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Spectral gaps without the pressure condition

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Abstract

For all convex co-compact hyperbolic surfaces, we prove the existence of an essential spectral gap, that is, a strip beyond the unitarity axis in which the Selberg zeta function has only finitely many zeroes. We make no assumption on the dimension δ of the limit set; in particular, we do not require the pressure condition $\delta \leq \frac{1}{2}$. This is the first result of this kind for quantum Hamiltonians.

Our proof follows the strategy developed by Dyatlov and Zahl. The main new ingredient is the fractal uncertainty principle for δ-regular sets with $\delta < 1$, which may be of independent interest.

1. Introduction

Let $M = \Gamma\backslash\mathbb{H}^2$ be a (noncompact) convex co-compact hyperbolic surface. The Selberg zeta function $Z_M(s)$ is a product over the set $\mathcal{L}_M$ of all primitive closed geodesics

$$Z_M(s) = \prod_{\ell\in\mathcal{L}_M}\prod_{k=0}^{\infty}\left(1 - e^{-(s+k)\ell}\right), \quad \operatorname{Re} s \gg 1,$$

and it extends meromorphically to $s \in \mathbb{C}$. From the spectral description of Z_M it is known that $Z_M(s)$ has only finitely many zeroes in $\{\operatorname{Re} s > \frac{1}{2}\}$, which correspond to small eigenvalues of the Laplacian. The situation in $\{\operatorname{Re} s \leq \frac{1}{2}\}$ is more complicated since the zeroes of Z_M are no longer given by a self-adjoint spectral problem on $L^2(M)$; they instead correspond to scattering resonances of M and are related to decay of waves.

A natural question is whether M has an *essential spectral gap*; that is, does there exist $\beta > 0$ such that $Z_M(s)$ has only finitely many zeroes in $\{\operatorname{Re} s > \frac{1}{2} - \beta\}$? The known answers so far depend on the exponent of convergence

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of the Poincaré series of the group, denoted $\delta \in [0, 1)$. Patterson [Pat76] and Sullivan [Sul79] proved that there is a gap of size $\beta = \frac{1}{2} - \delta$ when $\delta < \frac{1}{2}$, and Naud [Nau05] showed there is a gap of size $\beta > \frac{1}{2} - \delta$ when $0 < \delta \leq \frac{1}{2}$. The present paper removes the restrictions on δ:

Theorem 1. *Every convex co-compact hyperbolic surface has an essential spectral gap.*

Spectral gaps for hyperbolic surfaces have many important applications, such as diophantine problems (see Bourgain–Gamburd–Sarnak [BGS11], Oh–Winter [OW16], Magee–Oh–Winter [MOW], and the lecture notes by Sarnak [Sar14]) and remainders in the prime geodesic theorem (see, for instance, the book of Borthwick [Bor16, §14.6]). Moreover, hyperbolic surfaces are a standard model for more general open quantum chaotic systems, where spectral gaps have been studied since the work of Lax–Phillips [LP67], Ikawa [Ika88] and Gaspard–Rice [GR89]; see Section 1.1 below.

Theorem 1 can also be viewed in terms of the scattering resolvent

$$R(\lambda) = \Big(-\Delta_M - \frac{1}{4} - \lambda^2\Big)^{-1} : \begin{cases} L^2(M) \to H^2(M), & \operatorname{Im}\lambda > 0, \\ L^2_{\mathrm{comp}}(M) \to H^2_{\mathrm{loc}}(M), & \lambda \in \mathbb{C}, \end{cases}$$

where $\Delta_M \leq 0$ is the Laplace–Beltrami operator of M. The family $R(\lambda)$ is meromorphic, as proved by Mazzeo–Melrose [MM87], Guillopé–Zworski [GZ95], and Guillarmou [Gui05]. Its poles, called *resonances*, correspond to the zeroes of $Z_M(s)$, $s := \frac{1}{2} - i\lambda$; see, for instance, [Bor16, Ch. 10]. Therefore, Theorem 1 says that there are only finitely many resonances with $\operatorname{Im}\lambda > -\beta$. Since our proof uses [DZ16] and a fractal uncertainty principle (Theorem 3), we obtain a polynomial resolvent bound:

Theorem 2. *Let M be as in Theorem* 1, *and take $\beta = \beta(M) > 0$ given by Theorem* 3 *below. Then for each $\varepsilon > 0$, there exists $C_0 > 0$ such that for all $\varphi \in C_0^\infty(M)$,*

$$\|\varphi R(\lambda)\varphi\|_{L^2\to L^2} \leq C|\lambda|^{-1-2\min(0,\operatorname{Im}\lambda)+\varepsilon}, \quad |\lambda| > C_0,\ \operatorname{Im}\lambda \in [-\beta+\varepsilon, 1], \tag{1.1}$$

where the constant C depends on ε, φ, but not on λ.

Remarks. 1. We see from Theorem 2 that there is an essential spectral gap of size β for all $\beta < \beta(M)$, where $\beta(M)$ is given by Theorem 3, but not necessarily for $\beta = \beta(M)$. However, this is irrelevant since Theorem 3 does not specify the value of $\beta(M)$.

2. Spectral gaps for convex co-compact hyperbolic surfaces were studied numerically by Borthwick [Bor14] and Borthwick–Weich [BW16]; see also [Bor16, §16.3.2] and Figure 1.

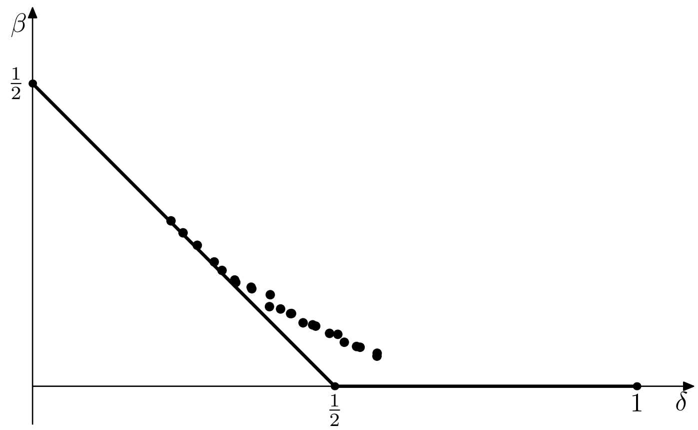

Figure 1. Numerically computed essential spectral gaps β for symmetric 3-funneled and 4-funneled surfaces from [BW16, Fig. 14] (specifically, $G_{100}^{I_1}$ in the notation of [BW16]; data used with permission of authors). Each point corresponds to one surface and has coordinates (δ, β). The solid line is the standard gap $\beta = \max(0, \frac{1}{2} - \delta)$.

1.1. *Systems with hyperbolic trapping.* The case of convex co-compact hyperbolic surfaces studied here belongs to the more general class of *open systems with uniformly hyperbolic trapped sets which have fractal structure*; see the reviews of Nonnenmacher [Non11] and Zworski [Zwo17] for the definition of these systems and an overview of the history of the spectral gap problem. Another example of such systems is given by scattering in the exterior of several convex obstacles under the no-eclipse condition, where spectral gaps were studied by Ikawa [Ika88], Gaspard–Rice [GR89], and Petkov–Stoyanov [PS10] and observed experimentally by Barkhofen et al. [BWP$^+$13].

For general hyperbolic systems, resolvent bounds of type (1.1) have important applications to dispersive partial differential equations, including (the list of references below is by no means extensive)

- exponential local energy decay $\mathcal{O}(e^{-\beta t})$ of linear waves modulo a finite dimensional space corresponding to resonances with $\operatorname{Im} \lambda > -\beta$ (see Christianson [Chr09] and Guillarmou–Naud [GN09]);
- exponential stability for nonlinear wave equations (see Hintz–Vasy [HV18]);
- local smoothing estimates (see Datchev [Dat09]);
- Strichartz estimates (see Burq–Guillarmou–Hassell [BGH10] and Wang [Wan17]).

Theorem 2 is the first unconditional spectral gap result for quantum chaotic Hamiltonians with fractal hyperbolic trapped sets. It is a step towards the following general spectral gap conjecture:

CONJECTURE ([Zwo17, §3.2, Conj. 3]). *Suppose that P is an operator for which the scattering resolvent admits a meromorphic continuation* (*e.g.*, $P = -\Delta_M$, *where* (M, g) *is a complete Riemannian manifold with Euclidean or asymptotically hyperbolic infinite ends*). *Assume that the underlying classical flow* (*e.g.*, *the geodesic flow on* (M, g)) *has a compact hyperbolic trapped set.*

Then there exists $\beta > 0$ such that (M, g) has an essential spectral gap of size β; *that is*, *the scattering resolvent has only finitely many poles with* $\operatorname{Im}\lambda > -\beta$.

We give a brief overview of some of the previous works related to this conjecture, as well as some recent results. We remark that the question of which scattering systems have exponential wave decay has been studied since the work of Lax–Phillips [LP67]; see [LP89, epilogue] for an overview of the history of this question.

- A spectral gap of size $\beta = -P(\frac{1}{2})$ under the *pressure condition* $P(\frac{1}{2}) < 0$ was proved for obstacle scattering by Ikawa [Ika88], computed in the physics literature by Gaspard–Rice [GR89], and proved for general hyperbolic trapped sets by Nonnenmacher–Zworski [NZ09a], [NZ09b]. Here $P(\sigma)$ is the *topological pressure* of the system; see, for instance, [Non11, (14)] or [Zwo17, (3.28)]. For the case of hyperbolic surfaces considered here, we have $P(\sigma) = \delta - \sigma$, so the pressure condition is $\delta < \frac{1}{2}$ and the pressure gap is the Patterson–Sullivan gap.
- An improved spectral gap $\beta > -P(\frac{1}{2})$ under the relaxed pressure condition $P(\frac{1}{2}) \leq 0$ was proved by Naud [Nau05] for convex co-compact hyperbolic surfaces, Stoyanov [Sto11] for more general cases of Ruelle zeta functions including higher-dimensional convex co-compact hyperbolic manifolds, and Petkov–Stoyanov [PS10] for obstacle scattering. The above papers rely on the method originally developed by Dolgopyat [Dol98].
- Jakobson–Naud [JN12] conjectured a gap of size $-\frac{1}{2}P(1) = \frac{1-\delta}{2}$ for hyperbolic surfaces and obtained upper bounds on the size of the gap.
- Dyatlov–Zahl [DZ16] reduced the spectral gap question for convex co-compact hyperbolic manifolds to a *fractal uncertainty principle* (see Section 1.2 below) and showed an improved gap $\beta > \frac{1}{2} - \delta$ for surfaces with $\delta = \frac{1}{2}$ and for nearby surfaces using methods from additive combinatorics. The size of the gap in [DZ16] decays superpolynomially as a function of the regularity constant C_R (defined in Section 1.3 below). Dyatlov–Jin [DJ18] adapted the methods of [Dol98], [Nau05] to obtain an improved gap for $0 < \delta \leq \frac{1}{2}$ which depends polynomially on C_R. Later Bourgain–Dyatlov [BD17] gave an improved gap $\beta > \frac{1}{2} - \delta$ which depends only on $\delta > 0$ and not on C_R. The present paper is in some sense orthogonal to [DJ18], [BD17] since it

gives a gap $\beta > 0$. Thus the result of the present paper is interesting when $\delta \geq \frac{1}{2}$, and the results of [DJ18] and [BD17] are interesting when $\delta \leq \frac{1}{2}$.

- In a related setting of open quantum baker's maps, Dyatlov–Jin [DJ17] used a fractal uncertainty principle to show that every such system has a gap, and they obtained quantitative bounds on the size of the gap.
- We finally discuss the case of scattering on finite area hyperbolic surfaces with cusps. An example is the modular surface $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2$, where the zeroes of the Selberg zeta function fall into two categories:
  (1) infinitely many *embedded eigenvalues* on the line $\{\operatorname{Re} s = \frac{1}{2}\}$;
  (2) the rest, corresponding to the zeroes of the Riemann zeta function.

  In particular, the modular surface has no essential spectral gap. The same is true for any finite area surface; that is, there are infinitely many resonances in the half-plane $\{\operatorname{Re} s > \frac{1}{2} - \beta\}$ for all $\beta > 0$. This follows from the fact that the number of resonances in a ball of radius T grows like T^2, together with the following bound proved by Selberg [Sel90, Th. 1]:

$$\sum_{\substack{s \text{ resonance} \\ |\operatorname{Im} s| \leq T}} \left(\frac{1}{2} - \operatorname{Re} s\right) = \mathcal{O}(T \log T) \quad \text{as } T \to \infty.$$

  However, the question of how close resonances can lie to the critical line $\operatorname{Re} s = \frac{1}{2}$ for a *generic* finite area surface is more complicated, in particular, embedded eigenvalues are destroyed by generic conformal perturbations of the metric (see Colin de Verdière [CdV82, CdV83]) and by generic perturbations within the class of hyperbolic surfaces (see Phillips–Sarnak [PS85]).

  Note that the present paper does not apply to the finite area case for two reasons: (1) the methods of [DZ16] do not apply to manifolds with cusps, in particular, because the trapped set is not compact, and (2) finite area surfaces have $\Lambda_\Gamma = \mathbb{S}^1$ and thus $\delta = 1$.

1.2. *Uncertainty principle for hyperbolic limit sets.* The proof of Theorem 1 uses the strategy of [DZ16], which reduced the spectral gap question to a fractal uncertainty principle. To state it, define the operator $\mathcal{B}_\chi = \mathcal{B}_\chi(h)$ on $L^2(\mathbb{S}^1)$ by

$$\mathcal{B}_\chi f(y) = (2\pi h)^{-1/2} \int_{\mathbb{S}^1} |y - y'|^{2i/h} \chi(y, y') f(y')\, dy', \tag{1.2}$$

where $|y - y'|$ denotes the Euclidean distance on $\mathbb{R}^2$ restricted to the unit circle $\mathbb{S}^1$ and

$$\chi \in C_0^\infty(\mathbb{S}^1_\Delta), \quad \mathbb{S}^1_\Delta := \{(y, y') \in \mathbb{S}^1 \times \mathbb{S}^1 \mid y \neq y'\}.$$

The semiclassical parameter $h > 0$ corresponds to the inverse of the frequency and also to the inverse of the spectral parameter: $h \sim |\lambda|^{-1}$. We will be interested in the limit $h \to 0$. The operator $\mathcal{B}_\chi$ is bounded on $L^2(\mathbb{S}^1)$ uniformly in h; see [DZ16, §5.1]. We can view $\mathcal{B}_\chi$ as a hyperbolic analogue of the (semiclassically rescaled) Fourier transform.

A key object associated to the surface M is the *limit set* $\Lambda_\Gamma \subset \mathbb{S}^1$; see, for instance, [Bor16, §2.2.1] or [DZ16, (4.11)] for the definition. Theorems 1 and 2 follow by combining [DZ16, Th. 3] with the following uncertainty principle for Λ_Γ:

THEOREM 3. *Let $M = \Gamma\backslash\mathbb{H}^2$ be a convex co-compact hyperbolic surface, and denote by $\Lambda_\Gamma(h^\rho) \subset \mathbb{S}^1$ the h^ρ-neighborhood of the limit set. Then there exist $\beta > 0$ and $\rho \in (0,1)$ depending only on M such that for all $\chi \in C_0^\infty(\mathbb{S}^1_\Delta)$ and $h \in (0,1)$,*

$$\| \mathbb{1}_{\Lambda_\Gamma(h^\rho)} \mathcal{B}_\chi(h) \mathbb{1}_{\Lambda_\Gamma(h^\rho)} \|_{L^2(\mathbb{S}^1)\to L^2(\mathbb{S}^1)} \le Ch^\beta, \tag{1.3}$$

where the constant C depends on M, χ, but not on h.

Remarks. 1. We call (1.3) an uncertainty principle because it implies that no quantum state can be microlocalized h^ρ close to $\Gamma_\pm \subset S^*M$, where S^*M denotes the co-sphere bundle of M and $\Gamma_\pm$ are the incoming/outgoing tails, consisting of geodesics trapped in the future (Γ_-) or in the past (Γ_+). The lifts of $\Gamma_\pm$ to $S^*\mathbb{H}^2$ can be expressed in terms of Λ_Γ. See [DZ16, §§1.1, 4.1.2] for details — in particular, for how to define microlocalization to an h^ρ-neighborhood of $\Gamma_\pm$.

2. Recent work of Dyatlov–Zworski [DZ18] provides an alternative to [DZ16] for showing that Theorem 3 implies Theorem 1, using transfer operator techniques.

3. The value of β depends only on δ and the regularity constant C_R of the set Λ_Γ; see Sections 1.3 and 4.3. Recently Jin–Zhang [JZ17, Th. 1.3] obtained an estimate on β in terms of δ, C_R which has the form (here $\mathbf{K}$ is a large universal constant)

$$\beta = \exp\Big[-\mathbf{K}(C_R\delta^{-1}(1-\delta)^{-1})^{\mathbf{K}(1-\delta)^{-3}}\Big].$$

The parameter ρ will be taken very close to 1 depending on δ, C_R; see (4.19).

4. If we vary M within the moduli space $\mathscr{M}$ of convex co-compact hyperbolic surfaces, then δ changes continuously (in fact, real analytically). Moreover, as shown in [BD17, Lemma 2.12], the regularity constant C_R can be estimated explicitly in terms of the disks and group elements in a Schottky representation of M and thus is bounded locally uniformly on $\mathscr{M}$. Therefore, the value of β is bounded away from 0 as long as M varies in a compact subset of $\mathscr{M}$.

1.3. *Uncertainty principle for regular fractal sets.* In order to prove Theorem 3 we exploit the fractal structure of the limit set Λ_Γ. For simplicity, we make the illegal choice of $\rho := 1$ in the informal explanations below.

The (Hausdorff and Minkowski) dimension of Λ_Γ is equal to $\delta \in [0,1)$, so the volume of $\Lambda_\Gamma(h)$ decays like $h^{1-\delta}$ as $h \to 0$. For $\delta < \frac{1}{2}$, this implies (using the $L^1 \to L^\infty$ estimate on $\mathcal{B}_\chi(h)$ together with Hölder's inequality) the uncertainty principle (1.3) with $\beta = \frac{1}{2} - \delta$ and thus recovers the Patterson–Sullivan gap; see [DZ16, §5.1].

However, for $\delta \geq \frac{1}{2}$, one cannot obtain (1.3) by using only the volume of the set $\Lambda_\Gamma(h)$. Indeed, if we replace $\Lambda_\Gamma(h)$ by an interval of size $h^{1/2}$, then a counterexample to (1.3) is given by a Gaussian wavepacket of width $h^{1/2}$. Therefore, one needs to exploit finer fractal structure of the limit set. For us such structure is given by *Ahlfors–David regularity*, which roughly speaking states that Λ_Γ has dimension δ at each point on each scale:

Definition 1.1. Let $X \subset \mathbb{R}$ be a nonempty closed set and $\delta \in [0,1]$, $C_R \geq 1$, $0 \leq \alpha_0 \leq \alpha_1 \leq \infty$. We say that X is *δ-regular with constant C_R on scales α_0 to α_1* if there exists a Borel measure μ_X on $\mathbb{R}$ such that

- μ_X is supported on X, that is $\mu_X(\mathbb{R} \setminus X) = 0$;
- for each interval I of size $|I| \in [\alpha_0, \alpha_1]$, we have $\mu_X(I) \leq C_R |I|^\delta$;
- if additionally I is centered at a point in X, then $\mu_X(I) \geq C_R^{-1} |I|^\delta$.

Remarks. 1. The condition that μ_X is supported on X is never used in this paper (and the measure μ_X is referred to explicitly only in Section 2.2), however we keep it to make the definition compatible with [DJ18].

2. In estimates regarding regular sets, it will be important that the constants involved may depend on δ, C_R, but not on α_0, α_1. Thus it is useful to think of δ, C_R as fixed and α_1/α_0 as large.

3. As indicated above, the limit set Λ_Γ is δ-regular on scales 0 to 1 where $\delta \in [0,1)$ is the exponent of convergence of the Poincaré series of the group Γ; see Section 4.3.

The key component of the proof of Theorem 3 is the following fractal uncertainty principle for the Fourier transform and general δ-regular sets; it is a result of independent interest. In Section 4 we show that Theorem 4 implies Theorem 3 by linearizing the phase of the operator (1.2). (This makes the value of the exponent β smaller; see the remark following Proposition 4.3.)

THEOREM 4. *Let* $0 \leq \delta < 1$, $C_R \geq 1$, $N \geq 1$, *and assume that*

- $X \subset [-1,1]$ *is δ-regular with constant C_R on scales N^{-1} to* 1; *and*
- $Y \subset [-N,N]$ *is δ-regular with constant C_R on scales* 1 *to* N.

Then there exist $\beta > 0, C$ depending only on δ, C_R such that for all $f \in L^2(\mathbb{R})$,

$$\operatorname{supp} \hat{f} \subset Y \quad \Longrightarrow \quad \|f\|_{L^2(X)} \le CN^{-\beta}\|f\|_{L^2(\mathbb{R})}. \tag{1.4}$$

Here $L^2(X)$ is defined using the Lebesgue measure.

Remark. Since X is only required to be δ-regular down to scale N^{-1}, rather than 0, it may contain intervals of size N^{-1} and thus have positive Lebesgue measure. In fact it is useful to picture X as a union of intervals of size N^{-1} distributed in a fractal way, and similarly picture Y as a union of intervals of size 1. See also Lemma 2.3.

The proof of Theorem 4 is given in Section 3. We give here a brief outline. The key component is the following nonstandard quantitative unique continuation result, Proposition 3.3: for each $c_1 > 0$, there exists $c_3 > 0$ depending only on δ, C_R, c_1 such that

$$f \in L^2(\mathbb{R}), \quad \operatorname{supp} \hat{f} \subset Y \quad \Longrightarrow \quad \|f\|_{L^2(U')} \ge c_3\|f\|_{L^2(\mathbb{R})}, \tag{1.5}$$

where Y is as in Theorem 4 and $U' = \bigcup_{j\in\mathbb{Z}} I'_j$, where each $I'_j \subset [j, j+1]$ is an (arbitrarily chosen) subinterval of size c_1. It is important that c_3, as well as other constants in the argument, does not depend on the large parameter N.

Theorem 4 follows from (1.5) by iteration on scale. Here δ-regularity of X with $\delta < 1$ is used to obtain the missing subinterval property (see Lemmas 2.6 and 2.10): there exists $c_1 = c_1(\delta, C_R) > 0$ such that for all $j \in \mathbb{Z}$, the set $[j, j+1] \setminus X$ contains some interval I'_j of size c_1, and same is true for dilates αX when $1 \le \alpha \ll N$. The lower bound (1.5) gives an upper bound on the L^2 norm of f on $\mathbb{R} \setminus U' \supset X$, which iterated $\sim \log N$ times gives the power improvement in (1.4). See Section 3.4 for details.

To prove (1.5), we first show a similar bound where the support condition on $\hat{f}$ is replaced by a decay condition: for $\theta(\xi) := \log(10 + |\xi|)^{-\frac{1+\delta}{2}}$ and all $f \in L^2(\mathbb{R})$,

$$\left\| \exp\big(\theta(\xi)|\xi|\big)\hat{f}(\xi)\right\|_{L^2(\mathbb{R})} \le C_1\|f\|_{L^2(\mathbb{R})} \quad \Longrightarrow \quad \|f\|_{L^2(U')} \ge c_3\|f\|_{L^2(\mathbb{R})}, \tag{1.6}$$

where c_3 depends only on δ, c_1, C_1. The proof uses estimates on harmonic measures for domains of the form $\{|\operatorname{Im} z| < r\} \setminus I'_j \subset \mathbb{C}$. See the remark following the statement of Lemma 3.2.

Coming back to (1.5), we construct a function $\psi \not\equiv 0$ which is compactly supported, more precisely $\operatorname{supp} \psi \subset [-\frac{c_1}{10}, \frac{c_1}{10}]$, and satisfies the Fourier decay bound

$$|\widehat{\psi}(\xi)| \le \exp\big(- c_2\theta(\xi)|\xi|\big) \quad \text{for all } \xi \in Y, \tag{1.7}$$

where $c_2 > 0$ depends only on δ, C_R, c_1. To do that, we use δ-regularity of Y with $\delta < 1$ to construct a weight $\omega : \mathbb{R} \to (0,1]$ such that

$$\sup |\partial_\xi \log \omega| \le C_0, \quad \int_{\mathbb{R}} \frac{|\log \omega(\xi)|}{1+\xi^2}\, d\xi \le C_0,$$
$$\omega(\xi) \le \exp\big(-\theta(\xi)|\xi|\big) \quad \text{for all } \xi \in Y,$$

where C_0 depends only on δ, C_R. By a quantitative version of the Beurling–Malliavin Multiplier Theorem (see Lemma 2.11), there exists $\psi \not\equiv 0$ with the required support property and $|\widehat{\psi}(\xi)| \le \omega(\xi)^{c_2}$ for all $\xi \in \mathbb{R}$, thus (1.7) holds. See Section 3.1 for details.

Finally, we put

$$g := f * \psi \in L^2(\mathbb{R}), \quad \hat{g}(\xi) = \hat{f}(\xi)\widehat{\psi}(\xi).$$

If $\operatorname{supp} \hat{f} \subset Y$, then by (1.7) we have

$$\Big\| \exp\big(c_2\theta(\xi)|\xi|\big)\hat{g}(\xi)\Big\|_{L^2} \le \|f\|_{L^2}. \tag{1.8}$$

On the other hand, if $U'' := \bigcup_{j\in\mathbb{Z}} I_j''$, where $I_j'' \subset I_j'$ is the interval with the same center as I_j' and size $c_1/2$, then the support condition on ψ implies that $g = (\mathbb{1}_{U'} f) * \psi$ on U'' and thus $\|g\|_{L^2(U'')} \le \|f\|_{L^2(U')}$. We revise the proof of (1.6) with f replaced by g, U' by U'', and the Fourier decay bound replaced by (1.8), to obtain (1.5) and thus finish the proof of Theorem 4. In the process we apply the argument with Y replaced by its translates $Y+\ell$, $\ell \in \mathbb{Z}$, $|\ell| \le N$; to each translate corresponds its own multiplier ψ. See Section 3.3 for details.

2. **Preliminaries**

2.1. *Notation.* We first introduce the notation used in the paper. For two sets $X, Y \subset \mathbb{R}$, define $X + Y := \{x + y \mid x \in X,\ y \in Y\}$. For $\lambda \ge 0$, denote $\lambda X := \{\lambda x \mid x \in X\}$. For an interval $I = x_0 + [-r, r] \subset \mathbb{R}$ with $r \ge 0$, denote by $|I| := 2r$ the size of I and say that x_0 is the center of I. For $X \subset \mathbb{R}$ and $\alpha \ge 0$, define the α-neighborhood of X by

$$X(\alpha) := X + [-\alpha, \alpha] \ \subset\ \mathbb{R}. \tag{2.1}$$

For $X \subset \mathbb{R}$, denote by $\mathbf{1}_X \in L^\infty(\mathbb{R})$ the indicator function of X and by $\mathbb{1}_X : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the corresponding multiplication operator. For each $\xi \in \mathbb{R}$, denote

$$\langle \xi \rangle := \sqrt{1 + |\xi|^2}.$$

We use the following convention for the Fourier transform of $f \in L^1(\mathbb{R})$:

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)\, dx. \tag{2.2}$$

One advantage of this convention is that $\mathcal{F}$ extends to a unitary operator on $L^2(\mathbb{R})$. Recall the Fourier inversion formula

$$f(x) = \mathcal{F}^*\hat{f}(x) = \int_{\mathbb{R}} e^{2\pi i x\xi}\hat{f}(\xi)\,d\xi \tag{2.3}$$

and the convolution formula

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\cdot\hat{g}(\xi). \tag{2.4}$$

For $s \in \mathbb{R}$, define the Sobolev space $H^s(\mathbb{R})$ with the norm

$$\|f\|_{H^s} := \|\langle\xi\rangle^s\hat{f}(\xi)\|_{L^2}. \tag{2.5}$$

We also use the unitary semiclassical Fourier transform $\mathcal{F}_h$ on $L^2(\mathbb{R})$ defined by

$$\mathcal{F}_h f(\xi) = h^{-1/2}\int_{\mathbb{R}} e^{-2\pi i x\xi/h} f(x)\,dx = h^{-1/2}\hat{f}\Big(\frac{\xi}{h}\Big), \quad h > 0. \tag{2.6}$$

The following identity holds for all $X, Y \subset \mathbb{R}$, $x_0, y_0 \in \mathbb{R}$, and h, and follows directly from the fact that $\mathcal{F}_h^*$ conjugates shifts to multiplication operators:

$$\|\,\mathbb{1}_{X+x_0}\,\mathcal{F}_h^*\,\mathbb{1}_{Y+y_0}\,\|_{L^2\to L^2} = \|\,\mathbb{1}_X\,\mathcal{F}_h^*\,\mathbb{1}_Y\,\|_{L^2\to L^2}. \tag{2.7}$$

We also note the following corollary of the triangle inequality:

$$X \subset \bigcup_j X_j, \quad Y \subset \bigcup_k Y_k \implies \|\,\mathbb{1}_X\,\mathcal{F}_h^*\,\mathbb{1}_Y\,\|_{L^2\to L^2} \le \sum_{j,k}\|\,\mathbb{1}_{X_j}\,\mathcal{F}_h^*\,\mathbb{1}_{Y_k}\,\|_{L^2\to L^2}. \tag{2.8}$$

Finally, we record the following version of Hölder's inequality:

$$\sum_j a_j^\kappa\cdot b_j^{1-\kappa} \le \Big(\sum_j a_j\Big)^\kappa\cdot\Big(\sum_j b_j\Big)^{1-\kappa}, \quad a_j, b_j \ge 0,\ \kappa\in(0,1). \tag{2.9}$$

2.2. *Regular sets.* We now establish properties of δ-regular sets (see Definition 1.1), some of which have previously appeared in [DZ16]. For the reader's convenience, we first give a few examples:

- $\{0\}$ is 0-regular on scales 0 to ∞ with constant $C_R = 1$;
- $[0,1]$ is 1-regular on scales 0 to 1 with constant 2;
- the mid-third Cantor set $\mathcal{C} \subset [0,1]$ is $\log_2 3$-regular on scales 0 to 1 with constant 100 (see [DJ18, §5.2] for examples of more general Cantor sets);
- the set $[0,1] \sqcup \{2\}$ cannot be δ-regular on scales 0 to 1 with any constant for any δ;
- the set $[0, h^{1/2}]$ cannot be δ-regular on scales h to 1 with any h-independent constant for any δ (here $0 < h \ll 1$).

We next show that certain operations preserve the class of δ-regular sets if we allow to increase the regularity constant and shrink the scales on which regularity is imposed. The precise dependence of the new regularity constant on the original one, though specified in the lemmas below, is not important for our later proofs.

LEMMA 2.1 (Affine transformations). *Let X be a δ-regular set with constant C_R on scales α_0 to α_1. Fix $\lambda > 0$ and $y \in \mathbb{R}$. Then the set $\widetilde{X} := y + \lambda X$ is δ-regular with constant C_R on scales $\lambda\alpha_0$ to $\lambda\alpha_1$.*

Proof. This is straightforward to verify, taking the measure

$$\mu_{\widetilde{X}}(A) := \lambda^\delta \mu_X\big(\lambda^{-1}(A - y)\big). \qquad \square$$

LEMMA 2.2 (Increasing the upper scale). *Let X be a δ-regular set with constant C_R on scales α_0 to α_1. Fix $T \geq 1$. Then X is δ-regular with constant $\widetilde{C}_R := 2TC_R$ on scales α_0 to $T\alpha_1$.*

Proof. Let I be an interval such that $\alpha_0 \leq |I| \leq T\alpha_1$. We first show the upper bound $\mu_X(I) \leq \widetilde{C}_R|I|^\delta$. For $\alpha_0 \leq |I| \leq \alpha_1$, this is immediate, so we may assume that $\alpha_1 < |I| \leq T\alpha_1$. Then I can be covered by $\lceil T \rceil \leq 2T$ intervals of size α_1 each, and therefore

$$\mu_X(I) \ \leq\ 2T \cdot C_R \alpha_1^\delta \ \leq\ \widetilde{C}_R |I|^\delta.$$

Now, assume that I is centered at a point in X. We show the lower bound $\mu_X(I) \geq \widetilde{C}_R^{-1}|I|^\delta$. As before, we may assume that $\alpha_1 < |I| \leq T\alpha_1$. Let $I' \subset I$ be the interval with the same center and $|I'| = \alpha_1$. Then

$$\mu_X(I) \ \geq\ \mu_X(I') \ \geq\ C_R^{-1}\alpha_1^\delta \ \geq\ \widetilde{C}_R^{-1}|I|^\delta. \qquad \square$$

LEMMA 2.3 (Neighborhoods). *Let X be a δ-regular set with constant C_R on scales α_0 to $\alpha_1 \geq 2\alpha_0$. Fix $T \geq 1$. Then the neighborhood $X(T\alpha_0) = X + [-T\alpha_0, T\alpha_0]$ is δ-regular with constant $\widetilde{C}_R := 4TC_R$ on scales $2\alpha_0$ to α_1.*

Proof. Put $\widetilde{X} := X(T\alpha_0)$, and define the measure $\mu_{\widetilde{X}}$ supported on $\widetilde{X}$ by convolution:

$$\mu_{\widetilde{X}}(A) := \frac{1}{T\alpha_0}\int_{-T\alpha_0}^{T\alpha_0} \mu_X(A + y)\, dy.$$

Let I be an interval such that $2\alpha_0 \leq |I| \leq \alpha_1$. Then

$$\mu_{\widetilde{X}}(I) \ \leq\ 2C_R|I|^\delta \ \leq\ \widetilde{C}_R|I|^\delta.$$

Now, assume additionally that I is centered at a point $x_1 \in \widetilde{X}$. Take $x_0 \in X$ such that $|x_0 - x_1| \leq T\alpha_0$, and let I' be the interval of size $\frac{1}{2}|I|$ centered at x_0. Then $\mu_X(I') \geq (2C_R)^{-1}|I|^\delta$. Let $J = x_0 - x_1 + [-\frac{1}{2}\alpha_0, \frac{1}{2}\alpha_0]$, then $J \cap [-T\alpha_0, T\alpha_0]$ is an interval of size at least $\frac{1}{2}\alpha_0$, and for each $y \in J$, we have $I' \subset I + y$. It follows that

$$\mu_{\widetilde{X}}(I) \ \geq\ \frac{1}{2T}\mu_X(I') \ \geq\ \widetilde{C}_R^{-1}|I|^\delta. \qquad \square$$

LEMMA 2.4 (Nonlinear transformations). *Assume that* $F : \mathbb{R} \to \mathbb{R}$ *is a* C^1 *diffeomorphism such that for some constant* $C_F \geq 1$,

$$C_F^{-1} \ \leq \ |\partial_x F| \ \leq \ C_F.$$

Let X *be a* δ*-regular set with constant* C_R *on scales* α_0 *to* $\alpha_1 \geq C_F^2 \alpha_0$. *Then* $F(X)$ *is a* δ*-regular set with constant* $\widetilde{C}_R := C_F C_R$ *on scales* $C_F\alpha_0$ *to* $C_F^{-1}\alpha_1$.

Proof. Put $\widetilde{X} := F(X)$, and define the measure $\mu_{\widetilde{X}}$ supported on $\widetilde{X}$ as a pullback:

$$\mu_{\widetilde{X}}(A) := \mu_X(F^{-1}(A)).$$

Let $\widetilde{I}$ be an interval with $C_F\alpha_0 \leq |\widetilde{I}| \leq C_F^{-1}\alpha_1$. Take the interval $I := F^{-1}(\widetilde{I})$. Then

$$C_F^{-1}|\widetilde{I}| \ \leq \ |I| \ \leq \ C_F|\widetilde{I}|.$$

In particular, $\alpha_0 \leq |I| \leq \alpha_1$. Therefore,

$$\mu_{\widetilde{X}}(\widetilde{I}) \ = \ \mu_X(I) \ \leq \ C_R|I|^\delta \ \leq \ \widetilde{C}_R|\widetilde{I}|^\delta.$$

If additionally $\widetilde{I}$ is centered at a point $\tilde{x} \in \widetilde{X}$, then I contains the interval I' of size $C_F^{-1}|\widetilde{I}|$ centered at $F^{-1}(\tilde{x}) \in X$. Therefore,

$$\mu_{\widetilde{X}}(\widetilde{I}) \ \geq \ \mu_X(I') \ \geq \ C_R^{-1}|I'|^\delta \ \geq \ \widetilde{C}_R^{-1}|\widetilde{I}|^\delta. \qquad \square$$

LEMMA 2.5 (Intersections with intervals). *Let* X *be a* δ*-regular set with constant* C_R *on scales* α_0 *to* α_1. *Fix two different intervals* $J \subset J'$ *with the same center and* $|J'| - |J| \geq \alpha_0$. *Assume that* $X \cap J$ *is nonempty and* $X \cap J' \subset J$. *Then* $X \cap J$ *is* δ*-regular with constant* C_R *on scales* α_0 *to* $\tilde{\alpha}_1 := \min(\alpha_1, |J'| - |J|)$.

Proof. Put $\widetilde{X} := X \cap J = X \cap J'$, and consider the measure $\mu_{\widetilde{X}}(A) := \mu_X(A \cap J')$ supported on $\widetilde{X}$. Let I be an interval with $\alpha_0 \leq |I| \leq \tilde{\alpha}_1$. Then

$$\mu_{\widetilde{X}}(I) \ \leq \ \mu_X(I) \ \leq \ C_R|I|^\delta.$$

Now, assume that I is centered at some $x \in \widetilde{X}$. Then $x \in J$ and thus $I \subset J'$, giving

$$\mu_{\widetilde{X}}(I) \ = \ \mu_X(I) \ \geq \ C_R^{-1}|I|^\delta. \qquad \square$$

We now establish further properties of δ-regular sets, starting with a quantitative version of the fact that every δ-regular set with $\delta < 1$ is nowhere dense:

LEMMA 2.6 (The missing subinterval property). *Let* X *be a* δ*-regular set with constant* C_R *on scales* α_0 *to* α_1, *and* $0 \leq \delta < 1$. *Fix an integer*

$$L \geq (3C_R)^{\frac{2}{1-\delta}}. \tag{2.10}$$

Assume that I is an interval with $\alpha_0 \le |I|/L < |I| \le \alpha_1$ and $I_1, \dots, I_L$ is the partition of I into intervals of size $|I|/L$. Then there exists ℓ such that $X \cap I_\ell = \emptyset$.

Proof. Using Lemma 2.1, we reduce to the case $I = [0, L]$, $\alpha_0 \le 1 < L \le \alpha_1$. Then $I_\ell = [\ell - 1, \ell]$. We argue by contradiction, assuming that each I_ℓ intersects X. Then $I'_\ell := [\ell - 3/2, \ell + 1/2]$ contains a size 1 interval centered at a point in X and thus

$$\mu_X(I'_\ell) \ge C_R^{-1} \quad \text{for all } \ell = 1, \dots, L.$$

On the other hand, $\bigcup_{\ell=1}^L I'_\ell = [-1/2, L + 1/2]$ can be covered by two intervals of size L and each point lies in at most three of the intervals I'_ℓ. Therefore,

$$C_R^{-1} L \ \le\ \sum_{\ell=1}^{L} \mu_X(I'_\ell) \ \le\ 3\mu_X\Big(\bigcup_{\ell=1}^{L} I'_\ell\Big) \ \le\ 6 C_R L^\delta,$$

which contradicts (2.10). □

We next obtain the following fact used in Section 4.1:

LEMMA 2.7 (Splitting into smaller regular sets). *Let X be a δ-regular set with constant C_R on scales α_0 to α_1, and assume that $0 \le \delta < 1$ and $(4C_R)^{\frac{2}{1-\delta}} \alpha_0 \le \rho \le \alpha_1$. Then there exists a collection of disjoint intervals $\mathcal{J}$ such that*

$$X = \bigsqcup_{J \in \mathcal{J}} (X \cap J); \quad (4C_R)^{-\frac{2}{1-\delta}} \rho \le |J| \le \rho \quad \textit{for all } J \in \mathcal{J}, \tag{2.11}$$

and each $X \cap J$ is δ-regular with constant $\widetilde{C}_R := (4C_R)^{\frac{2}{1-\delta}} C_R$ on scales α_0 to ρ.

Proof. Fix an integer L satisfying (2.10) and $L \le (4C_R)^{\frac{2}{1-\delta}}$. Consider the intervals

$$I_\ell := \frac{\rho}{L}[\ell, \ell + 1], \quad \ell \in \mathbb{Z}.$$

By Lemma 2.6, for each ℓ, at least one of the intervals $I_\ell, I_{\ell+1}, \dots, I_{\ell+L-1}$ does not intersect X. Define the collection $\mathcal{J}$ as follows: $J \in \mathcal{J}$ if and only if $J = I_\ell \cup \dots \cup I_r$ for some $\ell \le r$, each of the intervals $I_\ell, \dots, I_r$ intersects X, but $I_{\ell-1}, I_{r+1}$ do not intersect X. Then (2.11) holds.

For each $J = I_\ell \cup \dots \cup I_r \in \mathcal{J}$, take $J' := I_{\ell-1} \cup \dots \cup I_{r+1}$. Then $X \cap J' \subset J$ and $|J'| - |J| = 2\rho/L$. By Lemma 2.5, $X \cap J$ is δ-regular with constant C_R on scales α_0 to $2\rho/L$. Then by Lemma 2.2, $X \cap J$ is δ-regular with constant $\widetilde{C}_R$ on scales α_0 to ρ. □

The following covering statement is used in the proof of Lemma 3.1:

LEMMA 2.8 (The small cover property). *Let X be a δ-regular set with constant C_R on scales α_0 to α_1. Let I be an interval, and let $\rho > 0$ satisfy $\alpha_0 \le \rho \le |I| \le \alpha_1$. Then there exists a nonoverlapping collection*[1] $\mathcal{J}$ *of* $N_{\mathcal{J}}$ *intervals of size ρ each such that*

$$X \cap I \ \subset \ \bigcup_{J \in \mathcal{J}} J, \quad N_{\mathcal{J}} \le 12 C_R^2 \Big(\frac{|I|}{\rho} \Big)^{\delta}.$$

Proof. Let $\mathcal{J}$ consist of all intervals of the form $\rho[j, j+1]$, $j \in \mathbb{Z}$ which intersect $X \cap I$. Then $X \cap I \subset \bigcup_{J \in \mathcal{J}} J$. It remains to prove the upper bound on $N_{\mathcal{J}}$. For this we use an argument similar to the one in Lemma 2.6.

For each $J \in \mathcal{J}$, let $J' \supset J$ be the interval with the same center and $|J'| = 2\rho$. Since J intersects X, J' contains an interval of size ρ centered at a point in X. Therefore,

$$\mu_X(J') \ge C_R^{-1} \rho^{\delta}.$$

On the other hand, $\bigcup_{J \in \mathcal{J}} J' \subset I(\frac{3}{2}\rho)$ can be covered by four intervals of size $|I|$ and each point lies in at most three of the intervals J'. Therefore,

$$N_{\mathcal{J}} \cdot C_R^{-1} \rho^{\delta} \ \le \ \sum_{J \in \mathcal{J}} \mu_X(J') \ \le \ 3\mu_X\Big(\bigcup_{J \in \mathcal{J}} J' \Big) \ \le \ 12 C_R |I|^{\delta},$$

which implies the upper bound on $N_{\mathcal{J}}$. □

LEMMA 2.9 (The Lebesgue measure of a regular set). *Let $X \subset [-\alpha_1, \alpha_1]$ be a δ-regular set with constant C_R on scales $\alpha_0 > 0$ to α_1. Then the Lebesgue measure of X satisfies*

$$\mu_L(X) \le 24 C_R^2 \alpha_1^{\delta} \alpha_0^{1-\delta}. \tag{2.12}$$

Proof. Applying Lemma 2.8 with $I := [0, \alpha_1]$, $\rho := \alpha_0$, we cover $X \cap I$ with at most $12 C_R^2 (\alpha_1/\alpha_0)^{\delta}$ intervals of size α_0 each. It follows that

$$\mu_L(X \cap I) \ \le \ 12 C_R^2 \Big(\frac{\alpha_1}{\alpha_0} \Big)^{\delta} \cdot \alpha_0 \ = \ 12 C_R^2 \alpha_1^{\delta} \alpha_0^{1-\delta}.$$

Repeating the argument with $I := [-\alpha_1, 0]$ and combining the resulting two bounds, we get (2.12). □

We finally describe a tree discretizing a δ-regular set. (This tree is simpler than the one used in [DZ16] and [DJ18] because we do not merge consecutive intervals.) Let $X \subset \mathbb{R}$ be a set, and fix an integer $L \ge 2$, the base of the discretization. Put

$$V_n(X) := \Big\{ I = \Big[\frac{j}{L^n}, \frac{j+1}{L^n} \Big] \ \Big| \ j \in \mathbb{Z}, \ I \cap X \ne \emptyset \Big\}, \quad n \in \mathbb{Z}. \tag{2.13}$$

[1] A collection of intervals is nonoverlapping if the intersection of each two different intervals is either empty or consists of one point.

Note that $X \subset \bigcup_{I \in V_n(x)} I$ for all n. Moreover, each $I' \in V_n(X)$ is contained in exactly one $I \in V_{n-1}(X)$; we say that I is the *parent* of I' and I' is a *child* of I. Each interval has at most L children.

The next lemma, used in Section 3.4, follows immediately from Lemma 2.6:

LEMMA 2.10 (Each parent is missing a child). *Let X be a δ-regular set with constant C_R on scales α_0 to α_1, and $0 \le \delta < 1$. Let L satisfy* (2.10) *and take $n \in \mathbb{Z}$ such that $\alpha_0 \le L^{-n-1} \le L^{-n} \le \alpha_1$. Then each $I \in V_n(X)$ has at most $L - 1$ children.*

2.3. *The multiplier theorem.* We next present the multiplier theorem originally due to Beurling–Malliavin [BM62], which is the key harmonic analysis tool in the proof of Theorem 4. It will be used in the proof of Lemma 3.1 below, with a weight ω tailored to the fractal set Y. We refer the reader to Mashreghi–Nazarov–Havin [MNK05] for a discussion of the history of this theorem and recent results.

THEOREM 5 ([MNK05, Th. BM1]). *Let $\omega \in C^1(\mathbb{R}; (0, 1])$ satisfy the conditions*

$$\int_{\mathbb{R}} \frac{|\log \omega(\xi)|}{1 + \xi^2}\, d\xi < \infty, \tag{2.14}$$

$$\sup |\partial_\xi \log \omega| < \infty. \tag{2.15}$$

Then for each $c_0 > 0$, there exists a function $\psi \in L^2(\mathbb{R})$ such that

$$\operatorname{supp} \psi \subset [-c_0, c_0], \quad |\widehat{\psi}| \le \omega, \quad \psi \not\equiv 0. \tag{2.16}$$

Remark. Condition (2.14) states that $\omega(\xi)$ does not come too close to 0 too often as $|\xi| \to \infty$. This condition is necessary to have a compactly supported $\psi \not\equiv 0$ with $|\widehat{\psi}| \le \omega$, see (3.6).

In Section 3.1 we will use the following quantitative refinement of Theorem 5:

LEMMA 2.11. *For all $C_0, c_0 > 0$, there exists $c = c(C_0, c_0) > 0$ such that the following holds: Let $\omega \in C^1(\mathbb{R}; (0, 1])$ be a weight function satisfying*

$$\int_{\mathbb{R}} \frac{|\log \omega(\xi)|}{1 + \xi^2}\, d\xi \le C_0, \tag{2.17}$$

$$\sup |\partial_\xi \log \omega| \le C_0. \tag{2.18}$$

Then there exists a function $\psi \in L^2(\mathbb{R})$ such that

$$\operatorname{supp} \psi \subset [-c_0, c_0], \quad |\widehat{\psi}| \le \omega^c, \quad \|\widehat{\psi}\|_{L^2(-1,1)} \ge c. \tag{2.19}$$

Proof. We argue by contradiction. Fix $C_0, c_0 > 0$ such that Lemma 2.11 does not hold. Then there exists a sequence of weights $\omega_1, \omega_2, \dots$ each satisfying (2.17), (2.18) and such that for each $\psi \in L^2(\mathbb{R})$,

$$\text{(2.20)} \qquad \operatorname{supp} \psi \subset [-c_0, c_0], \quad |\widehat{\psi}| \leq (\omega_n)^{2^{-n}} \implies \|\widehat{\psi}\|_{L^2(-1,1)} \leq 2^{-n}.$$

Define the weight ω by

$$\omega := \prod_{n=1}^{\infty} (\omega_n)^{2^{-n}}.$$

Then ω satisfies (2.14) and (2.15). By Theorem 5 there exists $\psi \in L^2(\mathbb{R})$ satisfying (2.16). For each n, we have $|\widehat{\psi}| \leq \omega \leq (\omega_n)^{2^{-n}}$. Then (2.20) implies that $\|\widehat{\psi}\|_{L^2(-1,1)} \leq 2^{-n}$ for all n and thus $\widehat{\psi} = 0$ on $(-1, 1)$. However, since ψ is compactly supported, $\widehat{\psi}$ is real analytic and thus $\psi \equiv 0$, which contradicts (2.16). ☐

2.4. *Harmonic measures on slit domains.* We finally review the facts we need from the theory of harmonic measures, referring the reader to Conway [Con95, Ch. 21], Aleman–Feldman–Ross [AFR09], and Itô–McKean [IM74, §7] for more details. These facts are used in Section 3.2 below.

Let $\Omega \subset \mathbb{C}$ be a bounded open domain with smooth boundary $\partial\Omega$, and take $t \in \Omega$. The *harmonic measure of* Ω *centered at* t, denoted μ_t^Ω, is defined as follows. Let $f \in C(\partial\Omega)$, and let u be the harmonic extension of f, namely, the unique function such that $u \in C(\overline{\Omega})$, $u|_{\partial\Omega} = f$, and u is harmonic in Ω. Then for all f, we have

$$u(t) = \int_{\partial\Omega} f \, d\mu_t^\Omega.$$

Such μ_t^Ω is a (nonnegative) probability measure; indeed, nonnegativity follows from the maximum principle and $\mu_t^\Omega(\partial\Omega) = 1$ since 1 is a harmonic function. Moreover, since $\partial\Omega$ is smooth, μ_t^Ω is absolutely continuous with respect to the arclength measure μ_L on $\partial\Omega$. We denote by $\frac{d\mu_t^\Omega}{d\mu_L}$ the corresponding Radon–Nikodym derivative.

Since harmonic functions are invariant under conformal transformations, we have the following fact: if Ω' is another bounded domain with smooth boundary and $\varkappa : \Omega \to \Omega'$ is a conformal transformation extending to a homeomorphism $\overline{\Omega} \to \overline{\Omega}'$, then

$$\text{(2.21)} \qquad \mu_t^\Omega(A) = \mu_{\varkappa(t)}^{\Omega'}(\varkappa(A)) \quad \text{for all } t \in \Omega,\ A \subset \partial\Omega.$$

Another interpretation of harmonic measure is as follows (see, for instance, [IM74, §7.12]): let $(W_\tau)_{\tau \geq 0}$ be the Brownian motion starting at $W_0 = t$. Then μ_t^Ω is the probability distribution of the point on $\partial\Omega$ through which W_τ exits Ω first.

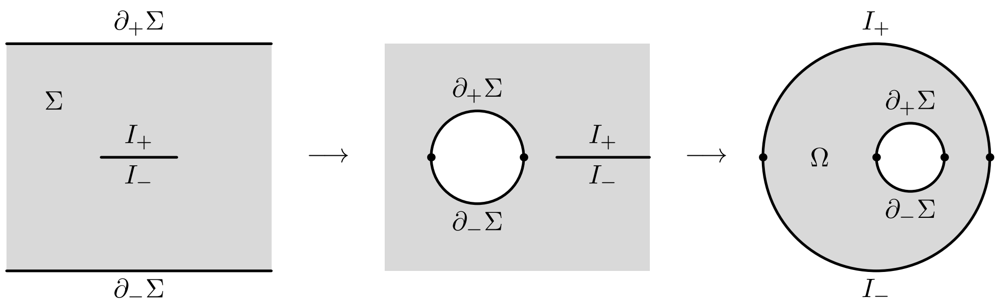

Figure 2. A conformal transformation $\varkappa$ from the domain Σ defined in (2.22) onto an annulus shaped domain Ω with smooth boundary, depicted here as a composition of two transformations. We mark the images of the components of $\partial\Sigma$. If, for simplicity, we put $r := \pi/2$, $I_0 := [0, \log 2]$, then the first transformation is $\zeta = \frac{e^z-1}{2-e^z}$ and the second one is $w = \frac{1}{1-i\sqrt{\zeta}}$, where we take the branch of the square root which sends $\mathbb{C} \setminus [0,\infty)$ to the upper half-plane.

We henceforth consider the following domain in $\mathbb{C}$:

$$\Sigma := \{x + iy \mid x \in \mathbb{R},\ |y| < r\} \setminus I_0, \tag{2.22}$$

where $I_0 \subset \mathbb{R}$ is an interval with $0 < |I_0| \leq 1$ and $r \in (0, 1)$. The domain Σ is unbounded, and it does not have a smooth boundary because of the slit I_0, however one can still define the harmonic measure μ_t^Σ for each $t \in \Sigma$. One way to see this is by taking a conformal transformation $\varkappa$ which maps Σ onto an annulus shaped domain Ω with smooth boundary (see Figure 2 and [AFR09, §2.3]), and define μ_t^Σ by (2.21). However $\varkappa$ does not extend to a homeomorphism $\partial\Sigma \to \partial\Omega$ since the images of sequences approaching the same point of I_0 from the top and from the bottom have different limits. To fix this issue, we redefine $\partial\Sigma$ as consisting of two lines

$$\partial_\pm\Sigma := \{x \pm ir \mid x \in \mathbb{R}\} \tag{2.23}$$

and two copies $I_\pm$ of the interval I_0 corresponding to limits as $\operatorname{Im} z \to \pm 0$. This is in agreement with the Brownian motion interpretation as it encodes in which direction W_τ crosses I_0.

The importance of harmonic measures in this paper is due to the following

LEMMA 2.12. *Assume that the function F is holomorphic and bounded on Σ and extends continuously to $\partial\Sigma$. Then for each $t \in \Sigma$, we have*

$$\log |F(t)| \leq \int_{\partial\Sigma} \log |F| \, d\mu_t^\Sigma. \tag{2.24}$$

Proof. The function $\log|F|$ is subharmonic and bounded above on Σ. Then (2.24) follows from Perron's construction of solutions to the Dirichlet problem via subharmonic functions; see, for instance, [Con95, §19.7]. $\square$

We now show several estimates on the harmonic measure μ_t^Σ for the domain (2.22). Denote by $d(t,I_0)$ the distance from $t\in\mathbb{R}$ to the interval I_0. In Lemmas 2.13–2.15 below, the precise dependence of the bounds on I_0 is irrelevant. However, the dependence on r is important.

LEMMA 2.13. *Assume that $t\in\Sigma\cap\mathbb{R}$ satisfies $d(t,I_0)\geq\frac{1}{10}|I_0|$. Then*

$$\Big\|\frac{d\mu_t^\Sigma}{d\mu_L}\Big\|_{L^p(I_\pm)}\leq C_p\quad\textit{for all } p\in[1,2), \tag{2.25}$$

where $C_p>0$ depends only on $|I_0|$ and p.

Proof. Without loss of generality, we may assume that $I_0=[0,\ell]$, where $0<\ell\leq 1$. We have $\Sigma\subset\widetilde\Sigma$, where

$$\widetilde\Sigma:=\mathbb{C}\setminus I_0.$$

Then (see, for instance, [Con95, Cor. 21.1.14])

$$\mu_t^\Sigma|_{I_\pm}\ \leq\ \mu_t^{\widetilde\Sigma}|_{I_\pm}. \tag{2.26}$$

We can also interpret (2.26) in stochastic terms: every trajectory of the Brownian motion starting at t which hits some $A\subset I_\pm$ before hitting $\partial\Sigma\setminus A$ also has the property that it hits A before $\partial\widetilde\Sigma\setminus A$.

To compute $\mu_t^{\widetilde\Sigma}$ we use the conformal transformation

$$z\mapsto w=\sqrt{\frac{t-\ell}{t}\cdot\frac{z}{\ell-z}},$$

which maps $\widetilde\Sigma$ to the upper half-plane, $I_\pm$ to $\pm[0,\infty)$, and t to i. Using the well-known formula for the harmonic measure of the upper half-plane, we get

$$\mu_t^{\widetilde\Sigma}|_{I_\pm}=\frac{|dw|}{\pi(1+w^2)}=\sqrt{\frac{t(t-\ell)}{z(\ell-z)}}\cdot\frac{|dz|}{2\pi|t-z|}.$$

Since $|t-z|\geq\frac{\ell}{10}$ for all $z\in[0,\ell]$, it follows that

$$\frac{d\mu_t^\Sigma}{d\mu_L}(z)\leq\frac{1}{\sqrt{z(\ell-z)}}\quad\text{for all } z\in I_\pm,$$

which implies (2.25). $\square$

LEMMA 2.14. *Assume that $t\in\Sigma\cap\mathbb{R}$ satisfies $d(t,I_0)\leq 1$. Then*

$$\frac{d\mu_t^\Sigma}{d\mu_L}(x\pm ir)\leq\frac{2}{r}e^{-d(x,I_0)},\quad x\in\mathbb{R}.$$

Proof. We have $\Sigma \subset \widetilde{\Sigma}$, where

$$\widetilde{\Sigma} := \{(x + iy) \mid x \in \mathbb{R},\ |y| < r\}.$$

Therefore, similarly to (2.26) we have $\mu_t^\Sigma|_{\partial_\pm\Sigma} \le \mu_t^{\widetilde{\Sigma}}|_{\partial_\pm\Sigma}$. The conformal transformation

$$z \mapsto w = i \exp\left(\frac{\pi(z - t)}{2r}\right)$$

maps $\widetilde{\Sigma}$ to the upper half-plane, $\partial_\pm\Sigma$ to $\mp[0, \infty)$, and t to i. Therefore,

$$\mu_t^{\widetilde{\Sigma}}|_{\partial_\pm\Sigma} = \frac{|dw|}{\pi(1 + w^2)} = \frac{|dz|}{4r \cosh\left(\pi \frac{\operatorname{Re} z - t}{2r}\right)}.$$

It follows that

$$\left|\frac{d\mu_t^\Sigma}{d\mu_L}(x \pm ir)\right| \ \le\ \frac{1}{2r} e^{-\frac{\pi|x-t|}{2r}} \ \le\ \frac{1}{2r} e^{-|x-t|} \ \le\ \frac{2}{r} e^{-d(x,I_0)}. \qquad \square$$

Lemma 2.15. *Assume that $t \in \Sigma \cap \mathbb{R}$ satisfies $d(t, I_0) \le 1$. Then*

$$\mu_t^\Sigma(I_\pm) \ge \frac{|I_0|}{8} e^{-2/r}.$$

Proof. Without loss of generality, we may assume that $I_0 = [-\ell, 0]$, where $0 < \ell \le 1$ and $0 < t \le 1$. We have $\Sigma \supset \widetilde{\Sigma}$, where

$$\widetilde{\Sigma} = \{x + iy \mid x \in \mathbb{R},\ |y| < r\} \setminus (-\infty, 0].$$

Similarly to (2.26), we have $\mu_t^\Sigma(I_\pm) \ge \mu_t^{\widetilde{\Sigma}}(I_\pm)$. The conformal transformation

$$z \mapsto w = \sqrt{\frac{1 - e^{\pi z/r}}{e^{\pi t/r} - 1}}$$

maps $\widetilde{\Sigma}$ to the upper half-plane, t to i, and

$$I_\pm \ \mapsto\ \mp\left[0, \sqrt{\frac{1 - e^{-\pi\ell/r}}{e^{\pi t/r} - 1}}\right] \ \supset\ \mp\left[0, \frac{\sqrt{\ell}}{2} e^{-\frac{\pi}{2r}}\right].$$

It follows that

$$\mu_t^\Sigma(I_\pm) \ \ge\ \mu_t^{\widetilde{\Sigma}}(I_\pm) \ \ge\ \frac{1}{\pi} \arctan\left(\frac{\sqrt{\ell}}{2} e^{-\frac{\pi}{2r}}\right) \ \ge\ \frac{\ell}{8} e^{-2/r}. \qquad \square$$

3. The general fractal uncertainty principle

In this section, we prove Theorem 4. We establish the components of the argument in Sections 3.1 and 3.2 and combine them in Section 3.3 to obtain a unique continuation estimate for functions with Fourier supports in regular sets. In Section 3.4, we iterate this estimate to finish the proof.

3.1. *An adapted multiplier.* We first construct a compactly supported function whose Fourier transform decays much faster than $\exp(-|\xi|/\log|\xi|)$ as $|\xi| \to \infty$ on a δ-regular set. We henceforth denote

$$\theta(\xi) := \log(10 + |\xi|)^{-\frac{1+\delta}{2}}. \tag{3.1}$$

The function ψ constructed in the lemma below is used as a convolution kernel in the proof of Lemma 3.4. We remark that α_1 is a finite but large parameter, and it is important that the constants in the estimates do not depend on α_1.

LEMMA 3.1. *Assume that $Y \subset [-\alpha_1, \alpha_1]$ is a δ-regular set with constant C_R on scales 2 to α_1, and $\delta \in (0,1)$. Fix $c_1 > 0$. Then there exist a constant $c_2 > 0$ depending only on δ, C_R, c_1 and a function $\psi \in L^2(\mathbb{R})$ such that*

$$\operatorname{supp}\psi \subset \Big[-\frac{c_1}{10}, \frac{c_1}{10}\Big], \tag{3.2}$$

$$\|\widehat{\psi}\|_{L^2([-1,1])} \geq c_2, \tag{3.3}$$

$$|\widehat{\psi}(\xi)| \leq \exp(-c_2\langle\xi\rangle^{1/2}) \quad \textit{for all } \xi \in \mathbb{R}, \tag{3.4}$$

$$|\widehat{\psi}(\xi)| \leq \exp\big(-c_2\theta(\xi)|\xi|\big) \quad \textit{for all } \xi \in Y. \tag{3.5}$$

Remarks. 1. It is essential that condition (3.5) be imposed only on Y. Indeed, it is a standard fact in harmonic analysis (see, for instance, [HJ94, §1.5.4]) that every compactly supported $\psi \in L^2(\mathbb{R})$ with $\psi \not\equiv 0$ satisfies

$$\int_{\mathbb{R}} \frac{\log|\widehat{\psi}(\xi)|}{1+|\xi|^2}\, d\xi > -\infty, \tag{3.6}$$

which would contradict (3.5) if Y were replaced by $\mathbb{R}$.

2. In (3.4) one could replace $\exp(-\langle\xi\rangle^{1/2})$ by any weight satisfying (2.14), (2.15) and decaying as $|\xi| \to \infty$ faster than any negative power of $|\xi|$. Also, the proof below works with $\frac{1+\delta}{2}$ replaced by 1, though in that case (3.5) would not suffice for our application.

3. Recently Jin–Zhang [JZ17] have shown that the Hilbert transform of the logarithm of the weight ω constructed in the proof below has uniformly bounded Lipschitz constant. Then Lemma 3.1 can be proved using a weaker (and considerably easier to prove) version of the Beurling–Malliavin Theorem [MNK05, Th. 1].

Proof. We will use Lemma 2.11. For this we construct a weight adapted to the set Y. Define $n_1 \in \mathbb{N}$ by the inequality $2^{n_1} \leq \alpha_1 < 2^{n_1+1}$. For every $n \in \mathbb{N}$, $n \leq n_1$, put

$$A_n := [-2^{n+1}, -2^n] \sqcup [2^n, 2^{n+1}], \quad \rho_n := n^{-\frac{1+\delta}{2}} \cdot 2^n \geq 2.$$

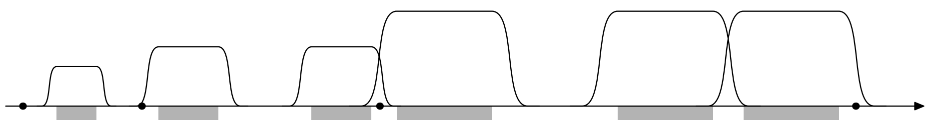

Figure 3. The functions χ_J featured in (3.7), where the intervals J are shaded. The dots mark powers of 2.

Using Lemma 2.8, construct a nonoverlapping collection $\mathcal{J}_n$ of N_n intervals of size ρ_n each such that all elements of $\mathcal{J}_n$ intersect A_n and

$$Y \cap A_n \subset \bigcup_{J \in \mathcal{J}_n} J, \quad N_n \leq 24 C_R^2 \cdot \left(\frac{2^n}{\rho_n}\right)^\delta .$$

Fix a cutoff function

$$\chi(\xi) \in C^1(\mathbb{R}; [0,1]), \quad \sup |\partial_\xi \chi| \leq 10, \quad \operatorname{supp} \chi \subset [-1,1], \quad \chi = 1 \text{ on } \Big[-\frac{1}{2}, \frac{1}{2}\Big].$$

For an interval J with center ξ_J, define the function $\chi_J \in C^1(\mathbb{R})$ by

$$\chi_J(\xi) = |J| \cdot \chi\left(\frac{\xi - \xi_J}{|J|}\right),$$

so that

$$0 \leq \chi_J \leq |J|, \quad \sup |\partial_\xi \chi_J| \leq 10,$$
$$\operatorname{supp} \chi_J \subset \widetilde{J} := \xi_J + \Big[-|J|, |J|\Big], \quad \chi_J = |J| \quad \text{on } J.$$

Now, define the weight $\omega \in C^1(\mathbb{R}; (0,1])$ by (see Figure 3)

$$\omega(\xi) := \exp(-2\langle\xi\rangle^{1/2}) \cdot \prod_{n=1}^{n_1} \prod_{J \in \mathcal{J}_n} \exp(-10\chi_J). \tag{3.7}$$

For each $\xi \in Y$, $|\xi| \geq 2$, there exist $n \in [1, n_1]$ and $J \in \mathcal{J}_n$ such that $\xi \in J$. Also, $\exp(-2\langle\xi\rangle^{1/2}) \leq \exp(-\theta(\xi)|\xi|)$ for $|\xi| \leq 2$. Therefore,

$$\omega(\xi) \leq \exp(-\langle\xi\rangle^{1/2}) \qquad \text{for all } \xi \in \mathbb{R}, \tag{3.8}$$

$$\omega(\xi) \leq \exp\big(-\theta(\xi)|\xi|\big) \quad \text{for all } \xi \in Y. \tag{3.9}$$

Since each ξ lies in at most 500 intervals in $\bigcup_{n=1}^{n_1} \bigcup_{J \in \mathcal{J}_n} \widetilde{J}$, we have

$$\sup |\partial_\xi \log \omega| \leq 10^5.$$

Next,

$$\begin{aligned}\int_{\mathbb{R}} \frac{|\log \omega(\xi)|}{1+\xi^2}\, d\xi &\leq 100 + 10^5 \sum_{n=1}^{n_1} N_n \cdot 2^{-2n} \rho_n^2 \\ &\leq 10^5 + 10^7 C_R^2 \sum_{n=1}^{\infty} \left(\frac{2^n}{\rho_n}\right)^{\delta-2} =: C_0,\end{aligned}$$

where C_0 depends only on δ, C_R. Here we use the formula for ρ_n and the inequality $(1+\delta)(1-\delta/2) > 1$ valid for all $\delta \in (0,1)$.

We have verified that the weight ω satisfies (2.17) and (2.18). Applying Lemma 2.11 with $c_0 := c_1/10$, we construct $\psi \in L^2(\mathbb{R})$ satisfying (3.2) and

$$|\widehat{\psi}| \leq \omega^{c_2}, \quad \|\widehat{\psi}\|_{L^2(-1,1)} \geq c_2,$$

where the constant c_2 depends only on δ, C_R, c_1. By (3.8) and (3.9), ψ satisfies (3.4) and (3.5). $\square$

3.2. *A bound on functions with compact Fourier support.* We next use the harmonic measure estimates from Section 2.4 to obtain the following quantitative unique continuation estimate which is used in the proof of Lemma 3.4 below.

LEMMA 3.2. *Assume that $\mathcal{I}$ is a nonoverlapping collection of intervals of size 1 each, and for each $I \in \mathcal{I}$, we choose a subinterval $I'' \subset I$ with $|I''| = c_0 > 0$ independent of I. Then there exists a constant C depending only on c_0 such that for all $r \in (0,1)$, $0 < \kappa \leq e^{-C/r}$, and $f \in L^2(\mathbb{R})$ with $\hat{f}$ compactly supported, we have*

$$\sum_{I \in \mathcal{I}} \|f\|^2_{L^2(I)} \leq \frac{C}{r} \bigg(\sum_{I \in \mathcal{I}} \|f\|^2_{L^2(I'')} \bigg)^{\kappa} \cdot \|e^{2\pi r|\xi|} \hat{f}(\xi)\|^{2(1-\kappa)}_{L^2(\mathbb{R})}. \tag{3.10}$$

Remark. The bound (1.6) in the introduction follows from (3.10). To see this, take large K to be chosen later and decompose $f = f_1 + f_2$, where $\operatorname{supp} \hat{f}_1 \subset [-K, K]$, $\operatorname{supp} \hat{f}_2 \subset \mathbb{R} \setminus (-K, K)$. Put $r := \frac{1}{10}\theta(K)$, and apply (3.10) to f_1 (with I' taking the role of I''):

$$\|f_1\|^2_{L^2(\mathbb{R})} \leq \frac{C}{\theta(K)} \|f_1\|^{2\kappa}_{L^2(U')} \cdot \|f\|^{2(1-\kappa)}_{L^2(\mathbb{R})}, \tag{3.11}$$

where we use that $\|e^{2\pi r|\xi|} \hat{f}_1(\xi)\|_{L^2} \leq \|\exp(\theta(\xi)|\xi|) \hat{f}(\xi)\|_{L^2} \leq C_1 \|f\|_{L^2}$. Moreover,

$$\|f_2\|_{L^2} \leq e^{-\theta(K)K} \|\exp(\theta(\xi)|\xi|) \hat{f}(\xi)\|_{L^2} \leq C_1 e^{-\theta(K)K} \|f\|_{L^2}. \tag{3.12}$$

We have $\|f_1\|^{2\kappa}_{L^2(U')} \leq C(\|f\|^{2\kappa}_{L^2(U')} + \|f_2\|^{2\kappa}_{L^2(\mathbb{R})})$. Combining (3.11) with (3.12), we get

$$\|f\|^2_{L^2} = \|f_1\|^2_{L^2} + \|f_2\|^2_{L^2} \leq \frac{C}{\theta(K)} \left(\|f\|^{2\kappa}_{L^2(U')} \cdot \|f\|^{2(1-\kappa)}_{L^2} + e^{-2\theta(K)\kappa K} \|f\|^2_{L^2} \right), \tag{3.13}$$

where the constant C depends only on c_1, C_1. Since $\delta < 1$, we have

$$e^{-2\theta(K)\kappa K} / \theta(K) \to 0$$

as $K \to \infty$. We then fix K large enough depending on δ, c_1, C_1 to remove the last term on the right-hand side of (3.13), giving (1.6). The proof of the unique continuation bound in Section 3.3 is inspired by the above argument.

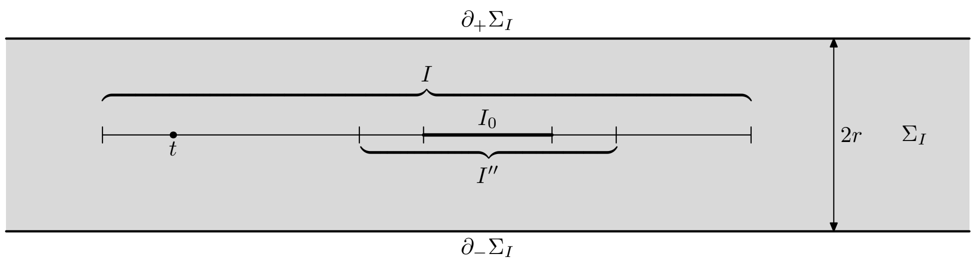

Figure 4. The slit domain Σ_I with the intervals $I_0 \subset I'' \subset I$.

Proof of Lemma 3.2. Since $\hat{f}$ is compactly supported, f has a holomorphic continuation F given by (2.3):

$$F(z) = \int_{\mathbb{R}} e^{2\pi i z\xi} \hat{f}(\xi)\, d\xi, \quad z \in \mathbb{C}; \quad f = F|_{\mathbb{R}}.$$

The function $F(z)$ is bounded on $\{|\operatorname{Im} z| \le r\}$ and

$$\int_{\mathbb{R}} |F(x \pm ir)|^2\, dx = \int_{\mathbb{R}} |e^{\mp 2\pi r\xi} \hat{f}(\xi)|^2\, d\xi \le \|e^{2\pi r|\xi|}\hat{f}(\xi)\|_{L^2}^2. \tag{3.14}$$

For each $I \in \mathcal{I}$, let $I_0 \Subset I''$ be the interval with the same center as I'' and $|I_0| = \frac{1}{2}c_0$. Define the slit domain (see Figure 4)

$$\Sigma_I := \{x + iy \mid x \in \mathbb{R},\ |y| < r\} \setminus I_0.$$

For each $t \in I \setminus I'' \subset \Sigma_I$, let $\mu_t = \mu_t^{\Sigma_I}$ be the harmonic measure of Σ_I on

$$\partial\Sigma_I = I_0 \sqcup \partial_-\Sigma_I \sqcup \partial_+\Sigma_I, \quad \partial_\pm\Sigma_I = \{x \pm ir \mid x \in \mathbb{R}\}$$

centered at t. Here we put together the top and bottom copies $I_\pm$ of I_0 (see the paragraph following (2.23)); that is, for $A \subset I_0$, we have

$$\mu_t(A) = \mu_t(A \cap I_+) + \mu_t(A \cap I_-).$$

By Lemma 2.15, we have

$$\kappa_I := \mu_t(I_0) \ \ge\ \frac{c_0}{8}e^{-2/r} \ \ge\ e^{-C/r} \ \ge\ \kappa,$$

where C denotes a constant depending only on c_0 (whose value might differ in different parts of the proof). By Lemma 2.12 we estimate

$$\begin{aligned} 2\log|f(t)| \le \int_{\partial\Sigma_I} 2\log|F(z)|\, d\mu_t(z) = 4\kappa_I \cdot \frac{1}{\kappa_I}\int_{I_0} \frac{\log|f(x)|}{2}\, d\mu_t(x) \\ + (1-\kappa_I)\cdot \frac{1}{1-\kappa_I}\int_{\partial_-\Sigma_I \sqcup \partial_+\Sigma_I} 2\log|F(z)|\, d\mu_t(z). \end{aligned}$$

Since the exponential function is convex and $\frac{1}{\kappa_I}\mu_t|_{I_0}$, $\frac{1}{1-\kappa_I}\mu_t|_{\partial_-\Sigma_I\sqcup\partial_+\Sigma_I}$ are probability measures, we obtain

$$|f(t)|^2 \le \Big(\frac{1}{\kappa_I}\int_{I_0}|f(x)|^{1/2}\,d\mu_t(x)\Big)^{4\kappa_I} \cdot \Big(\frac{1}{1-\kappa_I}\int_{\partial_-\Sigma_I\sqcup\partial_+\Sigma_I}|F(z)|^2\,d\mu_t(z)\Big)^{1-\kappa_I}.$$

Since $\kappa\le\kappa_I<1$ and $\lambda^{-\lambda}\le\exp(1/e)$ for all $\lambda>0$, it follows that

$$\begin{aligned}|f(t)|^2 \le 10\Big(\int_{I_0}|f(x)|^{1/2}\,d\mu_t(x)\Big)^{4\kappa}&\\ \cdot\Big(\Big(\int_{I_0}|f(x)|^{1/2}\,d\mu_t(x)\Big)^4+\int_{\partial_-\Sigma_I\sqcup\partial_+\Sigma_I}|F(z)|^2\,d\mu_t(z)\Big)^{1-\kappa}&.\end{aligned}\tag{3.15}$$

Recall that $t\in I\setminus I''$. By Lemma 2.13 with $p=4/3$ and Hölder's inequality, we have

$$\Big(\int_{I_0}|f(x)|^{1/2}\,d\mu_t(x)\Big)^4\le C\|f\|^2_{L^2(I_0)},\tag{3.16}$$

and by Lemma 2.14,

$$\int_{\partial_-\Sigma_I\sqcup\partial_+\Sigma_I}|F(z)|^2\,d\mu_t(z)\le\frac{C}{r}\int_{\operatorname{Im}z\in\{\pm r\}}e^{-d(\operatorname{Re}z,I)}|F(z)|^2dz.\tag{3.17}$$

Combining (3.15)–(3.17), we get

$$|f(t)|^2\le\frac{C}{r}\|f\|^{2\kappa}_{L^2(I_0)}\cdot\Big(\int_{\operatorname{Im}z\in\{0,\pm r\}}e^{-d(\operatorname{Re}z,I)}|F(z)|^2\,dz\Big)^{1-\kappa}.$$

Integrating in $t\in I\setminus I''$ and using Hölder's inequality (2.9), we estimate

$$\begin{aligned}\sum_{I\in\mathcal{I}}\|f\|^2_{L^2(I\setminus I'')}&\le\frac{C}{r}\Big(\sum_{I\in\mathcal{I}}\|f\|^2_{L^2(I_0)}\Big)^{\kappa}\\&\quad\cdot\Big(\sum_{I\in\mathcal{I}}\int_{\operatorname{Im}z\in\{0,\pm r\}}e^{-d(\operatorname{Re}z,I)}|F(z)|^2\,dz\Big)^{1-\kappa}\\&\le\frac{C}{r}\Big(\sum_{I\in\mathcal{I}}\|f\|^2_{L^2(I'')}\Big)^{\kappa}\cdot\Big(\int_{\operatorname{Im}z\in\{0,\pm r\}}|F(z)|^2\,dz\Big)^{1-\kappa}.\end{aligned}$$

Combining this with (3.14) and the bound

$$\sum_{I\in\mathcal{I}}\|f\|^2_{L^2(I'')}\le\Big(\sum_{I\in\mathcal{I}}\|f\|^2_{L^2(I'')}\Big)^{\kappa}\cdot\|e^{2\pi r|\xi|}\hat f(\xi)\|^{2(1-\kappa)}_{L^2},$$

we obtain (3.10). □

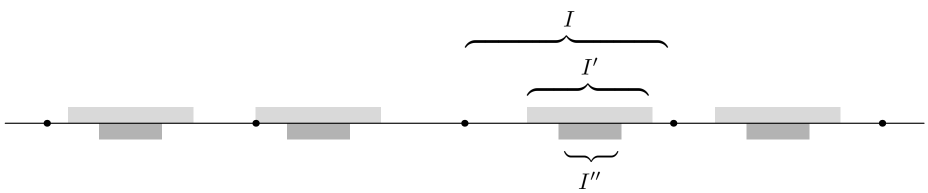

Figure 5. The sets U' (light shaded) and U'' (dark shaded) used in Proposition 3.3 and Lemma 3.4, with endpoints of the intervals $I \in \mathcal{I}$ denoted by dots.

3.3. *The iterative step.* The key component of the proof of Theorem 4 is the following unique continuation property for functions with Fourier support in a δ-regular set:

PROPOSITION 3.3. *Assume that $Y \subset [-\alpha_1, \alpha_1]$ is δ-regular with constant C_R on scales 1 to α_1, and $\delta \in (0,1)$. Take*

$$\mathcal{I} := \Big\{[j, j+1] \mid j \in \mathbb{Z}\Big\}, \tag{3.18}$$

and assume that for each $I \in \mathcal{I}$, we are given a subinterval $I' \subset I$ with $|I'| = c_1 > 0$ independent of I; see Figure 5. Define

$$U' := \bigcup_{I \in \mathcal{I}} I'.$$

Then there exists $c_3 > 0$ depending only on δ, C_R, c_1 such that for all $f \in L^2(\mathbb{R})$ with $\operatorname{supp} \hat{f} \subset Y$, we have

$$\|f\|_{L^2(U')} \geq c_3 \|f\|_{L^2(\mathbb{R})}. \tag{3.19}$$

Remark. It is important that U' be the union of infinitely many intervals, rather than a single interval. Indeed, the following estimate is false:

$$f \in L^2(\mathbb{R}), \quad \operatorname{supp} \hat{f} \subset [-1,1] \quad \Longrightarrow \quad \|f\|_{L^2(-1,1)} \geq c\|f\|_{L^2(-2,2)}$$

as can be seen by taking $f(x) = x^N \chi(x)$, where χ is a Schwartz function with $\operatorname{supp} \hat{\chi} \subset [-1,1]$, and letting $N \to \infty$.

Henceforth in this section C denotes a constant which only depends on δ, C_R, c_1 (whose value may differ in different places). Recall the definition (3.1) of $\theta(\xi)$. For $f \in L^2$, denote by $\|f\|_{H^{-10}}$ its Sobolev norm defined in (2.5); it will be useful for summing over different phase shifts of f in (3.30) below.

The main ingredient of the proof of Proposition 3.3 is the following lemma, which combines the results of Sections 3.1 and 3.2. It is proved by splitting f into two pieces, one which lives on frequencies $\leq K$ and the other, on frequencies $\geq K$.

LEMMA 3.4. *Assume that* $Z \subset [-\alpha_1, \alpha_1]$ *is* δ*-regular with constant* C_R *on scales* 1 *to* $\alpha_1 \geq 2$, *and* $\delta \in (0,1)$. *Let* $\{I'\}_{I\in\mathcal{I}}$ *be as in Proposition* 3.3. *Then we have for all* $f \in L^2(\mathbb{R})$ *with* $\operatorname{supp}\hat{f} \subset Z$, *all* $K > 10$, *and* $\kappa := \exp(-C/\theta(K))$,

$$\|\hat{f}\|^2_{L^2(-1,1)} \leq CK^{21}\Big(\|\,\mathbb{1}_{U'}f\|^2_{H^{-10}} + \exp\Big(-C^{-1}\theta(K)K\Big)\|f\|^2_{H^{-10}}\Big)^{\kappa}\cdot\|f\|^{2(1-\kappa)}_{H^{-10}}.$$

Proof. For each $I \in \mathcal{I}$, let $I'' \Subset I'$ be the interval with the same center as I' and $|I''| = \frac{1}{2}c_1$. Denote

$$U'' := \bigcup_{I\in\mathcal{I}} I''.$$

Let ψ be the function constructed in Lemma 3.1 for Y replaced by

$$Z(2) := Z + [-2,2].$$

Here $Z(2) \subset [-(\alpha_1+2), \alpha_1+2]$ is a δ-regular set with constant $100C_R$ on scales 2 to $\alpha_1 + 2$ by Lemmas 2.3 and 2.2. By (3.3)–(3.5) we have, for some $c_2 \in (0,1)$ depending only on δ, C_R, c_1,

$$\|\widehat{\psi}\|_{L^2(-1,1)} \geq c_2, \tag{3.20}$$

$$|\widehat{\psi}(\xi)| \leq \exp(-c_2\langle\xi\rangle^{1/2}) \qquad \text{for all } \xi\in\mathbb{R}, \tag{3.21}$$

$$|\widehat{\psi}(\xi)| \leq \exp\Big(-c_2\theta(\xi)|\xi|\Big) \quad \text{for all } \xi \in Z(2). \tag{3.22}$$

Take arbitrary $\eta \in [-2,2]$, and let $f_\eta(x) := e^{2\pi i\eta x}f(x)$, so that $\hat{f}_\eta(\xi) = \hat{f}(\xi-\eta)$. The freedom of choice in η will be useful in (3.29) below; for simplicity, the reader can consider the case $\eta = 0$. Put

$$g_\eta := f_\eta * \psi \in L^2(\mathbb{R}).$$

By the support condition (3.2),

$$g_\eta = (\mathbb{1}_{U'}f_\eta) * \psi \quad \text{on } U''. \tag{3.23}$$

By (2.4), (3.22), and since $\operatorname{supp}\hat{f}_\eta \subset Z + \eta \subset Z(2)$, we have

$$|\hat{g}_\eta(\xi)| \leq \exp\Big(-c_2\theta(\xi)|\xi|\Big)\cdot|\hat{f}_\eta(\xi)| \quad \text{for all } \xi\in\mathbb{R}. \tag{3.24}$$

Put

$$r := \frac{c_2}{10}\theta(K) \in (0,1).$$

Since $\theta(\xi)$ is decreasing for $\xi \geq 0$, we have

$$\sup_{|\xi|\leq K} e^{2\pi r|\xi|}\exp\Big(-c_2\theta(\xi)|\xi|\Big) \leq 1. \tag{3.25}$$

We now decompose g_η into low and high frequencies:

$$g_\eta = g_1 + g_2, \quad g_1, g_2 \in L^2, \quad \operatorname{supp}\hat{g}_1 \subset \{|\xi| \leq K\}, \quad \operatorname{supp}\hat{g}_2 \subset \{|\xi| \geq K\}.$$

Then by (3.24) and (3.25),

$$\|e^{2\pi r|\xi|}\hat{g}_1(\xi)\|_{L^2} \le CK^{10}\|f\|_{H^{-10}}, \tag{3.26}$$

$$\|g_2\|_{L^2} \le C\exp\big(-C^{-1}\theta(K)K\big)\|f\|_{H^{-10}}. \tag{3.27}$$

Applying Lemma 3.2 to the function g_1 and using (3.26), we get

$$\|g_1\|^2_{L^2} \le \frac{CK^{20}}{r}\|g_1\|^{2\kappa}_{L^2(U'')}\cdot\|f\|^{2(1-\kappa)}_{H^{-10}}, \quad \kappa := e^{-C/r}. \tag{3.28}$$

By (3.23), (2.4), and (3.21),

$$\begin{aligned}\|g_1\|_{L^2(U'')} &\le \|g_\eta\|_{L^2(U'')} + \|g_2\|_{L^2} \le \|(\mathbb{1}_{U'}f_\eta)*\psi\|_{L^2} + \|g_2\|_{L^2}\\ &\le C\|\mathbb{1}_{U'}f\|_{H^{-10}} + \|g_2\|_{L^2}.\end{aligned}$$

Then by (3.27) and (3.28) and since $r^{-1} \le CK$, we have for all $\eta \in [-2,2]$,

$$\begin{aligned}\|g_\eta\|^2_{L^2} &= \|g_1\|^2_{L^2} + \|g_2\|^2_{L^2}\\ &\le CK^{21}\Big(\|\mathbb{1}_{U'}f\|^2_{H^{-10}} + \exp\big(-C^{-1}\theta(K)K\big)\|f\|^2_{H^{-10}}\Big)^{\kappa}\cdot\|f\|^{2(1-\kappa)}_{H^{-10}}.\end{aligned}$$

It remains to use the following corollary of (3.20):

$$\begin{aligned}\|\hat{f}\|^2_{L^2(-1,1)} &\le c_2^{-2}\int_{[-1,1]^2}|\hat{f}(\zeta)\widehat{\psi}(\xi)|^2\,d\xi d\zeta\\ &\le c_2^{-2}\int_{-2}^{2}\int_{\mathbb{R}}|\hat{f}(\xi-\eta)\widehat{\psi}(\xi)|^2\,d\xi d\eta\\ &= c_2^{-2}\int_{-2}^{2}\|g_\eta\|^2_{L^2}\,d\eta,\end{aligned} \tag{3.29}$$

where $\hat{g}_\eta(\xi) = \hat{f}(\xi-\eta)\widehat{\psi}(\xi)$ by (2.4). □

Armed with Lemma 3.4, we now give

Proof of Proposition 3.3. Take $\ell \in \mathbb{Z}$ such that $|\ell| \le \alpha_1$. By Lemmas 2.1 and 2.2 the set $Y+\ell \subset [-2\alpha_1, 2\alpha_1]$ is δ-regular with constant $4C_R$ on scales 1 to $2\alpha_1$. Put

$$f_\ell(x) := e^{2\pi i\ell x}f(x).$$

Then $\hat{f}_\ell(\xi) = \hat{f}(\xi-\ell)$, and thus $\operatorname{supp}\hat{f}_\ell \subset Y+\ell$. By Lemma 3.4 applied to f_ℓ and $Z := Y+\ell$, for all $K > 10$ and $\kappa := \exp(-C/\theta(K))$, we have

$$\|\hat{f}_\ell\|^2_{L^2(-1,1)} \le CK^{21}\Big(\|\mathbb{1}_{U'}f_\ell\|^2_{H^{-10}} + \exp\big(-C^{-1}\theta(K)K\big)\|f_\ell\|^2_{H^{-10}}\Big)^{\kappa}\cdot\|f_\ell\|^{2(1-\kappa)}_{H^{-10}}.$$

Since $\operatorname{supp} \hat{f} \subset Y \subset [-\alpha_1, \alpha_1]$, using Hölder's inequality (2.9) we obtain

$$\begin{aligned}\|f\|_{L^2}^2 &\leq \sum_{\ell \in \mathbb{Z}:\ |\ell| \leq \alpha_1} \|\hat{f}_\ell\|_{L^2(-1,1)}^2 \\ &\leq CK^{21} \Big(\sum_\ell \| \mathbb{1}_{U'} f_\ell\|_{H^{-10}}^2 + \exp\big(-C^{-1}\theta(K)K\big) \sum_\ell \|f_\ell\|_{H^{-10}}^2 \Big)^\kappa \\ &\quad \cdot \Big(\sum_\ell \|f_\ell\|_{H^{-10}}^2 \Big)^{1-\kappa}.\end{aligned}$$

Since

$$\sum_\ell \|f_\ell\|_{H^{-10}}^2 \leq C\|f\|_{L^2}^2, \quad \sum_\ell \| \mathbb{1}_{U'} f_\ell\|_{H^{-10}}^2 \leq C\|f\|_{L^2(U')}^2 \tag{3.30}$$

and by the Minkowski inequality $(a+b)^\kappa \leq a^\kappa + b^\kappa$, $a, b \geq 0$, we have

$$\|f\|_{L^2}^2 \leq CK^{21}\|f\|_{L^2(U')}^{2\kappa} \cdot \|f\|_{L^2}^{2(1-\kappa)} + CK^{21} \exp\big(-C^{-1}\theta(K)\kappa K\big)\|f\|_{L^2}^2.$$

Recalling that $\kappa = \exp(-C/\theta(K))$, $\delta < 1$, and the definition (3.1) of $\theta(K)$, we have $\kappa K \geq C^{-1}\sqrt{K}$ and thus

$$\lim_{K \to \infty} K^{21} \exp\big(-C^{-1}\theta(K)\kappa K\big) = 0.$$

Therefore, fixing K large enough depending only on δ, C_R, c_1, we have

$$\|f\|_{L^2}^2 \leq CK^{21}\|f\|_{L^2(U')}^{2\kappa} \cdot \|f\|_{L^2}^{2(1-\kappa)},$$

which implies (3.19) with $c_3 = (CK^{21})^{-\frac{1}{2\kappa}}$. □

3.4. *The iteration argument.* We now finish the proof of Theorem 4 by iterating Proposition 3.3. Let δ, C_R, N, X, Y satisfy the assumptions of Theorem 4.

First of all, Lemma 2.9 gives the Lebesgue measure bounds

$$\mu_L(X) \leq 24C_R^2 N^{\delta-1}, \quad \mu_L(Y) \leq 24C_R^2 N^\delta.$$

Applying Hölder's inequality twice and using (2.3), we see that for each $f \in L^2(\mathbb{R})$ with $\operatorname{supp} \hat{f} \subset Y$,

$$\begin{aligned}\|f\|_{L^2(X)} &\leq \sqrt{\mu_L(X)}\|f\|_{L^\infty} \leq \sqrt{\mu_L(X)}\|\hat{f}\|_{L^1} \\ &\leq \sqrt{\mu_L(X)\mu_L(Y)}\|\hat{f}\|_{L^2} \leq 24C_R^2 N^{\delta-\frac{1}{2}}\|f\|_{L^2},\end{aligned} \tag{3.31}$$

where we used the Lebesgue measure to define $\|f\|_{L^2(X)}$. This implies (1.4) for $\delta < 1/2$ with $\beta = 1/2 - \delta$. Therefore, we henceforth assume that $1/2 \leq \delta < 1$ (though we will only use that $0 < \delta < 1$).

Put

$$L := \big\lceil (3C_R)^{\frac{2}{1-\delta}} \big\rceil \in \mathbb{N}$$

so that (2.10) holds. Let $V_n(X)$, $n \in \mathbb{Z}$, be the elements of the tree of intervals covering X constructed in (2.13). Because of our choice of L, the tree $V_n(X)$

satisfies the missing child property, Lemma 2.10, which will be used in the proof of Lemma 3.6 below. Define the coarse-graining of X on the scale L^{-n}:

$$U_n := \bigcup_{I\in V_n(X)} I\Big(\frac{1}{10L^n}\Big) \ \supset\ X\Big(\frac{1}{10L^n}\Big). \tag{3.32}$$

Here we use the notation (2.1) for neighborhoods of sets.

We use the sets U_n to construct a family of weights. Let φ be a nonnegative Schwartz function such that

$$\operatorname{supp}\widehat{\varphi} \subset [-1,1], \quad \int_{\mathbb{R}} \varphi(x)\,dx = 1,$$

and for $n\in\mathbb{Z}$, define

$$\varphi_n(x) := L^n\cdot\varphi(L^n x), \quad \widehat{\varphi}_n(\xi) = \widehat{\varphi}(L^{-n}\xi).$$

Take $T\in\mathbb{N}$, and for $n\in\mathbb{Z}$, define the following weight (see Figure 6):

$$\Psi_n := \mathbf{1}_{U_{n+1}} * \varphi_{n+T}.$$

We will later fix T independently of N (see (3.38)) and take $0\le n\lesssim \log N$. Note that Ψ_n is a Schwartz function and $0\le\Psi_n\le 1$.

The fattening of the intervals in the definition of U_n and the need for the parameter T are explained by the following lemma, which is used at the end of the proof:

LEMMA 3.5. *There exists a constant C_φ depending only on φ such that for all n,*

$$\Psi_n \ge 1 - \frac{C_\varphi}{L^{T-1}} \quad \textit{on } X. \tag{3.33}$$

Proof. Let $x\in X$. Then by (3.32),

$$\Big[x-\frac{1}{10L^{n+1}}, x+\frac{1}{10L^{n+1}}\Big] \ \subset\ U_{n+1}.$$

We have

$$\Psi_n(x) = \int_{\mathbb{R}} \mathbf{1}_{U_{n+1}}(x-L^{-n-T}y)\varphi(y)\,dy \ge \int_{-L^{T-1}/10}^{L^{T-1}/10} \varphi(y)\,dy,$$

and (3.33) follows since φ is a Schwartz function of integral 1. □

Next, Proposition 3.3 implies that when $\operatorname{supp}\hat f \subset Y(2L^n)$, a positive proportion of the L^2 mass of f is removed when multiplying by the weight Ψ_n. (This is similar to restricting f to U_{n+1}, which is the coarse-graining of X on the scale L^{-n-1}.)

LEMMA 3.6. *There exists $\tau>0$ depending only on δ, C_R such that for all $T\in\mathbb{N}$, $n\in\mathbb{N}$ such that $L^{n+1}\le N$ and*

$$f\in L^2(\mathbb{R}), \quad \operatorname{supp}\hat f \subset Y(2L^n),$$

we have $\operatorname{supp}\widehat{\Psi_n f} \subset Y(2L^{n+T})$ *and*

$$\|\Psi_n f\|_{L^2} \le (1-\tau)\|f\|_{L^2}. \tag{3.34}$$

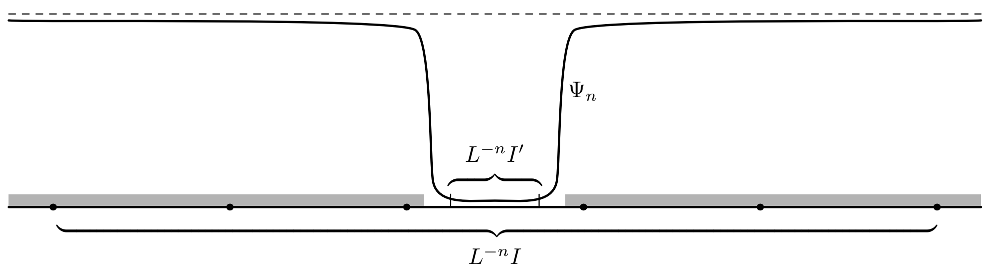

Figure 6. The weight Ψ_n for large T and the interval $L^{-n}I'$ used in the proof of Lemma 3.6. The dashed line corresponds to the constant function 1 and the dots mark $L^{-n-1}\mathbb{Z}$; the shaded region is U_{n+1}.

Proof. By (2.4), we have $\operatorname{supp}\widehat{\Psi}_n \subset \operatorname{supp}\widehat{\varphi}_{n+T} \subset [-L^{n+T}, L^{n+T}]$. Since $\widehat{\Psi_n f} = \widehat{\Psi}_n * \hat{f}$,

$$\operatorname{supp}\widehat{\Psi_n f} \ \subset\ \operatorname{supp}\hat{f} + [-L^{n+T}, L^{n+T}],$$

which gives the Fourier support condition on $\Psi_n f$.

It remains to show (3.34). Define the following rescaling of f:

$$\widetilde{f}(x) := L^{-n/2} \cdot f(L^{-n}x).$$

Then $\|\widetilde{f}\|_{L^2(A)} = \|f\|_{L^2(L^{-n}\cdot A)}$ for any set A and, with $\mathcal{F}$ denoting the Fourier transform,

$$\operatorname{supp}\mathcal{F}\widetilde{f} \ \subset\ \widetilde{Y} := \frac{1}{L^n}Y + [-2,2] \ \subset\ [-\alpha_1, \alpha_1], \quad \alpha_1 := \frac{10N}{L^n}.$$

By Lemmas 2.1–2.3 the set $\widetilde{Y}$ is δ-regular with constant $1000C_R$ on scales 1 to α_1.

Let $\mathcal{I}$ be the partition of $\mathbb{R}$ into size 1 intervals defined in (3.18). For each $I \in \mathcal{I}$, choose an interval $I' \subset I$ of size $c_1 := (2L)^{-1}$ as follows. There exists $j \in \mathbb{Z}$ such that $I_j := L^{-1}[j, j+1]$ is contained in I and satisfies $L^{-n}I_j \notin V_{n+1}(X)$. Indeed, for $L^{-n}I \notin V_n(X)$, this is obvious (as one can take any I_j contained in I) and for $L^{-n}I \in V_n(X)$, it follows by Lemma 2.10. We then let $I' \subset I_j$ have the same center as I_j and size $|I'| = c_1 = \frac{1}{2}|I_j|$. Note that the intervals $L^{-n}I'$ are relatively far from X; more precisely, (see Figure 6),

$$L^{-n}I' \cap U_{n+1}\Big(\frac{1}{10L^{n+T}}\Big) = \emptyset. \tag{3.35}$$

Applying Proposition 3.3 to the function $\widetilde{f}$, the set $\widetilde{Y}$, and the subintervals I' described in the previous paragraph, for some $c_3 > 0$ depending only on δ, C_R, we obtain

$$\|\widetilde{f}\|_{L^2(U')} \geq c_3\|\widetilde{f}\|_{L^2}, \quad U' := \bigcup_{I\in\mathcal{I}} I',$$

and therefore

$$\|f\|_{L^2(L^{-n}\cdot U')} \geq c_3\|f\|_{L^2}. \tag{3.36}$$

By (3.35) we have

$$\Psi_n \leq c_\varphi \quad \text{on } L^{-n}U', \quad c_\varphi := \int_{\mathbb{R}\setminus[-\frac{1}{10},\frac{1}{10}]} \varphi(x)\,dx < 1.$$

Since $0 \leq \Psi_n \leq 1$, together with (3.36) this implies

$$\begin{aligned}\|\Psi_n f\|_{L^2}^2 &\leq c_\varphi^2\|f\|_{L^2(L^{-n}\cdot U')}^2 + \|f\|_{L^2(\mathbb{R}\setminus(L^{-n}\cdot U'))}^2\\ &= \|f\|_{L^2}^2 - (1-c_\varphi^2)\|f\|_{L^2(L^{-n}\cdot U')}^2\\ &\leq \|f\|_{L^2}^2 - (1-c_\varphi^2)c_3^2\|f\|_{L^2}^2.\end{aligned}$$

This gives (3.34), where τ is defined by $(1-\tau)^2 = 1-(1-c_\varphi^2)c_3^2$. □

We are now ready to finish the proof of Theorem 4. To do this we iterate Lemma 3.6 $\sim \log N$ times. At each next step we use coarser information on frequency (that is, the Fourier support $\operatorname{supp}\hat{f}_m$ is contained in larger neighborhoods of Y) and finer information on position (that is, f_m involves cutoffs to smaller neighborhoods of X).

Proof of Theorem 4. Assume that $f \in L^2(\mathbb{R})$ and $\operatorname{supp}\hat{f} \subset Y$. Fix large $T \in \mathbb{N}$ to be chosen below. For $m \in \mathbb{N}$, define

$$f_m := \Bigg(\prod_{\ell=1}^{m-1}\Psi_{\ell T}\Bigg)f.$$

Iterating Lemma 3.6, we see that for each $m \in \mathbb{N}$ such that $L^{(m-1)T+1} \leq N$, we have

$$\|f_m\|_{L^2} \leq (1-\tau)^{m-1}\|f\|_{L^2}, \quad \operatorname{supp}\hat{f}_m \subset Y(2L^{mT}).$$

Then by Lemma 3.5, for all $m \in \mathbb{N}$ such that $L^{(m-1)T+1} \leq N$,

$$\|f\|_{L^2(X)} \leq \left(1-\frac{C_\varphi}{L^{T-1}}\right)^{1-m}\|f_m\|_{L^2} \leq \left(1-\frac{C_\varphi}{L^{T-1}}\right)^{1-m}(1-\tau)^{m-1}\|f\|_{L^2}. \tag{3.37}$$

Fix T large enough depending only on δ, C_R so that

$$\left(1-\frac{C_\varphi}{L^{T-1}}\right)^{-1}(1-\tau) \leq 1-\frac{\tau}{2}. \tag{3.38}$$

Then (3.37) gives

$$\|f\|_{L^2(X)} \leq \left(1-\frac{\tau}{2}\right)^{m-1}\|f\|_{L^2}.$$

Taking m such that $L^{(m-1)T+1} \leq N \leq L^{mT+1}$, we get (1.4) with

$$\beta = -\frac{\log\left(1-\frac{\tau}{2}\right)}{T\log L},$$

finishing the proof. □

4. **The hyperbolic fractal uncertainty principle**

In this section, we generalize Theorem 4 first by allowing a variable amplitude (Section 4.1) and then by taking a general phase (Section 4.2). Both generalizations are stated using the semiclassical parameter $h > 0$ corresponding to the inverse of the frequency. In Section 4.3, we apply the result of Section 4.2 to prove Theorem 3.

4.1. *The uncertainty principle with variable amplitude.* We first prove the following semiclassical rescaling of Theorem 4 which also relaxes the assumptions on the sets X, Y. In particular, it allows for unbounded X, Y but takes their intersections with bounded intervals, which is a more convenient assumption for applications. Recall the notation (2.1) for neighborhoods of sets and the semiclassical Fourier transform (2.6).

PROPOSITION 4.1. *Let $0 \le \delta < 1$, $C_R, C_I \ge 1$, and assume that $X, Y \subset \mathbb{R}$ are δ-regular with constant C_R on scales 0 to 1. Let $I_X, I_Y \subset \mathbb{R}$ be intervals with $|I_X|, |I_Y| \le C_I$. Then there exists $\beta > 0$ depending only on δ, C_R and $C > 0$ depending only on δ, C_R, C_I such that for all $h \in (0,1)$,*

$$\| \mathbb{1}_{X(h)\cap I_X}\, \mathcal{F}_h^*\, \mathbb{1}_{Y(h)\cap I_Y} \|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le C h^\beta. \tag{4.1}$$

Proof. Without loss of generality, we may assume that h is small depending on δ, C_R. By Lemma 2.3, the sets $X(h), Y(h)$ are δ-regular with constant $8C_R$ on scales h to 1. By Lemma 2.7 there exist collections of disjoint intervals $\mathcal{J}_X, \mathcal{J}_Y$ such that

$$\begin{aligned} X(h) &= \bigsqcup_{J\in\mathcal{J}_X} X_J, \quad X_J := X(h)\cap J,\\ Y(h) &= \bigsqcup_{J'\in\mathcal{J}_Y} Y_{J'}, \quad Y_{J'} := Y(h)\cap J',\\ (32C_R)^{-\frac{2}{1-\delta}} &\le |J| \le 1 \quad \text{for all } J\in\mathcal{J}_X\cup\mathcal{J}_Y, \end{aligned}$$

and the sets $X_J, Y_{J'}$ are δ-regular with constant $\widetilde{C}_R := (100C_R)^{\frac{2}{1-\delta}} C_R$ on scales h to 1.

We have the following estimate for each $J \in \mathcal{J}_X$, $J' \in \mathcal{J}_Y$, where $\beta, C > 0$ depend only on δ, C_R:

$$\| \mathbb{1}_{X_J}\, \mathcal{F}_h^*\, \mathbb{1}_{Y_{J'}} \|_{L^2\to L^2} \le C h^\beta. \tag{4.2}$$

Indeed, since $X_J, Y_{J'}$ have diameter no more than 1, we may shift $X_J, Y_{J'}$ to make them lie inside $[-1,1]$. By (2.7) this does not change the left-hand side of (4.2); by Lemma 2.1 it does not change δ-regularity. Take arbitrary $g \in L^2(\mathbb{R})$ and put $f := \mathcal{F}_h^*\, \mathbb{1}_{Y_{J'}}\, g$ and $N := h^{-1}$. Then $\operatorname{supp}\hat f$ lies in $N \cdot Y_{J'}$, which by Lemma 2.1 is δ-regular with constant $\widetilde{C}_R$ on scales 1 to N. Applying

Theorem 4, we obtain

$$\| \mathbb{1}_{X_J} \mathcal{F}_h^* \mathbb{1}_{Y_{J'}} g\|_{L^2} \;=\; \| \mathbb{1}_{X_J} f\|_{L^2} \;\leq\; Ch^\beta \|f\|_{L^2} \;\leq\; Ch^\beta \|g\|_{L^2},$$

implying (4.2).

Next, the number of intervals in $\mathcal{J}_X$ intersecting I_X is bounded as follows:

$$\#\{J \in \mathcal{J}_X \mid J \cap I_X \neq \emptyset\} \;\leq\; (32C_R)^{\frac{2}{1-\delta}} C_I + 2$$

and a similar estimate holds for the number of intervals in $\mathcal{J}_Y$ intersecting I_Y. Combining these estimates with (4.2) and using the triangle inequality (2.8), we obtain (4.1), finishing the proof. □

We now prove a fractal uncertainty principle for operators $A = A(h) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ of the form

$$Af(x) = h^{-1/2} \int_{\mathbb{R}} e^{2\pi i x\xi/h} a(x,\xi) f(\xi)\, d\xi, \tag{4.3}$$

where $a(x,\xi) \in C_0^\infty(\mathbb{R}^2)$ satisfies

$$\sup |\partial_x^k a| \leq C_k, \quad \operatorname{diam} \operatorname{supp} a \leq C_a \tag{4.4}$$

for each k and some constants C_k, C_a. In the statement below it is convenient to replace neighborhoods of size h by those of size h^ρ, where $\rho \in (0,1)$. In practice we will take ρ very close to 1 so that the resulting losses do not negate the gain h^β. The proof of Proposition 4.2 relies on Proposition 4.1 and the fact that for $\rho < 1$, functions in the range of $A(h)\, \mathbb{1}_{Y(h^\rho)}$ are concentrated on $Y(2h^\rho)$ in the semiclassical Fourier space.

PROPOSITION 4.2. *Let $0 \leq \delta < 1$, $C_R \geq 1$ and assume that $X, Y \subset \mathbb{R}$ are δ-regular with constant C_R on scales 0 to 1 and (4.4) holds. Then there exists $\beta > 0$ depending only on δ, C_R such that for all $\rho \in (0,1)$ and $h \in (0,1)$,*

$$\| \mathbb{1}_{X(h^\rho)}\, A(h)\, \mathbb{1}_{Y(h^\rho)} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq Ch^{\beta - 2(1-\rho)}, \tag{4.5}$$

where the constant C depends only on $\delta, C_R, \{C_k\}, C_a, \rho$.

Proof. Denote by C constants which depend only on $\delta, C_R, \{C_k\}, C_a, \rho$. (The value of C may differ in different parts of the proof.) We note that[2]

$$\|A\|_{L^2 \to L^2} \leq C. \tag{4.6}$$

To see this, we compute the integral kernel of A^*A:

$$\mathcal{K}_{A^*A}(\xi,\eta) = h^{-1} \int_{\mathbb{R}} e^{2\pi i x(\eta-\xi)/h} \overline{a(x,\xi)} a(x,\eta)\, dx.$$

[2]If all (x,ξ)-derivatives of a are bounded, then $\mathcal{F}_h A$ is a pseudodifferential operator and (4.6) follows from the Calderón–Vaillancourt Theorem.

Using (4.4) and repeated integration by parts in x, we obtain

$$|\mathcal{K}_{A^*A}(\xi,\eta)| \le Ch^{-1}\Big\langle\frac{\xi-\eta}{h}\Big\rangle^{-10},$$

which by Schur's inequality (see, e.g., [Zwo12, Th. 4.21]) gives $\|A^*A\|_{L^2\to L^2}\le C$, and thus (4.6) holds.

Take intervals I_X, I_Y such that $\operatorname{supp} a\subset I_X\times I_Y$ and $|I_X|,|I_Y|\le C_a$. We write

$$\begin{aligned}\mathbb{1}_{X(h^\rho)}\,A\,\mathbb{1}_{Y(h^\rho)} = \mathbb{1}_{X(h^\rho)\cap I_X}\,A\,\mathbb{1}_{Y(h^\rho)\cap I_Y} &= \mathbb{1}_{X(h^\rho)\cap I_X}\,\mathcal{F}_h^*A_1\\ &\quad + A_2\mathcal{F}_hA\,\mathbb{1}_{Y(h^\rho)\cap I_Y},\\ A_1 &:= \mathbb{1}_{\mathbb{R}\setminus(Y(2h^\rho)\cap I_Y(1))}\,\mathcal{F}_hA\,\mathbb{1}_{Y(h^\rho)\cap I_Y},\\ A_2 &:= \mathbb{1}_{X(h^\rho)\cap I_X}\,\mathcal{F}_h^*\,\mathbb{1}_{Y(2h^\rho)\cap I_Y(1)},\end{aligned}$$

so that by (4.6)

$$\|\,\mathbb{1}_{X(h^\rho)}\,A\,\mathbb{1}_{Y(h^\rho)}\,\|_{L^2\to L^2}\le\|A_1\|_{L^2\to L^2}+C\|A_2\|_{L^2\to L^2}. \tag{4.7}$$

The operator $\mathcal{F}_hA$ is pseudodifferential, thus its integral kernel is rapidly decaying once we step h^ρ away from the diagonal. Since the sets $\mathbb{R}\setminus(Y(2h^\rho)\cap I_Y(1))$ and $Y(h^\rho)\cap I_Y$ are distance h^ρ away from each other, this implies

$$\|A_1\|_{L^2\to L^2}\le Ch^{10}. \tag{4.8}$$

More precisely, to show (4.8) we compute the integral kernel of A_1:

$$\mathcal{K}_{A_1}(\xi,\eta)=\mathbf{1}_{\mathbb{R}\setminus(Y(2h^\rho)\cap I_Y(1))}(\xi)\mathbf{1}_{Y(h^\rho)\cap I_Y}(\eta)\cdot h^{-1}\int_{\mathbb{R}}e^{2\pi ix(\eta-\xi)/h}a(x,\eta)\,dx.$$

Note that $|\xi-\eta|\ge h^\rho$ on $\operatorname{supp}\mathcal{K}_{A_1}$. Using (4.4) and repeated integration by parts in x, for each $M\in\mathbb{N}_0$ we obtain

$$|\mathcal{K}_{A_1}(\xi,\eta)|\le C_Mh^{-1}\Big\langle\frac{\xi-\eta}{h}\Big\rangle^{-M-1},$$

which implies (4.8) by another application of Schur's inequality as soon as $M\ge\frac{10}{1-\rho}$.

We now estimate $\|A_2\|$. By Proposition 4.1 there exists $\beta>0$ depending only on δ, C_R such that

$$\|\,\mathbb{1}_{X(h)\cap I_X(1)}\,\mathcal{F}_h^*\,\mathbb{1}_{Y(h)\cap I_Y(2)}\,\|_{L^2\to L^2}\le Ch^\beta.$$

We cover $X(h^\rho)\cap I_X$, $Y(2h^\rho)\cap I_Y(1)$ as follows:

$$\begin{aligned}X(h^\rho)\cap I_X &\subset \bigcup_{\substack{p\in h\mathbb{Z}\\ |p|\le h^\rho}}\big(X(h)\cap I_X(1)\big)+p,\\ Y(2h^\rho)\cap I_Y(1) &\subset \bigcup_{\substack{q\in h\mathbb{Z}\\ |q|\le 2h^\rho}}\big(Y(h)\cap I_Y(2)\big)+q.\end{aligned}$$

Each of the above unions has at most $10h^{\rho-1}$ elements, therefore by (2.7) and the triangle inequality (2.8) we get

$$\|A_2\|_{L^2\to L^2} \le Ch^{\beta-2(1-\rho)}. \tag{4.9}$$

Combining (4.7)–(4.9), we obtain (4.5). □

4.2. *Uncertainty principle with general phase.* We next prove a fractal uncertainty principle for operators $B = B(h) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ of the form

$$Bf(x) = h^{-1/2} \int_{\mathbb{R}} e^{i\Phi(x,y)/h} b(x,y)\, f(y)\, dy, \tag{4.10}$$

where for some open set $U \subset \mathbb{R}^2$,

$$\Phi \in C^\infty(U;\mathbb{R}), \quad b \in C_0^\infty(U), \quad \partial^2_{xy}\Phi \neq 0 \quad \text{on } U. \tag{4.11}$$

The condition $\partial^2_{xy}\Phi \neq 0$ ensures that locally we can write the graph of the twisted gradient of Φ in terms of some symplectomorphism $\varkappa$ of open subsets of $T^*\mathbb{R}$:

$$(x,\xi) = \varkappa(y,\eta) \quad \Longleftrightarrow \quad \xi = \partial_x\Phi(x,y),\ \eta = -\partial_y\Phi(x,y). \tag{4.12}$$

Then B is a Fourier integral operator associated to $\varkappa$; see, for instance, [DZ16, §2.2]. Note that symplectomorphisms of the form (4.12) satisfy the following transversality condition: each vertical leaf $\{y = \text{const}\} \subset T^*\mathbb{R}^2$ is mapped by $\varkappa$ to a curve which is transversal to all vertical leaves $\{x = \text{const}\}$. Proposition 4.3 below can be interpreted in terms of the theory of Fourier integral operators, however we give a proof which is self-contained and does not explicitly rely on this theory.

PROPOSITION 4.3. *Let* $0 \le \delta < 1$, $C_R \ge 1$, *and assume that* $X, Y \subset \mathbb{R}$ *are* δ*-regular with constant* C_R *on scales* 0 *to* 1 *and* (4.11) *holds. Then there exist* $\beta > 0$, $\rho \in (0,1)$ *depending only on* δ, C_R *and* $C > 0$ *depending only on* δ, C_R, Φ, b *such that for all* $h \in (0,1)$,

$$\|\, \mathbb{1}_{X(h^\rho)}\, B(h)\, \mathbb{1}_{Y(h^\rho)}\, \|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le Ch^\beta. \tag{4.13}$$

Remark. The value of β in Proposition 4.3 (and in Theorem 3) is smaller than the one in Theorem 4 and Propositions 4.1–4.2. Denoting the latter by $\tilde\beta$, our argument gives (4.13) with $\beta = \tilde\beta/4$; see (4.19) below. By taking ρ sufficiently close to 1, one can get any $\beta < \tilde\beta/2$. However, since we do not specify the value of β, this difference is irrelevant to the final result.

We first note that it is enough to prove Proposition 4.3 under the assumption

$$1 \;<\; |\partial^2_{xy}\Phi| \;<\; 2 \quad \text{on } U. \tag{4.14}$$

Indeed, assume that Proposition 4.3 is established for all Φ satisfying (4.14). Then it also holds for all $\lambda_\Phi > 0$ and Φ satisfying

$$\lambda_\Phi \ < \ |\partial^2_{xy}\Phi| \ < \ 2\lambda_\Phi \quad \text{on } U, \tag{4.15}$$

where β, ρ do not depend on λ_Φ but C does. Indeed, put $\widetilde{\Phi} := \lambda_\Phi^{-1}\Phi$; then $\widetilde{\Phi}$ satisfies (4.14). If $\widetilde{B}(h)$ is given by (4.10) with Φ replaced by $\widetilde{\Phi}$, then $B(h) = \widetilde{B}(\lambda_\Phi^{-1}h)$, and thus by slightly increasing ρ we see that Proposition 4.3 for $\widetilde{\Phi}$ implies it for Φ. Finally, for the case of general Φ, we use a partition of unity for b and shrink U accordingly to split B into the sum of finitely many operators of the form (4.10), each of which has a phase function satisfying (4.15) for some λ_Φ.

The proof of Proposition 4.3 relies on the following statement which fattens the set X by $h^{\rho/2}$, intersects $Y(h^\rho)$ with a size $h^{1/2}$ interval, and is proved by making a change of variables and taking the semiclassical parameter $\tilde{h} := h^{1/2}$ in Proposition 4.2:

Lemma 4.4. *Assume* (4.14) *holds. Then there exist* $\beta > 0$, $\rho \in (0,1)$ *depending only on* δ, C_R *and* $C > 0$ *depending only on* δ, C_R, Φ, b *such that for all* $h \in (0,1)$ *and all intervals* J *of size* $h^{1/2}$,

$$\| \mathbb{1}_{X(h^{\rho/2})}\, B(h)\, \mathbb{1}_{Y(h^\rho)\cap J} \|_{L^2\to L^2} \leq C h^\beta. \tag{4.16}$$

Proof. Fix $\rho \in (\frac{1}{2}, 1)$ to be chosen later. Breaking the symbol b into pieces using a partition of unity, we may assume that

$$\operatorname{supp} b \ \subset \ I_X \times I_Y' \ \subset \ I_X \times I_Y \ \subset \ U,$$

where I_X, I_Y, I_Y' are some intervals with $I_Y' \Subset I_Y$. We may assume that $J \subset I_Y$; indeed, otherwise the operator in (4.16) is equal to 0 for h small enough. Let y_0 be the center of J and define the function

$$\varphi : I_X \to \mathbb{R}, \quad \varphi(x) = \frac{1}{2\pi}\partial_y \Phi(x, y_0).$$

By (4.14) we have

$$\frac{1}{2\pi} \ < \ |\partial_x \varphi| \ < \ \frac{1}{\pi} \quad \text{on } I_X. \tag{4.17}$$

In particular, $\varphi : I_X \to \varphi(I_X)$ is a diffeomorphism. We extend φ to a diffeomorphism of $\mathbb{R}$ such that (4.17) holds on the entire $\mathbb{R}$. Let $\Psi \in C^\infty(I_X \times I_Y)$ be the remainder in Taylor's formula for Φ, defined by

$$\Phi(x,y) = \Phi(x,y_0) + 2\pi(y-y_0)\varphi(x) + (y-y_0)^2\Psi(x,y), \quad x \in I_X,\ y \in I_Y.$$

Consider the isometries $W_X, W_Y : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$W_X f(x) = e^{-i\Phi(\varphi^{-1}(x),y_0)/h}\Big|\partial_x(\varphi^{-1})(x)\Big|^{1/2} f(\varphi^{-1}(x)),$$
$$W_Y f(y) = h^{-1/4} f\Big(\frac{y-y_0}{h^{1/2}}\Big).$$

Here we extend $\Phi(\varphi^{-1}(x), y_0)$ from $\varphi(I_X)$ to a real-valued function on $\mathbb{R}$. We also fix a function $\chi \in C_0^\infty((-1,1);[0,1])$ such that $\chi = 1$ near $[-\frac{1}{2}, \frac{1}{2}]$, and define the cutoff χ_J by

$$\chi_J(y) = \chi\Big(\frac{y-y_0}{h^{1/2}}\Big), \quad \chi_J = 1 \quad \text{on } J.$$

Put $A = A(h) := W_X B(h) \chi_J W_Y$, and then we write A in the form (4.3):

$$Af(x) = \tilde{h}^{-1/2} \int_{\mathbb{R}} e^{2\pi i x\xi/\tilde{h}} a(x,\xi;\tilde{h}) f(\xi)\, d\xi,$$

where $\tilde{h} := h^{1/2}$ and

$$a(x,\xi;\tilde{h}) = e^{i\xi^2\Psi(\varphi^{-1}(x),y_0+\tilde{h}\xi)}\Big|\partial_x(\varphi^{-1})(x)\Big|^{1/2} b(\varphi^{-1}(x), y_0+\tilde{h}\xi)\chi(\xi).$$

The amplitude a satisfies (4.4) with the constants C_k, C_a depending only on Φ, b. We now have

$$\begin{aligned}\|\, \mathbb{1}_{X(h^{\rho/2})}\, B\, \mathbb{1}_{Y(h^\rho)\cap J}\,\|_{L^2\to L^2} &\le \|W_X\, \mathbb{1}_{X(h^{\rho/2})}\, B\chi_J\, \mathbb{1}_{Y(h^\rho)}\, W_Y\|_{L^2\to L^2} \\ &\le \|\, \mathbb{1}_{\widetilde{X}(C_\Phi \tilde{h}^\rho)}\, A\, \mathbb{1}_{\widetilde{Y}(\tilde{h}^{2\rho-1})}\,\|_{L^2\to L^2},\end{aligned} \tag{4.18}$$

where $\widetilde{X} := \varphi(X)$, $\widetilde{Y} := h^{-1/2}(Y - y_0)$. By Lemmas 2.4 and 2.2 the set $\widetilde{X}$ is δ-regular with constant $\widetilde{C}_R := 8\pi^2 C_R$ on scales 0 to 1. By Lemma 2.1 the set $\widetilde{Y}$ has the same property. Applying Proposition 4.2 we obtain

$$\|\, \mathbb{1}_{\widetilde{X}(\tilde{h}^{2\rho-1})}\, A\, \mathbb{1}_{\widetilde{Y}(\tilde{h}^{2\rho-1})}\,\|_{L^2\to L^2} \le C\tilde{h}^{\tilde{\beta}-4(1-\rho)} = Ch^{\frac{\tilde{\beta}}{2}-2(1-\rho)},$$

where $\tilde{\beta} > 0$ depends only on δ, C_R and C depends only on $\delta, C_R, \Phi, b, \rho$. Fixing

$$\rho := 1 - \frac{1}{8}\tilde{\beta}, \quad \beta := \frac{\tilde{\beta}}{4}, \tag{4.19}$$

taking h small enough so that $C_\Phi \tilde{h}^\rho \le \tilde{h}^{2\rho-1}$, and using (4.18), we obtain (4.16). $\square$

We now finish the proof of Proposition 4.3 using almost orthogonality similarly to [DZ16, §5.2]:

Proof of Proposition 4.3. Denote by C constants which depend only on δ, C_R, Φ, b. Since $\partial^2_{xy}\Phi \neq 0$ on U, after using a partition of unity for b and shrinking U we may assume that

$$|\partial_x\Phi(x,y) - \partial_x\Phi(x,y')| \ge C^{-1}|y-y'| \quad \text{for all } (x,y),(x,y') \in U. \tag{4.20}$$

Take $\beta > 0$, $\rho \in (0,1)$ defined in Lemma 4.4. By [DZ16, Lemma 3.3], there exists $\psi = \psi(x;h) \in C^\infty(\mathbb{R};[0,1])$ such that for some global constants $C_{k,\psi}$,

$$\psi = 1 \quad \text{on } X(h^\rho), \quad \operatorname{supp}\psi \subset X(h^{\rho/2}), \tag{4.21}$$

$$\sup|\partial_x^k \psi| \le C_{k,\psi} h^{-\rho k/2}. \tag{4.22}$$

Take the smallest interval I_Y such that $\operatorname{supp} b \subset \mathbb{R} \times I_Y$. Take a maximal set of $\frac{1}{2}h^{1/2}$-separated points

$$y_1, \dots, y_N \in Y(h^\rho) \cap I_Y, \quad N \le Ch^{-1/2},$$

and let J_n be the interval of size $h^{1/2}$ centered at y_n. Define the operators

$$B_n := \sqrt{\psi}\, B\, \mathbb{1}_{Y(h^\rho)\cap J_n}, \quad n = 1, \dots, N.$$

Then by Lemma 4.4 we have, uniformly in n,

$$\|B_n\|_{L^2\to L^2} \;\le\; \|\,\mathbb{1}_{X(h^{\rho/2})}\, B\, \mathbb{1}_{Y(h^\rho)\cap J_n}\,\|_{L^2\to L^2} \;\le\; Ch^\beta. \tag{4.23}$$

On the other hand, $Y(h^\rho) \cap I_Y \subset \bigcup_n (Y(h^\rho) \cap J_n)$, and thus

$$\|\,\mathbb{1}_{X(h^\rho)}\, B\, \mathbb{1}_{Y(h^\rho)}\,\|_{L^2\to L^2} \;\le\; \Big\|\sqrt{\psi}\, B\, \mathbb{1}_{Y(h^\rho)\cap I_Y}\Big\|_{L^2\to L^2} \;\le\; \Big\|\sum_{n=1}^N B_n\Big\|_{L^2\to L^2}. \tag{4.24}$$

We will estimate the right-hand side of (4.24) by the Cotlar–Stein Theorem [Zwo12, Th. C.5]. We say that two points y_n, y_m are *close* if $|y_n - y_m| \le 10h^{1/2}$ and are *far* otherwise. Each point is close to at most 100 other points. The following estimates hold when y_n, y_m are far:

$$B_n B_m^* = 0, \tag{4.25}$$

$$\|B_n^* B_m\|_{L^2\to L^2} \le Ch^{10}. \tag{4.26}$$

Indeed, (4.25) follows immediately since $J_n \cap J_m = \emptyset$. To show (4.26), we compute the integral kernel of $B_n^* B_m$:

$$\begin{aligned}&\mathcal{K}_{B_n^* B_m}(y,y')\\ &= \mathbb{1}_{Y(h^\rho)\cap J_n}(y)\, \mathbb{1}_{Y(h^\rho)\cap J_m}(y') \cdot h^{-1} \int_{\mathbb{R}} e^{\frac{i}{h}(\Phi(x,y')-\Phi(x,y))}\overline{b(x,y)}b(x,y')\psi(x)\,dx.\end{aligned}$$

Since y_n, y_m are far, we have $|y - y'| \ge h^{1/2}$ on $\operatorname{supp}\mathcal{K}_{B_n^* B_m}$. We now repeatedly integrate by parts in x. Each integration produces a gain of $h^{1/2}$ due to (4.20) and a loss of $h^{-\rho/2}$ due to (4.22). Since $\rho < 1$, after finitely many steps we obtain (4.26). See the proof of [DZ16, Lemma 5.2] for details.

Now (4.23), (4.25), and (4.26) imply by the Cotlar–Stein Theorem that

$$\Big\|\sum_{n=1}^N B_n\Big\|_{L^2\to L^2} \le Ch^\beta,$$

which gives (4.13) because of (4.24). $\square$

4.3. *Proof of Theorem* 3. We parametrize the circle by $\theta \in \mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$. Let $\Lambda_\Gamma \subset \mathbb{S}^1$ be the limit set of Γ; we lift it to a 2π-periodic subset of $\mathbb{R}$, denoted by X.

The set $X \subset \mathbb{R}$ is δ-regular with some constant C_R on scales 0 to 1, where we can take as μ_X the Hausdorff measure of dimension δ or equivalently the lift of the Patterson–Sullivan measure; see, for example, [Sul79, Th. 7] and [Bor16, Lemma 14.13 and Th. 14.14]. Here $\delta \in [0,1]$ is the exponent of convergence of Poincaré series of the group and $\delta < 1$ when $M = \Gamma\backslash\mathbb{H}^2$ is convex co-compact but not compact; see, for instance, [Bor16, §2.5.2] and [Bea71, Th. 2].

Let $\mathcal{B}_\chi(h)$ be the operator defined in (1.2). By partition of unity, we may assume that $\operatorname{supp}\chi$ lies in the product of two half-circles. Then for all $h \in (0,1)$, $\rho \in (0,1)$,

$$\| \mathbb{1}_{\Lambda_\Gamma(h^\rho)}\, \mathcal{B}_\chi(h)\, \mathbb{1}_{\Lambda_\Gamma(h^\rho)} \|_{L^2(\mathbb{S}^1)\to L^2(\mathbb{S}^1)} = \| \mathbb{1}_{X(h^\rho)}\, B(h)\, \mathbb{1}_{X(h^\rho)} \|_{L^2(\mathbb{R})\to L^2(\mathbb{R})},$$

where $B = B(h)$ has the form (4.10):

$$Bf(\theta) = h^{-1/2} \int_{\mathbb{R}} e^{i\Phi(\theta,\theta')/h} b(\theta,\theta') f(\theta')\, d\theta'.$$

Here, denoting $U := \{(\theta,\theta') \mid \theta - \theta' \notin 2\pi\mathbb{Z}\}$, the function $b \in C_0^\infty(U)$ is a compactly supported lift of $(2\pi)^{-1/2}\chi$ and $\Phi \in C^\infty(U;\mathbb{R})$ is given by

$$\Phi(\theta,\theta') = \log 4 + 2\log\left|\sin\left(\frac{\theta-\theta'}{2}\right)\right|, \quad \theta,\theta' \in \mathbb{R}.$$

We have

$$\partial^2_{\theta\theta'}\Phi = \frac{1}{2\sin^2\left(\frac{\theta-\theta'}{2}\right)} \neq 0 \quad \text{on } U.$$

By Proposition 4.3 there exist $\beta > 0$ and $\rho \in (0,1)$ depending only on δ, C_R and $C > 0$ depending on δ, C_R, χ such that for all $h \in (0,1)$,

$$\| \mathbb{1}_{X(h^\rho)}\, B(h)\, \mathbb{1}_{X(h^\rho)} \|_{L^2\to L^2} \le Ch^\beta,$$

which implies (1.3) and finishes the proof of Theorem 3.

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References

[AFR09] A. Aleman, N. S. Feldman, and W. T. Ross, *The Hardy Space of a Slit Domain*, *Frontiers in Math.*, Birkhäuser Verlag, Basel, 2009. MR 2548414. Zbl 1188.30003. https://doi.org/10.1007/978-3-0346-0098-9.

[BWP+13] S. Barkhofen, T. Weich, A. Potzuweit, H.-J. Stöckmann, U. Kuhl, and M. Zworski, Experimental observation of the spectral gap in microwave n-disk systems, *Phys. Rev. Lett.* **110** no. 16 (2013), 164102. https://doi.org/10.1103/PhysRevLett.110.164102.

[Bea71] A. F. Beardon, Inequalities for certain Fuchsian groups, *Acta Math.* **127** (1971), 221–258. MR 0286996. Zbl 0235.30022. https://doi.org/10.1007/BF02392054.

[BM62] A. Beurling and P. Malliavin, On Fourier transforms of measures with compact support, *Acta Math.* **107** (1962), 291–309. MR 0147848. Zbl 0127.32601. https://doi.org/10.1007/BF02545792.

[Bor14] D. Borthwick, Distribution of resonances for hyperbolic surfaces, *Exp. Math.* **23** no. 1 (2014), 25–45. MR 3177455. Zbl 1321.58021. https://doi.org/10.1080/10586458.2013.857282.

[Bor16] D. Borthwick, *Spectral Theory of Infinite-Area Hyperbolic Surfaces*, second ed., *Progr. Math.* **318**, Birkhäuser/Springer, [Cham], 2016. MR 3497464. Zbl 1351.58001. https://doi.org/10.1007/978-3-319-33877-4.

[BW16] D. Borthwick and T. Weich, Symmetry reduction of holomorphic iterated function schemes and factorization of Selberg zeta functions, *J. Spectr. Theory* **6** no. 2 (2016), 267–329. MR 3485944. Zbl 1366.37053. https://doi.org/10.4171/JST/125.

[BD17] J. Bourgain and S. Dyatlov, Fourier dimension and spectral gaps for hyperbolic surfaces, *Geom. Funct. Anal.* **27** no. 4 (2017), 744–771. MR 3678500. Zbl 06766918. https://doi.org/10.1007/s00039-017-0412-0.

[BGS11] J. Bourgain, A. Gamburd, and P. Sarnak, Generalization of Selberg's $\frac{3}{16}$ theorem and affine sieve, *Acta Math.* **207** no. 2 (2011), 255–290. MR 2892611. Zbl 1276.11081. https://doi.org/10.1007/s11511-012-0070-x.

[BGH10] N. Burq, C. Guillarmou, and A. Hassell, Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics, *Geom. Funct. Anal.* **20** no. 3 (2010), 627–656. MR 2720226. Zbl 1206.58009. https://doi.org/10.1007/s00039-010-0076-5.

[Chr09] H. Christianson, Applications of cutoff resolvent estimates to the wave equation, *Math. Res. Lett.* **16** no. 4 (2009), 577–590. MR 2525026. Zbl 1189.58012. https://doi.org/10.4310/MRL.2009.v16.n4.a3.

[Con95] J. B. Conway, *Functions of One Complex Variable. II*, *Grad. Texts in Math.* **159**, Springer-Verlag, New York, 1995. MR 1344449. Zbl 0887.30003. https://doi.org/10.1007/978-1-4612-0817-4.

[Dat09] K. Datchev, Local smoothing for scattering manifolds with hyperbolic trapped sets, *Comm. Math. Phys.* **286** no. 3 (2009), 837–850. MR 2472019. Zbl 1189.58016. https://doi.org/10.1007/s00220-008-0684-1.

[Dol98] D. Dolgopyat, On decay of correlations in Anosov flows, *Ann. of Math.* (2) **147** no. 2 (1998), 357–390. MR 1626749. Zbl 0911.58029. https://doi.org/10.2307/121012.

[DJ17] S. Dyatlov and L. Jin, Resonances for open quantum maps and a fractal uncertainty principle, *Comm. Math. Phys.* **354** no. 1 (2017), 269–316. MR 3656519. Zbl 1372.81101. https://doi.org/10.1007/s00220-017-2892-z.

[DJ18] S. Dyatlov and L. Jin, Dolgopyat's method and the fractal uncertainty principle, to appear in *Anal. PDE*, 2018.

[DZ16] S. Dyatlov and J. Zahl, Spectral gaps, additive energy, and a fractal uncertainty principle, *Geom. Funct. Anal.* **26** no. 4 (2016), 1011–1094. MR 3558305. Zbl 06656978. https://doi.org/10.1007/s00039-016-0378-3.

[DZ18] S. Dyatlov and M. Zworski, Fractal uncertainty for transfer operators, 2018, to appear in *IMRN*, 2018. arXiv 1710.05430.

[GR89] P. Gaspard and S. A. Rice, Scattering from a classically chaotic repellor, *J. Chem. Phys.* **90** no. 4 (1989), 2225–2241. MR 0980392. https://doi.org/10.1063/1.456017.

[Gui05] C. Guillarmou, Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds, *Duke Math. J.* **129** no. 1 (2005), 1–37. MR 2153454. Zbl 1099.58011. https://doi.org/10.1215/S0012-7094-04-12911-2.

[GN09] C. Guillarmou and F. Naud, Wave decay on convex co-compact hyperbolic manifolds, *Comm. Math. Phys.* **287** no. 2 (2009), 489–511. MR 2481747. Zbl 1196.58011. https://doi.org/10.1007/s00220-008-0706-z.

[GZ95] L. Guillopé and M. Zworski, Polynomial bounds on the number of resonances for some complete spaces of constant negative curvature near infinity, *Asymptotic Anal.* **11** no. 1 (1995), 1–22. MR 1344252. Zbl 0859.58028. https://doi.org/10.3233/ASY-1995-11101.

[HJ94] V. Havin and B. Jöricke, *The Uncertainty Principle in Harmonic Analysis*, *Ergeb. Math. Grenzgeb.* **28**, Springer-Verlag, Berlin, 1994. MR 1303780. Zbl 0827.42001. https://doi.org/10.1007/978-3-642-78377-7.

[HV18] P. Hintz and A. Vasy, The global non-linear stability of the Kerr–de Sitter family of black holes, 2018, to appear in *Acta Math.*, 2018. arXiv 1606.04014.

[Ika88] M. Ikawa, Decay of solutions of the wave equation in the exterior of several convex bodies, *Ann. Inst. Fourier* (*Grenoble*) **38** no. 2 (1988), 113–146. MR 0949013. Zbl 0636.35045. https://doi.org/10.5802/aif.1137.

[IM74] K. Itô and H. P. McKean, Jr., *Diffusion Processes and their Sample Paths*, *Grundlehren Math. Wiss.* **125**, Springer-Verlag, Berlin-New York, 1974, second printing, corrected. MR 0345224. Zbl 0285.60063.

[JN12] D. Jakobson and F. Naud, On the critical line of convex co-compact hyperbolic surfaces, *Geom. Funct. Anal.* **22** no. 2 (2012), 352–368. MR 2929068. Zbl 1284.30035. https://doi.org/10.1007/s00039-012-0154-y.

[JZ17] L. Jin and R. Zhang, Fractal uncertainty principle with explicit exponent, 2017. arXiv 1710.00250v1.

[LP67] P. D. Lax and R. S. Phillips, *Scattering Theory*, *Pure Appl. Math.* **26**, Academic Press, New York, 1967. MR 0217440. Zbl 0186.16301.

[LP89] P. D. Lax and R. S. Phillips, *Scattering Theory*, *Pure Appl. Math.* **26**, 1989, second edition, Academic Press, New York. MR 1037774. Zbl 0697.35004.

[MOW] M. Magee, H. Oh, and D. Winter, Uniform congruence counting for Schottky semigroups in $\mathrm{SL}_2(\mathbb{Z})$, *J. Reine Angew. Math.*, with an appendix by Jean Bourgain, Alex Kontorovich, and Michael Magee, published online 2017-01-12. https://doi.org/10.1515/crelle-2016-0072.

[MNK05] D. Mashregi, F. L. Nazarov, and V. P. Khavin, The Beurling-Malliavin multiplier theorem: The seventh proof, *Algebra i Analiz* **17** no. 5 (2005), 3–68. MR 2241422. https://doi.org/10.1090/S1061-0022-06-00926-5.

[MM87] R. R. Mazzeo and R. B. Melrose, Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, *J. Funct. Anal.* **75** no. 2 (1987), 260–310. MR 0916753. Zbl 0636.58034. https://doi.org/10.1016/0022-1236(87)90097-8.

[Nau05] F. Naud, Expanding maps on Cantor sets and analytic continuation of zeta functions, *Ann. Sci. École Norm. Sup.* (4) **38** no. 1 (2005), 116–153. MR 2136484. Zbl 1110.37021. https://doi.org/10.1016/j.ansens.2004.11.002.

[Non11] S. Nonnenmacher, Spectral problems in open quantum chaos, *Nonlinearity* **24** no. 12 (2011), R123–R167. Zbl 1229.35223. https://doi.org/10.1088/0951-7715/24/12/R02.

[NZ09a] S. Nonnenmacher and M. Zworski, Quantum decay rates in chaotic scattering, *Acta Math.* **203** no. 2 (2009), 149–233. MR 2570070. Zbl 1226.35061. https://doi.org/10.1007/s11511-009-0041-z.

[NZ09b] S. Nonnenmacher and M. Zworski, Semiclassical resolvent estimates in chaotic scattering, *Appl. Math. Res. Express. AMRX* **2009** no. 1 (2009), 74–86. MR 2581379. Zbl 1181.81055. https://doi.org/10.1093/amrx/abp003.

[OW16] H. Oh and D. Winter, Uniform exponential mixing and resonance free regions for convex cocompact congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, *J. Amer. Math. Soc.* **29** no. 4 (2016), 1069–1115. MR 3522610. Zbl 1360.37083. https://doi.org/10.1090/jams/849.

[Pat76] S. J. Patterson, The limit set of a Fuchsian group, *Acta Math.* **136** no. 3-4 (1976), 241–273. MR 0450547. Zbl 0336.30005. https://doi.org/10.1007/BF02392046.

[PS10] V. Petkov and L. Stoyanov, Analytic continuation of the resolvent of the Laplacian and the dynamical zeta function, *Anal. PDE* **3** no. 4 (2010), 427–489. MR 2718260. Zbl 1251.37031. https://doi.org/10.2140/apde.2010.3.427.

[PS85] R. S. Phillips and P. Sarnak, On cusp forms for co-finite subgroups of $psl(2, \mathbb{R})$, *Invent. Math.* **80** no. 2 (1985), 339–364. MR 0788414. Zbl 0558.10017. https://doi.org/10.1007/BF01388610.

[Sar14] P. Sarnak, Notes on thin matrix groups, in *Thin Groups and Superstrong Approximation*, *Math. Sci. Res. Inst. Publ.* **61**, Cambridge Univ. Press, Cambridge, 2014, pp. 343–362. MR 3220897. Zbl 1365.11039.

[Sel90] A. Selberg, Remarks on the distribution of poles of Eisenstein series, in *Festschrift in Honor of I. I. Piatetski-Shapiro on the Occasion of his Sixtieth Birthday, Part II* (Ramat Aviv, 1989), *Israel Math. Conf. Proc.* **3**, Weizmann, Jerusalem, 1990, pp. 251–278. MR 1159119. Zbl 0712.11034.

[Sto11] L. Stoyanov, Spectra of Ruelle transfer operators for Axiom A flows, *Nonlinearity* **24** no. 4 (2011), 1089–1120. MR 2776112. Zbl 1230.37040. https://doi.org/10.1088/0951-7715/24/4/005.

[Sul79] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, *Inst. Hautes Études Sci. Publ. Math.* **50** no. 1 (1979), 171–202. MR 0556586. Zbl 0439.30034. https://doi.org/10.1007/BF02684773.

[CdV82] Y. Colin de Verdière, Pseudo-laplaciens. I, *Ann. Inst. Fourier* (*Grenoble*) **32** no. 3 (1982), xiii, 275–286. MR 0688031. Zbl 0489.58034. https://doi.org/10.5802/aif.890.

[CdV83] Y. Colin de Verdière, Pseudo-laplaciens. II, *Ann. Inst. Fourier* (*Grenoble*) **33** no. 2 (1983), 87–113. MR 0699488. Zbl 0496.58016. https://doi.org/10.5802/aif.917.

[Wan17] J. Wang, Strichartz estimates for convex co-compact hyperbolic surfaces, 2017. arXiv 1707.06310.

[Zwo12] M. Zworski, *Semiclassical Analysis*, *Grad. Stud. Math.* **138**, Amer. Math. Soc., Providence, RI, 2012. MR 2952218. Zbl 1252.58001. https://doi.org/10.1090/gsm/138.

[Zwo17] M. Zworski, Mathematical study of scattering resonances, *Bull. Math. Sci.* **7** no. 1 (2017), 1–85. MR 3625851. Zbl 1368.35230. https://doi.org/10.1007/s13373-017-0099-4.

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