Semipositivity theorems for moduli problems

By Osamu Fujino

Abstract

We prove some semipositivity theorems for singular varieties coming from graded polarizable admissible variations of mixed Hodge structure. As an application, we obtain that the moduli functor of stable varieties is semipositive in the sense of Kollár. This completes Kollár's projectivity criterion for the moduli spaces of higher-dimensional stable varieties.

Contents

1. Introduction

The main purpose of this paper is to give a proof of the following "folklore statement" (see, for example, [Ale96], [Kar00], [Kov09], [AH11], [Kol13a], and [Fn11b]) based on [FF14] (see also [FFS14]). In general, the (quasi-) projectivity of some moduli space is a subtle problem and is harder than it looks (see, for example, [Kol06], and [Vie10]). We note that the coarse moduli space of stable varieties was first constructed in the category of algebraic spaces.

THEOREM 1.1 (Projectivity of moduli spaces of stable varieties). *Every closed complete subspace of the coarse moduli space of stable varieties is* projective.

To the best knowledge of the author, Theorem 1.1 is new for stable n-folds with $n \geq 3$. For the details, see the comments in 1.8 below. Note that a *stable n-fold* is an n-dimensional projective semi-log-canonical variety with ample canonical divisor and is called an *n-dimensional semi-log-canonical model* in [Kol13a]. For the details of semi-log-canonical varieties, see, for example, [Fn14] and [Kol13b]. Theorem 1.1 is a direct consequence of Theorem 1.2 below by Kollár's projectivity criterion (see [Kol90, §§2 and 3]).

THEOREM 1.2 (Semipositivity of $\mathcal{M}^{\text{stable}}$). *Let $\mathcal{M}^{\text{stable}}$ be the moduli functor of stable varieties. Then $\mathcal{M}^{\text{stable}}$ is semipositive in the sense of Kollár.*

For the reader's convenience, let us recall the definition of the semipositivity of $\mathcal{M}^{\text{stable}}$, which is a special case of [Kol90, 2.4. Definition].

Definition 1.3 (see [Kol90, 2.4. Definition]). The moduli functor $\mathcal{M}^{\text{stable}}$ of stable varieties is said to be *semipositive (in the sense of Kollár)* if the following condition holds:

There is a fixed positive integer m_0 such that if C is a smooth projective curve and $(f : X \to C) \in \mathcal{M}^{\text{stable}}(C)$, then $f_* \omega_{X/C}^{[mm_0]}$ is a nef locally free sheaf on C for every positive integer m.

As the culmination of the works of several authors (see, for example, [Ale94], [Ale96], [AM04], [HMX14], [HX13], [KM97], [Kol08], [Kol16], [KSB88]), we have

COROLLARY 1.4. *The moduli functor $\mathcal{M}_H^{\text{stable}}$ of stable varieties with Hilbert function H is coarsely represented by a* projective *algebraic scheme.*

As an easy consequence of Corollary 1.4, we obtain

COROLLARY 1.5 (see [Vie95, Th. 1.11]). *The moduli functor $\mathcal{M}_H$ of canonically polarized smooth projective varieties with Hilbert function H is coarsely represented by a* quasi-projective *algebraic scheme.*

More generally, we have

COROLLARY 1.6. *The moduli functor $\mathcal{M}_H^{\text{can}}$ of canonically polarized normal projective varieties having only canonical singularities with Hilbert function H is coarsely represented by a* quasi-projective *algebraic scheme.*

Theorem 1.2 follows almost directly from the definition of the semipositivity of $\mathcal{M}^{\text{stable}}$ in Definition 1.3 (see [Kol90, 2.4. Def.]) and the following semipositivity theorem.

THEOREM 1.7 (Semipositivity theorem I). *Let X be an equidimensional variety that satisfies Serre's S_2 condition and is a normal crossing in codimension one. Let $f : X \to C$ be a projective surjective morphism onto a smooth projective curve C such that every irreducible component of X is dominant*

onto C. Assume that there exists a non-empty Zariski open set U of C such that $f^{-1}(U)$ has only semi-log-canonical singularities. Then $f_*\omega_{X/C}$ is nef.

Assume further that $\omega_{X/C}^{[k]}$ is locally free and f-generated for some positive integer k. Then $f_*\omega_{X/C}^{[m]}$ is nef for every $m \geq 1$.

1.8 (Comments). By the recent developments of the minimal model program, Theorem 1.7 seems to be a reasonable formulation of [Kol90, 4.12. Theorem] for higher-dimensional singular varieties. Kollár has pointed out that the assumption that the fibers are surfaces was inadvertently omitted from its statement. He is really claiming [Kol90, 4.12. Theorem] for $f : Z \to C$ with $\dim Z = 3$ (see [Kol90, 1. Introduction]). Therefore, Theorem 1.7 is new when $\dim X \geq 4$. Likewise, Theorems 1.1 and 1.2 are new when the dimension of the stable varieties are greater than or equal to three. We feel that the arguments in [Kol90, 4.14] only work when the fibers are surfaces. In other words, we needed some new ideas and techniques to prove Theorem 1.7. Our arguments heavily depend on the recent advances on the semipositivity theorems of Hodge bundles ([FF14] and [FFS14]) and some ideas in [Fn14].

For the general theory of Kollár's projectivity criterion, see [Kol90, §§2 and 3] and [Vie95, Th. 4.34]. We do not discuss the technical details of the construction of moduli spaces of stable varieties in this paper. We mainly treat various semipositivity theorems. Note that the projectivity criterion discussed here is independent of the existence problem of moduli spaces. We recommend the reader to see [Kol90, §2] and [Kol13a, §§4 and 5] for Kollár's program for constructing moduli spaces of stable varieties (see also [Kol]). Our paper is related to the topic in [Kol13a, 5.5 (Projectivity)].

In this paper, we prove Theorem 1.7 in the framework of [Fn14] and [FF14], although we do not use the arguments in [Fn14] explicitly. Note that [Fn14] and [FF14] heavily depend on the theory of *mixed* Hodge structures on cohomology with compact support. A key ingredient of this paper is the following semipositivity theorem, which is essentially contained in [FF14] (see also [FFS14]). It is a generalization of Fujita's semipositivity theorem (see [Ft78, (0.6) Main Theorem]). We note that a Hodge theoretic approach to the original Fujita semipositivity theorem was introduced by Zucker (see [Zuc82]).

THEOREM 1.9 (Basic semipositivity theorem; see [FF14, §7]). *Let (X, D) be a simple normal crossing pair such that D is reduced. Let $f : X \to C$ be a projective surjective morphism onto a smooth projective curve C. Assume that every stratum of X is dominant onto C. Then $f_*\omega_{X/C}(D)$ is nef.*

Although we do not know what is the best formulation of the semipositivity theorem for moduli problems, we think Theorem 1.9 will be one of the most fundamental results for application to Kollár's projectivity criterion for

moduli spaces. We can prove Theorem 1.7 by using Theorem 1.9. By the same proof as that of Theorem 1.7, we obtain a generalization of Theorem 1.7, which implies both Theorems 1.7 and 1.9.

THEOREM 1.10 (Semipositivity theorem II). *Let X be an equidimensional variety that satisfies Serre's S_2 condition and is a normal crossing in codimension one. Let $f : X \to C$ be a projective surjective morphism onto a smooth projective curve C such that every irreducible component of X is dominant onto C. Let D be a reduced Weil divisor on X such that no irreducible component of D is contained in the singular locus of X. Assume that there exists a non-empty Zariski open set U of C such that $(f^{-1}(U), D|_{f^{-1}(U)})$ is a semi-log-canonical pair. Then $f_*\omega_{X/C}(D)$ is nef.*

We further assume that $\mathcal{O}_X(k(K_X + D))$ is locally free and f-generated for some positive integer k. Then $f_*\mathcal{O}_X(m(K_{X/C}+D))$ is nef for every $m \geq 1$.

By combining Theorem 1.10 with Viehweg's covering trick, we obtain Theorem 1.11, which is an answer to the question in [Ale96, 5.6]. Although we do not discuss the moduli spaces of *stable pairs* here, Theorems 1.10 and 1.11 play important roles in the proof of the projectivity of the moduli spaces of stable pairs (see [Ale96], [FP97], [Has03], [KP17], and 4.3 below).

THEOREM 1.11 (Semipositivity theorem III). *Let X be an equidimensional variety that satisfies Serre's S_2 condition and is a normal crossing in codimension one. Let $f : X \to C$ be a projective surjective morphism onto a smooth projective curve C such that every irreducible component of X is dominant onto C. Let Δ be an effective $\mathbb{Q}$-Weil divisor on X such that no irreducible component of the support of Δ is contained in the singular locus of X. Assume that there exists a non-empty Zariski open set U of C such that $(f^{-1}(U), \Delta|_{f^{-1}(U)})$ is a semi-log-canonical pair. We further assume that $\mathcal{O}_X(k(K_X + \Delta))$ is locally free and f-generated for some positive integer k. Then $f_*\mathcal{O}_X(k(K_{X/C} + \Delta))$ is nef. Therefore, $f_*\mathcal{O}_X(kl(K_{X/C} + \Delta))$ is nef for every $l \geq 1$.*

Recently, Kovács and Patakfalvi generalized Theorem 1.1 for stable pairs in [KP17]. Note that one of the main ingredients of [KP17] is Theorem 1.11. For the details, we recommend that the reader see [KP17]. Theorem 1.11 is also a key result for the proof of the ampleness of the CM line bundle on the moduli space of canonically polarized varieties in [PX17]. In any case, we expect our semipositivity theorems established in [FF14] to play an important role in the study of higher-dimensional complex algebraic varieties.

Remark 1.12. In this paper, we do not use algebraic spaces for the proof of the semipositivity theorems. We only treat *projective* varieties. Note that Theorem 1.9 follows from the theory of variations of mixed Hodge structure.

The variations of mixed Hodge structure discussed in [FF14] (see also [FFS14]) are *graded polarizable* and *admissible*. Therefore, we cannot directly apply the results in [FF14] to the variations of (mixed) Hodge structure arising from families of algebraic spaces. We need some polarization to obtain various semipositivity theorems in our framework. We also note that the admissibility assures us of the existence of canonical extensions of Hodge bundles, which does not always hold for abstract graded polarizable variations of mixed Hodge structure (see [FF14, Exam. 1.5]).

We do not use the Fujita–Zucker–Kawamata semipositivity theorem coming from the theory of polarized variations of Hodge structure (see [Fn17c]).

Remarks 1.13. (1) As explained in [Kol90] and [Kol11], it is difficult to directly check the *quasi-projectivity* of non-complete singular spaces. This is because there is no good ampleness criterion for non-complete spaces. In this paper, we adopt Kollár's framework in [Kol90, §§2 and 3], where we use the Nakai–Moishezon criterion to check the projectivity of complete algebraic spaces. Note that Viehweg discusses the *quasi-projectivity* of *non-complete* moduli spaces (see [Vie95] and [Vie10]). On the other hand, Kollár and we prove the *projectivity* of *complete* moduli spaces (see [Kol90]).

(2) In general, we have to formulate and prove semipositivity theorems for non-normal (reducible) varieties X even if we are mainly interested in the moduli spaces of smooth (or normal) stable varieties. Let $\mathcal{M}_g$ (resp. $\overline{\mathcal{M}}_g$) be the moduli functor of smooth projective curves (resp. stable curves) with $g \geq 2$. We consider $(f : X \to C) \in \overline{\mathcal{M}}_g(C)$ such that C is contained in $\overline{M}_g \setminus M_g$, where M_g (resp. $\overline{M}_g$) is the coarse moduli space of $\mathcal{M}_g$ (resp. $\overline{\mathcal{M}}_g$). Then a general fiber of $f : X \to C$ may be non-normal and reducible. We can prove the projectivity of $\overline{M}_g$ by Kollár's projectivity criterion. However, we cannot directly prove the quasi-projectivity of M_g.

(3) Although we repeatedly use Viehweg's covering arguments, we do not use the notion of *weak positivity*, which was introduced by Viehweg and plays crucial roles in his works (see [Vie83], [Vie95], and [Vie10]). We just treat the *semipositivity* on smooth projective curves (see [Kol90]).

(4) From the Hodge theoretic viewpoint, our approach is based on the theory of *mixed* Hodge structures (see [FF14] and [FFS14]). The arguments in [Vie95], [Kol90], and [Vie10] use only *pure* Hodge structures. It is one of the main differences between our approach and the others.

(5) In this paper, we use the theory of variations of mixed Hodge structure only in the proof of Theorem 1.9. Moreover, for the proof of Theorem 1.9, we only need the theory of variations of mixed Hodge structure in the case where the base space is a curve. If we assume that the base space is a curve, then the theory described in [FF14] becomes much simpler than the general case.

We summarize the contents of this paper. In Section 2, we collect some basic definitions. In Section 3, we quickly review the moduli functor $\mathcal{M}^{\mathrm{stable}}$ of stable varieties and its coarse moduli space. Section 4 is the main part of this paper, where we prove the theorems in Section 1. Our proofs depend on [FF14], some ideas in [Fn14], and Viehweg's covering arguments. In Section 5, we prove the corollaries in Section 1.

We will work over $\mathbb{C}$, the complex number field, throughout this paper. Note that, by the Lefschetz principle, all the results in this paper hold over any algebraically closed field k of characteristic zero. We will freely use the notation and terminology in [FF14] and [Fn14]. For the standard notation and conventions of the log minimal model program, see [Fn11a] and [Fn17a].

2. **Preliminaries**

Let us recall the definition of *nef locally free sheaves*. For the details, see, for example, [Vie95, §2].

Definition 2.1 (Nef locally free sheaves). A locally free sheaf of finite rank $\mathcal{E}$ on a complete variety X is *nef* if the following equivalent conditions are satisfied:

 (i) $\mathcal{E} = 0$ or $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is nef on $\mathbb{P}_X(\mathcal{E})$.
(ii) For every map from a smooth projective curve $f : C \to X$, every quotient line bundle of $f^*\mathcal{E}$ has non-negative degree.

A nef locally free sheaf was originally called a (*numerically*) *semipositive* sheaf in the literature.

In this paper, we only discuss various semipositivity theorems for locally free sheaves on a smooth projective curve. The following well-known lemma is very useful. We omit the proof of Lemma 2.2 because it is an easy exercise.

LEMMA 2.2. *Let C be a smooth projective curve, and let $\mathcal{E}_i$ be a locally free sheaf on C for $i = 1, 2$. Assume that $\mathcal{E}_1 \subset \mathcal{E}_2$, $\mathcal{E}_1$ is nef and that $\mathcal{E}_1$ coincides with $\mathcal{E}_2$ over some non-empty Zariski open set of C. Then $\mathcal{E}_2$ is nef.*

The following lemma is more or less known to the experts. However, we cannot find it explicitly in the literature. So we describe it here for the reader's convenience.

LEMMA 2.3. *Let $\mathcal{E}$ be a locally free sheaf of finite rank on a smooth projective irreducible curve C. Let Σ be a fixed Zariski closed set of C with $\Sigma \subsetneq C$. Assume that there exists some positive integer μ such that, for every finite surjective morphism $\pi : C' \to C$ from a smooth projective irreducible curve C' that is étale over some open neighborhood of Σ and for every ample line bundle $\mathcal{H}'$ on C', $\pi^*\mathcal{E} \otimes (\mathcal{H}')^\mu$ is nef on C'. Then $\mathcal{E}$ is nef.*

Proof. We take an ample line bundle $\mathcal{H}$ on C. For any positive integer α, we can construct a finite covering $\pi : C' \to C$ from a smooth projective irreducible curve C' such that $\pi^*\mathcal{H} = (\mathcal{H}')^{1+2\alpha\mu}$ for some ample line bundle $\mathcal{H}'$ on C'. We can make π étale over some open neighborhood of Σ and assume that π is Galois (see [KMM87, Th. 1-1-1]). We note the trace map splits the natural inclusion $\mathcal{O}_C \to \pi_*\mathcal{O}_{C'}$. Since $\pi^*\mathcal{E} \otimes (\mathcal{H}')^\mu$ is nef, there is some positive integer β such that

$$S^{(2\alpha)\beta}(\pi^*\mathcal{E} \otimes (\mathcal{H}')^\mu) \otimes (\mathcal{H}')^\beta = \pi^*(S^{2\alpha\beta}(\mathcal{E}) \otimes \mathcal{H}^\beta)$$

is generated by its global sections (see [Vie95, Prop. 2.9]). Therefore, we have a surjective morphism

$$\bigoplus_{\text{finite}} \mathcal{O}_{C'} \to \pi^*(S^{2\alpha\beta}(\mathcal{E}) \otimes \mathcal{H}^\beta).$$

Thus the induced morphism

$$\left(\bigoplus_{\text{finite}} \pi_*\mathcal{O}_{C'}\right) \otimes \mathcal{H}^\beta \to S^{2\alpha\beta}(\mathcal{E}) \otimes \mathcal{H}^{2\beta} \otimes \pi_*\mathcal{O}_{C'} \to S^{2\alpha\beta}(\mathcal{E}) \otimes \mathcal{H}^{2\beta}$$

is surjective. By replacing β by some multiple, we may assume that $\pi_*\mathcal{O}_{C'} \otimes \mathcal{H}^\beta$ is generated by its global sections. In this case, $S^{2\alpha\beta}(\mathcal{E}) \otimes \mathcal{H}^{2\beta}$ is generated by its global sections. This implies that $\mathcal{E}$ is nef (see [Vie95, Prop. 2.9]). $\square$

We need the notion of *simple normal crossing pairs* for Theorem 1.9. Note that a simple normal crossing pair is sometimes called a *semi-snc* pair in the literature (see [BVP13, Def. 1.1]).

Definition 2.4 (Simple normal crossing pairs). We say that the pair (X, D) is a *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety Y, where Y has regular system of parameters $(x_1, \ldots, x_p, y_1, \ldots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$D = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, D) is a *simple normal crossing pair* if it is a simple normal crossing at every point of X. We sometimes say that D is a *simple normal crossing divisor* on X if (X, D) is a simple normal crossing pair and D is reduced. If $(X, 0)$ is a simple normal crossing pair, then we simply say that X is a *simple normal crossing variety*. Let X be a simple normal crossing variety, and let $X = \sum_{i \in I} X_i$ be the irreducible decomposition. A *stratum* of X is an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \ldots, i_k\} \subset I$.

Definition 2.5 (Stratum). Let (X, D) be a simple normal crossing pair such that D is reduced. Let $\nu : X^\nu \to X$ be the normalization. We put $K_{X^\nu} + \Theta = \nu^*(K_X + D)$; that is, Θ is the sum of the inverse images of D and the singular locus of X. A *stratum* of (X, D) is an irreducible component of X or the ν-image of a log canonical center of (X^ν, Θ). This definition is compatible with Definition 2.4.

For the reader's convenience, we recall the notion of *semi-log-canonical pairs*.

Definition 2.6 (Semi-log-canonical pairs). Let X be an equidimensional algebraic variety that satisfies Serre's S_2 condition and is a normal crossing in codimension one. Let Δ be an effective $\mathbb{R}$-divisor on X such that no irreducible component of $\operatorname{Supp} \Delta$ is contained in the singular locus of X. The pair (X, Δ) is called a *semi-log-canonical pair* (an *slc pair*, for short) if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier; and
(2) (X^ν, Θ) is log canonical, where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$; that is, Θ is the sum of the inverse images of Δ and the conductor of X.

If $(X, 0)$ is a semi-log-canonical pair, then we simply say that X is a *semi-log-canonical variety* or X has only *semi-log-canonical singularities*.

For the details of semi-log-canonical pairs and the basic notation, see [Fn14] and [Kol13b].

2.7 ($\mathbb{Q}$-divisors). Let D be a $\mathbb{Q}$-divisor on an equidimensional variety X; that is, D is a finite formal $\mathbb{Q}$-linear combination

$$D = \sum_i d_i D_i$$

of irreducible reduced subschemes D_i of codimension one such that $D_i \neq D_j$ for $i \neq j$. We define the *round-up* $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$ (resp. *round-down* $\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i$), where for every real number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined

by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We set

$$D^{<0} = \sum_{d_i < 0} d_i D_i, \quad D^{>0} = \sum_{d_i > 0} d_i D_i, \quad \text{and} \quad D^{=1} = \sum_{d_i = 1} D_i.$$

2.8 (Demi-normal variety). Let X be an equidimensional variety that satisfies Serre's S_2 condition and is a normal crossing in codimension one. Then X is sometimes said to be *demi-normal* (see [Kol13b, Def. 5.1]).

Let $\pi : Y \to X$ be a finite surjective morphism between demi-normal varieties, and let D be a $\mathbb{Q}$-divisor on X such that no irreducible component of D is contained in the singular locus of X. Then there is a Zariski open set U of X such that $\mathrm{codim}_X(X \setminus U) \geq 2$ and that $D|_U$ is $\mathbb{Q}$-Cartier on U. Assume that π is étale over the generic point of any irreducible component of $\mathrm{Supp}\, D$. Then we have a well-defined $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\pi^*(D|_U)$ on $\pi^{-1}(U)$. In this situation, $\pi^{-1} D$ denotes the $\mathbb{Q}$-divisor on Y that is the closure of $\pi^*(D|_U)$. By construction, no irreducible component of $\pi^{-1} D$ is contained in the singular locus of Y.

We recall the definition of $\omega_{X/C}^{[m]}$.

Definition 2.9. In Theorem 1.7, $\omega_{X/C}^{[m]}$ is the *m-th reflexive power* of $\omega_{X/C}$. It is the double dual of the m-th tensor power of $\omega_{X/C}$:

$$\omega_{X/C}^{[m]} := (\omega_{X/C}^{\otimes m})^{**}.$$

For the details of divisors and divisorial sheaves on demi-normal varieties, see [Kol13b, §5.1].

2.10 (Lemmas on resolution of singularities for reducible varieties). We prepare the following lemmas on resolution of singularities. We will use them in Section 4.

LEMMA 2.11. *Let (Y, Δ) be a simple normal crossing pair. Let D be a Weil (resp. $\mathbb{Q}$-Weil or $\mathbb{R}$-Weil) divisor on Y such that $\mathrm{Supp}\, D \subset \mathrm{Supp}\, \Delta$. Then there exists a sequence of blow-ups*

$$Y = Y_0 \xleftarrow{\pi_1} Y_1 \xleftarrow{\pi_2} Y_2 \xleftarrow{\pi_3} \cdots \xleftarrow{\pi_k} Y_k$$

with the following properties:

(1) $(Y_0, \Delta_0) = (Y, \Delta)$.
(2) *Let U be the largest open subset of Y such that $(U, D|_U)$ is a simple normal crossing pair. Then π_i is an isomorphism over U for every $i \geq 1$.*
(3) *D_i is the strict transform of D on Y_i for every $i \geq 1$.*
(4) *π_i is a blow-up whose center is a stratum of $(Y_{i-1}, \mathrm{Supp}\, \Delta_{i-1})$ and is located outside U for every $i \geq 1$.*

(5) $\Delta_i = (\pi_i)_*^{-1}\Delta_{i-1} + E_i$, *where E_i is the π_i-exceptional divisor on Y_i for every $i \geq 1$.*

(6) (Y_k, D_k) *is a simple normal crossing pair; that is, D_k is Cartier (resp. $\mathbb{Q}$-Cartier or $\mathbb{R}$-Cartier).*

We note that (Y_i, Δ_i) is a simple normal crossing pair for every $i \geq 1$.

Proof. This is a direct consequence of [BVP13, 8. The non-reduced case].
□

LEMMA 2.12. *Let Y be an equidimensional variety that satisfies Serre's S_2 condition and is a simple normal crossing in codimension one. Let Δ_Y be an effective $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor) on Y such that no irreducible component of Δ_Y is contained in the singular locus of Y. Assume that there exists a non-empty dense Zariski open set Y_0 of Y such that (Y_0, Δ_{Y_0}) is semi-log-canonical, where $\Delta_{Y_0} = (\Delta_Y)|_{Y_0}$. Then there exists a projective surjective birational morphism $\pi : Y' \to Y$, which is a composite of blow-ups, from a simple normal crossing variety Y' with the following properties:*

(1) $\operatorname{Sing} Y'$ *maps birationally onto the closure of $\operatorname{Sing} Y^{\mathrm{snc}}$ by π, where Y^{snc} is the open subset of Y that has only smooth points and simple normal crossing points. Note that $\operatorname{Sing} Y'$ (resp. $\operatorname{Sing} Y^{\mathrm{snc}}$) is the singular locus of Y' (resp. Y^{snc}).*

(2) $\Delta_{Y'}$ *is a subboundary $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor) on Y', that is, $\Delta_{Y'} = (\Delta_{Y'})^{\leq 1}$, such that $(Y', \Delta_{Y'})$ is a simple normal crossing pair with $K_{Y'_0} + \Delta_{Y'_0} = \pi_0^*(K_{Y_0} + \Delta_{Y_0})$, where $Y'_0 = \pi^{-1}(Y_0)$, $\Delta_{Y'_0} = (\Delta_{Y'})|_{Y'_0}$, and $\pi_0 = \pi|_{Y'_0}$.*

(3) $\Delta_{Y'}$ *is the closure of $\Delta_{Y'_0}$ in Y'.*

(4) π *is an isomorphism over U, where U is the largest open subset of Y_0 such that $(U, (\Delta_Y)|_U)$ is a simple normal crossing pair.*

Proof. By [BVP13, Th. 1.4], we can construct a projective surjective birational morphism $\pi^\dagger : Y^\dagger \to Y$, which is a composite of blow-ups, from a simple normal crossing variety $Y^\dagger$ such that $\operatorname{Sing} Y^\dagger$ maps birationally onto the closure of $\operatorname{Sing} Y^{\mathrm{snc}}$ and that $(Y^\dagger, (\pi^\dagger)_*^{-1}\Delta_Y + E^\dagger)$ is a simple normal crossing pair. We note that $E^\dagger$ is a reduced Weil divisor on $Y^\dagger$ whose support coincides with the exceptional locus of $\pi^\dagger$. Of course, $\pi^\dagger$ is an isomorphism over $\widetilde{U}(\supset U)$ by construction, where $\widetilde{U}$ is the largest open subset of Y such that $(\widetilde{U}, (\Delta_Y)|_{\widetilde{U}})$ is a simple normal crossing pair. We put

$$K_{Y_0^\dagger} + \Delta_{Y_0^\dagger} = (\pi_0^\dagger)^*(K_{Y_0} + \Delta_{Y_0}),$$

where $Y_0^\dagger = (\pi^\dagger)^{-1}(Y_0)$ and $\pi_0^\dagger = \pi^\dagger|_{Y_0}$. Note that $\Delta_{Y_0^\dagger}$ is a subboundary $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor) on $Y_0^\dagger$ since (Y_0, Δ_{Y_0}) is semi-log-canonical. By

construction, we see that $\operatorname{Supp} \Delta_{Y_0^\dagger} \subset \operatorname{Supp}((\pi^\dagger)_*^{-1}\Delta_Y + E^\dagger)$. Then the closure of $\Delta_{Y_0^\dagger}$ is contained in a simple normal crossing divisor $\operatorname{Supp}((\pi^\dagger)_*^{-1}\Delta_Y + E^\dagger)$. By Lemma 2.11, we can take some sequence of blow-ups $\pi' : Y' \to Y^\dagger$, which is an isomorphism over $Y_0^\dagger$ and is an isomorphism over the generic point of every stratum of $Y^\dagger$, such that $(Y', \Delta_{Y'})$ is a simple normal crossing pair, where $\Delta_{Y'}$ is the closure of $\Delta_{Y_0'} = \Delta_{Y_0^\dagger}$ and is a subboundary $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor) on Y'. By construction, we see that this is what we wanted. $\qquad\square$

Let us consider blow-ups of simple normal crossing varieties.

LEMMA 2.13. *Let Y be a projective simple normal crossing variety. Let $\pi : Y' \to Y$ be a blow-up whose center S is a stratum of Y with $\operatorname{codim}_Y S \geq 2$. We take a Weil divisor $K_{Y'}$ on Y' such that $\omega_{Y'} \simeq \mathcal{O}_{Y'}(K_{Y'})$ and that no irreducible component of $K_{Y'}$ is contained in the singular locus of Y'. Then $\omega_Y \simeq \mathcal{O}_Y(K_Y)$ and $K_{Y'} + E = \pi^* K_Y$ hold, where $K_Y = \pi_* K_{Y'}$ and $E = \operatorname{Exc}(\pi)$; that is, E is a reduced Weil divisor on Y' whose support coincides with the exceptional locus of π. Let Δ be a Cartier divisor on Y. Then the inclusion*

$$\pi_* \mathcal{O}_{Y'}(kK_{Y'} + \Delta') \subset \mathcal{O}_Y(kK_Y + \Delta)$$

holds for every positive integer k, where $\Delta' = \pi^* \Delta$.

Proof. Since $\operatorname{codim}_Y S \geq 2$, the natural inclusion $\mathcal{O}_Y \hookrightarrow \pi_* \mathcal{O}_{Y'}$ is an isomorphism. We put $K_Y = \pi_* K_{Y'}$. Then K_Y satisfies $\omega_Y \simeq \mathcal{O}_Y(K_Y)$. By definition, $K_{Y'} - \pi^* K_Y$ is contained in the exceptional locus of π. By considering the normalizations of Y' and Y, we see that $K_{Y'} + E = \pi^* K_Y$. Since $\mathcal{O}_Y \simeq \pi_* \mathcal{O}_{Y'}$, we get the desired inclusion

$$\pi_* \mathcal{O}_{Y'}(kK_{Y'} + \Delta') \subset \pi_* \mathcal{O}_{Y'}(k(K_{Y'} + E) + \Delta') \simeq \mathcal{O}_Y(kK_Y + \Delta)$$

for every positive integer k since E is an effective divisor on Y'. $\qquad\square$

LEMMA 2.14. *Let Y be a projective simple normal crossing variety. Let $\pi : Y' \to Y$ be a blow-up whose center S is a minimal stratum of Y with $\operatorname{codim}_Y S = 1$. We take a Weil divisor K_Y on Y such that $\omega_Y \simeq \mathcal{O}_Y(K_Y)$ and that no irreducible component of K_Y is contained in the singular locus of Y. Then $\omega_{Y'} \simeq \mathcal{O}_{Y'}(K_{Y'})$ holds, where $K_{Y'} = \pi^* K_Y - E$ and $E = \operatorname{Exc}(\pi)$; that is, E is a reduced Weil divisor on Y' whose support coincides with the exceptional locus of π. Let Δ be a Cartier divisor on Y. Then the inclusion*

$$\pi_* \mathcal{O}_{Y'}(kK_{Y'} + \Delta') \subset \mathcal{O}_Y(kK_Y + \Delta)$$

holds for every positive integer k, where $\Delta' = \pi^* \Delta$.

Proof. By assumption, S is a minimal stratum of Y with $\operatorname{codim}_Y S = 1$. This means that there are two irreducible components Y_1 and Y_2 of Y such that

S is an irreducible component of $Y_1 \cap Y_2$ and that $S \cap Y_i = \emptyset$ for every irreducible component Y_i other than Y_1 and Y_2. Therefore, $\pi : Y' \to Y$ is nothing but the normalization in a neighborhood of S. Thus we see that $K_{Y'} = \pi^* K_Y - E$ satisfies $\omega_{Y'} \simeq \mathcal{O}_{Y'}(K_{Y'})$. By considering the dual of the natural inclusion $\mathcal{O}_Y \hookrightarrow \pi_* \mathcal{O}_{Y'}$, we get a generically isomorphic injection $\pi_* \omega_{Y'} \hookrightarrow \omega_Y$. By projection formula and $\pi^* K_Y = K_{Y'} + E$, we obtain the desired inclusion

$$
\begin{aligned}
\pi_* \mathcal{O}_{Y'}(kK_{Y'} + \Delta') &\subset \pi_* \mathcal{O}_{Y'}(K_{Y'} + (k-1)\pi^* K_Y + \pi^* \Delta) \\
&\simeq \pi_* \mathcal{O}_{Y'}(K_{Y'}) \otimes \mathcal{O}_Y((k-1)K_Y + \Delta) \\
&\subset \mathcal{O}_Y(kK_Y + \Delta)
\end{aligned}
$$

for every positive integer k since $\pi_* \mathcal{O}_{Y'}(K_{Y'}) \subset \mathcal{O}_Y(K_Y)$ as mentioned above. $\square$

3. A quick review of $\mathcal{M}^{\mathrm{stable}}$

In this section, we quickly review the moduli space of stable varieties (see [Kol13a]). First, let us recall the definition of *stable varieties*.

Definition 3.1 (Stable varieties). Let X be a connected projective semi-log-canonical variety with ample canonical divisor. Then X is called a *stable variety* or a *semi-log-canonical model*.

In order to obtain the boundedness of the moduli functor of stable varieties, we have to fix some numerical invariants. So we introduce the notion of the *Hilbert function* for stable varieties.

Definition 3.2 (Hilbert function of stable varieties). Let X be a stable variety. The *Hilbert function* of X is

$$
H_X(m) := \chi(X, \omega_X^{[m]}),
$$

where $\omega_X^{[m]} = (\omega_X^{\otimes m})^{**} \simeq \mathcal{O}_X(mK_X)$. By [Fn14, Cor. 1.9], we see that

$$
H_X(m) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mK_X)) \geq 0
$$

for every $m \geq 2$.

The following definition of the moduli functor of stable varieties is mainly due to Kollár. Note that a stable variety X is not necessarily Cohen–Macaulay when $\dim X \geq 3$. We think that it is one of the main difficulties when we treat families of stable varieties.

Definition 3.3 (Moduli functor of stable varieties). Let $H(m)$ be a $\mathbb{Z}$-valued function. The moduli functor of stable varieties with Hilbert function

H is

$$\mathcal{M}_H^{\mathrm{stable}}(S) := \left\{ \begin{array}{c} \text{Flat, proper families } X \to S, \text{ fibers are stable} \\ \text{varieties with ample canonical divisor} \\ \text{and Hilbert function } H(m), \; \omega_{X/S}^{[m]} \text{ is flat over } S \\ \text{and commutes with base change for every } m, \\ \text{modulo isomorphisms over } S \end{array} \right\}.$$

Remark 3.4. We consider $(f : X \to S) \in \mathcal{M}_H^{\mathrm{stable}}(S)$. By the base change theorem and [Fn14, Cor. 1.9], we obtain that $f_*\omega_{X/S}^{[m]}$ is a locally free sheaf on S with $\mathrm{rank}\, f_*\omega_{X/S}^{[m]} = H(m)$ for every $m \geq 2$.

Let us quickly review the construction of the coarse moduli space of stable varieties following [Kol13a].

3.5 (Coarse moduli space of $\mathcal{M}^{\mathrm{stable}}$). Let us consider the moduli functor

$$\mathcal{M}^{\mathrm{stable}}(S) := \left\{ \begin{array}{c} \text{Flat, proper families } X \to S, \text{ fibers are stable} \\ \text{varieties with ample canonical divisor,} \\ \omega_{X/S}^{[m]} \text{ is flat over } S \\ \text{and commutes with base change for every } m, \\ \text{modulo isomorphisms over } S \end{array} \right\}$$

of stable varieties. It is obvious that $\mathcal{M}_H^{\mathrm{stable}}$ is an open and closed subfunctor of $\mathcal{M}^{\mathrm{stable}}$. It is known that the moduli functor $\mathcal{M}^{\mathrm{stable}}$ is well behaved; that is, $\mathcal{M}^{\mathrm{stable}}$ is locally closed. For the details, see [Kol08, Cor. 25]. We already know that the moduli functor $\mathcal{M}^{\mathrm{stable}}$ satisfies the valuative criterion of separatedness and the valuative criterion of properness by Kollár's gluing theory and the existence of log canonical closures (see [Kol16, Th. 24] and [HX13, §7]). Moreover, it is well known that the automorphism group $\mathrm{Aut}(X)$ of a stable variety X is a finite group. For a more general result, see [Fn14, Cor. 6.17]. Then, by using [KM97, 1.2 Corollary], we obtain a coarse moduli space M^{stable} of $\mathcal{M}^{\mathrm{stable}}$ in the category of algebraic spaces (see, for example, [Kol13a, 5.3 (Existence of coarse moduli spaces)]). Note that M^{stable} is a separated algebraic space which is locally of finite type. Since $\mathcal{M}^{\mathrm{stable}}$ satisfies the valuative criterion of properness, M_H^{stable} is proper if and only if it is of finite type.

4. **Proof of theorems**

Let us start the proof of Theorem 1.9. Theorem 1.9 is essentially contained in [FF14, §7] (see also [FFS14]). We need no extra assumptions on D and local monodromies since C is a curve.

Proof of Theorem 1.9. In Step 1, we will reduce the problem to a simpler case by using [BVP13]. In Step 2, we will prove the desired nefness by using [FF14].

Step 1. There is a closed subset Σ of C such that every stratum of (X, D) is smooth over $C_0 = C \setminus \Sigma$. This means that every stratum of (X_0, D_0) is smooth over C_0, where $X_0 = f^{-1}(C_0)$ and $D_0 = D|_{X_0}$. Apply [BVP13, Th. 1.4] to $(X, \mathrm{Supp}(D + f^*\Sigma))$. Then we obtain a birational morphism $g : X' \to X$ from a projective simple normal crossing variety X' such that g is an isomorphism outside $\mathrm{Supp}\, f^*\Sigma$ and that $g_*^{-1}D + g^*f^*\Sigma$ has a simple normal crossing support on X'. Let D' be the horizontal part of $g_*^{-1}D$. By taking some more blow-ups, we may further assume that D' is a Cartier divisor on X' (see Lemma 2.11). We note that $g : X' \to X$ is an isomorphism over X_0 by construction. Therefore, (X', D') is a simple normal crossing pair. By construction, we have $\omega_{X'}(D') \simeq g^*\omega_X(D) \otimes \mathcal{O}_{X'}(E)$ such that $f \circ g(E) \subset \Sigma$ and that the effective part of E is g-exceptional. Thus, we have $g_*\omega_{X'}(D') \subset \omega_X(D)$. Hence we obtain an inclusion

$$f_*g_*\omega_{X'/C}(D') \to f_*\omega_{X/C}(D),$$

which is an isomorphism over C_0. Therefore, it is sufficient to prove that $f_*g_*\omega_{X'/C}(D')$ is nef by Lemma 2.2. By replacing (X, D) with (X', D'), we may assume that every stratum of (X, D) is dominant onto C. Of course, by assumption, every stratum of (X_0, D_0) is smooth over C_0.

Step 2. By [FF14, Th. 4.15], $R^d f_{0*}\iota_!\mathbb{Q}_{X_0 \setminus D_0}$, where $d = \dim X - 1$, $X_0 = f^{-1}(C_0)$, $f_0 = f|_{X_0}$, $D_0 = D|_{X_0}$, and $\iota : X_0 \setminus D_0 \hookrightarrow X_0$, underlines a graded polarizable admissible variation of $\mathbb{Q}$-mixed Hodge structure. In particular, every local monodromy on $R^d f_{0*}\iota_!\mathbb{Q}_{X_0 \setminus D_0}$ around Σ is quasi-unipotent (see [FF14, Def. 3.11]). Moreover, we can consider (upper and lower) canonical extensions of Hodge bundles (see [FF14, Def. 3.11 and Remark 7.4]). By [FF14, Th. 7.3(a)], $R^d f_*\mathcal{O}_X(-D)$ is characterized as the lower canonical extension of

$$\mathrm{Gr}_F^0(R^d f_{0*}\iota_!\mathbb{Q}_{X_0 \setminus D_0} \otimes \mathcal{O}_{C_0}).$$

We note that we can freely replace C_0 with its non-empty Zariski open set. We take a unipotent reduction $\pi : C' \to C$ of $R^d f_{0*}\iota_!\mathbb{Q}_{X_0 \setminus D_0}$ (see [Kaw81, Th. 17 and Corollary 18]). We may assume that π is a finite Galois cover (see [KMM87, Th. 1-1-1]). By shrinking C_0, we may further assume that $\pi : C' \to C$ is étale over C_0. We note that the local system $\pi_0^* R^d f_{0*}\iota_!\mathbb{Q}_{X_0 \setminus D_0}$ on $C_0' = \pi^{-1}(C_0)$ underlies a graded polarizable admissible variation of $\mathbb{Q}$-mixed Hodge structure since $R^d f_{0*}\iota_!\mathbb{Q}_{X_0 \setminus D_0}$ underlies a graded polarizable admissible variation of $\mathbb{Q}$-mixed Hodge structure, where $\pi_0 = \pi|_{C_0'} : C_0' \to C_0$. Let $\mathcal{G}$ be the canonical extension of

$$\mathrm{Gr}_F^0(\pi_0^* R^d f_{0*}\iota_!\mathbb{Q}_{X_0 \setminus D_0} \otimes \mathcal{O}_{C_0'}).$$

Then $\mathcal{G}$ is locally free and $\mathcal{G}^*$ is a nef locally free sheaf on C' (see [FF14, Cor. 5.23], [FFS14], [FF17], [Fs17], and so on). Since $R^d f_* \mathcal{O}_X(-D) \simeq (\pi_* \mathcal{G})^G$, where G is the Galois group of $\pi : C' \to C$, we obtain a nontrivial map $\pi^* R^d f_* \mathcal{O}_X(-D) \to \mathcal{G}$, which is an isomorphism on C'_0. Note that $R^d f_* \mathcal{O}_X(-D)$ is the lower canonical extension of

$$\mathrm{Gr}_F^0(R^d f_{0*} \iota_! \mathbb{Q}_{X_0 \setminus D_0} \otimes \mathcal{O}_{C_0}).$$

Therefore, by taking the dual, we obtain an inclusion $0 \to \mathcal{G}^* \to \pi^* f_* \omega_{X/C}(D)$, which is an isomorphism on C'_0. Thus, $\pi^* f_* \omega_{X/C}(D)$ is nef by Lemma 2.2. So we obtain that $f_* \omega_{X/C}(D)$ is nef because π is surjective.

Anyway, we obtain the desired nefness of $f_* \omega_{X/C}(D)$. $\qquad\square$

For an alternative proof of Theorem 1.9 based on the Kollár–Ohsawa type vanishing theorem for semi-log-canonical pairs, see [Fn15]. We note that [Fn15] depends on the theory of mixed Hodge structures on cohomology with compact support.

Remark 4.1. When X is smooth in Theorem 1.9, the semipositivity theorem obtained in [Fn04, Th. 3.9] is sufficient for the proof of Theorem 1.9. Note that [Fn04, Th. 3.9] also follows from the theory of graded polarizable admissible variations of mixed Hodge structure.

Let us prove Theorem 1.10 since it contains Theorem 1.7 as a special case.

Proof of Theorem 1.10. In Step 1, we will prove the nefness of $f_* \omega_{X/C}(D)$. In Step 2, we will treat $f_* \mathcal{O}_X(m(K_{X/C} + D))$.

Step 1. We take a double cover $\pi : \widetilde{X} \to X$ due to Kollár (see [Kol13b, 5.23]). Then $\omega_X(D)$ is a direct summand of $\pi_* \omega_{\widetilde{X}}(\pi^{-1}D)$. By replacing X and f with $\widetilde{X}$ and $f \circ \pi$, respectively, we may further assume that X is a simple normal crossing in codimension one.

We note that X and D satisfy the assumptions in Lemma 2.12 and that $(f^{-1}(U), D|_{f^{-1}(U)})$ is semi-log-canonical by assumption. Therefore, we can apply Lemma 2.12. Then we can construct a projective surjective birational morphism $h : Z \to X$, which is a composite of blow-ups, from a simple normal crossing variety Z with the following properties:

(1) There exists a subboundary $\mathbb{Q}$-divisor B on Z, that is, $B^{\leq 1} = B$, such that (Z, B) is a simple normal crossing pair.
(2) $\mathrm{Sing}\, Z$ maps birationally onto the closure of $\mathrm{Sing}\, X^{\mathrm{snc}}$ by h, where $\mathrm{Sing}\, Z$ (resp. $\mathrm{Sing}\, X^{\mathrm{snc}}$) is the singular locus of Z (resp. X^{snc}). Note that X^{snc} is the open subset of X that has only smooth points and simple normal crossing points.

(3) $K_V + B|_V = g^* \left(K_{f^{-1}(U)} + D|_{f^{-1}(U)} \right)$, where $V = (f \circ h)^{-1}(U)$ and $g = h|_V : V \to f^{-1}(U)$.

We note that the natural inclusion $\mathcal{O}_X \hookrightarrow h_* \mathcal{O}_Z$ is an isomorphism and that $Z \setminus V$ contains no irreducible components of B by construction. Thus we have

$$g_* \omega_V ((B|_V)^{=1}) \simeq \omega_{f^{-1}(U)}(D|_{f^{-1}(U)}) \quad \text{and} \quad h_* \omega_Z(B^{=1}) \subset \omega_X(D).$$

Therefore, it is sufficient to prove that $(f \circ h)_* \omega_{Z/C}(B^{=1})$ is nef by Lemma 2.2 in order to prove the nefness of $f_* \omega_{X/C}(D)$. By shrinking U suitably, we may assume that every stratum of Z maps to a point in $C \setminus U$ or is dominant onto C. If $Z \setminus V$ contains an irreducible component of B, then we take some blow-ups outside V and replace B with the closure of $B|_V$ (see Lemma 2.11). Thus, we can always assume that $Z \setminus V$ contains no irreducible components of B.

Assume that there exists a stratum S of Z in $(f \circ h)^{-1}(\Sigma)$, where $\Sigma = C \setminus U$. If $\operatorname{codim}_Z S \geq 2$ or S is a minimal stratum of Z with $\operatorname{codim}_Z S = 1$, then we take a blow-up $\pi : Z' \to Z$ of Z along S as in Lemmas 2.13 and 2.14. We note that $\pi^* B = \pi_*^{-1} B$ and that $(\pi^* B)^{=1} = \pi_*^{-1}(B^{=1}) = \pi^*(B^{=1})$. By Lemmas 2.13 and 2.14, there is a generically isomorphic inclusion

$$\pi_* \omega_{Z'}((\pi^* B)^{=1}) \subset \omega_Z(B^{=1}),$$

which is an isomorphism over U. Therefore, by Lemma 2.2, it is sufficient to prove that $(f \circ h \circ \pi)_* \omega_{Z'/C}((\pi^* B)^{=1})$ is nef. This means that we can replace (Z, B) with $(Z', \pi^* B)$. By repeating this process finitely many times, we may assume that $(f \circ h)^{-1}(\Sigma)$ contains no strata of Z. In this case, the nefness of $(f \circ h)_* \omega_{Z/C}(B^{=1})$ follows from Theorem 1.9. Anyway, we obtain that $f_* \omega_{X/C}(D)$ is a nef locally free sheaf on C.

Step 2 (see Proof of [Vie95, Cor. 2.45]). By Viehweg's clever covering trick, we can prove that $f_* \mathcal{O}_X(m(K_{X/C} + D))$ is nef for every $m \geq 2$ by using the case where $m = 1$. Here, we closely follow the proof of [Vie95, Cor. 2.45].

Let $\mathcal{H}$ be an ample line bundle on C. We set

$$r = \min \left\{ \mu \in \mathbb{Z}_{>0} \,\middle|\, (f_* \mathcal{O}_X(k(K_{X/C} + D))) \otimes \mathcal{H}^{\mu k - 1} \text{ is nef} \right\}.$$

By assumption, we have that the natural map

$$f^* f_* \mathcal{O}_X(k(K_{X/C} + D)) \to \mathcal{O}_X(k(K_{X/C} + D))$$

is surjective. Since $(f_* \mathcal{O}_X(k(K_{X/C} + D))) \otimes \mathcal{H}^{rk-1}$ is a nef locally free sheaf, $(f_* \mathcal{O}_X(k(K_{X/C} + D))) \otimes \mathcal{H}^{rk}$ is ample. Therefore, we see that

$$S^N((f_* \mathcal{O}_X(k(K_{X/C} + D))) \otimes \mathcal{H}^{rk})$$

is generated by its global sections for some positive integer N. Hence we see that $\mathcal{O}_X(K_{X/C} + D) \otimes f^* \mathcal{H}^r$ is semi-ample. More precisely, $\mathcal{O}_X(k(K_{X/C} + D)) \otimes f^* \mathcal{H}^{rk}$ is locally free and semi-ample. By the usual covering argument (see

Remark 4.2), $(f_*\mathcal{O}_X(k(K_{X/C}+D)))\otimes\mathcal{H}^{r(k-1)}$ is nef (see [Vie95, Prop. 2.43]). This is only possible if $(r-1)k-1 < r(k-1)$. It is equivalent to $r \leq k$. Therefore, $(f_*\mathcal{O}_X(k(K_{X/C}+D)))\otimes\mathcal{H}^{k^2-1}$ is nef. The same holds true if we take any base change by $\pi : C' \to C$ such that π is a finite morphism from a smooth projective curve and ramifies only over general points of C. Therefore, $f_*\mathcal{O}_X(k(K_{X/C}+D))$ is nef (see Lemma 2.3). By the same argument as above, we see that $\mathcal{O}_X(K_{X/C}+D)\otimes f^*\mathcal{H}$ is semi-ample since $f_*\mathcal{O}_X(k(K_{X/C}+D))$ is nef. More precisely, $\mathcal{O}_X(k(K_{X/C}+D))\otimes f^*\mathcal{H}^k$ is locally free and semi-ample in the usual sense. By the covering argument (see Remark 4.2), $(f_*\mathcal{O}_X(m(K_{X/C}+D)))\otimes\mathcal{H}^{m-1}$ is nef for every $m > 0$ (see [Vie95, Prop. 2.43]). The same holds true if we take any base change by $\pi : C' \to C$ such that π is a finite morphism from a smooth projective curve and ramifies only over general points of C. Therefore, $f_*\mathcal{O}_X(m(K_{X/C}+D))$ is nef for every $m > 0$ (see Lemma 2.3).

We have completed the proof of Theorem 1.10. $\square$

Remark 4.2 (see [Kol90, 4.15. Lemma and 4.16]). Let $\varphi : X' \to X$ be a cyclic cover associated to a general member $A \in |\mathcal{O}_X(kl(K_{X/C}+D))\otimes f^*\mathcal{H}^{rkl}|$ (resp. $|\mathcal{O}_X(kl(K_{X/C}+D))\otimes f^*\mathcal{H}^{kl}|$) for some positive integer l. Then $f' := f\circ\varphi : X' \to C$ and $\varphi^{-1}D$ satisfy all the assumptions for $f : X \to C$ and D. Therefore, we see that $f'_*\omega_{X'/C}(\varphi^{-1}D)$ is nef by Step 1 in the proof of Theorem 1.10. By construction, $\mathcal{O}_X(k(K_{X/C}+D))\otimes f^*\mathcal{H}^{r(k-1)}$ (resp. $\mathcal{O}_X(m(K_{X/C}+D))\otimes f^*\mathcal{H}^{m-1}$) is a direct summand of $\varphi_*\omega_{X'/C}(\varphi^{-1}D)$. Thus, we obtain that $f_*\mathcal{O}_X(k(K_{X/C}+D))\otimes\mathcal{H}^{r(k-1)}$ (resp. $f_*\mathcal{O}_X(m(K_{X/C}+D))\otimes\mathcal{H}^{m-1}$) is a nef locally free sheaf on C.

Theorem 1.2 is almost obvious by Theorem 1.7.

Proof of Theorem 1.2. We consider $(f : X \to C) \in \mathcal{M}^{\text{stable}}(C)$, where C is a smooth projective curve. By Kawakita's inversion of adjunction [Kaw07, theorem] (see also [Pat16, Lemma 2.10 and Cor. 2.11]), X itself is a semi-log-canonical variety. By the definition of $\mathcal{M}^{\text{stable}}$, we can find a positive integer k such that $\omega_{X/C}^{[k]}$ is locally free and f-ample. Hence $f_*\omega_{X/C}^{[m]}$ is nef for every $m \geq 1$ by Theorem 1.7. It implies that $\mathcal{M}^{\text{stable}}$ is semipositive in the sense of Kollár. $\square$

By Kollár's results (see [Kol13a, §§2 and 3]), Theorem 1.1 follows from Theorem 1.2. For some technical details, see also [Vie95, Ths. 4.34 and 9.25].

Proof of Theorem 1.1. It is sufficient to prove this theorem for connected subspaces. Let Z be a connected closed complete subspace of M^{stable}. It is obvious that M^{stable} has an open subspace of finite type that contains Z. By replacing $\mathcal{M}^{\text{stable}}$ with the subfunctor given by this subspace, we get a new

functor $\mathcal{N}$ that is bounded. By recalling the construction of the coarse moduli space, we know that there is a locally closed subscheme S of $\mathrm{Hilb}(\mathbb{P}^N)$ for some N such that Z is obtained as the geometric quotient $S/\mathrm{Aut}(\mathbb{P}^N)$. Let $f : X \to S$ be the universal family. By the proof of [Kol90, 2.6. Theorem], we see that $\det(f_*\omega_{X/S}^{[k]})^p$ descends to an ample line bundle on Z for a sufficiently large and divisible positive integer k and a sufficiently divisible positive integer p (see [Vie95, Lemma 9.26]). Note that [Kol90, 2.6. Theorem] needs the semi-positivity of $\mathcal{M}^{\mathrm{stable}}$. For the details, see [Kol90, §§2 and 3] and [Vie95]. $\square$

Theorem 1.10 is useful for the projectivity of the moduli space of *stable maps* (see [FP97]). For some related topics, see also [Ale96] and [KP17].

4.3 (Projectivity of the space of stable maps (see [FP97])). We freely use the notation in [FP97]. Let $\mathcal{F} = (\pi, \mathcal{C} \to S, \{p_i\}, \mu)$ be a *stable family of maps* over S to $\mathbb{P}^r$. For the definition, see [FP97, 1.1. Definitions]. We set

$$E_k(\pi) = \pi_* \left(\omega_\pi^k \left(\sum_{i=1}^n kp_i \right) \otimes \mu^*(\mathcal{O}(3k)) \right).$$

In [FP97, Lemma 3], it is proved that $E_k(\pi)$ is a nef locally free sheaf on S for $k \geq 2$ by using the results in [Kol90, §4]. This nefness is used for the projectivity of the moduli space of stable maps in [FP97, 4.3. Projectivity]. The nefness of $E_k(\pi)$ can be checked as follows:

Since $k \geq 2$, by the base change theorem, we may assume that S is a smooth projective curve. We take a general member H of $|\mu^*\mathcal{O}(3)|$. Then $(\mathcal{C}, \sum_{i=1}^n p_i + H)$ is a semi-log-canonical surface and $K_{\mathcal{C}/S} + \sum_{i=1}^n p_i + H$ is π-ample. Therefore

$$\pi_*\mathcal{O}_\mathcal{C} \left(k \left(K_{\mathcal{C}/S} + \sum_{i=1}^n p_i + H \right) \right) \simeq E_k(\pi)$$

is nef for every $k \geq 2$ by Theorem 1.10.

Let us start the proof of Theorem 1.11.

Proof of Theorem 1.11. We will use Viehweg's covering arguments (see [Vie83]) and a special case of Theorem 1.10.

We take a double cover $\pi : \widetilde{X} \to X$ due to Kollár (see [Kol13b, 5.23]). We put $\widetilde{\Delta} = \pi^{-1}\Delta$. Then $\mathcal{O}_{\widetilde{X}}(k(K_{\widetilde{X}} + \widetilde{\Delta}))$ is locally free and $f \circ \pi$-generated. Moreover, $\mathcal{O}_X(kl(K_{X/C} + \Delta))$ is a direct summand of $\pi_*\mathcal{O}_{\widetilde{X}}(kl(K_{\widetilde{X}} + \widetilde{\Delta}))$ for every $l \geq 1$. Therefore, by replacing X and Δ with $\widetilde{X}$ and $\widetilde{\Delta}$, respectively, we may further assume that $\widetilde{X}$ is a simple normal crossing in codimension one.

Since X and Δ satisfy the assumptions in Lemma 2.12, we can apply Lemma 2.12. Then there is a projective surjective birational morphism h :

$Z \to X$, which is a composite of blow-ups, from a simple normal crossing variety Z with the following properties:

(1) There is a subboundary $\mathbb{Q}$-divisor Δ_Z on Z, that is, $\Delta_Z^{\leq 1} = \Delta_Z$, such that (Z, Δ_Z) is a simple normal crossing pair.

(2) $\operatorname{Sing} Z$ maps birationally onto the closure of $\operatorname{Sing} X^{\mathrm{snc}}$ by h, where $\operatorname{Sing} Z$ (resp. $\operatorname{Sing} X^{\mathrm{snc}}$) is the singular locus of Z (resp. X^{snc}). Note that X^{snc} is the open subset of X that has only smooth points and simple normal crossing points.

(3) $K_V + \Delta_V = (h_V)^*(K_{f^{-1}(U)} + \Delta|_{f^{-1}(U)})$, where $V = (f \circ h)^{-1}(U)$, $h_V = h|_V : V \to f^{-1}(U)$, and $\Delta_V = (\Delta_Z)|_V$.

Note that the natural inclusion $\mathcal{O}_X \hookrightarrow h_*\mathcal{O}_Z$ is an isomorphism and that $Z \setminus V$ contains no irreducible components of Δ_Z by construction. Therefore, we have

$$(h_V)_*\mathcal{O}_V(k(K_V + \Delta_V^{>0})) \simeq \mathcal{O}_{f^{-1}(U)}(k(K_X + \Delta))$$

and

$$h_*\mathcal{O}_Z(k(K_Z + \Delta_Z^{>0})) \subset \mathcal{O}_X(k(K_X + \Delta)).$$

Hence it is sufficient to prove that $(f \circ h)_*\mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{>0}))$ is nef by Lemma 2.2 for the proof of the nefness of $f_*\mathcal{O}_X(k(K_{X/C} + \Delta))$. By shrinking U, we may assume that every stratum of $(Z, \operatorname{Supp} \Delta_Z)$ is smooth over U or maps to a point in $C \setminus U$. If $Z \setminus V$ contains an irreducible component of Δ_Z, then we take some blow-ups outside V and replace Δ_Z with the closure of Δ_V (see Lemma 2.11). Therefore, we can assume that $Z \setminus V$ contains no irreducible components of Δ_Z. We may further assume that $Z \setminus V$ contains no strata of $(Z, \operatorname{Supp} \Delta_Z)$ contained in $\operatorname{Supp} \Delta_Z$ by taking some more blow-ups.

Assume that there exists a stratum S of Z in $(f \circ h)^{-1}(\Sigma)$, where $\Sigma = C \setminus U$. If $\operatorname{codim}_Z S \geq 2$ or S is a minimal stratum of Z with $\operatorname{codim}_Z S = 1$, then we take a blow-up $\pi : Z' \to Z$ of Z along S as in Lemmas 2.13 and 2.14. We note that $k\Delta_Z^{\geq 0}$ is Cartier and $\pi^*\Delta_Z = \pi_*^{-1}\Delta_Z$. Of course, $\pi^*(k\Delta_Z^{\geq 0}) = k(\pi_*^{-1}\Delta_Z)^{>0}$ is Cartier. By Lemmas 2.13 and 2.14, we have a natural inclusion

$$\pi_*\mathcal{O}_{Z'}(k(K_{Z'} + (\pi^*\Delta_Z)^{>0})) \subset \mathcal{O}_Z(k(K_Z + \Delta_Z^{\geq 0})),$$

which is an isomorphism over U. Therefore, we can replace (Z, Δ_Z) with $(Z', \pi^*\Delta_Z)$ by Lemma 2.2 in order to prove the nefness of $(f \circ h)_*\mathcal{O}_Z(k(K_{Z/C} + \Delta_Z))$. By repeating this process finitely many times, we may assume that $Z \setminus V$ contains no strata of Z.

LEMMA 4.4. *Let $\pi : Z' \to Z$ be a projective surjective birational morphism from a simple normal crossing variety Z', which is a composite of blow-ups whose centers are outside V. Assume that $(Z', \pi_*^{-1}(\Delta_Z^{\geq 0}))$ is a simple normal crossing pair. Then we can write*

$$K_{Z'} + \pi_*^{-1}(\Delta_Z^{\geq 0}) = \pi^*(K_Z + \Delta_Z^{\geq 0}) + E'$$

for some effective π-exceptional $\mathbb{Q}$-divisor E' on Z'.

Proof of Lemma 4.4. This follows from the fact that $Z \setminus V$ contains no strata of Z and no strata of $(Z, \operatorname{Supp} \Delta_Z)$. □

By the above construction, we have

$$K_V + \Delta_V^{>0} = h^*(K_{f^{-1}(U)} + \Delta_{f^{-1}(U)}) + (-\Delta_V^{<0}).$$

We will apply the covering argument discussed in [Cam04, §4.4] (see also [Fn17b, §8]), which is a modification of Viehweg's covering argument in [Vie83, Lemma 5.1 and Cor. 5.2]. We set $g = f \circ h$. By taking more blow-ups over $Z \setminus V$ (see [BVP13] and Lemmas 2.11 and 4.4), we may assume that

$$\mathcal{F} := \operatorname{Image}\left(g^* g_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \to \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0}))\right)$$

is a line bundle that is g-generated and that

$$\mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \simeq \mathcal{F} \otimes \mathcal{O}_Z(E)$$

such that E is an effective Cartier divisor on Z and that $\operatorname{Supp} E$ is a simple normal crossing divisor on Z. We note that E is equal to $-k\Delta_V^{\leq 0}$ over U by construction. We may further assume that $\operatorname{Supp} E \cup \operatorname{Supp} \Delta_Z^{\geq 0}$ is a simple normal crossing divisor on Z. Let

$$g : Z \xrightarrow{\psi} \widetilde{C} \longrightarrow C$$

be the Stein factorization. Note that $g_* \mathcal{O}_Z$ is a torsion-free sheaf on C since every irreducible component of Z is dominant onto C. We also note that $\widetilde{C}$ is normal (see [Fn14, Lemma 3.6]). Then

$$(4.1) \qquad \mathcal{F} = \operatorname{Image}\left(\psi^* \psi_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \to \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0}))\right).$$

Let $\mathcal{H}$ be an ample line bundle on C. We set

$$r = \min\left\{\mu \in \mathbb{Z}_{>0} \,\Big|\, g_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \otimes \mathcal{H}^{\mu k - 1} \text{ is nef}\right\}.$$

The following lemma is essentially contained in [Vie83, Lemma 5.1 and Cor. 5.2].

LEMMA 4.5 (see [Cam04, Lemma 4.19] and [Fn17b, Lemma 8.2]). *Let $g : Z \to C$ be as above. Let $\mathcal{A}$ be an ample line bundle on C. Assume that*

$$S^N\left(g_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \otimes \mathcal{A}^k\right)$$

is generated by its global sections for some positive integer N. Then

$$g_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \otimes \mathcal{A}^{k-1}$$

is nef on C.

Proof of Lemma 4.5. By the definition of $\mathcal{F}$ and (4.1), we have

$$\psi_*\mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \simeq \psi_*\mathcal{F}.$$

This implies that

$$g_*\mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \simeq g_*\mathcal{F}.$$

Therefore, $S^N(g_*\mathcal{F} \otimes \mathcal{A}^k)$ is generated by its global sections by assumption. Hence $|(\mathcal{F} \otimes g^*\mathcal{A}^k)^N|$ is a free linear system on Z. Note that $\mathcal{F}$ is g-generated. We set

$$\mathcal{L} = \mathcal{O}_Z\left(K_{Z/C} + \Delta_Z^{=1} + k\{\Delta_Z^{\geq 0}\}\right) \otimes g^*\mathcal{A}.$$

Then we have

$$\mathcal{L}^k = \mathcal{O}_Z\left(E + (k-1)k\{\Delta_Z^{\geq 0}\}\right) \otimes \mathcal{F} \otimes g^*\mathcal{A}^k.$$

Let H be a general member of the free linear system $|(\mathcal{F} \otimes g^*\mathcal{A}^k)^N|$. Then we obtain

$$\mathcal{L}^{kN} = \mathcal{O}_Z\left(H + NE + N(k-1)k\{\Delta_Z^{\geq 0}\}\right).$$

We take a (kN)-fold cyclic cover $p : \widetilde{Z} \to Z$ associated to

$$\mathcal{L}^{kN} = \mathcal{O}_Z\left(H + NE + N(k-1)k\{\Delta_Z^{\geq 0}\}\right).$$

Note that $(\widetilde{Z}, p^*\Delta_Z^{=1})$ is a semi-log-canonical pair (see [Kol16, Th. 24]). More explicitly, $\widetilde{Z}$ can be written as follows:

$$\widetilde{Z} = \mathrm{Spec}_Z \bigoplus_{i=0}^{kN-1} \left(\mathcal{L}^{(i)}\right)^{-1},$$

where

$$\left(\mathcal{L}^{(i)}\right)^{-1} = \mathcal{L}^{-i} \otimes \mathcal{O}_Z\left(\left\lfloor \frac{i}{k}\left(E + (k-1)k\{\Delta_Z^{\geq 0}\}\right)\right\rfloor\right).$$

For the details of cyclic covers, see, for example, [EV92, §3], although [EV92, §3] only treats the case where Z is smooth. Since

$$p_*\omega_{\widetilde{Z}} \simeq \mathcal{H}om_{\mathcal{O}_Z}\left(p_*\mathcal{O}_{\widetilde{Z}}, \omega_Z\right) \simeq \mathcal{H}om_{\mathcal{O}_Z}\left(\bigoplus_{i=0}^{kN-1}\left(\mathcal{L}^{(i)}\right)^{-1}, \omega_Z\right) \simeq \bigoplus_{i=0}^{kN-1} \omega_Z \otimes \mathcal{L}^{(i)},$$

$\omega_Z \otimes \mathcal{L}^{(k-1)}$ is a direct summand of $p_*\omega_{\widetilde{Z}}$, where

$$\mathcal{L}^{(k-1)} = \mathcal{L}^{k-1} \otimes \mathcal{O}_Z\left(-\left\lfloor \frac{k-1}{k}\left(E + (k-1)k\{\Delta_Z^{\geq 0}\}\right)\right\rfloor\right).$$

We note that

$$\mathcal{O}_Z\left(K_{Z/C} + \Delta_Z^{=1}\right) \otimes \mathcal{L}^{(k-1)}$$
$$= \mathcal{O}_Z\left(k\left(K_{Z/C} + \Delta_Z^{\geq 0}\right) - \left\lfloor \frac{k-1}{k}E\right\rfloor\right) \otimes g^*\mathcal{A}^{k-1}$$
$$\subset \mathcal{O}_Z\left(k\left(K_{Z/C} + \Delta_Z^{\geq 0}\right)\right) \otimes g^*\mathcal{A}^{k-1}$$

and that E is the relative base locus of $\mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0}))$. Therefore, we have a natural isomorphism

$$g_* \left(\mathcal{O}_Z(K_{Z/C} + \Delta_{\overline{Z}}^{=1}) \otimes \mathcal{L}^{(k-1)} \right) \simeq g_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \otimes \mathcal{A}^{k-1}$$

since

$$\left\lfloor \frac{k-1}{k} E \right\rfloor \leq E$$

(see the proof of [Fn17b, Lemma 8.2]). We note that $\mathcal{O}_Z(K_Z + \Delta_{\overline{Z}}^{=1}) \otimes \mathcal{L}^{(k-1)}$ is a direct summand of $p_* \mathcal{O}_{\widetilde{Z}}(K_{\widetilde{Z}} + p^* \Delta_{\overline{Z}}^{=1})$. By a special case of Theorem 1.10, we obtain that $(g \circ p)_* \omega_{\widetilde{Z}}(p^* \Delta_{\overline{Z}}^{=1})$ is nef. Therefore, the direct summand $g_* \left(\mathcal{O}_Z(k(K_{Z/C} + \Delta_{\overline{Z}}^{=1}) \otimes \mathcal{L}^{(k-1)} \right)$ is also a nef locally free sheaf on C. This means that $g_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \otimes \mathcal{A}^{k-1}$ is a nef locally free sheaf on C. □

By the definition of r, $g_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \otimes \mathcal{H}^{rk-1}$ is nef. Therefore,

$$S^N \left(g_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \otimes \mathcal{H}^{rk} \right)$$

is generated by its global sections for some positive integer N. Then, by Lemma 4.5, we obtain that

$$g_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \otimes \mathcal{H}^{r(k-1)}$$

is nef. This is only possible if $(r-1)k - 1 < r(k-1)$. It is equivalent to $r \leq k$. Therefore,

$$g_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0})) \otimes \mathcal{H}^{k^2-1}$$

is nef. The same holds true if we take any base change by $\pi : C' \to C$ such that π is a finite morphism from a smooth projective curve and ramifies only over general points of C. Therefore,

$$g_* \mathcal{O}_Z(k(K_{Z/C} + \Delta_Z^{\geq 0}))$$

is nef (see Lemma 2.3).

Hence we obtain that $f_* \mathcal{O}_X(k(K_{X/C} + \Delta))$ is a nef locally free sheaf on C. Since $\mathcal{O}_X(kl(K_X + \Delta))$ is f-generated, by replacing k with kl in the above arguments, we obtain that $f_* \mathcal{O}_X(kl(K_{X/C} + \Delta))$ is nef for every positive integer l. □

We close this section with comments on Kollár's arguments in [Kol90, §4] for the reader's convenience.

4.6 (Comments on Kollár's arguments in [Kol90, §4]). In [Kol90, §4], Kollár essentially claims Theorem 1.7 when $\dim X = 3$. However, it is not so obvious to follow his arguments. In the last part of [Kol90, 4.14], he says

> As in the proof of 4.13 the kernel of δ is a direct summand and thus semipositive.

In [Kol90, 4.14], E is not always smooth. Therefore, it is not clear what kind of variations of Hodge structure should be considered. The map

$$(f \circ g)_* \omega_{E/C} \xrightarrow{\delta} R^1 (f \circ g)_* \omega_{X/C}$$

in [Kol90, 4.14] is different from the map

$$\delta' : (f \circ g)_* \omega_{D'/C} \to R^1 (f \circ g)_* \omega_{Z'/C}$$

in the proof of [Kol90, 4.13. Lemma] from the Hodge theoretic viewpoint. Note that D' and Z' are smooth by construction. In general, E and X are *singular* in [Kol90, 4.14]. Kollár informed the author that the nefness of $(f \circ g)_* \omega_{X/C}(E)$ can be checked with the aid of the classification of semi-log-canonical surface singularities. Note that his arguments only work for the case where the fibers are surfaces. Anyway, we do not pursue them here because the nefness of $(f \circ g)_* \omega_{X/C}(E)$ is a special case of Theorem 1.10 when $f \circ g$ is projective.

5. **Proof of corollaries**

In this final section, we prove the corollaries in Section 1.

Proof of Corollary 1.4. We know the moduli functor $\mathcal{M}_H^{\mathrm{stable}}$ is bounded by [HMX14]. Therefore, we have the coarse moduli space M_H^{stable} of $\mathcal{M}_H^{\mathrm{stable}}$ that is a complete algebraic space. Note that the moduli functor $\mathcal{M}_H^{\mathrm{stable}}$ satisfies the valuative criterion of separatedness and the valuative criterion of properness. Since the moduli functor $\mathcal{M}_H^{\mathrm{stable}}$ is semipositive in the sense of Kollár by Theorems 1.2 and 1.7, we see that M_H^{stable} is a projective algebraic scheme (see [Kol90]). □

Corollaries 1.5 and 1.6 are easy consequences of Corollary 1.4.

Proof of Corollary 1.5. The moduli functor $\mathcal{M}_H$ is an open subfunctor of $\mathcal{M}_H^{\mathrm{stable}}$ because the smoothness is an open condition. Therefore Corollary 1.5 follows from Corollary 1.4. □

Proof of Corollary 1.6. Note that any small deformations of canonical singularities are canonical (see [Kaw99, Main Theorem]). Therefore the moduli functor $\mathcal{M}_H^{\mathrm{can}}$ is an open subfunctor of $\mathcal{M}_H^{\mathrm{stable}}$. Hence, Corollary 1.6 follows from Corollary 1.4. □

References

[AH11]   D. ABRAMOVICH and B. HASSETT, Stable varieties with a twist, in *Classification of Algebraic Varieties*, *EMS Ser. Congr. Rep.*, Eur. Math. Soc., Zürich, 2011, pp. 1–38. MR 2779465. Zbl 1223.14039. https://doi.org/10.4171/007-1/1.

[Ale94]   V. Alexeev, Boundedness and K^2 for log surfaces, *Internat. J. Math.* **5** no. 6 (1994), 779–810. MR 1298994. Zbl 0838.14028. https://doi.org/10. 1142/S0129167X94000395.

[Ale96]   V. Alexeev, Moduli spaces $M_{g,n}(W)$ for surfaces, in *Higher-Dimensional Complex Varieties* (Trento, 1994), de Gruyter, Berlin, 1996, pp. 1–22. MR 1463171. Zbl 0896.14014. https://doi.org/10.1515/9783110814736.1.

[AM04]   V. Alexeev and S. Mori, Bounding singular surfaces of general type, in *Algebra, Arithmetic and Geometry with Applications* (West Lafayette, IN, 2000), Springer, Berlin, 2004, pp. 143–174. MR 2037085. Zbl 1103.14021.

[BVP13]   E. Bierstone and F. Vera Pacheco, Resolution of singularities of pairs preserving semi-simple normal crossings, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM* **107** no. 1 (2013), 159–188. MR 3031268. Zbl 1285.14014.

[Cam04]   F. Campana, Orbifolds, special varieties and classification theory, *Ann. Inst. Fourier* (*Grenoble*) **54** no. 3 (2004), 499–630. MR 2097416. Zbl 1062. 14014. https://doi.org/10.5802/aif.2027.

[EV92]   H. Esnault and E. Viehweg, *Lectures on Vanishing Theorems*, DMV Seminar **20**, Birkhäuser Verlag, Basel, 1992. MR 1193913. Zbl 0779.14003. https://doi.org/10.1007/978-3-0348-8600-0.

[Fn04]   O. Fujino, Higher direct images of log canonical divisors, *J. Differential Geom.* **66** no. 3 (2004), 453–479. MR 2106473. Zbl 1072.14019. https://doi.org/10.4310/jdg/1098137840.

[Fn11a]   O. Fujino, Fundamental theorems for the log minimal model program, *Publ. Res. Inst. Math. Sci.* **47** no. 3 (2011), 727–789. MR 2832805. Zbl 1234.14013. https://doi.org/10.2977/PRIMS/50.

[Fn11b]   O. Fujino, Recent developments in minimal model theory, *Sugaku Expositions* **24** no. 2 (2011), 205–237. MR 2882844.

[Fn14]   O. Fujino, Fundamental theorems for semi log canonical pairs, *Algebr. Geom.* **1** no. 2 (2014), 194–228. MR 3238112. Zbl 1296.14014. https://doi.org/10.14231/AG-2014-011.

[Fn15]   O. Fujino, Vanishing and semipositivity theorems for semi-log canonical pairs, 2015. arXiv 1509.00531.

[Fn17a]   O. Fujino, *Foundations of the Minimal Model Program*, MSJ Memoirs **35**, Mathematical Society of Japan, Tokyo, 2017. MR 3643725. Zbl 06726987. https://doi.org/10.2969/msjmemoirs/035010000.

[Fn17b]   O. Fujino, Notes on the weak positivity theorems, in *Algebraic Varieties and Automorphism Groups*, Adv. Stud. Pure Math. **75**, Math. Soc. Japan, Tokyo, 2017, pp. 73–118.

[Fn17c]   O. Fujino, On semipositivity, injectivity, and vanishing theorems, in *Hodge Theory and L^2-Analysis*, Adv. Lect. Math. **39**, Int. Press, Somerville, MA, 2017, pp. 245–282. Zbl 06817379.

[FF14]   O. Fujino and T. Fujisawa, Variations of mixed Hodge structure and semipositivity theorems, *Publ. Res. Inst. Math. Sci.* **50** no. 4 (2014), 589–661. MR 3273305. Zbl 1305.14004. https://doi.org/10.4171/PRIMS/145.

[FF17]   O. Fujino and T. Fujisawa, On semipositivity theorems, 2017, preprint. arXiv 1701.02039.

[FFS14]   O. Fujino, T. Fujisawa, and M. Saito, Some remarks on the semipositivity theorems, *Publ. Res. Inst. Math. Sci.* **50** no. 1 (2014), 85–112. MR 3167580. Zbl 1326.14018. https://doi.org/10.4171/PRIMS/125.

[Fs17]   T. Fujisawa, A remark on semipositivity theorems, 2017. arXiv 1710.01008.

[Ft78]   T. Fujita, On Kähler fiber spaces over curves, *J. Math. Soc. Japan* **30** no. 4 (1978), 779–794. MR 0513085. Zbl 0393.14006. https://doi.org/10.2969/jmsj/03040779.

[FP97]   W. Fulton and R. Pandharipande, Notes on stable maps and quantum cohomology, in *Algebraic Geometry* (Santa Cruz 1995), *Proc. Sympos. Pure Math.* **62**, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96. MR 1492534. Zbl 0898.14018. https://doi.org/10.1090/pspum/062.2/1492534.

[HMX14]   C. D. Hacon, J. McKernan, and C. Xu, Boundedness of moduli of varieties of general type, 2014, to appear in *J. Eur. Math. Soc.* (*JEMS*). arXiv 1412.1186.

[HX13]   C. D. Hacon and C. Xu, Existence of log canonical closures, *Invent. Math.* **192** no. 1 (2013), 161–195. MR 3032329. Zbl 1282.14027. https://doi.org/10.1007/s00222-012-0409-0.

[Has03]   B. Hassett, Moduli spaces of weighted pointed stable curves, *Adv. Math.* **173** no. 2 (2003), 316–352. MR 1957831. Zbl 1072.14014. https://doi.org/10.1016/S0001-8708(02)00058-0.

[Kar00]   K. Karu, Minimal models and boundedness of stable varieties, *J. Algebraic Geom.* **9** no. 1 (2000), 93–109. MR 1713521. Zbl 0980.14008.

[Kaw07]   M. Kawakita, Inversion of adjunction on log canonicity, *Invent. Math.* **167** no. 1 (2007), 129–133. MR 2264806. Zbl 1114.14009. https://doi.org/10.1007/s00222-006-0008-z.

[Kaw81]   Y. Kawamata, Characterization of abelian varieties, *Compositio Math.* **43** no. 2 (1981), 253–276. MR 0622451. Zbl 0471.14022. Available at http://www.numdam.org/item?id=CM_1981__43_2_253_0.

[Kaw99]   Y. Kawamata, Deformations of canonical singularities, *J. Amer. Math. Soc.* **12** no. 1 (1999), 85–92. MR 1631527. Zbl 0906.14001. https://doi.org/10.1090/S0894-0347-99-00285-4.

[KMM87]   Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the minimal model problem, in *Algebraic Geometry* (Sendai, 1985), *Adv. Stud. Pure Math.* **10**, North-Holland, Amsterdam, 1987, pp. 283–360. MR 0946243. Zbl 0672.14006.

[KM97]   S. Keel and S. Mori, Quotients by groupoids, *Ann. of Math.* (2) **145** no. 1 (1997), 193–213. MR 1432041. Zbl 0881.14018. https://doi.org/10.2307/2951828.

[KSB88]   J. Kollár and N. I. Shepherd-Barron, Threefolds and deformations of surface singularities, *Invent. Math.* **91** no. 2 (1988), 299–338. MR 0922803. Zbl 0642.14008. https://doi.org/10.1007/BF01389370.

[Kol]   J. Kollár, Families of varieties of general type, in preparation.

[Kol90]   J. Kollár, Projectivity of complete moduli, *J. Differential Geom.* **32** no. 1 (1990), 235–268. MR 1064874. Zbl 0684.14002. https://doi.org/10.4310/jdg/1214445046.

[Kol06]   J. Kollár, Non-quasi-projective moduli spaces, *Ann. of Math.* (2) **164** no. 3 (2006), 1077–1096. MR 2259254. Zbl 1140.14011. https://doi.org/10.4007/annals.2006.164.1077.

[Kol08]   J. Kollár, Hulls and husks, 2008. arXiv 0805.0576.

[Kol11]   J. Kollár, Two examples of surfaces with normal crossing singularities, *Sci. China Math.* **54** no. 8 (2011), 1707–1712. MR 2824967. Zbl 1241.14013. https://doi.org/10.1007/s11425-010-4161-x.

[Kol13a]   J. Kollár, Moduli of varieties of general type, in *Handbook of Moduli. Vol. II*, *Adv. Lect. Math.* **25**, Int. Press, Somerville, MA, 2013, pp. 131–157. MR 3184176. Zbl 1322.14006.

[Kol13b]   J. Kollár, *Singularities of the Minimal Model Program*, *Cambridge Tracts in Math.* **200**, Cambridge University Press, Cambridge, 2013, With a collaboration of Sándor Kovács. MR 3057950. Zbl 1282.14028. https://doi.org/10.1017/CBO9781139547895.

[Kol16]   J. Kollár, Sources of log canonical centers, in *Minimal Models and Extremal Rays* (Kyoto, 2011), *Adv. Stud. Pure Math.* **70**, Math. Soc. Japan, [Tokyo], 2016, pp. 29–48. MR 3617777. Zbl 1369.14013.

[Kov09]   S. J. Kovács, Young person's guide to moduli of higher dimensional varieties, in *Algebraic Geometry. Part* 2 (Seattle 2005), *Proc. Sympos. Pure Math.* **80**, Amer. Math. Soc., Providence, RI, 2009, pp. 711–743. MR 2483953. Zbl 1182.14034. https://doi.org/10.1090/pspum/080.2/2483953.

[KP17]   S. J. Kovács and Z. Patakfalvi, Projectivity of the moduli space of stable log-varieties and subadditivity of log-Kodaira dimension, *J. Amer. Math. Soc.* **30** no. 4 (2017), 959–1021. MR 3671934. Zbl 06750372. https://doi.org/10.1090/jams/871.

[Pat16]   Z. Patakfalvi, Fibered stable varieties, *Trans. Amer. Math. Soc.* **368** no. 3 (2016), 1837–1869. MR 3449226. Zbl 1375.14118. https://doi.org/10.1090/tran/6386.

[PX17]   Z. Patakfalvi and C. Xu, Ampleness of the CM line bundle on the moduli space of canonically polarized varieties, *Algebr. Geom.* **4** no. 1 (2017), 29–39. MR 3592464. Zbl 06741285. https://doi.org/10.14231/AG-2017-002.

[Vie83]   E. Viehweg, Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces, in *Algebraic Varieties and Analytic Varieties* (Tokyo, 1981), *Adv. Stud. Pure Math.* **1**, North-Holland, Amsterdam, 1983, pp. 329–353. MR 0715656. Zbl 0513.14019.

[Vie95]   E. Viehweg, *Quasi-Projective Moduli for Polarized Manifolds*, *Ergeb. Math. Grenzgeb.* **30**, Springer-Verlag, Berlin, 1995. MR 1368632. Zbl 0844.14004. https://doi.org/10.1007/978-3-642-79745-3.

[Vie10]   E. Viehweg, Compactifications of smooth families and of moduli spaces of polarized manifolds, *Ann. of Math.* (2) **172** no. 2 (2010), 809–910. MR 2680483. Zbl 1238.14009. https://doi.org/10.4007/annals.2010.172.809.

[Zuc82]   S. Zucker, Remarks on a theorem of Fujita, *J. Math. Soc. Japan* **34** no. 1 (1982), 47–54. MR 0639804. Zbl 0503.14002. https://doi.org/10.2969/jmsj/03410047.

(Received: October 29, 2012)
(Revised: November 2, 2017)

Department of Mathematics, Graduate School of Science,
Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail: fujino@math.sci.osaka-u.ac.jp