

The André-Oort conjecture for \mathcal{A}_g

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Abstract

We give a proof of the André-Oort conjecture for \mathcal{A}_g — the moduli space of principally polarized abelian varieties. In particular, we show that a recently proven “averaged” version of the Colmez conjecture yields lower bounds for Galois orbits of CM points. The André-Oort conjecture then follows from previous work of Pila and the author.

1. Introduction

Recall the statement of the André-Oort conjecture:

CONJECTURE 1.1. *Let S be a Shimura variety, and let V be an irreducible closed algebraic subvariety of S . Then V contains only finitely many maximal special subvarieties.*

For the definition of the terms “Shimura variety” and “special subvariety,” as well as a brief history of the origins of the conjecture, see [21]. In the past few decades, there has been an enormous amount of work on the André-Oort conjecture. Notably, a full proof of the conjecture under the assumption of the Generalized Riemann Hypothesis (GRH) for CM fields has been given by Klingler, Ullmo, and Yafaev [11], [21].

Following a strategy introduced by Pila and Zannier, in previous work [16] it was shown that the André-Oort conjecture for the coarse moduli space of principally polarized abelian varieties \mathcal{A}_g follows once one establishes that the sizes of the Galois orbits of special points are “large.” This is what we prove in this paper. More specifically, we prove the following:

THEOREM 1.2. *There exists $\delta_g > 0$ such that if Φ is a primitive CM type for a CM field E , and if A is any g -dimensional polarized abelian variety over $\overline{\mathbb{Q}}$ with endomorphism ring equal to the full ring of integers \mathcal{O}_E and CM type Φ ,*

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then the field of moduli¹ $\mathbb{Q}(A)$ of A satisfies

$$[\mathbb{Q}(A) : \mathbb{Q}] \gg |\text{Disc}(E)|^{\delta_g}.$$

In [16, Th. 7.1] it is proved that Theorem 1.2² implies the André-Oort conjecture for \mathcal{A}_g . Thus, we obtain the following:

THEOREM 1.3. *The André-Oort Conjecture holds for \mathcal{A}_g for any $g \geq 1$.*

Additionally, we point out that Ziyang Gao has recently shown [9] that the André-Oort conjecture for any mixed Shimura variety whose pure part is a Shimura subvariety of \mathcal{A}_g follows from Theorem 4.2.

Though it is explained in detail in [16] and other wonderful survey papers how the general strategy for André-Oort works, in an effort to be as self-contained as possible we give a short sketch of the main story in Section 6. The reader who is already familiar with this story or is interested only in the proof of Theorem 1.2 may safely skip this section.

As the argument for Theorem 1.2 is relatively short, let us first give a quick sketch of it.

1.1. *Proof sketch of Theorem 1.2.* Let E be a CM field with totally real subfield E_0 , and let Φ be a CM type for E . Now let $S(E, \Phi)$ denote a complete set of representatives up to isomorphism for the complex abelian varieties A with complex multiplication by (\mathcal{O}_E, Φ) ; that is, \mathcal{O}_E acts on A in such a way that the induced representation of E on $T_0A(\mathbb{C})$ is given by Φ . The abelian varieties in $S(E, \Phi)$ are pairwise isogenous, and the field of moduli K of all these abelian varieties is the same. Moreover, it is elementary that there are at least $|\text{Disc}(E)|^{1/4-o(1)}$ elements in $S(E, \Phi)$ — we note that the exponent of $1/4$ is irrelevant for us; we only care that it is some positive constant.

Next, Colmez [5] has proven that the Faltings heights of all the elements in $S(E, \phi)$ are the same. Moreover, he has conjectured a precise formula for the Faltings height. All that matters for our purposes is that his conjectural formula is subpolynomial in $|\text{Disc}(E)|$. Now, while Colmez’s conjecture is not yet proved, an averaged version has been announced by Andreatta, Goren, Howard, and Madapusi Pera and more recently by Yuan and Zhang, and this is sufficient to establish our desired upper bound for the Faltings height.

Finally, a theorem of Masser and Wüstholz now says that all the elements of $S(E, \phi)$ have isogenies between them of degree at most $\max(h_{\text{Fal}}, [K : \mathbb{Q}])^{c_g}$,

¹By “field of moduli” here we mean the intersection of all number fields over which the polarized abelian variety A has a model, or alternatively the field over which the point A is defined in the moduli space.

²Actually it is shown that a related statement, Theorem 5.2 of Section 5, implies the conjecture.

where h_{Fal} denotes their common Faltings height, and c_g is some positive constant depending only on the degree $2g$ of E . Since there are at most polynomially many isogenies of a given degree N , it follows that $[K : \mathbb{Q}]$ must grow at least polynomially in $|\text{Disc}(E)|$.

1.2. *Paper outline.* In Section 2 we gather the basic facts we need about Faltings heights, CM varieties, and the theorem of Masser-Wüstholz. In Section 3 we show how the average version of the Colmez conjecture yields upper bounds for Faltings heights of CM abelian varieties. In Section 4 we use these upper bounds together with the theorem of Masser and Wüstholz to prove our desired lower bounds on Galois orbits. In Section 5 we recall how these lower bounds imply the André-Oort conjecture for \mathcal{A}_g . In Section 6, we give a brief sketch for the interested reader of the complete proof of André-Oort, putting together the ingredients in the literature to explain the complete argument for the interested reader.

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2. Background

2.1. *Faltings height.* Let A be an abelian variety over $\overline{\mathbb{Q}}$. Let K be any subfield of $\overline{\mathbb{Q}}$ such that A is definable over K with everywhere semistable reduction. Now let $\pi : \mathcal{A} \rightarrow \text{Spec } \mathcal{O}_K$ be the Néron model of A , and take ω to be any nonzero global section of $\mathcal{L} := \pi_* \Omega_{\mathcal{A}/\text{Spec } \mathcal{O}_K}^g$. The (stable) Faltings height of A is defined as follows [8]:

$$h_{\text{Fal}}(A) := \frac{1}{[K : \mathbb{Q}]} \left(\log |H^0(\text{Spec } \mathcal{O}_K, \mathcal{L}) : (\mathcal{O}_K \cdot \omega)| - \frac{1}{2} \sum_{\sigma: K \rightarrow \mathbb{C}} \log \left| \int_{A(\mathbb{C})} \sigma(\omega \wedge \overline{\omega}) \right| \right).$$

This turns out to be a well-defined quantity independent of the choice of K and ω . It can be thought of as a measure of the arithmetic complexity of A . We shall need the following theorem of Bost [3]:

THEOREM 2.1. *There exists a constant c_g depending on g alone such that for A an abelian variety over $\overline{\mathbb{Q}}$ of dimension g , $h_{\text{Fal}}(A) \geq c_g$. In fact, one can take c_g to be linear in g .*

2.2. Complex multiplication. Let E be a CM field with totally real subfield E_0 , set $g = [E_0 : \mathbb{Q}]$, and let Φ be a CM type of E . That is, Φ is a set of embeddings $E \hookrightarrow \mathbb{C}$ such that the set of all embeddings is given by the disjoint union of Φ and $\overline{\Phi}$. Next, define $S(E, \Phi)$ to be the set of isomorphism classes of g -dimensional complex abelian varieties A together with an embedding $\mathcal{O}_E \rightarrow \text{End}_{\mathbb{C}}(A)$ such that the induced action of E on the tangent space $T_0A(\mathbb{C}) \cong \mathbb{C}^g$ is given by Φ . It is known that the number of elements in $S(E, \Phi)$ is given by [17, Prop. 7.17]

$$|S(E, \Phi)| = |\text{Cl}(E)|,$$

where $\text{Cl}(F)$ denotes the class number of a field F .

In general, class numbers of algebraic number fields can be very small, but this is not so for CM fields. The Brauer-Siegel theorem [4], the fact that the regulators of E and E_0 are the same, and the fact that $|\text{Disc}(E)| \geq |\text{Disc}(E_0)|^2$ imply that

$$|S(E, \Phi)| \gg_g \frac{|\text{Cl}(E)|}{|\text{Cl}(E_0)|} \gg_g |\text{Disc}(E)|^{1/4 - o_g(1)}.$$

Now, suppose that Φ is a primitive CM type, which means that it is not induced from a CM type of a strict CM subfield of E . Then $\text{End}_{\mathbb{C}}(A) \cong \mathcal{O}_E$ for any $A \in S(E, \Phi)$, so isogenies between abelian varieties in $S(E, \Phi)$ necessarily respect the \mathcal{O}_E action. Finally, all the elements in $S(E, \Phi)$ are isogenous, and isogenies correspond to ideals in \mathcal{O}_E in the following way [17, §7.6, Prop. 22]:

Fix $A \in S(E, \Phi)$, and for a nonzero integral ideal I , let T_I be the kernel of I acting on A . Then the map sending I to the isogeny $A \rightarrow A/T_I$ is a bijection between the set of nonzero integral ideals and the set of isomorphism classes of pairs (B, ψ) , where $B \in S(E, \Phi)$ and ψ is an isogeny from A to B . Moreover, the norm of I equals the degree of the isogeny arising from I . Since there are $n^{o_g(1)}$ ideals of norm n , we have proven

PROPOSITION 2.2. *For a primitive CM type Φ , there are two elements in $S(E, \Phi)$ such that the lowest degree isogeny between them has degree at least $|\text{Disc}(E)|^{1/4 - o_g(1)}$.*

2.3. Masser-Wüstholz isogeny theorem. We shall make heavy use of the following theorem of Masser-Wüstholz [12]:

THEOREM 2.3. *Let A, B be abelian varieties of dimension g over a number field k , and suppose that there exists an isogeny between them over \mathbb{C} . Then if we let N be the minimal degree of an isogeny between them over \mathbb{C} , we have*

the bound

$$N \ll_g \max(h_{\text{Fal}}(A), [k : \mathbb{Q}])^{c_g},$$

where c_g is a positive constant depending only on g .

This theorem is proved in [12], even though it is stated a little differently; see Theorem II and the last paragraph on page 23 there.

3. Colmez’s conjecture

It is a theorem of Colmez [5, Th. II.2.10,(ii)] that all the elements in $S(E, \Phi)$ have the same Faltings height denoted by $h_{\text{Fal}}(E, \Phi)$. Moreover, he conjectured a precise formula [5, Conj. II.2.11] for this height as follows:

CONJECTURE 3.1. *We have the identity*

$$h_{\text{Fal}}(E, \Phi) = \sum_{\rho} c_{\rho, \Phi} \left(\frac{L'(0, \rho)}{L(0, \rho)} + \frac{\log f_{\rho}}{2} \right),$$

where ρ ranges over irreducible complex representations of the Galois group of the normal closure of E for which $L(0, \rho)$ does not vanish, $c_{\rho, \Phi}$ are rational numbers depending only on the finite combinatorial data given by Φ and the Galois group of the normal closure of E , and f_{ρ} is the Artin conductor of ρ .

While this conjecture is still open, the following “averaged” version has been proven by Andreatta, Goren, Howard and Madapusi-Pera [2] (for their analogue in the orthogonal case, see [1]), and independently by Yuan and Zhang [22].

THEOREM 3.2. *Colmez’s conjecture holds if one averages over all CM types, up to a small error. Precisely,*

$$\sum_{\Phi} h_{\text{Fal}}(E, \Phi) = \sum_{\Phi} \left(\sum_{\rho} c_{\rho, \Phi} \left(\frac{L'(0, \rho)}{L(0, \rho)} + \frac{\log f_{\rho}}{2} \right) \right),$$

where the outer sum is over all 2^g CM types of E .

We shall use the above theorem only to prove the following corollary:

COROLLARY 3.3. *Let E be a CM field with $[E : \mathbb{Q}] = 2g$, Φ be a primitive CM type, and $A \in S(E, \Phi)$. We have the bound*

$$h_{\text{Fal}}(A) \leq |\text{Disc}(E)|^{o_g(1)}.$$

Proof. Applying Theorem 2.1 to every term in $\sum_{\Phi} h(E, \Phi)$ except the one corresponding to the CM type of A shows that

$$h_{\text{Fal}}(A) \leq -(2^g - 1)c_g + \sum_{\Phi} h_{\text{Fal}}(E, \Phi),$$

where c_g is a positive constant depending only on g . Thus it suffices to show that

$$\sum_{\Phi} h_{\text{Fal}}(E, \Phi) \leq |\text{Disc}(E)|^{o_g(1)}.$$

We shall do this by showing that every term on the right-hand side of Theorem 3.2 is bounded above by $|\text{Disc}(E)|^{o_g(1)}$.

Firstly, for any irreducible Artin representation ρ , we have $f_{\rho} \leq |\text{Disc}(E)|$, and so

$$\log f_{\rho} \leq |\text{Disc}(E)|^{o_g(1)}.$$

Next, if we logarithmically differentiate the functional equation for the Artin L -function, we obtain that $\frac{L'(1, \rho)}{L(1, \rho)} + \frac{L'(0, \bar{\rho})}{L(0, \bar{\rho})} = O_g(\log f_{\rho})$, and thus it suffices to bound $\left| \frac{L'(1, \bar{\rho})}{L(1, \bar{\rho})} \right|$.

By Brauer’s theorem on induced characters, every Artin L -function is a product of quotients of Hecke L -functions, and so by the Brauer-Siegel theorem we have the estimate

$$L(1, \bar{\rho}) = |\text{Disc}(E)|^{o_g(1)}.$$

Finally, by Cauchy’s theorem we can express $L'(1, \bar{\rho})$ as an average of $L(s, \bar{\rho})$ over an arbitrary small circle centered around 1 in the complex plane. Taking the circle to have radius $\varepsilon > 0$ and using the standard convexity estimate for $L(s, \bar{\rho})$ [10, (5.2)] now yields

$$L'(1, \bar{\rho}) \ll_{\varepsilon} |\text{Disc}(E)|^{\varepsilon + o_g(1)}.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof. □

4. Lower bounds for Galois orbits

We are now ready to prove Theorem 1.2 (restated as Theorem 4.2 here for the reader’s convenience). We begin with the following lemma, which we are grateful to the referee for suggesting:

LEMMA 4.1. *Let A be an abelian variety, and let $\mathbb{Q}(A)$ be the field of moduli of A . Then there exists a field $\mathbb{Q}(A)'$ such that all the endomorphisms and polarizations of A are defined over $\mathbb{Q}(A)'$ and $[\mathbb{Q}(A)' : \mathbb{Q}(A)] \leq 2 \cdot 3^{4g^2}$.*

Proof. Let $\mathbb{Q}(A)'$ be the compositum of $\mathbb{Q}(e^{2\pi i/3})$ and the field of moduli of A equipped with a basis for $A[3]$. Since A equipped with a basis of $A[3]$ is rigid, A is definable over $\mathbb{Q}(A)'$. Then if B is the dual abelian variety to A , since $B[3] \cong \text{Hom}(A[3], \mu_3)$, the points of $B[3]$ are also defined over $\mathbb{Q}(A)'$. The lemma now follows from [19, Prop 2.3]. □

THEOREM 4.2. *There exists $\delta_g > 0$ such that if E is a CM field of degree $2g$, Φ is a primitive CM type for E , and A is an abelian variety in $S(E, \Phi)$, then the field of moduli $\mathbb{Q}(A)$ of A satisfies*

$$[\mathbb{Q}(A) : \mathbb{Q}] \gg |\text{Disc}(E)|^{\delta_g}.$$

Proof. Note that by Proposition 2.2, there are two elements A and B in $S(E, \Phi)$ such that the minimal isogeny between them is of degree at least $|\text{Disc}(E)|^{1/4 - o_g(1)}$. Let $\mathbb{Q}(A)', \mathbb{Q}(B)'$ be as in Lemma 4.1. Now we can write B as A/T_I for some ideal $I \subset \mathcal{O}_E$, where T_I is the kernel of I acting on A . Since A with its endomorphisms is defined over $\mathbb{Q}(A)'$, it follows that B can be defined over $\mathbb{Q}(A)'$ as well, so $\mathbb{Q}(B) \subset \mathbb{Q}(A)'$. Thus $K_{A,B} := \mathbb{Q}(A)'\mathbb{Q}(B)'$ is a field over which A, B together with a basis for their 3-torsion can be defined, and so by [19, Prop 2.3] all elements of $\text{Hom}(A, B)$ are defined over $K_{A,B}$. Thus applying Theorem 2.3 and Corollary 3.3 we learn that $[K_{A,B} : \mathbb{Q}]^{c_g} \geq |\text{Disc}(E)|^{1/4 - o(1)}$. Since $[K_{A,B} : \mathbb{Q}(A)] \leq 4 \cdot 3^{8g^2}$, this proves the result for any $\delta_g < \frac{1}{4c_g}$. \square

5. The André-Oort Conjecture

Recall the statement of the André-Oort conjecture for \mathcal{A}_g :

CONJECTURE 5.1. *Let V be an irreducible closed algebraic subvariety of \mathcal{A}_g . Then V contains only finitely many maximal special subvarieties.*

For a point $x \in \mathcal{A}_g(\overline{\mathbb{Q}})$, let A_x denote the corresponding g -dimensional principally polarized abelian variety, $R_x = Z(\text{End}(A_x))$ the centre of the endomorphism ring of A_x , and $\text{Disc}(R_x)$ the discriminant of R_x . In general we have the following lower bound conjectured by Edixhoven in [7]:

THEOREM 5.2. *Let $g \geq 1$. There exists a constant $b_g > 0$ such that, for a special point $x \in \mathcal{A}_g$,*

$$|\text{Disc}(R_x)| \ll_g |\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x|^{b_g}$$

(with the implied constants depending on g).

Proof. Let K be a CM field of degree $2g$ with a CM type ϕ , and let L be the normal closure of K over \mathbb{Q} . Let (K^*, ϕ^*) denote the dual CM field and dual CM type of (K, ϕ) . Then there is a reciprocity homomorphism between ideal groups $t_{K,\phi} : I_{K^*} \rightarrow I_K$ defined as follows: let ϕ_L denote the set of embeddings $L \rightarrow \overline{K}$ extending all the embeddings in ϕ . Then the Galois group $\text{Gal}(L/K^*)$ is the right-stabilizer in $\text{Gal}(L/\mathbb{Q})$ of ϕ_L . Then if I is a fractional ideal of K^* , let $t_{K,\phi}(I)$ be the unique fractional ideal in K satisfying

$$t_{K,\phi}(I)\mathcal{O}_L = \prod_{\sigma \in \phi_L/\text{Gal}(L/K^*)} \sigma(I\mathcal{O}_L).$$

That such an ideal exists can be seen as follows: the map

$$t'_{K,\phi}(\alpha) = \prod_{\sigma \in \phi_L / \text{Gal}(L/K^*)} \sigma(\alpha)$$

is a map from $(K^*)^\times$ to K^\times , and $t_{K,\phi}$ is the induced map on ideals.

Then $t_{K,\phi}$ descends to give a map $r_{K,\phi}$ on class groups. Let \widetilde{r}_K denote the composition $r_{K,\phi} \circ \text{Nm}_{L/K^*}$, where Nm_{L/K^*} is the norm map on class groups. In [20, Th. 7.1] it is shown that Theorem 5.2 follows from the following purely field-theoretic statement: there exists a positive constant $\delta(g)$ depending only on g such that for any pair (K, ϕ) with ϕ a primitive CM-type,

$$(1) \quad |\text{im}(\widetilde{r}_K)| \gg_{g,\varepsilon} \text{Disc}(K)^{\delta(g)-\varepsilon}.$$

Moreover, for $A \in S(K, \phi)$, by [17, §15, Main Theorem 1], $[\mathbb{Q}(A) : \mathbb{Q}]$ and $|\text{im}(r_{K,\phi})|$ are almost the same size. Since we cannot find an adequate reference, we explain this point in some detail:

First, by class field theory the homomorphism Nm_{L/K^*} has index in $\text{Cl}(K^*)$ bounded by $[L : K^*]$ which is $O_g(1)$, so we focus on the homomorphism $r_{K,\phi}$.

Define the subgroup H of ideal classes $[I]$ in $\text{Cl}(K^*)$ such that there exists an element $a \in K^\times$ with $t_{K,\phi}(I) = (a)$ and $\text{Norm}_{K/\mathbb{Q}}(I) = a\bar{a}$ for some $a \in K^\times$. Note that the condition is independent of which ideal representative for $[I]$ one chooses. Then $[\mathbb{Q}(A) : \mathbb{Q}] = |\text{Cl}(K^*)|/|H|$ by [17, §15, Main Theorem 1].

Now, clearly H is contained in the kernel of $r_{K,\phi}$. Moreover, for $[I]$ in the kernel of $r_{K,\phi}$, we can find $a \in K^\times$ such that $t_{K,\phi}(I) = (a)$. This implies that $\text{Norm}_{K/\mathbb{Q}}(I) = r_{K,\phi}(I)\overline{r_{K,\phi}(I)} = a\bar{a}\mathcal{O}_K$, so that $\frac{a\bar{a}}{|\mathcal{O}_K/I|}$ is a totally positive unit of K . Moreover, a is well defined up to an element of the unit group U_K , and so $\frac{a\bar{a}}{\text{Norm}_{K/\mathbb{Q}}(I)}$ is well defined up to an element of $U_K^+/N(U_K)$, where U_K^+ denote the totally positive units and $N(z) = z\bar{z}$. Therefore, H is the kernel of the homomorphism from $\ker r_{K,\phi}$ to $U_K^+/N(U_K)$. This latter group is a quotient of $U_K^+/(U_K^+)^2$, which is of size at most 2^g . Thus, we learn that the index of H in the kernel of $r_{K,\phi}$ is $O_g(1)$, and so

$$[\mathbb{Q}(A) : \mathbb{Q}] = |\text{Cl}(K^*)|/|H| \ll_g |\text{Cl}(K^*)|/|\ker r_{K,\phi}| = |\text{im}(r_{K,\phi})|.$$

The theorem now follows from Theorem 1.2 combined with the fact that $S(K, \phi)$ is not empty. □

In [16, Th. 7.1] it is proved that Theorem 5.2 implies the André-Oort conjecture for \mathcal{A}_g . Thus, we obtain the following:

THEOREM 5.3. *The André-Oort Conjecture holds for \mathcal{A}_g for any $g \geq 1$.*

6. Sketch of the complete proof of André-Oort

In this section, for the reader’s convenience we outline the proof of Theorem 1.3 from “first principles,” quoting all the big results we need from the literature, pointing out the use of the main ingredients [16], [14] and Theorem 1.2, which, as we have seen, relies on the Masser-Wüstholz theorem and on the averaged Colmez conjecture proved in [2] and in [22].

6.1. *Setup.* Let V be an irreducible subvariety of \mathcal{A}_g , and suppose for the sake of contradiction that V violates Conjecture 5.1. Since CM points are defined over $\overline{\mathbb{Q}}$, by replacing V with the irreducible components of the Zariski closure of its CM points we may assume that V is defined over $\overline{\mathbb{Q}}$ as well. Let L be a number field over which V is defined. We adopt the following notation: if F and G are positive functions depending on various variables, we write $F \prec_S G$ if there exist constants $A, B > 0$ depending only on the set S , such that it is identically true that $F \leq A \cdot G^B$.

We begin with the observation that \mathcal{A}_g has a uniformization by the Siegel upper half plane \mathbb{H}_g of symmetric $g \times g$ matrices whose imaginary part is positive definite. Namely, there is a covering map $\pi : \mathbb{H}_g \rightarrow \mathcal{A}_g$, and an action of $\Gamma := \mathrm{Sp}_{2g}(\mathbb{Z})$ on \mathbb{H}_g such that π is invariant under the action of Γ and induces an isomorphism $\pi : \Gamma \backslash \mathbb{H}_g \rightarrow \mathcal{A}_g$. Now \mathbb{H}_g is a symmetric space and there is a well-known fundamental domain [18, §VI] $F \subset \mathbb{H}_g$ for the action of Γ with nice properties, so that π induces a homeomorphism from F to an open dense subset (in the complex analytic topology) of \mathcal{A}_g .

6.2. *Getting polynomially many points of small height in $\pi^{-1}(V) \cap F$.* Denote by \overline{F} the closure of F in \mathbb{H}_g . Now, even though the map π is highly transcendental, the pullback under $\pi|_{\overline{F}}$ of a CM point x of \mathcal{A}_g is an algebraic point y of degree bounded by $2g$ (or perhaps $O(g)$ such points if the preimage lies on the boundary of \overline{F}). Moreover, this pullback is not “too big” — in the sense that the naive height $H(y)$ is polynomially bounded in $|\mathrm{Disc}R_x|$ (see Theorem 3.1 in [15]). Now take a CM point $x \in V$, and set $X = H(y)$. Then the orbit of x under the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ is also in V . Now, we have

$$H(y) \prec |\mathrm{Disc}R_x| \prec \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x \ll_L \mathrm{Gal}(\overline{\mathbb{Q}}/L) \cdot x,$$

where the second inequality follows from Theorem 5.2. Finally, note that if x' and x are in the same $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ -orbit, then $R_{x'} \cong R_x$. So if $Y := \{y' \in F \mid \pi(y') \in \mathrm{Gal}(\overline{\mathbb{Q}}/L) \cdot x\}$, then $H(y') \prec_L Y$ for every $y' \in Y$.

For any set S , let $N(S, X)$ be the number of degree $2g$ algebraic points of S whose height is at most X . If x and Y are as above, and $X := \max_{y' \in Y} H(y')$, then $N(\pi^{-1}(V) \cap F, X) \geq |Y|$, which is at least a fixed positive power of X by the inequality above. Since V contains infinitely many CM points x by assumption, the heights of the bounded-degree points $y \in F$ mapping to

them must tend to infinity by Northcott's theorem, so X can be taken to be arbitrarily large. Thus we have shown that there are arbitrarily large X such that $N(\pi^{-1}(V) \cap F, X)$ is bounded below by a fixed power of X .

6.3. *Getting an algebraic subvariety in $\pi^{-1}(V) \cap F$.* Now, while $\pi^{-1}(V) \cap F$ is not an algebraic or even semialgebraic set, it turns out that it can be defined using subanalytic functions, together with the exponential function. That is to say, it is definable in the structure $\mathbb{R}_{\text{an,exp}}$ [13]. This is in some sense a very weak property of a set, but it turns out to be surprisingly useful. The key is that $\mathbb{R}_{\text{an,exp}}$ is an o-minimal structure. We will not say more on this topic here, and the interested reader should see [6]. We shall only need the following extremely powerful theorem of Pila and Wilkie [14]:

THEOREM 6.1. *For any set T definable in $\mathbb{R}_{\text{an,exp}}$, let T^{alg} denote the union of all connected, positive dimensional semialgebraic sets in T . Then $N(T \setminus T^{\text{alg}}, X)$ grows subpolynomially in X .*

Combining the above theorem with the result of the previous section, it follows that $\pi^{-1}(V) \cap F$ contains a semialgebraic set W .

6.4. *Getting a special subvariety in V .* We are now in the strange situation that the pullback of an algebraic set $\pi^{-1}(V)$ contains a semialgebraic set W , even though π is transcendental. It turns out that this can happen only if a special subvariety is involved. Formally, V contains a special subvariety S such that $\pi^{-1}(S)$ contains W . This is known as the hyperbolic Ax-Lindemann theorem, and is the main result of [16, Th. 6.1].

6.5. *Finishing up.* By applying the above arguments to only those special points not lying on positive-dimensional special subvarieties we have proved that all but a finite number of the CM points on V lie on a positive-dimensional special subvariety of V . To handle the positive dimensional special subvarieties, we reduce to the case of special points as follows.

Since special subvarieties in V are images of group orbits in \mathbb{H}_g , an o-minimality argument quickly shows that such orbits occur in finitely many algebraic families. This implies that all the maximal special subvarieties of V occur in finitely many algebraic families. More precisely, one has finitely many finite maps $V_0 \times S \rightarrow V$, where S is a Shimura variety, V_0 is a subvariety of another Shimura variety S' such that V_0 contains no special subvarieties of S' , and the maximal special subvarieties of V occur as $\{x\} \times S$, where $\{x\}$ is a special point of V_0 . Theorem 1.3 thus follows by applying what we have proven so far to the subvariety V_0 of the Shimura variety S' . See [16, §7] for this argument done in full.

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