E) > \frac{d}{2} \Rightarrow |\Delta(E)| > 0.$$

Here $|\cdot|$ denotes the Lebesgue measure and $\dim(\cdot)$ is the Hausdorff dimension.

Following a scheme due to Mattila (cf. [14, Prop. 2.3]), Theorem 2.3 implies the following result towards Falconer's conjecture. When $d = 2, 3$, this recovers the previous best known results of Wolff (d=2, [32]) and Du-Guth-Ou-Wang-Wilson-Zhang (d=3, [14]), via a different approach. In the case $d \geq 4$, this improves the previous best known result in [14]:

THEOREM 2.6. *Let $d \geq 2$ and $E \subset \mathbb{R}^d$ be a compact set with*

$$\dim(E) > \frac{d^2}{2d-1} = \frac{d}{2} + \frac{1}{4} + \frac{1}{8d-4}.$$

Then $|\Delta(E)| > 0$.

By applying a very recent work of Liu [20, Th. 1.4], Theorem 2.3 also implies the following result for the pinned distance set problem, with the same threshold.

THEOREM 2.7. *Let $d \geq 2$, and let $E \subset \mathbb{R}^d$ be a compact set with*

$$\dim(E) > \frac{d^2}{2d-1} = \frac{d}{2} + \frac{1}{4} + \frac{1}{8d-4}.$$

Then there exists $x \in E$ such that its pinned distance set

$$\Delta_x(E) := \{|x - y| : y \in E\}$$

has positive Lebesgue measure.

(III) *Spherical average Fourier decay rates of fractal measures.* Let $\beta_d(\alpha)$ denote the supremum of the numbers β for which

$$\|\widehat{\mu}(R\cdot)\|^2_{L^2(\mathbb{S}^{d-1})} \leq C_{\alpha,\mu} R^{-\beta} \tag{2.7}$$

whenever $R > 1$ and μ is an α-dimensional measure in $\mathbb{R}^d$. The problem of identifying the precise value of $\beta_d(\alpha)$ was proposed by Mattila [26].

A lower bound of $\beta_d(\alpha)$ as in Theorem 2.8 follows from Theorem 2.3 (cf. [14, Rem. 2.5]). When $d = 2$, this recovers the sharp result of Wolff [32]. When $d = 3$ and $\alpha \in (\frac{3}{2}, 2]$, this recovers the previous best known result of Du-Guth-Ou-Wang-Wilson-Zhang [14]. In the case $d = 3, \alpha \in (2, 3)$ or $d \geq 4, \alpha \in (d/2, d)$, this improves the previous best known result in [14].

THEOREM 2.8. *Let* $d \geq 2$ *and* $\alpha \in (\frac{d}{2}, d)$. *Then*

$$\beta_d(\alpha) \geq \frac{(d-1)\alpha}{d}.$$

The proofs of Theorems 2.2 and 2.3 are entirely similar and we only do the proof of the former here, which is slightly more involved.

Proof of Theorem 2.2. Denote $e^{it\Delta} f(x)$ by $Ef(x,t)$ and (x,t) by $\tilde{x}$. Since $\operatorname{supp}\widehat{f} \subseteq B^n(0,1)$, we have $\operatorname{supp}\widehat{Ef} \subseteq B^{n+1}(0,1)$. Thus there exists a Schwartz bump function ψ on $\mathbb{R}^{n+1}$ (we require $\widehat{\psi} \equiv 1$ on $B^{n+1}(0,100)$) such that $(Ef)^2 = (Ef)^2 * \psi$.

The function $\max_{|\tilde{y}-\tilde{x}| \leq e^{100n}} |\psi(\tilde{y})|$ is rapidly decaying. We call it $\psi_1(\tilde{x})$. Note also that any (x,t) in $\mathbb{R}^{n+1}$ belongs to a unique integral lattice cube whose center we denote by $\tilde{m} = (m, m_{n+1}) = (m_1, \ldots, m_{n+1}) = \tilde{m}(x,t)$.

Then we have

(2.8)
$$\begin{aligned}
&\left\| \sup_{0<t<R} |e^{it\Delta} f| \right\|^2_{L^2(B^n(0,R);d\mu_R)} \\
&= \int_{B^n(0,R)} \sup_{0<t<R} |Ef(x,t)|^2 d\mu_R(x) \\
&\leq \int_{B^n(0,R)} \sup_{0<t<R} \left(|Ef|^2 * |\psi|\right)(x,t) d\mu_R(x) \\
&\leq \int_{B^n(0,R)} \sup_{0<t<R} \left(|Ef|^2 * \psi_1\right)(\tilde{m}(x,t)) d\mu_R(x) \\
&\leq \sum_{\substack{m=(m_1,\ldots,m_n)\in\mathbb{Z}^n, \\ |m_i| \leq R}} \left(\int_{|x-m|\leq 10} d\mu_R(x) \right) \cdot \sup_{\substack{m_{n+1}\in\mathbb{Z} \\ 0\leq m_{n+1}\leq R}} (|Ef|^2 * \psi_1)(m, m_{n+1}).
\end{aligned}$$

For each $m \in \mathbb{Z}^n$, let $b(m)$ be an integer in $[0, R]$ such that

$$\sup_{\substack{m_{n+1}\in\mathbb{Z} \\ 0\leq m_{n+1}\leq R}} (|Ef|^2 * \psi_1)(m, m_{n+1}) = (|Ef|^2 * \psi_1)(m, b(m)).$$

Also we assume $\|f\|_2 = 1$ so $|e^{it\Delta}f|$ is uniformly bounded pointwisely. For each $m \in \mathbb{Z}^n$, we define

$$\nu_m := \int_{|x-m|\le 10} d\mu_R(x) \lesssim 1.$$

By (2.8), we have

$$\begin{aligned} &\left\| \sup_{0<t<R} |e^{it\Delta}f| \right\|^2_{L^2(B^n(0,R);d\mu_R)} \\ \lesssim &\sum_{\substack{\nu \text{ dyadic} \\ \nu\in[R^{-100n},1]}} \sum_{\substack{m\in\mathbb{Z}^n, |m_i|\le R \\ \nu_m\sim\nu}} \nu \cdot (|Ef|^2 * \psi_1)(m, b(m)) + R^{-90n}. \end{aligned} \tag{2.9}$$

For each dyadic ν, denote $A_\nu = \{m \in \mathbb{Z}^n : |m_i| \le R, \nu_m \sim \nu\}$. Performing a dyadic pigeonholing over ν we see that there exists a dyadic $\nu \in [R^{-100n}, 1]$ such that for any small $\varepsilon > 0$,

$$\begin{aligned} &\left\| \sup_{0<t<R} |e^{it\Delta}f| \right\|^2_{L^2(B^n(0,R);d\mu_R)} \\ \lessapprox &\sum_{m\in A_\nu} \nu \cdot (|Ef|^2 * \psi_1)(m, b(m)) + R^{-89n} \\ \lesssim &\sum_{m\in A_\nu} \nu \cdot \left(\int_{B^{n+1}((m,b(m)),R^\varepsilon)} |Ef|^2 \right) + R^{-89n} \\ \lesssim &\nu \cdot \int_{\bigcup_{m\in A_\nu} B^{n+1}((m,b(m)),R^\varepsilon)} |Ef|^2 + R^{-89n}. \end{aligned} \tag{2.10}$$

Consider the set $X_\nu = \bigcup_{m\in A_\nu} B^{n+1}((m, b(m)), R^\varepsilon)$. It is a union of a collection of distinct R^ε-balls and at the same time, it is also a union of unit balls. These balls' projection onto the $(x_1, \ldots, x_n)$-plane are essentially disjoint. (A point can be covered $\lesssim R^\varepsilon$ times.) For every $r > R^{2\varepsilon}$, by the definition of A_ν, the intersection of X_ν and any r-ball can be contained in no more than $R^{10n\varepsilon}\nu^{-1}r^\alpha$ disjoint R^ε-balls. Hence we can apply Corollary 1.7 to X_ν with $\gamma \lesssim R^{100n\varepsilon}\nu^{-1}$ and α. With (2.10) this gives

$$\left\| \sup_{0<t<R} |e^{it\Delta}f| \right\|^2_{L^2(B^n(0,R);d\mu_R)} \lessapprox \nu^{\frac{n-1}{n+1}} R^{\frac{\alpha}{n+1}} \|f\|_2^2 \lesssim R^{\frac{\alpha}{n+1}} \|f\|_2^2. \tag{2.11}$$

This concludes the proof. □

3. Main inductive proposition and proof of Theorem 1.6

To prove Theorem 1.6, we will use a broad-narrow analysis, which involves inductions. To make everything work we introduce another parameter K and state the theorem in a slightly different way. We say that a collection of quantities are dyadically constant if all the quantities are in the same interval of the form $[2^j, 2^{j+1}]$, where j is an integer. This is our main inductive proposition:

PROPOSITION 3.1. *Let $n \geq 1$. For any $0 < \varepsilon < 1/100$, there exist constants C_ε and $0 < \delta = \delta(\varepsilon) \ll \varepsilon$ (e.g. $\delta = \varepsilon^{100}$) such that the following holds for all $R \geq 1$ and all f with $\operatorname{supp}\widehat{f} \subset B^n(0,1)$. Let $p = \frac{2(n+1)}{n-1}$ ($p = \infty$ when $n = 1$). Suppose that $Y = \bigcup_{k=1}^M B_k$ is a union of lattice K^2-cubes in $B^{n+1}(0,R)$ and each lattice $R^{1/2}$-cube intersecting Y contains $\sim \lambda$ many K^2-cubes in Y, where $K = R^\delta$. Suppose that*

$$\|e^{it\Delta}f\|_{L^p(B_k)} \text{ is dyadically a constant in } k = 1, 2, \ldots, M.$$

Let $1 \leq \alpha \leq n+1$ and γ be given by

$$\gamma := \max_{\substack{B^{n+1}(x',r) \subset B^{n+1}(0,R) \\ x' \in \mathbb{R}^{n+1}, r \geq K^2}} \frac{\#\{B_k : B_k \subset B(x',r)\}}{r^\alpha}. \tag{3.1}$$

Then

$$\|e^{it\Delta}f\|_{L^p(Y)} \leq C_\varepsilon M^{-\frac{1}{n+1}} \gamma^{\frac{2}{(n+1)(n+2)}} \lambda^{\frac{n}{(n+1)(n+2)}} R^{\frac{\alpha}{(n+1)(n+2)}+\varepsilon} \|f\|_2. \tag{3.2}$$

Theorem 1.6 follows from Proposition 3.1 by a dyadic pigeonholing argument:

Proof of (Proposition 3.1 $\Longrightarrow$ Theorem 1.6). Given $X = \bigcup_k B_k$, a union of lattice unit cubes satisfying the assumptions of Theorem 1.6, we sort these unit cubes B_k according to the value of $\|e^{it\Delta}f\|_{L^p(B_k)}$. Assuming $\|f\|_2 = 1$, there are only $O(\log R)$ significant dyadic choices for this value. Therefore, we can choose $X' \subset X$, a union of unit cubes B, such that

$$\left\{\|e^{it\Delta}f\|_{L^p(B)} : B \in X'\right\} \text{ are dyadically constant}$$

and

$$\|e^{it\Delta}f\|_{L^2(X)} \lessapprox \|e^{it\Delta}f\|_{L^2(X')}.$$

Let M be the total number of unit cubes B in X'. Since f has Fourier support in the unit ball, by locally constant property, $|e^{it\Delta}f|$ is essentially constant on unit balls. Therefore, the estimate (1.9) is equivalent to

$$\|e^{it\Delta}f\|_{L^p(X')} \lessapprox M^{-\frac{1}{n+1}} \gamma^{\frac{2}{(n+1)(n+2)}} \lambda^{\frac{n}{(n+1)(n+2)}} R^{\frac{\alpha}{(n+1)(n+2)}} \|f\|_2, \tag{3.3}$$

where $p = \frac{2(n+1)}{n-1}$, and γ, λ are as in the assumptions of Theorem 1.6.

We further sort the unit cubes B in X' as follows:

(1) Let β be a dyadic number and $\mathcal{B}_\beta$ a sub-collection of the unit cubes in X' such that for each B in $\mathcal{B}_\beta$, the lattice K^2-cube $\tilde{B}$ containing B satisfies

$$\|e^{it\Delta}f\|_{L^p(\tilde{B})} \sim \beta.$$

Denote the collection of relevant K^2-cubes by $\tilde{\mathcal{B}}_\beta$.

(2) Fix β. Let λ' be a dyadic number and $\mathcal{B}_{\beta,\lambda'}$ a sub-collection of $\mathcal{B}_\beta$ such that for each $B \in \mathcal{B}_{\beta,\lambda'}$, the lattice $R^{1/2}$-cube Q containing B contains $\sim \lambda'$ many K^2-cubes from $\tilde{\mathcal{B}}_\beta$. Denote the collection of relevant K^2-cubes by $\tilde{\mathcal{B}}_{\beta,\lambda'}$.

Since there are only $O(\log R)$ many significant choices for all dyadic numbers β, λ', we can choose some β and λ' so that $\#\mathcal{B}_{\beta,\lambda'} \gtrapprox M$. Then it follows easily by definition that

$$M' := \#\tilde{\mathcal{B}}_{\beta,\lambda'} \gtrapprox M, \quad \lambda' \le \lambda$$

and

$$\gamma' := \max_{\substack{B^{n+1}(x',r)\subset B^{n+1}(0,R)\\ x'\in\mathbb{R}^{n+1}, r\ge K^2}} \frac{\#\{\tilde{B} \in \tilde{\mathcal{B}}_{\beta,\lambda'} : \tilde{B} \subset B(x',r)\}}{r^\alpha} \le \gamma.$$

Applying Proposition 3.1 to $\|e^{it\Delta}f\|_{L^p(Y)}$ with $Y = \bigcup_{\tilde{B}\in\tilde{\mathcal{B}}_{\beta,\lambda'}} \tilde{B}$ and parameters M', γ', λ', we get

$$\|e^{it\Delta}f\|_{L^p(X)} \lessapprox \|e^{it\Delta}f\|_{L^p(Y)} \lessapprox M^{-\frac{1}{n+1}} \gamma^{\frac{2}{(n+1)(n+2)}} \lambda^{\frac{n}{(n+1)(n+2)}} R^{\frac{\alpha}{(n+1)(n+2)}} \|f\|_2,$$

as desired. □

The rest of this section is devoted to a proof of Proposition 3.1. Note that when the radius R is $\lesssim 1$, the estimate (3.2) is trivial. So we can assume that R is sufficiently large compared to any constant depending on ε. We will induct on radius R in our proof.

In the proof, we will sometimes have paragraphs starting with "*Intuition.*" We hope that these will help the readers understand what we do next.

Intuition. For our union Y of K^2-cubes, we want to use decoupling theory on each K^2-cube. This will relate the whole $e^{it\Delta}f$ to its contributions $e^{it\Delta}f_\tau$ from various $1/K$-caps τ in the frequency space. Instead of doing decoupling in dimension $n+1$, we are going to do a broad-narrow analysis following Bourgain-Guth [6], Bourgain [3], Bourgain-Demeter [5] and Guth [19]. For each K^2-cube, one of the following two has to happen:

(i) It is broad in the sense that there are $n+1$ contributing caps that are transversal. In this case the function is controlled by multilinear estimates that are usually strong enough.

(ii) It is narrow (i.e., not broad). In this case all the contributing caps have normal directions close to a hyperplane, which enables us to use decoupling in dimension n.

Either way we get better estimates than a direct $(n+1)$-dimensional decoupling. We control the broad part directly and do an induction on the narrow part. Our induction has its roots in the proof of the refined Strichartz estimate in [12], [13].

Throughout this section we fix $p = \frac{2(n+1)}{n-1}$. In the frequency space we decompose $B^n(0,1)$ into disjoint K^{-1}-cubes τ. Denote the set of K^{-1}-cubes τ by $\mathcal{S}$. For a function f with $\mathrm{supp}\widehat{f} \subset B^n(0,1)$, we have $f = \sum_\tau f_\tau$, where $\widehat{f_\tau}$ is $\widehat{f}$ restricted to τ. Given a K^2-cube B, we define its *significant* set as

$$\mathcal{S}(B) := \left\{ \tau \in \mathcal{S} : \|e^{it\Delta} f_\tau\|_{L^p(B)} \geq \frac{1}{100(\#\mathcal{S})} \|e^{it\Delta} f\|_{L^p(B)} \right\}.$$

Note that due to the triangle inequality,

$$\Big\| \sum_{\tau \in \mathcal{S}(B)} e^{it\Delta} f_\tau \Big\|_{L^p(B)} \sim \|e^{it\Delta} f\|_{L^p(B)}.$$

We say that a K^2-cube B is *narrow* if there is an n-dimensional subspace V such that for all $\tau \in \mathcal{S}(B)$,

$$\mathrm{Angle}(G(\tau), V) \leq \frac{1}{100nK},$$

where $G(\tau) \subset S^n$ is a spherical cap of radius $\sim K^{-1}$ given by

$$G(\tau) := \left\{ \frac{(-2\xi, 1)}{|(-2\xi, 1)|} \in S^n : \xi \in \tau \right\},$$

and $\mathrm{Angle}(G(\tau), V)$ denotes the smallest angle between any non-zero vector $v \in V$ and $v' \in G(\tau)$. Otherwise, we say the K^2-cube B is *broad*. It follows from this definition that for any broad B, there exist $\tau_1, \ldots, \tau_{n+1} \in \mathcal{S}(B)$ such that for any $v_j \in G(\tau_j)$,

$$|v_1 \wedge v_2 \wedge \cdots \wedge v_{n+1}| \gtrsim K^{-n}. \tag{3.4}$$

Denote the union of broad K^2-cubes B_k in Y by Y_{broad} and the union of narrow K^2-cubes B_k in Y by Y_{narrow}. We call it the broad case if Y_{broad} contains $\geq M/2$ many K^2-cubes, and the narrow case otherwise. We will deal with the broad case in Section 3.1, using the multilinear refined Strichartz estimate from [13]. We handle the narrow case in Section 3.2 by an inductive argument via the Bourgain-Demeter l^2 decoupling theorem [5] and induction on scales.

3.1. *Broad case.* Recall that $K = R^\delta$. A key tool we are using in the broad case is the following multilinear refined Strichartz estimate from [13], which is proved using l^2 decoupling, induction on scales and multilinear Kakeya estimates (see [1], [17]).

THEOREM 3.2 (cf. [13, Th. 4.2]). *Let $q = \frac{2(n+2)}{n}$. Let f be a function with Fourier support in $B^n(0,1)$. Suppose that $\tau_1, \ldots, \tau_{n+1} \in \mathcal{S}$ and (3.4) holds for any $v_j \in G(\tau_j)$. Suppose that $Q_1, Q_2, \ldots, Q_N$ are lattice $R^{1/2}$-cubes in B_R^{n+1}, so that*

$$\|e^{it\Delta} f_{\tau_i}\|_{L^q(Q_j)} \text{ is dyadically a constant in } j \text{ for each } i = 1, 2, \ldots, n+1.$$

Let Y denote $\bigcup_{j=1}^{N} Q_j$. Then for any $\epsilon > 0$,

$$\left\| \prod_{i=1}^{n+1} \left| e^{it\Delta} f_{\tau_i} \right|^{\frac{1}{n+1}} \right\|_{L^q(Y)} \leq C_\varepsilon R^\varepsilon N^{-\frac{n}{(n+1)(n+2)}} \|f\|_2. \tag{3.5}$$

Throughout the remainder of this subsection we will prove Proposition 3.1 in the broad case. In the broad case, there are $\sim M$ many broad K^2-cubes B. Denote the collection of $(n+1)$-tuple of transverse caps by Γ:

$$\Gamma := \{\tilde{\tau} = (\tau_1, \ldots, \tau_{n+1}) : \tau_j \in \mathcal{S} \text{ and (3.4) holds for any } v_j \in G(\tau_j)\}.$$

Then for each broad B,

$$\left\| e^{it\Delta} f \right\|_{L^p(B)}^p \leq K^{O(1)} \prod_{j=1}^{n+1} \left(\int_B \left| e^{it\Delta} f_{\tau_j} \right|^p \right)^{\frac{1}{n+1}} \tag{3.6}$$

for some $\tilde{\tau} = (\tau_1, \ldots, \tau_{n+1}) \in \Gamma$. In order to exploit the transversality, we want to bound the above geometric average of integrals by an integral of geometric average up to a loss of $K^{O(1)}$. We can do this by using translations and locally constant property. Given a K^2-cube B, denote its center by x_B. We break B into finitely overlapping balls of the form $B(x_B + v, 2)$, where $v \in B(0, K^2) \cap \mathbb{Z}^{n+1}$. For each τ_j, we can view $|e^{it\Delta} f_{\tau_j}|$ essentially as constant on each $B(x_B + v, 2)$. Choose $v_j \in B(0, K^2) \cap \mathbb{Z}^{n+1}$ such that $\|e^{it\Delta} f_{\tau_j}\|_{L^\infty(B)}$ is attained in $B(x_B + v_j, 2)$. Denote $v_j = (x_j, t_j)$, and define f_{τ_j, v_j} by

$$\widehat{f_{\tau_j, v_j}}(\xi) := \widehat{f_{\tau_j}}(\xi) e^{i(x_j \cdot \xi + t_j |\xi|^2)}.$$

Then

$$e^{it\Delta} f_{\tau_j, v_j}(x) = e^{i(t+t_j)\Delta} f_{\tau_j}(x + x_j),$$

and $|e^{it\Delta} f_{\tau_j, v_j}(x)|$ attains $\|e^{it\Delta} f_{\tau_j}\|_{L^\infty(B)}$ in $B(x_B, 2)$. Therefore,

$$\int_B \left| e^{it\Delta} f_{\tau_j} \right|^p \leq K^{O(1)} \int_{B(x_B, 2)} \left| e^{it\Delta} f_{\tau_j, v_j} \right|^p. \tag{3.7}$$

Now for each broad B, we find some $\tilde{\tau} = (\tau_1, \ldots, \tau_{n+1}) \in \Gamma$ and $\tilde{v} = (v_1, \ldots, v_{n+1})$ such that

$$\begin{aligned} \left\| e^{it\Delta} f \right\|_{L^p(B)}^p &\leq K^{O(1)} \prod_{j=1}^{n+1} \left(\int_{B(x_B, 2)} \left| e^{it\Delta} f_{\tau_j, v_j} \right|^p \right)^{\frac{1}{n+1}} \\ &\leq K^{O(1)} \int_{B(x_B, 2)} \prod_{j=1}^{n+1} \left| e^{it\Delta} f_{\tau_j, v_j} \right|^{\frac{p}{n+1}}. \end{aligned} \tag{3.8}$$

Since there are only $K^{O(1)}$ choices for $\tilde{\tau}$ and $\tilde{v}$, we can choose some $\tilde{\tau}$ and $\tilde{v}$ such that (3.8) holds for at least $K^{-C} M$ broad balls B. From now on, fix $\tilde{\tau}$ and $\tilde{v}$, and let f_j denote f_{τ_j, v_j}. Next we further sort the collection $\mathcal{B}$ of remaining broad balls as follows:

(1) For a dyadic number A, let $\mathcal{B}_A$ be a sub-collection of $\mathcal{B}$ in which for each B, we have

$$\left\| \prod_{j=1}^{n+1} \left| e^{it\Delta} f_j \right|^{\frac{1}{n+1}} \right\|_{L^\infty(B(x_B,2))} \sim A.$$

(2) Fix A, and for dyadic numbers $\tilde{\lambda}, \iota_1, \ldots, \iota_{n+1}$, let $\mathcal{B}_{A,\tilde{\lambda},\iota_1,\ldots,\iota_{n+1}}$ be a sub-collection of $\mathcal{B}_A$ in which for each B, the $R^{1/2}$-cube Q containing B contains $\sim \tilde{\lambda}$ cubes from $\mathcal{B}_A$ and

$$\|e^{it\Delta} f_j\|_{L^q(Q)} \sim \iota_j, \quad j = 1, 2, \ldots, n+1.$$

Here $q = \frac{2(n+2)}{n}$.

Recall that $p > q$, where $p = \frac{2(n+1)}{n-1}$ is the sharp exponent for decoupling in dimension n, and $q = \frac{2(n+2)}{n}$ is the exponent for which the multilinear refined Strichartz estimate in dimension $n+1$ holds. The first dyadic pigeonholing together with the locally constant property enables us to dominate L^p-norm by L^q-norm using reverse Hölder. The second dyadic pigeonholing allows us to apply the multilinear refined Strichartz estimate to control the L^q-norm.

We can assume that $\|f\|_2 = 1$. Then all the above dyadic numbers making significant contributions can be assumed to be between R^{-C} and R^C for a large constant C. Therefore, there exist some dyadic numbers $A, \tilde{\lambda}, \iota_1, \ldots, \iota_{n+1}$ such that $\mathcal{B}_{A,\tilde{\lambda},\iota_1,\ldots,\iota_{n+1}}$ contains $\geq K^{-C}M$ many cubes B. Fix a choice of $A, \tilde{\lambda}, \iota_1, \ldots, \iota_{n+1}$, and denote $\mathcal{B}_{A,\tilde{\lambda},\iota_1,\ldots,\iota_{n+1}}$ by $\mathcal{B}$ for convenience (a mild abuse of notation). Then, in the broad case, it follows from (3.8) and our choice of A that

$$\begin{aligned} \|e^{it\Delta} f\|_{L^p(Y)} \leq & K^{O(1)} \left\| \prod_{j=1}^{n+1} \left| e^{it\Delta} f_j \right|^{\frac{1}{n+1}} \right\|_{L^p(\cup_{B\in\mathcal{B}} B(x_B,2))} \\ \leq & K^{O(1)} M^{\frac{1}{p}-\frac{1}{q}} \left\| \prod_{j=1}^{n+1} \left| e^{it\Delta} f_j \right|^{\frac{1}{n+1}} \right\|_{L^q(\cup_{B\in\mathcal{B}} B(x_B,2))} \\ \leq & K^{O(1)} M^{-\frac{1}{(n+1)(n+2)}} \left\| \prod_{j=1}^{n+1} \left| e^{it\Delta} f_j \right|^{\frac{1}{n+1}} \right\|_{L^q(\cup_{Q\in\mathcal{Q}} Q)}, \end{aligned} \tag{3.9}$$

where $\mathcal{Q}$ is the collection of relevant $R^{1/2}$-cubes Q when we define $\mathcal{B}$. Note that

$$(\#\mathcal{Q})\lambda \geq (\#\mathcal{Q})\tilde{\lambda} \sim \#\mathcal{B} \geq K^{-C}M,$$

so

$$\tilde{N} := \#\mathcal{Q} \geq K^{-C}\frac{M}{\lambda}. \tag{3.10}$$

Applying Theorem 3.2, we get

$$\left\| \prod_{j=1}^{n+1} \left| e^{it\Delta} f_j \right|^{\frac{1}{n+1}} \right\|_{L^q(\cup_{Q\in\mathcal{Q}} Q)} \le K^{O(1)} \left(\frac{M}{\lambda}\right)^{-\frac{n}{(n+1)(n+2)}} \|f\|_2,$$

and therefore by (3.9),

$$\|e^{it\Delta} f\|_{L^p(Y)} \le K^{O(1)} M^{-\frac{1}{n+2}} \lambda^{\frac{n}{(n+1)(n+2)}} \|f\|_2.$$

Note that

$$M^{-\frac{1}{n+2}} \lambda^{\frac{n}{(n+1)(n+2)}} \le K^{O(1)} M^{-\frac{1}{n+1}} \gamma^{\frac{2}{(n+1)(n+2)}} \lambda^{\frac{n}{(n+1)(n+2)}} R^{\frac{\alpha}{(n+1)(n+2)}}$$

holds if and only if $M \le K^{O(1)} \gamma^2 R^\alpha$. Indeed, by definition (3.1) of γ, we have $M \le \gamma R^\alpha$ and $\gamma \ge K^{-2\alpha}$. So the broad case is done.

3.2. *Narrow case.* For each narrow ball, we have the following lemma, which is a consequence of the l^2 decoupling theorem in dimension n and Minkowski's inequality. This argument is essentially contained in Bourgain-Demeter's proof of the l^2 decoupling conjecture, and we omit the details. (See the proof of Proposition 5.5 in [5].)

LEMMA 3.3. *Suppose that B is a narrow K^2-cube in $\mathbb{R}^{n+1}$. Then for any $\varepsilon > 0$,*

$$\|e^{it\Delta} f\|_{L^p(B)} \le C_\varepsilon K^\varepsilon \left(\sum_{\tau\in\mathcal{S}} \left\| e^{it\Delta} f_\tau \right\|^2_{L^p(\omega_B)} \right)^{1/2}.$$

Here $p = \frac{2(n+1)}{n-1}$, $\mathcal{S}$ denotes the set of K^{-1}-cubes that tile $B^n(0,1)$, and ω_B is a weight function that is essentially a characteristic function on B. More precisely, ω_B has Fourier support in $B(0, K^{-2})$ and satisfies

$$1_B(\tilde{x}) \lesssim \omega_B(\tilde{x}) \le \left(1 + \frac{|\tilde{x} - C(B)|}{K^2}\right)^{-1000n}.$$

For each $\tau \in \mathcal{S}$, we will deal with $e^{it\Delta} f_\tau$ by parabolic rescaling and induction on radius. In order to do so, we need to further decompose f in physical space and perform dyadic pigeonholing several times to get the right setup for our inductive hypothesis at scale $R_1 := R/K^2$ after rescaling.

Intuition. For each $1/K$-cap τ, all wave packets associated with f_τ through a given point have to lie in a common box that has one side length R and other side lengths R/K. Every single box of this type will become an R/K^2-ball if we perform a parabolic rescaling to transform τ into the standard 1-cap. We want to use the inductive hypothesis for radius R/K^2 in an efficient way. A few dyadic pigeonholing steps will be needed.

First, we break the physical ball $B^n(0,R)$ into R/K-cubes D. For each pair (τ, D), let $f_{\Box_{\tau,D}}$ be the function formed by cutting off f on the cube D (with a Schwartz tail) in physical space and the cube τ in Fourier space. Note that $e^{it\Delta}f_{\Box_{\tau,D}}$, restricted to $B^{n+1}(R)$, is essentially supported on an $R/K \times \cdots \times R/K \times R$-box,[3] which we denote by $\Box_{\tau,D}$. The box $\Box_{\tau,D}$ is in the direction given by $(-2c(\tau),1)$ and intersects $t=0$ at the cube D, where $c(\tau)$ is the center of τ. For a fixed τ, the different boxes $\Box_{\tau,D}$ tile $B^{n+1}(0,R)$. In particular, for each τ, a given K^2-cube B lies in exactly one box $\Box_{\tau,D}$. We write $f=\sum_{\Box} f_{\Box}$ for abbreviation. By Lemma 3.3, for each narrow K^2-cube B,

$$\|e^{it\Delta}f\|_{L^p(B)} \lesssim K^{\varepsilon^4}\left(\sum_{\Box}\left\|e^{it\Delta}f_{\Box}\right\|^2_{L^p(\omega_B)}\right)^{1/2}. \tag{3.11}$$

We will have a gain $\frac{1}{K^{2\varepsilon}}$ from induction on radius. Therefore, in (3.11) we are allowed to lose a small power of K. This small power depends on ε and should be smaller than 2ε. It could be $\varepsilon^2, \varepsilon^3, \varepsilon^4$, etc.

Next, we perform a dyadic pigeonholing to get our inductive hypothesis for each $f_\Box$. Recall that $K=R^\delta$, where $\delta=\varepsilon^{100}$. Denote

$$R_1 := R/K^2 = R^{1-2\delta}, \quad K_1 := R_1^\delta = R^{\delta-2\delta^2}.$$

Tile $\Box$ by $KK_1^2 \times \cdots \times KK_1^2 \times K^2K_1^2$-tubes S, and also tile $\Box$ by $R^{1/2} \times \cdots \times R^{1/2} \times KR^{1/2}$-tubes S' (all running parallel to the long axis of $\Box$). To understand these scales, see Figure 1 for the change in physical space (3.20) during the process of parabolic rescaling. In particular, after rescaling the $\Box$ becomes an R_1-cube, and the tubes S' and S become lattice $R_1^{1/2}$-cubes and K_1^2-cubes respectively. We apply the following to regroup tubes S and S' inside each $\Box$:

(1) Sort those tubes S that intersect Y according to the value $\|e^{it\Delta}f_\Box\|_{L^p(S)}$ and the number of narrow K^2-cubes contained in it. For dyadic numbers η, β_1, we use $\mathbb{S}_{\Box,\eta,\beta_1}$ to stand for the collection of tubes $S\subset\Box$, each of which containing $\sim\eta$ narrow K^2-cubes in Y_{narrow} and $\|e^{it\Delta}f_\Box\|_{L^p(S)}\sim\beta_1$.
(2) For fixed η,β_1, we sort the tubes $S'\subset\Box$ according to the number of tubes $S\in\mathbb{S}_{\Box,\eta,\beta_1}$ contained in it. For dyadic number λ_1, let $\mathbb{S}_{\Box,\eta,\beta_1,\lambda_1}$ be the subcollection of $\mathbb{S}_{\Box,\eta,\beta_1}$ such that for each $S\in\mathbb{S}_{\Box,\eta,\beta_1,\lambda_1}$, the tube S' containing S contains $\sim\lambda_1$ tubes from $\mathbb{S}_{\Box,\eta,\beta_1}$.
(3) For fixed η,β_1,λ_1, we sort the boxes $\Box$ according to the value $\|f_\Box\|_2$, the number $\#\mathbb{S}_{\Box,\eta,\beta_1,\lambda_1}$ and the value γ_1 defined below. For dyadic numbers β_2, M_1, γ_1, let $\mathbb{B}_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1}$ denote the collection of boxes $\Box$, each of

[3] In reality, our boxes will have edge length slightly larger, say being larger by $K^{\varepsilon^{100}}$ times. See, e.g., the wave packet decomposition theorem in [18]. This would not hurt us in any way, and we omit this technicality for reading convenience.

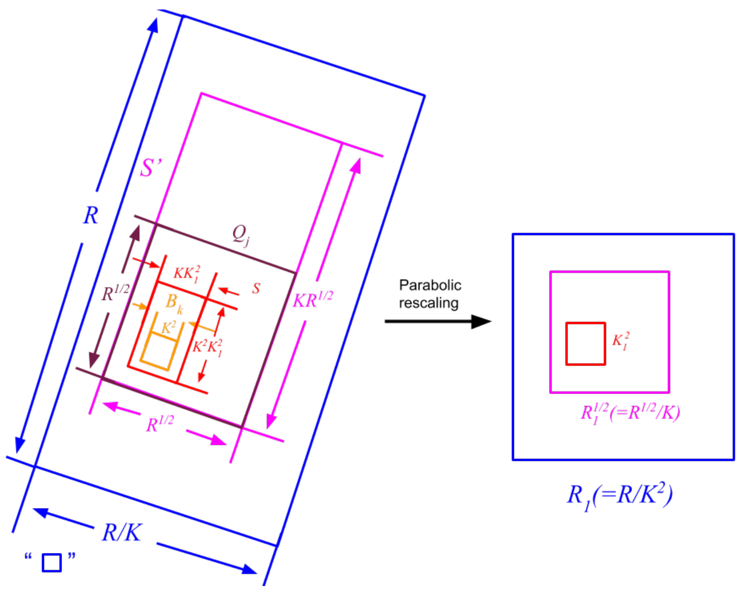

Figure 1. Tubes of different scales in the □.

which satisfying that

$$\|f_\square\|_2 \sim \beta_2, \quad \#\mathbb{S}_{\square,\eta,\beta_1,\lambda_1} \sim M_1$$

and

$$\max_{T_r \subset \square : r \geq K_1^2} \frac{\#\{S \in \mathbb{S}_{\square,\eta,\beta_1,\lambda_1} : S \subset T_r\}}{r^\alpha} \sim \gamma_1, \tag{3.12}$$

where T_r are $Kr \times \cdots \times Kr \times K^2 r$-tubes in □ running parallel to the long axis of □.

Let $Y_{\square,\eta,\beta_1,\lambda_1}$ be the union of the tubes S in $\mathbb{S}_{\square,\eta,\beta_1,\lambda_1}$, and $\chi_{Y_{\square,\eta,\beta_1,\lambda_1}}$ the corresponding characteristic function. Then on Y_{narrow}, we can write

$$e^{it\Delta} f = \sum_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1} \left(\sum_{\square \in \mathbb{B}_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1}} e^{it\Delta} f_\square \cdot \chi_{Y_{\square,\eta,\beta_1,\lambda_1}} \right) + O(R^{-1000n}) \|f\|_2.$$

The small error term $O(R^{-1000n})\|f\|_2$ will prove to be harmless in our computations. We will neglect this term in the sequel. Again, to make the statement really rigorous one needs to increase the side lengths of □ by a tiny power of R, say $R^{\delta^{100}} \sim K^{\delta^{99}}$. As before, we choose to ignore this technicality in order to facilitate the main exposition.

In particular, on each narrow B we have

$$e^{it\Delta} f = \sum_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1} \left(\sum_{\substack{\Box\in\mathbb{B}_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1}\\ B\subset Y_{\Box,\eta,\beta_1,\lambda_1}}} e^{it\Delta} f_\Box \right). \tag{3.13}$$

Without loss of generality, we assume that $\|f\|_2 = 1$. Then we can further assume that the dyadic numbers above are in reasonable ranges, say

$$1 \le \eta \le K^{O(1)}, \quad R^{-C} \le \beta_1 \le K^{O(1)}, \quad 1 \le \lambda_1 \le R^{O(1)}$$

and

$$R^{-C} \le \beta_2 \le 1, \quad 1 \le M_1 \le R^{O(1)}, \quad K^{-2n} \le \gamma_1 \le R^{O(1)},$$

where C is a large constant such that the contributions from those β_1 and β_2 less than R^{-C} are negligible. Therefore, there are only $O(\log R)$ significant choices for each dyadic number. Because of (3.11) and (3.13), by pigeonholing, we can choose $\eta, \beta_1, \lambda_1, \beta_2, M_1, \gamma_1$ so that

$$\|e^{it\Delta} f\|_{L^p(B)} \lesssim (\log R)^6 K^{\varepsilon^4} \left(\sum_{\substack{\Box\in\mathbb{B}_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1}\\ B\subset Y_{\Box,\eta,\beta_1,\lambda_1}}} \|e^{it\Delta} f_\Box\|^2_{L^p(\omega_B)} \right)^{1/2} \tag{3.14}$$

holds for a fraction $\gtrsim (\log R)^{-6}$ of all narrow K^2-cubes B.

We fix $\eta, \beta_1, \lambda_1, \beta_2, M_1, \gamma_1$ for the rest of the proof. Let $Y_\Box$ and $\mathbb{B}$ stand for the abbreviations of $Y_{\Box,\eta,\beta_1,\lambda_1}$ and $\mathbb{B}_{\eta,\beta_1,\lambda_1,\beta_2,M_1,\gamma_1}$ respectively. Finally we sort the narrow balls B satisfying (3.14) by $\#\{\Box \in \mathbb{B} : B \subset Y_\Box\}$. Let $Y' \subset Y_{\text{narrow}}$ be a union of narrow K^2-cubes B, each of which obeying

$$\|e^{it\Delta} f\|_{L^p(B)} \lesssim (\log R)^6 K^{\varepsilon^4} \left(\sum_{\Box\in\mathbb{B}: B\subset Y_\Box} \|e^{it\Delta} f_\Box\|^2_{L^p(\omega_B)} \right)^{1/2} \tag{3.15}$$

and

$$\#\{\Box \in \mathbb{B} : B \subset Y_\Box\} \sim \mu \tag{3.16}$$

for some dyadic number $1 \le \mu \le K^{O(1)}$. Moreover, the number of K^2-cubes B in Y' is $\gtrsim (\log R)^{-7} M$.

Now we are done with dyadic pigeonholing argument. Let us put all these together. By our assumption that $\|e^{it\Delta} f\|_{L^p(B_k)}$ is essentially constant in $k = 1, 2, \ldots, M$, in the narrow case we have

$$\|e^{it\Delta} f\|^p_{L^p(Y)} \lesssim (\log R)^7 \sum_{B\subset Y'} \|e^{it\Delta} f\|^p_{L^p(B)}. \tag{3.17}$$

For each $B \subset Y'$, it follows from (3.15), (3.16) and Hölder's inequality that

$$\|e^{it\Delta} f\|_{L^p(B)}^p \lesssim (\log R)^{6p} K^{\varepsilon^4 p} \mu^{\frac{p}{2}-1} \sum_{\square \in \mathbb{B}: B \subset Y_\square} \|e^{it\Delta} f_\square\|_{L^p(\omega_B)}^p. \tag{3.18}$$

Putting (3.17) and (3.18) together and, as before, omitting the rapidly decaying tails,

$$\|e^{it\Delta} f\|_{L^p(Y)} \lesssim (\log R)^{13} K^{\varepsilon^4} \mu^{\frac{1}{n+1}} \left(\sum_{\square \in \mathbb{B}} \left\| e^{it\Delta} f_\square \right\|_{L^p(Y_\square)}^p \right)^{1/p}. \tag{3.19}$$

Next, to each $\|e^{it\Delta} f_\square\|_{L^p(Y_\square)}$ we apply parabolic rescaling and induction on radius. For each $1/K$-cube $\tau = \tau_\square$ in $B^n(0,1)$, we write $\xi = \xi_0 + K^{-1}\zeta \in \tau$, where ξ_0 is the center of τ. Then

$$|e^{it\Delta} f_\square(x)| = K^{-n/2} |e^{i\tilde{t}\Delta} g(\tilde{x})|$$

for some function g with Fourier support in the unit cube and $\|g\|_2 = \|f_\square\|_2$, where the new coordinates $(\tilde{x}, \tilde{t})$ are related to the old coordinates (x,t) by

$$\begin{cases} \tilde{x} = K^{-1}x + 2tK^{-1}\xi_0, \\ \tilde{t} = K^{-2}t. \end{cases} \tag{3.20}$$

For simplicity, denote the above relation by $(\tilde{x}, \tilde{t}) = F(x,t)$. Therefore

$$\|e^{it\Delta} f_\square(x)\|_{L^p(Y_\square)} = K^{\frac{n+2}{p} - \frac{n}{2}} \|e^{i\tilde{t}\Delta} g(\tilde{x})\|_{L^p(\tilde{Y})} = K^{-\frac{1}{n+1}} \|e^{i\tilde{t}\Delta} g(\tilde{x})\|_{L^p(\tilde{Y})}, \tag{3.21}$$

where $\tilde{Y}$ is the image of $Y_\square$ under the new coordinates.

Note that we can apply our inductive hypothesis (3.2) at scale $R_1 = R/K^2$ to $\|e^{i\tilde{t}\Delta} g(\tilde{x})\|_{L^p(\tilde{Y})}$ with new parameters $M_1, \gamma_1, \lambda_1, R_1$. More precisely, $\tilde{Y} = F(Y_\square)$ consists of $\sim M_1$ distinct K_1^2-cubes $F(S)$ in an R_1-ball $F(\square)$, and the K_1^2-cubes $F(S)$ are organized into $R_1^{1/2}$-cubes $F(S')$ such that each cube $F(S')$ contains $\sim \lambda_1$ cubes $F(S)$. Moreover, $\|e^{i\tilde{t}\Delta} g(\tilde{x})\|_{L^p(F(S))}$ is dyadically a constant in $S \subset Y_\square$. By our choice of γ_1, we have

$$\max_{\substack{B^{n+1}(x',r) \subset F(\square) \\ x' \in \mathbb{R}^{n+1}, r \geq K_1^2}} \frac{\#\{F(S) : F(S) \subset B(x',r)\}}{r^\alpha} \sim \gamma_1.$$

Henceforth, by (3.21) and inductive hypothesis (3.2), at scale R_1 we have

$$\begin{aligned} &\|e^{it\Delta} f_\square(x)\|_{L^p(Y_\square)} \\ \lesssim& K^{-\frac{1}{n+1}} M_1^{-\frac{1}{n+1}} \gamma_1^{\frac{2}{(n+1)(n+2)}} \lambda_1^{\frac{n}{(n+1)(n+2)}} \left(\frac{R}{K^2}\right)^{\frac{\alpha}{(n+1)(n+2)}+\varepsilon} \|f_\square\|_2. \end{aligned} \tag{3.22}$$

From (3.19) and (3.22) we obtain

(3.23)
$$\begin{aligned}
&\|e^{it\Delta}f\|_{L^p(Y)}\\
\lesssim &K^{2\varepsilon^4}\mu^{\frac{1}{n+1}}K^{-\frac{1}{n+1}}M_1^{-\frac{1}{n+1}}\gamma_1^{\frac{2}{(n+1)(n+2)}}\lambda_1^{\frac{n}{(n+1)(n+2)}}\left(\frac{R}{K^2}\right)^{\frac{\alpha}{(n+1)(n+2)}+\varepsilon}\left(\sum_{\Box\in\mathbb{B}}\|f_\Box\|_2^p\right)^{1/p}\\
\lesssim &K^{2\varepsilon^4}\left(\frac{\mu}{\#\mathbb{B}}\right)^{\frac{1}{n+1}}K^{-\frac{1}{n+1}}M_1^{-\frac{1}{n+1}}\gamma_1^{\frac{2}{(n+1)(n+2)}}\lambda_1^{\frac{n}{(n+1)(n+2)}}\left(\frac{R}{K^2}\right)^{\frac{\alpha}{(n+1)(n+2)}+\varepsilon}\|f\|_2,
\end{aligned}$$

where the last inequality follows from orthogonality $\sum_\Box \|f_\Box\|_2^2 \lesssim \|f\|_2^2$ and the assumption that $\|f_\Box\|_2 \sim$ constant in $\Box \in \mathbb{B}$.

Intuition. To finish our inductive argument, we have to relate the old and new parameters. Our setup allows us to do this in a nice way: Given M_1, λ_1 and γ_1, if η is small, i.e., each S contains very few narrow K^2-cubes, then M is relatively small; if η is large, i.e., each S contains a lot of narrow K^2-cubes, then λ and γ are relatively large. Both make the right-hand side of what we want to prove reasonably large. This is the reason why one could believe the numerology will work out.

Consider the cardinality of the set $\{(\Box, B) : \Box \in \mathbb{B}, B \subset Y_\Box \cap Y'\}$. By our choice of μ as in (3.16), there is a lower bound

$$\#\{(\Box, B) : \Box \in \mathbb{B}, B \subset Y_\Box \cap Y'\} \gtrsim (\log R)^{-7} M\mu.$$

On the other hand, by our choices of M_1 and η, for each $\Box \in \mathbb{B}$, $Y_\Box$ contains $\sim M_1$ tubes S and each S contains $\sim \eta$ narrow cubes in Y_{narrow}, so

$$\#\{(\Box, B) : \Box \in \mathbb{B}, B \subset Y_\Box \cap Y'\} \lesssim (\#\mathbb{B}) M_1 \eta.$$

Therefore, we get

$$\frac{\mu}{\#\mathbb{B}} \lesssim \frac{(\log R)^7 M_1 \eta}{M}. \tag{3.24}$$

Next by our choices of γ_1 as in (3.12) and η,

$$\begin{aligned}
&\gamma_1 \cdot \eta\\
\sim &\max_{T_r \subset \Box : r \ge K_1^2} \frac{\#\{S : S \subset Y_\Box \cap T_r\}}{r^\alpha}\\
&\cdot \#\{B : B \subset S \cap Y_{\text{narrow}} \text{ for any fixed } S \subset Y_\Box\}\\
\lesssim &\max_{T_r \subset \Box : r \ge K_1^2} \frac{\#\{B \subset Y : B \subset T_r\}}{r^\alpha} \le \frac{K\gamma(Kr)^\alpha}{r^\alpha} = \gamma K^{\alpha+1},
\end{aligned}$$

where the last inequality follows from the definition (3.1) of γ and the fact that we can cover a $Kr \times \cdots \times Kr \times K^2 r$-tube T_r by $\sim K$ finitely overlapping

Kr-balls. Hence,

$$\eta \lesssim \frac{\gamma K^{\alpha+1}}{\gamma_1}. \tag{3.25}$$

Finally we relate λ_1 and λ by considering the number of narrow K^2-balls in each relevant $R^{1/2} \times \cdots \times R^{1/2} \times KR^{1/2}$-tube S'. Recall that each relevant S' contains $\sim \lambda_1$ tubes S in $Y_\Box$, and each such S contains $\sim \eta$ narrow balls. On the other hand, we can cover S' by $\sim K$ finitely overlapping $R^{1/2}$-balls and by assumption each $R^{1/2}$-ball contains $\lesssim \lambda$ many K^2-cubes in Y. Thus it follows that

$$\lambda_1 \lesssim \frac{K\lambda}{\eta}. \tag{3.26}$$

By inserting (3.24) and (3.26) into (3.23),

$$\begin{aligned}
&\|e^{it\Delta} f\|_{L^p(Y)} \\
\lesssim& \frac{K^{3\varepsilon^4}}{K^{2\varepsilon}} \left(\frac{\eta\gamma_1}{K^{\alpha+1}}\right)^{\frac{2}{(n+1)(n+2)}} M^{-\frac{1}{n+1}} \lambda^{\frac{n}{(n+1)(n+2)}} R^{\frac{\alpha}{(n+1)(n+2)}+\varepsilon} \|f\|_2 \\
\lesssim& \frac{K^{3\varepsilon^4}}{K^{2\varepsilon}} M^{-\frac{1}{n+1}} \gamma^{\frac{2}{(n+1)(n+2)}} \lambda^{\frac{n}{(n+1)(n+2)}} R^{\frac{\alpha}{(n+1)(n+2)}+\varepsilon} \|f\|_2,
\end{aligned}$$

where the last inequality follows from (3.25). Since $K = R^\delta$ and R can be assumed to be sufficiently large compared to any constant depending on ε, we have $\frac{K^{3\varepsilon^4}}{K^{2\varepsilon}} \ll 1$ and the induction closes for the narrow case. This completes the proof of Proposition 3.1. □

3.3. *Remark*. In Section 2, we have seen that Corollary 1.7 is a direct result of Theorem 1.6, and they are equally useful in the applications to the sharp L^2 estimate of the Schrödinger maximal function. We can also prove Corollary 1.7 from scratch using a similar argument as in this section, which is slightly easier in two aspects compared to that of Theorem 1.6. First, in the broad case, it is sufficient to use multilinear restriction estimates and not necessary to invoke the multilinear refined Strichartz. Secondly, because there is one parameter less, the dyadic pigeonholing argument in the narrow case would be slightly reduced; for example, see Figure 2 for tubes of different scales in the $\Box$ under the setting of Corollary 1.7.

In fact, an adaptation of some arguments in the work [32] of Wolff on the Falconer distance set problem in dimension 2 can already imply Corollary 1.7 when $n = 1$. In the special case $n = 1$, the broad versus narrow dichotomy becomes the one on bilinear versus linear. To handle the linear part, the idea of induction on scales and splitting the ball into rectangular boxes "$\Box$" of size $R \times R/K$ in our proof already existed in Wolff's paper. We thank Hong Wang for pointing this out to us.

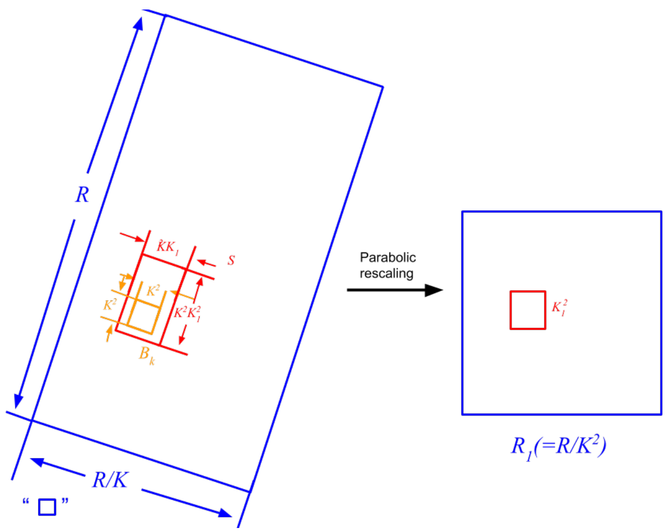

Figure 2. Tubes of different scales in the □ (in inductive argument for Corollary 1.7).

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