Annals of Mathematics **186** (2017), 647–648
https://doi.org/10.4007/annals.2017.186.2.7

Erratum on “On the number of generators of ideals in polynomial rings”

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Abstract

We explain a mistake that occurred in the proof of Murthy's conjecture by the author.

Introduction

The purpose of this erratum is to explain a counter-example to [Fas16, Th. 3.2.7]. It follows that the proof of Murthy's conjecture stated in [Fas16] no longer holds, i.e., that the conjecture is still open. We would like to thank Mrinal Das for spotting the mistake in [Fas16, Lemma 3.2.3].

1. **The counter-example**

In [Fas16, Th. 3.2.7], the following result was stated.

Theorem. *Let k be an infinite field of characteristic different from* 2, *and let R be an essentially smooth k-algebra. Moreover*, *let $n \geq 2$, $v \in Q_{2n}(R)$ and $v_0 = (0, \ldots, 0) \in Q_{2n}(R)$. The strong lifting property holds for the row v if and only if $v \in v_0 EO_{2n+1}(R)$.*

However, this result is incorrect. It follows from [MPM98, Ex. 2.4] that the conclusion cannot hold as we now show.

Let $R = \mathbb{C}[X, Y]$, and let $f = X^3 + Y^3 - 1 \in R$. There exists a matrix $\sigma \in SL_2(R/I)$ whose class in $SK_1(R/I)$ is nontrivial. Let $(\overline{a}_1, \overline{a}_2) = e_1\sigma$ be the first row of σ, and let $I = (f) \subset R$. It is easy to see that $(\overline{a_1 f}, \overline{a_2 f})$ generate I/I^2. Now, we have a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ I & \longrightarrow & I/I^2 \end{array}$$

Keywords: polynomial rings, number of generators of ideals, complete intersections
AMS Classification: Primary: 13A15, 13C40, 14M10; Secondary: 14R10, 19M05, 19G38.

whose vertical maps are isomorphisms. If the set of generators $(\overline{a_1 f}, \overline{a_2 f})$ of I/I^2 lifts to a set of generators of I, we then see that the matrix σ lifts to a matrix in $SL_2(R)$, contradicting the fact that its class in $SK_1(R/I)$ is nontrivial.

Let us show now that this example contradicts [Fas16, Th. 3.2.7]. Let $a_1, a_2 \in R$ be such that their classes modulo I are respectively $\overline{a}_1, \overline{a}_2$. By assumption, there exists $b_1, b_2, r \in R$ such that $a_1 b_1 + a_2 b_2 = 1 - rf$. This yields $a_1 f b_1 r + a_2 f b_2 r = rf - r^2 f^2$ and thus a row $v = (a_1 f, a_2 f, rf) \in Q_4(R)$ such that $I(v) = I = (f)$. As $R = \mathbb{C}[X, Y]$, it follows that $v \in v_0 EO_5(R)$. Yet, the strong lifting property is not satisfied since $(\overline{a_1 f}, \overline{a_2 f})$ do not lift to generators of I.

The mistake in the proof of [Fas16, Th. 3.2.7] lies in Lemma [Fas16, Lemma 3.2.3]. If the ideal $I(vM)$ is indeed generated by the set of elements given there, it is not true that it satisfies the strong lifting property. As a consequence, the proof of [Fas16, Th. 3.2.9] collapses.

References

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(Received: December 12, 2016)
(Revised: June 6, 2017)

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