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Nodal sets of Laplace eigenfunctions: proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture

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Abstract

Let u be a harmonic function in the unit ball $B(0,1) \subset \mathbb{R}^n$, $n \geq 3$, such that $u(0) = 0$. Nadirashvili conjectured that there exists a positive constant c, depending on the dimension n only, such that

$$H^{n-1}(\{u = 0\} \cap B) \geq c.$$

We prove Nadirashvili's conjecture as well as its counterpart on C^∞-smooth Riemannian manifolds. The latter yields the lower bound in Yau's conjecture. Namely, we show that for any compact C^∞-smooth Riemannian manifold M (without boundary) of dimension n, there exists $c > 0$ such that for any Laplace eigenfunction φ_λ on M, which corresponds to the eigenvalue λ, the following inequality holds: $c\sqrt{\lambda} \leq H^{n-1}(\{\varphi_\lambda = 0\})$.

1. **Introduction.**

Let M be a C^∞ smooth Riemannian manifold (with or without boundary) of dimension n. Let B be a geodesic ball on M with radius 1. Assume $\lambda > 0$. Consider any solution of the equation $\Delta u + \lambda u = 0$ in B (the boundary conditions for u do not matter), and denote the zero set of u by Z_u. We prove the following result:

Theorem 1.1. *There exist $c > 0$ and λ_0, depending on M and B only, such that if $\lambda > \lambda_0$, then*

$$c\sqrt{\lambda} \leq H^{n-1}(Z_u \cap B).$$

We prove a similar result for harmonic functions, which was conjectured by Nadirashvili ([12]):

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Theorem 1.2. *There exists $c > 0$, depending on M and B only, such that for any harmonic function h on B that vanishes at the center of B, the following estimate holds*:

$$c \leq H^{n-1}(Z_h \cap B).$$

As an immediate corollary from Theorem 1.2 we obtain that if h is a non-constant harmonic function in $\mathbb{R}^3$, then the zero set of h has an infinite area. Apparently, this is also a new result.

Theorems 1.1 and 1.2 are related to each other by a standard trick that allows us to pass from Laplace eigenfunctions to harmonic functions. If u satisfies $\Delta u + \lambda u = 0$ on M, then one can consider the harmonic function

$$h(x,t) = u(x) \exp(\sqrt{\lambda} t)$$

on the product manifold $M \times \mathbb{R}$. The zero set of h and the zero set of u are related by

$$Z_h = Z_u \times \mathbb{R}.$$

Then Theorem 1.1 will follow in a straightforward way from the $\frac{1}{\sqrt{\lambda}}$-scaled version of Theorem 1.2 and the fact that Z_u is $\frac{\text{const}}{\sqrt{\lambda}}$ dense in B. The latter fact, which is well known, is the corollary of the Harnack inequality for harmonic functions. For the reader's convenience, we present the proof of this fact in Section 8, where we also deduce Theorem 1.1 from the scaled version of Theorem 1.2.

Most of this paper is devoted to the proof of Nadirashvili's conjecture.

Nadirashvili's conjecture was motivated by the question of Yau, who conjectured that if M is a compact C^∞-smooth Riemannian manifold with no boundary, then there exist $c, C > 0$, depending on M only, such that the Laplace eigenfunctions φ_λ on M (φ_λ corresponds to the eigenvalue λ) satisfy

$$c\sqrt{\lambda} \leq H^{n-1}(\varphi_\lambda = 0) \leq C\sqrt{\lambda}.$$

The lower bound for Yau's conjecture in dimension 2, which is not difficult, was proved by Brüning and also by Yau. In dimension $n \geq 3$ the lower bound for Yau's conjecture follows now from Theorem 1.1.

For the case of real-analytic metrics, the Yau conjecture was proved by Donnelly and Fefferman [3]. Theorems 1.1 and 1.2 do not follow from the Donnelly-Fefferman argument and are new in the case $M = \mathbb{R}^n$, $n \geq 3$, endowed with the standard Euclidean metric. Roughly speaking, Nadirashvili's conjecture implies the lower bound for Yau's conjecture and gives additional information on small scales. The assumption of real analyticity of the metric seems to be of no help for the question of Nadirashvili, but it was exploited by Donnelly and Fefferman to establish the lower and upper bounds in Yau's conjecture.

Concerning the upper bounds for Yau's conjecture without real-analyticity assumptions, Donnelly and Fefferman ([4]) proved that in dimension $n = 2$ the following estimate holds:

$$H^1(u = 0) \leq C\lambda^{3/4}.$$

Recently this upper bound was refined to $C\lambda^{3/4-\varepsilon}$ in [10], which we advise be read before this paper.

In higher dimensions Hardt and Simon ([7]) showed that

$$H^{n-1}(u = 0) \leq C\lambda^{C\sqrt{\lambda}}.$$

Recently an upper bound with polynomial growth was obtained in [9]:

$$H^{n-1}(u = 0) \leq C\lambda^{C}.$$

In this paper we use techniques of propagation of smallness developed in [10] and [9].

We refer to [14] and [12] for the interesting conjectures on Laplace eigenfunctions and harmonic functions. See also [1], [2], [13], [3] for the previous results on the lower bounds. This paper is self contained with the exception of Theorem 5.1, which was borrowed from [9].

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2. **Almost monotonicity of the frequency**

Given a point O on a Riemannian manifold M, let us consider normal coordinates with center at O. We will identify a neighborhood of O on M with a neighborhood of the origin in the Euclidean space. Now, we have two metrics: the Euclidean metric, which we will denote by $d(x, y)$, and the Riemannian metric $d_g(x,y)$. The symbol $B(x,r)$ will denote the ball with center at x and radius r in Euclidean metric. while $B_g(x,r)$ is used for the geodesic ball with respect to g. The radius r will always be smaller than the injectivity radius. Due to the choice of normal coordinates for any $\varepsilon > 0$, there is a sufficiently small $R_0 = R_0(\varepsilon, M, g, O) > 0$ such that

$$\frac{d(x,y)}{d_g(x,y)} \in (1-\varepsilon, 1+\varepsilon) \tag{1}$$

for any two distinct points x, y in $B_g(O, R_0)$. We will always assume that R_0 is sufficiently small. In particular, we assume that (1) holds with $\varepsilon = 1/2$.

Throughout the paper the words "cube" and "box" (hyperrectangle) will be used in the standard Euclidean sense. The reason why we need two metrics, but not one, is because we will frequently partition cubes into smaller cubes, and the combinatorial geometry ideas are easier to describe in $\mathbb{R}^n$ than on a manifold. We kindly advise the reader to think that M is $\mathbb{R}^n$, to throw away half of the used notations, and to remove the ε error term in the monotonicity property for the frequency function defined below.

Let u be a harmonic function on M. Given a ball $B_g(x,r)$, define the function

$$H(x,r) = \int_{\partial B_g(x,r)} |u|^2 dS_r,$$

where S_r is the surface measure on $\partial B_g(x,r)$.

We will use a slightly non-standard definition of the frequency function:

$$\beta(x,r) = \frac{rH'(x,r)}{2H(x,r)}.$$

Our definition is slightly different from the one in [6], [5], and [8]; in particular, in the case of ordinary harmonic functions in $\mathbb{R}^n$ we do not normalize $H(r)$ by the total surface area $|S_r|$. Sometimes we will specify the dependence of β and H on u and write $H_u(x,r)$ and $\beta_u(x,r)$. The frequency is almost monotonic in the following sense:

For any $\varepsilon > 0$, there exists $R_0 > 0$ such that if $r_1 < r_2 < R_0$ and $d_g(x, O) < R_0$, then

$$\beta(x, r_1) \le \beta(x, r_2)(1+\varepsilon).$$

See also Remark (3) to Theorem 2.2 in [11].

It follows directly from the definition that

$$H(x,r_2)/H(x,r_1) = \exp\left(2\int_{r_1}^{r_2} \beta(x,r) d\log r\right) \tag{2}$$

and by the almost monotonicity property that

$$\left(\frac{r_2}{r_1}\right)^{2\beta(x,r_1)/(1+\varepsilon)} \le \frac{H(x,r_2)}{H(x,r_1)} \le \left(\frac{r_2}{r_1}\right)^{2\beta(x,r_2)(1+\varepsilon)}. \tag{3}$$

3. A lemma on monotonic functions

Lemma 3.1. *Let f be a non-negative, monotonic and non-decreasing function on the interval $[a, b]$. Assume that $f \geq e$ on this interval. Then there exist a point $x \in [a, \frac{a+b}{2})$ and $N \geq e$ such that*

$$N \leq f(t) \leq eN \tag{4}$$

for any $t \in (x - \frac{b-a}{20 \log^2 f(x)}, x + \frac{b-a}{20 \log^2 f(x)}) \subset [a, b]$.

Proof. Without loss of generality we can assume $a = 0, b = 1$. Define a sequence of numbers $x_i \in [0, 1)$ such that $x_1 = 0$ and $x_{i+1} = x_i + \frac{1}{10 \log^2 f(x_i)}$ as long as $x_{i+1} < 1/2$. The sequence might be finite. Assume that (4) fails for $x = \frac{x_{i+1}+x_i}{2}$ and $N = f(x_i)$. Then $f(x_{i+1}) \geq ef(x_i)$. Assuming this for all such x, we obtain $f(x_i) \geq e^i$. Hence $x_{i+1} - x_i = \frac{1}{10 \log^2 f(x_i)} \leq \frac{1}{10i^2}$. Since $\sum_{i=1}^{\infty} \frac{1}{10i^2} < 1/2$, we see that $x_i < 1/2$ for all integers i and $f(x_i) \leq f(1/2)$ while $f(x_i) \to \infty$ as $i \to \infty$. $\square$

We want to apply Lemma 3.1 to a modified frequency function:

$$\widetilde{\beta}(p, r) := \sup_{t \in (0, r]} \beta(p, t).$$

We note that $\widetilde{\beta}$ is monotonic and β and $\widetilde{\beta}$ are comparable due to almost monotonicity of the frequency:

$$\beta(p, r) \leq \widetilde{\beta}(p, r) \leq (1 + \varepsilon)\beta(p, r), \tag{5}$$

if $B_g(p, r) \subset B_g(O, R_0)$, where $R_0 = R_0(\varepsilon, O, M, g)$. Hereafter we will work in a small neighborhood of O and always assume that (5) holds with $\varepsilon = 1$.

Lemma 3.2. *Consider a ball $B_g(p, 2r) \subset B_g(O, R_0)$, and assume that $\beta(p, r/2) > 10$. Then there exist $s \in [r, \frac{3}{2}r)$ and $N \geq 5$ such that*

$$N \leq \beta(p, t) \leq 2eN \tag{6}$$

for any $t \in (s(1 - \frac{1}{1000 \log^2 N}), s(1 + \frac{1}{1000 \log^2 N}))$.

Proof. Indeed, we can apply Lemma 3.1 for $\widetilde{\beta}(p, t)$ on $[r, 2r)$ and find such s and N that

$$2N < \widetilde{\beta}(p, t) \leq 2eN$$

for $t \in (s - \frac{r}{20 \log^2(2N)}, s + \frac{r}{20 \log^2(2N)})$. By (5) we have $N < \beta(p, t) \leq 2eN$ for t on the same interval.

Since $\beta(p, r/2) > 10$, we have $2N \geq 10$. Recall that $s \in [r, 2r)$. These two observations imply

$$\left(s\left(1 - \frac{1}{1000\log^2 N}\right), s\left(1 + \frac{1}{1000\log^2 N}\right)\right)$$
$$\subset \left(s - \frac{r}{20\log^2(2N)}, s + \frac{r}{20\log^2(2N)}\right). \qquad \square$$

4. Behavior near the maximum

In this section we study the behavior of a harmonic function in the spherical layer of width $\sim \frac{1}{\log^2 N}$ from Lemma 3.2, where the frequency is comparable to N. We will consider a sphere within this spherical layer and collect several estimates for growth of u near the point, where the maximum is attained on that sphere.

The same notation as in Lemma 3.2 is used here: we consider a ball $B(p, 2r) \subset B(O, R_0)$ with $\beta(p, r/2) \geq 10$ and a number $s \in [r, 2r)$ such that the following holds: For any t in the interval

$$I := \left(s\left(1 - \frac{1}{1000\log^2 N}\right), s\left(1 + \frac{1}{1000\log^2 N}\right)\right),$$

the frequency is estimated by $N < \beta(p, t) \leq 2eN$. We will always assume that N is larger than 5.

By $c, c_1, C, C_1, C_2, \dots$ we will denote positive constants that depend on M, g, n, O, R_0 only. These constants are allowed to vary from line to line.

Consider the function $H(p, t) = \int_{\partial B_g(p,t)} u^2$. By (2) and (6) we have

$$(t_2/t_1)^{2N} \leq \frac{H(p, t_2)}{H(p, t_1)} \leq (t_2/t_1)^{4eN} \tag{7}$$

for any $t_1 < t_2$ in I.

Consider a point x on $\partial B_g(p, s)$ such that the maximum of $|u|$ on $\overline{B}_g(p, s)$ is attained at x, and define $K = |u(x)|$. Let us fix numbers

$$A = 10^6, \delta \in [\frac{1}{A\log^{100} N}, \frac{1}{A\log^2 N}], s_{-\delta} = s(1-\delta), s_\delta = s(1+\delta). \tag{8}$$

Note that $s_{-\delta} < s < s_\delta$ and $\delta < 1/10^6$.

LEMMA 4.1. *There exist $c > 0$ and $C > 0$, depending on M, g, n, O, R_0 only, such that*

$$\sup_{B_g(p, s_{-\delta})} |u| \leq CK2^{-c\delta N}, \tag{9}$$

$$\sup_{B_g(p, s_\delta)} |u| \leq CK2^{C\delta N}. \tag{10}$$

Proof. We will prove only (9); the same argument works for the second inequality (10).

By the standard estimate of L^2-norm of a function, by L^∞-norm and by (7) we have

$$K^2 \geq C_1 s^{-n+1} H(p,s) \geq C_1 s^{-n+1} H(p, s_{-\delta/2})(1+\delta/2)^{2N}.$$

We need an estimate that compares L^2-norm of a harmonic function on the boundary of a ball and L^2-norm in the ball:

$$sH(p, s_{-\delta/2}) = s\int_{\partial B_g(p,s_{-\delta/2})} |u|^2 \geq C_1 \int_{B_g(p,s_{-\delta/2})} |u|^2.$$

Let $\widetilde{x}$ be a point on $\partial B_g(p, s_{-\delta})$, where the maximum is attained. Define $\widetilde{K} = |u(\widetilde{x})|$. Since the volume $|B_g(\widetilde{x}, \frac{\delta}{2}s)| \geq C_2(\delta s)^n$, we have

$$\int_{B_g(p,s_{-\delta/2})} |u|^2 \geq \int_{B_g(\widetilde{x},\frac{\delta}{2}s)} |u|^2 \geq C_2(\delta s)^n \fint_{B_g(\widetilde{x},\frac{\delta}{2}s)} |u|^2.$$

One can estimate the value of a harmonic function u in the center of a ball by a constant multiple of the average of $|u|$ over the ball, so

$$\fint_{B_g(\widetilde{x},\frac{\delta}{2}s)} |u|^2 \geq (\fint_{B_g(\widetilde{x},\frac{\delta}{2}s)} |u|)^2 \geq C_3 |u|^2(\widetilde{x}) = C_3 \widetilde{K}^2.$$

Combining the estimates above one has

$$K^2 \geq C_4 \delta^n (1+\delta/2)^{2N} \widetilde{K}^2. \tag{11}$$

Note that $\log(1+\delta/2) \geq \delta/4$ for $\delta \in (0, 1/10^6)$, so

$$(1+\delta/2)^{2N}\delta^n \geq \exp(N\delta/2 + n\log\delta) = \exp(N\delta/4)\exp(N\delta/4 + n\log\delta).$$

Using that $\delta \in [\frac{1}{A\log^{100} N}, \frac{1}{A\log^2 N}]$ it is easy to show that

$$C_5 + N\delta/4 + n\log\delta > 0$$

for sufficiently large $C_5 = C_5(n)$. Thus $K^2 \geq C_6 \exp(N\delta/4)\widetilde{K}^2$. $\square$

Now, we can estimate the doubling index near x. Define $\mathcal{N}(x,r)$ by

$$2^{\mathcal{N}(x,r)} = \frac{\sup\limits_{B_g(x,2r)} |u|}{\sup\limits_{B_g(x,r)} |u|}.$$

One can estimate the growth of a harmonic function in terms of the doubling index. For any ε, there exist $R_0 > 0$ and $C > 0$ such that for any positive numbers r_1, r_2 with $2r_1 \leq r_2$ and $B_g(x, r_2) \subset B_g(O, R_0)$, the following version of the logarithmic convexity property holds (see [9]):

$$\left(\frac{r_2}{r_1}\right)^{\mathcal{N}(x,r_1)(1-\varepsilon)-C} \leq \frac{\sup_{B_g(x,r_2)}|u|}{\sup_{B_g(x,r_1)}|u|} \leq \left(\frac{r_2}{r_1}\right)^{\mathcal{N}(x,r_2)(1+\varepsilon)+C}. \tag{12}$$

In particular, the doubling index is almost monotonic in the following sense:

$$\mathcal{N}(x, r_1)(1-\varepsilon) - C \le \mathcal{N}(x, r_2)(1+\varepsilon) + C.$$

LEMMA 4.2. *There exists $C = C(M, g, n, O, R_0) > 0$ such that*

$$\sup_{B_g(x,\delta s)} |u| \le K 2^{C\delta N + C}, \tag{13}$$

and for any $\widetilde{x}$ with $d(x, \widetilde{x}) \le \frac{\delta}{4} s$,

$$\mathcal{N}(\widetilde{x}, \frac{\delta}{4} s) \le C\delta N + C, \tag{14}$$

$$\sup_{B_g(\widetilde{x}, \frac{\delta s}{10N})} |u| \ge K 2^{-C\delta N \log N - C}. \tag{15}$$

Proof. The first estimate (13) immediately follows from (10) since

$$B_g(x, \delta s) \subset B_g(p, s(1+\delta)).$$

To establish (14) we note that

$$2^{\mathcal{N}(\widetilde{x}, \frac{\delta}{4} s)} = \frac{\sup_{B_g(\widetilde{x}, \delta s/2)} |u|}{\sup_{B_g(\widetilde{x}, \delta s/4)} |u|} \le \frac{\sup_{B_g(x,\delta s)} |u|}{K} \le 2^{C\delta N + C}.$$

It remains to obtain (15). We will use (14) and almost monotonicity (12) of the doubling index:

$$\frac{\sup_{B_g(\widetilde{x}, \delta s/4)} |u|}{\sup_{B_g(\widetilde{x}, \frac{\delta s}{10N})} |u|} \le (40N)^{C_1 \mathcal{N}(\widetilde{x}, \frac{\delta}{4} s) + C_1} \le 2^{C_2 \delta N \log N + C_2 \log N} \le 2^{C_3 \delta N \log N + C_3}.$$

In the last inequality we used that $\delta \in [\frac{1}{A \log^{100} N}, \frac{1}{A \log^2 N}]$. Since $\sup_{B_g(\widetilde{x}, \delta s/4)} |u| \ge |u|(x) = K$, the proof of (15) will be completed if we take $C = C_3$. $\square$

5. Number of cubes with big doubling index

Given a cube Q, we will denote

$$\sup_{x \in Q, r \le \operatorname{diam}(Q)} \log \frac{\sup_{B_g(x, 10n \cdot r)} |u|}{\sup_{B_g(x,r)} |u|}$$

by $N(Q)$ and call it the doubling index of Q. This definition is different from a doubling index for balls but more convenient in the following sense. If a cube q is contained in a cube Q, then $N(q) \le N(Q)$. Furthermore, if a cube q is covered by cubes Q_i with $\operatorname{diam}(Q_i) \ge \operatorname{diam}(q)$, then $N(Q_i) \ge N(q)$ for some Q_i.

The following result was proved in [9], where it was applied to upper estimates of the volume of nodal sets. However this result appears to be useful for lower bounds as well.

THEOREM 5.1. *There exist a constant $c > 0$ and an integer $A > 1$, depending on the dimension n only, and positive numbers $N_0 = N_0(M, g, n, O)$, $R = R(M, g, n, O)$ such that for any cube $Q \subset B(O, R)$, the following holds: if we partition Q into A^n equal subcubes, then the number of subcubes with doubling index greater than $\max(N(Q)/(1 + c), N_0)$ is less than $\frac{1}{2}A^{n-1}$.*

Further, we will partition the cube Q into A^{nk} subcubes (k will tend to infinity) and iterate the Theorem 5.1 for the subcubes.

Notation. Let $A > 1$ be the integer from Theorem 5.1. Given an Euclidean n-dimensional cube Q , we partition Q into A^n equal subcubes with $1/A$ smaller size than Q, we denote these cubes by Q_{i_1}, $i_1 = 1, 2, \ldots, A^n$, then partition each Q_{i_1} into A^n equal subcubes Q_{i_1,i_2}, $i_2 = 1, 2, \ldots, A^n$ and so on. We denote the collection of all subcubes $Q_{i_1,i_2,\ldots,i_k}$ of all sizes by $\mathbb{A}$.

By C_k^i we denote the binomial coefficients $\frac{k!}{i!(k-i)!}$.

Let $j_1, j_2, j_3, \ldots$ be independent and identically distributed random variables such that

$$\mathbb{P}(j_k = i) = 1/A^n \text{ for } i = 1, 2, \ldots, A^n.$$

We make a remark that we use the probabilistic notation because they are simpler than writing "the number of subcubes with."

LEMMA 5.2. *Let c,N_0 be positive numbers. Let N be a function from the set of subcubes $\mathbb{A}$ to $\mathbb{R}_+$ with the following properties:*

(i) *N is monotonic with respect to inclusion: if $q_1, q_2 \in$ Q and $q_1 \subset q_2$, then $N(q_1) \leq N(q_2)$;*

(ii) *for any cube $\widetilde{Q} \in \mathbb{A}$,*

$$\mathbb{P}\left(N(\widetilde{Q}_{j_1}) \geq \max(\frac{N(\widetilde{Q})}{1+c}, N_0)\right) \leq \frac{1}{2A}.$$

Then for any integers l, k with $0 \leq l \leq k$, $k \geq 1$, the following holds:

(16)
$$\mathbb{P}\left(N(Q_{j_1,j_2,\ldots,j_{k-1},j_k}) \leq \max(\frac{N(Q)}{(1+c)^l}, N_0)\right) \geq \sum_{i=l}^{k} C_k^i \left(\frac{1}{2A}\right)^{k-i} \left(1 - \frac{1}{2A}\right)^i,$$

and for any $\varepsilon > 0$, there exist $\sigma > 0$ and an integer k_0 such that

$$\mathbb{P}\left(N(Q_{j_1,j_2,\ldots,j_k}) \geq \max(\frac{N(Q)}{(1+c)^{\sigma k/\log k}}, N_0)\right) \leq \left(\frac{1}{2A}\right)^{k(1-\varepsilon)} \tag{17}$$

for all positive integers $k > k_0$.

Before we start the proof of Lemma 5.2 we give some informal explanations.

Heuristics. Let $p = \frac{1}{2A}$. Suppose we have k independent and identically distributed variables y_i and that each y_i takes value 0 with probability p and value 1 with probability $(1-p)$. Then

$$\mathbb{P}\Big(\sum_{i=1}^{k} y_i \geq l\Big) = \sum_{i=l}^{k} C_k^i p^{k-i} (1-p)^i$$

for $0 \leq l \leq k$.

Suppose now that y_i are independent variables, each y_i takes only two values 0 and 1, $\mathbb{P}(y_i = 0) \leq p$ and $\mathbb{P}(y_i = 1) \geq 1-p$. Now, y_i are not assumed to be identically distributed. Then

$$\mathbb{P}\Big(\sum_{i=1}^{k} y_i \geq l\Big) \geq \sum_{i=l}^{k} C_k^i p^{k-i} (1-p)^i . \tag{18}$$

The proof of (16) is parallel to (18) with the exception that we have to always add words “or smaller than N_0.” Namely, starting with a cube $Q_{i_1,\dots,i_k}$ and choosing randomly its subcube $Q_{i_1,\dots,i_k,j_{k+1}}$ the doubling index of the latter is either $(1+c)$ times smaller than the doubling index of $Q_{i_1,\dots,i_k}$ or smaller than N_0 with probability at least $1-p$.

Inequality (17) will be proved with the help of the following fact:

Claim. *Let $p \in (0,1)$ be a fixed number. Then for any $\varepsilon > 0$, there are $\sigma > 0$ and $k_0 > 0$ such that*

$$\sum_{i=0}^{l-1} C_k^i p^{k-i} (1-p)^i \leq p^{k(1-\varepsilon)} \tag{19}$$

for any $k > k_0$ and $l \in [0, \sigma k/\log k]$.

Proof of the claim. Note that $C_k^i \leq k^l$ for $i \leq l$. Hence

$$\sum_{i=0}^{l-1} C_k^i p^{k-i} (1-p)^i \leq l k^l p^{k-l}.$$

It sufficient to choose $\sigma > 0$ so that

$$l k^l p^{k-l} \leq p^{k(1-\varepsilon)}$$

for large k, which is equivalent to

$$l k^l (1/p)^l \leq (1/p)^{\varepsilon k}.$$

Since $l \leq \sigma k/\log k$, we have

$$l \leq e^{\log k} \leq (1/p)^{\frac{\varepsilon}{3}k}, \quad k^l \leq e^{\sigma k} \leq (1/p)^{\frac{\varepsilon}{3}k}, \quad (1/p)^l \leq (1/p)^{\sigma k/\log k} \leq (1/p)^{\frac{\varepsilon}{3}k}$$

for k large enough and $\sigma < \frac{\varepsilon}{3}\log(1/p)$. Multiplying the three inequalities above we finish the proof of the claim. $\square$

Proof of Lemma 5.2. We are going to prove inequality (16) by induction on k. For $k=1$, it is true due to assumption (ii). Assume that (16) holds for $(k-1)$ in place of k. We want to show that (16) holds for k. For $k>l>0$, define the disjoint events

$$E_{l,k} := \left\{N(Q_{j_1,j_2,\dots,j_{k-1},j_k}) \in \left(\max\left(\frac{N(Q)}{(1+c)^{l+1}}, N_0\right), \max\left(\frac{N(Q)}{(1+c)^{l}}, N_0\right)\right]\right\},$$
$$E_{0,k} := \left\{N(Q_{j_1,j_2,\dots,j_{k-1},j_k}) \in \left(\max\left(\frac{N(Q)}{(1+c)}, N_0\right), \max(N(Q), N_0)\right]\right\},$$
$$E_{k,k} := \left\{N(Q_{j_1,j_2,\dots,j_{k-1},j_k}) \le \max\left(\frac{N(Q)}{(1+c)^{k}}, N_0\right)\right\}.$$

If $l<0$ or $l>k$, we will denote the empty event by $E_{l,k}$.

The doubling index of any cube is non-strictly greater than the doubling index of any its subcube. Hence $E_{i,k} \subset \cup_{j=0}^{i} E_{j,k-1}$ and $E_{j,k-1} \subset \cup_{i=j}^{k} E_{i,k}$, where both unions are disjoint. Hence

$$\mathbb{P}\left(N(Q_{j_1,j_2,\dots,j_{k-1},j_k}) \le \max\left(\frac{N(Q)}{(1+c)^{l}}, N_0\right)\right) = \sum_{i=l}^{k} \mathbb{P}(E_{i,k})$$

We start to prove by induction on k that

$$\sum_{i=l}^{k} \mathbb{P}(E_{i,k}) \ge \sum_{i=l}^{k} C_k^i \left(\frac{1}{2A}\right)^{k-i} \left(1-\frac{1}{2A}\right)^{i}.$$

Indeed,

$$\begin{aligned}
\sum_{i=l}^{k} \mathbb{P}(E_{i,k}) &= \sum_{i=l}^{k}\sum_{j=0}^{i} \mathbb{P}(E_{j,k-1}\cap E_{i,k}) = \sum_{j=0}^{k}\sum_{i=\max(l,j)}^{k} \mathbb{P}(E_{j,k-1}\cap E_{i,k}) \\
&\ge \sum_{j=l}^{k-1}\sum_{i=j}^{k} \mathbb{P}(E_{j,k-1}\cap E_{i,k}) + \sum_{i=l}^{k} \mathbb{P}(E_{l-1,k-1}\cap E_{i,k}) \\
&= \sum_{j=l}^{k-1} \mathbb{P}(E_{j,k-1}) + \sum_{i=l}^{k} \mathbb{P}(E_{l-1,k-1}\cap E_{i,k}) = I + II.
\end{aligned}$$

It follows from (ii) that

$$\mathbb{P}(E_{l-1,k-1}\cap E_{l-1,k}) \le \frac{1}{2A}\mathbb{P}(E_{l-1,k-1}).$$

Since $\sum_{i=l-1}^{k} \mathbb{P}(E_{l-1,k-1}\cap E_{i,k}) = \mathbb{P}(E_{l-1,k-1})$, we obtain

$$II = \mathbb{P}(E_{l-1,k-1}) - \mathbb{P}(E_{l-1,k-1}\cap E_{l-1,k}) \ge \left(1-\frac{1}{2A}\right)\mathbb{P}(E_{l-1,k-1}).$$

Hence

$$I + II \geq \sum_{j=l}^{k-1} \mathbb{P}(E_{j,k-1}) + \left(1 - \frac{1}{2A}\right) \mathbb{P}(E_{l-1,k-1})$$
$$= \frac{1}{2A} \sum_{j=l}^{k-1} \mathbb{P}(E_{j,k-1}) + \left(1 - \frac{1}{2A}\right) \sum_{j=l-1}^{k-1} \mathbb{P}(E_{j,k-1}).$$

By the induction hypothesis for $k-1$, we can estimate the latter amount from below by

$$\sum_{i=l}^{k-1} C_{k-1}^{i} \left(\frac{1}{2A}\right)^{k-i} \left(1 - \frac{1}{2A}\right)^{i} + \sum_{i=l-1}^{k-1} C_{k-1}^{i} \left(\frac{1}{2A}\right)^{k-1-i} \left(1 - \frac{1}{2A}\right)^{i+1}$$
$$= \sum_{i=l}^{k} (C_{k-1}^{i} + C_{k-1}^{i-1}) \left(\frac{1}{2A}\right)^{k-i} \left(1 - \frac{1}{2A}\right)^{i} = \sum_{i=l}^{k} C_{k}^{i} \left(\frac{1}{2A}\right)^{k-i} \left(1 - \frac{1}{2A}\right)^{i}.$$

Inequality (16) is proved, which implies

$$\mathbb{P}\left(N(Q_{j_1,j_2,\dots,j_{k-1},j_k}) > \max(\frac{N(Q)}{(1+c)^l}, N_0)\right) \leq \sum_{i=0}^{l-1} C_{k}^{i} \left(\frac{1}{2A}\right)^{k-i} \left(1 - \frac{1}{2A}\right)^{i}.$$

It remains to prove (17). By (19) applied for $p = \frac{1}{2A}$ we have

$$\sum_{i=0}^{l-1} C_{k}^{i} \left(\frac{1}{2A}\right)^{k-i} \left(1 - \frac{1}{2A}\right)^{i} \leq \left(\frac{1}{2A}\right)^{k(1-\varepsilon)}$$

for $0 \leq l \leq \sigma k / \log k$, $k \geq k_0$. □

Now, we ready to formulate the corollary of Theorem 5.1 and Lemma 5.2, which will be used in the next section.

THEOREM 5.3. *There exist constants $c_1, c_2, C > 0$ and a positive integer B_0, depending on the dimension n only, and positive numbers $N_0 = N_0(M, g, n, O)$, $R = R(M, g, n, O)$ such that for any cube $Q \subset B(O, R)$, the following holds: if we partition Q into B^n equal subcubes, where $B > B_0$, then the number of subcubes with doubling index greater than*

$$\max(N(Q) 2^{-c_1 \log B / \log \log B}, N_0)$$

is less than CB^{n-1-c_2}.

Proof. Let us fix c, A, N_0, R from Theorem 5.1. Fix a cube $Q \subset B(O, R)$ and partition it into A^n equal subcubes Q_{i_1}, then partition each Q_{i_1} into A^n subcubes Q_{i_1,i_2} and so on. We denote by $\mathbb{A}$ the collection of all subcubes $Q_{i_1,i_2,\dots,i_k}$ of all sizes.

First, we will consider the case $B = A^k$, where k is sufficiently large. In this case Theorem 5.3 follows from Lemma 5.2. Let us first check assumptions

(i) and (ii) of Lemma 5.2. The monotonicity property (i) for the doubling index of cubes is clear from the definition. The second assumption (ii) follows from Theorem 5.1. Now fix $\varepsilon > 0$ so small that

$$\left(\frac{1}{2A}\right)^{1-\varepsilon} = \left(\frac{1}{A}\right)^{1+c_2}$$

for some $c_2 > 0$.

The conclusion (17) of Lemma 5.2 for this ε claims that the number of subcubes $Q_{i_1,i_2,\dots,i_k}$ with $N(Q_{i_1,i_2,\dots,i_k}) \geq \max(\frac{N(Q)}{(1+c)^{\sigma k/\log k}}, N_0)$ is smaller than

$$A^{nk}\left(\frac{1}{2A}\right)^{k(1-\varepsilon)} = A^{k(n-1-c_2)} = B^{n-1-c_2}.$$

Note that $\log B = k \log A$. We therefore can choose $c_1 > 0$ so small that

$$\log(1+c)\cdot \sigma k/\log k \geq c_1 \log 2 \cdot \log B/\log\log B$$

for all sufficiently large $B = A^k$. This is done to provide

$$(1+c)^{\sigma k/\log k} \geq 2^{c_1 \log B/\log\log B}.$$

We have proved Theorem 5.3 in the case $B = A^k$.

Now, let $B \in [A^k, A^{k+1}]$ and define $\widetilde{B} = A^k$. There are two partitions of Q into equal subcubes, say $Q = \cup Q_i$, $i = 1 \cdots B^n$, and $Q = \cup \widetilde{Q}_i$, $i = 1 \cdots \widetilde{B}^n$. We know that the number of cubes $\widetilde{Q}_i$ with doubling index greater than $\max(N(Q)2^{-c_1 \log \widetilde{B}/\log\log \widetilde{B}}, N_0)$ is less than $\widetilde{B}^{n-1-c_2}$. Each cube Q_i is covered by a finite number, which depends on dimension n and on $A = A(n)$ only, of cubes $\widetilde{Q}_j$, which have a smaller diameter. If $N(Q_i)$ is greater than $\max(N(Q)2^{-c_1 \log \widetilde{B}/\log\log \widetilde{B}}, N_0)$, then one of $\widetilde{Q}_j$ that covers Q_i also has $N(\widetilde{Q}_j)$ greater than $\max(N(Q)2^{-c_1 \log \widetilde{B}/\log\log \widetilde{B}}, N_0)$. Thus the number of cubes Q_i with doubling index greater than $\max(N(Q)2^{-c_1 \log \widetilde{B}/\log\log \widetilde{B}}, N_0)$ is less than $C\widetilde{B}^{n-1-c_2}$. We can decrease c_1 and increase C to replace $\widetilde{B}$ by B in the previous sentence. □

Remark 5.4. Here we collected several informal remarks to orient the reader. The goal of this paper is to estimate the Hausdorff measure of dimension $n-1$ of zero sets of harmonic functions from below. If a harmonic function is zero at the center of a cube and the doubling index of this cube is bounded by a fixed constant, then it is not difficult and well known that there is a lower bound for the volume of the zero set in this cube. Unfortunately, the bound depends on the doubling index, and it is not clear why the lower estimate does not become worse as the doubling index becomes large.

In the next section there will be an argument that works for the case of the large doubling index of the original cube. Speaking non-formally the argument will show that for a proper choice of B, the number of subcubes, which contain zeroes, is larger than B^{n-1} , and the argument severely exploits that the number of bad subcubes with large doubling index is smaller than B^{n-1}. We do not specify here what the words "smaller" and "larger" mean.

If we partition the cube with zero at the center into B^n equal subcubes, there can be some subcubes with small doubling index, which intersect the zero set, but there also can be bad subcubes with large doubling index, where we have no good a priori estimate. The estimate for the number of bad subcubes appears to be useful.

In Theorem 5.1 the number of subcubes A^n is fixed, and it shows that all except at most $\frac{1}{2}A^{n-1}$ of the subcubes have constant times smaller doubling index than a big cube. For the estimates of the volume of the nodal set, it is crucial that the number of exceptions is smaller than A^{n-1}. In Theorem 5.3 the number of subcubes B^n tends to infinity, but the bigger B the smaller the doubling index for the most of the subcubes becomes, and we still want the number of bad subcubes with big doubling index to be smaller than B^{n-1}. Theorem 5.3 is the iterated version of Theorem 5.1; the iteration procedure is similar to the independent flips of the coin. The quantity $k/\log k \sim \log B/\log\log B$ in Theorem 5.3 comes from the simple estimate of the tails of the binomial distribution (19).

We also note that for the purposes of this paper, a weaker estimate than the conclusion of Theorem 5.3 would be sufficient. Namely, it is sufficient to know that the number of subcubes with doubling index greater than

$$\max(N(Q)/(\log B)^{\kappa}, N_0)$$

is less than $B^{n-1}/(\log B)^{\kappa}$, where $\kappa > 0$ is a sufficiently large constant depending only on the dimension.

6. **A tunnel with controlled growth**

This section contains a geometrical construction that allows us to find many disjoint balls with sign changes of the harmonic function (Proposition 6.1). It appears to be useful for lower estimates for the nodal sets. The construction is using the estimates for the number of cubes with big doubling index and requires a look at several statements of the previous sections. The whole section consists of the proof of one proposition.

PROPOSITION 6.1. *Fix a point O on the Riemannian manifold M equipped with Riemannian metric g. There is a sufficiently small radius $R_0 > 0$ such that for any ball $B_g(p, 2r) \subset B_g(O, R_0)$ and for any harmonic function u on $B_g(p, 2r)$, the following holds: If $\beta(p, r)$ is sufficiently large, then there is a number N with*

$$\beta(p, r)/10 \leq N \leq 2\beta\left(p, \frac{3}{2}r\right)$$

and at least $[\sqrt{N}]^{n-1}2^{c_3 \log N/\log\log N}$ *disjoint balls* $B_g(x_i, \frac{r}{\sqrt{N}}) \subset B(p, 2r)$ *such that* $u(x_i) = 0$.

Proof. According to Section 4 we can find a spherical layer where the frequency does not grow too fast: there exist numbers $s \in [r, \frac{3}{2}r]$ and $N \geq 5$ such that

$$N \leq \beta(p,t) \leq 2eN$$

for any $t \in (s(1 - \frac{1}{1000\log^2 N}), s(1 + \frac{1}{1000\log^2 N}))$.

By the monotonicity property of the frequency, we have

$$\beta(p,r) \leq (1+\varepsilon)\beta(p,t) \leq 10N$$

and

$$N \leq \beta(p,s) \leq (1+\varepsilon)\beta(p, \frac{3}{2}r) \leq 2\beta\left(p, \frac{3}{2}r\right).$$

Until the end of this section we will assume that N is sufficiently large.

Fix a point $x \in \partial B_g(p,s)$ such that $\sup\limits_{\partial B_g(p,s)} |u| = |u(x)|$. Put

$$\delta = \frac{1}{10^8 n^2 \log^2 N}. \tag{20}$$

Consider a point $\widetilde{x} \in \partial B_g(p, s(1-\delta))$ such that $d_g(x,\widetilde{x}) = \delta s$. In other words, $\widetilde{x}$ is the nearest point to x on $\partial B_g(p, s(1-\delta))$. Note that

$$C_1(n)\frac{r}{\log^2 N} \leq d(x,\widetilde{x}) \leq C_2(n)\frac{r}{\log^2 N}. \tag{21}$$

Let us consider a box T (a hyperrectangle in the Euclidean space) such that x and $\widetilde{x}$ are the centers of the opposite faces of T, one side of T is equal to $d(x,\widetilde{x})$ and $n-1$ other sides are equal to $\frac{d(x,\widetilde{x})}{[\log N]^4}$, where $[\cdot]$ denotes the integer part of a number.

Let us divide T into equal boxes T_i, $i = 1, 2, \ldots, [\sqrt{N}]^{n-1}$, so that each T_i has one side of length $d(x,\widetilde{x})$ and $(n-1)$ sides of length $\frac{d(x,\widetilde{x})}{[\sqrt{N}][\log N]^4}$. We partition each T_i into equal cubes $q_{i,t}$, $t = 1, 2, \ldots, [\sqrt{N}][\log N]^4$, with side $\frac{d(x,\widetilde{x})}{[\sqrt{N}][\log N]^4}$, and the cubes $q_{i,t}$ are arranged in t so that $d(q_{i,t}, x) \geq d(q_{i,t+1}, x)$. We will call the boxes T_i "tunnels."

Note that

$$d_g(p, q_{i,1}) \leq d_g(p,\widetilde{x}) + d_g(\widetilde{x}, q_{i,1}) \leq s(1-\delta) + C\frac{\delta s\sqrt{n}}{[\log N]^4} \leq s(1-\delta/2).$$

Hence $q_{i,1} \subset B_g(p, s(1-\delta/4))$. Recall that $|u(x)| = K$. Then by (9),

$$\sup_{q_{i,1}} |u| \leq K 2^{-c_1 \frac{N}{\log^2 N} + C_1}. \tag{22}$$

Applying (14) with δ, which is $100n^2$ times larger than δ defined by (20), we obtain that for any point $y \in T$,

$$\mathcal{N}(y, 10n\delta s) \leq C\delta N + C \leq N/100. \tag{23}$$

The center of $q_{i,t}$ will be denoted by $x_{i,t}$.

Now, let $t = [\sqrt{N}][\log N]^4$. We can inscribe a geodesic ball $B_{i,t}$ in $\frac{1}{2}q_{i,t}$ with center at $x_{i,t}$ and radius $\frac{s}{N}$. Taking into account

$$d_g(x_{i,t}, x) \leq \frac{C_2 s}{[\log N]^6},$$

we deduce from (15), applied with $\widetilde{x} = x_{i,t}$, that

$$\sup_{B_{i,t}} |u| \geq K 2^{-C_3 \frac{N}{\log^5 N} - C_3},$$

and therefore

$$\sup_{\frac{1}{2}q_{i,[\sqrt{N}][\log N]^4}} |u| \geq K 2^{-C_3 \frac{N}{\log^5 N} - C_3}. \tag{24}$$

Inequalities (22) and (24) imply the following estimate: there exist positive c, C such that

$$\sup_{\frac{1}{2}q_{i,[\sqrt{N}][\log N]^4}} |u| \geq \sup_{\frac{1}{2}q_{i,1}} |u| 2^{cN/\log^2 N - C}. \tag{25}$$

The next step in the proof of Proposition 6.1 is the following claim:

Claim 6.2. *There exist $c > 0$, $N_0 > 0$ such that at least half of tunnels T_i have the following property*:

$$N(q_{i,t}) \leq \max\left(\frac{N}{2^{c \log N/\log\log N}}, N_0\right) \tag{26}$$

for all $t = 1, 2, \ldots [\sqrt{N}][\log N]^4$.

Proof of the claim. We will assume that N is sufficiently big. Let us call a cube $q_{i,t}$ bad if $N(q_{i,t}) > N 2^{-c_1 \log N/\log\log N}$, where a constant c_1 is from Theorem 5.3. It is sufficient to show that the number of bad cubes is less than the half of the number of tunnels T_i, i.e., $\frac{1}{2}[\sqrt{N}]^{n-1}$.

Let us partition T into equal Euclidean cubes $Q_t, t = 1, 2, \ldots, [\log N]^4$ with side $\frac{d(x,\widetilde{x})}{[\log N]^4}$. For any point $y \in T$, we have

$$d_g(x, y) \leq 2d(x, y) \leq 4d(x, \widetilde{x}) \leq \frac{s}{10^7 \log^2 N} =: \rho.$$

By (14) we have

$$\frac{\sup_{B_g(y,\rho)} |u|}{\sup_{B_g(y,\rho/2)} |u|} \leq 2^{CN/\log^2 N + C}.$$

The last observation implies that

$$N(Q_t) \leq N$$

for $t = 1, 2, \dots, [\log N]^4$.

It follows from Theorem 5.3 with $B = [\sqrt{N}]$ that the number of bad cubes in Q_t is less than $C[\sqrt{N}]^{n-1-c_2}$. Thus the number of all bad cubes is less than

$$C[\sqrt{N}]^{n-1-c_2}[\log N]^4 \leq \frac{1}{2}[\sqrt{N}]^{n-1}. \qquad \square$$

We will call a tunnel T_i good if (26) holds.

The next step in the proof of Proposition 6.1 is the following claim.

Claim 6.3. *There exists $c_2 > 0$ such that if N is sufficiently large and T_i is a good tunnel, then there are at least $2^{c_2 \log N/\log\log N}$ closed cubes $\overline{q_{i,t}}$ that contain zero of u.*

Proof of the claim. By (26) we know that

$$\log \frac{\sup_{\frac{1}{2}q_{i,t+1}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} \leq \log \frac{\sup_{4q_{i,t}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} \leq \frac{N}{2^{c_1 \log N/\log\log N}} \tag{27}$$

for any $t = 1, 2, \dots, [\sqrt{N}][\log N]^4 - 1$.

Let us split the set $\{1, 2, \dots, [\sqrt{N}][\log N]^4 - 1\}$ into two subsets S_1, S_2. The set S_1 is the set of all t such that u does not change the sign in $\overline{q_{i,t}} \cup \overline{q_{i,t+1}}$ and $S_2 = \{1, 2, \dots, [\sqrt{N}][\log N]^4 - 1\} \setminus S_1$. By the Harnack inequality for $t \in S_1$ we have

$$\log \frac{\sup_{\frac{1}{2}q_{i,t+1}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} \leq C_1, \tag{28}$$

and for any $t \in S_2$ the inequality (27) holds. We therefore have

$$\begin{aligned} \log \frac{\sup_{\frac{1}{2}q_{i,[\sqrt{N}][\log N]^4}} |u|}{\sup_{\frac{1}{2}q_{i,1}} |u|} &= \sum_{S_1} \log \frac{\sup_{\frac{1}{2}q_{i,t+1}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} + \sum_{S_2} \log \frac{\sup_{\frac{1}{2}q_{i,t+1}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} \\ &\leq |S_1| C_1 + |S_2| \frac{N}{2^{c_1 \log N/\log\log N}}. \end{aligned}$$

By (25),

$$cN/\log^2 N - C \leq \log \frac{\sup_{\frac{1}{2}q_{i,[\sqrt{N}][\log N]^4}} |u|}{\sup_{\frac{1}{2}q_{i,1}} |u|}. \tag{29}$$

Hence

$$cN/\log^2 N - C \leq |S_1|C_1 + |S_2|\frac{N}{2^{c_1 \log N/\log\log N}}.$$

Note that

$$|S_1|C_1 \leq C_1[\sqrt{N}]\log^4 N \leq \frac{c}{2}N/\log^2 N - C$$

for N large enough. Thus

$$|S_2| \geq \frac{2^{c_1 \log N/\log\log N}}{2\log^2 N} \geq 2^{\frac{c_1}{2}\log N/\log\log N}. \qquad \square$$

We continue the proof of Proposition 6.1. At least half of the tunnels T_i are good by Claim 6.2. Hence the number of cubes $\overline{q_{i,t}}$ where u changes a sign is at least $\frac{1}{2}[\sqrt{N}]^{n-1}2^{c_2 \log N/\log\log N}$. For any such cube, let us fix a point $x_{i,t} \in \overline{q_{i,t}}$ such that $u(x_{i,t}) = 0$. We had find many disjoint cubes with sign changes. To replace the cubes by balls is not difficult.

The fact that the side of $\overline{q_{i,t}}$ is comparable to $\frac{r}{\sqrt{N}\log^6 N}$ shows that each ball $B_g(x_{i_0,t_0}, \frac{r}{\sqrt{N}})$ intersects not greater than $C_3[\log N]^{6n}$ other balls $B_g(x_{i,t}, \frac{r}{\sqrt{N}})$. We can choose the maximal set of disjoint balls $B_g(x_{i,t}, \frac{r}{\sqrt{N}})$. Since the number of $x_{i,t}$ is at least $\frac{1}{2}[\sqrt{N}]^{n-1}2^{c_2 \log N/\log\log N}$ and the number of intersections for each ball is bounded by $C_3[\log N]^{6n}$, the maximal set of disjoint balls $B_g(x_{i,t}, \frac{r}{\sqrt{N}})$ will consist of at least $\frac{c_3[\sqrt{N}]^{n-1}2^{c_2 \log N/\log\log N}}{[\log N]^{6n}}$ balls. We can choose $c_4 \in (0, c_2)$ such that

$$\frac{c_3[\sqrt{N}]^{n-1}2^{c_2 \log N/\log\log N}}{[\log N]^{6n}} \geq [\sqrt{N}]^{n-1}2^{c_4 \log N/\log\log N}$$

for large enough N. $\square$

Remark 6.4. The following remark will not be used later but shows the flexibility of the construction. In the statement and in the proof of Proposition 6.1 one can replace $\sqrt{N}$ by N^α with any $\alpha \in (0,1)$ and the statement will remain true.

We fix a point O on the Riemannian manifold M equipped with Riemannian metric g. There is a sufficiently small radius $R_0 > 0$ such that for any ball $B_g(p, 2r) \subset B_g(O, R_0)$ and for any harmonic function u on $B_g(p, 2r)$, the following holds: If $\beta(p, r)$ is sufficiently large, then there is a number N with

$$\beta(p,r)/10 \leq N \leq 2\beta\left(p, \frac{3}{2}r\right)$$

and at least $N^{\alpha(n-1)}2^{c \log N/\log\log N}$ disjoint balls $B_g(x_i, \frac{r}{N^\alpha}) \subset B(p, 2r)$ such that $u(x_i) = 0$.

7. Estimate of the volume of the nodal set

In this section we prove Theorem 1.2. We formulate it in the scaled form. Define the function

$$F(N) := \inf \frac{H^{n-1}(\{u=0\} \cap B_g(x,\rho))}{\rho^{n-1}},$$

where the infimum is taken over all balls $B_g(x,\rho)$ within $B_g(O,R_0)$ and all harmonic functions u on M with respect to metric g such that $u(x)=0$ and $N(B_g(O,R_0)) \leq N$. Here we denote by the $N(B_g(O,R_0))$ the supremum of $\beta(x,r)$ over all $B_g(x,r) \subset B_g(O,R_0)$. Recall that the radius $R_0 = R_0(M,g,n,O)$ is a sufficiently small positive number.

THEOREM 7.1. *There exists $c>0$ such that $F(N) \geq c$ for all positive N.*

Proof. Let u be a harmonic function that vanishes at x, $B_g(x,\rho) \subset B(O,R_0)$ and $\beta_u(x,\rho) \leq N$ for $B_g(x,\rho) \subset B(O,R_0)$. By the almost monotonicity property of the doubling index we know that

$$N \geq \frac{1}{2} \lim_{\rho \to 0} \beta(x,\rho) \geq 1/2.$$

Hence N is separated from zero. Furthermore, let us assume that $F(N)$ is almost attained on u:

$$\frac{H^{n-1}(\{u=0\} \cap B_g(x,\rho))}{\rho^{n-1}} \leq 2F(N). \tag{30}$$

We start with a naive and well-known estimate that gives some lower bound for $F(N)$. There exists $c_1>0$ such that

$$\frac{H^{n-1}(\{u=0\} \cap B_g(x,\rho))}{\rho^{n-1}} \geq \frac{c_1}{(\beta(x,\rho/2))^{n-1}} \geq \frac{c_2}{N^{n-1}}. \tag{31}$$

This estimate follows from the fact that if a harmonic function u vanishes at x and has the frequency (or the doubling index) of $B_g(x,\rho/2)$ smaller than N, then one can inscribe in $B_g(x,\rho/2)$ a ball of radius $\sim \frac{\rho}{N}$ where u is positive and a ball of radius $\sim \frac{\rho}{N}$ where u is negative. For instance, see [10] for the details.

We can use the estimate (31) to bound $F(N)$ from below for small $\beta(x,\rho/2)$. Now, we will assume that N is sufficiently big and will show that $\beta(x,\rho/2)$ is bounded.

We argue by assuming the contrary. Let $\beta(x,\rho/2)$ be sufficiently big. Then we can apply Proposition 6.1 for the ball $B_g(x,2r) = B_g(x,\rho)$ and find a number

$$\widetilde{N} \geq \beta(x,\rho/2)/10$$

and $[\sqrt{\widetilde{N}}]^{n-1}2^{c_3\log\widetilde{N}/\log\log\widetilde{N}}$ disjoint balls $B_g(x_i,\frac{r}{\sqrt{\widetilde{N}}})$ within $B(x,2r)$ such that $u(x_i)=0$. For each i, we know that

$$H^{n-1}(\{u=0\}\cap B_g(x_i,\frac{r}{\sqrt{\widetilde{N}}}))\geq F(N)\left(\frac{r}{\sqrt{\widetilde{N}}}\right)^{n-1}.$$

Since the number of such balls is at least $[\sqrt{\widetilde{N}}]^{n-1}2^{c_3\log\widetilde{N}/\log\log\widetilde{N}}$ and these balls are disjoint and contained in $B_g(x,\rho)$, we get

$$H^{n-1}(\{u=0\}\cap B_g(x,\rho))\geq F(N)\left[\sqrt{\widetilde{N}}\ \right]^{n-1}2^{c_3\log\widetilde{N}/\log\log\widetilde{N}}\left(\frac{\rho/2}{\sqrt{\widetilde{N}}}\right)^{n-1}.$$

We can decrease c_3 to a smaller positive constant c_4 such that

$$H^{n-1}(\{u=0\}\cap B_g(x,\rho))\geq 2^{c_4\log\widetilde{N}/\log\log\widetilde{N}}F(N)\rho^{n-1} \tag{32}$$

if N is sufficiently large. The last observation contradicts to (30).

We have proved that $\widetilde{N}$ is bounded from above by some positive constant N_0, and we can use (31) with $\rho=r$ to obtain the uniform bound $F(N)\geq\frac{c_5}{N_0^{n-1}}$. □

Remark 7.2. Now, we know that $F(N)$ is uniformly bounded from below. Since $\beta(x,\rho/2)/10\leq\widetilde{N}$, the inequality (32) implies the following estimate of the volume of the nodal set:

$$\frac{H^{n-1}(\{u=0\}\cap B_g(x,\rho))}{\rho^{n-1}}\geq 2^{c_6\log\beta(x,\rho/2)/\log\log\beta(x,\rho/2)}$$

for $\beta(x,\rho/2)>\beta_0$.

8. **The lower bound in Yau's conjecture**

In this section we prove Theorem 1.1. Let B be a geodesic ball of fixed radius on a Riemannian manifold M. Consider a function u on B that satisfies $\Delta u+\lambda u=0$ in B and the harmonic extension of u

$$h(x,t)=u(x)\exp(\sqrt{\lambda}t).$$

The following lemma is well known, but for the convenience of the reader we give the proof below.

LEMMA 8.1. *There exists $C_1>0$ and $\lambda_0>0$, depending on M and B only such that if $\lambda>\lambda_0$, then Z_u is $\frac{C_1}{\sqrt{\lambda}}$ dense in B.*

Proof. Let y be a point in $B\times[-1,1]$. Denote the geodesic ball with center at y and radius r on $M\times\mathbb{R}$ by $B_{M\times\mathbb{R}}(y,r)$. The Harnack inequality for harmonic functions says that there exist $C_1(M,B)>1$ and $r_0(M,B)>0$

such that if $0 < r < r_0$ and h is positive on $B_{M\times\mathbb{R}}(y,r)$, then for any $\widetilde{y} \in B_{M\times\mathbb{R}}(y,r/2)$, the following inequality holds:

$$h(\widetilde{y}) < C_1 h(y).$$

Let us formulate the Harnack inequality in the following form: if $|h(\widetilde{y})| \geq C_1|h(y)|$, then h changes sign in $B_{M\times\mathbb{R}}(y,r)$.

Let $C_2 = \log C_1$. Consider a point $y = (x,0)$, $x \in B$ and the point $\widetilde{y} = (x, C_2/\sqrt{\lambda})$. Since $h(\widetilde{y}) = C_1 h(y)$, by the Harnack inequality we know that if λ is sufficiently big and $B_{M\times\mathbb{R}}(y, 3C_2/\sqrt{\lambda}) \subset B \times [-1,1]$, then there is a point $\widetilde{\widetilde{y}} \in B\times[-1,1]$ such that $h(\widetilde{\widetilde{y}}) = 0$ and $d_g(y,\widetilde{\widetilde{y}}) \leq 3C_2/\sqrt{\lambda}$. In other words, Z_h is $\frac{\text{const}}{\sqrt{\lambda}}$ dense in $B\times[-1,1]$. Since $Z_h = Z_u \times \mathbb{R}$, the zero set Z_u is also $\frac{C}{\sqrt{\lambda}}$ dense in B. □

Now, it is a straightforward matter to prove Theorem 1.1.

Proof. By Lemma 8.1, if $\lambda > \lambda_0$, then Z_u is $\frac{C}{\sqrt{\lambda}}$ dense in B and we can find $c(\sqrt{\lambda})^n$ disjoint balls $B_M(x_i, C/\sqrt{\lambda})$ such that $u(x_i) = 0$. It is sufficient to show that

$$H^{n-1}(Z_u \cap B_M(x_i, C/\sqrt{\lambda})) \geq c_1 \lambda^{-\frac{n-1}{2}} \tag{33}$$

for some $c_1 = c_1(M,B) > 0$. Indeed, since the balls are disjoint, it would immediately give $H^{n-1}(Z_u \cap B) \geq c_2\sqrt{\lambda}$.

We can apply Theorem 7.1 for the function h to see that

$$H^n(Z_h \cap B_{M\times\mathbb{R}}((x_i,0), \frac{C}{\sqrt{\lambda}})) \geq c_3 \lambda^{-n/2}.$$

In view of $Z_h = Z_u \times \mathbb{R}$, that gives (33). □

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