Annals of Mathematics **186** (2017), 641–646
https://doi.org/10.4007/annals.2017.186.2.6

Total Betti numbers of modules of finite projective dimension

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Abstract

The Buchsbaum-Eisenbud-Horrocks Conjecture predicts that the i^{th} Betti number $\beta_i(M)$ of a nonzero module M of finite length and finite projective dimension over a local ring R of dimension d should be at least $\binom{d}{i}$. It would follow from the validity of this conjecture that $\sum_i \beta_i(M) \geq 2^d$. We prove the latter inequality holds in a large number of cases and that, when R is a complete intersection in which 2 is invertible, equality holds if and only if M is isomorphic to the quotient of R by a regular sequence of elements.

1. Introduction

We recall a long-standing conjecture (see [3, 1.4] and [6, Prob. 24]):

Conjecture (Buchsbaum-Eisenbud-Horrocks Conjecture). *Let R be a commutative Noetherian ring such that* $\operatorname{Spec}(R)$ *is connected, and let M be a nonzero, finitely generated R-module of finite projective dimension. For any finite projective resolution $0 \to P_d \to \cdots \to P_1 \to P_0 \to M \to 0$ of M, we have*

$$\operatorname{rank}_R(P_i) \geq \binom{c}{i},$$

where $c = \operatorname{height}_R(\operatorname{ann}_R(M))$, *the height of the annihilator ideal of M.*

The validity of the Buchsbaum-Eisenbud-Horrocks Conjecture would imply that the "total rank" of any projective resolution of M is at least 2^c. In this paper, we prove this latter inequality holds in a large number of cases:

Theorem 1. *Assume R, M, and $P_\cdot$ are as in the Buchsbaum-Eisenbud-Horrocks Conjecture and, in addition, that*

Keywords: Betti numbers, Buchsbaum-Eisenbud-Horrocks Conjecture
AMS Classification: Primary: 13D02.
This work was partially supported by grant #318705 from the Simons Foundation.

(1) *R is locally a complete intersection and M is* 2*-torsion free*, *or*
(2) *R contains* $\mathbb{Z}/p$ *as a subring for an odd prime p*.

Then $\sum_i \operatorname{rank}_R(P_i) \geq 2^c$, *where* $c = \operatorname{height}_R(\operatorname{ann}_R(M))$.

Theorem 2 below is the special case of Theorem 1 in which we assume R is a local ring and M has finite length. We record it as a separate theorem since Theorem 1 follows immediately from it and also because in the local situation we can say a bit more.

For a local ring R and a finitely generated R-module M, let $\beta_i(M)$ be the i^{th} Betti number of R, defined to be the rank of the i^{th} free module in the minimal free resolution of M.

Theorem 2. *Assume* $(R, \mathfrak{m}, k)$ *is a local* (*Noetherian*, *commutative*) *ring of Krull dimension d and that M is a nonzero R-module of finite length and finite projective dimension. If either*

(1) *R is the quotient of a regular local ring by a regular sequence of elements and* 2 *is invertible in R*, *or*
(2) *R contains* $\mathbb{Z}/p$ *as a subring for an odd prime p*,

then $\sum_i \beta_i(M) \geq 2^d$.

Moreover, *if the assumptions in* (1) *hold and* $\sum_i \beta_i(M) = 2^d$, *then M is isomorphic to the quotient of R by a regular sequence of d elements.*

To see that Theorem 1 follows from Theorem 2, with the notation of the first theorem, let $\mathfrak{p}$ be a minimal prime containing $\operatorname{ann}_R(M)$ of height c. Then $\dim(R_{\mathfrak{p}}) = c$, $M_{\mathfrak{p}}$ has finite length, and $\beta_i(M_{\mathfrak{p}}) \leq \operatorname{rank}_R(P_i)$ for all i. Moreover, if M is 2-torsion free, then $2 \notin \mathfrak{p}$ and hence is invertible in $R_{\mathfrak{p}}$.

I thank Seth Lindokken, Michael Brown, Claudia Miller, Peder Thompson and Luchezar Avramov for useful conversations about this paper.

2. Complete intersections of residual characteristic not 2

In this section we prove part (1) of Theorem 2 and the assertion concerning when the equation $\sum_i \beta_i(M) = 2^d$ holds; see Theorem 2.4 below.

For any local ring $(R, \mathfrak{m}, k)$, let $\operatorname{Perf}^{\mathsf{fl}}(R)$ be the category of bounded complexes of finite rank free R-modules $F.$ such that $H_i(F.)$ has finite length for all i, and define $K_0^{\mathsf{fl}}(R)$ to be the Grothendieck group of $\operatorname{Perf}^{\mathsf{fl}}(R)$. Recall that $K_0^{\mathsf{fl}}(R)$ is generated by isomorphism classes of objects of $\operatorname{Perf}^{\mathsf{fl}}(R)$, modulo relations coming from short exact sequences and quasi-isomorphisms.

Let $\psi^2 : K_0^{\mathsf{fl}}(R) \to K_0^{\mathsf{fl}}(R)$ be the 2^{nd} Adams operation, as defined by Gillet-Soulé [4]. Gillet-Soulé's definition involves the Dold-Kan correspondence between complexes and simplicial modules, but if 2 is invertible in R, then ψ^2 admits a simpler description: For $F. \in \operatorname{Perf}^{\mathsf{fl}}(R)$, let $T^2(F.)$ denote its second tensor power $F. \otimes_R F.$ endowed with the action of the symmetric group $\Sigma_2 = \langle \tau \rangle$

given by

$$\tau \cdot (x \otimes y) = (-1)^{|x||y|} y \otimes x.$$

Since $\frac{1}{2} \in R$, we have a direct sum decomposition $T^2(F.) = S^2(F.) \oplus \Lambda^2(F.)$, where $S^2(F.) := \ker(\tau - \mathrm{id})$ and $\Lambda^2(F.) := \ker(\tau + \mathrm{id})$. By [1, 6.14] we have

$$\psi^2[F.] = [S^2(F.)] - [\Lambda^2(F.)] \in K_0^{\mathsf{fl}}(R). \tag{2.1}$$

Let ℓ_R denote the length of an R-module, and write $\chi : K_0^{\mathsf{fl}}(R) \to \mathbb{Z}$ for the Euler characteristic map: $\chi([F.]) = \sum_i (-1)^i \, \ell_R \, H_i(F.)$.

Proposition 2.2 (Gillet-Soulé; see [4, 7.1]). *If R is a local complete intersection of dimension d, then $\chi \circ \psi^2 = 2^d \cdot \chi$.*

Definition 2.3. A local ring $(R, \mathfrak{m}, k)$ of dimension d such that 2 is invertible in R will be called a *quasi-Roberts ring* if there we have an equality of maps $\chi \circ \psi^2 = 2^d \cdot \chi$.

Theorem 2.4. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d such that 2 is invertible in R. If R is a quasi-Roberts ring, then for any nonzero R-module M of finite length and finite projective dimension, we have $\sum_i \beta_i(M) \geq 2^d$.*

Moreover, if $\sum_i \beta_i(M) = 2^d$, then $M \cong R/(y_1, \ldots, y_d)$ for some regular sequence of elements $y_1, \ldots, y_d \in \mathfrak{m}$.

Proof. Let $F.$ be the minimal free resolution of M, so that $\chi(F.) = \ell_R(M)$ and $\mathrm{rank}_R(F_i) = \beta_i(M)$. Using (2.1) we get

(2.5)
$$\begin{aligned} 2^d \cdot \ell_R(M) = \chi(\psi^2(F)) &= \sum_i (-1)^i \, \ell_R \, H_i(S^2(F.)) - \sum_j (-1)^j \, \ell_R \, H_j(\Lambda^2(F.)) \\ &\leq \sum_{i \text{ even}} \ell_R \, H_i(S^2(F.)) + \sum_{i \text{ odd}} \ell_R \, H_i(\Lambda^2(F.)). \end{aligned}$$

Since $S^2(F.)$ and $\Lambda^2(F.)$ are direct summands of $F. \otimes_R F.$,

$$\sum_{i \text{ even}} \ell_R \, H_i(S^2(F.)) + \sum_{i \text{ odd}} \ell_R \, H_i(\Lambda^2(F.)) \leq \sum_i \ell_R \, H_i(F. \otimes_R F.). \tag{2.6}$$

For each i, $H_i(F. \otimes_R F.) \cong H_i(F. \otimes_R M)$ is a subquotient of $F_i \otimes_R M$ and thus

$$\ell_R \, H_i(F. \otimes_R M) \leq \ell_R(F_i \otimes_R M) = \mathrm{rank}(F_i) \cdot \ell_R(M) = \beta_i(M) \cdot \ell_R(M). \tag{2.7}$$

Putting the inequalities (2.5), (2.6), and (2.7) together yields

$$2^d \cdot \ell_R(M) \leq \ell_R(M) \cdot \sum_i \beta_i(M),$$

and since $\ell_R(M) > 0$, we conclude $\sum_i \beta_i(M) \geq 2^d$.

Now suppose $\sum_i \beta_i(M) = 2^d$. Then the inequalities (2.5), (2.6), and (2.7) must actually be equalities, which means that $H_i(S^2(F.)) = 0$ for all odd i, $H_j(\Lambda^2(F.)) = 0$ for all even j, and $F. \otimes_R M$ has trivial differential. Since

$H_0(\Lambda^2(F.)) \cong \Lambda^2(M)$ is the classical second exterior power, M must be cyclic, i.e., of the form R/I for some ideal I. Since $F. \otimes_R R/I$ has trivial differential, $I/I^2 \cong \mathrm{Tor}_1^R(R/I, R/I)$ is free as an R/I-module, and thus a result of Ferrand and Vasconcelos (see [2, 2.2.8]) gives that I is generated by a regular sequence of elements. □

3. **Rings of odd characteristic**

In this section we prove part (2) of Theorem 2. The main idea is to replace the Euler characteristic χ occurring in the proof of part (1) with the *Dutta multiplicity*.

Definition 3.1. Assume $(R, \mathfrak{m}, k)$ is a complete local ring of dimension d that contains $\mathbb{Z}/p$ as a subring for some prime p and that k is a perfect field. For $F. \in \mathrm{Perf}^{\mathsf{fl}}(R)$, define

$$\chi_\infty(F.) = \lim_{e\to\infty} \frac{\chi(\varphi^e F.)}{p^{de}},$$

where φ^e denotes extension of scalars along the e^{th} iterate of the Frobenius endomorphism of R. The limit is known to exist by, e.g., [9, 7.3.3].

Proof of Theorem 2 *part* (2). There is a faithfully flat map $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ of local rings such that $\mathfrak{m} \cdot R' = \mathfrak{m}'$, R' is complete and k' is algebraically closed; see [5, 0.10.3.1]. Letting $M' := M \otimes_R R'$, we have that M' is a nonzero R'-module of finite length and finite projective dimension, $\beta_i^{R'}(M') = \beta_i^R(M)$ for all i, and $\dim(R') = \dim(R)$. We may therefore assume R is complete with algebraically closed residue field.

Let $F.$ be the minimal free resolution of M. Since R is complete with perfect residue field, a result of Roberts [9, 7.3.5] gives

$$\chi_\infty(F.) > 0 \tag{3.2}$$

and a result of Kurano-Roberts [7, 3.1] gives (using (2.1))

$$\chi_\infty(S^2(F.)) - \chi_\infty(\Lambda^2(F.)) = \chi_\infty(\psi^2(F.)) = 2^d \cdot \chi_\infty(F.). \tag{3.3}$$

For each $e \geq 0$, we have $\varphi^e S^2(F.) \cong S^2(\varphi^e F.)$ and $\varphi^e \Lambda^2(F.)) \cong \Lambda^2(\varphi^e F.)$, and thus

$$\chi_\infty(S^2(F.)) = \lim_{e\to\infty} \frac{1}{p^{de}} \sum_i (-1)^i \,\ell_R\, H_i(S^2(\varphi^e F.)),$$

$$\chi_\infty(\Lambda^2(F.)) = \lim_{e\to\infty} \frac{1}{p^{de}} \sum_i (-1)^i \,\ell_R\, H_i(\Lambda^2(\varphi^e F.)).$$

As in the proof of Theorem 2.4, for a fixed e, we have

$$\frac{1}{p^{de}}\sum_i(-1)^i\,\ell_R\,H_i(S^2(\varphi^e F.)) - \frac{1}{p^{de}}\sum_i(-1)^i\,\ell_R\,H_i(\Lambda^2(\varphi^e F.)) \leq \sum_j \ell_R\,H_j(T^2(\varphi^e F.)).$$

By [8, 1.7], the complex $\varphi^e(F.)$ is the minimal free resolution of the finite length module $\varphi^e(M)$ for each $e \geq 0$. As in the proof of Theorem 2.4, for each i, we have

$$\ell_R\,H_i(T^2(\varphi^e F.)) \leq \operatorname{rank}(\varphi^e F_i)\cdot \ell_R(\varphi^e M) = \beta_i(M)\cdot\chi(\varphi^e F.).$$

We have proven that

$$\frac{1}{p^{de}}\sum_i(-1)^i\,\ell_R\,H_i(\varphi^e S^2(F.)) - \frac{1}{p^{de}}\sum_i(-1)^i\,\ell_R\,H_i(\varphi^e\Lambda^2(F.)) \leq \frac{1}{p^{de}}\chi(\varphi^e F.)\cdot\sum_i \beta_i(M)$$

holds for each $e \geq 0$. Taking limits and using (3.3) gives

$$2^d\cdot\chi_\infty(F.) \leq \chi_\infty(F.)\cdot\sum_i\beta_i(M).$$

Since $\chi_\infty(F.) > 0$ by (3.2), we conclude $\sum_i \beta_i(M) \geq 2^d$. □

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(Received: April 7, 2017)
(Revised: May 22, 2017)

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