Annals of Mathematics **187** (2018), 667–719
https://doi.org/10.4007/annals.2018.187.3.2

Palindromic subshifts and simple periodic groups of intermediate growth

By Volodymyr Nekrashevych

Abstract

We describe a new class of groups of Burnside type, by giving a procedure transforming an arbitrary non-free minimal action of the dihedral group on a Cantor set into an orbit-equivalent action of an infinite finitely generated periodic group. We show that if the associated Schreier graphs are linearly repetitive, then the group is of intermediate growth. In particular, this gives first examples of simple groups of intermediate growth.

1. **Introduction**

Let a be a homeomorphism of period two (an involution) of a Cantor set $\mathcal{X}$. Choose a finite group A of homeomorphisms of $\mathcal{X}$ such that for all $h \in A$, $\zeta \in \mathcal{X}$, we have $h(\zeta) = \zeta$ or $h(\zeta) = a(\zeta)$, and for every $\zeta \in \mathcal{X}$, there exists $h \in A$ such that $h(\zeta) = a(\zeta)$. We say that A is a *fragmentation* of a.

Suppose that we have a minimal action on the Cantor set of the infinite dihedral group D_∞ generated by two involutions a, b. (An action is said to be *minimal* if all its orbits are dense.) Let A and B be fragmentations of a and b. We are interested in the group $\langle A \cup B\rangle$ generated by A and B.

Examples of such groups are the first Grigorchuk group [Gri80] and every group from the family of Grigorchuk groups defined in [Gri85]. They are all obtained by fragmenting one particular minimal action of the dihedral group (associated with the binary Gray code; see Example 3.3 below).

We show that under rather general conditions on the fragmentation, the group $G = \langle A \cup B\rangle$ possesses interesting properties, pertinent to three classical problems of group theory (Burnside's problem on periodic groups, Day's problem on amenable groups, and Milnor's problem on intermediate growth). For example, *every* non-free minimal action of D_∞ can be fragmented to produce

Keywords: groups of intermediate growth, torsion groups, Burnside groups, simple groups, minimal subshifts, topological full groups

AMS Classification: Primary: 20F69, 20F50; Secondary: 20E32, 37B05, 37B20.

a finitely generated infinite periodic group. Moreover, if the action of D_∞ is expansive, then one can fragment it (in uncountably many different ways) to get a simple group. Actions of low complexity (for example, coming from palindromic minimal substitutional subshifts) can be fragmented to produce groups of intermediate growth, including simple ones.

Namely, we prove the following (see Theorem 4.1).

THEOREM 1.1. *Suppose that ξ is a fixed point of a and that for every $h \in A$ such that $h(\xi) = \xi$, the interior of the set of fixed points of h accumulates on ξ. Then the group $\langle A \cup B\rangle$ is periodic and infinite.*

If the action of $D_\infty = \langle a, b\rangle$ is expansive, then the *topological full group* of $G = \langle A \cup B\rangle$ contains an infinite finitely generated simple subgroup $\mathsf{A}(G, \mathcal{X})$, see [Nek15]. The group $\mathsf{A}(G, \mathcal{X})$ is a subgroup of a (possibly bigger than G) fragmentation $\langle A_1 \cup B_1\rangle$ to which Theorem 1.1 is also applicable, so that the group $\mathsf{A}(G, \mathcal{X})$ is periodic. For a definition of the topological full group and the group $\mathsf{A}(G, \mathcal{X})$, see Section 2.2.

Here a group G is said to be *periodic* (or *torsion*) if for every $g \in G$, there exists n such that g^n is the identity. The question of existence of finitely generated infinite periodic groups (groups of *Burnside type*) is the classical *Burnside problem* [Bur02]. A harder version (*bounded Burnside problem*) asks for a group with bounded order n of elements. The general problem (without a bound on the order of elements) was solved by E. Golod and I. Shafarevich [Gol64] in 1964. The bounded Burnside problem was solved by S. Adyan and P. Novikov [NA68] in 1968. The *restricted Burnside problem* (the bounded Burnside problem in the class of residually finite groups) was solved (in the negative) in 1989 by E. Zelmanov [Zel90]. For a survey of the Burnside problem and related topics, see [GL02].

Previously known examples of infinite periodic finitely generated groups can be split into three classes. One class consists of solutions of the bounded Burnside problem. They are constructed using some versions of the small cancellation theory: combinatorial as in the original Adyan-Novikov proof (see [Adi79]), geometrical due to A. Ol′shanskii (see [Ol′91], [Iva94], [Lys96]) and E. Rips [Rip82], and via M. Gromov's theory of hyperbolic groups (see [Gro87], [Ol′93]).

The second class is the Golod-Shafarevich groups. The third class is groups generated by automata and groups defined by their action on rooted trees. The first examples in this class were constructed by S. Aleshin [Ale72], V. Sushchanskii [Suš79], R. Grigorchuk [Gri80], and N. Gupta and S. Sidki [GS83]. Related constructions (self-similar groups, branch groups, etc.) became a very active area of research; see [BGŠ03], [Gri05], [Nek05]. But periodic groups in this class remain to be more or less isolated examples.

Note that the groups in the latter two classes are necessarily residually finite. In fact, most methods of study of these groups heavily rely on their residual finiteness (e.g., on their action on a rooted tree). The groups in the first class may be simple, e.g., the Ol′shanskii-Tarski monsters [Ol′82].

Theorem 1.1 produces a new large class of groups of Burnside type. It includes the Grigorchuk group, so it intersects with the above mentioned third class of periodic groups, but it also contains many new groups. For instance, we produce the first example of a group of Burnside type generated by piecewise isometries of a polygon (with a finite number of pieces) and examples of simple groups.

Any non-free (i.e., such that some non-trivial elements have fixed points) action of D_∞ can be fragmented so that it satisfies the conditions of Theorem 1.1. For example, if $\mathcal{S} \subset \mathsf{X}^{\mathbb{Z}}$ is a minimal *palindromic* subshift (such that the elements of $\mathcal{S}$ contain arbitrarily long palindromes), then the transformations

$$a(w)(n) = w(-n), \qquad b(w)(n) = w(1-n)$$

for $w \in \mathcal{S}$ generate a minimal action of D_∞ such that a or b (depending on the parity of the lengths of arbitrarily long palindromes) has a fixed point. Then it is easy to fragment the corresponding generators so that the conditions of Theorem 1.1 are satisfied. Then, after passing to the group $\mathsf{A}(G, \mathcal{S})$, we get a finitely generated simple periodic group.

Minimal palindromic shifts are classical objects in Dynamics; see, for example, [BG13, §§4.3–4.4] and references therein. See also [GLN17], where spectral properties of a substitutional system associated with the Grigorchuk group are studied.

Answers to the Burnside problem were important examples for the theory of amenable groups. Amenability was defined by J. von Neumann [vN29] in his analysis of the Banach-Tarski paradox. He noted that a group containing a non-commutative free group is non-amenable, and he showed that amenability is preserved under some group-theoretic operations (extensions, direct limits, passing to a subgroup and to a quotient). So, there are "obviously non-amenable" groups (groups containing a free subgroup) and "obviously amenable" groups (groups that can be constructed from finite and commutative groups using the above operations). The "obviously amenable" groups are called *elementary amenable*. For more on amenability, see [Wag93], [Pat88], [Gre69].

Groups of Burnside type are never obviously non-amenable, since they do not contain free subgroups. They are also never elementary amenable; see [Cho80, Th. 2.3]. The fact that the class of groups without free subgroups and the class of elementary amenable groups are distinct is proved in [Cho80] precisely using the existence of groups of Burnside type.

Groups of Burnside type were the first examples to show that neither class (groups without free subgroups and elementary amenable groups) coincides with the class of amenable groups.

They were the first examples of non-amenable groups without free subgroups (free Burnside groups and Tarski monsters; see [Ol′80], [Ady82]). All known examples of finitely generated infinite groups of bounded exponent are non-amenable.

The groups in the second class (the Golod-Shafarevich groups) are all non-amenable by a result of M. Ershov [Ers11].

Groups of Burnside type (the Grigorchuk groups [Gri83], [Gri85]) were also the first examples of non-elementary amenable groups. In fact, for a long time the only known examples of non-elementary amenable groups were based on the Grigorchuk groups. Later, other examples were constructed [BV05], [BKN10], [AAV13], but all of them where defined by their actions on rooted trees, so, in particular, they were residually finite. For a long time the question if there exist infinite finitely generated simple amenable groups was open. It was answered by K. Juschenko and N. Monod in [JM13]. They showed that the *topological full group* of a minimal homeomorphism of the Cantor set is amenable (confirming a conjecture of R. Grigorchuk and K. Medynets). Here the topological full group of a (cyclic in this case) group G acting on a Cantor set $\mathcal{X}$ is the group of all homeomorphisms $h : \mathcal{X} \longrightarrow \mathcal{X}$ such that for every $\zeta \in \mathcal{X}$, there exists a neighborhood U of ζ and an element $g \in G$ such that $g|_U = h|_U$. In other words, it is obtained by "fragmenting" a minimal action of $\mathbb{Z}$ in a way similar to our definition of a fragmentation of D_∞. Our definition is different, however, as we do not require the sets where the action of an element $h \in A$ coincides with the action of a to be open.

It was proved earlier in [Mat06] and [BM08] that if τ is a minimal homeomorphism of the Cantor set, then the topological full group of $\langle\tau\rangle$ has simple derived subgroup, and if the homeomorphism is expansive (i.e., is conjugate to a subshift), then the derived subgroup is finitely generated. See also [CJN16], where a similar result is proved for $\mathbb{Z}^n$-actions. H. Matui proved in [Mat13] that the derived subgroups of the full groups of minimal subshifts are of exponential growth.

The methods of [JM13] were generalized in [JNdlS16] to cover a wide (including almost all known examples) class of non-elementary amenable groups.

For arbitrary fragmentations A, B of the generators a, b of a minimal action of the dihedral group, the group $\langle A \cup B\rangle$ can be embedded into the topological full group of a minimal subshift. This was observed for the first time (for the Grigorchuk group) by N. Matte Bon in [MB15]. It follows that all groups generated by fragmenting a minimal dihedral group are amenable. Theorem 1.1 produces, therefore, the first examples of simple amenable groups of Burnside type.

The only previously known simple groups of Burnside type were of bounded exponent and non-amenable (e.g., Ol′shanskii-Tarski monsters [Ol′82]).

If G is a group generated by a finite set S, then its *growth function* $\gamma(n)$ is the number of elements of G that can be written as products of at most n elements of $S \cup S^{-1}$. J. Milnor asked in [Mil68] whether there exists a group with growth function eventually bigger than any polynomial and eventually smaller than any exponential function. Such groups are called groups of *intermediate growth*, and the first example of such a group is the Grigorchuk group from [Gri80], [Gri83]. Amenability of the Grigorchuk group follows from its intermediate growth.

Recently, L. Bartholdi and A. Erschler developed a technique of *inverted orbits* and used it to construct a great variety of groups with a prescribed intermediate growth; see [BE12], [BE14a], [BE14b] and [Bar17].

Until now all constructions of groups of intermediate growth used as a starting point the groups from the family of Grigorchuk groups defined in [Gri85], or groups close to them (see [BŠ01], [BP06], [Ers06], [KP13], [Bri14]). This imposes some restrictions on the type of groups that can be obtained this way. In particular, the following problem, asked by R. Grigorchuk in 1984 (see Problem 9.8 in the "Kourovka notebook" [MK14]) remained to be open.

Problem. Does there exist a finitely generated simple group of intermediate growth?

See also [Man12, p. 132], [MK14, Prob. 15.17], [Gri14, Prob. 2], and [BM07], [BE14b]. In fact, it was even an open question for a long time whether all groups of intermediate growth are residually finite (see [Gri91, 8.4]). Note that it follows from M. Gromov's Theorem [Gro81] that groups of polynomial growth are residually finite. The first examples of groups of intermediate growth that are not residually finite were constructed by A. Erschler in [Ers04]. L. Bartholdi and A. Erschler showed in [BE14b] that every countable group not containing a group of exponential growth can be embedded into a group of intermediate growth. In particular, one can embed, using their result, any locally finite simple group into a group of intermediate growth.

Let $G = \langle A \cup B \rangle$ be a group satisfying the conditions of Theorem 1.1. Let $\zeta \in \mathcal{X}$ be a generic point of the Cantor set. Denote by Γ_ζ its *orbital* graph. Its set of vertices is the G-orbit of ζ. For every vertex η and every generator $s \in A \cup B$, we have an edge connecting η with $s(\eta)$, labeled by s. Since the generators $s \in A \cup B$ act on each point either trivially or as one of the homeomorphisms a, b, the graph Γ_ζ is just a "decorated" version of the orbital graph of ζ for the dihedral group $D_\infty = \langle a, b \rangle$. The latter is a bi-infinite chain, whose edges are alternatively labeled by a and b. The orbital graph Γ_ζ is obtained from it by replacing every edge labeled by a or b by a collection of edges labeled by some elements of A or B, respectively, and adding loops labeled by elements of $A \cup B$. Therefore, the graphs Γ_ζ are naturally represented by bi-infinite sequences $w_\zeta = \cdots x_{-1}x_0x_1\cdots$ over some finite alphabet.

Minimality of the action implies that the graphs Γ_ζ (equivalently, the sequences w_ζ) are *repetitive* for a generic ζ: for every finite subgraph Σ of Γ_ζ, there exists $R_\Sigma \in \mathbb{N}$ such that for every vertex η of Γ_ζ, there exists an isomorphic copy (as a labeled graph) of Σ at distance not more than R_Σ from η in Γ_ζ.

We say that G has *linearly repetitive orbits* if there exists a constant C such that R_Σ is bounded from above by C times the diameter of Σ. We prove the following (see Theorem 6.6).

THEOREM 1.2. *Let $\langle A \cup B\rangle$ be a group satisfying the conditions of Theorem* 1.1. *If it has linearly repetitive orbits of generic points*, *then it is of intermediate growth. If the action of the dihedral group D_∞ is expansive*, *then the corresponding group $\mathsf{A}(\langle A \cup B\rangle, \mathcal{X})$ is finitely generated*, *simple*, *periodic*, *and has intermediate growth.*

Thus the answer to Problem 1 is positive. For properties of linearly repetitive (also called *linearly recurrent*) dynamical systems and quasi-crystals, and applications to spectral theory of Schrödinger operators, see [KLS15, Ch.s 6 and 9] and [DL06]. Linear repetitiveness is closely related to the so-called *Boshernitzan condition*; see [Bos92].

Linear repetitivity is a stronger condition than *linear complexity.* An infinite sequence w has linear complexity if the number of different subwords of length n of the sequence w is bounded from above by Cn for some n. If a group G generated by a fragmentation of a minimal action of the dihedral group has orbital graphs of linear complexity, then G is Liouville, by a theorem of N. Matte Bon [MB14] (which is applicable to a more general type of groups and with a weaker condition on the sequences). The Liouville condition (absence of non-constant bounded harmonic functions) is stronger than amenability, but weaker than subexponential growth.

Our method of proving periodicity and intermediate growth is substantially different from the original proofs of periodicity and intermediate growth of the Grigorchuk group, since we cannot use an action on a rooted tree. All previous proofs of intermediate growth of a group used "length reduction" of automorphisms of rooted trees, as in the original paper of R. Grigorchuk, or used intermediate growth of the Grigorchuk groups.

We study how points travel inside the orbital graphs Γ_ζ under the action of positive powers of one element (to prove periodicity) or under the action of a long product of generators (to prove intermediate growth). In both cases we use one-dimensional structure of the orbit, i.e., the fact that a trajectory starting in one vertex and ending in another has to pass through all the vertices between them. Small neighborhoods of the special point ξ from Theorem 1.1 act as "reflectors": trajectories approaching them often "bounce" and change their direction. This is used in Theorem 1.1 to prove that a sequence $g^k(\zeta)$, $k \geq 1$, must eventually come back to ζ, thus getting periodicity of g. A similar idea shows that in the case when Γ_ζ is linearly repetitive, the trajectory $\zeta, g_1(\zeta), g_2g_1(\zeta), \ldots, g_n \cdots g_2g_1(\zeta)$ of a vertex ζ under a long product $g_n \cdots g_2g_1$ of generators of G tends to change its direction often, so that it rarely goes far away. This gives, using the techniques of inverted orbits of [BE12], a subexponential estimate of the form $C_1 \exp\left(\frac{n}{\exp(C_2\sqrt{\log n})}\right)$ on the total number of elements $g_n \cdots g_2g_1 \in G$, thus proving Theorem 1.2.

It is interesting to note that the proof of the main result of [JM13], as analyzed in [JNdlS16] and [JMMdlS18], is also using a similar idea: trajectories of a random walk on the orbital graph Γ_ζ eventually return back to ζ with probability one. This, together with the one-dimensional structure of the graph, implies amenability of the group. Here we do not need the "reflectors" produced by a special point of $\mathcal{X}$, since we need only a probabilistic result to prove amenability.

Orbital graphs of the Grigorchuk groups were studied in great detail by Y. Vorobets in [Vor12]; an important part of our construction is based on his results.

Section 2 contains preliminary general facts on groups acting on topological spaces, orbital graphs, graphs of germs, and minimal actions of the dihedral group. In Section 3 we define fragmentations of dihedral groups and study their orbital graphs and graphs of germs. Section 4 contains the proof of Theorem 1.1. Theorem 1.2 is proved in Section 6. Finally, Section 8 describes in detail one particular example: a fragmentation of the substitutional Fibonacci shift. We include it here to give an explicit example of a finitely generated simple periodic group of intermediate growth, without relying on the somewhat indirect proof of finite generation of $\mathsf{A}(G, \mathcal{X})$ in [Nek15].

Acknowledgements. The author is very grateful to Laurent Bartholdi, Justin Cantu, Yves de Cornulier, Rostislav Grigorchuk, Mikhail Hlushchanka, Kate Juschenko, Rostislav Kravchenko, Nicolas Matte Bon, Mark Sapir, Said Sidki, and the referees for remarks and suggestions.

The paper is based upon work supported by the National Science Foundation under Grant DMS-1709480.

2. **Preliminaries on group actions**

We use left actions, so in a product a_1a_2 the transformation a_2 is performed before a_1. We denote the identity transformation and the identity element of a group by ε (except for $\mathbb{Z}/2\mathbb{Z}$, where the trivial element is naturally denoted 0). The symmetric and the alternating group acting on a set A are denoted $\mathsf{S}(A)$ and $\mathsf{A}(A)$, respectively.

For a finite alphabet X, we denote by X^ω the space of infinite one-sided sequences $x_1x_2\cdots$ of elements of X, and by $\mathsf{X}^{\mathbb{Z}}$ the space of two-sided sequences $\cdots x_{-2}x_{-1}\,.\,x_0x_1\cdots$. Both spaces are endowed with the direct product topology, where X is discrete. We denote by X^* the set of all finite words over the alphabet X, i.e., the free monoid generated by X.

2.1. *Graphs of actions.* All graphs in this section are oriented, and loops and multiple edges are allowed. Their edges are labeled. Distances between vertices in such graphs are measured ignoring the orientation. Similarly, connectedness and connected components are also defined ignoring the orientation. Isomorphisms must preserve orientation and labeling. A graph is called *rooted* if one vertex, called the *root*, is marked. Every morphism of rooted graphs must map the root to the root.

We denote a ball of radius r with center in a vertex v of a graph Γ by $B_v(r)$. It is considered to be a rooted graph (with the root v). Its set of edges is the set of all edges of Γ connecting the vertices of $B_v(r)$. The orientation and labeling are inherited from Γ.

Let G be a group generated by a finite set S and acting by homeomorphisms on a compact metrizable space $\mathcal{X}$. For $\zeta \in \mathcal{X}$, the *orbital graph* Γ_ζ is the graph with the set of vertices equal to the orbit $G\zeta$ of ζ, in which for every $\eta \in G\zeta$ and every $s \in S$, there is an arrow from η to $s(\eta)$ labeled by s.

The graph Γ_ζ is naturally isomorphic to the *Schreier graph* of the group G modulo the stabilizer G_ζ. The Schreier graph of G modulo a subgroup H is, by definition, the graph with the set of vertices equal to the set of cosets gH, $g \in G$, in which for every coset gH and every generator $s \in S$, there is an arrow from gH to sgH labeled by s.

Denote by $G_{(\zeta)}$ the subgroup of elements of G acting trivially on a neighborhood of ζ, i.e., the subgroup of all elements $g \in G$ such that ζ is an interior point of the set of fixed points of g. The *graph of germs* $\widetilde{\Gamma}_\zeta$ is the Schreier graph of G modulo $G_{(\zeta)}$. Note that $G_{(\zeta)}$ is a normal subgroup of G_ζ, hence the map $hG_{(\zeta)} \mapsto hG_\zeta$ induces a Galois covering of graphs $\widetilde{\Gamma}_\zeta \longrightarrow \Gamma_\zeta$ with the group of deck transformations $G_\zeta/G_{(\zeta)}$. We call $G_\zeta/G_{(\zeta)}$ the *group of germs* of the point ζ.

The vertices of $\widetilde{\Gamma}_\zeta$ are identified with *germs* of elements of G at ζ. Here a germ is an equivalence class of a pair (g, ζ), where two pairs (g_1, ζ) and (g_2, ζ) are equivalent if there exists a neighborhood U of ζ such that $g_1|_U = g_2|_U$.

If $g_2(\zeta_2) = \zeta_1$, then the composition $(g_1, \zeta_1)(g_2, \zeta_2)$ is well defined and is equal to (g_1g_2, ζ_2). The inverse of the germ (g, ζ) is the germ $(g^{-1}, g(\zeta))$. The set of all germs of an action is a *groupoid* with respect to these operations, i.e., a small category of isomorphisms. It has a natural topology with the basis consisting of the sets of the form $\mathcal{U}_{g,U} = \{(g, \zeta) \, : \, \zeta \in U\}$, where $g \in G$ and U is an open subset of $\mathcal{X}$.

Definition 2.1. A point $\zeta \in \mathcal{X}$ is said to be *G-regular* if its group of germs is trivial, i.e., if every element $g \in G$ fixing ζ acts identically on a neighborhood of ζ. If ζ is not G-regular, then we say that it is *singular*.

Note that for every $g \in G$, the set of points $\zeta \in \mathcal{X}$ such that $g(\zeta) = \zeta$ but $g \notin G_{(\zeta)}$ is equal to the boundary of the set of fixed points of g. It follows that this set is closed and nowhere dense. Consequently, if G is countable (in particular, if G is finitely generated), then the set of G-regular points is co-meager (residual).

Note also that $gG_\zeta g^{-1} = G_{g(\zeta)}$ and $gG_{(\zeta)}g^{-1} = G_{(g(\zeta))}$ for all $\zeta \in \mathcal{X}$ and $g \in G$, which implies that the set of G-regular points is G-invariant.

Depending on the separation conditions for the elements of the group of germs $G_\zeta/G_{(\zeta)}$ with respect to the natural topology on the groupoid of germs, singular points can be classified in the following way.

Definition 2.2. Suppose that $\zeta \in \mathcal{X}$ is a singular point. We say that ζ is a *Hausdorff singularity* if for every $g \in G_\zeta \setminus G_{(\zeta)}$, the interior of the set of fixed points of g does not accumulate on ζ. Otherwise, ζ is a *non-Hausdorff singularity.*

We say that ζ is a *purely non-Hausdorff singularity* if for every $g \in G_\zeta$, the interior of the set of fixed points of g accumulates on ζ.

Let (Γ_1, v_1), (Γ_2, v_2) be connected rooted labeled graphs, where v_i are the roots. Define the distance $d((\Gamma_1, v_1), (\Gamma_2, v_2))$ between them as $2^{-(R+1)}$, where R is the maximal integer such that the balls $B_{v_1}(R) \subset \Gamma_1$ and $B_{v_2}(R) \subset \Gamma_2$ of radius R with centers in v_1 and v_2 are isomorphic as rooted graphs. Fix a finite set of labels S and a positive integer k. Let $\mathcal{G}_{S,k}$ be the set of all isomorphism classes of connected oriented rooted graphs edge-labeled by elements of S and such that every vertex is adjacent to at most k edges. Then the metric d defines a compact topology on $\mathcal{G}_{S,k}$.

PROPOSITION 2.3. *The set of points of continuity of the map $\zeta \mapsto (\Gamma_\zeta, \zeta)$ from $\mathcal{X}$ to the space of labeled rooted graphs is equal to the set of regular points.*

The set of points of continuity of the map $\zeta \mapsto (\widetilde{\Gamma}_\zeta, \zeta)$ is equal to the union of the set of regular points and the set of Hausdorff singularities.

The statement about the regular points was proved in [Vor12].

Proof. The ball $B_\zeta(r)$ in Γ_ζ can be described by a finite system of equations and inequalities of the form $g_1(\zeta) = g_2(\zeta)$ or $g_1(\zeta) \neq g_2(\zeta)$ for pairs of elements $g_1, g_2 \in G$ of length at most r. If the point ζ is regular, then every such equality or inequality holds for all points of some neighborhood of ζ. It follows that there exists a neighborhood N of ζ such that for every $\eta \in N$, the balls $B_\zeta(r)$ and $B_\eta(r)$ of the corresponding orbital graphs are isomorphic as rooted labeled graphs. This implies that the map $\eta \mapsto (\Gamma_\eta, \eta)$ is continuous at η.

Conversely, suppose that $\eta \mapsto (\Gamma_\eta, \eta)$ is continuous at ζ, and let g be an element of G_ζ. Write it as a product of generators and their inverses. This product will correspond to a path in (Γ_ζ, ζ) starting and ending in ζ. By continuity of the map $\eta \mapsto (\Gamma_\eta, \eta)$, and the definition of the topology on the space of rooted graphs, there exists a neighborhood N of ζ such that for every $\eta \in N$, an isomorphic path starting and ending in η appears in Γ_η. This implies that $g(\eta) = \eta$ for all $\eta \in N$, and hence ζ is regular.

The proof of the statement about the map $\zeta \mapsto (\widetilde{\Gamma}_\zeta, \zeta)$ is similar. One has just to use the fact that ζ belongs to the union of the set of regular points and the set of Hausdorff singularities if and only if for every $g \in G$ such that $(g, \zeta) = (\varepsilon, \zeta)$ and every $h \in G$, there exists a neighborhood U of ζ such that either $h|_U = g|_U$ or $(h, \eta) \neq (g, \eta)$ for all $\eta \in U$. □

Definition 2.4. The action of G on $\mathcal{X}$ is said to be *minimal* if all G-orbits are dense in $\mathcal{X}$.

PROPOSITION 2.5. *Suppose that the action of G on $\mathcal{X}$ is minimal. Then for every $r > 0$, there exists $R(r) > 0$ such that for every G-regular point $\zeta \in \mathcal{X}$ and for every $\eta \in \mathcal{X}$, there exists a vertex η' of Γ_η such that $d(\eta, \eta') \le R(r)$ and the rooted balls $B_\zeta(r) \subset \Gamma_\zeta$ and $B_{\eta'}(r) \subset \Gamma_\eta$ are isomorphic.*

Proof. By Proposition 2.3, if ζ is G-regular, then for every $r > 0$, there exists a neighborhood N of ζ such that for every $\eta' \in N$, the balls $B_\zeta(r)$ and $B_{\eta'}(r)$ of the corresponding orbital graphs are isomorphic as rooted labeled graphs.

For every point $\eta \in \mathcal{X}$, there exists an element $g \in G$ such that $g(\eta) \in N$. The set of sets of the form $g^{-1}(N)$ covers $\mathcal{X}$ and, by compactness, there exists a finite subcover $g_1^{-1}(N), g_2^{-1}(N), \ldots, g_n^{-1}(N)$. Let R be the maximal length of the elements g_i with respect to the generating set S. Then for every $\eta \in \mathcal{X}$, there exists g_i such that $g_i(\eta) \in N$, and hence the balls $B_\zeta(r)$ and $B_{\eta'}(r)$ are isomorphic for $\eta' = g_i(\eta)$. Distance from η to η' is not more than R. Since the number of isomorphism classes of balls of radius r in the orbital graphs of G is finite, we can find an estimate $R(r)$ independent of ζ. □

Definition 2.6. We say that the action of G on $\mathcal{X}$ is *linearly repetitive* if there exists $K > 1$ such that the function $R(r)$ from Proposition 2.5 satisfies $R(r) < Kr$ for all $r \ge 1$.

2.2. *Topological full groups.* Let G be a group acting on a Cantor set $\mathcal{X}$. The *topological full group* $\mathsf{F}(G, \mathcal{X})$ of the action is the group of all homeomorphisms $h : \mathcal{X} \longrightarrow \mathcal{X}$ such that for every $\zeta \in \mathcal{X}$, there exist a neighborhood U

of ζ and an element $g \in G$ such that $h|_U = g|_U$. Topological full groups were introduced in [GPS99]. (See also [Kri80], where an earlier particular example has appeared.) See the papers [Mat06], [Mat13], [Mat15], [Mat12] and the survey [dC14] for various properties of topological full groups of group actions and étale groupoids.

Let $U \subset \mathcal{X}$ be a non-empty clopen set, and let $g_1, g_2, \ldots, g_n \in G$ be such that the sets $U_1 = g_1(U), U_2 = g_2(U), \ldots, U_n = g_n(U)$ are pairwise disjoint. Then for every permutation $\alpha \in \mathsf{S}_n = \mathsf{S}(\{1, 2, \ldots, n\})$, we get the corresponding element h_α of the topological full group acting by the rule

$$h_\alpha(\zeta) = \begin{cases} g_j g_i^{-1}(\zeta) & \text{if } \zeta \in U_i \text{ and } \alpha(i) = j, \\ \zeta & \text{if } \zeta \notin \bigcup_{i=1}^n U_i. \end{cases}$$

The map $\alpha \longrightarrow h_\alpha$ is a monomorphism from S_n to $\mathsf{F}(G)$. Denote by $\mathsf{A}(G, \mathcal{X})$ the subgroup generated by the images of the alternating subgroups $\mathsf{A}_n < \mathsf{S}_n$ for all such monomorphisms.

The following is proved in [Nek15].

THEOREM 2.7. *If the action of G on $\mathcal{X}$ is minimal, then $\mathsf{A}(G, \mathcal{X})$ is simple and is contained in every non-trivial normal subgroup of $\mathsf{F}(G, \mathcal{X})$. If the action of G on $\mathcal{X}$ is expansive and every G-orbit has cardinality at least* 5, *then $\mathsf{A}(G, \mathcal{X})$ is finitely generated.*

Definition 2.8. An action of G on $\mathcal{X}$ is said to be *expansive* if there exists $\delta > 0$ such that $d(g(\zeta_1), g(\zeta_2)) < \delta$ for all $g \in G$ implies $\zeta_1 = \zeta_2$ (where d is a metric on $\mathcal{X}$ compatible with the topology).

An action $(G, \mathcal{X})$ on a Cantor set is expansive if and only if there exists a G-equivariant homeomorphism from $\mathcal{X}$ to a closed G-invariant subset of A^G for some finite alphabet A.

2.3. *Minimal actions of the dihedral group.* When a set of generators S of a group G consists of elements of order two, then we will consider the orbital graphs and graphs of germs as non-oriented, so that an edge connecting two vertices v_1 and v_2 labeled by $s \in S$ replaces two arrows labeled by s: one from v_1 to v_2, and one from v_2 to v_1 (if the edge is not a loop).

Let a, b be homeomorphisms of period two of a Cantor set $\mathcal{X}$ such that the dihedral group $\langle a, b\rangle$ acts minimally on $\mathcal{X}$.

LEMMA 2.9. *The orbital graphs of $\langle a, b\rangle$ are either one-ended or two-ended infinite chains. The graphs of germs are two-ended infinite chains.*

Proof. The Schreier graphs of the infinite dihedral group D_∞ are either infinite chains (one-ended or two-ended), or finite chains, or finite cycles. The latter two cases are impossible, since then we have a finite orbit, which contradicts the minimality.

Suppose that a graph of germs is a one-ended infinite chain. Then the endpoint of the chain is a fixed point of one of the generators, and there are no other fixed points of the generators in this orbit. Since this is a graph of germs, it follows that the generator fixes this point together with every point of a neighborhood. But then, by minimality, there are other points of the orbit where the germ of the generator is trivial, which is a contradiction. □

COROLLARY 2.10. *If the stabilizer $\langle a, b\rangle_\xi$ is non-trivial, then there exists a unique point ξ' in the orbit of ξ such that $\langle a, b\rangle_{\xi'}$ is equal either to $\langle a\rangle$ or to $\langle b\rangle$.*

Let us show how minimality and expansivity conditions for D_∞- and $\mathbb{Z}$-actions are related.

PROPOSITION 2.11. *Let a and b be homeomorphisms of period two of a Cantor set $\mathcal{X}$. If the action the dihedral group $\langle a, b\rangle$ is minimal, then either the action of $\langle ab\rangle$ is minimal, or $\mathcal{X}$ is split into a disjoint union of two clopen $\langle ab\rangle$-invariant sets S_1, S_2 such that the action of $\langle ab\rangle$ on each of these sets is minimal, and $a(S_1) = b(S_1) = S_2$, $a(S_2) = b(S_2) = S_1$.*

In particular, if the action of D_∞ is non-free (has non-trivial stabilizers of some points), then the D_∞-minimality is equivalent to the $\mathbb{Z}$-minimality.

Proof. Suppose that the $\langle a, b\rangle$-action is minimal. If $A \subset \mathcal{X}$ is a closed non-empty $\langle ab\rangle$-invariant set, then $a(A)$ is also a closed $\langle ab\rangle$-invariant set (since $(ab)a(A) = a(ba)A = a(ab)^{-1}(A) = a(A)$). It follows that $a(A) \cap A$ and $a(A) \cup A$ are closed and $\langle a, b\rangle$-invariant. Consequently, $a(A) \cup A = \mathcal{X}$, and either $a(A) \cap A = \mathcal{X}$, or $a(A) \cap A = \emptyset$, which finishes the proof. □

PROPOSITION 2.12. *Let a and b be homeomorphisms of period two of a Cantor set $\mathcal{X}$. Suppose that they generate an expansive action of D_∞. Then there exists a finite alphabet A, a permutation $\iota : A \longrightarrow A$ such that $\iota^2 = \varepsilon$, and a $\mathbb{Z}$-subshift $\mathcal{S} \subset A^{\mathbb{Z}}$ such that there exists a homeomorphism $\mathcal{X} \longrightarrow \mathcal{S}$ conjugating the action of the generators a and b with the homeomorphisms of $\mathcal{S}$ given by the formulas*

$$a(w)(n) = \iota(w(-n)), \qquad b(w)(n) = \iota(w(1-n))$$

for every $w \in \mathcal{S}$ and $n \in \mathbb{Z}$.

Recall that a *subshift* is a closed $\mathbb{Z}$-invariant subset of $A^{\mathbb{Z}}$.

Proof. There exists a partition $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ of $\mathcal{X}$ into clopen sets such that every point $\zeta \in \mathcal{X}$ is uniquely determined by its *itinerary*, which is defined as the map $I_\zeta : D_\infty \longrightarrow \mathcal{U}$ given by the condition $I_\zeta(g) \ni g(\zeta)$. We may assume that $\mathcal{U}$ is a-invariant, i.e., that for every $U \in \mathcal{U}$, the set $a(U)$ belongs to $\mathcal{U}$. Otherwise, we can replace $\mathcal{U}$ by the partition induced by $\mathcal{U}$ and

$a(\mathcal{U})$: two points ζ_1, ζ_2 belong to one piece of the induced partition if and only if they belong to one piece of $\mathcal{U}$ and to one piece of $a(\mathcal{U})$.

Then for every $\zeta \in \mathcal{X}$ and $g \in D_\infty$, we have $I_\zeta(g) = a(I_\zeta(ag))$, so that I_ζ, and hence ζ, are uniquely determined by the sequence $I_\zeta((ab)^n)$, $n \in \mathbb{Z}$. Let us denote $J_\zeta(n) = I_\zeta((ab)^n)$.

The set of sequences of the form $J_\zeta(n)$ is obviously a closed shift-invariant subset of the full shift $\mathcal{U}^{\mathbb{Z}}$.

Let us describe the action of a and b on the sequences $J_\zeta(n)$. We have $J_{a(\zeta)}(n) = I_\zeta((ab)^n a) = I_\zeta(a(ba)^n) = a(I_\zeta((ba)^n)) = a(J_\zeta(-n))$ and $J_{b(\zeta)}(n) = I_\zeta((ab)^n b) = I_\zeta(a(ba)^{n-1}) = a(I_\zeta((ba)^{n-1}) = a(J_\zeta(1-n))$. We can therefore define the permutation ι of $\mathcal{U}$ equal to the action of a on $\mathcal{U}$. $\square$

If the permutation ι in Proposition 2.12 is identical, then the transformations a and b are "central symmetries" of the infinite sequences from the subshift $\mathcal{A}$. The transformation a flips the sequences around the zeroth letter, while b flips them around the space between the zeroth and first letters. The subshift $\mathcal{S}$ has to be invariant under a (and then it will be invariant under b). Such subshifts are called *palindromic*. A minimal subshift is palindromic if and only if a sequence $w \in \mathcal{S}$ contains arbitrarily long palindromes as subwords. So, every palindromic minimal subshift is associated with a natural minimal expansive action of D_∞.

Example 2.13. Let τ be the substitution (i.e., an endomorphism of the free monoid $\{0,1\}^*$) given by

$$\tau : 0 \mapsto 01, \qquad 1 \mapsto 10.$$

The words $\tau^n(0)$ converge to an infinite sequence $0110100110010110\cdots$ called the *Thue-Morse sequence*. Let $\mathcal{S}$ be the set of all bi-infinite sequences $w = \cdots x_{-1}x_0x_1\cdots$ such that every subword of w is a subword of $\lim_{n\to\infty} \tau^n(0)$. It is known that $\mathcal{S}$ is a minimal subshift (see [AS03, Example 10.9.3]).

Note that the words $\tau^2(0)$ and $\tau^2(1)$ are palindromes:

$$\tau^2(0) = 0110, \qquad \tau^2(1) = 1001.$$

It follows by induction that $\tau^{2n}(0)$ and $\tau^{2n}(1)$ are palindromes for all $n \geq 1$. Consequently, the shift $\mathcal{S}$ is palindromic, and the central symmetries a and b around the position number 0 and the space between positions number 0 and 1 generate a minimal expansive action of D_∞.

Example 2.14. Consider the alphabet $\mathsf{X} = \{1, 1^*, 2, 2^*\}$, the involution $\iota : x \leftrightarrow x^*$, $x \in \{1, 2\}$, and the substitution

$$\tau : 1 \mapsto 2, \quad 1^* \mapsto 2^*, \quad 2 \mapsto 1^*2^*, \quad 2^* \mapsto 21.$$

We have $\iota \circ \tau = \tau \circ \iota$ on X^*, where $\iota(x_1x_2\cdots x_n) = \iota(x_n)\iota(x_{n-1})\cdots\iota(x_1)$.

Let $\mathcal{S} \subset \mathsf{X}^{\mathbb{Z}}$ be the subshift generated by τ, similarly to the previous example. Since ι commutes with τ, all words $\tau^n(2^*2)$ are ι-invariant:

$$2^*|2, \quad 21|1^*2^*, \quad 1^*2^*2|2^*21, \quad \dots .$$

It follows that the shift $\mathcal{S}$ is invariant under the transformations a and b defined as in Proposition 2.12.

2.4. *Odometer actions.* In some sense the opposite condition to expansiveness is *residual finiteness* of the action. We say that an action of a group G on a Cantor set $\mathcal{X}$ is residually finite if the G-orbit of every clopen subset of $\mathcal{X}$ is finite. An action is residually finite if and only if there exists a homeomorphism $\Phi : \mathcal{X} \longrightarrow \partial T$ of $\mathcal{X}$ with the boundary of a locally finite rooted tree T and an action of G on T by automorphisms such that Φ is G-equivariant (with respect to the action of G on ∂T induced by the action on T); see [GNS00, Prop. 6.4].

Every minimal residually finite action of $\mathbb{Z}$ on a Cantor set is topologically conjugate to an *odometer*, i.e., the transformation $\alpha : x \mapsto x + 1$ on the projective limit $\overline{\mathbb{Z}}$ of a sequence $\mathbb{Z}/(d_1 d_2 \cdots d_n)\mathbb{Z}$ of finite cyclic groups.

Proposition 2.15. *Consider a non-free minimal residually finite action of the dihedral group D_∞ on a Cantor set $\mathcal{X}$. Then there exists a homeomorphism of $\mathcal{X}$ with a projective limit $\overline{\mathbb{Z}}$ of finite cyclic groups conjugating the action of D_∞ with the action generated by the homeomorphisms*

$$a(x) = 1 - x, \qquad b(x) = -x.$$

Proof. Let a and b be the generators of D_∞ of order 2. By Corollary 2.10, one of the generators a, b has a fixed point, and by Proposition 2.11 the action of ab is minimal. Then the homeomorphism $\alpha = ab$ is an odometer, i.e., is conjugate to the action of $x \mapsto x + 1$ on some projective limit $\overline{\mathbb{Z}}$ of finite cyclic groups.

We have $b\alpha b = \alpha^{-1}$. If b_1, b_2 are order two homeomorphisms of the Cantor set such that $b_i \alpha b_i = \alpha^{-1}$, then $b_1 b_2$ commutes with α. By continuity and minimality of the action of α on $\overline{\mathbb{Z}}$, the homeomorphism $b_1 b_2$ commutes with every transformation of the form $x \mapsto x + h$ for $h \in \overline{\mathbb{Z}}$. It follows that $b_1 b_2(h) = b_1 b_2(h + 0) = h + b_1 b_2(0)$ for every $h \in \overline{\mathbb{Z}}$, i.e., that $b_1 b_2$ is of the form $x \mapsto x + g$ for some $g \in \overline{\mathbb{Z}}$.

The transformation $b_0(x) = -x$ satisfies $b_0 \alpha b_0 = \alpha^{-1}$. It follows that b is of the form $b(x) = -x + g$ for some $g \in \overline{\mathbb{Z}}$. Then $a = \alpha b$ is given by $a(x) = -x + g + 1$.

If all cyclic groups $d_1 d_2 \cdots d_n \mathbb{Z}$ in the projective limit have odd order, then the equation $2x = g$ has a solution in $\overline{\mathbb{Z}}$ for every $g \in \overline{\mathbb{Z}}$. Otherwise, either the equation $2x = g$, or the equation $2x = g + 1$, has a solution. It follows that in the odd case both involutions a, b have a fixed point, while in the even case exactly one of them has a fixed point.

Let us assume that b has a fixed point $\xi \in \overline{\mathbb{Z}}$. Then, conjugating everything by the shift $x \mapsto x - \xi$, we may assume that $\xi = 0$. Then $a : x \mapsto -x + 1$ and $b : x \mapsto -x$. $\square$

For example, if $\mathcal{X}$ is the ring of dyadic integers, i.e., the projective limit of the cyclic groups $\mathbb{Z}/2^n\mathbb{Z}$, then the corresponding action of $a : x \mapsto 1 - x$ and $b : x \mapsto -x$ is conjugate to the following action on the space $\{0, 1\}^\omega$ of right-infinite binary sequences:

$$a(0w) = 1w, \qquad a(1w) = 0w,$$
$$b(0w) = 0a(w), \qquad b(1w) = 1b(w).$$

Figure 1 shows the corresponding action on the binary rooted tree.

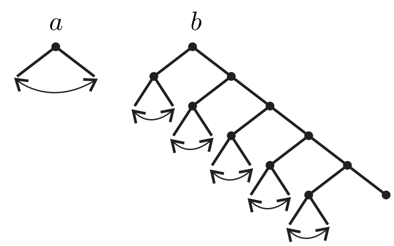

Figure 1. The Gray code action of D_∞.

This particular realization corresponds to the Gray code (see [Wil89, Ch. 1]). It is also the standard self-similar action of the iterated monodromy group of the Chebyshev polynomial $T_2 = 2x^2 - 1$. The iterated monodromy group of the degree d Chebyshev polynomial is conjugate to the natural action of D_∞ on the ring $\lim_\leftarrow \mathbb{Z}/d^n\mathbb{Z}$ of d-adic integers. For their standard self-similar actions, see [Nek05, Prop. 6.12.6].

3. **Fragmentations of dihedral groups**

Definition 3.1. Let a be a homeomorphism of period two of a Cantor set $\mathcal{X}$. A *fragmentation* of a is a finite group A of homeomorphisms of $\mathcal{X}$ such that for every $h \in A$ and $\zeta \in \mathcal{X}$, we have $h(\zeta) = \zeta$ or $h(\zeta) = a(\zeta)$, and for every $\zeta \in \mathcal{X}$, there exists $h \in A$ such that $h(\zeta) = a(\zeta)$.

Note that if h is an element of a fragmentation A, then the sets $E_{h,\varepsilon} = \{\zeta \in \mathcal{X} : h(\zeta) = \zeta\}$ and $E_{h,a} = \{\zeta \in \mathcal{X} : h(\zeta) = a(\zeta)\}$ are closed, a-invariant, their intersection is the set of fixed point of a, and we have $E_{h,\varepsilon} \cup E_{h,a} = \mathcal{X}$. If the set of fixed points of a has empty interior (e.g., if it is an element of a generating set of a minimal action of D_∞), then the interiors of $E_{h,\varepsilon}$ and $E_{h,a}$ are disjoint.

Suppose that the set of fixed points of a has empty interior. Choose for every $h \in A$ a set Q_h either equal to the interior of $E_{h,\varepsilon}$ or to the interior of $E_{h,a}$, and consider the intersection of the chosen sets. Let $\mathcal{P}$ be the set of all intersections that can be obtained this way. The set $\mathcal{P}$ has the following properties:

(1) $\mathcal{P}$ is finite;
(2) the elements of $\mathcal{P}$ are a-invariant, open, and pairwise disjoint;
(3) $\bigcup_{P\in\mathcal{P}} \overline{P} = \mathcal{X}$;
(4) for all $P_1, P_2 \in \mathcal{P}$ such that $P_1 \neq P_2$, the set $\overline{P_1} \cap \overline{P_2}$ consists of fixed points of a.

We call the elements of $\mathcal{P}$ the *pieces* of the fragmentation A. Every piece $P \in \mathcal{P}$ defines an epimorphism $\pi_P : A \longrightarrow \mathbb{Z}/2\mathbb{Z}$ by the rule

$$\pi_P(h) = \begin{cases} 0 & \text{if } P \subset E_{h,\varepsilon}, \\ 1 & \text{if } P \subset E_{h,a}. \end{cases}$$

In other words, $\pi_P(h) = 1$ if h acts on P as a, and $\pi_P(h) = 0$ if h acts on P as the identity. The map $(\pi_P)_{P\in\mathcal{P}}$ defines an embedding of A into $(\mathbb{Z}/2\mathbb{Z})^{\mathcal{P}}$.

Conversely, suppose that a collection $\mathcal{P}$ satisfies conditions (1)–(4). Then for any $\pi \in (\mathbb{Z}/2\mathbb{Z})^{\mathcal{P}}$, the map a_π defined by

$$a_\pi(\zeta) = \begin{cases} a(\zeta) & \text{if } \zeta \in \overline{P}, P \in \mathcal{P} \text{ and } \pi(P) = 1, \\ \zeta & \text{if } \zeta \in \overline{P}, P \in \mathcal{P} \text{ and } \pi(P) = 0 \end{cases}$$

is a homeomorphism, and the map $\pi \mapsto a_\pi$ is an isomorphism of $(\mathbb{Z}/2\mathbb{Z})^{\mathcal{P}}$ with a fragmentation of a.

The group $(\mathbb{Z}/2\mathbb{Z})^{\mathcal{P}}$ is, therefore, the maximal fragmentation of a with the set of pieces $\mathcal{P}$. Any subgroup $A \leq (\mathbb{Z}/2\mathbb{Z})^{\mathcal{P}}$ such that all homomorphisms $\pi_P : A \longrightarrow \mathbb{Z}/2\mathbb{Z}$ are surjective is a fragmentation with the set of pieces $\mathcal{P}$.

Our main subject is fragmentations with purely non-Hausdorff singularities; see Definition 2.2. Every non-free minimal action of D_∞ can be fragmented so that we get a purely non-Hausdorff singularity in the following way.

LEMMA 3.2. *Suppose that a has a fixed point ξ. Then for every $n \geq 1$ there exists a partition of $\mathcal{X} \setminus \{\xi\}$ into a disjoint union of open a-invariant subsets $P_1, P_2, \ldots, P_n$ such that each set P_i accumulates on ξ.*

Proof. Let $U_k, k \geq 0$, be a descending sequence of clopen neighborhoods of ξ such that $U_0 = \mathcal{X}$ and $\bigcap_{k\geq 0} U_k = \{\xi\}$. Then $V_k = U_k \cap a(U_k)$ is a descending sequence of clopen a-invariant neighborhoods of ξ such that $\bigcap_{k\geq 1} V_k = \{\xi\}$. Remove all repetitions, so that $V_k \neq V_{k+1}$ for every k.

Choose an arbitrary partition of the set of non-negative integers into n disjoint infinite subsets $I_1, I_2, \ldots, I_n$, and define $P_i = \bigcup_{k\in I_i} V_k \setminus V_{k+1}$. □

Suppose that $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ is as in Lemma 3.2, and suppose that the set of fixed points of a has empty interior. Choose a subgroup $A \le (\mathbb{Z}/2\mathbb{Z})^{\mathcal{P}}$ such that each homomorphism $\pi_{P_i} : A \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is surjective, but there is no element $h \in A$ such that $\pi_{P_i}(h) = 1$ for all P_i. Then A is a fragmentation of a such that ξ is a purely non-Hausdorff singularity. It is always possible to choose such an A if $n \ge 3$. For example, for $n = 3$, such a subgroup of $(\mathbb{Z}/2\mathbb{Z})^3$ is $\{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$.

Example 3.3. Consider the space $\{0,1\}^\omega$ of right-infinite sequences $x_1 x_2 \cdots$ over the binary alphabet $\{0,1\}$. Consider the Gray code transformations a and b, as defined in 2.4.

The homeomorphism b has a unique fixed point $\xi = 111\cdots$. The sets $W_n = \underbrace{11\cdots1}_{n \text{ times}}0\{0,1\}^\omega$ of sequences starting with exactly n ones form a partition of $\{0,1\}^\omega \setminus \{\xi\}$ into open b-invariant subsets.

Consider the partition

$$P_0 = \bigcup_{k=0}^{\infty} W_{3k}, \quad P_1 = \bigcup_{k=0}^{\infty} W_{3k+1}, \quad P_2 = \bigcup_{k=0}^{\infty} W_{3k+2}$$

of $\{0,1\}^\omega \setminus \{\xi\}$, and the subgroup $B = \{b_1, b_2, b_3, \varepsilon\}$, where b_1 acts as b on $P_0 \cup P_1$, b_2 acts as b on $P_0 \cup P_2$, and b_3 acts as b on $P_1 \cup P_2$. The group generated by a and B is the *first Grigorchuk group*, introduced in [Gri80]. Its generators a, b_1, b_2, b_3 are usually denoted a, b, c, d. See Figure 2 for a description of their action on the binary tree, where the boundary is naturally identified with the space $\{0,1\}^\omega$.

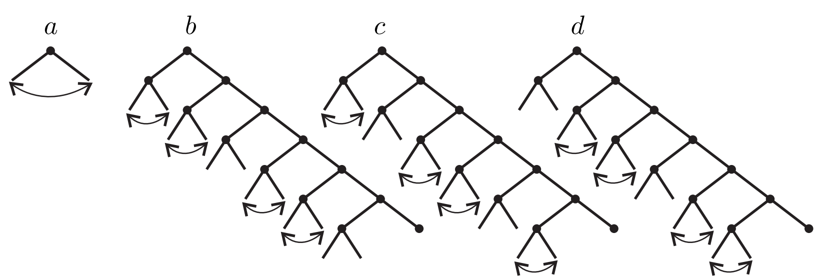

Figure 2. The Grigorchuk group.

Choosing different sets P_0, P_1, P_2 equal to unions of the sets W_k, we get all groups from the family of Grigorchuk groups G_w studied in [Gri85]. If we choose other number of pieces in a partition $\mathcal{P}$, then we get groups defined and studied by Z. Šunić in [Šun07].

3.1. *Orbital graphs of fragmented dihedral groups.*

Definition 3.4. Let a and b be homeomorphisms of period two of a Cantor set $\mathcal{X}$. A *fragmentation of the dihedral group* $\langle a, b\rangle$ is the group generated by $A \cup B$, where A and B are fragmentations of the homeomorphisms a and b, respectively.

Let $G = \langle A \cup B\rangle$ be a fragmentation of a minimal action of a dihedral group $\langle a, b\rangle$. Denote by $\mathcal{P}_A$ and $\mathcal{P}_B$ the sets of pieces of the fragmentations A and B.

For every $\zeta \in \mathcal{X}$ and $a' \in A$, we have $a'(\zeta) = a(\zeta)$ or $a'(\zeta) = \zeta$. For every $\zeta \in \mathcal{X}$, there exists $a' \in A$ such that $a'(\zeta) = a(\zeta)$. The same is true for B and b. It follows that the orbital graphs of G are just "decorated" versions of the orbital graphs of $D_\infty = \langle a, b\rangle$.

Namely, if ζ_1, ζ_2 are two different vertices of an orbital graph of $\langle a, b\rangle$ connected by an edge labeled by a, then ζ_1, ζ_2 belong to one piece $P \in \mathcal{P}_A$. These vertices are connected in the orbital graph of G by edges labeled by all elements $h \in A$ such that $\pi_P(h) = 1$. We will sometimes represent such a multiple edge by $\xrightarrow{P}$. The analogous statement is true for the edges labeled by b. Thus, we replace the labels a and b of the orbital graph of the dihedral group by pieces of the respective fragmentation. Note that all loops of an orbital graph can be reconstructed from the edges that are not loops.

See Figure 3, where an orbital graph of the Grigorchuk group and the corresponding orbital graph of the dihedral group are shown. Note that the edges labeled by b are replaced by multiple edges labeled by $\{b_1, b_2\}$, $\{b_3, b_1\}$, or $\{b_2, b_3\}$ (completed by the necessary loops). These sets of labels correspond to the pieces P_0, P_1, and P_2, respectively; see Example 3.3.

A *segment* Σ is a finite connected subgraph of an orbital graph Γ_ζ such that if v_1, v_2 are adjacent vertices of Σ, then all edges of Γ_ζ connecting v_1 and v_2 belong to Σ. We *do not*, however, include the loops of the endpoints of Σ into the segment, for a technical reason.

We will sometimes arbitrarily choose a direction (left/right) on a graph Γ_ζ. Orientation of subsegments of Γ_ζ will be induced from the orientation of Γ_ζ.

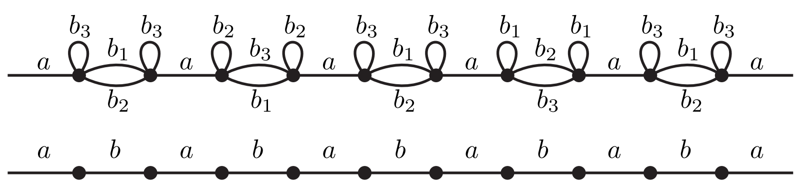

Figure 3. The orbital graphs of the Grigorchuk group and the dihedral group.

If Σ is an oriented segment, then we denote by Σ^{-1} the segment with the opposite orientation. We denote by $|\Sigma|$ the length of Σ, i.e., the number of its vertices minus one.

By Corollary 2.10, every non-trivial stabilizer $\langle a, b\rangle_\zeta$ of a point $\zeta \in \mathcal{X}$ is conjugate to a stabilizer equal to $\langle a\rangle$ or $\langle b\rangle$. It follows that every stabilizer G_ζ of a singular point is conjugate to the stabilizer of a fixed point of a or b.

Lemma 3.5. *Suppose that $\xi \in \mathcal{X}$ is a fixed point of a. The group of germs $G_\xi/G_{(\xi)}$ is naturally isomorphic to the quotient $H_\xi = A/A_{(\xi)}$ of A by the subgroup of elements acting trivially on a neighborhood of ξ. In other words, the epimorphism $G_\xi \longrightarrow G_\xi/G_{(\xi)}$ restricts to an epimorphism $A \longrightarrow G_\xi/G_{(\xi)}$.*

Proof. Consider a germ (g, ξ). We can write g as a product $b_1a_1b_2a_2 \cdots b_na_n$, where $a_i \in A$, $b_i \in B$, and a_i, b_i are all non-trivial except maybe for b_1 or a_n. There are no fixed points of b in the orbit of ξ, and the only fixed point of a in the orbit of ξ is the point ξ itself. It follows that the germs $(b_n, a_n(\xi))$, $(a_{n-1}, b_na_n(\xi))$, $\ldots$, $(b_1, a_1b_2a_2 \cdots b_na_n(\xi))$ are equal either to germs of the identity, or to the germs of the respective elements a or b. (We use the fact that points on the boundary of the pieces of A or B are fixed points of a or b, respectively.) Consequently, the germ (g, ξ) is equal to a germ of the form (ha_n, ξ), where $h \in \langle a, b\rangle$. In particular, the germ of an element of the stabilizer of ξ is equal to the germ of an element of A. This finishes the proof of the lemma. □

For the rest of the section, ξ is a fixed point of a. Consider the graph Ξ with the set of vertices equal to the direct product of H_ξ with the set of vertices of Γ_ξ. Two vertices (h_1, v_1) and (h_2, v_2) of Ξ are connected by an edge labeled by $h \in A \cup B$ if $h_1 = h_2$ and v_1 and v_2 are vertices of Γ_ξ connected by an edge labeled by h, or if $v_1 = v_2 = \xi$ and the image of h under the epimorphism $A \longrightarrow H_\xi$ is equal to h_1h_2. Informally speaking, we take $|H_\xi|$ copies of Γ_ξ and connect their roots ξ by the Cayley graph of H_ξ.

Proposition 3.6. *The graph of germs $\widetilde{\Gamma}_\xi$ is naturally isomorphic to Ξ. The action of the group of deck transformations $G_\xi/G_{(\xi)} \cong H_\xi$ of the covering $\widetilde{\Gamma}_\xi \longrightarrow \Gamma_\xi$ coincides with the natural action of H_ξ on Ξ.*

Proof. We know (see the proof of Lemma 3.5) that every germ (g, ξ) is equal to a germ of the form $(g'h, \xi)$, where $g' \in \{b, ab, bab, abab, \ldots\}$ and $h \in A$. Identify the germ $(g'h, \xi)$ with the vertex $(\tilde{h}, v) \in \Xi$, where $v = g'(\xi) = g(\xi)$ and $\tilde{h}$ is the image of h in H_ξ. It is easy to check that this identification is an isomorphism of graphs. The statement about the action by deck transformations also follows directly from the description of the germs (g, ξ). □

Let $P_1, P_2, \ldots, P_n \in \mathcal{P}_A$ be all pieces of the fragmentation A that accumulate on ξ. Then the maps $\pi_{P_i} : A \longrightarrow \mathbb{Z}/2\mathbb{Z}$ are naturally factored into the composition of epimorphisms $A \longrightarrow H_\xi$ and $H_\xi \longrightarrow \mathbb{Z}/2\mathbb{Z}$. We will denote the latter epimorphism also by π_{P_i}. Then $(\pi_{P_i})_{i=1}^n : H_\xi \longrightarrow (\mathbb{Z}/2\mathbb{Z})^n$ is an isomorphic embedding, since an element $h \in A$ is trivial in H_ξ if and only if $\pi_{P_i}(h) = 0$ for all $i = 1, 2, \ldots, n$.

Denote by Λ_i the quotient of $\widetilde{\Gamma}_\xi$ by the action of $\ker \pi_{P_i}$. It follows from Proposition 3.6 that Λ_i is the graph obtained by taking two copies $\{0\} \times \Gamma_\xi$ and $\{1\} \times \Gamma_\xi$ of Γ_ξ and connecting the endpoints $(0, \xi)$ and $(1, \xi)$ by $\xrightarrow{P_i}$.

Denote by $\lambda_i : \widetilde{\Gamma}_\xi \longrightarrow \Lambda_i$ the natural covering map. In terms of Ξ and Λ_i, it is given by the rule $\lambda_i(h, v) = (\pi_{P_i}(h), v)$.

See Figure 4, where the graphs $\widetilde{\Gamma}_\xi$, Λ_i, and Γ_ξ for the Grigorchuk group are shown. The corresponding covering maps $\lambda_i : \widetilde{\Gamma}_\xi \longrightarrow \Lambda_i$ and $\Lambda_i \longrightarrow \Gamma_\xi$ map a vertex on one graph to the vertex of the graph below on the same vertical line. The graph Λ_i corresponds to the piece P_i on which b_3 acts identically, and b_1, b_2 as the corresponding generator of the dihedral group (in this case b).

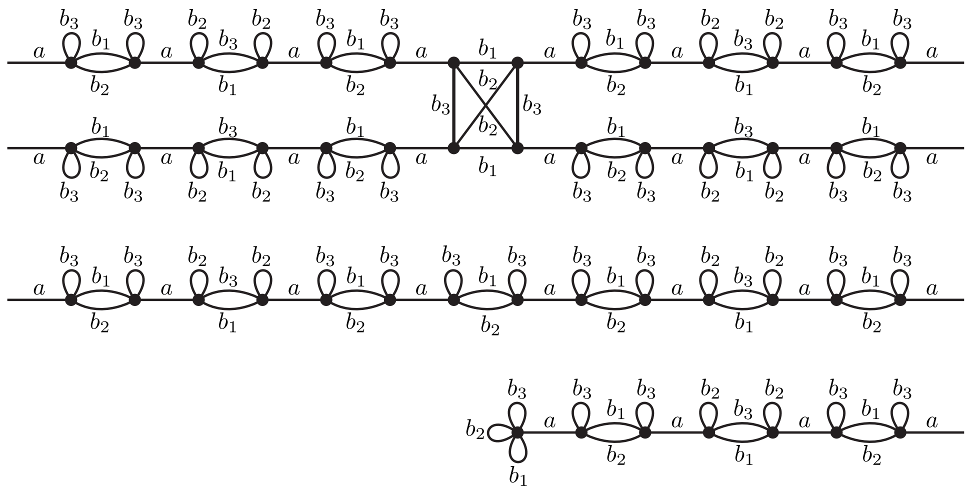

Figure 4. The graphs $\widetilde{\Gamma}_\xi$, Λ_i, and Γ_ξ.

PROPOSITION 3.7. *If $\zeta_n \in P_i$, $n \geq 1$, is a sequence converging to ξ, then the rooted orbital graphs Γ_{ζ_n} converge to Λ_i in the space of rooted labeled graphs.*

For the definition of the space of rooted graphs, see Section 2.1.

Proof. For a given positive integer r, consider the ball $B_{(\varepsilon,\xi)}(r)$ of radius r in the graph of germs $\widetilde{\Gamma}_\xi$. It is given by a set of equalities and inequalities of germs of the form $(g_1, \xi) = (g_2, \xi)$ or $(g_1, \xi) \neq (g_2, \xi)$ for elements $g_1, g_2 \in G$ of length at most r. If $(g_1, \xi) = (g_2, \xi)$, then $g_1(\zeta) = g_2(\zeta)$ for all ζ belonging to a neighborhood of ξ. If $g_1(\xi) \neq g_2(\xi)$, then we also have $g_1(\zeta) \neq g_2(\zeta)$ for all ζ in a neighborhood of ξ. Suppose that $(g_1, \xi) \neq (g_2, \xi)$ but $g_1(\xi) = g_2(\xi)$. Then

$(g_1, \xi) = (gh_1, \xi)$ and $(g_2, \xi) = (gh_2, \xi)$ for some $g \in \langle a, b\rangle$ and $h_1, h_2 \in A$. If $\pi_{P_i}(h_1h_2) = 0$, then $h_1|_{P_i} = h_2|_{P_i}$, hence $g_1(\zeta) = g_2(\zeta)$ for all $\zeta \in N \cap P_i$ for some neighborhood N of ξ. If $\pi_{P_i}(h_1h_2) = 1$, then $g_1(\zeta) \neq g_2(\zeta)$ for all points $\zeta \in N \cap P_i$ for some neighborhood N of ξ, since $h_1h_2|_{P_i} = a|_{P_i}$.

We see that for all points $\zeta \in N \cap P_i$, where N is a sufficiently small neighborhood of ξ, the ball $B_\zeta(r)$ of the orbital graph Γ_ζ is equal to the quotient of the ball $B_{(\varepsilon,\xi)}(r) \subset \widetilde{\Gamma}_\xi$ by the action of the kernel of the projection π_{P_i}. □

Corollary 3.8. *Every segment of an orbital graph of G is isomorphic to a segment of the orbital graph of a regular point. In particular, for every segment Σ of an orbital graph of G, an isomorphic copy of Σ is contained in every orbital graph of G at some bounded distance $R(\Sigma)$ from every vertex of the orbital graph.*

Proof. Take an arbitrary limit $\Lambda_i = \lim_{n\to\infty} \Gamma_{\zeta_n}$ of orbital graphs of regular points, where ζ_n converges to ξ. We have shown that Λ_i is obtained by taking two copies of Γ_ξ and joining the copies of ξ by an edge. (Note that this will change the set of loops at ξ.) It follows that every segment of Γ_ξ is isomorphic to a segment of Λ_i. (Recall that segments do not contain the loops at the endpoints, by the definition of segments.) Consequently, it is isomorphic to a segment of the orbital graph Γ_{ζ_n} for all sufficiently big n. □

Corollary 3.9. *For every oriented segment Σ of an orbital graph of G, there exist isomorphic copies of Σ and Σ^{-1} in every oriented orbital graph.*

Proof. A copy $\phi(\Sigma)$ of the segment Σ is contained in Γ_ξ. It follows that every Λ_i contains the copies $\{0\} \times \phi(\Sigma)$ and $\{1\} \times \phi(\Sigma)$ of Σ. They have opposite orientation and are contained in a segment Σ' of Λ_i. A copy of the segment Σ' is contained in every orbital graph, and inside it we have two copies of Σ in opposite orientations. □

4. **Periodicity**

Theorem 4.1. *Let G be a fragmentation of a minimal dihedral group action on a Cantor set $\mathcal{X}$. If there exists a purely non-Hausdorff singularity $\xi \in \mathcal{X}$, then G is periodic.*

Proof. We may assume that ξ is a fixed point of a. Let $g \in G$. Let m be the length of g as a product of elements of $A \cup B$. Then for every $\zeta \in \mathcal{X}$, the image $g(\zeta)$ belongs to the ball $B_\zeta(m)$ in the orbital graph Γ_ζ and is uniquely determined by the labels of the edges of $B_\zeta(m)$.

Lemma 4.2. *Let Δ and Σ be subsegments of an orbital graph of G such that Δ has length m and Σ contains the $(m+1)$-neighborhood of Δ. Then for every vertex v of Δ, there exists an embedding φ of Σ into an orbital graph of a regular point and an integer $k \geq 1$ such that $g^k(\varphi(v)) \in \varphi(\Delta)$.*

Proof. Suppose that it is not true for some Σ, Δ, $v \in \Delta$, i.e., that for every orbital graph Γ of a regular point and every embedding $\varphi : \Sigma \longrightarrow \Gamma$, the sequence $g^k(\varphi(v))$, $k \geq 1$, does not come back to $\varphi(\Delta)$. Since for every vertex u the distance from u to $g(u)$ is not more than m, the sequence $g^k(\varphi(v))$, $k \geq 1$, always stays in one of the two connected components of $\Gamma \setminus \varphi(\Delta)$. It follows that $g^k(\varphi(v))$ converges to one of the two ends of the graph Γ. This end is on the same side of $\varphi(\Delta)$ as $g(\varphi(v))$.

There exist embeddings $\Sigma \longrightarrow \Gamma_\xi$ in both orientations. In particular, there exists an embedding $\varphi : \Sigma \longrightarrow \Gamma_\xi$ such that $\varphi(g(v))$ is on the same side of $\varphi(\Delta)$ as ξ. Consider the corresponding copy $\varphi_0 : \Sigma \longrightarrow \{\varepsilon\} \times \Gamma_\xi$ of Σ in the ray $\{\varepsilon\} \times \Gamma_\xi$ of the graph of germs $\widetilde{\Gamma}_\xi = \Xi$.

Consider the image $\lambda_i \circ \varphi_0(\Sigma)$ of $\varphi_0(\Sigma)$ in any Λ_i. It belongs to the ray $\{0\} \times \Gamma_\xi$ of Λ_i. Since $\varphi_0(g(v))$ is closer to (ε, ξ) than $\varphi_0(\Delta)$, the sequence $g^k(\lambda_i \circ \varphi_0(v))$ will converge to the infinite end of the ray $\{1\} \times \Gamma_\xi$ of Λ_i.

It follows that the sequence $g^k(\varphi_0(v))$ will converge in $\widetilde{\Gamma}_\xi$ to an end $\{h\} \times \Gamma_\xi$ different from $\{\varepsilon\} \times \Gamma_\xi$.

Since ξ is a purely non-Hausdorff singularity, there exists a projection $\lambda_j : \widetilde{\Gamma}_\xi \longrightarrow \Lambda_j$ such that $\lambda_j(\{h\} \times \Gamma_\xi) = \lambda_j(\{\varepsilon\} \times \Gamma_\xi) = \{0\} \times \Gamma_\xi$. Then the sequence $\lambda_j(g^k(\varphi_0(w)))$ will move from one connected component of $\Lambda_j \setminus \lambda_j(\varphi_0(\Delta)$ to another, which is a contradiction, as Λ_j is a limit of orbital graphs of regular points. See Figure 5, where projections of $\widetilde{\Gamma}_\xi$ onto Λ_i and Λ_j are shown. $\square$

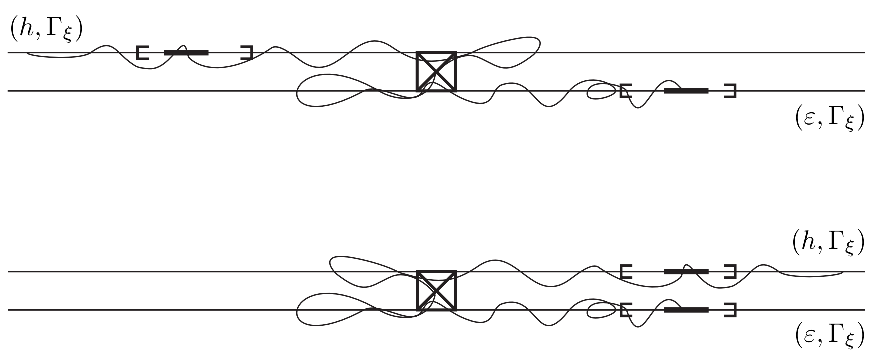

Figure 5. Coming back.

Let Σ and Δ be as in Lemma 4.2, and let $v_0, v_1, \ldots, v_m$ be the list of the vertices of Δ. According to the lemma, there exists a copy of Δ in an orbital graph Γ of a regular point such that $g^{k_0}(v_0) \in \Delta$ for some $k_0 \geq 1$. Let Σ_0 be a sufficiently big segment of Γ containing Δ such that the $(m+1)$-neighborhood of the sequence $g^k(v_0)$ for $k = 0, 1, \ldots, k_0$ belongs to Σ_0. Then $g^{k_0}(v_0) \in \Delta$ in every copy of Σ_0 in every orbital graph.

Now apply Lemma 4.2 for $\Sigma = \Sigma_0$ and for the vertex v_1 of Δ. We will find an orbital graph with a copy of Σ_0 in which both sequences $g^k(v_0)$ and $g^k(v_1)$

eventually return back to Δ. Therefore there exists a segment Σ_1 containing Δ such that $g^k(v_0)$ and $g^k(v_1)$ return to Δ in every orbital graph containing Σ_1. Continuing this way we will find a segment Σ_m such that every vertex of Δ returns inside Σ_m back to Δ under some positive power of g. It follows that the orbit of every vertex of $\Delta \subset \Sigma_m$ is finite and contained in Σ_m.

Let Γ be an orbital graph of a regular point. By Proposition 2.5, there exists $R > 0$ such that for every vertex u of Γ, there exists a copy of Σ_m on both sides of u at distances at most R. Let M be the number of vertices of Σ_m. Then for every vertex u of Γ, either the sequence $g^k(u)$ includes a point of one of the neighboring copies of Δ, or it always stays between them. In the first case the length of the orbit is not more than M, in the second case it is less than $2R + 2M$. It follows that the lengths of all g-orbits of vertices of Γ are uniformly bounded, hence there exists n such that g^n acts trivially on the vertices of Γ. But the set of vertices of Γ is dense in $\mathcal{X}$, so $g^n = \varepsilon$, which finishes the proof of the theorem. $\square$

5. **Amenability and simplicity**

PROPOSITION 5.1. *Let G be a fragmentation of a minimal action of the dihedral group. Then G can be embedded into the topological full group of a minimal action of $\mathbb{Z}$ on a Cantor set, and hence it is amenable.*

Proof. We repeat the argument of [MB15]. Let Γ_ζ be the orbital graph of a regular point $\zeta \in \mathcal{X}$. It is a bi-infinite chain. Choose an arbitrary bijective identification of the edges of the chain with integers such that adjacent edges are identified with integers n, m such that $|n - m| = 1$. Let $w_\zeta = (a_n)_{n\in\mathbb{Z}}$ be the corresponding sequence of elements of $\mathcal{P}_A \cup \mathcal{P}_B$ describing the connections between the adjacent vertices; see 3.1.

Let $\mathcal{W}$ be the set of all sequences w such that every finite subword of w is a subword of w_ζ. The set $\mathcal{W}$ is obviously a closed shift-invariant set. Note that for every finite subword u of w_ζ, there exists $R > 0$ such that for every $i \in \mathbb{Z}$, there exists $j \in \mathbb{Z}$ such that $|i - j| \le R$ and $a_j a_{j+1} \cdots a_{j+|u|-1} = u$; see Corollary 3.9. This in turn implies that the action of the shift on $\mathcal{W}$ is minimal. Denote by $\sigma : \mathcal{W} \longrightarrow \mathcal{W}$ the shift, which is given by $\sigma(w)(n) = w(n + 1)$.

The action of every element $s \in A \cup B$ on a vertex η of Γ_ζ is uniquely determined by the labels of the two edges adjacent to η. This defines a natural action of s on $\mathcal{W}$ given by the rule

$$s(w) = \begin{cases} \sigma(w) & \text{if } \pi_{w(0)}(s) = 1, \\ \sigma^{-1}(w) & \text{if } \pi_{w(-1)}(s) = 1, \\ w & \text{otherwise.} \end{cases}$$

If w describes the orbital graph Γ_ζ, then $s(w)$ represents the orbital graph of $\Gamma_{s(\zeta)}$.

It is easy to see that the action of s on $\mathcal{W}$ is by an element of the full group $\mathsf{F}(\langle\sigma\rangle, \mathcal{W})$, so that we get an isomorphic embedding of G into $\mathsf{F}(\langle\sigma\rangle, \mathcal{W})$. The result of K. Juschenko and N. Monod from [JM13] implies now amenability of G. □

PROPOSITION 5.2. *Suppose that the action of $\langle a, b\rangle$ on $\mathcal{X}$ is expansive. Let G be a fragmentation of the dihedral group. Then the action of G on $\mathcal{X}$ is also expansive, and the group $\mathsf{A}(G, \mathcal{X})$ is simple and finitely generated.*

Note that if $\mathsf{A}(G, \mathcal{X})$ is finitely generated, then it is a subgroup of a fragmentation of the dihedral group with the same groups of germs of points as for G. In particular, if there is a purely non-Hausdorff singularity $\xi \in \mathcal{X}$, then $\mathsf{A}(G, \mathcal{X})$ is periodic by Theorem 4.1 and amenable by Proposition 5.1.

Proof. Let $\delta > 0$ be such that $d(g(\zeta), g(\eta)) < \delta$ for all $g \in \langle a, b\rangle$ implies $\zeta = \eta$. Consider an arbitrary pair P_1, P_2 of pieces of the fragmentation A. There exist $a_i \in A$ such that $a_i|_{P_i} = a|_{P_i}$. Either $a_1|_{P_2} = a|_{P_2}$, or $a_1|_{P_2} = \varepsilon|_{P_2}$. We also have that either $a_2|_{P_1} = a|_{P_1}$, or $a_2|_{P_1} = \varepsilon|_{P_1}$. It follows that for some $a' \in \{a_1, a_2, a_1a_2\}$, we have $a'|_{P_1\cup P_2} = a|_{P_1\cup P_2}$. We then have $a'|_{\overline{P_1}\cup\overline{P_2}} = a|_{\overline{P_1}\cup\overline{P_2}}$.

It follows that for every two points $\zeta, \eta \in \mathcal{X}$, there exists $a' \in A$ such that $a'(\zeta) = a(\zeta)$ and $a'(\eta) = a(\eta)$. Similarly, there exists $b' \in B$ such that $b'(\zeta) = b(\zeta)$ and $b'(\eta) = b(\eta)$. Consequently, for every $g \in \langle a, b\rangle$ there exists $g' \in G$ such that $g'(\zeta) = g(\zeta)$ and $g'(\eta) = g(\eta)$.

Suppose that $d(g(\zeta), g(\eta)) < \delta$ for all $g \in G$. Then, by the above, we have $d(g(\zeta), g(\eta)) < \delta$ for all $g \in \langle a, b\rangle$, which implies, by expansivity of $(\langle a, b\rangle, \mathcal{X})$, that $\zeta = \eta$. Thus, $(G, \mathcal{X})$ is also expansive. Properties of $\mathsf{A}(G, \mathcal{X})$ follow now from Theorem 2.7. □

6. Intermediate growth

6.1. *Inverted orbits.* Let S be a finite symmetric generating set of a group G acting on a set $\mathcal{X}$. Choose a point $\xi \in \mathcal{X}$.

Let $g = g_1g_2\cdots g_n$, $g_i \in S$, be a word over S (i.e., an element of the free monoid S^*). Following [BE12], we define the *inverted orbit* $\mathcal{O}_\xi(g)$ as the set

$$\mathcal{O}_\xi(g) = \{g_1(\xi), g_1g_2(\xi), g_1g_2g_3(\xi), \ldots, g_1g_2\cdots g_n(\xi)\},$$

where the corresponding products of g_i are considered to be elements of G.

Definition 6.1. Let $g = g_1g_2\cdots g_n$ be an element of S^*. We say that a pair (i, j) of indices $1 \le i < j \le n$ is a *first return* of ξ in the word g if $g_{i+1}g_{i+2}\cdots g_j(\xi) = \xi$ and $g_{k+1}\cdots g_j(\xi) \neq \xi$ for all $i < k < j$. The number $j - i$ is called the *length* of the first return.

For example, if $g_i(\xi) = \xi$, then $(i-1, i)$ is a first return of length 1. Note that we do not include the cases $g_1 g_2 \cdots g_j(\xi) = \xi$ as first returns.

LEMMA 6.2. *The number of first returns of* ξ *in* $g = g_1 g_2 \cdots g_n$ *is equal to* $n - |\mathcal{O}_\xi(g)|$.

Proof. Denote

$$\xi_1 = g_1(\xi), \quad \xi_2 = g_1 g_2(\xi), \quad \ldots, \quad \xi_n = g_1 g_2 \cdots g_n(\xi).$$

A pair (i, j) is a first return if and only if $\xi_i = \xi_j$ and $\xi_k \neq \xi_j$ for $i < k < j$. It follows that if $\xi_{i_1} = \xi_{i_2} = \cdots = \xi_{i_m}$ is the list of all instances of a given element of $\mathcal{O}_\xi(g)$, and $i_1 < i_2 < \cdots i_m$, then (i_1, i_2), $(i_2, i_3), \ldots, (i_{m-1}, i_m)$ are first returns, and that every first return appears in this way exactly once. Note that the number of the first returns in this list is equal to $m - 1$. It follows that the total number of first the returns is equal to $n - |\mathcal{O}_\xi(g)|$. □

Denote

$$\nu_\xi(n) = \max_{g = g_1 g_2 \cdots g_n \in S^*} |\mathcal{O}_\xi(g)|.$$

The function $\nu_\xi(n)$ is obviously non-decreasing.

LEMMA 6.3. *For all* $m, n \geq 0$, *we have*

$$\nu_\xi(m+n) \leq \nu_\xi(m) + \nu_\xi(n).$$

Proof. Consider a word $g_1 g_2 \cdots g_{n+m} \in S^*$ of length $n+m$. Then

$$\begin{aligned} \mathcal{O}_\xi(g_1 g_2 \cdots g_{n+m}) &= \{g_1(\xi), g_1 g_2(\xi), \ldots, g_1 g_2 \cdots g_m(\xi)\} \\ &\quad \cup g_1 g_2 \cdots g_m \left(\{g_{m+1}(\xi), g_{m+2} g_{m+1}(\xi), \ldots, g_{m+1} g_{m+2} \cdots g_{m+n}(\xi)\}\right) \\ &= \mathcal{O}_\xi(g_1 g_2 \cdots g_m) \cup g_1 g_2 \cdots g_m(\mathcal{O}_\xi(g_{m+1} g_{m+2} \cdots g_{m+n})). \end{aligned}$$

It follows that

$$(1) \quad |\mathcal{O}_\xi(g_1 \cdots g_{n+m})| \leq |\mathcal{O}_\xi(g_1 \cdots g_m)| + |\mathcal{O}_\xi(g_{m+1} \cdots g_{m+n})| \leq \nu_\xi(n) + \nu_\xi(m)$$

for every word $g_1 g_2 \cdots g_{m+n}$, hence $\nu_\xi(m+n) \leq \nu_\xi(m) + \nu_\xi(n)$. □

We will also need the following general fact.

LEMMA 6.4. *Suppose that a function* $f : \mathbb{N} \longrightarrow \mathbb{N}$ *is non-decreasing and satisfies* $f(n+m) \leq f(n) + f(m)$ *for all* $m, n \in \mathbb{N}$. *Then for all* $n \geq m$, *we have* $\frac{f(n)}{n} \leq \frac{2f(m)}{m}$.

Proof. There exist $q \in [0, n/m] \cap \mathbb{N}$ and $r \in 0, 1, \ldots, m-1$ such that $n = qm + r$. Then

$$f(n) = f(qm + r) \leq q f(m) + f(r),$$

hence

$$\begin{aligned}\frac{f(n)}{n} &\leq \frac{qf(m)+f(r)}{n}\\ &\leq \frac{nf(m)/m+f(m)}{n} = \frac{f(m)}{m} + \frac{f(m)}{n}\\ &= \frac{f(m)}{m}\left(1+\frac{m}{n}\right) \leq \frac{2f(m)}{m}.\end{aligned}$$

□

6.2. *Inverted orbits of linearly repetitive actions.* Let $G = \langle A \cup B\rangle$ be a fragmentation of a minimal action of the dihedral group $\langle a, b\rangle$.

PROPOSITION 6.5. *If the action of G is linearly repetitive and there exists a purely non-Hausdorff singularity $\xi \in \mathcal{X}$, then there exist positive constants C_1, C_2 such that*

$$\nu_\zeta(n) \leq C_1 n e^{-C_2\sqrt{\log n}}$$

for all $\zeta \in \mathcal{X}$ and $n \geq 1$.

Proof. We may assume that the purely non-Hausdorff singularity ξ is a fixed point of one of the generators a, b (see Corollary 2.10). Let it be a. We orient Γ_ξ so that the vertex ξ is on the left, and Γ_ξ is infinite to the right. If Σ is a segment or a ray infinite to the right, then a *left subsegment* of Σ is a subsegment Σ' such that the left end of Σ' coincides with the left end of Σ. In a similar way the notion of a *right subsegment* is defined.

Let $\{P_0, P_1, \ldots, P_{d-1}\}$ be the set of all pieces of the fragmentation A that accumulate on ξ. Let Λ_i be the limit of the orbital graphs Γ_{ζ_n} for regular points ζ_n converging to ξ inside P_i. Then Λ_i is isomorphic to $\Gamma_\xi^{-1} \xrightarrow{P_i} \Gamma_\xi$; see Proposition 3.7. For a natural number n, we denote by P_n the label P_i for $i \equiv n \pmod d$.

Take an arbitrary left subsegment Z_0 of Γ_ξ, and define inductively segments Z_n in the following way. Suppose that we have defined Z_n, and let N be the length of Z_n. Then there exists a copy L_n of $Z_n^{-1} \xrightarrow{P_n} Z_n$ in Γ_ξ such that the right end of L_n is at distance at most KN from ξ for some fixed constant K (coming from the estimate of linear repetitivity of orbital graphs). Define Z_{n+1} to be the smallest segment of Γ_ξ containing ξ and L_n; see Figure 6.

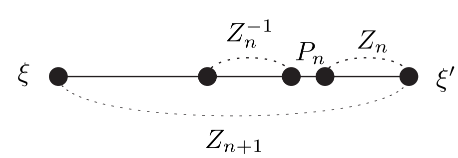

Figure 6. Definition of segments Z_n.

The length of Z_{n+1} is between $2N$ and KN. It follows that the length of Z_n is between $2^n|Z_0|$ and $K^n|Z_0|$.

For every n, there exists a regular point ζ_n such that Z_n is isomorphic to a segment I of Γ_{ζ_n} such that the right end of I is equal to ζ_n. Passing to a convergent subsequence of ζ_n as $n \to \infty$, and using the fact that Z_n is a right subsegment of Z_{n+1}, we will find a point $\xi' \in \mathcal{X}$ such that for every n, there is an isomorphic copy of $Z_n^{-1} \xrightarrow{P_n} Z_n$ in $\Gamma_{\xi'}$ with the right end equal to ξ'.

Take an arbitrary $n > 1$, and let $k_0 > 0$ be such that $|Z_{k_0-1}| \le n < |Z_{k_0}|$. Then $2^{k_0-1}|Z_0| < n \le K^{k_0}|Z_0|$, hence there exists positive constants c_1, c_2 not depending on n such that $c_1 \log n \le k_0 \le c_2 \log n$ for all $n \ge 1$.

Denote by S the generating set $A \cup B$, and let S^* be the free monoid generated by S.

Let $M, n > 1$, and consider an arbitrary word $g_1 g_2 \cdots g_{Mn} \in S^*$. Let r_ξ and $r_{\xi'}$ be the numbers of first returns of length at most n of ξ and ξ', respectively, in the word $g_1 g_2 \cdots g_{Mn}$, and let R_ξ, $R_{\xi'}$ be the respective numbers of first returns of length more than n. By Lemma 6.2, we have

$$|\mathcal{O}_\xi(g_1 g_2 \cdots g_{Mn})| = Mn - r_\xi - R_\xi, \quad |\mathcal{O}_{\xi'}(g_1 g_2 \cdots g_{Mn})| = Mn - r_{\xi'} - R_{\xi'}.$$

We also have, for every $s = 0, 1, \ldots, M-1$, that the number

$$|\mathcal{O}_\xi(g_{sn+1} \cdots g_{(s+1)n})|$$

is equal to n minus the number of first returns of ξ in the word

$$g_{sn+1} g_{sn+2} \cdots g_{(s+1)n}.$$

Every such first return is a first return of ξ in the word $g_1 g_2 \cdots g_{Mn}$ and its length is not more than n. Since the words $g_{sn+1} g_{sn+2} \cdots g_{(s+1)n}$ do not overlap, we get

$$\begin{aligned} Mn - r_\xi \le\ & |\mathcal{O}_\xi(g_1 \cdots g_n)| \\ & + |\mathcal{O}_\xi(g_{n+1} \cdots g_{2n})| + \cdots + |\mathcal{O}_\xi(g_{(M-1)n+1} \cdots g_{Mn})|, \end{aligned}$$

hence

$$\begin{aligned} |\mathcal{O}_\xi(g_1 g_2 \cdots g_{Mn})| \le\ & |\mathcal{O}_\xi(g_1 \cdots g_n)| \\ & + |\mathcal{O}_\xi(g_{n+1} \cdots g_{2n})| + \cdots + |\mathcal{O}_\xi(g_{(M-1)n+1} \cdots g_{Mn})| - R_\xi, \end{aligned}$$

and the same inequality holds for ξ'.

Denote by $\nu(n)$ the maximum of $|\mathcal{O}_\xi(h_1 \cdots h_n)| + |\mathcal{O}_{\xi'}(h_1 \cdots h_n)|$ for all words $h_1 \cdots h_n \in S^*$ of length n. We then have

$$|\mathcal{O}_\xi(g_1 g_2 \cdots g_{Mn})| + |\mathcal{O}_{\xi'}(g_1 g_2 \cdots g_{Mn})| \le M\nu(n) - (R_\xi + R_{\xi'}). \tag{2}$$

Note also that it follows from the inequality (1) in the proof of Lemma 6.3 that the function ν is subadditive.

There exist isomorphic copies of $Z_{k_0+d}^{-1} \xrightarrow{P_{k_0}} Z_{k_0+d}$ in Γ_ξ (resp., in $\Gamma_{\xi'}$) that are contained in the $K_1 n$-neighborhood of ξ (resp., ξ'), where K_1 is a fixed constant.

Note that if $\Gamma_{\xi'}$ is bi-infinite, we can find two copies of $Z_{k_0+d}^{-1} \xrightarrow{P_{k_0}} Z_{k_0+d}$ on both sides of ξ' and both inside the $K_1 n$-neighborhood of ξ'. Let us denote the copy of $Z_{k_0+d}^{-1} \xrightarrow{P_{k_0}} Z_{k_0+d}$ in Γ_ξ by Δ.

The inverted orbit $\mathcal{O}_\xi(g_1 g_2 \cdots g_{Mn})$ contains at least

$$|\mathcal{O}_\xi(g_1 g_2 \cdots g_{Mn})| - K_1 n$$

elements outside the $K_1 n$-neighborhood of ξ.

Suppose that $\zeta = g_1 g_2 \cdots g_t(\xi)$ is one of them. We will show now how ζ "produces" a long first return of either ξ or ξ', and we will use it to prove that there are many long first returns. Consider the path

$$\gamma = (\xi, g_t(\xi), g_{t-1} g_t(\xi), \ldots, g_1 g_2 \cdots g_t(\xi))$$

in Γ_ξ. It starts in ξ and traverses Δ. Let s_1 be the smallest index (i.e., the last moment) such that $g_{s_1} g_{s_1+1} \cdots g_t(\xi)$ is the left end of Δ. Let s_2 be the largest index (i.e., the first moment) such that $s_2 < s_1$ and $g_{s_2} g_{s_2+1} \cdots g_t(\xi)$ is the right end of Δ. Then

$$\gamma_1 = (g_{s_1} \cdots g_t(\xi), g_{s_1-1} g_{s_1} \cdots g_t(\xi), \ldots, g_{s_2} g_{s_2+1} \cdots g_t(\xi))$$

is a path starting in the left end of Δ, ending in the right end of Δ, staying all the time inside Δ, and touching its endpoints only in the first and the last moments.

Recall that the graph of germs $\widetilde{\Gamma}_\xi$ is isomorphic to the graph Ξ with the set of vertices $H_\xi \times \Gamma_\xi$, as it is described in Proposition 3.6.

Consider the covering $\lambda_i : \Xi \longrightarrow \Lambda_i$, where $i \in \{0, 1, \ldots, d-1\}$ is the residue of k_0 modulo d, and let $\widetilde{\Delta}$ be the lift the central part $Z_{k_0+d}^{-1} \xrightarrow{P_{k_0}} Z_{k_0+d} \cong \Delta$ of Λ_i to Ξ. (Recall that Z_{k_0+d} is a left subsegment of Γ_ξ.) Let $\widetilde{\gamma}$ be the lift of γ_1 to $\widetilde{\Delta}$ starting in the branch $\{\varepsilon\} \times \Gamma_\xi$ of Ξ.

The end of $\widetilde{\gamma}$ belongs to a branch $\{h\} \times \Gamma_\xi$ of Ξ for some $h \in H_\xi \setminus \{\varepsilon\}$. There exists $i' \in \{0, 1, \ldots, d-1\}$ such that $\pi_{P_{i'}}(h) = 0$, since ξ is a purely non-Hausdorff singularity. Let $k' \in \{k_0+1, k_0+2, \ldots, k_0+d\}$ be such that $k' \equiv i' \pmod{d}$.

Denote by Δ' the central part of Δ isomorphic to $Z_{k'}^{-1} \xrightarrow{P_{k_0}} Z_{k'}$. (It exists, since $Z_{k'}$ is a left subsegment of Z_{k_0+d}.) Denote the full preimage of Δ' in Ξ by $\widetilde{\Delta}'$.

The path $\widetilde{\gamma}$ must enter and exit $\widetilde{\Delta}'$. It enters $\widetilde{\Delta}'$ in the branch $\{\varepsilon\} \times \Gamma_\xi$. Consider the segment $\widetilde{\gamma}'$ of $\widetilde{\gamma}$ from the last entering of $\widetilde{\Delta}'$ in the branch $\{\varepsilon\} \times \Gamma_\xi$ to the first touching the exit from $\widetilde{\Delta}'$ after that. The exit must be in a different branch, since otherwise $\widetilde{\gamma}$ must touch the entrance of $\widetilde{\Delta}'$ one more time (again inside the branch $\{\varepsilon\} \times \Gamma_\xi$). The path $\widetilde{\gamma}'$ always stays inside $\widetilde{\Delta}'$ and touches the entrance and the exit of $\widetilde{\Delta}'$ precisely once each. The image of $\widetilde{\gamma}'$ in Δ is of the form

$$(g_{l_1} \cdots g_t(\xi), \quad g_{l_1-1} g_{l_1} \cdots g_t(\xi), \quad \ldots, \quad g_{l_2} \cdots g_t(\xi)),$$

for some $s_2 \leq l_2 < l_1 \leq s_1$, where the only point in the path equal to the left end of Δ' is $g_{l_1} \cdots g_t(\xi_i)$ and the only point equal to the right end of Δ' is $g_{l_2} \cdots g_t(\xi_i)$.

We have the covering map $\widetilde{\Delta}' \longrightarrow (Z_{k'}^{-1} \xrightarrow{P_{k'}} Z_{k'})$ equal to the restriction of $\lambda_{i'} : \Xi \longrightarrow \Lambda_{i'}$. Note that the segment of $\Gamma_{\xi'}$ of the form $Z_{k'}^{-1} \xrightarrow{P_{k'}} Z_{k'}$ with the right end equal to ξ' is isomorphic as a labeled graph to the corresponding central part of $\Lambda_{i'}$. Let ψ be this isomorphism (from the central part of $\Lambda_{i'}$ to the segment of $\Gamma_{\xi'}$). There are two such isomorphisms, and we choose the one mapping the segment $\{0\} \times Z_{k'}$ of $\Lambda_{i'}$ to the right half (the one containing ξ') of the subsegment $Z_{k'}^{-1} \xrightarrow{P_{k'}} Z_{k'}$ of $\Gamma_{\xi'}$.

If the path $\widetilde{\gamma}'$ exits $\widetilde{\Delta}'$ for the first time in the branch $\{h\} \times \Gamma_\xi$, then its image under $\psi \circ \lambda_{i'}$ is a path starting in ξ', touching a vertex of the middle edge $P_{k'}$, and then coming back to ξ' always staying inside $Z_{k'}^{-1} \xrightarrow{P_{k'}} Z_{k'}$. The path $\psi \circ \lambda_{i'}(\widetilde{\gamma}')$ is equal to

$$(\xi', \quad g_{l_1-1}(\xi'), \quad \ldots, \quad g_{l_2} \cdots g_{l_1-1}(\xi') = \xi'),$$

where ξ' is equal only to the first and to the last vertex in the path; see Figure 7.

We get a first return $(l_2 - 1, l_1 - 1)$ of length at least $2|Z_{k'}| > n$. We know that $g_{l_2} \cdots g_t(\xi)$ is equal to the right end of Δ', hence the right end of Δ' and the value of l_2 uniquely determines ζ. We see that one such return is produced by at most d points of the inverted orbit $\mathcal{O}(g_1 g_2 \cdots g_{Mn})$.

If the path $\widetilde{\gamma}'$ exits $\widetilde{\Delta}'$ for the first time in a branch $\{h'\} \times \Gamma_\xi$ labeled by $h' \neq h$, then it has to traverse $\widetilde{\Delta}'$ at least one more time. It follows that $\widetilde{\gamma}$ has a subpath $\widetilde{\gamma}'$ starting at $(h', \xi) \in \Xi$, reaching a preimage of the right end of Δ', and some time after that coming back to (h', ξ). Take the shortest subpath of this form. Its length is at least $2|Z_{k'}| > n$. Then the image of this subpath under λ_i in Γ_ξ produces a first return $(l_2 - 1, l_1 - 1)$ of ξ of length at least n; see Figure 8. In the same way as in the first case, the image under $g_1 g_2 \cdots g_{l_2-1}$ of one of the endpoints of the central edge of $\Delta' = (Z_{k'}^{-1} \xrightarrow{P_{k_0}} Z_{k'})$ is equal to ζ. It follows that at most two points of $\mathcal{O}_{\xi_i}(g_1 g_2 \cdots g_{Mn})$ can produce the same first return this way.

We see that each $\zeta \in \mathcal{O}_\xi(g_1 g_2 \cdots g_{Mn})$ outside the $K_1 n$-neighborhood of ξ produces either a long first return of ξ' or a long first return of ξ, and each such return is produced by at most d points ζ.

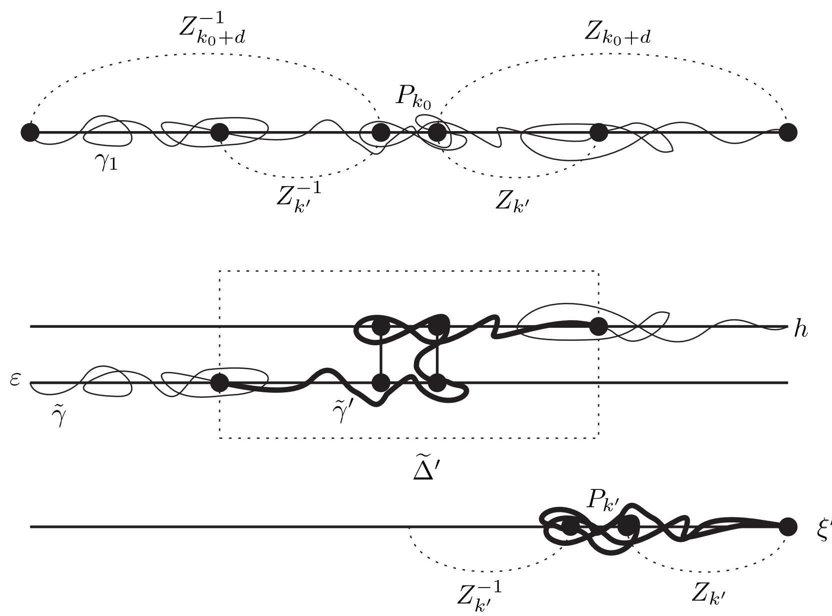

Figure 7. Finding a first return of ξ'.

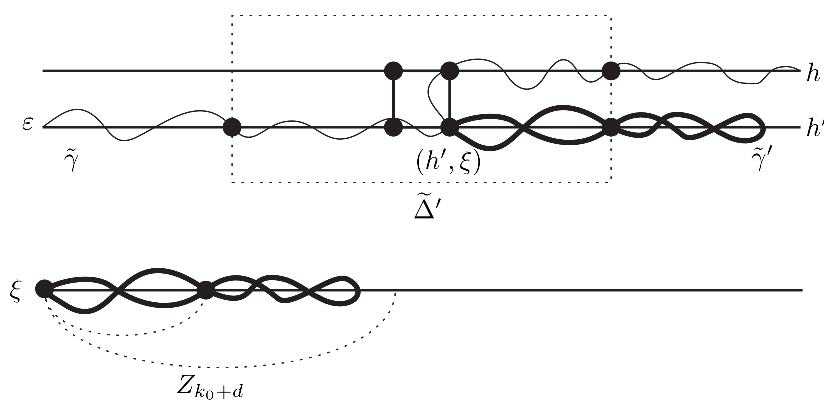

Figure 8. Finding a first return of ξ.

The same argument shows that each point of $\mathcal{O}_{\xi'}(g_1 g_2 \cdots g_{Mn})$ outside of the $K_1 n$ neighborhood of ξ' produces a long first return of ξ' or ξ, and each such first return is produced by at most $2d$ points of the inverted orbit. (We have to multiply by 2, since we have to consider both sides of ξ'.)

It follows that the number $R_\xi + R_{\xi'}$ of long first returns satisfies

$$R_\xi + R_{\xi'} \geq \frac{1}{3d}\Big(|\mathcal{O}_\xi(g_1 g_2 \cdots g_{Mn})| - K_1 n + |\mathcal{O}_{\xi'}(g_1 g_2 \cdots g_{Mn})| - 2K_1 n\Big).$$

Consequently, by (2),

$$\begin{aligned}|\mathcal{O}_\xi(g_1g_2\cdots g_{Mn})| + |\mathcal{O}_{\xi'}(g_1g_2\cdots g_{Mn})| &\le M\nu(n) - (R_\xi + R_{\xi'})\\ &\le M\nu(n) - \frac{1}{3d}\Big(|\mathcal{O}_\xi(g_1g_2\cdots g_{Mn})| + |\mathcal{O}_{\xi'}(g_1g_2\cdots g_{Mn})| - 3K_1 n\Big),\end{aligned}$$

hence

$$\frac{3d+1}{3d}\Big(|\mathcal{O}_\xi(g_1g_2\cdots g_{Mn})| + |\mathcal{O}_{\xi'}(g_1g_2\cdots g_{Mn})|\Big) \le M\nu(n) + \frac{K_1}{d}n.$$

Since $g_1g_2\cdots g_{Mn}$ was arbitrary, we have

$$\frac{3d+1}{3d}\nu(Mn) \le M\nu(n) + \frac{K_1}{d}n.$$

Let us denote $\delta(n) = \frac{\nu(n)}{n}$. Multiplying the last inequality by $\frac{3d}{(3d+1)Mn}$, we get

$$\delta(Mn) \le \frac{3d}{3d+1}\delta(n) + \frac{K_2}{M}$$

for $K_2 = 3K_1/(3d+1)$.

Let $M = \left\lceil \frac{K_3}{\delta(n)} \right\rceil$ for $K_3 = 2(3d+1)K_2$. Then

$$\delta(Mn) \le \frac{3d}{3d+1}\delta(n) + \frac{1}{2(3d+1)}\delta(n) = \frac{6d+1}{6d+2}\delta(n).$$

Denote $\rho = \frac{6d+1}{6d+2}$. It is only important that $0 < \rho < 1$.

Fix n_0, and define inductively a sequence n_k by the rule

$$n_{k+1} = \left\lceil K_3\delta(n_k)^{-1} \right\rceil n_k.$$

Then $\delta(n_{k+1}) \le \rho\delta(n_k)$ for every k. Choosing a bigger K_3 in advance, if necessary, we may assume that n_k is strictly increasing.

We may assume that $K_3 > 1$, and then

$$n_{k+1} = \left\lceil K_3\delta(n_k)^{-1} \right\rceil n_k \le K_4\delta(n_k)^{-1}\cdot n_k,$$

for $K_4 = K_3 + 1$, since $\delta(n) \le 1$ for all n. Then

$$\begin{aligned} n_k &\le n_0 K_4^k \delta(n_0)^{-1}\delta(n_1)^{-1}\cdots\delta(n_{k-1})^{-1}\\ &\le K_4^k\rho^k\delta(n_k)^{-1}\rho^{k-1}\delta(n_k)^{-1}\cdots\rho\delta(n_k)^{-1} = K_4^k\rho^{k(k+1)/2}\delta(n_k)^{-k}.\end{aligned}$$

Raising the inequality to the power $1/k$, we get

$$n_k^{1/k} \le K_4\rho^{\frac{k+1}{2}}\delta(n_k)^{-1},$$

hence

$$\delta(n_k) \le K_4\rho^{\frac{k+1}{2}} n_k^{-1/k} \le K_5\rho_1^k n_k^{-1/k}$$

for $\rho_1 = \sqrt{\rho}$ and $K_5 = K_4\rho_1$.

Take an arbitrary n. Let k be such that $n_k \le n < n_{k+1}$. We have

$$\begin{aligned}\delta(n_k) \le K_5\rho_1^k n_k^{-1/k} \le K_5 n_k^{-1/k} &\le K_5\left(K_4^{-1}\delta(n_k)n_{k+1}\right)^{-1/k}\\ &\le K_5K_4\left(\delta(n_k)n_{k+1}\right)^{-1/k},\end{aligned}$$

hence

$$\delta(n_k)^{1+1/k} \le K_6 n_{k+1}^{-1/k} < K_6 n^{-1/k}$$

for $K_6 = K_4K_5$, so

$$\delta(n_k) \le (K_6 n^{-1/k})^{\frac{k}{k+1}} \le K_6 n^{-1/(k+1)}.$$

Therefore, using Lemma 6.4, we get

$$\delta(n) \le 2\delta(n_k) \le 2K_6 n^{-1/(k+1)}$$

and

$$\delta(n) \le 2\delta(n_k) \le 2\delta(n_0)\rho^k.$$

Suppose that $k \le \sqrt{\log n}$. Then

$$\delta(n) \le 2K_6 n^{-1/(k+1)} \le 2K_6 n^{-1/(\sqrt{\log n}+1)} = 2K_6 e^{-\frac{\log n}{1+\sqrt{\log n}}} \le K_7 e^{-K_8\sqrt{\log n}}$$

for some positive constants K_7 and K_8.

Suppose that $k > \sqrt{\log n}$. Then

$$\delta(n) \le 2\delta(n_0)\rho^k \le 2\delta(n_0)\rho^{\sqrt{\log n}} = 2\delta(n_0)e^{\log\rho\sqrt{\log n}}.$$

We see that in both cases we have

$$\delta(n) \le C_1 e^{-C_2\sqrt{\log n}}$$

for $C_1 = \max\{K_7, 2\delta(n_0)\}$ and $C_2 = \min\{K_8, -\log\rho\}$.

We have $\nu_\xi(n) \le \nu(n)$, hence

$$\frac{\nu_\xi(n)}{n} \le \delta(n) \le C_1 e^{-C_2\sqrt{\log n}}$$

for all n.

Now let $\zeta_0 \in \mathcal{X}$ be arbitrary. Let $n \ge 1$ be a natural number, and let k be such that $|Z_{k-1}| \le n < |Z_k|$. Then for every k, the vertex ζ_0 of Γ_{ζ_0} is contained in a segment Σ isomorphic to a segment of the form $Z_k^{-1}IZ_k$, where $|I| \le K_1 n$ for some fixed K_1. (Recall that $|Z_k|/|Z_{k-1}|$ is bounded.)

We may assume that the distance from ζ_0 to the copies of Z_k and Z_k^{-1} in Σ is more than n. Consider a word $g = g_1g_2\cdots g_{Mn}$ for some $M > 1$. Split g into subwords $h_1 = g_1g_2\cdots g_n$, $h_2 = g_{n+1}g_{n+2}\cdots g_{2n}$, $\ldots, h_M = g_{(M-1)n+1}g_{(M-1)n+2}\cdots g_{Mn}$.

Suppose that $\zeta_1 \in \mathcal{O}_{\zeta_0}(g_1g_2\cdots g_{Mn}) \setminus \Sigma$. Then for some t, we have $\zeta_1 = g_1g_2\cdots g_t(\zeta_0)$. Without loss of generality, let us assume that ζ_1 is to the right of ζ_0.

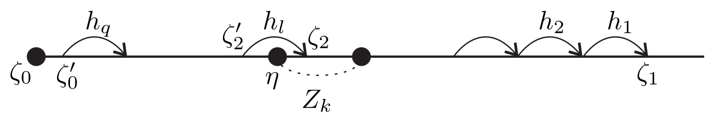

Figure 9. A uniform estimate on $\nu(n)$.

Represent $t = qn + r$, where $r \in \{0, 1, \dots, n-1\}$. Then

$$\zeta_1 = h_1 h_2 \cdots h_q g_{qn+1} \cdots g_{qn+r}(\zeta_0).$$

Denote $\zeta_0' = g_{qn+1} \cdots g_{qn+r}(\zeta_0)$. We have the sequence

$$(\zeta_0, \quad \zeta_0', \quad h_q(\zeta_0'), \quad h_{q-1}h_q(\zeta_0'), \quad \dots, \quad h_1 h_2 \cdots h_q(\zeta_0') = \zeta_1)$$

such that the distance between consecutive terms in the sequence is not more than n. It follows that one of the elements of the sequence belongs to the right subsegment Z_k of Σ; see Figure 9. Let

$$\zeta_2 = h_l h_{l+1} \cdots h_q(\zeta_0') = g_{(l-1)n+1} g_{(l-1)n+2} \cdots g_t(\zeta_0)$$

be the first such point. Then $\zeta_2' = h_{l+1} \cdots h_q(\zeta_0')$ is to the left of Z_k. Consider the path

$$(\zeta_2', \quad g_{ln}(\zeta_2'), \quad g_{ln-1}g_{ln}(\zeta_2'), \quad \dots, \quad g_{(l+1)n+1} \cdots g_{ln}(\zeta_2') = \zeta_2)$$

It passes through the left end η of the subsegment Z_k of Σ. Let

$$g_s g_{s+1} \cdots g_{(l+1)n}(\zeta_2')$$

be the last entry of the sequence equal to η. Note that $(l-1)n + 1 \le s \le ln$.

Then the path

$$(\xi, \quad g_{s-1}(\xi), \quad \dots, \quad g_{(l-1)n+1} g_{(l-1)n+2} \cdots g_{s-1}(\xi))$$

stays inside Z_k. It follows that if we map the copy of $Z_k \subset \Sigma$ to the original place of Z_k (the left end of Γ_ξ), then ζ_2 will be moved to a point belonging to $\mathcal{O}_\xi(g_{(l-1)n+1} g_{(l-1)n+2} \cdots g_{ln}) = \mathcal{O}_\xi(h_l)$. It follows that for every value of l, there are not more than $\nu_\xi(n)$ possible values of ζ_2.

We have $\zeta_1 = h_1 h_2 \cdots h_{l-1}(\zeta_2)$. Consequently, for each l, there are not more than $\nu_\xi(n)$ possible values of ζ_1. It follows that the total number of possible values of ζ_1 is not more than $M\nu_\xi(n)$.

We proved that

$$\nu_{\zeta_0}(Mn) \le (K_1 n + 2|Z_k|) + 2M\nu_\xi(n) \le K_2 n + 2M\nu_\xi(n)$$

for arbitrary n and M, where $K_2 > 0$ is fixed. Take $n = M$. Then

$$\nu_{\zeta_0}(n^2) \le K_2 n + 2C_1 n^2 e^{-C_2\sqrt{\log n}} \le C_1' n^2 e^{-C_2'\sqrt{\log n^2}}$$

for some $C_1', C_2' > 0$.

For every $n \ge 1$, there exists k such that $k^2/4 \le n \le k^2$. Then

$$\nu_{\zeta_0}(n) \le \nu_{\zeta_0}(k^2) \le C_1' k^2 e^{-C_2'\sqrt{\log(k^2)}} \le C_1'' n e^{-C_2''\sqrt{\log n}}$$

for some positive constants C_1'' and C_2'', which finishes the proof. $\square$

6.3. *Intermediate growth.*

THEOREM 6.6. *Let G be a fragmentation of a minimal action of a dihedral group on a Cantor set. Suppose that there exists a purely non-Hausdorff singularity $\xi \in \mathcal{X}$ and that the orbital graphs of the action of G on orbits of generic points are linearly repetitive. Then the growth of G is intermediate. It is eventually larger than any polynomial function and is bounded from above by* $\exp\left(C_1 n e^{-C_2\sqrt{\log n}}\right)$ *for some positive constants C_1 and C_2.*

Proof. Choose a point $\xi_0 \in \mathcal{X}$. Denote by I_k the segment of Γ_{ξ_0} of length k such that ξ_0 is its left end. Let K be a big number that we will choose later.

Choose α such that $1 < \alpha < \frac{K+1}{K}$. Define $\Sigma_k = I_{\lfloor \alpha^k \rfloor}$ for $k \geq 1$, and denote by ζ_k the right end of Σ_k. Then

$$\frac{\alpha^{k+1}-1}{\alpha^k} = \alpha - \alpha^{-k} < \frac{|\Sigma_{k+1}|}{|\Sigma_k|} < \frac{\alpha^{k+1}}{\alpha^k - 1} = \frac{\alpha}{1-\alpha^{-k}}.$$

Let $g = g_1 g_2 \cdots g_n \in S^*$ be a word of length n in generators $S = A \cup B$. For an arbitrary regular point $\zeta \in \mathcal{X}$, consider the set

$$W_\zeta = \{\zeta, g_n(\zeta), g_{n-1}g_n(\zeta), \ldots, g_1 g_2 \cdots g_n(\zeta)\}.$$

It is a segment of Γ_ζ since it is the range of a path in Γ_ζ. Denote by l_ζ its length. Let W'_ζ be the subsegment of Γ_ζ consisting of W_ζ and the two adjacent edges. There exists an isomorphic copy $\phi_\zeta(W'_\zeta)$ of W'_ζ in the right half of Γ_{ξ_0} such that its left end is at distance at most Kl_ζ from ξ_0 (if K is big enough).

Let k be the smallest positive integer such that $\lfloor \alpha^k \rfloor$ is larger than the distance from ξ_0 to the left end of $\phi_\zeta(W_\zeta)$. Then ζ_k is to the right of the left end of $\phi_\zeta(W_\zeta)$. Suppose that ζ_k is to the right of the right end of $\phi_\zeta(W_\zeta)$. Let m be the distance from ξ_0 to the left end of $\phi_\zeta(W_\zeta)$. Then $\lfloor \alpha^k \rfloor \geq m + l$, $\lfloor \alpha^{k-1} \rfloor \leq m$, and $m \leq Kl$. It follows that

$$\frac{\alpha}{1-\alpha^{-k+1}} > \frac{\lfloor \alpha^k \rfloor}{\lfloor \alpha^{k-1} \rfloor} \geq \frac{m+l}{m} = 1 + \frac{l}{m} \geq 1 + \frac{1}{K} = \frac{K+1}{K},$$

which is a contradiction for all k bigger than some fixed k_0.

It follows that if k is big enough, then the right end ζ_k of Σ_k belongs to $\phi_\zeta(W_\zeta)$. It follows that there exists s such that $g_s \cdots g_n(\phi_\zeta(\zeta)) = \zeta_k$, or

$$\phi_\zeta(\zeta) = g_n g_{n-1} \cdots g_s(\zeta_k).$$

It follows that $\phi_\zeta(\zeta) \in \mathcal{O}_{\zeta_k}(g_n g_{n-1} \cdots g_1)$.

Consider the set $\mathcal{L}_g$ of all triples $(\phi_\zeta(W'_\zeta), \phi_\zeta(\zeta), \phi_\zeta(g(\zeta)))$ for all regular $\zeta \in \mathcal{X}$. If we know $\mathcal{L}_g$, then we know g, since for every $\zeta \in \mathcal{X}$, an isomorphic copy $\phi_\zeta(W'_\zeta)$ of W'_ζ will appear as the first component of an element of $\mathcal{L}_g$, and then the second and the third components $\phi_\zeta(\zeta)$ and $\phi_\zeta(g(\zeta))$ of that element will determine $g(\zeta)$.

Note that since the length of W_ζ is not more than n, the left end of $\phi(W_\zeta)$ is at the distance at most Kn from ξ_0, hence k is bounded above by $\log(Kn)/\log\alpha \asymp \log n$.

It follows that the cardinality of $\mathcal{L}_g$ is not greater than $C_1 n \log n e^{-C_2\sqrt{\log n}}$ for some positive constants C_1 and C_2, by Proposition 6.5.

Let us estimate now the number of possible sets $\mathcal{L}_g$ for all words $g \in S^*$ of length n. Each element of $\mathcal{L}_g$ consists of a point of $I_{(K+1)n}$, a segment of length at most n containing this point, and a point in this segment. It follows that the number of possibilities for each element of $\mathcal{L}_g$ is bounded above by Cn^4 for some constant C. Consequently, the number of possible sets $\mathcal{L}_g$ is less than

$$\begin{aligned}
&(Cn^4)^{C_1 n \log n e^{-C_2\sqrt{\log n}}}\\
&\quad = \exp\left(C_1 n \log n e^{-C_2\sqrt{\log n}}(4\log n + \log C)\right)\\
&\quad \le \exp\left(C_1' n (\log n)^2 e^{-C_2\sqrt{\log n}}\right) = \exp\left(C_1' n (e^{2\log\log n - C_2\sqrt{\log n}})\right)\\
&\quad \le \exp\left(C_1' n e^{-C_2'\sqrt{\log n}}\right)
\end{aligned}$$

for some $C_1', C_2' > 0$ and all n big enough, since $\frac{\log\log n}{\sqrt{\log n}} \to 0$ as $n \to \infty$. As the element g is uniquely determined by $\mathcal{L}_g$, this gives the necessary subexponential estimate of the growth of G. The group G cannot be of polynomial growth, since it is finitely generated, infinite, and periodic, which excludes the possibility of a polynomial growth, by M. Gromov's Theorem [Gro81]. $\square$

7. **Examples**

7.1. *Substitutional systems.* Let X be a finite alphabet. Let $\tau : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ be an endomorphism of the free monoid X^*. It is uniquely determined by the restriction $\tau : \mathsf{X} \longrightarrow \mathsf{X}^*$, which is usually called a *substitution.* The associated subshift $\mathcal{X}_\tau \subset \mathsf{X}^{\mathbb{Z}}$ is the set of all bi-infinite sequences w such that for every finite subword v of w, there exists $n \ge 0$ and $x \in \mathsf{X}$ such that v is a subword of $\tau^n(x)$. It is non-empty if and only if there exists $x \in \mathsf{X}$ such that the length of $\tau^n(x)$ goes to infinity as $n \to \infty$.

D. Damanik and D. Lenz in [DL06] proved that a substitutional shift is linearly repetitive if and only if it is minimal and gave a criterion of minimality in terms of the substitution.

Let us illustrate how substitutional dynamical systems can be used to construct periodic simple groups of intermediate growth on the example of the Thue-Morse substitution.

Consider the action of the dihedral group from Example 2.13. It acts on the shift $\mathcal{S}$ generated by the substitution

$$\tau(0) = 01, \quad \tau(1) = 10.$$

Let us introduce new symbols t, B, C, D and modify the substitution τ^2:

$$\tau'(0) = 0t1D1t0, \quad \tau'(1) = 1t0D0t1,$$
$$\tau'(D) = C, \quad \tau'(C) = B, \quad \tau'(B) = D, \quad \tau'(t) = t.$$

Let $\mathcal{S}'$ be the set of sequences in the shift generated by τ' that have the letters $1, 2$ on the even positions, and letters B, C, D, t on the odd positions. We have a natural map $\kappa : \mathcal{S}' \longrightarrow \mathcal{S}$ erasing the letters B, C, D. One can show that for every $w \in \mathcal{S}$, the set $\kappa^{-1}(w)$ consists of a single element, except for w equal to a shift of one of the two infinite palindromes

$$\cdots 10010110\,.\,01101001 \cdots$$

and

$$\cdots 01101001\,.\,10010110 \cdots,$$

when $\kappa^{-1}(w)$ has three elements that differ from each other only by the central letter B, C, or D.

Let a be the transformation of $\mathcal{S}'$ flipping a sequence around the letter on the zeroth position, and let b, c, d, and t, respectively, be the transformations flipping a sequence around the letter on the first position if it is C or D, B or D, B or C, and t, respectively, and acting trivially otherwise. Then the action of a, b, c, d, t on $\mathcal{S}'$ lifts by κ to an action on $\mathcal{S}$ in a unique way. The sequences from $\mathcal{S}'$ are naturally interpreted as the orbital graphs of regular points, and the limits Λ_i of regular orbital graphs in the case of a singular point of the action of $G = \langle a, b, c, d, t\rangle$ on $\mathcal{S}$. The group $\mathsf{A}(G, \mathcal{S})$ is a finitely generated simple periodic group of intermediate growth.

7.2. *Groups of polygon rearrangements.* A nice class of examples illustrating Theorem 4.1 was suggested to the author by Yves de Cornulier. Consider the torus $\mathbb{R}^2/\mathbb{Z}^2$ and two central symmetries $a : x \mapsto -x+v$ and $b : x \mapsto -x$ for some $v \in \mathbb{R}^2/\mathbb{Z}^2$. Suppose that v is represented by $(x, y) \in \mathbb{R}^2$, such that $1, x, y$ are linearly independent over $\mathbb{Q}$. Then, by the classical Kronecker's theorem [Kro84], the action of $\mathbb{Z}$ generated by the composition $x \mapsto x + v$ of the two symmetries is minimal on the torus.

Let us split the torus into three b-invariant parts P_1, P_2, P_3 (e.g., each equal to a union of some polygons) such that the fixed point 0 of b belongs to the boundary of each of the parts. Consider then the transformations b_1, b_2, b_3 (defined up to a set of measure zero) of the torus acting trivially on P_1, P_2, P_3, respectively, and acting as b on their complements. We may also cut the torus open and represent it as a polygon, so that then a, b_1, b_2, b_3 act on

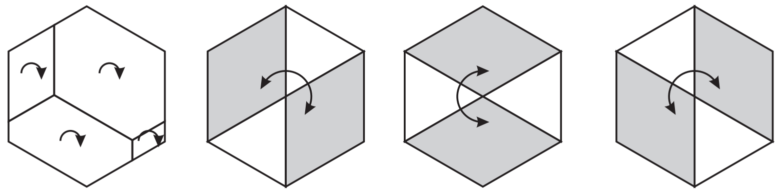

Figure 10. A periodic group acting on a hexagon.

the polygon by piecewise isometries. We can lift this action to an action by homeomorphisms of the Cantor set satisfying the conditions of Theorem 4.1: one has to double all points lying on the sides of the polygons (except for 0, which has to remain common to all three pieces P_i) and then propagate this doubling by the action of the group.

For example, we can consider the group generated by piecewise isometries of the regular hexagon, shown on Figure 10. The first transformation a rotates each of the four shown polygons by 180 degrees. The remaining three transformations rotate the shaded areas by 180 degrees around the center of the hexagon and fix the white areas. Theorem 4.1 implies that this group is periodic, provided the first generator is sufficiently generic (i.e., such that its composition with the 180 degree rotation around the center of the hexagon acts minimally on the torus).

8. **Fragmenting the golden mean dihedral group**

8.1. *The construction.* Let us describe an explicit example of a finitely generated simple periodic group of intermediate growth.

Denote by φ the golden mean $\frac{1+\sqrt{5}}{2}$. Let T_1 and T_2 be the transformations

$$T_1(x) = \varphi^{-1}x, \qquad T_2(x) = 1 - \varphi^{-2}x$$

of $[0,1]$. The ranges of T_1 and T_2 are the intervals $[0, \varphi - 1]$ and $[\varphi - 1, 1]$, respectively. They do not overlap and cover the circle $\mathbb{R}/\mathbb{Z}$.

For every infinite sequence $w = x_1x_2\cdots \in \mathsf{X}^\omega$ over the alphabet $\mathsf{X} = \{1, 2\}$, the intersection of the ranges of $T_{x_1} \circ T_{x_2} \circ \cdots \circ T_{x_n}$ is a single point.

Denote by a and b the transformations of the circle $\mathbb{R}/\mathbb{Z}$ given by

$$a(x) = \varphi - x, \qquad b(x) = 1 - x.$$

Then ba is the rotation $x \mapsto x + \varphi$ of the circle. We get a minimal action of the dihedral group $\langle a, b\rangle$ on the circle.

Direct computations show that

$$\begin{aligned} a \circ T_1(x) &= T_1 \circ b(x),\\ a \circ T_2(x) &= T_2 \circ b(x) \end{aligned}$$

and

$$\begin{aligned} b \circ T_1 \circ T_1(x) &= T_2(x), \\ b \circ T_2(x) &= T_1 \circ T_1(x), \\ b \circ T_1 \circ T_2(x) &= T_1 \circ T_2 \circ b(x) \end{aligned}$$

for all $x \in [0, 1]$, where a acts on $[0, \varphi - 1]$ by $x \mapsto \varphi - x - 1$ and on $[\varphi - 1, 1]$ by $x \mapsto \varphi - x$.

We get the following associated action on the sequences $x_1 x_2 \cdots \in \mathsf{X}^\omega$:

$$\begin{gathered} a(1w) = 1b(w), \qquad a(2w) = 2b(w), \\ b(11w) = 2w, \quad b(2w) = 11w, \quad b(12w) = 12b(w). \end{gathered}$$

More formally, we have the natural map $\kappa : \mathsf{X}^\omega \longrightarrow \mathbb{R}/\mathbb{Z}$ mapping a sequence $x_1 x_2 \cdots$ to the unique intersection point of the ranges of $T_{x_1} \circ T_{x_2} \circ \cdots \circ T_{x_n}$. Then the map κ is a semiconjugacy of the transformations a, b acting on X^ω with the transformations a, b acting on $\mathbb{R}/\mathbb{Z}$.

As usual, we will identify X^ω with the boundary of the rooted tree X^*. However, it is more natural to change the metric on the tree in the following way. The *weight* of the letter 1 is equal to 1, the weight of the letter 2 is equal to 2. The weight of a word $v \in \mathsf{X}^*$ is equal to the sum of the weights of its letters.

Denote by L_n the set of words of weight n. We denote by $L_n v$ for $v \in \mathsf{X}^*$ the set of words of the form uv for $u \in L_n$. Similarly, if A is a subset of X^*, then we denote by $A\mathsf{X}^\omega$ the set of all sequences $w \in \mathsf{X}^\omega$ such that a beginning of w belongs to A.

We obviously have

$$L_n = L_{n-1}1 \sqcup L_{n-2}2, \tag{3}$$

and $L_0 = \{\varnothing\}$, $L_1 = \{1\}$, so that $|L_n|$, for $n = 0, 1, 2, \ldots$, is the Fibonacci sequence $1, 1, 2, 3, 5, \ldots$.

The transformation b has one fixed point $(12)^\omega$ (encoding the point $1/2$; the point 0 has two encodings, which are interchanged by b). The transformation a has two fixed points $1(12)^\omega$ and $2(12)^\omega$ (encoding the points $\varphi/2$ and $(\varphi + 1)/2$ of the circle). Let W_n, for $n \geq 0$, be the set of sequences starting by $(12)^n 2$ or by $(12)^n 11$. Define $P_i = \bigcup_{k=0}^\infty W_{3k+i}$ for $i = 0, 1, 2$. The sets P_i form an open partition of $\mathsf{X}^\omega \setminus \{(12)^\omega\}$.

Define, similarly to the Grigorchuk group, the homeomorphisms b_0, c_0, d_0 of X^ω acting trivially on P_2, P_1, P_0, respectively, and as b on their complements. Let a_0 be the homeomorphism interchanging $11\mathsf{X}^\omega$ with $2\mathsf{X}^\omega$. More explicitly,

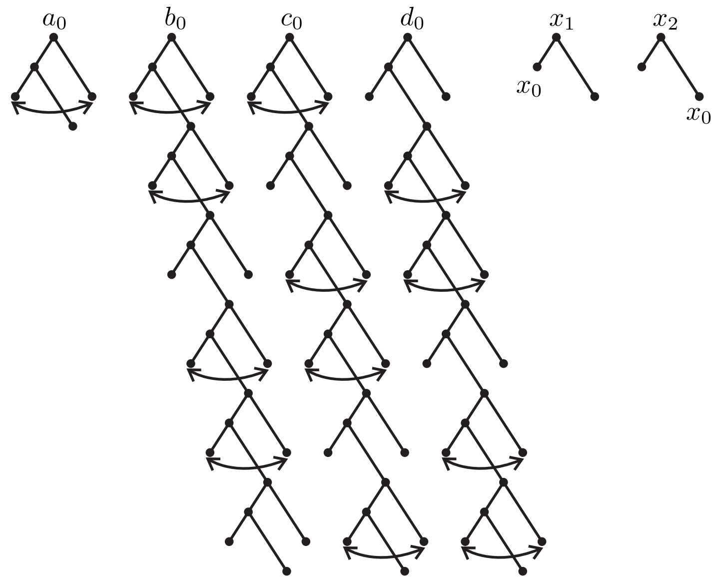

Figure 11. The generators of F.

the homeomorphisms a_0, b_0, c_0, d_0 are given by

$$\begin{aligned}
a_0(11w) &= 2w, & a_0(2w) &= 11w, & a_0(12w) &= 12w,\\
b_0(11w) &= 2w, & b_0(2w) &= 11w, & b_0(12w) &= 12c_0(w),\\
c_0(11w) &= 2w, & c_0(2w) &= 11w, & c_0(12w) &= 12d_0(w),\\
d_0(11w) &= 11w, & d_0(2w) &= 2w, & d_0(12w) &= 12b_0(w).
\end{aligned}$$

Note that a_0 belongs to the full topological group of $\langle b_0\rangle$.

Let us also fragment the homeomorphism a around its fixed points $1(12)^\omega$ and $2(12)^\omega$, in the same way as we fragmented the transformation b around $(12)^\omega$. Namely, for every letter $x \in \{a, b, c, d\}$, define

$$\begin{aligned}
x_1(1w) &= 1x_0(w), & x_1(2w) &= 2w,\\
x_2(1w) &= 1w, & x_2(2w) &= 2g_0(w).
\end{aligned}$$

See Figure 11 for a description of the action of the generators on the boundary of the tree X^*.

The homeomorphisms b_i, c_i, d_i are examples of homeomorphisms defined by *finite asynchronous automata*; see [GN00], [GNS00].

Let F be the group generated by a_i, b_i, c_i, d_i, $i = 0, 1, 2$. The goal of this section is to prove the following.

Theorem 8.1. *The group F coincides with its topological full group. It is periodic and of intermediate growth. Its derived subgroup $[F, F]$ is simple and has finite index in F.*

Periodicity follows from Theorem 4.1. Note that the set $\{b_0, c_0, d_0, \varepsilon\}$ is a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. The element a_0 commutes with this subgroup, hence the elements a_0, b_0, c_0, d_0 generate a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. The elements $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ pairwise commute and generate a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^6$. In particular, F is a quotient of the free product $(\mathbb{Z}/2\mathbb{Z})^3 * (\mathbb{Z}/2\mathbb{Z})^6$. It follows that $F/[F,F]$ is a quotient of $(\mathbb{Z}/2\mathbb{Z})^9$, hence is finite. In fact, one can show that the abelianization epimorphism $F \longrightarrow F/[F,F]$ is induced by the epimorphism $(\mathbb{Z}/2\mathbb{Z})^3 * (\mathbb{Z}/2\mathbb{Z})^6 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^9$, so that $F/[F,F] \cong (\mathbb{Z}/2\mathbb{Z})^9$, but we do not need it here.

Consequently, it is enough to prove subexponential growth, the equality $F = \mathsf{F}(F, \mathsf{X}^\omega)$, and that $[F,F] = \mathsf{A}(F, \mathsf{X}^\omega)$.

8.2. *Locally finite groups* S_ω *and* A_ω.

LEMMA 8.2. *For every $n \geq 1$, we have*

$$\mathsf{X}^\omega = L_n\mathsf{X}^\omega \sqcup L_{n-1}2\mathsf{X}^\omega.$$

Proof. It is true for $n = 1$: we have $L_1 = \{1\}$ and $L_0 = \{\varnothing\}$ and $\mathsf{X}^\omega = 1\mathsf{X}^\omega \sqcup 2\mathsf{X}^\omega$. Suppose it is true for n. Then we have

$$\begin{aligned}\mathsf{X}^\omega &= L_n\mathsf{X}^\omega \sqcup L_{n-1}2\mathsf{X}^\omega = L_n1\mathsf{X}^\omega \sqcup L_n2\mathsf{X}^\omega \sqcup L_{n-1}2\mathsf{X}^\omega \\ &= (L_n1 \cup L_{n-1}2)\mathsf{X}^\omega \cup L_n2\mathsf{X}^\omega = L_{n+1}\mathsf{X}^\omega \cup L_n2\mathsf{X}^\omega.\end{aligned}$$ $\square$

We say that two finite words v_1, v_2 are *incomparable* if neither of them is a beginning of the other. For a set A of pairwise incomparable words, we denote by $\mathsf{S}(A)$ (resp. $\mathsf{A}(A)$) the group of all (resp. even) permutations of A seen as homeomorphisms of X^ω. If α is a permutation of A, then the corresponding homeomorphism of X^ω acts by the rule $\alpha(vw) = \alpha(v)w$ for $v \in A$, and $\alpha(w) = w$ for $w \notin A\mathsf{X}^\omega$.

The groups $\mathsf{S}(L_nt)$, $t \in \{1,2\}$, are naturally isomorphic to $\mathsf{S}(L_n)$, where the isomorphism is induced by the bijection $v \mapsto vt$.

By Lemma 8.2, the groups $\mathsf{S}(L_n)$ and $\mathsf{S}(L_{n-1}2)$ act on disjoint subsets of X^*, hence they commute. Denote $\mathsf{S}(L_n) \oplus \mathsf{S}(L_{n-1}2) = \langle \mathsf{S}(L_n) \cup \mathsf{S}(L_{n-1}2)\rangle$. The group $\mathsf{A}(L_n) \oplus \mathsf{A}(L_{n-1}2)$ is defined the same way.

Note that

$$\mathsf{S}(L_n) \oplus \mathsf{S}(L_{n-1}2) < \mathsf{S}(L_{n+1}) \oplus \mathsf{S}(L_n2),$$

where $\mathsf{S}(L_n)$ is embedded diagonally into the direct sum by the homomorphism induced by the natural maps

$$v \mapsto v1 : L_n \longrightarrow L_n1 \subset L_{n+1}$$

and

$$v \mapsto v2 : L_n \longrightarrow L_n2,$$

and $\mathsf{S}(L_{n-1}2)$ is embedded isomorphically to the factor $\mathsf{S}(L_{n+1})$ by the natural inclusion $L_{n-1}2 \subset L_{n+1}$. The same is true for the embedding

$$\mathsf{A}(L_n) \oplus \mathsf{A}(L_{n-1}2) < \mathsf{A}(L_{n+1}) \oplus \mathsf{A}(L_n 2).$$

Denote by S_ω and A_ω the unions of the groups $\mathsf{S}(L_n) \oplus \mathsf{S}(L_{n-1}2)$ and $\mathsf{A}(L_n) \oplus \mathsf{A}(L_{n-1}2)$, respectively, i.e., the direct limit of the described embeddings.

PROPOSITION 8.3. *The quotient* $\mathsf{S}_\omega/\mathsf{A}_\omega$ *is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^2$ *and is equal to the set of images of* $a_0, a_1, a_2, 1$.

Proof. The quotient $(\mathsf{S}(L_n) \oplus \mathsf{S}(L_{n-1}2))/(\mathsf{A}(L_n) \oplus \mathsf{A}(L_{n-1}2)$ is naturally isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. It follows from the description of the embedding $\mathsf{S}(L_n) \oplus \mathsf{S}(L_{n-1}2) \hookrightarrow \mathsf{S}(L_{n+1}) \oplus \mathsf{S}(L_n 2)$ that if (x, y) is the image of an element $g \in \mathsf{S}(L_n) \oplus \mathsf{S}(L_{n-1}2)$ in the quotient $(\mathsf{S}(L_n) \oplus \mathsf{S}(L_{n-1}2))/(\mathsf{A}(L_n) \oplus \mathsf{A}(L_{n-1}2)$, then the image of the same element in the quotient $(\mathsf{S}(L_{n+1}) \oplus \mathsf{S}(L_n 2))/(\mathsf{A}(L_{n+1}) \oplus \mathsf{A}(L_n 2))$ is $(x + y, x)$.

Note that the map $(x, y) \mapsto (x + y, x)$ is an automorphism of $(\mathbb{Z}/2\mathbb{Z})^2$ and that the orbit of any non-zero element of $(\mathbb{Z}/2\mathbb{Z})^2$ belongs to the cycle $(1, 0) \mapsto (1, 1) \mapsto (0, 1) \mapsto (1, 0)$.

Let $g \in \mathsf{S}_\omega$ and n be such that $g \in \mathsf{S}(L_n) \oplus \mathsf{S}(L_{n-1}2)$. Consider then the sequence $\xi_g = (t_i)_{i \geq n}$ of the images of g in the quotients $(\mathsf{S}(L_i) \oplus \mathsf{S}(L_{i-1}2))/(\mathsf{A}(L_i) \oplus \mathsf{A}(L_{i-1}2)$. Note that the sequence ξ_g is defined only starting from some coordinate. We identify two sequences if they are equal in all coordinates where both of them are defined.

Then for every $g \in \mathsf{S}_\omega$, the sequence ξ_g is either equivalent to the constant zero sequence, or to one of the three shifts of the sequence

$$(1, 0), (1, 1), (0, 1), (1, 0), (1, 1), (0, 1), \ldots .$$

The sequence ξ_g is equivalent to the constant zero sequence if and only if $g \in \mathsf{A}_\omega$. It follows that $\mathsf{S}_\omega/\mathsf{A}_\omega$ is isomorphic to group of equivalence classes of the sequences

$$\begin{gathered}((0, 0), (0, 0), (0, 0), \ldots),\\ ((1, 0), (1, 1), (0, 1), \ldots),\\ ((1, 1), (0, 1), (1, 0), \ldots),\\ ((0, 1), (1, 0), (1, 1), \ldots),\end{gathered}$$

which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

The elements a_0, a_1, a_2 are equal to the permutations $(11, 2) \in \mathsf{S}(L_2)$, $(111, 12) \in \mathsf{S}(L_3)$, $(211, 22) \in \mathsf{S}(L_4)$, and hence the corresponding sequences

$\xi_{a_0}, \xi_{a_1}, \xi_{a_2}$ are

$$\begin{array}{cccccccc} *, & (1,0), & (1,1), & (0,1), & (1,0), & (1,1), & \dots \\ *, & *, & (1,0), & (1,1), & (0,1), & (1,0), & \dots \\ *, & *, & *, & (1,0), & (1,1), & (0,1), & \dots, \end{array}$$

where asterisk marks the places where the sequence is not defined. We see that the images of a_0, a_1, a_2 are all non-trivial elements of $\mathsf{S}_\omega/\mathsf{A}_\omega$. □

8.3. *The action of the elements of F.* Let x be one of the letters a, b, c, d, and let $v \in \mathsf{X}^*$. We then denote by x_v the homeomorphism of X^ω defined by the rule

$$x_v(w) = \begin{cases} v x_0(u) & \text{if } w = vu, \\ w & \text{if } w \notin v\mathsf{X}^\omega. \end{cases}$$

Denote also, for every non-negative integer k,

$$x_{3k} = x_{(12)^k}, \qquad x_{3k+1} = x_{1(12)^k}, \qquad x_{3k+2} = x_{2(12)^k}.$$

Note that this definition agrees with the original definitions of x_0, x_1, x_2.

We have $a_n \in \mathsf{S}(L_{n+2}) < \mathsf{S}(L_{n+2}) \oplus \mathsf{S}(L_{n+1}2)$ and

$$b_n = a_n c_{n+3}, \qquad c_n = a_n d_{n+3}, \qquad d_n = b_{n+3}.$$

It follows that we have the following equalities:

$$b_i = a_i c_{i+3} = a_i a_{i+3} d_{i+6} = a_i a_{i+3} b_{i+9} = a_i a_{i+3} a_{i+9} c_{i+12} = \cdots, \tag{4}$$

$$c_i = a_i d_{i+3} = a_i b_{i+6} = a_i a_{i+6} c_{i+9} = a_i a_{i+6} a_{i+9} d_{i+12} = \cdots, \tag{5}$$

$$d_i = b_{i+3} = a_{i+3} c_{i+6} = a_{i+3} a_{i+6} d_{i+9} = a_{i+3} a_{i+6} b_{i+12} = \cdots, \tag{6}$$

We will need the following direct corollary of equations (4)–(6).

LEMMA 8.4. *Let n be a positive integer, and let $i \in \{0, 1, 2\}$ be such that $n \equiv i \pmod 3$. Let $y = b$ if $n - i \equiv 6 \pmod 9$, $y = c$ if $n - i \equiv 3 \pmod 9$, and $y = d$ if $n - i \equiv 0 \pmod 9$.*

Then $y_i = h a_{n-3} d_n$, where $h \in \mathsf{S}(L_{n-4}) \oplus \mathsf{S}(L_{n-5}2)$.

We say that two sequences $w_1, w_2 \in \mathsf{X}^\omega$ are *cofinal* if there exist finite words $v_1, v_2 \in \mathsf{X}^*$ of equal weight and an infinite word $w \in \mathsf{X}^\omega$ such that $w_1 = v_1 w$ and $w_2 = v_1 w$.

Note that no two sequences from the set $R = \{(12)^\omega, 1(12)^\omega, 2(12)^\omega\}$ are cofinal, but every sequence of the form $v(12)^\omega$, where $v \in \mathsf{X}^*$, is cofinal to one of the sequences from the set R.

It follows directly from the definition of the generators of F that elements of F preserve cofinality classes of sequences. The next description of local action of elements of F on X^ω is easy to prove by induction on the length of g.

Proposition 8.5. *Let $g \in F$ be an arbitrary element. If $u \in \mathsf{X}^\omega$ is not cofinal to any of the elements of $R = \{(12)^\omega, 1(12)^\omega, 2(12)^\omega\}$, then there exists a finite beginning $v_1 \in \mathsf{X}^*$ of u and a word $v_2 \in \mathsf{X}^*$ of weight equal to the weight of v_1 such that*

$$g(v_1 w) = v_2 w$$

for all $w \in \mathsf{X}^\omega$.

If $u \in \mathsf{X}^\omega$ is cofinal to an element of R, then there exists a finite beginning $v_1 \in \mathsf{X}^*$ of u, a word $v_2 \in \mathsf{X}^*$ of weight equal to the weight of v_1, and an element $h \in \{\varepsilon, b_0, c_0, d_0\}$ such that

$$g(v_1 w) = v_2 h(w)$$

for all $w \in \mathsf{X}^\omega$.

Corollary 8.6. *Let $g \in F$. Then there exist finite sequences*

$$v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n \in \mathsf{X}^*,$$

$$h_1, h_2, \ldots, h_n \in \{\varepsilon, b_0, c_0, d_0\},$$

such that $\{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_n\}$ are maximal sets of pairwise incomparable words, weight of v_i is equal to the weight of u_i, and for every $w \in \mathsf{X}^\omega$, we have

$$g(v_i w) = u_i h_i(w).$$

The sequences $v_1, v_2, \ldots, v_n$, $u_1, u_2, \ldots, u_n$, and $h_1, h_2, \ldots, h_n$ uniquely describe the element g.

8.4. *The recursive structure of the orbital graphs of F.* The edges of the orbital graphs of F belong to one of the following types:

$$\begin{aligned} (12)^k 11w &\xrightarrow{\ e_{3k}\ } (12)^k 2w, \\ 1(12)^k 11w &\xrightarrow{\ e_{3k+1}\ } 1(12)^k 2w, \\ 2(12)^k 11w &\xrightarrow{\ e_{3k+2}\ } 2(12)^k 2w, \end{aligned}$$

where $w \in \mathsf{X}^\omega$.

The labels e_k stand for the following labelings by the generators

$$e_{3k+i} = \begin{cases} a_i, b_i, c_i & \text{if } k = 0, \\ b_i, c_i & \text{if } k \equiv 0 \pmod 3 \text{ and } k > 0, \\ b_i, d_i & \text{if } k \equiv 1 \pmod 3, \\ c_i, d_i & \text{if } k \equiv 2 \pmod 3, \end{cases} \tag{7}$$

where $i = 0, 1, 2$. Thus, for $k \geq 3$, the label e_k is determined by the residue of k modulo 9.

We will now define graphs I_k with the vertex set L_k. Each of the graphs I_k will be a chain with a fixed choice of the left/right direction.

For a finite or infinite word v, we denote by $I_k v$ the graph obtained from I_k by appending v to the name of each vertex of I_k.

The graphs I_0 and I_1 are single vertices $\varnothing$ and 1, respectively. Inductively define I_k by the rule

$$I_k = I_{k-2}^{-1}2 \overset{e_{k-2}}{\text{———}} I_{k-1}^{-1}1.$$

Proposition 8.7. *For every infinite word w, an orbital graph of F contains $I_n w$, and every finite segment of every orbital graph of F is contained in $I_n w$ for some n and w. Denote by P_n and Q_n the left and the right endpoints of the chain I_n, respectively. Then*

$$P_n = (12)^{n/3}, \quad 1(12)^{(n-1)/3}, \quad \textit{or } 2(12)^{(n-2)/3},$$

and

$$Q_n = (21)^{n/3}, \quad 1(21)^{(n-1)/3}, \quad \textit{or } 11(21)^{(n-2)/3},$$

depending on the residue $i = 0, 1, 2$ of n modulo 3.

Proof. Induction on n. □

Proposition 8.8. *The orbital graphs of F are linearly repetitive, hence the group F has subexponential growth.*

Proof. Consider the substitution τ given in Example 2.14, add new letters e_k, $k \geq 0$, to the alphabet, and modify τ as follows:

$$\tilde{\tau} : 1 \mapsto 2, \quad 1^* \mapsto 2^*, \quad 2 \mapsto 1^* e_0 2^*, \quad 2^* \mapsto 2e_0 1, \quad e_k \mapsto e_{k+1}.$$

Compare it to the Thue-Morse example in 7.1. We extend the involution * to the new alphabet by setting $e_i^* = e_i$, and to the set of all finite words by $(x_1 x_2 \cdots x_n)^* = x_n^* \cdots x_2^* x_1^*$. Note that the actions of $\tilde{\tau}$ and * on the set of finite words commute.

Let us prove by induction that the sequence obtained from $\tilde{\tau}^k(1)$ by deleting all the letters $1, 1^*, 2, 2^*$ coincides with the sequence of the edge labels in the segment I_k.

The segments I_0, I_1 have no edges, and the words $1, \tilde{\tau}(1) = 2$ also have no letters e_i. The segment I_2 is $2 \overset{e_0}{\text{———}} 11$, and the word $\tilde{\tau}^2(1)$ is $1^* e_0 2^*$. Suppose that the statement is true for all segments I_i for $i < k$. We have

$$\begin{aligned} \tilde{\tau}^k(1) &= \tilde{\tau}^{k-2}(1^* e_0 2^*) = (\tilde{\tau}^{k-2}(1))^* e_{k-2} (\tilde{\tau}^{k-2}(2))^* \\ &= (\tilde{\tau}^{k-2}(1))^* e_{k-2} (\tilde{\tau}^{k-1}(1))^*, \end{aligned}$$

which agrees with the recursive definition of the segments I_k and finishes the proof by induction.

Recall that the labels of the orbital graphs of F corresponding to the symbol e_k depend only on the residue of k modulo 9, if $k \geq 3$. It follows that the sequences describing the labels of bi-infinite orbital graphs of F belong to a substitutional shift. It is minimal by Proposition 2.5. Minimality of substitutional shifts is equivalent to their linear repetitivity (see [DL06]), hence the orbital graphs of regular points for the action of F are linearly repetitive. Theorem 6.6 then shows that F has intermediate growth. $\square$

8.5. *The proof of Theorem* 8.1. It follows from Corollary 8.6 that the topological full group $\mathsf{F}(F, \mathsf{X}^\omega)$ is the group of all transformations g that are defined by the rules of the form $g(v_i w) = u_i h_i(w)$, where $v_1, \ldots, v_n, u_1, \ldots, u_n \in \mathsf{X}^*$, and $h_1, h_2, \ldots, h_n \in \{b_0, c_0, d_0, \varepsilon\}$, as in Corollary 8.6.

PROPOSITION 8.9. *The group F coincides with its topological full group.*

Proof. Let us prove at first that F contains A_ω, i.e., that F contains $\mathsf{A}(L_n) \oplus \mathsf{A}(L_{n-1}2)$ for every $n \geq 1$.

The groups $\mathsf{A}(L_n)$ and $\mathsf{A}(L_n 2)$ are trivial for $n = 0, 1, 2$. Let us prove by induction that $\mathsf{A}(L_k) \oplus \mathsf{A}(L_{k-1}2) < F$. Suppose that it is true for all $k < n$, and let us prove it for n.

Recall that we have

$$\mathsf{X}^\omega = L_n \mathsf{X}^\omega \sqcup L_{n-1} 2 \mathsf{X}^\omega$$

and

$$L_n \mathsf{X}^\omega = L_{n-1} 1 \mathsf{X}^\omega \sqcup L_{n-2} 2 \mathsf{X}^\omega.$$

By Lemma 8.4, for one of the letters $y \in \{b, c, d\}$ and $i' \in \{0, 1, 2\}$ such that $i' \equiv i + 1 \pmod 3$, we will have $y_{i'} = h a_{n-2} d_{n+1}$, where $h \in \mathsf{S}(L_{n-3}) \oplus \mathsf{S}(L_{n-4}2) < \mathsf{S}(L_{n-1}) \oplus \mathsf{S}(L_{n-2}2\mathsf{X}^\omega)$.

The element a_{n-2} interchanges the sets $P_{n-2}2\mathsf{X}^\omega$ and $Q_{n-1}1\mathsf{X}^\omega$ and acts trivially on the complement of their union (where P_n and Q_n are as in Proposition 8.7), since

$$P_{n-2}2 = 1(12)^{(n-3)/3}2, \quad Q_{n-1}1 = 11(21)^{(n-3)/3}1 = 1(12)^{(n-3)/3}11$$

if $i = 0$,

$$P_{n-2}2 = 2(12)^{(n-4)/3}2, \quad Q_{n-1}1 = (21)^{(n-1)/3}1 = 2(12)^{(n-4)/3}11$$

if $i = 1$, and

$$P_{n-2}2 = (12)^{(n-2)/3}2, \quad Q_{n-1}1 = 1(21)^{(n-2)/3}1 = (12)^{(n-2)/3}11$$

if $i = 2$.

The element d_{n+1} preserves each set of the form $v\mathsf{X}^\omega$ for $v \in L_{n-1}1 \cup L_{n-2}2 \cup L_{n-1}2$, and it acts identically on each of them, except for the set $Q_{n-2}2\mathsf{X}^\omega$, since $Q_{n-2}2$ is one of the sequences $(21)^{(n-2)/3}2 = 2(12)^{(n-2)/3}$, $1(21)^{(n-3)/3}2 = (12)^{n/3}$, $11(21)^{(n-4)/3}2 = 1(12)^{(n-1)/3}$.

It follows that $y_{i'} \mathsf{A}(L_{n-2}2) y_{i'} = \mathsf{A}\Big((L_{n-2}2 \setminus \{P_{n-2}2\}) \cup \{Q_{n-1}1\}\Big)$; see Figure 12.

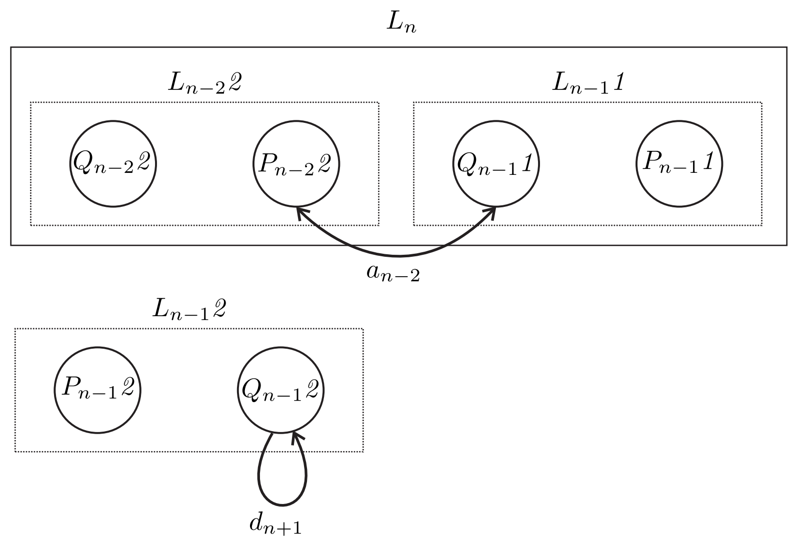

Figure 12. Generation of $\mathsf{A}(L_n) \oplus \mathsf{A}(L_{n-1}2)$.

Consequently, the group generated by $[\mathsf{A}(L_{n-1}), y_{i'}\mathsf{A}(L_{n-2}2)y_{i'}]$ is equal to $\mathsf{A}(L_n)$.

The elements of $\mathsf{A}(L_{n-1}) < F$ act on $L_{n-1}1\mathsf{X}^\omega \sqcup L_{n-1}2\mathsf{X}^\omega$. We have $L_{n-1}1 \subset L_n$, hence $\mathsf{A}(L_{n-1}2)$ is contained in the group generated by $\mathsf{A}(L_{n-1})$ and $\mathsf{A}(L_n)$, hence it is also contained in F.

This finishes the proof of the inclusion $\mathsf{A}_\omega < F$.

Lemma 8.10. *The group F contains S_ω and all elements of the form x_v for $x \in \{b, c, d\}$ and $v \in \mathsf{X}^*$.*

Proof. It was shown in Proposition 8.3 that $\mathsf{S}_\omega/\mathsf{A}_\omega$ is equal to the set of images of $a_0, a_1, a_2, \varepsilon$. Since $a_i \in F$, and $\mathsf{A}_\omega < F$, this implies that $\mathsf{S}_\omega < F$. In particular, all homeomorphisms a_n belong to F.

The relations $b_{n+3} = d_n, c_{n+3} = a_n b_n, d_{n+3} = a_n c_n$ imply by induction that all the homeomorphisms b_n, c_n, d_n belong to F.

Let $v \in \mathsf{X}^*$, and let n be the weight of v. Let $\sigma \in \mathsf{S}(L_n)$ be the transposition (v, u), where $u \in L_n$ is the unique sequence of the form $(12)^k, 1(12)^k$, or $2(12)^k$ of weight n. Then $x_v = \sigma x_n \sigma$. It follows that $x_v \in F$. □

Lemma 8.10 finishes the proof of Proposition 8.9. □

The next proposition finishes the proof of Theorem 8.1.

Proposition 8.11. *The derived subgroup $[F, F]$ coincides with $\mathsf{A}(F, \mathsf{X}^\omega)$.*

Proof. By Theorem 2.7 the group $\mathsf{A}(F, \mathsf{X}^\omega)$ is simple and is contained in every non-trivial normal subgroup of F. In particular, $\mathsf{A}(F, \mathsf{X}^\omega) \leq [F, F]$.

Therefore, it is enough to prove that $F/\mathsf{A}(F, \mathsf{X}^\omega)$ is commutative, i.e., that the generators of F commute modulo $\mathsf{A}(F, \mathsf{X}^\omega)$.

Note that $\mathsf{A}(F, \mathsf{X}^\omega)$ obviously contains A_ω. Note that $a_n = a_{n+3}$ modulo A_ω for all $n \geq 0$. We have $b_n = a_n a_{n+3} b_{n+9}$, $c_n = a_n a_{n+6} c_{n+9}$, $d_n = a_{n+3} a_{n+6} d_{n+9}$. It follows that $b_n = b_{n+9}$, $c_n = c_{n+9}$, and $d_n = d_{n+9}$ modulo $\mathsf{A}(F, \mathsf{X}^\omega)$. We have shown that for every $x \in \{a, b, c, d\}$, we have $x_n = x_{n+9}$ modulo $\mathsf{A}(F, \mathsf{X}^\omega)$.

If $n > 2$, and $v_1, v_2 \in L_n$, then there exists an element $\sigma \in \mathsf{A}(L_n)$ such that $\sigma(v_1) = v_2$. Then, for every letter $x \in \{a, b, c, d\}$, we have $\sigma x_{v_1} \sigma^{-1} = x_{v_2}$. It follows that $x_{v_1} = x_{v_2}$ modulo $\mathsf{A}(L_n)$.

For arbitrary words v_1, v_2 of sufficiently big weight, there exist incomparable words u_1, u_2 of the same weights as v_1, v_2, respectively. Then, for every $x, y \in \{a, b, c, d\}$, we have $x_{v_1} = x_{u_1}$ and $y_{v_2} = y_{u_2}$ modulo $\mathsf{A}(F, \mathsf{X}^\omega)$, and $[x_{u_1}, y_{u_2}] = 1$, hence $[x_{v_1}, y_{v_2}] \in \mathsf{A}(F, \mathsf{X}^\omega)$.

Let x_i, y_j be generators of F, where $x, y \in \{b, c, d\}$, $i, j \in \{0, 1, 2\}$. Then $x_i = x_{i+9k}$, $y_j = y_{j+9k}$ modulo $\mathsf{A}(F, \mathsf{X}^\omega)$ for all non-negative integers k and hence, by the above, x_i and y_j commute in $F/\mathsf{A}(F, \mathsf{X}^\omega)$. $\square$

References

[Adi79] S. I. Adian, *The Burnside Problem and Identities in Groups*, *Ergeb. Math. Grenzgeb.* **95**, Springer-Verlag, New York, 1979, translated from the Russian by John Lennox and James Wiegold. MR 0537580.

[Ady82] S. I. Adyan, Random walks on free periodic groups, *Izv. Akad. Nauk SSSR Ser. Mat.* **46** no. 6 (1982), 1139–1149, 1343. MR 0682486. Zbl 0512.60012. https://doi.org/10.1070/IM1983v021n03ABEH001799.

[Ale72] S. V. Alešin, Finite automata and the Burnside problem for periodic groups, *Mat. Zametki* **11** (1972), 319–328. MR 0301107. Zbl 0246.20024.

[AS03] J.-P. Allouche and J. Shallit, *Automatic Sequences. Theory, Applications, Generalizations*, Cambridge University Press, Cambridge, 2003. MR 1997038. Zbl 1086.11015. https://doi.org/10.1017/CBO9780511546563.

[AAV13] G. Amir, O. Angel, and B. Virág, Amenability of linear-activity automaton groups, *J. Eur. Math. Soc.* (*JEMS*) **15** no. 3 (2013), 705–730. MR 3085088. Zbl 1277.37019. https://doi.org/10.4171/JEMS/373.

[BG13] M. Baake and U. Grimm, *Aperiodic Order. Vol.* 1. *A Mathematical Invitation*, *Encyclopedia Math. Appl.* **149**, Cambridge University Press, Cambridge, 2013, with a foreword by Roger Penrose. MR 3136260. Zbl 1295.37001. https://doi.org/10.1017/CBO9781139025256.

[BM07] B. Bajorska and O. Macedońska, A note on groups of intermediate growth, *Comm. Algebra* **35** no. 12 (2007), 4112–4115. MR 2372323. Zbl 1132.20016. https://doi.org/10.1080/00927870701544914.

[Bar17] L. Bartholdi, Growth of groups and wreath products, in *Groups, Graphs and Random Walks*, *London Math. Soc. Lecture Note Ser.* **436**, Cambridge Univ. Press, Cambridge, 2017, pp. 1–76. MR 3644003. Zbl 1373.20054. https://doi.org/10.1017/9781316576571.

[BE12] L. Bartholdi and A. Erschler, Growth of permutational extensions, *Invent. Math.* **189** no. 2 (2012), 431–455. MR 2947548. Zbl 1286.20025. https://doi.org/10.1007/s00222-011-0368-x.

[BE14a] L. Bartholdi and A. Erschler, Groups of given intermediate word growth, *Ann. Inst. Fourier* (*Grenoble*) **64** no. 5 (2014), 2003–2036. MR 3330929. Zbl 1317.20043. https://doi.org/10.5802/aif.2902.

[BE14b] L. Bartholdi and A. Erschler, Imbeddings into groups of intermediate growth, *Groups Geom. Dyn.* **8** no. 3 (2014), 605–620. MR 3267517. Zbl 1335.20046. https://doi.org/10.4171/GGD/241.

[BGŠ03] L. Bartholdi, R. I. Grigorchuk, and Z. Šunić, Branch groups, in *Handbook of Algebra*, *Vol.* 3, North-Holland, Amsterdam, 2003, pp. 989–1112. MR 2035113. Zbl 1140.20306. https://doi.org/10.1016/S1570-7954(03)80078-5.

[BKN10] L. Bartholdi, V. A. Kaimanovich, and V. V. Nekrashevych, On amenability of automata groups, *Duke Math. J.* **154** no. 3 (2010), 575–598. MR 2730578. Zbl 1268.20026. https://doi.org/10.1215/00127094-2010-046.

[BŠ01] L. Bartholdi and Z. Šunić, On the word and period growth of some groups of tree automorphisms, *Comm. Algebra* **29** no. 11 (2001), 4923–4964. MR 1856923. Zbl 1001.20027.

[BV05] L. Bartholdi and B. Virág, Amenability via random walks, *Duke Math. J.* **130** no. 1 (2005), 39–56. MR 2176547. Zbl 1104.43002. https://doi.org/10.1215/S0012-7094-05-13012-5.

[BM08] S. Bezuglyi and K. Medynets, Full groups, flip conjugacy, and orbit equivalence of Cantor minimal systems, *Colloq. Math.* **110** no. 2 (2008), 409–429. MR 2353913. Zbl 1142.37011. https://doi.org/10.4064/cm110-2-6.

[Bos92] M. D. Boshernitzan, A condition for unique ergodicity of minimal symbolic flows, *Ergodic Theory Dynam. Systems* **12** no. 3 (1992), 425–428. MR 1182655. Zbl 0756.58030. https://doi.org/10.1017/S0143385700006866.

[Bri14] J. Brieussel, Growth behaviors in the range e^{r^α}, *Afr. Mat.* **25** no. 4 (2014), 1143–1163. MR 3277875. Zbl 1337.20041. https://doi.org/10.1007/s13370-013-0182-2.

[Bur02] W. Burnside, On an unsettled question in the theory of discontinuous groups, *Quart. J. Pure Appl. Math.* **33** (1902), 230–238. Zbl 33.0149.01.

[BP06] K.-U. Bux and R. Pérez, On the growth of iterated monodromy groups, in *Topological and Asymptotic Aspects of Group Theory*, *Contemp. Math.* **394**, Amer. Math. Soc., Providence, RI, 2006, pp. 61–76. MR 2216706. Zbl 1103.20038. https://doi.org/10.1090/conm/394/07434.

[CJN16] M. Chornyi, K. Juschenko, and V. Nekrashevych, On topological full groups of $\mathbb{Z}^d$-actions, 2016. arXiv 1602.04255.

[Cho80] C. Chou, Elementary amenable groups, *Illinois J. Math.* **24** no. 3 (1980), 396–407. MR 0573475. Zbl 0439.20017. Available at http://projecteuclid.org/euclid.ijm/1256047608.

[dC14] Y. de Cornulier, Groupes pleins-topologiques (d'après Matui, Juschenko, Monod, ...), *Astérisque* no. 361 (2014), Exp. No. 1064, viii, 183–223. MR 3289281. Zbl 1365.54001.

[DL06] D. Damanik and D. Lenz, Substitution dynamical systems: characterization of linear repetitivity and applications, *J. Math. Anal. Appl.* **321** no. 2 (2006), 766–780. MR 2241154. Zbl 1094.37007. https://doi.org/10.1016/j.jmaa.2005.09.004.

[Ers04] A. Erschler, Not residually finite groups of intermediate growth, commensurability and non-geometricity, *J. Algebra* **272** no. 1 (2004), 154–172. MR 2029029. Zbl 1049.20019. https://doi.org/10.1016/j.jalgebra.2002.11.005.

[Ers06] A. Erschler, Piecewise automatic groups, *Duke Math. J.* **134** no. 3 (2006), 591–613. MR 2254627. Zbl 1159.20019. https://doi.org/10.1215/S0012-7094-06-13435-X.

[Ers11] M. Ershov, Kazhdan quotients of Golod-Shafarevich groups, *Proc. Lond. Math. Soc.* (3) **102** no. 4 (2011), 599–636. MR 2793445. Zbl 1280.20037. https://doi.org/10.1112/plms/pdq022.

[GPS99] T. Giordano, I. F. Putnam, and C. F. Skau, Full groups of Cantor minimal systems, *Israel J. Math.* **111** (1999), 285–320. MR 1710743. Zbl 0942.46040. https://doi.org/10.1007/BF02810689.

[Gol64] E. S. Golod, On nil-algebras and finitely approximable p-groups, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 273–276. MR 0161878. Zbl 0215.39202.

[Gre69] F. P. Greenleaf, *Invariant Means on Topological Groups*, *Van Nostrand Mathematical Studies*, *No.* 16, Van Nostrand Reinhold, New York, 1969. MR 0251549. Zbl 0174.19001.

[GN00] R. I. Grigorchuk and V. V. Nekrashevich, The group of asynchronous automata and rational homeomorphisms of the Cantor set, *Mat. Zametki* **67** no. 5 (2000), 680–685. MR 1822615. Zbl 0988.37014. https://doi.org/10.1007/BF02676328.

[GNS00] R. I. Grigorchuk, V. V. Nekrashevich, and V. I. Sushchanskiĭ, Automata, dynamical systems, and groups, *Tr. Mat. Inst. Steklova* **231** no. Din. Sist., Avtom. i Beskon. Gruppy (2000), 134–214. MR 1841755. Zbl 1155.37311. Available at http://mi.mathnet.ru/eng/tm/v231/p134.

[Gri05] R. Grigorchuk, Solved and unsolved problems around one group, in *Infinite Groups: Geometric, Combinatorial and Dynamical Aspects*, *Progr. Math.* **248**, Birkhäuser, Basel, 2005, pp. 117–218. MR 2195454. Zbl 1165.20021. https://doi.org/10.1007/3-7643-7447-0_5.

[Gri14] R. Grigorchuk, On the gap conjecture concerning group growth, *Bull. Math. Sci.* **4** no. 1 (2014), 113–128. MR 3174281. Zbl 1323.20035. https://doi.org/10.1007/s13373-012-0029-4.

[GLN17] R. Grigorchuk, D. Lenz, and T. Nagnibeda, Schreier graphs of Grigorchuk's group and a subshift associated to a nonprimitive substitution, in *Groups, Graphs and Random Walks*, *London Math. Soc. Lecture Note Ser.* **436**, Cambridge Univ. Press, Cambridge, 2017, pp. 250–299. MR 3644012. Zbl 1371.05123. https://doi.org/10.1017/9781316576571.

[GL02] R. Grigorchuk and I. Lysenok, The Burnside problems, in *The Concise Handbook of Algebra*, Kluwer Academic Publishers, Dordrecht, 2002, Alexander V. Mikhalev and Günter F. Pilz, eds., pp. 111–115. https://doi.org/10.1007/978-94-017-3267-3_2.

[Gri80] R. I. Grigorchuk, On Burnside's problem on periodic groups, *Funktsional. Anal. i Prilozhen.* **14** no. 1 (1980), 53–54. MR 0565099. Zbl 0595.20029.

[Gri83] R. I. Grigorchuk, Milnor's problem on the growth of groups, *Sov. Math., Dokl.* **28** (1983), 23–26. Zbl 0547.20025. https://doi.org/10.1515/9781400851317-027.

[Gri85] R. I. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means, *Math. USSR Izv.* **25** no. 2 (1985), 259–300. Zbl 0583.20023. https://doi.org/10.1070/IM1985v025n02ABEH001281.

[Gri91] R. I. Grigorchuk, On growth in group theory, in *Proceedings of the International Congress of Mathematicians, Vol. I, II* (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 325–338. MR 1159221. Zbl 0749.20016.

[Gro87] M. Gromov, Hyperbolic groups, in *Essays in Group Theory*, *Math. Sci. Res. Inst. Publ.* **8**, Springer, New York, 1987, pp. 75–263. MR 0919829. Zbl 0634.20015. https://doi.org/10.1007/978-1-4613-9586-7_3.

[Gro81] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 53–73. MR 0623534. Zbl 0474.20018. Available at http://www.numdam.org/item?id=PMIHES_1981__53__53_0.

[GS83] N. Gupta and S. Sidki, On the Burnside problem for periodic groups, *Math. Z.* **182** no. 3 (1983), 385–388. MR 0696534. Zbl 0513.20024. https://doi.org/10.1007/BF01179757.

[Iva94] S. V. Ivanov, The free Burnside groups of sufficiently large exponents, *Internat. J. Algebra Comput.* **4** no. 1-2 (1994), ii+308. MR 1283947. Zbl 0822.20044. https://doi.org/10.1142/S0218196794000026.

[JMMdlS18] K. Juschenko, N. Matte Bon, N. Monod, and M. de la Salle, Extensive amenability and an application to interval exchanges, *Ergodic*

Theory Dynam. Systems **38** no. 1 (2018), 195–219. MR 3742543. https://doi.org/10.1017/etds.2016.32.

[JM13] K. Juschenko and N. Monod, Cantor systems, piecewise translations and simple amenable groups, *Ann. of Math.* (2) **178** no. 2 (2013), 775–787. MR 3071509. Zbl 1283.37011. https://doi.org/10.4007/annals.2013.178.2.7.

[JNdlS16] K. Juschenko, V. Nekrashevych, and M. de la Salle, Extensions of amenable groups by recurrent groupoids, *Invent. Math.* **206** no. 3 (2016), 837–867. MR 3573974. Zbl 06664763. https://doi.org/10.1007/s00222-016-0664-6.

[KP13] M. Kassabov and I. Pak, Groups of oscillating intermediate growth, *Ann. of Math.* (2) **177** no. 3 (2013), 1113–1145. MR 3034295. Zbl 1283.20027. https://doi.org/10.4007/annals.2013.177.3.7.

[KLS15] J. Kellendonk, D. Lenz, and J. Savinien, *Mathematics of Periodic Order*, *Progr. Math. Phys.* **309**, Birkhäuser/Springer, Basel, 2015. Zbl 1338.37005.

[Kri80] W. Krieger, On a dimension for a class of homeomorphism groups, *Math. Ann.* **252** no. 2 (1979/80), 87–95. MR 0593623. Zbl 0472.54028. https://doi.org/10.1007/BF01420115.

[Kro84] L. Kronecker, Näherunsgsweise ganzzahlige auflösung linearer gleichungen, in *Monatsberichte Königlich Preussischen Akademie der Wissenschaften zu Berlin*, 1884, pp. 1179–1193, 1271–1299. Zbl 16.0083.02.

[Lys96] I. G. Lysënok, Infinite Burnside groups of even period, *Izv. Ross. Akad. Nauk Ser. Mat.* **60** no. 3 (1996), 3–224. MR 1405529. Zbl 0926.20023. https://doi.org/10.1070/IM1996v060n03ABEH000077.

[Man12] A. Mann, *How Groups Grow*, *London Math. Soc. Lect. Note Ser.* **395**, Cambridge University Press, Cambridge, 2012. MR 2894945. Zbl 1253.20032. https://doi.org/10.1017/CBO9781139095129.

[MB14] N. Matte Bon, Subshifts with slow complexity and simple groups with the Liouville property, *Geom. Funct. Anal.* **24** no. 5 (2014), 1637–1659. MR 3261637. Zbl 1366.37040. https://doi.org/10.1007/s00039-014-0293-4.

[MB15] N. Matte Bon, Topological full groups of minimal subshifts with subgroups of intermediate growth, *J. Mod. Dyn.* **9** (2015), 67–80. MR 3395261. Zbl 1352.37037. https://doi.org/10.3934/jmd.2015.9.67.

[Mat06] H. Matui, Some remarks on topological full groups of Cantor minimal systems, *Internat. J. Math.* **17** no. 2 (2006), 231–251. MR 2205435. Zbl 1109.37008. https://doi.org/10.1142/S0129167X06003448.

[Mat12] H. Matui, Homology and topological full groups of étale groupoids on totally disconnected spaces, *Proc. Lond. Math. Soc.* (3) **104** no. 1 (2012), 27–56. MR 2876963. Zbl 1325.19001. https://doi.org/10.1112/plms/pdr029.

[Mat13] H. Matui, Some remarks on topological full groups of Cantor minimal systems II, *Ergodic Theory Dynam. Systems* **33** no. 5 (2013), 1542–1549. MR 3103094. Zbl 1293.37010. https://doi.org/10.1017/S0143385712000399.

[Mat15] H. Matui, Topological full groups of one-sided shifts of finite type, *J. Reine Angew. Math.* **705** (2015), 35–84. MR 3377390. Zbl 1372.22006. https://doi.org/10.1515/crelle-2013-0041.

[MK14] V. D. Mazurov and E. I. Khukhro, Unsolved problems in group theory. The Kourovka notebook. no. 18, 2014. arXiv 1401.0300.

[Mil68] J. Milnor, Problem 5603, *Amer. Math. Monthly* **75** (1968), 685–686.

[Nek05] V. Nekrashevych, *Self-Similar Groups*, *Math. Surveys Monogr.* **117**, American Mathematical Society, Providence, RI, 2005. MR 2162164. Zbl 1087.20032. https://doi.org/10.1090/surv/117.

[Nek15] V. Nekrashevych, Simple groups of dynamical origin, *Ergodic Theory Dynam. Systems* (2015), 26 pp., published online: 17 August 2017. https://doi.org/10.1017/etds.2017.47.

[vN29] J. von Neumann, Zur allgemeinen Theorie des Masses, in *Collected Works*, *Vol.*, *Fund. Math.* **13**, 1929, pp. 73–116, 333, 599–643. Zbl 55.0151.01.

[NA68] P. S. Novikov and S. I. Adian, Infinite periodic groups. I, *Izv. Akad. Nauk SSSR Ser. Mat.* **32** (1968), 212–244. MR 0240178. Zbl 0194.03301. https://doi.org/10.1070/IM1968v002n01ABEH000637.

[Ol′80] A. Y. Ol′shanskii, On the question of the existence of an invariant mean on a group, *Uspekhi Mat. Nauk* **35** no. 4(214) (1980), 199–200. MR 0586204. Zbl 0452.20032. https://doi.org/10.1070/RM1980v035n04ABEH001876.

[Ol′82] A. Y. Ol′shanskii, Groups of bounded period with subgroups of prime order, *Algebra i Logika* **21** no. 5 (1982), 553–618. MR 0721048.

[Ol′91] A. Y. Ol′shanskii, *Geometry of Defining Relations in Groups*, *Math. Appl.* (*Soviet Series*) **70**, Kluwer Academic Publishers Group, Dordrecht, 1991, translated from the 1989 Russian original by Yu. A. Bakhturin. MR 1191619. Zbl 0732.20019. https://doi.org/10.1007/978-94-011-3618-1.

[Ol′93] A. Y. Ol′shanskii, On residualing homomorphisms and G-subgroups of hyperbolic groups, *Internat. J. Algebra Comput.* **3** no. 4 (1993), 365–409. MR 1250244. Zbl 0830.20053. https://doi.org/10.1142/S0218196793000251.

[Pat88] A. L. T. Paterson, *Amenability*, *Math. Surveys Monogr.* **29**, American Mathematical Society, Providence, RI, 1988. MR 0961261. Zbl 0648.43001. https://doi.org/10.1090/surv/029.

[Rip82] E. Rips, Generalized small cancellation theory and applications. I. The word problem, *Israel J. Math.* **41** no. 1-2 (1982), 1–146. MR 0657850. Zbl 0508.20017. https://doi.org/10.1007/BF02760660.

[Šun07] Z. Šunić, Hausdorff dimension in a family of self-similar groups, *Geom. Dedicata* **124** (2007), 213–236. MR 2318546. Zbl 1169.20015. https://doi.org/10.1007/s10711-006-9106-8.

[Suš79] V. I. Suščans′kiĭ, Periodic p-groups of permutations and the unrestricted Burnside problem, *Dokl. Akad. Nauk SSSR* **247** no. 3 (1979), 557–561. MR 0545692. Zbl 0428.20023.

[Vor12] Y. Vorobets, Notes on the Schreier graphs of the Grigorchuk group, in *Dynamical Systems and Group Actions*, *Contemp. Math.* **567**, Amer. Math. Soc., Providence, RI, 2012, pp. 221–248. MR 2931920. Zbl 1255.05192. https://doi.org/10.1090/conm/567/11250.

[Wag93] S. Wagon, *The Banach-Tarski Paradox*, Cambridge University Press, Cambridge, 1993, with a foreword by Jan Mycielski, corrected reprint of the 1985 original. MR 1251963. Zbl 1372.43001.

[Wil89] H. S. Wilf, *Combinatorial Algorithms*: *An Update*, *CBMS-NSF Reg. Conf. Ser. Appl. Math.* **55**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989. MR 0993775. Zbl 0695.05002. https://doi.org/10.1137/1.9781611970166.

[Zel90] E. I. Zel′manov, Solution of the restricted Burnside problem for groups of odd exponent, *Izv. Akad. Nauk SSSR Ser. Mat.* **54** no. 1 (1990), 42–59, 221. MR 1044047. Zbl 0752.20017. https://doi.org/10.1070/IM1991v036n01ABEH001946.

(Received: March 7, 2016)
(Revised: November 16, 2017)

Texas A&M University, College Station, TX
E-mail: nekrash@math.tamu.edu
http://www.math.tamu.edu/~nekrash/