Annals of Mathematics **188** (2018), 965–1016
https://doi.org/10.4007/annals.2018.188.3.5

The Shimura–Waldspurger correspondence for Mp_{2n}

By Wee Teck Gan and Atsushi Ichino

Abstract

We generalize the Shimura–Waldspurger correspondence, which describes the generic part of the automorphic discrete spectrum of the metaplectic group Mp_2, to the metaplectic group Mp_{2n} of higher rank. To establish this, we transport Arthur's endoscopic classification of representations of the odd special orthogonal group SO_{2r+1} with $r \gg 2n$ by using a result of J.-S. Li on global theta lifts in the stable range.

1. Introduction

In a seminal 1973 paper [81], Shimura revolutionized the study of half integral weight modular forms by establishing a lifting

$$\{\text{Hecke eigenforms of weight } k + \tfrac{1}{2} \text{ and level } \Gamma_0(4)\}$$
$$\big\downarrow$$
$$\{\text{Hecke eigenforms of weight } 2k \text{ and level } \mathrm{SL}_2(\mathbb{Z})\}.$$

He proved this by using Weil's converse theorem and a Rankin–Selberg integral for the standard L-function of a half integral weight modular form. Subsequently, Niwa [68] and Shintani [82] explicitly constructed the Shimura lifting and its inverse by using theta series lifting. From a representation theoretic viewpoint, Howe [32] further developed the theory of theta series lifting by introducing the notion of reductive dual pairs, so that the lifting gives a correspondence between representations of one member of a reductive dual pair and those of the other member. Then, in two influential papers [88], [91], Waldspurger studied the Shimura correspondence in the framework of automorphic representations of the metaplectic group Mp_2, which is a nonlinear two-fold cover of $\mathrm{SL}_2 = \mathrm{Sp}_2$. Namely, he described the automorphic discrete

Keywords: automorphic representations, metaplectic groups, Shimura correspondence, theta lifts

AMS Classification: Primary: 11F70, 22E55.

spectrum of Mp_2 precisely in terms of that of $\mathrm{PGL}_2 = \mathrm{SO}_3$ via the global theta lifts between Mp_2 and (inner forms of) SO_3. Subordinate to this global result is the local Shimura correspondence, which is a classification of irreducible genuine representations of Mp_2 in terms of that of SO_3 and was also established by Waldspurger. For an expository account of Waldspurger's result, the reader can consult a lovely paper of Piatetski-Shapiro [69] and a more recent paper [16], which takes advantage of 30 years of hindsight and machinery.

The goal of the present paper is to establish a similar description of the automorphic discrete spectrum of Mp_{2n}, which is a nonlinear two-fold cover of Sp_{2n}, in terms of that of SO_{2n+1}. As in the case of Mp_2, it is natural to attempt to use the global theta lifts between Mp_{2n} and (inner forms of) SO_{2n+1} to relate these automorphic discrete spectra. However, we encounter a difficulty. For any irreducible cuspidal automorphic representation π of Mp_{2n}, there is an obstruction to the nonvanishing of its global theta lift to SO_{2n+1} given by the vanishing of the central L-value $L(\frac{1}{2}, \pi)$. Thus, if we would follow Waldspurger's approach, then we would need the nonvanishing of the central L-value $L(\frac{1}{2}, \pi, \chi)$ twisted by some quadratic Hecke character χ. In the case of Mp_2 (or equivalently PGL_2), Waldspurger [91] proved the existence of such χ by exploiting Flicker's result [13] on the correspondence between automorphic representations of GL_2 and those of its two-fold cover. Moreover, Jacquet [39] outlined a new proof of Waldspurger's nonvanishing result based on relative trace formulas, and Friedberg–Hoffstein [14] gave yet another proof and its extension to the case of GL_2. However, in the higher rank case, this seems to be a very difficult question in analytic number theory. The main novelty of this paper is to overcome this inherent difficulty when $L(\frac{1}{2}, \pi) = 0$.

1.1. *Near equivalence classes.* We now describe our results in more detail. Let F be a number field and $\mathbb{A}$ the adèle ring of F. Let n be a positive integer. We denote by Sp_{2n} the symplectic group of rank n over F and by $\mathrm{Mp}_{2n}(\mathbb{A})$ the metaplectic two-fold cover of $\mathrm{Sp}_{2n}(\mathbb{A})$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}_{2n}(\mathbb{A}) \longrightarrow \mathrm{Sp}_{2n}(\mathbb{A}) \longrightarrow 1.$$

Let $L^2(\mathrm{Mp}_{2n})$ be the subspace of $L^2(\mathrm{Sp}_{2n}(F)\backslash \mathrm{Mp}_{2n}(\mathbb{A}))$ on which $\{\pm 1\}$ acts as the nontrivial character, where we regard $\mathrm{Sp}_{2n}(F)$ as a subgroup of $\mathrm{Mp}_{2n}(\mathbb{A})$ via the canonical splitting. Then one of the basic problems is to understand the spectral decomposition of the unitary representation $L^2(\mathrm{Mp}_{2n})$ of $\mathrm{Mp}_{2n}(\mathbb{A})$, and our goal is to establish a description of its discrete spectrum

$$L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$$

in the style of Arthur's conjecture formulated in [19, Conj. 25.1], [18, §5.6].

We first describe the decomposition of $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$ into near equivalence classes (which are coarser than equivalence classes) of representations. Here we say that two irreducible representations $\pi = \bigotimes_v \pi_v$ and $\pi' = \bigotimes_v \pi'_v$ of

$\mathrm{Mp}_{2n}(\mathbb{A})$ are nearly equivalent if π_v and π'_v are equivalent for almost all places v of F. This decomposition will be expressed in terms of A-parameters defined as follows. Consider a formal (unordered) finite direct sum

$$\phi = \bigoplus_i \phi_i \boxtimes S_{d_i}, \tag{1.1}$$

where

- ϕ_i is an irreducible self-dual cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A})$;
- S_{d_i} is the unique d_i-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$.

We call ϕ an elliptic A-parameter for Mp_{2n} if

- $\sum_i n_i d_i = 2n$;
- if d_i is odd, then ϕ_i is symplectic, that is, the exterior square L-function $L(s, \phi_i, \wedge^2)$ has a pole at $s = 1$;
- if d_i is even, then ϕ_i is orthogonal, that is, the symmetric square L-function $L(s, \phi_i, \mathrm{Sym}^2)$ has a pole at $s = 1$;
- if $(\phi_i, d_i) = (\phi_j, d_j)$, then $i = j$.

If further $d_i = 1$ for all i, then we say that ϕ is generic. We remark that the basic analytic properties of exterior and symmetric square L-functions were established in [11], [40], [77], [78] and [12], [77], [78], respectively, and precisely one of $L(s, \phi_i, \wedge^2)$ and $L(s, \phi_i, \mathrm{Sym}^2)$ has a pole at $s = 1$ since the Rankin–Selberg L-function

$$L(s, \phi_i \times \phi_i^\vee) = L(s, \phi_i \times \phi_i) = L(s, \phi_i, \wedge^2) L(s, \phi_i, \mathrm{Sym}^2)$$

has a simple pole at $s = 1$ by [37, Prop. 3.6] and both of the two L-functions do not vanish at $s = 1$ by [77, Th. 5.1]. For each place v of F, let $\phi_v = \bigoplus_i \phi_{i,v} \boxtimes S_{d_i}$ be the localization of ϕ at v. Here we regard $\phi_{i,v}$ as an n_i-dimensional representation of L_{F_v} via the local Langlands correspondence [45], [30], [31], [76], where

$$L_{F_v} = \begin{cases} \text{the Weil group of } F_v & \text{if } v \text{ is archimedean,} \\ \text{the Weil–Deligne group of } F_v & \text{if } v \text{ is nonarchimedean.} \end{cases}$$

Note that ϕ_v gives rise to an A-parameter $\phi_v : L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$. We associate to it an L-parameter $\varphi_{\phi_v} : L_{F_v} \to \mathrm{Sp}_{2n}(\mathbb{C})$ by

$$\varphi_{\phi_v}(w) = \phi_v \left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{-\frac{1}{2}} \end{pmatrix} \right).$$

Our first result is

THEOREM 1.1. *Fix a nontrivial additive character ψ of $F \backslash \mathbb{A}$. Then there exists a decomposition*

$$L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n}) = \bigoplus_\phi L^2_{\phi,\psi}(\mathrm{Mp}_{2n}),$$

where the direct sum runs over elliptic A-parameters ϕ for Mp_{2n} and where $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ is a full near equivalence class of irreducible representations π in

$L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$ *such that the* L*-parameter of* π_v (*relative to* ψ_v—*see* Remark 5.3 *below*) *is* φ_{ϕ_v} *for almost all places* v *of* F.

Thus, to achieve our goal, it remains to describe the decomposition of $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ into equivalence classes of representations. In this paper, we carry this out when ϕ is generic.

Remark 1.2. It immediately follows from Theorem 1.1 that Mp_{2n} has no embedded eigenvalues, i.e., any family of eigenvalues of unramified Hecke algebras on the automorphic discrete spectrum of Mp_{2n} is distinct from that on the automorphic continuous spectrum of Mp_{2n}. This is an analog of Arthur's result [5, Th. 5], [6], which he needed to establish in the course of his proof of the classification of automorphic representations of orthogonal and symplectic groups. However, in our case, we first establish the classification (with the help of theta lifts) and then deduce from it the absence of embedded eigenvalues.

Remark 1.3. The dependence of $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ on ψ can be described as follows. Let ψ_a be another nontrivial additive character of $F\backslash\mathbb{A}$ given by $\psi_a(x) = \psi(ax)$ for some $a \in F^\times$. If ϕ is of the form (1.1), we define another elliptic A-parameter ϕ_a by

$$\phi_a = \bigoplus_i (\phi_i \otimes (\chi_a \circ \det)) \boxtimes S_{d_i},$$

where χ_a is the quadratic automorphic character of $\mathbb{A}^\times$ associated to $F(\sqrt{a})/F$ by class field theory. Then, for any irreducible summand π of $L^2_{\phi,\psi_a}(\mathrm{Mp}_{2n})$, the L-parameter of π_v (relative to ψ_v) is $\varphi_{\phi_{a,v}}$ for almost all v. In particular, we have

$$L^2_{\phi,\psi_a}(\mathrm{Mp}_{2n}) = L^2_{\phi_a,\psi}(\mathrm{Mp}_{2n}).$$

1.2. *Local Shimura correspondence.* As in Waldspurger's result [88], [91], our result will be expressed in terms of the local Shimura correspondence. Fix a place v of F, and assume for simplicity that v is nonarchimedean. For the moment, we omit the subscript v from the notation, so that F is a nonarchimedean local field of characteristic zero. Let $\mathrm{Irr}\,\mathrm{Mp}_{2n}$ be the set of equivalence classes of irreducible genuine representations of the metaplectic group Mp_{2n} over F. Then the local Shimura correspondence is a classification of $\mathrm{Irr}\,\mathrm{Mp}_{2n}$ in terms of $\mathrm{Irr}\,\mathrm{SO}(V)$, where V is a $(2n+1)$-dimensional quadratic space over F.

To be precise, recall that there are precisely two such quadratic spaces with trivial discriminant (up to isometry). We denote them by V^+ and V^- so that $\mathrm{SO}(V^+)$ is split over F. In [24], the first-named author and Savin showed that for any nontrivial additive character ψ of F, there exists a bijection

$$\theta_\psi : \mathrm{Irr}\,\mathrm{Mp}_{2n} \longleftrightarrow \mathrm{Irr}\,\mathrm{SO}(V^+) \sqcup \mathrm{Irr}\,\mathrm{SO}(V^-)$$

satisfying the following natural properties:

- θ_ψ preserves the square-integrability and the temperedness of representations;
- θ_ψ is compatible with the theory of R-groups (modulo square-integrable representations);
- θ_ψ is compatible with the Langlands classification (modulo tempered representations);
- θ_ψ preserves the genericity of tempered representations;
- θ_ψ preserves various invariants such as L- and ϵ-factors, Plancherel measures, and formal degrees [20].

Note that an analogous result in the archimedean case was proved by Adams–Barbasch [2], [3] more than 20 years ago.

On the other hand, the local Langlands correspondence, established by Arthur [6] for $\mathrm{SO}(V^+)$ and by Mœglin–Renard [62] for $\mathrm{SO}(V^-)$, gives a partition

$$\mathrm{Irr}\,\mathrm{SO}(V^+) \sqcup \mathrm{Irr}\,\mathrm{SO}(V^-) = \bigsqcup_\phi \Pi_\phi(\mathrm{SO}(V^\pm)),$$

where the disjoint union runs over equivalence classes of L-parameters $\phi : L_F \to \mathrm{Sp}_{2n}(\mathbb{C})$ and $\Pi_\phi(\mathrm{SO}(V^\pm))$ is the associated Vogan L-packet equipped with a bijection

$$\Pi_\phi(\mathrm{SO}(V^\pm)) \longleftrightarrow \hat{S}_\phi,$$

where S_ϕ is the component group of the centralizer of the image of ϕ in $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\hat{S}_\phi$ is the group of characters of S_ϕ. Composing this with the local Shimura correspondence, we obtain a local Langlands correspondence for Mp_{2n},

$$\mathrm{Irr}\,\mathrm{Mp}_{2n} = \bigsqcup_\phi \Pi_{\phi,\psi}(\mathrm{Mp}_{2n})$$

with

$$\Pi_{\phi,\psi}(\mathrm{Mp}_{2n}) \longleftrightarrow \hat{S}_\phi, \tag{1.2}$$

which depends on the choice of ψ, and which inherits various properties of the local Langlands correspondence for $\mathrm{SO}(V^\pm)$. We remark that the dependence of the L-packet $\Pi_{\phi,\psi}(\mathrm{Mp}_{2n})$ and the bijection (1.2) on ψ is described in [24, Th. 12.1]. We also emphasize that the L-packet $\Pi_{\phi,\psi}(\mathrm{Mp}_{2n})$ satisfies *endoscopic character relations*; see Section 1.5 below.

1.3. *Multiplicity formula for* Mp_{2n}. Suppose again that F is a number field. We now describe the multiplicity of any representation of $\mathrm{Mp}_{2n}(\mathbb{A})$ in $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ when ϕ is generic; i.e., ϕ is a multiplicity-free sum

$$\phi = \bigoplus_i \phi_i$$

of irreducible symplectic cuspidal automorphic representations ϕ_i of $\mathrm{GL}_{n_i}(\mathbb{A})$. We formally associate to ϕ a free $\mathbb{Z}/2\mathbb{Z}$-module

$$S_\phi = \bigoplus_i (\mathbb{Z}/2\mathbb{Z}) a_i$$

with a basis $\{a_i\}$, where a_i corresponds to ϕ_i. We call S_ϕ the global component group of ϕ. For any place v of F, this gives rise to a local L-parameter $\phi_v : L_{F_v} \to \mathrm{Sp}_{2n}(\mathbb{C})$ together with a canonical map $S_\phi \to S_{\phi_v}$. Thus, we obtain a compact group $S_{\phi,\mathbb{A}} = \prod_v S_{\phi_v}$ equipped with the diagonal map $\Delta : S_\phi \to S_{\phi,\mathbb{A}}$. Let $\hat{S}_{\phi,\mathbb{A}} = \bigoplus_v \hat{S}_{\phi_v}$ be the group of continuous characters of $S_{\phi,\mathbb{A}}$. For any $\eta = \bigotimes_v \eta_v \in \hat{S}_{\phi,\mathbb{A}}$, we may form an irreducible genuine representation

$$\pi_\eta = \bigotimes_v \pi_{\eta_v}$$

of $\mathrm{Mp}_{2n}(\mathbb{A})$, where $\pi_{\eta_v} \in \Pi_{\phi_v,\psi_v}(\mathrm{Mp}_{2n})$ is the representation of $\mathrm{Mp}_{2n}(F_v)$ associated to $\eta_v \in \hat{S}_{\phi_v}$. Note that η_v is trivial and π_{η_v} is unramified for almost all v. Also, by the Ramanujan conjecture for general linear groups, π_η is expected to be tempered. Finally, we define a quadratic character ϵ_ϕ of S_ϕ by setting

$$\epsilon_\phi(a_i) = \epsilon(\tfrac{1}{2}, \phi_i),$$

where $\epsilon(\frac{1}{2}, \phi_i) \in \{\pm 1\}$ is the root number of ϕ_i.

Our second result (under the hypothesis that Arthur's result [6] extends to the case of nonsplit odd special orthogonal groups; see Sections 3.1 and 6.2 below for more details) is

THEOREM 1.4. *Let ϕ be a generic elliptic A-parameter for* Mp_{2n}. *Then we have*

$$L^2_{\phi,\psi}(\mathrm{Mp}_{2n}) \cong \bigoplus_{\eta \in \hat{S}_{\phi,\mathbb{A}}} m_\eta \pi_\eta,$$

where

$$m_\eta = \begin{cases} 1 & \text{if } \Delta^*\eta = \epsilon_\phi, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.5. The description of the automorphic discrete spectrum of Mp_{2n} is formally similar to that of SO_{2n+1}, except that the condition $\Delta^*\eta = \epsilon_\phi$ is replaced by $\Delta^*\eta = \mathbf{1}$ in the case of SO_{2n+1}.

As an immediate consequence of Theorems 1.1 and 1.4, we obtain the following generalization of Waldspurger's result [88, p. 131].

COROLLARY 1.6. *The generic part of* $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$ (*which is defined as* $\bigoplus_\phi L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$, *where the direct sum runs over generic elliptic A-parameters ϕ for* Mp_{2n}) *is multiplicity-free.*

1.4. *Idea of the proof.* The main ingredients in the proof of Theorems 1.1 and 1.4 are

- Arthur's endoscopic classification [6] (which relies on, among other things, the fundamental lemma proved by Ngô [67] and the stabilization of the twisted trace formula established by Mœglin–Waldspurger [65], [66]);
- a result of J.-S. Li [49] on global theta lifts in the stable range.

It is natural to attempt to transport Arthur's result for the odd special orthogonal group SO_{2n+1} to the metaplectic group Mp_{2n} by using global theta lifts. However, as explained above, the difficulty arises when the central L-value vanishes.

To circumvent this difficulty, we consider the theta lift between Mp_{2n} and SO_{2r+1} with $r \gg 2n$, i.e., the one in the stable range. More precisely, let π be an irreducible summand of $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$. If π is cuspidal, then by the Rallis inner product formula, the global theta lift $\theta_\psi(\pi)$ to $\mathrm{SO}_{2r+1}(\mathbb{A})$ is always nonzero and square-integrable. Even if π is not necessarily cuspidal (so that the Rallis inner product formula is not available), J.-S. Li [49] has developed a somewhat unconventional method for lifting π to an irreducible summand $\theta_\psi(\pi)$ of $L^2_{\mathrm{disc}}(\mathrm{SO}_{2r+1})$, which is given by the spectral decomposition of the space of theta functions on $\mathrm{SO}_{2r+1}(\mathbb{A})$ associated to the reductive dual pair $(\mathrm{Mp}_{2n}, \mathrm{SO}_{2r+1})$. (Note that such theta functions are square-integrable by the stable range condition.) Then Arthur's result attaches an elliptic A-parameter ϕ' to $\theta_\psi(\pi)$, which turns out to be of the form

$$\phi' = \phi \oplus S_{2r-2n}$$

for some elliptic A-parameter ϕ for Mp_{2n}; see Proposition 3.1. We now define the A-parameter of π as ϕ, which proves Theorem 1.1.

To prove Theorem 1.4, we apply Arthur's result to the near equivalence class $L^2_{\phi'}(\mathrm{SO}_{2r+1})$ and transport its local-global structure to $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$. For this, we need the following multiplicity preservation: if

$$L^2_{\phi,\psi}(\mathrm{Mp}_{2n}) \cong \bigoplus_{\pi} m_\pi \pi,$$

then

$$L^2_{\phi'}(\mathrm{SO}_{2r+1}) \cong \bigoplus_{\pi} m_\pi \theta_\psi(\pi).$$

Since the above result of J.-S. Li amounts to the theta lift from Mp_{2n} to SO_{2r+1}, we need the theta lift from SO_{2r+1} to Mp_{2n} in the opposite direction. In fact, J.-S. Li [49] has also developed a method which allows us to lift an irreducible summand σ of $L^2_{\phi'}(\mathrm{SO}_{2r+1})$ (which is no longer cuspidal so that the conventional method does not work) to an irreducible subrepresentation $\theta_\psi(\sigma)$ of the space of automorphic forms on $\mathrm{Mp}_{2n}(\mathbb{A})$, which is realized as a Fourier–Jacobi coefficient of σ (as in the automorphic descent; see [27], [28]).

However, we do not know a priori that $\theta_\psi(\sigma)$ occurs in $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$. The key innovation in this paper is to show that $\theta_\psi(\sigma)$ is cuspidal if ϕ is generic (see Proposition 4.1). From this, we can deduce the multiplicity preservation and hence obtain a multiplicity formula for $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ when ϕ is generic (see Proposition 4.4).

However, there is still an issue: we do not know a priori that the local structure of $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ transported from $L^2_{\phi'}(\mathrm{SO}_{2r+1})$ agrees with the one defined via the local Shimura correspondence. In other words, we have to describe the local theta lift from SO_{2r+1} to Mp_{2n} in terms of the local Shimura correspondence (see Proposition 6.1). This is the most difficult part in the proof of Theorem 1.4 and will be proved as follows.

- We consider the theta lift of representations in the local A-packet $\Pi_{\phi'}(\mathrm{SO}_{2r+1})$ to Mp_{2n}, where ϕ' is a local A-parameter of the form
$$\phi' = \phi \oplus S_{2r-2n}$$
for some local L-parameter ϕ for Mp_{2n}. For our global applications, we may assume that ϕ is almost tempered. Then we can reduce the general case to the case of good L-parameters for smaller metaplectic groups, where we say that an L-parameter ϕ is good if any irreducible summand of ϕ is symplectic. This will be achieved by using irreducibility of some induced representations (see Lemma 5.5), which is due to Mœglin [54], [55], [57], [56] in the nonarchimedean case and to Mœglin–Renard [59] in the complex case, and which is proved in [22] in the real case.
- If ϕ is good, then we appeal to a global argument. As in our previous paper [21], we can find a global generic elliptic A-parameter Φ such that $\Phi_{v_0} = \phi$ for some v_0 and Φ_v is non-good for all $v \neq v_0$, and then apply Arthur's multiplicity formula (viewed as a product formula) to extract information at v_0 from the knowledge at all $v \neq v_0$. Strictly speaking, we need to impose more conditions on Φ, and the most crucial one is the nonvanishing of the central L-value $L(\frac{1}{2}, \Phi)$ (see Corollary 6.7), which makes the argument more complicated than that in [21].

Finally, we remark that when $n = 1$, our argument gives a new proof of the Shimura–Waldspurger correspondence for Mp_2 which is independent of Waldspurger's result [91, Th. 4] on the nonvanishing of central L-values.

1.5. *Endoscopy for* Mp_{2n}. In [50], [51], [52], W.-W. Li has developed the theory of endoscopy for Mp_{2n} and has stabilized the elliptic part of the trace formula for Mp_{2n}, which should yield a definition of local L-packets for Mp_{2n} satisfying endoscopic character relations. In this paper, the local L-packets for Mp_{2n} are defined via the local Shimura correspondence, and we do not know a priori that they satisfy the endoscopic character relations. However, this was established by Adams [1] and Renard [72] in the real case. Moreover, using a simple stable trace formula for Mp_{2n} established by W.-W. Li and the main result of this paper as key inputs, C. Luo [53], a student of the first-named author, has recently proved the endoscopic character relations in the nonarchimedean case.

Acknowledgments. We thank the referees for useful suggestions. The first-named author is partially supported by an MOE Tier one grant R-146-000-228-114. The second-named author is partially supported by JSPS KAKENHI Grant Number 26287003.

Notation. If F is a number field and G is a reductive algebraic group defined over F, we denote by $\mathcal{A}(G)$ the space of automorphic forms on $G(\mathbb{A})$, where $\mathbb{A}$ is the adèle ring of F. If $G = \mathrm{Mp}_{2n}$, we understand that $\mathcal{A}(G)$ consists only of genuine functions. We denote by $\mathcal{A}^2(G)$ and $\mathcal{A}_{\mathrm{cusp}}(G)$ the subspaces of square-integrable automorphic forms and cusp forms on $G(\mathbb{A})$, respectively.

If F is a local field and G is a reductive algebraic group defined over F, we denote by $\mathrm{Irr}\, G$ the set of equivalence classes of irreducible smooth representations of G, where we identify G with its group of F-valued points $G(F)$. If $G = \mathrm{Mp}_{2n}$, we understand that $\mathrm{Irr}\, G$ consists only of genuine representations.

For any irreducible representation π, we denote by $\pi^\vee$ its contragredient representation. For any abelian locally compact group S, we denote by $\hat{S}$ the group of continuous characters of S. For any positive integer d, we denote by S_d the unique d-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$.

2. Some results of J.-S. Li

In this section, we recall some results of J.-S. Li [47], [49] on theta lifts and unitary representations of low rank which will play a crucial role in this paper.

2.1. *Metaplectic and orthogonal groups.* Let F be either a number field or a local field of characteristic zero. Let n be a nonnegative integer and Sp_{2n} the symplectic group of rank n over F. (If $n = 0$, we interpret Sp_0 as the trivial group $\{1\}$.) If F is local, we denote by Mp_{2n} the metaplectic two-fold cover of Sp_{2n},

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}_{2n} \longrightarrow \mathrm{Sp}_{2n} \longrightarrow 1,$$

which is the unique (up to unique isomorphism) nonsplit two-fold topological central extension of Sp_{2n} when $n > 0$ and $F \neq \mathbb{C}$, and which is the trivial extension (i.e., $\mathrm{Mp}_{2n} \cong \mathrm{Sp}_{2n} \times \{\pm 1\}$) when $n = 0$ or $F = \mathbb{C}$. We may realize Mp_{2n} as the set $\mathrm{Sp}_{2n} \times \{\pm 1\}$ with multiplication law determined by Ranga Rao's two-cocycle [71]. If F is global, we denote by $\mathrm{Mp}_{2n}(\mathbb{A})$ the metaplectic two-fold cover of $\mathrm{Sp}_{2n}(\mathbb{A})$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}_{2n}(\mathbb{A}) \longrightarrow \mathrm{Sp}_{2n}(\mathbb{A}) \longrightarrow 1.$$

This cover splits over $\mathrm{Sp}_{2n}(F)$ canonically.

Let V be a quadratic space over F, i.e., a finite-dimensional vector space over F equipped with a nondegenerate quadratic form $q : V \to F$. We assume that V is odd-dimensional

$$\dim V = 2r + 1$$

and that the discriminant of q is trivial. We say that V is split if V is isometric to the orthogonal direct sum $\mathbb{H}^r \oplus F$, where $\mathbb{H}$ is the hyperbolic plane and F is equipped with a quadratic form $q(x) = x^2$. We denote by $\mathrm{O}(V)$ the orthogonal group of V and by $\mathrm{SO}(V) = \mathrm{O}(V)^0$ the special orthogonal group of V. Note that

$$\mathrm{O}(V) = \mathrm{SO}(V) \times \{\pm 1\}. \tag{2.1}$$

We write

$$\mathrm{SO}(V) = \mathrm{SO}_{2r+1}$$

if V is split. Then SO_{2r+1} is split over F. If F is local, we denote by $\varepsilon(V) \in \{\pm 1\}$ the normalized Hasse–Witt invariant of V; see [75, pp. 80–81]. In particular, $\varepsilon(V) = 1$ if V is split.

2.2. *Theta lifts.* Suppose first that F is local, and fix a nontrivial additive character ψ of F. Recall that with the above notation, the pair $(\mathrm{Mp}_{2n}, \mathrm{O}(V))$ is an example of a reductive dual pair. Let ω_ψ be the Weil representation of $\mathrm{Mp}_{2n} \times \mathrm{O}(V)$ with respect to ψ. For any irreducible genuine representation π of Mp_{2n}, the maximal π-isotypic quotient of ω_ψ is of the form

$$\pi \boxtimes \Theta_\psi(\pi)$$

for some representation $\Theta_\psi(\pi)$ of $\mathrm{O}(V)$. Then, by the Howe duality [35], [90], [26], $\Theta_\psi(\pi)$ has a unique irreducible (if nonzero) quotient $\theta_\psi(\pi)$. We regard $\theta_\psi(\pi)$ as a representation of $\mathrm{SO}(V)$ by restriction. By (2.1), $\theta_\psi(\pi)$ remains irreducible if nonzero.

Similarly, for any irreducible representation $\tilde{\sigma}$ of $\mathrm{O}(V)$, we define a representation $\Theta_\psi(\tilde{\sigma})$ of Mp_{2n} with its unique irreducible (if nonzero) quotient $\theta_\psi(\tilde{\sigma})$. We now assume that $r \geq n$. Let σ be an irreducible representation of $\mathrm{SO}(V)$. By the conservation relation [85], there exists at most one extension $\tilde{\sigma}$ of σ to $\mathrm{O}(V)$ such that $\theta_\psi(\tilde{\sigma})$ is nonzero, in which case we write $\theta_\psi(\tilde{\sigma}) = \theta_\psi(\sigma)$. (We interpret $\theta_\psi(\sigma)$ as zero if such $\tilde{\sigma}$ does not exist.)

Suppose next that F is global, and fix a nontrivial additive character ψ of $F\backslash\mathbb{A}$. Let $\pi = \bigotimes_v \pi_v$ be an abstract irreducible genuine representation of $\mathrm{Mp}_{2n}(\mathbb{A})$. Assume that the theta lift $\theta_{\psi_v}(\pi_v)$ of π_v to $\mathrm{SO}(V)(F_v)$ is nonzero for all places v of F. Then $\theta_{\psi_v}(\pi_v)$ is irreducible for all v and is unramified for almost all v. Hence we may define an abstract irreducible representation

$$\theta_\psi^{\mathrm{abs}}(\pi) = \bigotimes_v \theta_{\psi_v}(\pi_v)$$

of $\mathrm{SO}(V)(\mathbb{A})$. We call $\theta_\psi^{\mathrm{abs}}(\pi)$ the abstract theta lift of π to $\mathrm{SO}(V)(\mathbb{A})$. On the other hand, if π is an irreducible genuine cuspidal automorphic representation of $\mathrm{Mp}_{2n}(\mathbb{A})$, then we may define its global theta lift $\Theta_\psi^{\mathrm{aut}}(\pi)$ as the subspace of $\mathcal{A}(\mathrm{SO}(V))$ spanned by all automorphic forms of the form

$$\theta(f,\varphi)(h) = \int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Mp}_{2n}(\mathbb{A})} \theta(f)(g,h)\,\overline{\varphi(g)}\,dg$$

for $f \in \omega_\psi$ and $\varphi \in \pi$. Here ω_ψ is the Weil representation of $\mathrm{Mp}_{2n}(\mathbb{A}) \times \mathrm{O}(V)(\mathbb{A})$ with respect to ψ and $\theta(f)$ is the theta function associated to f, which is a slowly increasing function on $\mathrm{Sp}_{2n}(F)\backslash\mathrm{Mp}_{2n}(\mathbb{A}) \times \mathrm{O}(V)(F)\backslash\mathrm{O}(V)(\mathbb{A})$. If $\Theta_\psi^{\mathrm{aut}}(\pi)$ is nonzero and contained in $\mathcal{A}^2(\mathrm{SO}(V))$, then $\Theta_\psi^{\mathrm{aut}}(\pi)$ is irreducible and

$$\Theta_\psi^{\mathrm{aut}}(\pi) \cong \theta_\psi^{\mathrm{abs}}(\pi)$$

by [43, Cor. 7.1.3].

2.3. *Unitary representations of low rank.* The notion of rank for unitary representations was first introduced by Howe [34] in the case of symplectic groups and was extended by J.-S. Li [47] to the case of classical groups. Following [47, §4], we say that an irreducible unitary representation of SO_{2r+1} is of low rank if its rank (which is necessarily even) is less than $r-1$. Such representations are obtained by theta lifts as follows.

Assume $2n < r-1$. In particular, the reductive dual pair $(\mathrm{Mp}_{2n}, \mathrm{SO}_{2r+1})$ is in the stable range (see [48, Def. 5.1]). If F is local, then for any irreducible genuine representation π of Mp_{2n}, its theta lift $\theta_\psi(\pi)$ to SO_{2r+1} is nonzero. Moreover, if π is unitary, then so is $\theta_\psi(\pi)$ by [48]. In [47], J.-S. Li showed that this theta lift provides a bijection

(2.2)

$$\begin{gathered}\{\text{irreducible genuine unitary representations of } \mathrm{Mp}_{2n}\} \times \{\text{quadratic characters of } F^\times\} \\ \updownarrow \\ \{\text{irreducible unitary representations of } \mathrm{SO}_{2r+1} \text{ of rank } 2n\},\end{gathered}$$

which sends a pair (π,χ) in the first set to a representation $\theta_\psi(\pi) \otimes (\chi\circ\nu)$ of SO_{2r+1}, where $\nu : \mathrm{SO}_{2r+1} \to F^\times/(F^\times)^2$ is the spinor norm.

This result has a global analog. Let F be a number field and $\sigma = \bigotimes_v \sigma_v$ an irreducible unitary representation of $\mathrm{SO}_{2r+1}(\mathbb{A})$ which occurs as a subrepresentation of $\mathcal{A}(\mathrm{SO}_{2r+1})$. Then, by [33], [49, Lemma 3.2], the following are equivalent:

- σ is of rank $2n$;
- σ_v is of rank $2n$ for all v;
- σ_v is of rank $2n$ for some v.

Suppose that σ satisfies the above equivalent conditions. Then, for any v, there exist a unique irreducible genuine unitary representation π_v of $\mathrm{Mp}_{2n}(F_v)$ and a unique quadratic character χ_v of $F_v^\times$ such that

$$\sigma_v \cong \theta_{\psi_v}(\pi_v) \otimes (\chi_v \circ \nu).$$

By [49, Prop. 5.7], χ_v is unramified for almost all v and the abstract character $\chi = \bigotimes_v \chi_v$ of $\mathbb{A}^\times$ is in fact automorphic. This implies that $\theta_{\psi_v}(\pi_v)$ and hence π_v are unramified for almost all v. Hence we may define an abstract representation $\pi = \bigotimes_v \pi_v$ of $\mathrm{Mp}_{2n}(\mathbb{A})$, so that

$$\sigma \cong \theta_\psi^{\mathrm{abs}}(\pi) \otimes (\chi \circ \nu).$$

2.4. *Some inequalities.* Finally, we recall a result of J.-S. Li [49] which allows us to lift square-integrable (but not necessarily cuspidal) automorphic representations of $\mathrm{Mp}_{2n}(\mathbb{A})$ to $\mathrm{SO}_{2r+1}(\mathbb{A})$. For any irreducible genuine representation π of $\mathrm{Mp}_{2n}(\mathbb{A})$, we define its multiplicities $m(\pi)$ and $m_{\mathrm{disc}}(\pi)$ by

$$m(\pi) = \dim \mathrm{Hom}_{\mathrm{Mp}_{2n}(\mathbb{A})}(\pi, \mathcal{A}(\mathrm{Mp}_{2n})),$$
$$m_{\mathrm{disc}}(\pi) = \dim \mathrm{Hom}_{\mathrm{Mp}_{2n}(\mathbb{A})}(\pi, \mathcal{A}^2(\mathrm{Mp}_{2n})).$$

Obviously, $m_{\mathrm{disc}}(\pi) \le m(\pi)$. Likewise, if σ is an irreducible representation of $\mathrm{SO}_{2r+1}(\mathbb{A})$, we have its multiplicities $m(\sigma)$ and $m_{\mathrm{disc}}(\sigma)$.

THEOREM 2.1 (J.-S. Li [49]). *Assume that $2n < r - 1$. Let π be an irreducible genuine unitary representation of $\mathrm{Mp}_{2n}(\mathbb{A})$ and $\theta_\psi^{\mathrm{abs}}(\pi)$ its abstract theta lift to $\mathrm{SO}_{2r+1}(\mathbb{A})$. Then we have*

$$m_{\mathrm{disc}}(\pi) \le m_{\mathrm{disc}}(\theta_\psi^{\mathrm{abs}}(\pi)) \le m(\theta_\psi^{\mathrm{abs}}(\pi)) \le m(\pi).$$

3. Near equivalence classes and A-parameters

In this section, we attach an A-parameter to each near equivalence class contained in $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$.

3.1. *The automorphic discrete spectrum of* $\mathrm{SO}(V)$. We first describe the automorphic discrete spectrum

$$L^2_{\mathrm{disc}}(\mathrm{SO}(V)) = L^2_{\mathrm{disc}}(\mathrm{SO}(V)(F)\backslash \mathrm{SO}(V)(\mathbb{A})),$$

where V is a $(2r+1)$-dimensional quadratic space over a number field F with trivial discriminant. If $\mathrm{SO}(V)$ is split over F, then it follows from Arthur's result [6] that

$$L^2_{\mathrm{disc}}(\mathrm{SO}(V)) = \bigoplus_\phi L^2_\phi(\mathrm{SO}(V)),$$

where the direct sum runs over elliptic A-parameters ϕ for $\mathrm{SO}(V)$ (or equivalently those for Mp_{2r}) and $L^2_\phi(\mathrm{SO}(V))$ is a full near equivalence class of irreducible representations σ in $L^2_{\mathrm{disc}}(\mathrm{SO}(V))$ such that the L-parameter of σ_v is φ_{ϕ_v} for almost all places v of F. Even if $\mathrm{SO}(V)$ is not necessarily split over F, this decomposition is expected to hold.

3.2. *Attaching A-parameters.* Let C be a near equivalence class contained in $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$. Then C gives rise to a collection of L-parameters

$$\varphi_v : L_{F_v} \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})$$

for almost all v such that for any irreducible summand π of C, the L-parameter of π_v (relative to ψ_v) is φ_v for almost all v. Here ψ is the fixed nontrivial additive character of $F\backslash\mathbb{A}$. Then we have

PROPOSITION 3.1. *There exists a unique elliptic A-parameter ϕ for* Mp_{2n} *such that $\varphi_{\phi_v} = \varphi_v$ for almost all v.*

Proof. To prove the existence of ϕ, we fix an integer $r > 2n+1$ and consider the abstract theta lift from $\mathrm{Mp}_{2n}(\mathbb{A})$ to $\mathrm{SO}_{2r+1}(\mathbb{A})$. Choose any irreducible summand π of C. Since $m_{\mathrm{disc}}(\pi) \geq 1$, we deduce from Theorem 2.1 that $m_{\mathrm{disc}}(\theta^{\mathrm{abs}}_\psi(\pi)) \geq 1$; i.e., $\theta^{\mathrm{abs}}_\psi(\pi)$ occurs in $L^2_{\mathrm{disc}}(\mathrm{SO}_{2r+1})$. Hence, as explained in Section 3.1 above, Arthur's result [6] attaches an elliptic A-parameter ϕ' to $\theta^{\mathrm{abs}}_\psi(\pi)$.

We show that ϕ' contains S_{2r-2n} as a direct summand. Consider the partial L-function $L^S(s, \theta^{\mathrm{abs}}_\psi(\pi))$ of $\theta^{\mathrm{abs}}_\psi(\pi)$ relative to the standard representation of $\mathrm{Sp}_{2r}(\mathbb{C})$, where S is a sufficiently large finite set of places of F. If we write $\phi' = \bigoplus_i \phi_i \boxtimes S_{d_i}$ as in (1.1), then

$$L^S(s, \theta^{\mathrm{abs}}_\psi(\pi)) = \prod_i \prod_{j=1}^{d_i} L^S\Big(s + \frac{d_i+1}{2} - j, \phi_i\Big). \tag{3.1}$$

Note that $L^S(s, \phi_i)$ is holomorphic for $\mathrm{Re}\, s > 1$ for all i, and it has a pole at $s = 1$ if and only if ϕ_i is the trivial representation of $\mathrm{GL}_1(\mathbb{A})$. On the other hand, by the local theta correspondence for unramified representations, the L-parameter of $\theta_{\psi_v}(\pi_v)$ is

$$\varphi_v \oplus \Big(\bigoplus_{j=1}^{2r-2n} |\cdot|^{r-n+\frac{1}{2}-j} \Big) \tag{3.2}$$

for almost all v. Hence

$$L^S(s, \theta^{\mathrm{abs}}_\psi(\pi)) = L^S_\psi(s, \pi) \prod_{j=1}^{2r-2n} \zeta^S(s + r - n + \tfrac{1}{2} - j), \tag{3.3}$$

where $L^S_\psi(s,\pi)$ is the partial L-function of π relative to ψ and the standard representation of $\mathrm{Sp}_{2n}(\mathbb{C})$. Since $L^S_\psi(s,\pi)$ is holomorphic for $\mathrm{Re}\, s > n+1$ (see, e.g., [93, Th. 9.1]), it follows from (3.3) that $L^S(s,\theta^{\mathrm{abs}}_\psi(\pi))$ is holomorphic for $\mathrm{Re}\, s > r-n+\frac{1}{2}$ but has a pole at $s = r-n+\frac{1}{2}$. This and (3.1) imply that ϕ' contains S_{2t} as a direct summand for some $t \geq r-n$. If t is the largest such integer, then $L^S(s,\theta^{\mathrm{abs}}_\psi(\pi))$ has a pole at $s = t+\frac{1}{2}$. This forces $t = r-n$.

Thus, we may write

$$\phi' = \phi \oplus S_{2r-2n}$$

for some elliptic A-parameter ϕ. This and (3.2) imply that $\varphi_{\phi_v} = \varphi_v$ for almost all v. Moreover, by the strong multiplicity one theorem [37, Th. 4.4], ϕ is uniquely determined by this condition. □

We now denote by $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ the near equivalence class C, where ϕ is the A-parameter attached to C by Proposition 3.1. Then we have a decomposition

$$L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n}) = \bigoplus_\phi L^2_{\phi,\psi}(\mathrm{Mp}_{2n}),$$

where the direct sum runs over elliptic A-parameters ϕ for Mp_{2n}. Note that $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ is possibly zero for some ϕ. This completes the proof of Theorem 1.1.

Remark 3.2. Suppose that

$$L^2_{\phi,\psi}(\mathrm{Mp}_{2n}) \cong \bigoplus_\pi m_\pi \pi,$$

where the direct sum runs over irreducible genuine representations π of $\mathrm{Mp}_{2n}(\mathbb{A})$ and m_π is the multiplicity of π in $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$. If $r > 2n+1$ and $\phi' = \phi \oplus S_{2r-2n}$, then it follows from Theorem 2.1 and the Howe duality that

$$L^2_{\phi'}(\mathrm{SO}_{2r+1}) \hookleftarrow \bigoplus_\pi m_\pi \theta^{\mathrm{abs}}_\psi(\pi).$$

In Corollary 4.2 below, we show that the above embedding is an isomorphism if ϕ is generic.

Remark 3.3. By Arthur's multiplicity formula [6, Th. 1.5.2], once we know that local A-packets are multiplicity-free, we have

$$m_{\mathrm{disc}}(\sigma) \leq 1$$

for all irreducible representations σ of $\mathrm{SO}_{2r+1}(\mathbb{A})$. On the other hand, the multiplicity-freeness of local A-packets was proved by Mœglin [54], [55], [57], [56] in the nonarchimedean case, and by Mœglin [58] and Mœglin–Renard [59] in the complex case, but is not fully known in the real case, though there has been progress by Arancibia–Mœglin–Renard [4] and Mœglin–Renard [61], [60]. Hence, by Theorem 2.1, $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$ is multiplicity-free at least when F is totally imaginary.

4. **The case of generic elliptic A-parameters**

In this section, we study the structure of $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ for generic elliptic A-parameters ϕ.

4.1. *A key equality.* We define the multiplicity $m_{\mathrm{cusp}}(\pi)$ by

$$m_{\mathrm{cusp}}(\pi) = \dim \mathrm{Hom}_{\mathrm{Mp}_{2n}(\mathbb{A})}(\pi, \mathcal{A}_{\mathrm{cusp}}(\mathrm{Mp}_{2n}))$$

for any irreducible genuine representation π of $\mathrm{Mp}_{2n}(\mathbb{A})$. Obviously, we have

$$m_{\mathrm{cusp}}(\pi) \leq m_{\mathrm{disc}}(\pi) \leq m(\pi).$$

Proposition 4.1. *Let ϕ be a generic elliptic A-parameter for* Mp_{2n}. *Let π be an irreducible genuine representation of* $\mathrm{Mp}_{2n}(\mathbb{A})$ *such that the L-parameter of π_v (relative to ψ_v) is ϕ_v for almost all v. Then we have*

$$m_{\mathrm{cusp}}(\pi) = m_{\mathrm{disc}}(\pi) = m(\pi).$$

Proof. It suffices to show that for any realization $\mathcal{V} \subset \mathcal{A}(\mathrm{Mp}_{2n})$ of π, we have $\mathcal{V} \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{Mp}_{2n})$. Suppose on the contrary that $\mathcal{V} \not\subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{Mp}_{2n})$ for some such $\mathcal{V}$. Then the image $\mathcal{V}_P$ of $\mathcal{V}$ under the constant term map

$$\mathcal{A}(\mathrm{Mp}_{2n}) \longrightarrow \mathcal{A}_P(\mathrm{Mp}_{2n})$$

is nonzero for some proper parabolic subgroup P of Sp_{2n}. Here $\mathcal{A}_P(\mathrm{Mp}_{2n})$ is the space of genuine automorphic forms on $N(\mathbb{A})M(F)\backslash\mathrm{Mp}_{2n}(\mathbb{A})$, where M and N are a Levi component and the unipotent radical of P, respectively. Assume that P is minimal with this property, so that $\mathcal{V}_P$ is contained in the space of cusp forms in $\mathcal{A}_P(\mathrm{Mp}_{2n})$. Then, as explained in [44, p. 205], π is a subrepresentation of

$$\mathrm{Ind}_{\tilde{P}(\mathbb{A})}^{\mathrm{Mp}_{2n}(\mathbb{A})}(\rho)$$

for some irreducible cuspidal automorphic representation ρ of $\tilde{M}(\mathbb{A})$, where $\tilde{P}(\mathbb{A})$ and $\tilde{M}(\mathbb{A})$ are the preimages of $P(\mathbb{A})$ and $M(\mathbb{A})$ in $\mathrm{Mp}_{2n}(\mathbb{A})$, respectively. If $M \cong \prod_i \mathrm{GL}_{k_i} \times \mathrm{Sp}_{2n_0}$ with $\sum_i k_i + n_0 = n$, then ρ is of the form

$$\rho \cong \Big(\bigotimes_i \tilde{\tau}_{i,\psi} \Big) \otimes \pi_0$$

for some irreducible cuspidal automorphic representations τ_i and π_0 of $\mathrm{GL}_{k_i}(\mathbb{A})$ and $\mathrm{Mp}_{2n_0}(\mathbb{A})$, respectively. Here, as in [20, §2.6], we define the twist $\tilde{\tau}_{i,\psi} = \tau_i \otimes \chi_\psi$ of τ_i by the genuine quartic automorphic character χ_ψ of the two-fold cover of $\mathrm{GL}_{k_i}(\mathbb{A})$. By Proposition 3.1, π_0 has a weak lift τ_0 to $\mathrm{GL}_{2n_0}(\mathbb{A})$. Hence π has a weak lift to $\mathrm{GL}_{2n}(\mathbb{A})$ of the form

$$\Big(\boxplus_i (\tau_i \boxplus \tau_i^\vee) \Big) \boxplus \tau_0, \tag{4.1}$$

where $\boxplus$ denotes the isobaric sum.

On the other hand, if we write $\phi = \bigoplus_i \phi_i$ for some pairwise distinct irreducible symplectic cuspidal automorphic representations ϕ_i of $\mathrm{GL}_{n_i}(\mathbb{A})$, then π has a weak lift to $\mathrm{GL}_{2n}(\mathbb{A})$ of the form

$$\boxplus_i \phi_i. \tag{4.2}$$

By the strong multiplicity one theorem [37, Th. 4.4], the two expressions (4.1) and (4.2) must agree. However, τ_i in (4.1) either is non-self-dual or is self-dual but occurs with multiplicity at least 2, whereas ϕ_i in (4.2) is self-dual and occurs with multiplicity 1. This is a contradiction. Hence we have $\mathcal{V} \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{Mp}_{2n})$ as required. □

4.2. *The multiplicity preservation.* As a consequence of Proposition 4.1, we deduce

COROLLARY 4.2. *Let ϕ be a generic elliptic A-parameter for* Mp_{2n}. *Suppose that*

$$L^2_{\phi,\psi}(\mathrm{Mp}_{2n}) \cong \bigoplus_\pi m_\pi \pi.$$

If $r > 2n+1$ and $\phi' = \phi \oplus S_{2r-2n}$, then

$$L^2_{\phi'}(\mathrm{SO}_{2r+1}) \cong \bigoplus_\pi m_\pi \theta^{\mathrm{abs}}_\psi(\pi).$$

Proof. For any irreducible genuine unitary representation π of $\mathrm{Mp}_{2n}(\mathbb{A})$ such that the L-parameter of π_v (relative to ψ_v) is ϕ_v for almost all v, we have

$$m_{\mathrm{disc}}(\pi) = m_{\mathrm{disc}}(\theta^{\mathrm{abs}}_\psi(\pi)) \tag{4.3}$$

by Theorem 2.1 and Proposition 4.1. In view of Remark 3.2, it remains to show that for any irreducible summand σ of $L^2_{\phi'}(\mathrm{SO}_{2r+1})$, there exists an irreducible summand π of $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ such that $\sigma \cong \theta^{\mathrm{abs}}_\psi(\pi)$. Since the L-parameter of σ_v is

$$\varphi_{\phi'_v} = \phi_v \oplus \Big(\bigoplus_{j=1}^{2r-2n} |\cdot|^{r-n+\frac{1}{2}-j} \Big)$$

for almost all v, it follows from the local theta correspondence for unramified representations that

$$\sigma_v \cong \theta_{\psi_v}(\pi_{\phi_v}) \tag{4.4}$$

for almost all v, where π_{ϕ_v} is the irreducible genuine unramified representation of $\mathrm{Mp}_{2n}(F_v)$ whose L-parameter (relative to ψ_v) is ϕ_v. Note that such π_{ϕ_v} is unitary (see [86] and Remark 5.3 below). As explained in Section 2.3, such σ_v is of rank $2n$, and hence so is σ. Hence there exist a unique irreducible genuine unitary representation π of $\mathrm{Mp}_{2n}(\mathbb{A})$ and a unique quadratic automorphic character χ of $\mathbb{A}^\times$ such that

$$\sigma \cong \theta_\psi^{\mathrm{abs}}(\pi) \otimes (\chi \circ \nu).$$

Then we have

$$\sigma_v \cong \theta_{\psi_v}(\pi_v) \otimes (\chi_v \circ \nu)$$

for all v. Recalling the bijection (2.2), we deduce from this and (4.4) that $\pi_v \cong \pi_{\phi_v}$ and χ_v is trivial for almost all v. Since χ is automorphic, it must be trivial, so that $\sigma \cong \theta_\psi^{\mathrm{abs}}(\pi)$. Hence, by (4.3), we have $m_{\mathrm{disc}}(\pi) = m_{\mathrm{disc}}(\sigma) > 0$. $\square$

4.3. *Multiplicity formula for* SO_{2r+1}. Corollary 4.2 allows us to infer a local-global structure of $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ with a multiplicity formula from Arthur's result [6], which we now recall. Let ϕ' be an elliptic A-parameter for SO_{2r+1}, and write

$$\phi' = \bigoplus_i \phi_i \boxtimes S_{d_i}$$

as in (1.1). Let

$$S_{\phi'} = \bigoplus_i (\mathbb{Z}/2\mathbb{Z}) a_i'$$

be a free $\mathbb{Z}/2\mathbb{Z}$-module with a basis $\{a_i'\}$, where a_i' corresponds to $\phi_i \boxtimes S_{d_i}$, and put

$$\bar{S}_{\phi'} = S_{\phi'}/\Delta(\mathbb{Z}/2\mathbb{Z}).$$

For each place v of F, we regard the localization ϕ'_v of ϕ' at v as a local A-parameter $\phi'_v : L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2r}(\mathbb{C})$. Let $S_{\phi'_v}$ be the component group of the centralizer of the image of ϕ'_v in $\mathrm{Sp}_{2r}(\mathbb{C})$, and put $\bar{S}_{\phi'_v} = S_{\phi'_v}/\langle z_{\phi'_v}\rangle$, where $z_{\phi'_v}$ is the image of $-1 \in \mathrm{Sp}_{2r}(\mathbb{C})$ in $S_{\phi'_v}$. Then we have a canonical map $\bar{S}_{\phi'} \to \bar{S}_{\phi'_v}$. Thus, we obtain a compact group $\bar{S}_{\phi',\mathbb{A}} = \prod_v \bar{S}_{\phi'_v}$ equipped with the diagonal map $\Delta : \bar{S}_{\phi'} \to \bar{S}_{\phi',\mathbb{A}}$.

To each local A-parameter ϕ'_v, Arthur [6] assigned a finite set of (possibly zero, possibly reducible) semisimple representations of $\mathrm{SO}_{2r+1}(F_v)$ of finite length:

$$\Pi_{\phi'_v}(\mathrm{SO}_{2r+1}) = \{\sigma_{\eta'_v} \mid \eta'_v \in \hat{\bar{S}}_{\phi'_v}\}.$$

If ϕ' is generic, then $\Pi_{\phi'_v}(\mathrm{SO}_{2r+1})$ is simply the local L-packet associated to ϕ'_v (regarded as a local L-parameter) by the local Langlands correspondence. In particular, if further v is nonarchimedean or complex, then $\sigma_{\eta'_v}$ is nonzero and irreducible for any η'_v. However, if ϕ' is not generic, then Arthur's result [6] does not provide explicit knowledge of the representation $\sigma_{\eta'_v}$. Fortunately, in the nonarchimedean case, Mœglin's results [54], [55], [57], [56] provide an alternative explicit construction of $\Pi_{\phi'_v}(\mathrm{SO}_{2r+1})$ and rather precise knowledge of the properties of $\sigma_{\eta'_v}$ such as nonvanishing, multiplicity-freeness, and irreducibility. These results will be reviewed in Section 5 below.

For any $\eta' = \bigotimes_v \eta'_v \in \hat{\bar{S}}_{\phi',\mathbb{A}}$, we may form a semisimple representation $\sigma_{\eta'} = \bigotimes_v \sigma_{\eta'_v}$ of $\mathrm{SO}_{2r+1}(\mathbb{A})$. Let $\epsilon_{\phi'}$ be the quadratic character of $\bar{S}_{\phi'}$ defined by [6, (1.5.6)]. Then Arthur's multiplicity formula [6, Th. 1.5.2] asserts that

$$L^2_{\phi'}(\mathrm{SO}_{2r+1}) \cong \bigoplus_{\eta' \in \hat{\bar{S}}_{\phi',\mathbb{A}}} m_{\eta'} \sigma_{\eta'}, \tag{4.5}$$

where

$$m_{\eta'} = \begin{cases} 1 & \text{if } \Delta^* \eta' = \epsilon_{\phi'}, \\ 0 & \text{otherwise.} \end{cases}$$

4.4. *Structure of* $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$. Finally, with the help of Corollary 4.2, we can transfer the structure of $L^2_{\phi'}(\mathrm{SO}_{2r+1})$ to $L^2_{\phi,\psi}(\mathrm{Mp}_{2n})$ for any generic elliptic A-parameter ϕ for Mp_{2n}, where $\phi' = \phi \oplus S_{2r-2n}$ with $r > 2n+1$. If we write $\phi = \bigoplus_i \phi_i$ for some pairwise distinct irreducible symplectic cuspidal automorphic representations ϕ_i of $\mathrm{GL}_{n_i}(\mathbb{A})$, then S_ϕ and $S_{\phi'}$ are of the form

$$S_\phi = \bigoplus_i (\mathbb{Z}/2\mathbb{Z}) a_i, \qquad S_{\phi'} = S_\phi \oplus (\mathbb{Z}/2\mathbb{Z}) a'_0,$$

where a_i and a'_0 correspond to ϕ_i and S_{2r-2n}, respectively. In particular, the natural map

$$\iota : S_\phi \hookrightarrow S_{\phi'} \twoheadrightarrow \bar{S}_{\phi'}$$

is an isomorphism. Put $\epsilon_\phi = \iota^* \epsilon_{\phi'}$.

Lemma 4.3. *We have*

$$\epsilon_\phi(a_i) = \epsilon(\tfrac{1}{2}, \phi_i).$$

Moreover, if the central L*-value* $L(\frac{1}{2}, \phi)$ *does not vanish, then* ϵ_ϕ *is trivial.*

Proof. Recall that ϕ_i is symplectic and hence n_i is even; see [40, §9]. Let $\mathcal{L}_\phi = \prod_i \mathrm{Sp}_{n_i}(\mathbb{C})$ be a substitute for the hypothetical Langlands group of F as in [6, (1.4.4)], so that for any place v of F, we may regard ϕ_v as a local L-parameter $\phi_v : L_{F_v} \to \mathcal{L}_\phi$ via the local Langlands correspondence. Let $\tilde{\phi} : \mathcal{L}_\phi \to \mathrm{Sp}_{2n}(\mathbb{C})$ be a natural embedding. We define a homomorphism $\tilde{\phi}' : \mathcal{L}_\phi \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2r}(\mathbb{C})$ by

$$\tilde{\phi}' = \tilde{\phi} \oplus S_{2r-2n},$$

where we regard $\tilde{\phi}$ and $\tilde{\phi}'$ as representations of $\mathcal{L}_\phi$ and $\mathcal{L}_\phi \times \mathrm{SL}_2(\mathbb{C})$, respectively. Then we have a natural embedding $\kappa : S_{\phi'} \hookrightarrow \mathrm{Sp}_{2r}(\mathbb{C})$ whose image is equal to the centralizer of the image of $\tilde{\phi}'$ in $\mathrm{Sp}_{2r}(\mathbb{C})$. More explicitly, we have

$$\mathrm{Std} \circ (\kappa \times \tilde{\phi}') = \Big(\bigoplus_i \kappa_i \boxtimes \mathrm{Std}_i \Big) \oplus (\kappa'_0 \boxtimes S_{2r-2n})$$

as representations of $S_{\phi'} \times \mathcal{L}_\phi \times \mathrm{SL}_2(\mathbb{C})$, where Std is the standard representation of $\mathrm{Sp}_{2r}(\mathbb{C})$, Std_i is the composition of the projection $\mathcal{L}_\phi \to \mathrm{Sp}_{n_i}(\mathbb{C})$ with the standard representation of $\mathrm{Sp}_{n_i}(\mathbb{C})$, and κ_i and κ'_0 are the characters of $S_{\phi'}$ given by

$$\kappa_i(a_j) = \begin{cases} -1 & \text{if } i = j, \\ 1 & \text{if } i \neq j, \end{cases} \qquad \kappa_i(a'_0) = 1,$$
$$\kappa'_0(a_j) = 1, \qquad \kappa'_0(a'_0) = -1.$$

We consider the orthogonal representation $\mathrm{Ad} \circ (\kappa \times \tilde{\phi}')$ of $S_{\phi'} \times \mathcal{L}_\phi \times \mathrm{SL}_2(\mathbb{C})$, where Ad is the adjoint representation of $\mathrm{Sp}_{2r}(\mathbb{C})$, and we decompose it as

$$\mathrm{Ad} \circ (\kappa \times \tilde{\phi}') = \bigoplus_j \epsilon_j \boxtimes \rho_j \boxtimes S_{d_j}$$

for some characters ϵ_j of $S_{\phi'}$, some irreducible representations ρ_j of $\mathcal{L}_\phi$, and some positive integers d_j. In fact, ϵ_j descends to a character of $\bar{S}_{\phi'}$ for all j. If ρ_j is symplectic, we denote by

$$\epsilon(s, \phi, \rho_j) = \prod_v \epsilon(s, \rho_j \circ \phi_v, \psi_v)$$

the ϵ-function of ϕ relative to ρ_j, which is independent of the choice of ψ. Note that $\epsilon(\frac{1}{2}, \phi, \rho_j) \in \{\pm 1\}$. Then $\epsilon_{\phi'}$ is given by

$$\epsilon_{\phi'} = \prod_j \epsilon_j,$$

where the product runs over indices j such that ρ_j is symplectic and $\epsilon(\frac{1}{2}, \phi, \rho_j) = -1$. On the other hand, we have

$$\mathrm{Ad} \circ \tilde{\phi}' = (\mathrm{Ad} \circ \tilde{\phi}) \oplus (\tilde{\phi} \boxtimes S_{2r-2n}) \oplus S_3 \oplus S_7 \oplus \cdots \oplus S_{4r-4n-1}$$

as representations of $\mathcal{L}_\phi \times \mathrm{SL}_2(\mathbb{C})$. Since $\mathrm{Ad} \circ \tilde{\phi}$ is orthogonal, it does not contribute to $\epsilon_{\phi'}$. Moreover, we have

$$\tilde{\phi} \boxtimes S_{2r-2n} = \bigoplus_i \kappa_i \kappa'_0 \boxtimes \mathrm{Std}_i \boxtimes S_{2r-2n}$$

as representations of $S_{\phi'} \times \mathcal{L}_\phi \times \mathrm{SL}_2(\mathbb{C})$. Since Std_i is symplectic and $\epsilon(s, \phi, \mathrm{Std}_i) = \epsilon(s, \phi_i)$, this implies the first assertion.

If $L(\frac{1}{2}, \phi) \neq 0$, then since $L(\frac{1}{2}, \phi) = \prod_i L(\frac{1}{2}, \phi_i)$, we have $L(\frac{1}{2}, \phi_i) \neq 0$ and hence $\epsilon(\frac{1}{2}, \phi_i) = 1$ for all i. Thus, the second assertion follows from the first one. □

Similarly, for any place v of F, the natural map

$$\iota_v : S_{\phi_v} \hookrightarrow S_{\phi'_v} \twoheadrightarrow \bar{S}_{\phi'_v}$$

is an isomorphism. For any $\eta'_v \in \hat{\hat{S}}_{\phi'_v}$, we write

$$\sigma_{\eta'_v} = \bigoplus_i m_{\eta'_v,i} \sigma_{\eta'_v,i}$$

for some positive integers $m_{\eta'_v,i}$ and some pairwise distinct irreducible representations $\sigma_{\eta'_v,i}$ of $\mathrm{SO}_{2r+1}(F_v)$. Put $\eta_v = \iota_v^* \eta'_v$ and

$$\tilde{\pi}_{\eta_v} = \bigoplus_i m_{\eta'_v,i} \theta_{\psi_v}(\sigma_{\eta'_v,i}),$$

where $\theta_{\psi_v}(\sigma_{\eta'_v,i})$ is the theta lift to $\mathrm{Mp}_{2n}(F_v)$. Thus, for any $\eta = \bigotimes_v \eta_v \in \hat{S}_{\phi,\mathbb{A}}$, we may form a semisimple genuine representation $\tilde{\pi}_\eta = \bigotimes_v \tilde{\pi}_{\eta_v}$ of $\mathrm{Mp}_{2n}(\mathbb{A})$.

PROPOSITION 4.4. *Let ϕ be a generic elliptic A-parameter for* Mp_{2n}. *Then we have*

$$L^2_{\phi,\psi}(\mathrm{Mp}_{2n}) \cong \bigoplus_{\eta \in \hat{S}_{\phi,\mathbb{A}}} m_\eta \tilde{\pi}_\eta,$$

where

$$m_\eta = \begin{cases} 1 & \text{if } \Delta^*\eta = \epsilon_\phi, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that

$$L^2_{\phi'}(\mathrm{SO}_{2r+1}) \cong \bigoplus_\sigma m_\sigma \sigma.$$

Then, by Corollary 4.2 and the Howe duality, we have

$$L^2_{\phi,\psi}(\mathrm{Mp}_{2n}) \cong \bigoplus_\sigma m_\sigma \theta_\psi^{\mathrm{abs}}(\sigma).$$

This and the multiplicity formula (4.5) imply the assertion. □

Hence, to complete the proof of Theorem 1.4, it remains to describe $\tilde{\pi}_\eta$ in terms of the local Shimura correspondence. This will be established in Proposition 6.1 below.

Remark 4.5. In the above argument, we have fixed an integer $r > 2n+1$ and do not know a priori that $\tilde{\pi}_\eta$ is independent of the choice of r. This does not seem immediate and will be deduced from the description of $\tilde{\pi}_\eta$ in terms of the local Shimura correspondence. While this is a purely local problem, we will address it by a global argument in Section 6 below.

5. Local L- and A-packets

In this section, we review the representation theory of metaplectic and orthogonal groups over local fields. In particular, we state irreducibility of some induced representations which will play an important role in an inductive argument in Section 6 below.

5.1. *L-parameters.* Let F be a local field of characteristic zero and put

$$L_F = \begin{cases} \text{the Weil group of } F & \text{if } F \text{ is archimedean,} \\ \text{the Weil–Deligne group of } F & \text{if } F \text{ is nonarchimedean.} \end{cases}$$

Then the local Langlands correspondence [45], [30], [31], [76] provides a bijection

$$\mathrm{Irr}\,\mathrm{GL}_n \longleftrightarrow \{n\text{-dimensional representations of } L_F\}.$$

Let ϕ be an n-dimensional representation of L_F. We may regard ϕ as an L-parameter $\phi : L_F \to \mathrm{GL}_n(\mathbb{C})$. We say that

- ϕ is symplectic if there exists a nondegenerate antisymmetric bilinear form $b : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that $b(\phi(w)x, \phi(w)y) = b(x, y)$ for all $w \in L_F$ and $x, y \in \mathbb{C}^n$;
- ϕ is tempered if the image of the Weil group of F under ϕ is relatively compact in $\mathrm{GL}_n(\mathbb{C})$.

If ϕ is irreducible and symplectic, then it is tempered. Let τ be the irreducible representation of GL_n associated to ϕ. Then we have

- τ is essentially square-integrable if and only if ϕ is irreducible;
- τ is tempered if and only if ϕ is tempered.

Let ϕ be a $2n$-dimensional symplectic representation of L_F. We may regard ϕ as an L-parameter $\phi : L_F \to \mathrm{Sp}_{2n}(\mathbb{C})$. We decompose ϕ as

$$\phi = \bigoplus_i m_i \phi_i \tag{5.1}$$

for some positive integers m_i and some pairwise distinct irreducible representations ϕ_i of L_F. We say that

- ϕ is good if ϕ_i is symplectic for all i;
- ϕ is tempered if ϕ_i is tempered for all i;
- ϕ is almost tempered if $\phi_i| \cdot |^{-s_i}$ is tempered for some $s_i \in \mathbb{R}$ with $|s_i| < \frac{1}{2}$ for all i.

If ϕ is good, then it is tempered. Also, any localization of a global generic elliptic A-parameter for Mp_{2n} is almost tempered.

For any $2n$-dimensional symplectic representation ϕ of L_F, we may write

$$\phi = \varphi \oplus \varphi^\vee \oplus \phi_0, \tag{5.2}$$

where

- φ is a k-dimensional representation of L_F whose irreducible summands are all non-symplectic;
- ϕ_0 is a $2n_0$-dimensional good symplectic representation of L_F;
- $k + n_0 = n$.

More explicitly, if ϕ is of the form (5.1), then ϕ_0 is given by

$$\phi_0 = \bigoplus_{i \in I_0} m_i \phi_i,$$

where $I_0 = \{i \mid \phi_i \text{ is symplectic}\}$. Let S_ϕ be the component group of the centralizer of the image of ϕ in $\mathrm{Sp}_{2n}(\mathbb{C})$ and z_ϕ the image of $-1 \in \mathrm{Sp}_{2n}(\mathbb{C})$ in S_ϕ. Then S_ϕ is a free $\mathbb{Z}/2\mathbb{Z}$-module of the form

$$S_\phi = \bigoplus_{i \in I_0} (\mathbb{Z}/2\mathbb{Z}) a_i,$$

where a_i corresponds to ϕ_i, and $z_\phi = (m_i a_i)$. In particular, we have a natural identification $S_{\phi_0} = S_\phi$.

5.2. *Representation theory of* $\mathrm{SO}(V)$. The local Langlands correspondence [45], [79], [80], [6], [62] provides a partition

$$\mathrm{Irr}\, \mathrm{SO}(V) = \bigsqcup_\phi \Pi_\phi(\mathrm{SO}(V))$$

and bijections

$$\bigsqcup_V \Pi_\phi(\mathrm{SO}(V)) \longleftrightarrow \hat{S}_\phi, \tag{5.3}$$

where the first disjoint union runs over equivalence classes of $2n$-dimensional symplectic representations ϕ of L_F and the second disjoint union runs over isometry classes of $(2n+1)$-dimensional quadratic spaces V over F with trivial discriminant. For any $\sigma \in \Pi_\phi(\mathrm{SO}(V))$, we have

- σ is square-integrable if and only if ϕ is good and multiplicity-free;
- σ is tempered if and only if ϕ is tempered.

We write $\sigma = \sigma_\eta$ if σ corresponds to $\eta \in \hat{S}_\phi$ under the bijection (5.3). Let $\hat{S}_{\phi,V} \subset \hat{S}_\phi$ be the subset of all η such that σ_η is a representation of $\mathrm{SO}(V)$. Then we have

$$\hat{S}_{\phi,V} \subset \{\eta \in \hat{S}_\phi \mid \eta(z_\phi) = \varepsilon(V)\},$$

with equality when F is nonarchimedean or $F = \mathbb{C}$. Note that if $F = \mathbb{C}$, then $S_\phi = \{0\}$, so that the condition $\eta(z_\phi) = \varepsilon(V)$ disappears.

Write $\phi = \varphi \oplus \varphi^\vee \oplus \phi_0$ as in (5.2). We have $\Pi_\phi(\mathrm{SO}(V)) = \varnothing$ unless the F-rank of $\mathrm{SO}(V)$ is greater than or equal to k, in which case there exists a $(2n_0+1)$-dimensional quadratic space V_0 over F with trivial discriminant such that $V = \mathbb{H}^k \oplus V_0$. Let Q be a parabolic subgroup of $\mathrm{SO}(V)$ with Levi component $\mathrm{GL}_k \times \mathrm{SO}(V_0)$. Let τ be the irreducible representation of GL_k associated to φ. Then, by the inductive property of the local Langlands correspondence, for any $\eta \in \hat{S}_{\phi,V}$, σ_η is equal to an irreducible subquotient of

$$\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau \otimes \sigma_{\eta_0}),$$

where we write η as η_0 if we regard it as a character of S_{ϕ_0} via the identification $S_{\phi_0} = S_\phi$.

LEMMA 5.1. *If ϕ is almost tempered, then $\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau \otimes \sigma_0)$ is irreducible for any $\sigma_0 \in \Pi_{\phi_0}(\mathrm{SO}(V_0))$.*

Proof. If ϕ is tempered, then by definition, the L-packet $\Pi_\phi(\mathrm{SO}(V))$ consists of all irreducible summands of $\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau \otimes \sigma_0)$ for all $\sigma_0 \in \Pi_{\phi_0}(\mathrm{SO}(V_0))$. Hence the irreducibility of $\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau \otimes \sigma_0)$ is a consequence of the bijectivity of (5.3) and the fact that $S_{\phi_0} = S_\phi$.

Thus, if ϕ is almost tempered, then by induction in stages, it remains to show that $\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau \otimes \sigma_0)$ is irreducible if

- τ is an irreducible representation of GL_k whose L-parameter is of the form $\bigoplus_i \phi_i |\cdot|^{s_i}$ for some tempered representations ϕ_i of L_F and some $s_i \in \mathbb{R}$ with $0 < |s_i| < \frac{1}{2}$;
- σ_0 is an irreducible tempered representation of $\mathrm{SO}(V_0)$.

If F is archimedean, then this follows from a result of Speh–Vogan [83]; see also [87, Ch. 8]. If F is nonarchimedean, then this follows from a result of Mœglin–Waldspurger [64, §2.14] and a conjecture of Gross–Prasad and Rallis [29, Conj. 2.6], which is proved in [21, App. B]. $\square$

5.3. *Representation theory of* Mp_{2n}. Fix a nontrivial additive character ψ of F. The local Shimura correspondence [2], [3], [24] asserts that the theta lift (relative to ψ) induces a bijection

$$\theta_\psi : \mathrm{Irr}\,\mathrm{Mp}_{2n} \longleftrightarrow \bigsqcup_V \mathrm{Irr}\,\mathrm{SO}(V)$$

satisfying natural properties as stated in Section 1.2, where the disjoint union runs over isometry classes of $(2n+1)$-dimensional quadratic spaces V over F with trivial discriminant. Composing this with the local Langlands correspondence for $\mathrm{SO}(V)$, we obtain a partition

$$\mathrm{Irr}\,\mathrm{Mp}_{2n} = \bigsqcup_\phi \Pi_{\phi,\psi}(\mathrm{Mp}_{2n})$$

and bijections

$$\Pi_{\phi,\psi}(\mathrm{Mp}_{2n}) \longleftrightarrow \hat{S}_\phi, \tag{5.4}$$

where the disjoint union runs over equivalence classes of $2n$-dimensional symplectic representations ϕ of L_F. Since θ_ψ preserves the square-integrability and the temperedness of representations, for any $\pi \in \Pi_{\phi,\psi}(\mathrm{Mp}_{2n})$, we have

- π is square-integrable if and only if ϕ is good and multiplicity-free;
- π is tempered if and only if ϕ is tempered.

We write $\pi = \pi_\eta$ if π corresponds to $\eta \in \hat{S}_\phi$ under the bijection (5.4).

Write $\phi = \varphi \oplus \varphi^\vee \oplus \phi_0$ as in (5.2). Let P be a parabolic subgroup of Sp_{2n} with Levi component $\mathrm{GL}_k \times \mathrm{Sp}_{2n_0}$ and $\tilde{P}$ the preimage of P in Mp_{2n}. Let τ be the irreducible representation of GL_k associated to φ and $\tilde{\tau}_\psi = \tau \otimes \chi_\psi$ its twist by the genuine quartic character χ_ψ of the two-fold cover of GL_k given in [20, §2.6]. Then, by the inductive property of the local Shimura correspondence, for any $\eta \in \hat{S}_\phi$, π_η is equal to an irreducible subquotient of

$$\mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}_{2n}}(\tilde{\tau}_\psi \otimes \pi_{\eta_0}),$$

where we write η as η_0 if we regard it as a character of S_{ϕ_0} via the identification $S_{\phi_0} = S_\phi$.

Lemma 5.2. *If ϕ is almost tempered, then $\mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}_{2n}}(\tilde{\tau}_\psi \otimes \pi_0)$ is irreducible for any $\pi_0 \in \Pi_{\phi_0,\psi}(\mathrm{Mp}_{2n_0})$.*

Proof. If ϕ is tempered, then by the inductive property of the local Shimura correspondence, the L-packet $\Pi_{\phi,\psi}(\mathrm{Mp}_{2n})$ consists of all irreducible summands of $\mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}_{2n}}(\tilde{\tau}_\psi \otimes \pi_0)$ for all $\pi_0 \in \Pi_{\phi_0,\psi}(\mathrm{Mp}_{2n_0})$. Hence the irreducibility of $\mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}_{2n}}(\tilde{\tau}_\psi \otimes \pi_0)$ is a consequence of the bijectivity of (5.4) and the fact that $S_{\phi_0} = S_\phi$.

Thus, if ϕ is almost tempered, then by induction in stages, it remains to show that $\mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}_{2n}}(\tilde{\tau}_\psi \otimes \pi_0)$ is irreducible if

- τ is an irreducible representation of GL_k whose L-parameter is of the form $\bigoplus_i \phi_i |\cdot|^{s_i}$ for some tempered representations ϕ_i of L_F and some $s_i \in \mathbb{R}$ with $0 < |s_i| < \frac{1}{2}$;
- π_0 is an irreducible tempered representation of Mp_{2n_0}.

If $F = \mathbb{C}$, then this follows from a result of Speh–Vogan [83]; see also [87, Ch. 8]. If $F = \mathbb{R}$, then the argument in [83] should also work, but they only consider real reductive *linear* Lie groups. For the sake of completeness, we give a proof in [22]. Suppose that F is nonarchimedean. Then this follows from a result of Atobe [7, Th. 3.13], but we include the proof for the convenience of the reader. We may assume that $\pi = \mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}_{2n}}(\tilde{\tau}_\psi \otimes \pi_0)$ is a standard module. Let π' be the unique irreducible quotient of π. By the local Shimura correspondence, the theta lift $\sigma_0 = \theta_\psi(\pi_0)$ to $\mathrm{SO}(V_0)$ is nonzero and tempered for a unique $(2n_0+1)$-dimensional quadratic space V_0 over F with trivial discriminant. Put $\sigma = \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau \otimes \sigma_0)$, where $V = \mathbb{H}^k \oplus V_0$ and Q is the standard parabolic subgroup of $\mathrm{SO}(V)$ with Levi component $\mathrm{GL}_k \times \mathrm{SO}(V_0)$. As shown in the proof of Lemma 5.1, σ is irreducible. Then, by [24], we have

$$\theta_\psi(\pi') = \sigma.$$

On the other hand, as in the proof of [24, Th. 8.1], it follows from Kudla's filtration (see, e.g., [24, Prop. 7.3]) and [24, Lemma 7.4], which continues to hold for $s > -\frac{1}{2}$, that there exists a surjection

$$\mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}_{2n}}((\tau^\vee \otimes \chi_\psi) \otimes \Theta_\psi(\sigma_0)) \twoheadrightarrow \Theta_\psi(\sigma).$$

Since $\Theta_\psi(\sigma_0) = \theta_\psi(\sigma_0) = \pi_0$ by [24] and $\theta_\psi(\sigma) = \pi'$, this induces a surjection

$$\mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}_{2n}}((\tau^\vee \otimes \chi_\psi) \otimes \pi_0) \twoheadrightarrow \pi'.$$

Taking the contragredient and applying the MVW involution [63], [84], we obtain an injection

$$\pi' \hookrightarrow \mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}_{2n}}(\tilde{\tau}_\psi \otimes \pi_0).$$

Hence π' occurs in π as a quotient and as a subrepresentation. Since π' occurs in π with multiplicity one (see, e.g., [9, Rem. 4.5]), π must be irreducible. □

Remark 5.3. Suppose that F is nonarchimedean of odd residual characteristic and ψ is of order zero. Then we may regard $K = \mathrm{Sp}_{2n}(\mathcal{O})$ as a subgroup of Mp_{2n} via the standard splitting, where $\mathcal{O}$ is the integer ring of F. If an irreducible genuine representation π of Mp_{2n} is unramified, i.e., has a nonzero K-fixed vector, then it must be the unique unramified subquotient of a principal series representation

$$\mathrm{Ind}_{\tilde{B}}^{\mathrm{Mp}_{2n}}(\chi_\psi|\cdot|^{s_1} \otimes \cdots \otimes \chi_\psi|\cdot|^{s_n}), \tag{5.5}$$

where B is a Borel subgroup of Sp_{2n}, $\tilde{B}$ is the preimage of B in Mp_{2n}, and $s_1, \ldots, s_n \in \mathbb{C}$. In this case, by the local theta correspondence for unramified representations, the L-parameter of π (relative to ψ) is

$$\bigoplus_{i=1}^{n}(|\cdot|^{s_i} \oplus |\cdot|^{-s_i}).$$

Also, as shown in the proof of Lemma 5.2, the representation (5.5) is irreducible if $|\operatorname{Re} s_i| < \frac{1}{2}$ for all i. Hence, by [46, Lemma 3.3], the representation (5.5) is unitary if either $s_i \in \sqrt{-1}\mathbb{R}$ or $s_i \in \mathbb{R}$ with $|s_i| < \frac{1}{2}$ for all i.

5.4. *Some A-packets for* SO_{2r+1}. Let ϕ' be a $2r$-dimensional symplectic representation of $L_F \times \mathrm{SL}_2(\mathbb{C})$. We may regard ϕ' as an A-parameter $\phi' : L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2r}(\mathbb{C})$ and associate to it an L-parameter $\varphi_{\phi'} : L_F \to \mathrm{Sp}_{2r}(\mathbb{C})$ by

$$\varphi_{\phi'}(w) = \phi'\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{-\frac{1}{2}} \end{pmatrix}\right). \tag{5.6}$$

Let $S_{\phi'}$ be the component group of the centralizer of the image of ϕ' in $\mathrm{Sp}_{2r}(\mathbb{C})$, and put $\bar{S}_{\phi'} = S_{\phi'}/\langle z_{\phi'}\rangle$, where $z_{\phi'}$ is the image of $-1 \in \mathrm{Sp}_{2r}(\mathbb{C})$ in $S_{\phi'}$. Then Arthur [6] assigned to ϕ' an A-packet,

$$\Pi_{\phi'}(\mathrm{SO}_{2r+1}) = \{\sigma_{\eta'} \mid \eta' \in \hat{\bar{S}}_{\phi'}\},$$

where $\sigma_{\eta'}$ is a (possibly zero, possibly reducible) semisimple representation of SO_{2r+1} of finite length.

From now on, we only consider an A-parameter ϕ' of the form

$$\phi' = \phi \oplus S_{2r-2n}$$

for some $2n$-dimensional symplectic representation ϕ of L_F with $2n < r-1$. Then we have

Proposition 5.4. *Assume that ϕ is almost tempered. Then $\sigma_{\eta'}$ is nonzero and irreducible for any $\eta' \in \hat{\bar{S}}_{\phi'}$. Moreover, $\Pi_{\phi'}(\mathrm{SO}_{2r+1})$ is multiplicity-free; i.e., the $\sigma_{\eta'}$'s are pairwise distinct.*

This proposition is largely due to Mœglin [54], [55], [57], [56] when F is nonarchimedean, to Mœglin [58] and Mœglin–Renard [59] when $F = \mathbb{C}$, and to Arancibia–Mœglin–Renard [4] and Mœglin–Renard [61], [60] when $F = \mathbb{R}$ and ϕ is good. The reader can also consult an efficient and concise exposition of Mœglin's results by B. Xu [92]. In the next section, we will give a proof of Proposition 5.4 based on theta lifts. For this, we need the following irreducibility of some induced representations.

Write $\phi = \varphi \oplus \varphi^\vee \oplus \phi_0$ as in (5.2). Put $\phi_0' = \phi_0 \oplus S_{2r-2n}$, so that $\phi' = \varphi \oplus \varphi^\vee \oplus \phi_0'$. Let Q' be a parabolic subgroup of SO_{2r+1} with Levi component $\mathrm{GL}_k \times \mathrm{SO}_{2r-2k+1}$. Let τ be the irreducible representation of GL_k associated to φ. Then, by the definition and the inductive property of A-packets (see [6]), for any $\eta' \in \hat{\bar{S}}_{\phi'}$, $\sigma_{\eta'}$ is equal to the semisimplification of

$$\mathrm{Ind}_{Q'}^{\mathrm{SO}_{2r+1}}(\tau \otimes \sigma_{\eta_0'}),$$

where we write η' as η_0' if we regard it as a character of $\bar{S}_{\phi_0'}$ via the identification $\bar{S}_{\phi_0'} = \bar{S}_{\phi'}$.

Lemma 5.5. *Assume that ϕ is almost tempered. Then $\mathrm{Ind}_{Q'}^{\mathrm{SO}_{2r+1}}(\tau \otimes \sigma_0')$ is irreducible for any irreducible subrepresentation σ_0' of any representation in the A-packet $\Pi_{\phi_0'}(\mathrm{SO}_{2r-2k+1})$.*

Proof. The assertion was proved in a more general context by Mœglin [57, §3.2], [56, Prop. 5.1] when F is nonarchimedean, and by Mœglin–Renard [59, §6] when $F = \mathbb{C}$. In [22], we give a proof based on the Kazhdan–Lusztig algorithm when $F = \mathbb{R}$, noting that σ_0' has half integral infinitesimal character by [10, Lemme 3.4]. □

6. Comparison of local theta lifts

As explained in Section 4.4, to finish the proof of Theorem 1.4, it remains to describe the local theta lift from SO_{2r+1} to Mp_{2n} with $r > 2n+1$ in terms of the local Shimura correspondence. Namely, we need to compare the local theta correspondences for the following reductive dual pairs:

- $(\mathrm{Mp}_{2n}, \mathrm{SO}_{2n+1})$ in the equal rank case (and its inner forms);
- $(\mathrm{Mp}_{2n}, \mathrm{SO}_{2r+1})$ in the stable range.

To distinguish them, we keep using θ_ψ to denote the theta correspondence for the former but change it to ϑ_ψ for the latter. We emphasize that we need all inner forms of SO_{2n+1} in the former but only need the split form SO_{2r+1} in the latter.

Let F be a local field of characteristic zero, and fix a nontrivial additive character ψ of F. Let ϕ be a $2n$-dimensional symplectic representation of L_F, and put $\phi' = \phi \oplus S_{2r-2n}$ with $r > 2n+1$. Then we have a natural isomorphism

$$\iota : S_\phi \hookrightarrow S_{\phi'} \twoheadrightarrow \bar{S}_{\phi'}.$$

We now state the main result of this section.

Proposition 6.1. *Assume that ϕ is almost tempered. Then, for any $\eta' \in \hat{\bar{S}}_{\phi'}$, we have*

$$\vartheta_\psi(\sigma_{\eta'}) = \pi_\eta,$$

where $\eta = \iota^*\eta'$.

The rest of this section is devoted to the proof of Propositions 5.4 and 6.1.

6.1. *Reduction to the case of good L-parameters.* We proceed by induction on n.

Lemma 6.2. *Propositions 5.4 and 6.1 hold for $n = 0$.*

Proof. If $n = 0$, then $\phi' = S_{2r}$ and $\Pi_{\phi'}(\mathrm{SO}_{2r+1}) = \{\sigma_{\mathbf{1}}\}$, where $\sigma_{\mathbf{1}}$ is the trivial representation of SO_{2r+1}. Moreover, the theta lift $\vartheta_\psi(\sigma_{\mathbf{1}})$ to Mp_0 is the genuine 1-dimensional representation of $\mathrm{Mp}_0 = \{\pm 1\}$, which is also the theta lift of the trivial representation of $\mathrm{SO}_1 = \{1\}$. This completes the proof. □

From now on, we assume that $n > 0$.

Lemma 6.3. *Assume that Propositions 5.4 and 6.1 hold for all $2n_0$-dimensional good symplectic representations of L_F for all $n_0 < n$. Then they also hold for all $2n$-dimensional almost tempered non-good symplectic representations of L_F.*

Proof. Let ϕ be a $2n$-dimensional almost tempered non-good symplectic representation of L_F. Write $\phi = \varphi \oplus \varphi^\vee \oplus \phi_0$ as in (5.2). In particular, ϕ_0 is a $2n_0$-dimensional good symplectic representation of L_F with $n_0 < n$. Put $\phi'_0 = \phi_0 \oplus S_{2r-2n}$, so that $\phi' = \varphi \oplus \varphi^\vee \oplus \phi'_0$. Let $\eta' \in \hat{\bar{S}}_{\phi'}$, and put $\eta = \iota^*\eta'$. We write η' (resp. η) as η'_0 (resp. η_0) if we regard it as a character of $\bar{S}_{\phi'_0}$ (resp. S_{ϕ_0}) via the natural identification. Then, by definition, $\sigma_{\eta'}$ is the semisimplification of $\mathrm{Ind}_{Q'}^{\mathrm{SO}_{2r+1}}(\tau \otimes \sigma_{\eta'_0})$, where Q' is the standard parabolic subgroup of SO_{2r+1} with Levi component $\mathrm{GL}_k \times \mathrm{SO}_{2r-2k+1}$ and τ is the irreducible representation of GL_k associated to φ. Since $\sigma_{\eta'_0}$ is nonzero and irreducible by assumption, so is $\sigma_{\eta'}$ by Lemma 5.5. Moreover, the theta lift $\vartheta_\psi(\sigma_{\eta'_0})$ to Mp_{2n_0} is π_{η_0} by assumption. Hence it follows from the induction principle [42], [2, Cor. 3.21], [3, Th. 8.4], which easily extends to the case at hand, that there exists a nonzero equivariant map

$$\omega_\psi \longrightarrow \mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}_{2n}}(\tilde{\tau}_\psi \otimes \pi_{\eta_0}) \otimes \mathrm{Ind}_{Q'}^{\mathrm{SO}_{2r+1}}(\tau \otimes \sigma_{\eta'_0}),$$

where P is the standard parabolic subgroup of Sp_{2n} with Levi component $\mathrm{GL}_k \times \mathrm{Sp}_{2n_0}$. Since $\mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}_{2n}}(\tilde{\tau}_\psi \otimes \pi_{\eta_0})$ is irreducible by Lemma 5.2, this implies that the theta lift $\vartheta_\psi(\sigma_{\eta'})$ to Mp_{2n} is π_η. Thus, since $\Pi_{\phi,\psi}(\mathrm{Mp}_{2n})$ is multiplicity-free, so is $\Pi_{\phi'}(\mathrm{SO}_{2r+1})$. This completes the proof. □

We may now assume that Propositions 5.4 and 6.1 hold for all $2n$-dimensional almost tempered non-good symplectic representations of L_F. It remains to show that they also hold for all $2n$-dimensional good symplectic representations of L_F. In particular, we have finished the proof for $F = \mathbb{C}$ since any irreducible representation of $L_\mathbb{C}$ is 1-dimensional and hence non-symplectic.

Later, we also need the following description of the theta lift from SO_{2r+1} to Mp_{2n}.

Lemma 6.4. *Assume that F is nonarchimedean or $F = \mathbb{C}$ and that ϕ is almost tempered. Let $\varphi_{\phi'}$ be the L-parameter associated to ϕ' by* (5.6) *and σ' the irreducible representation in the L-packet $\Pi_{\varphi_{\phi'}}(\mathrm{SO}_{2r+1})$ associated to the trivial character of $S_{\varphi_{\phi'}}$. Then we have*

$$\vartheta_\psi(\sigma') = \pi_{\mathbf{1}},$$

where $\pi_{\mathbf{1}}$ is the irreducible representation in the L-packet $\Pi_{\phi,\psi}(\mathrm{Mp}_{2n})$ associated to the trivial character of S_ϕ.

Proof. If ϕ is non-good, then it follows from [6, Prop. 7.4.1] and Proposition 5.4 that σ' is the representation in the A-packet $\Pi_{\phi'}(\mathrm{SO}_{2r+1})$ associated to the trivial character of $\bar{S}_{\phi'}$. Hence, by Proposition 6.1, we have $\vartheta_\psi(\sigma') = \pi_{\mathbf{1}}$.

We may now assume that ϕ is good (and hence tempered), so that F is nonarchimedean. By definition, σ' is the unique irreducible quotient of the standard module

$$\mathrm{Ind}_{Q'}^{\mathrm{SO}_{2r+1}}(|\cdot|^{r-n-\frac{1}{2}} \otimes |\cdot|^{r-n-\frac{3}{2}} \otimes \cdots \otimes |\cdot|^{\frac{1}{2}} \otimes \sigma),$$

where Q' is the standard parabolic subgroup of SO_{2r+1} with Levi component $(\mathrm{GL}_1)^{r-n} \times \mathrm{SO}_{2n+1}$ and σ is the irreducible tempered representation in the L-packet $\Pi_\phi(\mathrm{SO}_{2n+1})$ associated to the trivial character of S_ϕ. Since the theta lift of σ to Mp_{2n} is $\pi_{\mathbf{1}}$, we have $\vartheta_\psi(\sigma') = \pi_{\mathbf{1}}$ by [25, Prop. 3.2]. □

6.2. *Multiplicity formula and globalization.* To finish the proof of Propositions 5.4 and 6.1, we appeal to a global argument. One of the global ingredients is Arthur's multiplicity formula (4.5) and the following variant.

Let $\mathbb{F}$ be a number field and $\mathbb{A}$ the adèle ring of $\mathbb{F}$. Let $\mathbb{V}$ be a $(2n+1)$-dimensional quadratic space over $\mathbb{F}$ with trivial discriminant. Recall the (expected) decomposition of $L^2_{\mathrm{disc}}(\mathrm{SO}(\mathbb{V}))$ into near equivalence classes described in Section 3.1. We only consider the near equivalence class $L^2_{\Phi}(\mathrm{SO}(\mathbb{V}))$ associated to a generic elliptic A-parameter Φ for $\mathrm{SO}(\mathbb{V})$. As in Section 1.3, we formally define the global component group S_Φ of Φ equipped with a canonical map $S_\Phi \to S_{\Phi_v}$ for each v and the diagonal map $\Delta : S_\Phi \to S_{\Phi,\mathbb{A}}$. For any $\eta = \bigotimes_v \eta_v \in \hat{S}_{\Phi,\mathbb{A}}$ such that $\eta_v \in \hat{S}_{\Phi_v,\mathbb{V}_v}$ for all v, we may form an irreducible representation $\Sigma_\eta = \bigotimes_v \Sigma_{\eta_v}$ of $\mathrm{SO}(\mathbb{V})(\mathbb{A})$, where Σ_{η_v} is the representation of $\mathrm{SO}(\mathbb{V})(F_v)$ associated to η_v by the local Langlands correspondence described in Section 5.2. Then Arthur's multiplicity formula (which has not been established if $\mathrm{SO}(\mathbb{V})$ is nonsplit but will be assumed in this paper) asserts that

$$L^2_{\Phi}(\mathrm{SO}(\mathbb{V})) \cong \bigoplus_\eta m_\eta \Sigma_\eta, \tag{6.1}$$

where the direct sum runs over continuous characters η of $S_{\Phi,\mathbb{A}}$ such that $\eta_v \in \hat{S}_{\Phi_v,\mathbb{V}_v}$ for all v and

$$m_\eta = \begin{cases} 1 & \text{if } \Delta^*\eta = \mathbf{1}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $L^2_{\Phi}(\mathrm{SO}(\mathbb{V}))$ is multiplicity-free and $m_{\mathrm{disc}}(\Sigma_\eta) = m_\eta$.

We will use Arthur's multiplicity formula

- to globalize a local representation to a global automorphic representation;
- to extract local information from a global product formula.

To make the argument work, we first globalize a local L-parameter to a global A-parameter suitably. For this, we need the following refinement of the globalization of Sakellaridis–Venkatesh [73, Th. 16.3.2], [36, Cor. A.8], which will be proved in Appendix A below.

Proposition 6.5. *Let S be a nonempty finite set of nonarchimedean places of $\mathbb{F}$ and v_0 a nonarchimedean place of $\mathbb{F}$ such that $v_0 \notin S$. For each $v \in S$, let ϕ_v be a $2n$-dimensional irreducible symplectic representation of $L_{\mathbb{F}_v}$. Then there exists an irreducible symplectic cuspidal automorphic representation Φ of $\mathrm{GL}_{2n}(\mathbb{A})$ such that*

- *$\Phi_v = \phi_v$ for all $v \in S$;*
- *Φ_v is a sum of 1-dimensional representations for all nonarchimedean $v \notin S \cup \{v_0\}$;*
- *$L(\frac{1}{2}, \Phi) \neq 0$.*

Here we regard Φ_v as a $2n$-dimensional representation of $L_{\mathbb{F}_v}$ via the local Langlands correspondence.

We also need the following globalization (noting that any irreducible symplectic representation of $L_{\mathbb{R}}$ is 2-dimensional), which will be proved in Appendix A below as well.

Proposition 6.6. *Assume that $\mathbb{F}$ is totally real. Let S be a finite set of places of $\mathbb{F}$ containing all archimedean places and v_0 a nonarchimedean place of $\mathbb{F}$ such that $v_0 \notin S$. For each $v \in S$, let ϕ_v be a 2-dimensional irreducible symplectic representation of $L_{\mathbb{F}_v}$. Then there exists an irreducible symplectic cuspidal automorphic representation Φ of $\mathrm{GL}_2(\mathbb{A})$ such that*

- *$\Phi_v = \phi_v$ for all $v \in S$;*
- *Φ_v is a sum of 1-dimensional representations for all $v \notin S \cup \{v_0\}$;*
- *$L(\frac{1}{2}, \Phi) \neq 0$.*

Here we regard Φ_v as a 2-dimensional representation of $L_{\mathbb{F}_v}$ via the local Langlands correspondence.

Later, we will use the following consequence of these globalizations.

Corollary 6.7. *Let F be a nonarchimedean local field of characteristic zero or $F = \mathbb{R}$. Let ϕ be a $2n$-dimensional good symplectic representation of L_F.*

(i) *Let $\mathbb{F}$ be a totally imaginary number field (resp. a real quadratic field) if F is nonarchimedean (resp. if $F = \mathbb{R}$) with two distinct places v_0, v_1 of $\mathbb{F}$ such that $\mathbb{F}_{v_0} = \mathbb{F}_{v_1} = F$. Then there exists a generic elliptic A-parameter Φ for Mp_{2n} such that*
   - *$\Phi_{v_0} = \Phi_{v_1} = \phi$;*
   - *the natural maps $S_\Phi \to S_{\Phi_{v_0}}$ and $S_\Phi \to S_{\Phi_{v_1}}$ agree;*
   - *$L(\frac{1}{2}, \Phi) \neq 0$.*

(ii) *Assume that ϕ is reducible. Let $\mathbb{F}$ be a totally imaginary number field (resp. $\mathbb{F} = \mathbb{Q}$) if F is nonarchimedean (resp. if $F = \mathbb{R}$) with a place v_0 of $\mathbb{F}$ such that $\mathbb{F}_{v_0} = F$. Then there exists a generic elliptic A-parameter Φ for Mp_{2n} such that*
   - *$\Phi_{v_0} = \phi$;*
   - *Φ_v is non-good for all $v \neq v_0$;*
   - *the natural map $S_\Phi \to S_{\Phi_{v_0}}$ is surjective;*
   - *the natural map $S_\Phi \to \prod_{v \neq v_0} S_{\Phi_v}$ is injective;*
   - *$L(\frac{1}{2}, \Phi) \neq 0$.*

Proof. Put $S_0 = \{v_0, v_1\}$ in case (i); $S_0 = \{v_0\}$ in case (ii). We may write $\phi = \bigoplus_i \phi_i$ for some (not necessarily distinct) $2n_i$-dimensional irreducible symplectic representations ϕ_i of L_F. For each i, choose a set $S_i = \{v_i, v_i'\}$ consisting of two distinct nonarchimedean places of $\mathbb{F}$ such that

- $S_0 \cap S_i = \varnothing$ for all i;
- $S_i \cap S_j = \varnothing$ for all $i \neq j$.

Since we can always find a $2n_i$-dimensional irreducible symplectic representation of $L_{\mathbb{F}_v}$ for all nonarchimedean v (see, e.g., [74, §2]) and any irreducible representation of $L_{\mathbb{C}}$ is 1-dimensional, it follows from Propositions 6.5 and 6.6 that there exists an irreducible symplectic cuspidal automorphic representation Φ_i of $\mathrm{GL}_{2n_i}(\mathbb{A})$ such that

- $\Phi_{i,v} = \phi_i$ for all $v \in S_0$;
- Φ_{i,v_i} is irreducible;
- $\Phi_{i,v}$ is a sum of 1-dimensional representations for all $v \notin S_0 \cup S_i$;
- $L(\frac{1}{2}, \Phi_i) \neq 0$.

In particular, the Φ_i's are pairwise distinct.

We show that $\Phi = \bigoplus_i \Phi_i$ satisfies the required conditions. In case (i), this is easy. In case (ii), since ϕ is reducible, Φ_v contains a 1-dimensional irreducible summand (which is non-symplectic) and hence is non-good for all $v \neq v_0$. Also, since Φ_{i,v_i} is irreducible but Φ_{j,v_i} is a sum of 1-dimensional representations for all $j \neq i$, the natural map $S_\Phi \to \prod_i S_{\Phi_{v_i}}$ is bijective, so that the natural map $S_\Phi \to \prod_{v \neq v_0} S_{\Phi_v}$ is injective. The remaining conditions can be easily verified. $\square$

6.3. *The case of good L-parameters.* We now prove Propositions 5.4 and 6.1 for any $2n$-dimensional good symplectic representation ϕ of L_F when F is nonarchimedean or $F = \mathbb{R}$. We give a somewhat roundabout argument below, though we could streamline it by using an explicit construction of A-packets by Mœglin [54], [55], [57], [56] and Mœglin–Renard [61], [60], which were not fully available when this paper was first written. Especially, when F is nonarchimedean, we could give a local proof by combining Mœglin's results [54], [55], [57], [56] with a result of Atobe and the first-named author [8] which describes local theta lifts explicitly, but we do not discuss it here.

Put $\phi' = \phi \oplus S_{2r-2n}$ with $r > 2n+1$. For any irreducible representation σ' of SO_{2r+1} and any character η' of $\bar{S}_{\phi'}$, we define the multiplicity $m(\sigma', \eta')$ by

$$m(\sigma', \eta') = \dim \mathrm{Hom}_{\mathrm{SO}_{2r+1}}(\sigma', \sigma_{\eta'}),$$

where $\sigma_{\eta'}$ is the representation in the A-packet $\Pi_{\phi'}(\mathrm{SO}_{2r+1})$ associated to η'.

Lemma 6.8. *Let σ' be an irreducible representation of SO_{2r+1}. Then, for any $\eta' \in \hat{\bar{S}}_{\phi'}$, we have*

$$m(\sigma', \eta') \leq 1,$$

with equality for at most one η'. Namely, $\Pi_{\phi'}(\mathrm{SO}_{2r+1})$ is multiplicity-free.

Proof. The assertion was proved in a more general context by Mœglin [57] when F is nonarchimedean, but it is not fully known when $F = \mathbb{R}$, though there has been progress by Mœglin–Renard [60]. Here we give a proof based on theta lifts.

We may assume that $m(\sigma', \eta') > 0$ for some η'. Let $\mathbb{F}, v_0, v_1$, and $\varPhi$ be as given in Corollary 6.7(i), so that $\varPhi_{v_0} = \varPhi_{v_1} = \phi$. Put $\varPhi' = \varPhi \oplus S_{2r-2n}$. Since $L(\frac{1}{2}, \varPhi) \neq 0$, it follows from Lemma 4.3 that the quadratic character $\epsilon_{\varPhi'}$ of $\bar{S}_{\varPhi'}$ is trivial. We define an abstract irreducible representation $\varSigma' = \bigotimes_v \varSigma'_v$ of $\mathrm{SO}_{2r+1}(\mathbb{A})$ by setting

- $\varSigma'_{v_0} = \varSigma'_{v_1} = \sigma'$;
- $\varSigma'_v$ to be the irreducible representation in the L-packet $\Pi_{\varphi_{\varPhi'_v}}(\mathrm{SO}_{2r+1})$ associated to the trivial character of $S_{\varphi_{\varPhi'_v}}$ if $v \neq v_0, v_1$.

By [6, Prop. 7.4.1], $\varSigma'_v$ is an irreducible summand of the representation in the A-packet $\Pi_{\varPhi'_v}(\mathrm{SO}_{2r+1})$ associated to the trivial character of $\bar{S}_{\varPhi'_v}$ if $v \neq v_0, v_1$. Since the pullback of $\eta' \otimes \eta'$ under the natural map

$$\bar{S}_{\varPhi'} \longrightarrow \bar{S}_{\varPhi'_{v_0}} \times \bar{S}_{\varPhi'_{v_1}} = \bar{S}_{\phi'} \times \bar{S}_{\phi'}$$

is trivial for any $\eta' \in \hat{\bar{S}}_{\phi'}$, we have an embedding

$$\Big(\bigoplus_{\eta' \in \hat{\bar{S}}_{\phi'}} (\sigma_{\eta'} \otimes \sigma_{\eta'}) \Big) \otimes \Big(\bigotimes_{v \neq v_0, v_1} \varSigma'_v \Big) \longrightarrow L^2_{\varPhi'}(\mathrm{SO}_{2r+1})$$

by the multiplicity formula (4.5). In particular,

$$m_{\mathrm{disc}}(\varSigma') \geq \sum_{\eta' \in \hat{\bar{S}}_{\phi'}} m(\sigma', \eta')^2 > 0.$$

We now consider theta lifts. Fix a nontrivial additive character $\varPsi$ of $\mathbb{F} \backslash \mathbb{A}$ such that $\varPsi_{v_0}$ and $\varPsi_{v_1}$ belong to the $(F^\times)^2$-orbit of the fixed nontrivial additive character ψ of F. Let $\varPi = \vartheta^{\mathrm{abs}}_{\varPsi}(\varSigma')$ be the abstract theta lift to $\mathrm{Mp}_{2n}(\mathbb{A})$ relative to $\varPsi$. Since $\varPhi$ is generic, it follows from Proposition 4.1 and Corollary 4.2 that $\varPi$ is an irreducible summand of $L^2_{\varPhi,\varPsi}(\mathrm{Mp}_{2n})$ and that

$$m_{\mathrm{cusp}}(\varPi) = m_{\mathrm{disc}}(\varPi) = m_{\mathrm{disc}}(\varSigma') > 0.$$

For any realization $\mathcal{V} \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{Mp}_{2n})$ of $\varPi$, let $\Theta^{\mathrm{aut}}_{\mathbb{V},\varPsi}(\mathcal{V})$ be the global theta lift to $\mathrm{SO}(\mathbb{V})(\mathbb{A})$ relative to $\varPsi$, where $\mathbb{V}$ is a $(2n+1)$-dimensional quadratic space over $\mathbb{F}$ with trivial discriminant. Since $\varPhi$ is generic, we deduce from the tower property and the local theta correspondence for unramified representations that $\Theta^{\mathrm{aut}}_{\mathbb{V},\varPsi}(\mathcal{V})$ is cuspidal (but possibly zero). Moreover, we have

$$L^S_{\varPsi}(\tfrac{1}{2}, \varPi) = L^S(\tfrac{1}{2}, \varPhi) \neq 0,$$

where S is a sufficiently large finite set of places of $\mathbb{F}$ and $L^S_{\varPsi}(s, \varPi)$ is the partial L-function of $\varPi$ relative to $\varPsi$ and the standard representation of $\mathrm{Sp}_{2n}(\mathbb{C})$.

Hence, by the Rallis inner product formula [43], [23], [93] and the local Shimura correspondence [2], [3], [24], there exists a unique $\mathbb{V}$ such that $\Theta^{\mathrm{aut}}_{\mathbb{V},\Psi}(\mathcal{V})$ is nonzero for any realization $\mathcal{V}$ of Π. For this unique $\mathbb{V}$, let $\Sigma = \theta^{\mathrm{abs}}_{\mathbb{V},\Psi}(\Pi)$ be the abstract theta lift to $\mathrm{SO}(\mathbb{V})(\mathbb{A})$ relative to Ψ. Then, by the multiplicity preservation [17, Prop. 2.6], we have

$$m_{\mathrm{disc}}(\Sigma) \geq m_{\mathrm{cusp}}(\Sigma) = m_{\mathrm{cusp}}(\Pi).$$

Also, it follows form the local theta correspondence for unramified representations that Σ is an irreducible summand of $L^2_{\Phi}(\mathrm{SO}(\mathbb{V}))$. Since Φ is generic, the multiplicity formula (6.1) implies that

$$m_{\mathrm{disc}}(\Sigma) = 1.$$

Thus, combining these (in)equalities, we obtain

$$1 \geq \sum_{\eta' \in \hat{\hat{S}}_{\phi'}} m(\sigma', \eta')^2 > 0.$$

This completes the proof. □

Let $\mathrm{JH}(\Pi_{\phi'}(\mathrm{SO}_{2r+1}))$ be the set of equivalence classes of irreducible representations σ' of SO_{2r+1} such that $m(\sigma', \eta') > 0$ for some η'.

Lemma 6.9. *The theta lift induces an injection*

$$\vartheta_\psi : \mathrm{JH}(\Pi_{\phi'}(\mathrm{SO}_{2r+1})) \hookrightarrow \Pi_{\phi,\psi}(\mathrm{Mp}_{2n}).$$

Proof. We retain the notation of the proof of Lemma 6.8. In particular, for any $\sigma' \in \mathrm{JH}(\Pi_{\phi'}(\mathrm{SO}_{2r+1}))$, we have an irreducible summand Σ' of $L^2_{\Phi'}(\mathrm{SO}_{2r+1})$ such that

- $\Sigma'_{v_0} = \sigma'$;
- $\Pi = \vartheta^{\mathrm{abs}}_{\Psi}(\Sigma')$ is an irreducible summand of $L^2_{\Phi,\Psi}(\mathrm{Mp}_{2n})$;
- $\Sigma = \theta^{\mathrm{abs}}_{\mathbb{V},\Psi}(\Pi)$ is an irreducible summand of $L^2_{\Phi}(\mathrm{SO}(\mathbb{V}))$.

Since the L-parameter of $\Sigma_{v_0} = \theta_{\Psi_{v_0}}(\Pi_{v_0})$ is $\Phi_{v_0} = \phi$, we have

$$\vartheta_\psi(\sigma') = \vartheta_{\Psi_{v_0}}(\Sigma'_{v_0}) = \Pi_{v_0} \in \Pi_{\phi,\psi}(\mathrm{Mp}_{2n}).$$

This and the Howe duality imply the assertion. □

We now show that the map ϑ_ψ in Lemma 6.9 is in fact surjective.

Lemma 6.10. *The theta lift induces a bijection*

$$\vartheta_\psi : \mathrm{JH}(\Pi_{\phi'}(\mathrm{SO}_{2r+1})) \longleftrightarrow \Pi_{\phi,\psi}(\mathrm{Mp}_{2n}).$$

Proof. Let $\eta' \in \hat{\hat{S}}_{\phi'}$, and put $\eta = \iota^*\eta'$. Let Σ_η be the irreducible representation in the L-packet $\Pi_\phi(\mathrm{SO}(V))$ associated to η, where V is the $(2n+1)$-dimensional quadratic space over F with trivial discriminant such that $\eta \in \hat{S}_{\phi,V}$. Let $\mathbb{F}, v_0, v_1$, and Φ be as given in Corollary 6.7(i), so that $\Phi_{v_0} = \Phi_{v_1} = \phi$.

Then there exists a unique $(2n+1)$-dimensional quadratic space $\mathbb{V}$ over $\mathbb{F}$ with trivial discriminant such that

- $\mathbb{V}_{v_0} = \mathbb{V}_{v_1} = V$;
- $\mathbb{V}_v$ is the split space with trivial discriminant for all $v \neq v_0, v_1$.

We define an abstract irreducible representation $\Sigma = \bigotimes_v \Sigma_v$ of $\mathrm{SO}(\mathbb{V})(\mathbb{A})$ by setting

- $\Sigma_{v_0} = \Sigma_{v_1} = \Sigma_\eta$;
- Σ_v to be the irreducible representation in the L-packet $\Pi_{\Phi_v}(\mathrm{SO}(\mathbb{V}))$ associated to the trivial character of S_{Φ_v} if $v \neq v_0, v_1$.

Then, by the multiplicity formula (6.1), Σ is an irreducible summand of $L^2_\Phi(\mathrm{SO}(\mathbb{V}))$. In fact, since Φ is generic, it follows from the argument in the proof of Proposition 4.1 that Σ is cuspidal.

Let $\Pi = \theta^{\mathrm{abs}}_\Psi(\Sigma)$ be the abstract theta lift to $\mathrm{Mp}_{2n}(\mathbb{A})$ (relative to Ψ as in the proof of Lemma 6.8), so that

$$\Pi_{v_0} = \theta_{\Psi_{v_0}}(\Sigma_{v_0}) = \theta_\psi(\Sigma_\eta) = \pi_\eta \in \Pi_{\phi,\psi}(\mathrm{Mp}_{2n}).$$

Since $L(\frac{1}{2}, \Phi) \neq 0$, we can deduce from the argument in the proof of Lemma 6.8 that the global theta lift $\Theta^{\mathrm{aut}}_\Psi(\Sigma)$ to $\mathrm{Mp}_{2n}(\mathbb{A})$ is nonzero and cuspidal, so that $\Theta^{\mathrm{aut}}_\Psi(\Sigma) \cong \Pi$ is an irreducible summand of $L^2_{\Phi,\Psi}(\mathrm{Mp}_{2n})$. Hence, by Corollary 4.2, the abstract theta lift $\Sigma' = \vartheta^{\mathrm{abs}}_\Psi(\Pi)$ to $\mathrm{SO}_{2r+1}(\mathbb{A})$ is an irreducible summand of $L^2_{\Phi'}(\mathrm{SO}_{2r+1})$, where $\Phi' = \Phi \oplus S_{2r-2n}$. Since $\Phi'_{v_0} = \phi'$, this implies that

$$\Sigma'_{v_0} \in \mathrm{JH}(\Pi_{\phi'}(\mathrm{SO}_{2r+1})).$$

On the other hand, we have

$$\vartheta_{\Psi_{v_0}}(\Sigma'_{v_0}) = \Pi_{v_0} = \pi_\eta.$$

Hence the map in Lemma 6.9 is surjective. □

Finally, we show that Propositions 5.4 and 6.1 hold for ϕ. We consider the irreducible case and the reducible case separately.

Lemma 6.11. *Assume that ϕ is irreducible. Then Propositions 5.4 and 6.1 hold for ϕ.*

Proof. Let $\sigma'_{\mathbf{1}}$ be the representation in the A-packet $\Pi_{\phi'}(\mathrm{SO}_{2r+1})$ associated to the trivial character of $\bar{S}_{\phi'}$. Then, by [6, Prop. 7.4.1], $\sigma'_{\mathbf{1}}$ contains the irreducible representation σ' in the L-packet $\Pi_{\varphi_{\phi'}}(\mathrm{SO}_{2r+1})$ associated to the trivial character of $S_{\varphi_{\phi'}}$. Since

$$\#\, \mathrm{JH}(\Pi_{\phi'}(\mathrm{SO}_{2r+1})) = 2$$

by the irreducibility of ϕ and Lemma 6.10, we may write $\mathrm{JH}(\Pi_{\phi'}(\mathrm{SO}_{2r+1})) = \{\sigma', \sigma''\}$ for some irreducible representation σ'' of SO_{2r+1}.

We show that $\sigma'_{\mathbf{1}}$ is irreducible. Suppose on the contrary that $\sigma'_{\mathbf{1}}$ is reducible. By Lemma 6.8, we have $\sigma'_{\mathbf{1}} = \sigma' \oplus \sigma''$. Let $\mathbb{F}, v_0, v_1$, and $\varPhi$ be as given in Corollary 6.7(i), so that $\varPhi_{v_0} = \varPhi_{v_1} = \phi$. Put $\varPhi' = \varPhi \oplus S_{2r-2n}$. Since $L(\frac{1}{2}, \varPhi) \neq 0$, it follows from Lemma 4.3 that the quadratic character $\epsilon_{\varPhi'}$ of $\bar{S}_{\varPhi'}$ is trivial. We define an abstract irreducible representation $\varSigma' = \bigotimes_v \varSigma'_v$ of $\mathrm{SO}_{2r+1}(\mathbb{A})$ by setting

- $\varSigma'_{v_0} = \sigma'$;
- $\varSigma'_{v_1} = \sigma''$;
- $\varSigma'_v$ to be the irreducible representation in the L-packet $\Pi_{\varphi_{\varPhi'_v}}(\mathrm{SO}_{2r+1})$ associated to the trivial character of $S_{\varphi_{\varPhi'_v}}$ if $v \neq v_0, v_1$.

Since $\sigma'_{\mathbf{1}} = \sigma' \oplus \sigma''$, it follows from the multiplicity formula (4.5) that $\varSigma'$ is an irreducible summand of $L^2_{\varPhi'}(\mathrm{SO}_{2r+1})$. Let $\varPi = \vartheta^{\mathrm{abs}}_{\varPsi}(\varSigma')$ be the abstract theta lift to $\mathrm{Mp}_{2n}(\mathbb{A})$ (relative to $\varPsi$ as in the proof of Lemma 6.8). Then, as shown in the proof of Lemma 6.8, there exists a unique $(2n+1)$-dimensional quadratic space $\mathbb{V}$ over $\mathbb{F}$ with trivial discriminant such that the abstract theta lift $\theta^{\mathrm{abs}}_{\mathbb{V},\varPsi}(\varPi)$ to $\mathrm{SO}(\mathbb{V})(\mathbb{A})$ is nonzero. On the other hand, if $v \neq v_0, v_1$ (so that v is not real), then by Lemma 6.4, $\varPi_v = \vartheta_{\varPsi_v}(\varSigma'_v)$ is the irreducible representation in the L-packet $\Pi_{\varPhi_v,\varPsi_v}(\mathrm{Mp}_{2n})$ associated to the trivial character of $S_{\varPhi_v}$, so that

$$\varepsilon(\mathbb{V}_v) = 1.$$

Also, by Lemma 6.10, we have

$$\{\varPi_{v_0}, \varPi_{v_1}\} = \{\vartheta_{\varPsi_{v_0}}(\varSigma'_{v_0}), \vartheta_{\varPsi_{v_1}}(\varSigma'_{v_1})\} = \{\vartheta_\psi(\sigma'), \vartheta_\psi(\sigma'')\} = \Pi_{\phi,\psi}(\mathrm{Mp}_{2n}),$$

so that

$$\varepsilon(\mathbb{V}_{v_0}) \cdot \varepsilon(\mathbb{V}_{v_1}) = -1.$$

This contradicts the fact that $\prod_v \varepsilon(\mathbb{V}_v) = 1$. Hence $\sigma'_{\mathbf{1}}$ is irreducible and $\sigma'_{\mathbf{1}} = \sigma'$.

By Lemmas 6.8 and 6.10, it remains to show that $\vartheta_\psi(\sigma')$ is associated to the trivial character of S_ϕ. If F is nonarchimedean, then this follows from Lemma 6.4. Suppose that $F = \mathbb{R}$ (so that $n = 1$) and that $\vartheta_\psi(\sigma')$ is associated to the nontrivial character of S_ϕ. From the above argument with the following modifications,

- $\mathbb{F} = \mathbb{Q}$,
- $\varPhi$ is a generic elliptic A-parameter for Mp_2 such that $\varPhi_\infty = \phi$ and $L(\frac{1}{2}, \varPhi) \neq 0$ (see Proposition 6.6),
- $\varSigma'_\infty = \sigma'$,
- $\varSigma'_v$ is associated to the trivial character of $S_{\varphi_{\varPhi'_v}}$ if v is nonarchimedean,

we can deduce that there exists a 3-dimensional quadratic space $\mathbb{V}$ over $\mathbb{F}$ with trivial discriminant such that $\varepsilon(\mathbb{V}_v) = 1$ for all nonarchimedean v but $\varepsilon(\mathbb{V}_\infty) = -1$. This is a contradiction and completes the proof. □

LEMMA 6.12. *Assume that ϕ is reducible. Then Propositions 5.4 and 6.1 hold for ϕ.*

Proof. Let $\eta' \in \hat{\bar{S}}_{\phi'}$, and put $\eta = \iota^* \eta'$. Let $\mathbb{F}, v_0$, and Φ be as given in Corollary 6.7(ii), so that $\Phi_{v_0} = \phi$. Since the natural map $(\prod_{v \neq v_0} S_{\Phi_v})\hat{} \to \hat{S}_\Phi$ is surjective, there exists a continuous character $\bigotimes_v \xi_v$ of $S_{\Phi,\mathbb{A}}$ such that

$$\xi_{v_0} = \eta, \tag{6.2}$$

$$\Big(\bigotimes_v \xi_v \Big) \circ \Delta = \mathbf{1}. \tag{6.3}$$

Then there exists a unique $(2n+1)$-dimensional quadratic space $\mathbb{V}$ over $\mathbb{F}$ with trivial discriminant such that $\xi_v \in \hat{S}_{\Phi_v, \mathbb{V}_v}$ for all v. We define an abstract irreducible representation $\Sigma = \bigotimes_v \Sigma_v$ of $\mathrm{SO}(\mathbb{V})(\mathbb{A})$ by setting

$$\Sigma_v = \Sigma_{\xi_v} \in \Pi_{\Phi_v}(\mathrm{SO}(\mathbb{V}))$$

for all v. Then, by the multiplicity formula (6.1), Σ is an irreducible summand of $L^2_\Phi(\mathrm{SO}(\mathbb{V}))$.

Let $\Pi = \theta^{\mathrm{abs}}_\Psi(\Sigma)$ be the abstract theta lift to $\mathrm{Mp}_{2n}(\mathbb{A})$ (relative to a fixed nontrivial additive character Ψ of $\mathbb{F}\backslash\mathbb{A}$ such that Ψ_{v_0} belongs to the $(F^\times)^2$-orbit of ψ), so that

$$\Pi_v = \theta_{\Psi_v}(\Sigma_v) = \theta_{\Psi_v}(\Sigma_{\xi_v}) = \pi_{\xi_v} \in \Pi_{\Phi_v, \Psi_v}(\mathrm{Mp}_{2n})$$

for all v. Then, as shown in the proof of Lemma 6.10, the abstract theta lift $\Sigma' = \vartheta^{\mathrm{abs}}_\Psi(\Pi)$ to $\mathrm{SO}_{2r+1}(\mathbb{A})$ is an irreducible summand of $L^2_{\Phi'}(\mathrm{SO}_{2r+1})$, where $\Phi' = \Phi \oplus S_{2r-2n}$. This implies that for any v, Σ'_v is an irreducible summand of $\sigma_{\xi'_v} \in \Pi_{\Phi'_v}(\mathrm{SO}_{2r+1})$ for some $\xi'_v \in \hat{\bar{S}}_{\Phi'_v}$. By Lemma 6.8, such ξ'_v is unique. In fact, if $v \neq v_0$ (so that v is not real), then since Φ_v is non-good, we can apply Propositions 5.4 and 6.1 to obtain $\Sigma'_v = \sigma_{\xi'_v}$ with

$$\iota_v^* \xi'_v = \xi_v, \tag{6.4}$$

where $\iota_v : S_{\Phi_v} \to \bar{S}_{\Phi'_v}$ is the natural isomorphism. On the other hand, since $L(\frac{1}{2}, \Phi) \neq 0$, it follows from Lemma 4.3 that the quadratic character $\epsilon_{\Phi'}$ of $\bar{S}_{\Phi'}$ is trivial. Hence, by the multiplicity formula (4.5), we must have

$$\Big(\bigotimes_v \xi'_v \Big) \circ \Delta = \mathbf{1}. \tag{6.5}$$

Thus, since the natural map $\hat{S}_{\Phi_{v_0}} \to \hat{S}_\Phi$ is injective, we deduce from (6.3), (6.4) and (6.5) that

$$\iota_{v_0}^* \xi'_{v_0} = \xi_{v_0}.$$

Hence we have $\iota_{v_0}^* \xi'_{v_0} = \eta$ by (6.2), so that

$$\xi'_{v_0} = \eta' \tag{6.6}$$

by the definition of η. In particular, $\sigma_{\eta'} \in \Pi_{\phi'}(\mathrm{SO}_{2r+1})$ is nonzero for any $\eta' \in \hat{\bar{S}}_{\phi'}$. By Lemmas 6.8 and 6.10, this implies that $\sigma_{\eta'}$ is irreducible for any η'. Moreover, by (6.6), we have

$$\vartheta_{\psi}(\sigma_{\eta'}) = \vartheta_{\Psi_{v_0}}(\Sigma'_{v_0}) = \Pi_{v_0} = \pi_{\eta}.$$

This completes the proof. □

This completes the proof of Propositions 5.4 and 6.1 and hence of Theorem 1.4.

Appendix A. Globalizations

In this appendix, we prove Propositions 6.5 and 6.6.

A.1. *Proof of Proposition* 6.5. Let F be a number field and $\mathbb{A}$ the adèle ring of F. Let S_∞ be the set of archimedean places of F. Let S be a nonempty finite set of nonarchimedean places of F and v_0 a nonarchimedean place of F such that $v_0 \notin S$. Then Proposition 6.5 is an immediate consequence of the following:

Proposition A.1. *For each $v \in S \cup \{v_0\}$, let τ_v be an irreducible square-integrable representation of $\mathrm{GL}_{2n}(F_v)$ such that $L(s, \tau_v, \wedge^2)$ has a pole at $s = 0$. Assume that τ_{v_0} is supercuspidal. Then there exists an irreducible cuspidal automorphic representation $\mathcal{T}$ of $\mathrm{GL}_{2n}(\mathbb{A})$ such that*

- *$\mathcal{T}_v = \tau_v$ for all $v \in S \cup \{v_0\}$;*
- *$\mathcal{T}_v$ is a principal series representation for all $v \notin S_\infty \cup S \cup \{v_0\}$;*
- *$L(s, \mathcal{T}, \wedge^2)$ has a pole at $s = 1$;*
- *$L(\frac{1}{2}, \mathcal{T}) \neq 0$.*

To prove this proposition, we globalize a generic representation of $\mathrm{Mp}_{2n}(F_v)$ to a globally generic automorphic representation of $\mathrm{Mp}_{2n}(\mathbb{A})$. More precisely, let N be the standard maximal unipotent subgroup of Sp_{2n}. We regard $N(\mathbb{A})$ as a subgroup of $\mathrm{Mp}_{2n}(\mathbb{A})$ via the canonical splitting. Fix a nontrivial additive character ψ of $F\backslash\mathbb{A}$. As in [19, §12], ψ gives rise to a generic character of $N(F)\backslash N(\mathbb{A})$, which we denote again by ψ.

Proposition A.2. *For each $v \in S \cup \{v_0\}$, let π_v be an irreducible genuine ψ_v-generic square-integrable representation of $\mathrm{Mp}_{2n}(F_v)$. Assume that π_{v_0} is supercuspidal and that if $n = 1$, then π_{v_0} is not the odd Weil representation relative to ψ_{v_0}. Then there exists an irreducible genuine globally ψ-generic cuspidal automorphic representation Π of $\mathrm{Mp}_{2n}(\mathbb{A})$ such that*

- *$\Pi_v = \pi_v$ for all $v \in S \cup \{v_0\}$;*
- *Π_v is a principal series representation for all $v \notin S_\infty \cup S \cup \{v_0\}$.*

Proposition A.1 can be easily deduced from Proposition A.2 and [36, Prop. 4.3]. We include the proof for the convenience of the reader.

Proof of Proposition A.1. For each $v \in S \cup \{v_0\}$, let π_v be the descent of τ_v to $\mathrm{Mp}_{2n}(F_v)$ relative to ψ_v; see [27], [28], [36, Th. 3.1]. Then π_v satisfies the conditions in Proposition A.2. Let $\mathit{\Pi}$ be as given in Proposition A.2. Let $\mathit{\Sigma} = \Theta_\psi^{\mathrm{aut}}(\mathit{\Pi})$ be the global theta lift to $\mathrm{SO}_{2n+1}(\mathbb{A})$ relative to ψ. Since $\theta_{\psi_{v_0}}(\pi_{v_0})$ is supercuspidal by [41, Th. 2.2], $\mathit{\Sigma}$ is cuspidal. By [15], $\mathit{\Sigma}$ is nonzero and globally generic. Hence $\mathit{\Sigma}$ is irreducible, and by the Rallis inner product formula [43], [23], [93], we have

$$L_\psi(\tfrac{1}{2}, \mathit{\Pi}) \neq 0,$$

where $L_\psi(s, \mathit{\Pi})$ is the L-function of $\mathit{\Pi}$ relative to ψ and the standard representation of $\mathrm{Sp}_{2n}(\mathbb{C})$. Moreover, it follows from the local Shimura correspondence [24] that

- $\mathit{\Sigma}_v$ is square-integrable for all $v \in S \cup \{v_0\}$;
- $\mathit{\Sigma}_v$ is a principal series representation for all $v \notin S_\infty \cup S \cup \{v_0\}$.

We now take $\mathcal{T}$ to be the functorial lift of $\mathit{\Sigma}$ to $\mathrm{GL}_{2n}(\mathbb{A})$. Then $L(s, \mathcal{T}, \wedge^2)$ has a pole at $s = 1$, and we have

$$L(\tfrac{1}{2}, \mathcal{T}) = L(\tfrac{1}{2}, \mathit{\Sigma}) = L_\psi(\tfrac{1}{2}, \mathit{\Pi}) \neq 0.$$

Also, since $\mathcal{T}_v$ is the functorial lift of $\mathit{\Sigma}_v$, it is a principal series representation for all $v \notin S_\infty \cup S \cup \{v_0\}$. Finally, by [36, Prop. 4.3], we have

$$\mathcal{T}_v = \tau_v$$

for all $v \in S \cup \{v_0\}$. In particular, $\mathcal{T}_{v_0}$ is supercuspidal and hence $\mathcal{T}$ is cuspidal. □

It remains to prove Proposition A.2, which is a refinement of the globalization of Sakellaridis–Venkatesh [73, Th. 16.3.2], [36, Cor. A.8]. We need to modify their argument to control the localization $\mathit{\Pi}_v$ at nonarchimedean v outside $S \cup \{v_0\}$.

Proof of Proposition A.2. We first introduce some notation. By abuse of notation, we write

$$G_v = \mathrm{Mp}_{2n}(F_v), \qquad G_S = \mathrm{Mp}_{2n}(F_S), \qquad G(\mathbb{A}) = \mathrm{Mp}_{2n}(\mathbb{A}),$$

where $F_S = \prod_{v \in S} F_v$. If F_v is nonarchimedean of odd residual characteristic, we regard $K_v = \mathrm{Sp}_{2n}(\mathcal{O}_v)$ as a subgroup of G_v via the standard splitting, where $\mathcal{O}_v$ is the integer ring of F_v. Also, we regard $G(F) = \mathrm{Sp}_{2n}(F)$ as a subgroup of $G(\mathbb{A})$ via the canonical splitting. Put

$$N_S = N(F_S), \qquad \psi_S = \bigotimes_{v \in S} \psi_v, \qquad \pi_S = \bigotimes_{v \in S} \pi_v.$$

Let $C_c^\infty(G_v)$ be the space of genuine smooth functions on G_v with compact support. Let $C_c^\infty(N_v\backslash G_v, \psi_v)$ be the space of genuine smooth functions f on G_v such that

- $\operatorname{supp} f$ is compact modulo N_v;
- $f(xg) = \psi_v(x) f(g)$ for all $x \in N_v$ and $g \in G_v$.

Then we have a map $\mathcal{P}_v : C_c^\infty(G_v) \to C_c^\infty(N_v\backslash G_v, \psi_v)$ defined by

$$(\mathcal{P}_v \tilde{f})(g) = \int_{N_v} \tilde{f}(xg)\overline{\psi_v(x)}\, dx.$$

Also, $C_c^\infty(N_v\backslash G_v, \psi_v)$ is equipped with a natural hermitian inner product $\langle\cdot,\cdot\rangle$. Let $C_c^\infty(N_S\backslash G_S, \psi_S)$ be defined similarly. For any automorphic form φ on $G(\mathbb{A})$, we define its Whittaker–Fourier coefficient $\mathcal{W}_\varphi$ by

$$\mathcal{W}_\varphi(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(xg)\overline{\psi(x)}\, dx.$$

Let $L^2(G)$ be the genuine part of $L^2(G(F)\backslash G(\mathbb{A}))$, which is equipped with the Petersson inner product $\langle\cdot,\cdot\rangle$. Let $L^2_{\mathrm{cusp}}(G)$ be the closure in $L^2(G)$ of the subspace of genuine cusp forms on $G(\mathbb{A})$. We define $L^2_{\mathrm{cusp},\psi\text{-gen}}(G)$ as the orthogonal complement in $L^2_{\mathrm{cusp}}(G)$ of the closure of the subspace of genuine cusp forms φ on $G(\mathbb{A})$ such that $\mathcal{W}_\varphi = 0$. Then, by the uniqueness of Whittaker models, $L^2_{\mathrm{cusp},\psi\text{-gen}}(G)$ is multiplicity-free. Fix a finite set S_0 of nonarchimedean places of F such that

- $S \cap S_0 = \varnothing$;
- $v_0 \notin S_0$;
- F_v is of odd residual characteristic and ψ_v is of order zero if $v \notin S_\infty \cup S \cup \{v_0\} \cup S_0$.

Let $\{\varPi_i\}$ be the set of irreducible summands of $L^2_{\mathrm{cusp},\psi\text{-gen}}(G)$ such that

- $\varPi_{i,v_0} = \pi_{v_0}$;
- $\varPi_{i,v}$ is a principal series representation for all $v \in S_0$;
- $\varPi_{i,v}$ is unramified (i.e., $\varPi_{i,v}$ has a nonzero K_v-fixed vector) for all $v \notin S_\infty \cup S \cup \{v_0\} \cup S_0$.

As in [73, §16.4], we show that π_S is weakly contained in the Hilbert space direct sum $\bigoplus_i \varPi_i$ (regarded as a representation of G_S). Since π_S belongs to the support of the Plancherel measure for $L^2(N_S\backslash G_S, \psi_S)$, it suffices to prove the following: for any compact subset $\Omega \subset G_S$ and any $f_S \in C_c^\infty(N_S\backslash G_S, \psi_S)$ with $\langle f_S, f_S\rangle = 1$, there exist $c_i \in \mathbb{R}$ with $c_i \geq 0$ and $\varphi_i \in \varPi_i$ with $\langle \varphi_i, \varphi_i\rangle = 1$ such that $\sum_i c_i = 1$ and

$$\langle R_y f_S, f_S\rangle = \sum_i c_i \langle R_y \varphi_i, \varphi_i\rangle$$

for all $y \in \Omega$. Here R_y is the right translation by y. For each $v \notin S_\infty \cup S$, we choose $f_v = \mathcal{P}_v \tilde{f}_v \in C_c^\infty(N_v\backslash G_v, \psi_v)$, where $\tilde{f}_v \in C_c^\infty(G_v)$ is given as follows:

- if $v = v_0$, then $\tilde{f}_{v_0}$ is a matrix coefficient of π_{v_0};
- if $v \in S_0$, then $\tilde{f}_v$ belongs to the wave packet associated to a Bernstein component consisting only of irreducible principal series representations (e.g., one containing a sufficiently ramified principal series representation);
- if $v \notin S_\infty \cup S \cup \{v_0\} \cup S_0$, then

$$\tilde{f}_v(g) = \begin{cases} \epsilon & \text{if } g \in \epsilon \cdot K_v \text{ with } \epsilon \in \{\pm 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

By [70, Lemma 4.4], we may further assume that $\langle f_v, f_v \rangle = 1$. For each $v \in S_\infty$, we choose $f_v \in C_c^\infty(N_v \backslash G_v, \psi_v)$ later. Put

$$f = f_S \otimes \Big(\bigotimes_{v \notin S} f_v \Big).$$

We define a Poincaré series P_f on $G(\mathbb{A})$ by

$$P_f(g) = \sum_{\gamma \in N(F) \backslash G(F)} f(\gamma g),$$

where the sum converges absolutely. Then we have

$$\begin{aligned} \langle P_f, P_f \rangle &\le \int_{N(F)\backslash G(\mathbb{A})} |f(g) P_f(g)| \, dg \\ &\le \int_{N(\mathbb{A})\backslash G(\mathbb{A})} |f(g)| \int_{N(F)\backslash N(\mathbb{A})} |P_f(xg)| \, dx \, dg < \infty, \end{aligned}$$

so that P_f is square-integrable over $G(F)\backslash G(\mathbb{A})$. Since π_{v_0} is supercuspidal, P_f is cuspidal by the condition on f_{v_0}. For any genuine cusp form φ on $G(\mathbb{A})$ such that $\mathcal{W}_\varphi = 0$, we have

$$\begin{aligned} \langle P_f, \varphi \rangle &= \int_{N(F)\backslash G(\mathbb{A})} f(g) \overline{\varphi(g)} \, dg \\ &= \int_{N(\mathbb{A})\backslash G(\mathbb{A})} f(g) \overline{\mathcal{W}_\varphi(g)} \, dg = 0. \end{aligned}$$

Hence $P_f \in L^2_{\mathrm{cusp},\psi\text{-gen}}(G)$. Moreover, by the conditions on f_v, we have $P_f \in \bigoplus_i \mathit{\Pi}_i$. Thus, as in [73, (16.8)], it remains to prove the following: there exists $f_v \in C_c^\infty(N_v \backslash G_v, \psi_v)$ for $v \in S_\infty$ such that

$$\langle R_y f_S, f_S \rangle = \langle R_y P_f, P_f \rangle$$

for all $y \in \Omega$. This was proved in [73, §16.4], but we include the proof for the convenience of the reader. We choose $f_v \in C_c^\infty(N_v \backslash G_v, \psi_v)$ for $v \in S_\infty$ such that $\langle f_v, f_v \rangle = 1$ and $\operatorname{supp} f_v$ is sufficiently small, so that there exists a compact subset $U \subset G(\mathbb{A})$ satisfying the following conditions:

- $U^{-1} = U$;
- $\Omega \subset U$;
- $\operatorname{supp} R_y f \subset N(\mathbb{A}) \cdot U$ for all $y \in \Omega$;

- $N(\mathbb{A}) \subset N(F) \cdot U$;
- $G(F) \cap U_4 \cdot N(\mathbb{A}) = N(F)$, where $U_4 = U \cdot U \cdot U \cdot U$.

For any $y \in \Omega$, we have

$$\begin{aligned}\langle R_y P_f, P_f \rangle &= \int_{N(F)\backslash G(\mathbb{A})} f(gy)\overline{P_f(g)}\, dg \\ &= \int_{N(F)\backslash G(\mathbb{A})} \sum_{\gamma \in N(F)\backslash G(F)} f(gy)\overline{f(\gamma g)}\, dg.\end{aligned}$$

Let $\gamma \in G(F)$. If $f(gy)\overline{f(\gamma g)} \neq 0$ for some $g \in G(\mathbb{A})$, then $gy \in N(\mathbb{A}) \cdot U$ and $\gamma g \in N(\mathbb{A}) \cdot U \subset N(F) \cdot U \cdot U$. Replacing γ by an element in $N(F) \cdot \gamma$ if necessary, we may assume that $\gamma g \in U \cdot U$. Then we have $\gamma = \gamma g \cdot y \cdot (gy)^{-1} \in U_4 \cdot N(\mathbb{A})$, so that $\gamma \in N(F)$. Hence

$$\langle R_y P_f, P_f \rangle = \int_{N(F)\backslash G(\mathbb{A})} f(gy)\overline{f(g)}\, dg = \langle R_y f, f \rangle = \langle R_y f_S, f_S \rangle.$$

Thus, we have shown that π_S is weakly contained in $\bigoplus_i \Pi_i$. As shown in the proof of Proposition A.1, the global theta lift $\Theta_\psi^{\mathrm{aut}}(\Pi_i)$ to $\mathrm{SO}_{2n+1}(\mathbb{A})$ is cuspidal. Hence, by [36, Prop. A.7], we have

$$\{\Pi_{i,S}\} \subset \mathrm{Irr}_{\mathrm{unit}, \psi_S\text{-gen}, \leq c}\, G_S$$

for some $0 \leq c < \frac{1}{2}$, where $\Pi_{i,S} = \bigotimes_{v \in S} \Pi_{i,v}$ and $\mathrm{Irr}_{\mathrm{unit}, \psi_S\text{-gen}, \leq c}\, G_S$ is the set of equivalence classes of irreducible genuine ψ_S-generic unitary representations of G_S whose exponents are bounded by c; see [36, p. 1328]. On the other hand, it follows from the analog of [36, Lemma A.2] for Mp_{2n} that π_S is isolated in $\mathrm{Irr}_{\mathrm{unit}, \psi_S\text{-gen}, \leq c}\, G_S$ with respect to the Fell topology. This forces $\pi_S = \Pi_{i,S}$ for some i and completes the proof, noting that $\Pi_{i,v}$ is an irreducible principal series representation as in Remark 5.3 for all $v \notin S_\infty \cup S \cup \{v_0\} \cup S_0$. □

A.2. *Proof of Proposition 6.6*. Let F be a totally real number field and $\mathbb{A}$ the adèle ring of F. Let S_∞ be the set of archimedean places of F. Let S be a finite set of places of F containing S_∞ and v_0 a nonarchimedean place of F such that $v_0 \notin S$. Then Proposition 6.6 is an immediate consequence of the following:

PROPOSITION A.3. *For each $v \in S$, let τ_v be an irreducible square-integrable representation of $\mathrm{GL}_2(F_v)$ with trivial central character. Then there exists an irreducible cuspidal automorphic representation $\mathcal{T}$ of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character such that*

- *$\mathcal{T}_v = \tau_v$ for all $v \in S$;*
- *$\mathcal{T}_v$ is a principal series representation for all $v \notin S \cup \{v_0\}$;*
- *$L(\frac{1}{2}, \mathcal{T}) \neq 0$.*

This proposition can be deduced from Waldspurger's result [91, Th. 4] on the nonvanishing of central L-values. (See also [14], in which Friedberg–Hoffstein gave a completely different proof.) Nevertheless, we give a more direct proof based on torus periods, so that the proof of the main result of this paper will be independent of Waldspurger's nonvanishing result.

We write

$$S = S_\infty \cup S_1 \cup S_2,$$

where S_1 (resp. S_2) is the set of nonarchimedean places v in S such that τ_v is not supercuspidal (resp. τ_v is supercuspidal). Put

$$S_0 = \begin{cases} S_\infty \cup S_1 & \text{if } \#(S_\infty \cup S_1) \text{ is even,} \\ S_\infty \cup S_1 \cup \{v_0\} & \text{if } \#(S_\infty \cup S_1) \text{ is odd.} \end{cases}$$

Let D be the quaternion division algebra over F which is ramified precisely at the places in S_0. For each $v \in S$, let π_v be the Jacquet–Langlands transfer of τ_v to $D_v^\times$. More explicitly, we have

- if $v \in S_\infty$, then π_v is odd-dimensional;
- if $v \in S_1$, then $\pi_v = \chi_v \circ \nu_v$ for some quadratic character χ_v of $F_v^\times$, where ν_v is the reduced norm on D_v;
- if $v \in S_2$, then π_v is supercuspidal.

Choose a totally imaginary quadratic extension E of F such that

- if $v \in S_1$ and χ_v is unramified, then E_v is the unramified quadratic extension of F_v;
- if $v \in S_1$ and χ_v is ramified, then E_v is the quadratic extension of F_v associated to χ_v by class field theory;
- if $v \in S_2$, then $E_v = F_v \times F_v$;
- if $v = v_0$, then $E_v \neq F_v \times F_v$.

Then E can be embedded into D. Fix an embedding $E \hookrightarrow D$.

We consider algebraic groups

$$G = D^\times / F^\times, \qquad H = E^\times / F^\times$$

defined over F, and regard H as a subgroup of G. Note that G is anisotropic over F and G_v is compact for all $v \in S_0$. Then, for any $v \in S$, π_v (regarded as a representation of G_v) is H-distinguished, i.e.,

$$\mathrm{Hom}_{H_v}(\pi_v, \mathbb{C}) \neq 0. \tag{A.1}$$

We globalize these distinguished representations to a globally distinguished automorphic representation.

Proposition A.4. *There exists an irreducible automorphic representation $\varPi$ of $G(\mathbb{A})$ such that*

- *$\varPi_v = \pi_v$ for all $v \in S$;*

- *Π_v is a principal series representation for all $v \notin S \cup \{v_0\}$;*
- *Π is not 1-dimensional;*
- *Π is globally H-distinguished, i.e.,*

$$\int_{H(F)\backslash H(\mathbb{A})} \varphi(h)\, dh \neq 0$$

for some $\varphi \in \Pi$.

Proposition A.3 can be easily deduced from Proposition A.4 and Waldspurger's result [89] on torus periods. Indeed, let Π be as given in Proposition A.4 and $\mathcal{T}$ the Jacquet–Langlands transfer of Π to $\mathrm{GL}_2(\mathbb{A})$. Since Π is not 1-dimensional, $\mathcal{T}$ is cuspidal. Moreover, since Π is globally H-distinguished, it follows from Waldspurger's result [89, Th. 2] (see also [38], in which Jacquet gave a new proof based on relative trace formulas) that

$$L(\tfrac{1}{2}, \mathcal{T}) \cdot L(\tfrac{1}{2}, \mathcal{T} \times \chi_E) \neq 0,$$

where χ_E is the quadratic automorphic character of $\mathbb{A}^\times$ associated to E/F by class field theory. Hence $\mathcal{T}$ satisfies the required conditions.

Proposition A.4 follows from the globalization of Prasad–Schulze-Pillot [70, Th. 4.1]. We include the proof for the convenience of the reader.

Proof of Proposition A.4. We first introduce some notation. If $v \notin S_0$, we denote by $K_v = \mathrm{PGL}_2(\mathcal{O}_v)$ the standard maximal compact subgroup of $G_v = \mathrm{PGL}_2(F_v)$, where $\mathcal{O}_v$ is the integer ring of F_v. Let $C_c^\infty(G_v)$ and $C_c^\infty(H_v\backslash G_v)$ be the spaces of smooth functions on G_v and $H_v\backslash G_v$ with compact support, respectively. Then we have a map $\mathcal{P}_v : C_c^\infty(G_v) \to C_c^\infty(H_v\backslash G_v)$ defined by

$$(\mathcal{P}_v \tilde{f})(g) = \int_{H_v} \tilde{f}(hg)\, dh.$$

Put $L^2(G) = L^2(G(F)\backslash G(\mathbb{A}))$, which is equipped with the Petersson inner product $\langle \cdot, \cdot \rangle$. We define $L^2_0(G)$ as the orthogonal complement in $L^2(G)$ of the closure of the subspace spanned by all automorphic characters of $G(\mathbb{A})$.

For each $v \neq v_0$, we choose $f_v = \mathcal{P}_v \tilde{f}_v \in C_c^\infty(H_v\backslash G_v)$, where $\tilde{f}_v \in C_c^\infty(G_v)$ is given as follows:

- if $v \in S$, then $\tilde{f}_v$ is a matrix coefficient of π_v;
- if $v \notin S \cup \{v_0\}$, then $\tilde{f}_v$ is the characteristic function of K_v.

By (A.1) and [70, Lemma 4.4] (which continues to hold for $v \in S_\infty$), we may further assume that $f_v(1) \neq 0$. For $v = v_0$, we choose $f_{v_0} \in C_c^\infty(H_{v_0}\backslash G_{v_0})$ later. Put $f = \bigotimes_v f_v$. We define a Poincaré series P_f on $G(\mathbb{A})$ by

$$P_f(g) = \sum_{\gamma \in H(F)\backslash G(F)} f(\gamma g),$$

where the sum converges absolutely. Obviously, $P_f \in L^2(G)$. Also, for any automorphic form φ on $G(\mathbb{A})$, we have
(A.2)
$$\langle P_f, \varphi\rangle = \int_{H(F)\backslash G(\mathbb{A})} f(g)\overline{\varphi(g)}\,dg = \int_{H(\mathbb{A})\backslash G(\mathbb{A})} f(g)\int_{H(F)\backslash H(\mathbb{A})} \overline{\varphi(hg)}\,dh\,dg.$$

We choose $f_{v_0} \in C_c^\infty(H_{v_0}\backslash G_{v_0})$ such that $\operatorname{supp} f_{v_0}$ is sufficiently small, so that

- $\operatorname{supp} f \cap G(F) \subset H(F)$;
- $\operatorname{supp} f_{v_0} \subset H_{v_0} \cdot \ker\mu$ for all characters μ of G_{v_0},

and such that $f_{v_0}(1) \neq 0$ and
$$\int_{H_{v_0}\backslash G_{v_0}} f_{v_0}(g)\,dg = 0.$$

Then we have
$$P_f(1) = f(1) \neq 0.$$

Let μ be an automorphic character of $G(\mathbb{A})$. By (A.2), we have $\langle P_f, \mu\rangle = 0$ unless μ is trivial on $H(\mathbb{A})$, in which case we have
$$\begin{aligned}\langle P_f, \mu\rangle &= \operatorname{vol}(H(F)\backslash H(\mathbb{A})) \cdot \int_{H(\mathbb{A})\backslash G(\mathbb{A})} f(g)\overline{\mu(g)}\,dg \\ &= \operatorname{vol}(H(F)\backslash H(\mathbb{A})) \cdot \prod_v \int_{H_v\backslash G_v} f_v(g_v)\overline{\mu_v(g_v)}\,dg_v = 0\end{aligned}$$
as well since
$$\int_{H_{v_0}\backslash G_{v_0}} f_{v_0}(g)\overline{\mu_{v_0}(g)}\,dg = \int_{H_{v_0}\backslash G_{v_0}} f_{v_0}(g)\,dg = 0.$$

Hence we have $P_f \in L_0^2(G)$.

We now take $\mathit{\Pi}$ to be any irreducible automorphic representation of $G(\mathbb{A})$ such that $\langle P_f, \varphi\rangle \neq 0$ for some $\varphi \in \mathit{\Pi}$. In particular, $\mathit{\Pi}$ is not 1-dimensional. Moreover, by (A.2), we have
$$\int_{H(F)\backslash H(\mathbb{A})} \varphi(hg)\,dh \neq 0$$
for some $g \in G(\mathbb{A})$, so that $\mathit{\Pi}$ is globally H-distinguished. Finally, by the conditions on f_v, we have

- $\mathit{\Pi}_v = \pi_v$ for all $v \in S$;
- $\mathit{\Pi}_v$ is unramified (i.e., $\mathit{\Pi}_v$ has a nonzero K_v-fixed vector) for all $v \notin S \cup \{v_0\}$.

Hence $\mathit{\Pi}$ satisfies the required conditions. $\square$

References

[1] J. Adams, Lifting of characters on orthogonal and metaplectic groups, *Duke Math. J.* **92** no. 1 (1998), 129–178. MR 1609329. Zbl 0983.11025. https://doi.org/10.1215/S0012-7094-98-09203-1.

[2] J. Adams and D. Barbasch, Reductive dual pair correspondence for complex groups, *J. Funct. Anal.* **132** no. 1 (1995), 1–42. MR 1346217. Zbl 0841.22014. https://doi.org/10.1006/jfan.1995.1099.

[3] J. Adams and D. Barbasch, Genuine representations of the metaplectic group, *Compositio Math.* **113** no. 1 (1998), 23–66. MR 1638210. Zbl 0913.11022. https://doi.org/10.1023/A:1000450504919.

[4] N. Arancibia, C. Mœglin, and D. Renard, Paquets d'Arthur des groupes classiques et unitaires,. arXiv 1507.01432.

[5] J. Arthur, The embedded eigenvalue problem for classical groups, in *On Certain L-Functions*, *Clay Math. Proc.* **13**, Amer. Math. Soc., Providence, RI, 2011, pp. 19–31. MR 2767508. Zbl 1223.22017.

[6] J. Arthur, *The Endoscopic Classification of Representations*: *Orthogonal and Symplectic Groups*, *Amer. Math. Soc. Colloq. Publ.* **61**, Amer. Math. Soc., Providence, RI, 2013. MR 3135650. Zbl 1310.22014. https://doi.org/10.1090/coll/061.

[7] H. Atobe, The local theta correspondence and the local Gan-Gross-Prasad conjecture for the symplectic-metaplectic case, *Math. Ann.* **371** no. 1-2 (2018), 225–295. MR 3788848. Zbl 06871952. https://doi.org/10.1007/s00208-017-1620-5.

[8] H. Atobe and W. T. Gan, Local theta correspondence of tempered representations and Langlands parameters, *Invent. Math.* **210** no. 2 (2017), 341–415. MR 3714507. Zbl 06805839. https://doi.org/10.1007/s00222-017-0730-8.

[9] D. Ban and C. Jantzen, The Langlands quotient theorem for finite central extensions of p-adic groups, *Glas. Mat. Ser. III* **48(68)** no. 2 (2013), 313–334. MR 3151110. Zbl 1304.22019. https://doi.org/10.3336/gm.48.2.07.

[10] N. Bergeron and L. Clozel, Quelques conséquences des travaux d'Arthur pour le spectre et la topologie des variétés hyperboliques, *Invent. Math.* **192** no. 3 (2013), 505–532. MR 3049928. Zbl 1309.22014. https://doi.org/10.1007/s00222-012-0415-2.

[11] D. Bump and S. Friedberg, The exterior square automorphic L-functions on $\mathrm{GL}(n)$, in *Festschrift in Honor of I. I. Piatetski-Shapiro on the Occasion of his Sixtieth Birthday*, *Part II* (Ramat Aviv, 1989), *Israel Math. Conf. Proc.* **3**, Weizmann, Jerusalem, 1990, pp. 47–65. MR 1159108. Zbl 0712.11030.

[12] D. Bump and D. Ginzburg, Symmetric square L-functions on $\mathrm{GL}(r)$, *Ann. of Math.* (2) **136** no. 1 (1992), 137–205. MR 1173928. Zbl 0753.11021. https://doi.org/10.2307/2946548.

[13] Y. Z. Flicker, Automorphic forms on covering groups of $\mathrm{GL}(2)$, *Invent. Math.* **57** no. 2 (1980), 119–182. MR 0567194. Zbl 0431.10014. https://doi.org/10.1007/BF01390092.

[14] S. Friedberg and J. Hoffstein, Nonvanishing theorems for automorphic L-functions on $\mathrm{GL}(2)$, *Ann. of Math.* (2) **142** no. 2 (1995), 385–423. MR 1343325. Zbl 0847.11026. https://doi.org/10.2307/2118638.

[15] M. Furusawa, On the theta lift from SO_{2n+1} to $\widetilde{\mathrm{Sp}}_n$, *J. Reine Angew. Math.* **466** (1995), 87–110. MR 1353315. Zbl 0827.11032. https://doi.org/10.1515/crll.1995.466.87.

[16] W. T. Gan, The Shimura correspondence à la Waldspurger, Notes of a short course given at the Postech Theta Festival, available on the author's website.

[17] W. T. Gan, The Saito-Kurokawa space of PGSp_4 and its transfer to inner forms, in *Eisenstein Series and Applications*, *Progr. Math.* **258**, Birkhäuser Boston, Boston, MA, 2008, pp. 87–123. MR 2402681. Zbl 1244.11054. https://doi.org/10.1007/978-0-8176-4639-4_3.

[18] W. T. Gan, A Langlands program for covering groups?, in *Proceedings of the Sixth International Congress of Chinese Mathematicians. Vol. I*, *Adv. Lect. Math.* (*ALM*) **36**, Int. Press, Somerville, MA, 2017, pp. 57–78. MR 3702069. Zbl 06817343.

[19] W. T. Gan, B. H. Gross, and D. Prasad, Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups, in *Sur les Conjectures de Gross et Prasad. I*, *Astérisque* **346**, Math. Soc. France, Paris, 2012, pp. 1–109. MR 3202556. Zbl 1280.22019.

[20] W. T. Gan and A. Ichino, Formal degrees and local theta correspondence, *Invent. Math.* **195** no. 3 (2014), 509–672. MR 3166215. Zbl 1297.22017. https://doi.org/10.1007/s00222-013-0460-5.

[21] W. T. Gan and A. Ichino, The Gross-Prasad conjecture and local theta correspondence, *Invent. Math.* **206** no. 3 (2016), 705–799. MR 3573972. Zbl 1358.11061. https://doi.org/10.1007/s00222-016-0662-8.

[22] W. T. Gan and A. Ichino, On the irreducibility of some induced representations of real reductive Lie groups, *Tunisian J. Math.* **1** no. 1 (2019), 73–107. https://doi.org/10.2140/tunis.2019.1.73.

[23] W. T. Gan, Y. Qiu, and S. Takeda, The regularized Siegel-Weil formula (the second term identity) and the Rallis inner product formula, *Invent. Math.* **198** no. 3 (2014), 739–831. MR 3279536. Zbl 1320.11037. https://doi.org/10.1007/s00222-014-0509-0.

[24] W. T. Gan and G. Savin, Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence, *Compos. Math.* **148** no. 6 (2012), 1655–1694. MR 2999299. Zbl 1325.11046. https://doi.org/10.1112/S0010437X12000486.

[25] W. T. Gan and S. Takeda, On the Howe duality conjecture in classical theta correspondence, in *Advances in the Theory of Automorphic Forms and their L-Functions*, *Contemp. Math.* **664**, Amer. Math. Soc., Providence, RI, 2016, pp. 105–117. MR 3502978. Zbl 06622631. https://doi.org/10.1090/conm/664/13063.

[26] W. T. Gan and S. Takeda, A proof of the Howe duality conjecture, *J. Amer. Math. Soc.* **29** no. 2 (2016), 473–493. MR 3454380. Zbl 1342.11051. https://doi.org/10.1090/jams/839.

[27] D. Ginzburg, S. Rallis, and D. Soudry, On a correspondence between cuspidal representations of GL_{2n} and $\widetilde{\mathrm{Sp}}_{2n}$, *J. Amer. Math. Soc.* **12**

no. 3 (1999), 849–907. MR 1671452. Zbl 0928.11027. https://doi.org/10.1090/S0894-0347-99-00300-8.

[28] D. Ginzburg, S. Rallis, and D. Soudry, Endoscopic representations of $\widetilde{\mathrm{Sp}}_{2n}$, *J. Inst. Math. Jussieu* **1** no. 1 (2002), 77–123. MR 1954940. Zbl 1052.11032. https://doi.org/10.1017/S1474748002000026.

[29] B. H. Gross and D. Prasad, On the decomposition of a representation of SO_n when restricted to SO_{n-1}, *Canad. J. Math.* **44** no. 5 (1992), 974–1002. MR 1186476. Zbl 0787.22018. https://doi.org/10.4153/CJM-1992-060-8.

[30] M. Harris and R. Taylor, *The Geometry and Cohomology of Some Simple Shimura Varieties*, *Ann. of Math. Stud.* **151**, Princeton Univ. Press, Princeton, NJ, 2001, with an appendix by Vladimir G. Berkovich. MR 1876802. Zbl 1036.11027. https://doi.org/10.1515/9781400837205.

[31] G. Henniart, Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique, *Invent. Math.* **139** no. 2 (2000), 439–455. MR 1738446. Zbl 1048.11092. https://doi.org/10.1007/s002220050012.

[32] R. Howe, θ-series and invariant theory, in *Automorphic Forms, Representations and L-Functions, Part* 1 (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), *Proc. Sympos. Pure Math.* **XXXIII**, Amer. Math. Soc., Providence, RI, 1979, pp. 275–285. MR 0546602. Zbl 0423.22016.

[33] R. Howe, Automorphic forms of low rank, in *Noncommutative Harmonic Analysis and Lie Groups* (Marseille, 1980), *Lecture Notes in Math.* **880**, Springer, Berlin-New York, 1981, pp. 211–248. MR 0644835. Zbl 0463.10015.

[34] R. Howe, On a notion of rank for unitary representations of the classical groups, in *Harmonic Analysis and Group Representations*, Liguori, Naples, 1982, pp. 223–331. MR 0777342.

[35] R. Howe, Transcending classical invariant theory, *J. Amer. Math. Soc.* **2** no. 3 (1989), 535–552. MR 0985172. Zbl 0716.22006. https://doi.org/10.2307/1990942.

[36] A. Ichino, E. Lapid, and Z. Mao, On the formal degrees of square-integrable representations of odd special orthogonal and metaplectic groups, *Duke Math. J.* **166** no. 7 (2017), 1301–1348. MR 3649356. Zbl 1378.11061. https://doi.org/10.1215/00127094-0000001X.

[37] H. Jacquet and J. A. Shalika, On Euler products and the classification of automorphic forms. II, *Amer. J. Math.* **103** no. 4 (1981), 777–815. MR 0623137. Zbl 0491.10020. https://doi.org/10.2307/2374050.

[38] H. Jacquet, Sur un résultat de Waldspurger, *Ann. Sci. École Norm. Sup.* (4) **19** no. 2 (1986), 185–229. MR 0868299. Zbl 0605.10015. https://doi.org/10.24033/asens.1506.

[39] H. Jacquet, On the nonvanishing of some L-functions, *Proc. Indian Acad. Sci. Math. Sci.* **97** no. 1-3 (1987), 117–155 (1988). MR 0983610. Zbl 0659.10031. https://doi.org/10.1007/BF02837819.

[40] H. Jacquet and J. Shalika, Exterior square L-functions, in *Automorphic Forms, Shimura Varieties, and L-Functions, Vol. II* (Ann Arbor, MI, 1988), *Perspect. Math.* **11**, Academic Press, Boston, MA, 1990, pp. 143–226. MR 1044830. Zbl 0695.10025.

[41] D. Jiang and D. Soudry, The local converse theorem for $\mathrm{SO}(2n+1)$ and applications, *Ann. of Math.* (2) **157** no. 3 (2003), 743–806. MR 1983781. Zbl 1049.11055. https://doi.org/10.4007/annals.2003.157.743.

[42] S. S. Kudla, On the local theta-correspondence, *Invent. Math.* **83** no. 2 (1986), 229–255. MR 0818351. Zbl 0583.22010. https://doi.org/10.1007/BF01388961.

[43] S. S. Kudla and S. Rallis, A regularized Siegel-Weil formula: the first term identity, *Ann. of Math.* (2) **140** no. 1 (1994), 1–80. MR 1289491. Zbl 0818.11024. https://doi.org/10.2307/2118540.

[44] R. P. Langlands, Automorphic representations, Shimura varieties, and motives. Ein Märchen, in *Automorphic Forms, Representations and L-Functions, Part* 2 (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), *Proc. Sympos. Pure Math.* **XXXIII**, Amer. Math. Soc., Providence, RI, 1979, pp. 205–246. MR 0546619. Zbl 0447.12009.

[45] R. P. Langlands, On the classification of irreducible representations of real algebraic groups, in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups, Math. Surveys Monogr.* **31**, Amer. Math. Soc., Providence, RI, 1989, pp. 101–170. MR 1011897. Zbl 0741.22009. https://doi.org/10.1090/surv/031/03.

[46] E. Lapid, G. Muić, and M. Tadić, On the generic unitary dual of quasisplit classical groups, *Int. Math. Res. Not.* no. 26 (2004), 1335–1354. MR 2046512. Zbl 1079.22015. https://doi.org/10.1155/S1073792804132832.

[47] J.-S. Li, On the classification of irreducible low rank unitary representations of classical groups, *Compositio Math.* **71** no. 1 (1989), 29–48. MR 1008803. Zbl 0694.22012. Available at http://www.numdam.org/item?id=CM_1989__71_1_29_0.

[48] J.-S. Li, Singular unitary representations of classical groups, *Invent. Math.* **97** no. 2 (1989), 237–255. MR 1001840. Zbl 0694.22011. https://doi.org/10.1007/BF01389041.

[49] J.-S. Li, Automorphic forms with degenerate Fourier coefficients, *Amer. J. Math.* **119** no. 3 (1997), 523–578. MR 1448215. Zbl 0871.11036. Available at http://muse.jhu.edu/journals/american_journal_of_mathematics/v119/119.3li.pdf.

[50] W.-W. Li, Transfert d'intégrales orbitales pour le groupe métaplectique, *Compos. Math.* **147** no. 2 (2011), 524–590. MR 2776612. Zbl 1216.22009. https://doi.org/10.1112/S0010437X10004963.

[51] W.-W. Li, La formule des traces stable pour le groupe métaplectique: les termes elliptiques, *Invent. Math.* **202** no. 2 (2015), 743–838. MR 3418244. Zbl 1386.11074. https://doi.org/10.1007/s00222-015-0577-9.

[52] W.-W. Li, Spectral transfer for metaplectic groups. I. Local character relations, *J. Inst. Math. Jussieu* (2016), published online 07 December 2016. https://doi.org/10.1017/S1474748016000384.

[53] C. Luo, Endoscopic character identities for metaplectic groups, 2018. arXiv 1801.10302.

[54] C. Mœglin, Sur certains paquets d'Arthur et involution d'Aubert-Schneider-Stuhler généralisée, *Represent. Theory* **10** (2006), 86–129. MR 2209850. Zbl 1139.22015. https://doi.org/10.1090/S1088-4165-06-00270-6.

[55] C. Mœglin, Paquets d'Arthur discrets pour un groupe classique p-adique, in *Automorphic Forms and L-Functions II. Local Aspects*, *Contemp. Math.* **489**, Amer. Math. Soc., Providence, RI, 2009, pp. 179–257. MR 2533005. Zbl 1181.22016. https://doi.org/10.1090/conm/489/09549.

[56] C. Mœglin, Image des opérateurs d'entrelacements normalisés et pôles des séries d'Eisenstein, *Adv. Math.* **228** no. 2 (2011), 1068–1134. MR 2822218. Zbl 1225.22016. https://doi.org/10.1016/j.aim.2011.06.003.

[57] C. Mœglin, Multiplicité 1 dans les paquets d'Arthur aux places p-adiques, in *On Certain L-Functions*, *Clay Math. Proc.* **13**, Amer. Math. Soc., Providence, RI, 2011, pp. 333–374. MR 2767522. Zbl 1181.22016.

[58] C. Mœglin, Paquets d'Arthur spéciaux unipotents aux places archimédiennes et correspondance de Howe, in *Representation Theory, Number Theory, and Invariant Theory*, *Progr. Math.* **323**, Birkhäuser/Springer, Cham, 2017, pp. 469–502. MR 3753920.

[59] C. Mœglin and D. Renard, Paquets d'Arthur des groupes classiques complexes, in *Around Langlands Correspondences*, *Contemp. Math.* **691**, Amer. Math. Soc., Providence, RI, 2017, pp. 203–256. MR 3666056. Zbl 1390.22013.

[60] C. Mœglin and D. Renard, Sur les paquets d'Arthur aux places réelles, translation, 2017. arXiv 1704.05096.

[61] C. Mœglin and D. Renard, Sur les paquets d'Arthur des groupes classiques réels, 2017. arXiv 1703.07226.

[62] C. Mœglin and D. Renard, Sur les paquets d'Arthur des groupes classiques et unitaires non quasi-déployés, 2018. arXiv 1803.07662.

[63] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un Corps p-adique*, *Lecture Notes in Math.* **1291**, Springer-Verlag, Berlin, 1987. MR 1041060. Zbl 0642.22002. https://doi.org/10.1007/BFb0082712.

[64] C. Mœglin and J.-L. Waldspurger, La conjecture locale de Gross-Prasad pour les groupes spéciaux orthogonaux: le cas général, in *Sur les Conjectures de Gross et Prasad. II*, *Astérisque* **347**, Math. Soc. France, Paris, 2012, pp. 167–216. MR 3155346. Zbl 1276.22007.

[65] C. Mœglin and J.-L. Waldspurger, *Stabilisation de la Formule des Traces Tordue, Volume* 1., *Progr. Math.* **316**, Birkhäuser/Springer, Cham, 2016. Zbl 1361.11004. https://doi.org/10.1007/978-3-319-30049-8.

[66] C. Mœglin and J.-L. Waldspurger, *Stabilisation de la Formule des Traces Tordue, Volume* 2., *Progr. Math.* **317**, Birkhäuser/Springer, Cham, 2016. Zbl 1381.11001. https://doi.org/10.1007/978-3-319-30058-0.

[67] B. C. Ngô, Le lemme fondamental pour les algèbres de Lie, *Publ. Math. Inst. Hautes Études Sci.* no. 111 (2010), 1–169. MR 2653248. Zbl 1200.22011. https://doi.org/10.1007/s10240-010-0026-7.

[68] S. Niwa, Modular forms of half integral weight and the integral of certain theta-functions, *Nagoya Math. J.* **56** (1975), 147–161. MR 0364106. Zbl 0303.10027. https://doi.org/10.1017/S0027763000016445.

[69] I. Piatetski-Shapiro, Work of Waldspurger, in *Lie Group Representations, II* (College Park, Md., 1982/1983), *Lecture Notes in Math.* **1041**, Springer, Berlin, 1984, pp. 280–302. MR 0748511. Zbl 0539.22017. https://doi.org/10.1007/BFb0073151.

[70] D. Prasad and R. Schulze-Pillot, Generalised form of a conjecture of Jacquet and a local consequence, *J. Reine Angew. Math.* **616** (2008), 219–236. MR 2369492. Zbl 1221.11129. https://doi.org/10.1515/CRELLE.2008.023.

[71] R. Ranga Rao, On some explicit formulas in the theory of Weil representation, *Pacific J. Math.* **157** no. 2 (1993), 335–371. MR 1197062. Zbl 0794.58017. Available at http://projecteuclid.org/euclid.pjm/1102634748.

[72] D. Renard, Endoscopy for $\mathrm{Mp}(2n, \mathbf{R})$, *Amer. J. Math.* **121** no. 6 (1999), 1215–1243. MR 1719818. Zbl 0942.22010. https://doi.org/10.1353/ajm.1999.0045.

[73] Y. Sakellaridis and A. Venkatesh, *Periods and Harmonic Analysis on Spherical Varieties*, *Astérisque* **396**, Math. Soc. France, Paris, 2017. MR 3764130. Zbl 06847674.

[74] G. Savin, Lifting of generic depth zero representations of classical groups, *J. Algebra* **319** no. 8 (2008), 3244–3258. MR 2408316. Zbl 1147.22008. https://doi.org/10.1016/j.jalgebra.2007.10.018.

[75] W. Scharlau, *Quadratic and Hermitian forms*, *Grundlehren Math. Wiss.* **270**, Springer-Verlag, Berlin, 1985. MR 0770063. Zbl 0584.10010. https://doi.org/10.1007/978-3-642-69971-9.

[76] P. Scholze, The local Langlands correspondence for GL_n over p-adic fields, *Invent. Math.* **192** no. 3 (2013), 663–715. MR 3049932. Zbl 1305.22025. https://doi.org/10.1007/s00222-012-0420-5.

[77] F. Shahidi, On certain L-functions, *Amer. J. Math.* **103** no. 2 (1981), 297–355. MR 0610479. Zbl 0467.12013. https://doi.org/10.2307/2374219.

[78] F. Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups, *Ann. of Math.* (2) **132** no. 2 (1990), 273–330. MR 1070599. Zbl 0780.22005. https://doi.org/10.2307/1971524.

[79] D. Shelstad, L-indistinguishability for real groups, *Math. Ann.* **259** no. 3 (1982), 385–430. MR 0661206. Zbl 0506.22014. https://doi.org/10.1007/BF01456950.

[80] D. Shelstad, Tempered endoscopy for real groups. III. Inversion of transfer and L-packet structure, *Represent. Theory* **12** (2008), 369–402. MR 2448289. Zbl 1159.22007. https://doi.org/10.1090/S1088-4165-08-00337-3.

[81] G. Shimura, On modular forms of half integral weight, *Ann. of Math.* (2) **97** (1973), 440–481. MR 0332663. Zbl 0266.10022. https://doi.org/10.2307/1970831.

[82] T. Shintani, On construction of holomorphic cusp forms of half integral weight, *Nagoya Math. J.* **58** (1975), 83–126. MR 0389772. Zbl 0316.10016. https://doi.org/10.1017/S0027763000016706.

[83] B. Speh and D. A. Vogan, Jr., Reducibility of generalized principal series representations, *Acta Math.* **145** no. 3-4 (1980), 227–299. MR 0590291. Zbl 0457.22011. https://doi.org/10.1007/BF02414191.

[84] B. Sun, Dual pairs and contragredients of irreducible representations, *Pacific J. Math.* **249** no. 2 (2011), 485–494. MR 2782681. Zbl 1215.22007. https://doi.org/10.2140/pjm.2011.249.485.

[85] B. Sun and C.-B. Zhu, Conservation relations for local theta correspondence, *J. Amer. Math. Soc.* **28** no. 4 (2015), 939–983. MR 3369906. Zbl 1321.22017. https://doi.org/10.1090/S0894-0347-2014-00817-1.

[86] M. Tadić, Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case), *Ann. Sci. École Norm. Sup.* (4) **19** no. 3 (1986), 335–382. MR 0870688. Zbl 0614.22005. https://doi.org/10.24033/asens.1510.

[87] D. A. Vogan, Jr., *Representations of Real Reductive Lie Groups*, *Progr. Math.* **15**, Birkhäuser, Boston, Mass., 1981. MR 0632407. Zbl 0469.22012.

[88] J.-L. Waldspurger, Correspondance de Shimura, *J. Math. Pures Appl.* (9) **59** no. 1 (1980), 1–132. MR 0577010. Zbl 0412.10019.

[89] J.-L. Waldspurger, Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie, *Compositio Math.* **54** no. 2 (1985), 173–242. MR 0783511. Zbl 0567.10021. Available at http://www.numdam.org/item?id=CM_1985__54_2_173_0.

[90] J.-L. Waldspurger, Démonstration d'une conjecture de dualité de Howe dans le cas p-adique, $p \neq 2$, in *Festschrift in Honor of I. I. Piatetski-Shapiro on the Occasion of his Sixtieth Birthday, Part I* (Ramat Aviv, 1989), *Israel Math. Conf. Proc.* **2**, Weizmann, Jerusalem, 1990, pp. 267–324. MR 1159105. Zbl 0722.22009.

[91] J.-L. Waldspurger, Correspondances de Shimura et quaternions, *Forum Math.* **3** no. 3 (1991), 219–307. MR 1103429. Zbl 0724.11026. https://doi.org/10.1515/form.1991.3.219.

[92] B. Xu, On Mœglin's parametrization of Arthur packets for p-adic quasisplit $Sp(N)$ and $SO(N)$, *Canad. J. Math.* **69** no. 4 (2017), 890–960. MR 3679701. Zbl 1373.22027. https://doi.org/10.4153/CJM-2016-029-3.

[93] S. Yamana, L-functions and theta correspondence for classical groups, *Invent. Math.* **196** no. 3 (2014), 651–732. MR 3211043. Zbl 1303.11054. https://doi.org/10.1007/s00222-013-0476-x.

(Received: June 1, 2017)
(Revised: August 3, 2018)

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076
E-mail: matgwt@nus.edu.sg

Department of Mathematics, Kyoto University, Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan
E-mail: ichino@math.kyoto-u.ac.jp