Annals of Mathematics **186** (2017), 237–275
https://doi.org/10.4007/annals.2017.186.1.6

Wilkie's conjecture for restricted elementary functions

By Gal Binyamini and Dmitry Novikov

To our teacher Askold Khovanskii, on the occasion of his 70th birthday.

Abstract

We consider the structure $\mathbb{R}^{\mathrm{RE}}$ obtained from $(\mathbb{R}, <, +, \cdot)$ by adjoining the restricted exponential and sine functions. We prove Wilkie's conjecture for sets definable in this structure: the number of rational points of height H in the transcendental part of any definable set is bounded by a polynomial in $\log H$. We also prove two refined conjectures due to Pila concerning the density of algebraic points from a fixed number field, or with a fixed algebraic degree, for $\mathbb{R}^{\mathrm{RE}}$-definable sets.

1. Introduction

1.1. *Statement of the main results.* Our main object of study is the structure

$$\mathbb{R}^{\mathrm{RE}} = (\mathbb{R}, <, +, \cdot, \exp|_{[0,1]}, \sin|_{[0,\pi]}). \tag{1}$$

The superscript RE stands for "restricted elementary". We consider the natural language for $\mathbb{R}^{\mathrm{RE}}$, where we also include constants for each real number. We will refer to formulas in this language as $\mathbb{R}^{\mathrm{RE}}$-formulas.

For a set $A \subset \mathbb{R}^m$, we define the *algebraic part* A^{alg} of A to be the union of all connected semi-algebraic subsets of A of positive dimension. We define the *transcendental part* A^{trans} of A to be $A \setminus A^{\mathrm{alg}}$.

Recall that the *height* of a (reduced) rational number $\frac{a}{b} \in \mathbb{Q}$ is defined to be $\max(|a|, |b|)$. More generally, for $\alpha \in \mathbb{Q}^{\mathrm{alg}}$ we denote by $H(\alpha)$ its absolute multiplicative height as defined in [4]. For a vector $\boldsymbol{\alpha}$ of algebraic numbers, we denote by $H(\boldsymbol{\alpha})$ the maximum among the heights of the coordinates.

Let $\mathcal{F} \subset \mathbb{C}$ denote a number field that will be fixed throughout this paper. For a set $A \subset \mathbb{C}^m$, we denote the set of $\mathcal{F}$-points of A by $A(\mathcal{F}) := A \cap \mathcal{F}^m$ and

Keywords: Pfaffian functions, rational points, definable sets, metric entropy
AMS Classification: Primary: 11G99, 03C64, 11U09.
© 2017 Department of Mathematics, Princeton University.

denote

$$A(\mathcal{F},H) := \{\mathbf{x} \in A(\mathcal{F}) : H(\mathbf{x}) \leqslant H\}. \tag{2}$$

The following is our main result.

THEOREM 1. *Let $A \subset \mathbb{R}^m$ be $\mathbb{R}^{\mathrm{RE}}$-definable. Then there exist integers $\kappa := \kappa(A)$ and $N = N(A, [\mathcal{F} : \mathbb{Q}])$ such that*

$$\#A^{\mathrm{trans}}(\mathcal{F},H) \leqslant N \cdot (\log H)^{\kappa}. \tag{3}$$

Theorem 1 establishes a conjecture of Wilkie [26, Conj. 1.11] for the case of the restricted exponential function. It also establishes a refined version due to Pila [24, Conj. 1.4], who conjectured that the exponent κ can be chosen to be independent of the field $\mathcal{F}$. For a statement of the full conjectures and an outline of the history of the problem, see Section 1.2.

We also prove an additional conjecture of Pila [24, Conj. 1.5] (in the case of the restricted exponential) on counting algebraic points of a fixed degree without restricting to a fixed number field. For $k \in \mathbb{N}$, we denote

$$A(k) := \{\mathbf{x} \in A : [\mathbb{Q}(x_1) : \mathbb{Q}], \dots, [\mathbb{Q}(x_m) : \mathbb{Q}] \leqslant k\}, \tag{4}$$

$$A(k,H) := \{\mathbf{x} \in A(k) : H(\mathbf{x}) \leqslant H\}. \tag{5}$$

Then we have the following.

THEOREM 2. *Let $A \subset \mathbb{R}^m$ be $\mathbb{R}^{\mathrm{RE}}$-definable. Then there exist integers $\kappa := \kappa(A,k)$ and $N = N(A,k)$ such that*

$$\#A^{\mathrm{trans}}(k,H) \leqslant N \cdot (\log H)^{\kappa}. \tag{6}$$

Note that in Theorem 2 the exponent κ may depend on the degree k.

1.2. *Background.* In [5], Bombieri and Pila considered the following problem: let $f : [0,1] \to \mathbb{R}$ be an analytic function and $X \subset \mathbb{R}^2$ its graph. What can be said about the number of integer points in the homothetic dilation tX? They showed that if f is transcendental, then for every $\varepsilon > 0$ there exists a constant $c(f,\varepsilon)$ such that $\#(tX \cap \mathbb{Z}^2) \leqslant c(f,\varepsilon)t^{\varepsilon}$ for all $t \geqslant 1$. The condition of transcendence is necessary, as can be observed by the simple example $f(x) = x^2$ satisfying $\#(tX \cap \mathbb{Z}^2) \simeq t^{1/2}$. The proof of [5] introduced a new method of counting integer points using certain interpolation determinants.

In [19] Pila extended the method of [5] to the problem of counting rational points on X. In particular, he proved that if f is transcendental, then for every $\varepsilon > 0$ there exists a constant $c(f,\varepsilon)$ such that $\#X(\mathbb{Q},H) \leqslant c(f,\varepsilon)H^{\varepsilon}$ for all $H \in \mathbb{N}$. In this generality, the asymptotic $O(H^{\varepsilon})$ is essentially the best possible, as illustrated by [20, Exam. 7.5].

Moving beyond the case of curves one encounters a new phenomenon: a set X may be transcendental while still containing algebraic curves, and in such a case (as illustrated by the graph of $x \to x^2$) one cannot expect the asymptotic $\#X(\mathbb{Q}, H) = O(H^\varepsilon)$. However, in [21, Th. 1.1] Pila showed that for compact subanalytic surfaces, this is the only obstruction. More precisely, for any compact subanalytic surface $X \subset \mathbb{R}^n$ and $\varepsilon > 0$ there exists a constant $C(X, \varepsilon)$ such that $\#X^{\text{trans}}(\mathbb{Q}, H) \leqslant C(X, \varepsilon)H^\varepsilon$. The same result for arbitrary compact subanalytic sets was conjectured in [20, Conj. 1.2]. In [26, Th. 1.8] Pila and Wilkie proved this conjecture in a considerably more general setting. Namely, they showed that if the set X is definable in *any* O-minimal structure and $\varepsilon > 0$, then there exists a constant $C(X, \varepsilon)$ such that $\#X^{\text{trans}}(\mathbb{Q}, H) \leqslant C(X, \varepsilon)H^\varepsilon$. This result contains, in particular, the case of compact subanalytic sets (and more generally globally subanalytic sets), obtained for the O-minimal structure of restricted analytic functions $\mathbb{R}_{\text{an}}$. It also contains much wider classes of definable sets, for instance those definable in the structure $\mathbb{R}_{\text{an,exp}}$ obtained by adjoining the graph of the *unrestricted* exponential function to $\mathbb{R}_{\text{an}}$. The Pila-Wilkie theorem in this generality turned out to have many important diophantine applications; see, e.g., [27] for a survey.

As mentioned earlier, the asymptotic $O(H^\varepsilon)$ is essentially the best possible if one allows arbitrary subanalytic sets (even analytic curves). However, one may hope that in more tame geometric contexts much better estimates can be obtained. In [26, Conj. 1.11] Wilkie conjectured that if X is definable in $\mathbb{R}_{\text{exp}}$, i.e., using the unrestricted exponential but without allowing arbitrary restricted analytic functions, then there exist constants $N(X)$ and $\kappa(X)$ such that

$$\#X^{\text{trans}}(\mathbb{Q}, H) \leqslant N(X) \cdot (\log H)^{\kappa(X)}. \tag{7}$$

In [24, Conjs. 1.4 and 1.5] Pila proposed two generalizations of this conjecture: namely, that for an arbitrary number field $\mathcal{F} \subset \mathbb{R}$, one has

$$\#X^{\text{trans}}(\mathcal{F}, H) \leqslant N(X, \mathcal{F}) \cdot (\log H)^{\kappa(X)}, \tag{8}$$

where only the constant $N(X, \mathcal{F})$ is allowed to depend on $\mathcal{F}$, and for $k \in \mathbb{N}$ one has

$$\#X^{\text{trans}}(k, H) \leqslant N(X, k) \cdot (\log H)^{\kappa(X,k)}, \tag{9}$$

where both constants are allowed to depend on k.

Some low-dimensional cases of the Wilkie conjecture have been established. In [22, Th. 1.3] Pila proved the analog of the Wilkie conjecture for graphs of Pfaffian functions (see Section 5 for the definition) or plane curves defined by the vanishing of a Pfaffian function. In [14, Cor. 5.5] Jones and Thomas have shown that the analog of the Wilkie conjecture holds for surfaces definable in the structure of restricted Pfaffian functions. In [24], [6] the Wilkie conjecture is confirmed for some special surfaces defined using the *unrestricted* exponential.

1.3. *Overview of the proof.*

1.3.1. *The Pfaffian category.* Our approach is based on an interplay between ideas of complex analytic geometry and the theory of Pfaffian functions. We briefly pause to comment on the latter. In [15] Khovanskii introduced the class of Pfaffian functions, defined as functions satisfying a type of triangular system of polynomial differential equations; see Section 5 for details. The Pfaffian functions enjoy good finiteness properties and have played a fundamental role in Wilkie's work on the model-completeness of $\mathbb{R}_{\exp}$ [28].

From the Pfaffian functions one can form the class of semi-Pfaffian sets, i.e., sets defined by a boolean combination of Pfaffian equalities and inequalities, and sub-Pfaffian sets, i.e., projections of semi-Pfaffian sets. Pfaffian functions have a natural notion of degree, and by works of Khovanskii [15] and Gabrielov and Vorobjov [11], [12], the number of connected components of any semi- or sub-Pfaffian set can be explicitly estimated from above in terms of the degrees of the Pfaffian functions involved; see Theorem 6. Moreover, for us it is important that these estimates are *polynomial* in the degrees.

1.3.2. *The holomorphic-Pfaffian category.* The theory of Pfaffian functions is an essentially real theory, based on topological ideas going back to the classical Rolle theorem. The holomorphic continuation of a real Pfaffian function on $\mathbb{R}^n$ is not, in general, a Pfaffian function on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. However, since our arguments are complex-analytic in nature, we restrict attention to *holomorphic-Pfaffian* functions: holomorphic functions whose graphs (in an appropriate domain) are sub-Pfaffian sets. It is a small miracle that the graph of the complex exponential e^z, and hence also of $\sin z$, is indeed a Pfaffian set when restricted to a strip. A similar feature is used in an essential way in the work of van den Dries [8], which we discuss below.

1.3.3. *The inductive scheme for counting rational points.* Fix some domain $\Omega \subset \mathbb{C}^n$. We begin by explaining our estimate for $\#X(\mathbb{Q}, H)$ when $X \subset \mathbb{C}^n$ is a *holomorphic-Pfaffian variety*, i.e., a set cut out by holomorphic-Pfaffian equations of Pfaffian degree β. For simplicity, we assume $X = X^{\mathrm{trans}}$. Our basic strategy is similar to the strategy used by Pila and Wilkie [26]: we seek to cover X by smaller pieces X_k, such that for each piece, one can find an algebraic hypersurface H_k with $X_k(\mathbb{Q}, H) \subset X_k \cap H_k$.

By way of comparison, in [26] the subdivision is performed in two steps. One first applies a reparametrization theorem to write X as the union of images of C^r-smooth maps $\phi_j : (0,1)^{\dim X} \to X$ with unit norms: this step is independent of H. One then subdivides each cube $(0,1)^{\dim X}$ into H^ε subcubes, and for each subcube, one constructs the hypersurface as above using a generalization of the Bombieri-Pila method [5]. Crucially for [26], the degrees of these hypersurfaces can be chosen to depend only on ε but not on H.

However, to go beyond the asymptotic $O(H^\varepsilon)$ it appears that one must allow the degrees to depend on H.

In our approach X is covered by $\mathrm{poly}(\beta)$ pieces $X_k := X \cap \Delta_k$, where Δ_k is a *Weierstrass polydisc* for X, a notion introduced below. (More accurately, we take Δ_k to be a Weierstrass polydisc shrunk by a factor of two.) We then construct a hypersurface H_k of degree $\mathrm{poly}(\beta, \log H)$ containing $X_k(\mathbb{Q}, H)$. Consequently, we replace X by $X \cap (\cup_k H_k)$, which is guaranteed to have strictly smaller dimension and degree polynomial in β and $\log H$. One can then finish the proof by induction on dimension and eventually obtain a zero-dimensional holomorphic-Pfaffian variety defined by equations of degree $\mathrm{poly}(\log H)$ and hence having at most $\mathrm{poly}(\log H)$ points. We proceed to explain the subdivision step and the construction of the algebraic hypersurfaces.

1.3.4. *Weierstrass polydiscs and holomorphic decompositions.* We define a *Weierstrass polydisc* $\Delta = \Delta_z \times \Delta_w$ for X to be a polydisc in some coordinate system, where $\dim X = \dim \Delta_z$ and $X \cap (\Delta_z \times \partial \Delta_w) = \emptyset$. It follows from this definition that the projection from $X \cap \Delta$ to Δ_z is a finite (ramified) covering map, and all fibers have the same number of points (counted with multiplicities). We denote this number by $e(X, \Delta)$ and call it the *multiplicity* of Δ. Weierstrass polydiscs are ubiquitous in complex analytic geometry: they are the basic sets where a complex analytic variety can be expressed as a finite cover of a polydisc.

By a variant of Weierstrass division we prove the following polynomial interpolation result: for any holomorphic function f on a neighborhood of Δ, there is a function P on Δ, holomorphic in the z-variables and polynomial of degree at most $e(X, \Delta)$ in each of the w-variables, such that $f \equiv P$ on $X \cap \Delta$ (see Proposition 7). Moreover, the norm of P can be estimated in terms of the norm of f. The existence of such a decomposition immediately implies that Δ is the domain of a *decomposition datum* in the sense of [3] (see Definition 8). Then the results of [3], themselves a complex-analytic analog of the Bombieri-Pila interpolation determinant method [5], imply that $(X \cap \Delta)(\mathbb{Q}, H)$ is contained in an algebraic hypersurface of degree $d = \mathrm{poly}(e(X, \Delta), \log H)$. (For a precise statement, see Proposition 12.) It will therefore suffice to cover X by $\mathrm{poly}(\beta)$ Weierstrass polydiscs Δ each satisfying $e(X, \Delta) = \mathrm{poly}(\beta)$.

1.3.5. *Covering by Weierstrass polydiscs.* The multiplicity $e(X, \Delta)$ is relatively easy to estimate using Pfaffian methods, being the number of isolated solutions of a system of Pfaffian equations and inequalities. The heart of the argument is therefore the covering by Weierstrass polydiscs. For this purpose we prove the following, somewhat stronger statement (see Theorem 7): if $B \subset \Omega$ is a ball of radius r around a point $p \in \Omega$, then there is a Weierstrass polydisc $\Delta \subset B$ for X with center p and polyradius at least $r/\mathrm{poly}(\beta)$. In other words, every point p is the center of a relatively large Weierstrass polydisc. From this it is easy to deduce that X can be covered by $\operatorname{poly}(\beta)$ Weierstrass polydiscs.

We briefly comment on the proof of Theorem 7. Suppose first that X has complex codimension 1. In this case we show, by a simple geometric argument, that the theorem can be reduced to finding a ball of radius $r/\operatorname{poly}(\beta)$ disjoint from $S^1 \cdot X$, where $S^1 = \{|\zeta| = 1\}$ acts by scalar multiplication on $\mathbb{C}^n$. The set $S^1 \cdot X$ is also sub-Pfaffian, now of real codimension 1. We use an argument involving metric entropy, specifically Vitushkin's formula (in the form given by Friedland and Yomdin [9]) to show that $S^1 \cdot X$ can be covered by relatively few balls of radius $r/\operatorname{poly}(\beta)$, and elementary considerations then show that it must be disjoint from one (in fact, many) such ball. For X of arbitrary codimension, we use an induction on codimension by repeated projections.

1.3.6. *From $\mathbb{R}^{\mathrm{RE}}$-definable sets to holomorphic-Pfaffian varieties.* At this point our review of the proof for holomorphic-Pfaffian X is essentially complete. We now briefly discuss the case of a general $\mathbb{R}^{\mathrm{RE}}$-definable set A. Let $I = [-1, 1]$, and assume that $A \subset I^m$. (The general case is easily reduced to this one.) Our approach for this case is based on a quantifier-elimination result of van den Dries [8] (itself a variant of the work of Denef and van den Dries [7] on subanalytic sets). In [8] it is shown, up to some minor variations in formulation, that any A as above is definable by a quantifier-free formula in a language L^D_{RE} that has a natural interpretation in the structure I. This language has an order relation $<$, m-ary operation symbols for certain special functions $f : I^m \to I$, and a binary operation D called restricted division, interpreted in I as

$$D(x,y) = \begin{cases} x/y & |x| \leqslant |y| \text{ and } y \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The crucial feature for us is that all functions appearing in the language extend as holomorphic-Pfaffian functions to a complex neighborhood of I^m. Therefore a set defined by quantifier-free L_{RE}-formulas, i.e., not involving the restricted division D, is essentially the real part of a holomorphic-Pfaffian variety (after some work to handle inequalities).

To handle a formula involving a restricted division $D(x, y)$, we replace it by three formulas: one for the case $|x| > |y|$ or $y = 0$ where we replace $D(x, y)$ by 0; one for the case $|x| = |y| \neq 0$ where we replace $D(x, y)$ by 1; and one for the case $|x| < |y|$ where we replace $D(x, y)$ by a new variable z and add the equation $zy = x$ (which is equivalent to $z = D(x, y)$ when $x < y$). Repeating this for every restricted division in the formula, we reduce the set A to a union of projections (forgetting the variables z) of sets B_j definable by quantifier-free L_{RE}-formulas. Moreover, each fiber of the projection $\pi : B_j \to A$ contains at most one point: this corresponds to the fact that we only add a variable z

under the restriction that $|x| < |y|$ and, in particular, $y \neq 0$, and under these conditions the equation $yz = x$ uniquely defines z. We call this type of projections *admissible* (see Definition 27).

It remains to study rational points in sets of the form $\pi(B)$, where B is defined by a quantifier-free L_{RE}-formula and π is an admissible projection. Up to some minor details involving the algebraic part of B, we may replace B by its complex-analytic germ — which is a holomorphic-Pfaffian variety. The strategy described above for holomorphic-Pfaffian varieties extends in a straightforward manner to their admissible projections. The relevant statement is Theorem 8.

1.3.7. *From rational points to $\mathcal{F}$-points and points of degree k.* The proof can be carried out for $\mathcal{F}$-points rather than $\mathbb{Q}$-points in exactly the same manner. (We follow the strategy of Pila [24, Th. 3.2].) The case of algebraic points of degree k requires some additional work. We essentially follow the proof of [23], with some minor additional details needed to obtain the necessary degree estimates.

1.4. *Contents of this paper.* This paper is organized as follows. In Section 2 we prove some preliminary results of polynomial interpolation in the complex setting; define the notion of a Weierstrass polydisc; and establish a result on holomorphic decompositions of functions over a Weierstrass polydisc. In Section 3 we give upper and lower bounds for interpolation determinants over a fixed Weierstrass polydisc, in analogy with the Bombieri-Pila determinant method. In Section 4 we recall the notion of ε-entropy and Vitushkin's bound and derive some simple consequences that are needed in the sequel. In Section 5 we recall the Pfaffian, semi-Pfaffian and sub-Pfaffian categories; introduce the holomorphic-Pfaffian category; and prove the key technical result on covering of holomorphic-Pfaffian varieties by Weierstrass polydiscs. In Section 6 we prove an analog of the Wilkie conjecture for holomorphic-Pfaffian varieties and their projections by an induction over dimension. In Section 7 we generalize the results of Section 6 to arbitrary $\mathbb{R}^{\mathrm{RE}}$-definable sets and prove the main Theorems 1 and 2. Finally in Section 8 we give some concluding remarks related to effectivity and uniformity of the bounds and discuss possible generalizations to other structures.

1.5. *Acknowledgments.* We are grateful to Yosef Yomdin for numerous invaluable discussions.

2. **Polynomial interpolation, Weierstrass polydiscs and holomorphic decomposition**

We fix some basic notation. All complex domains considered in this paper are assumed to be relatively compact with piecewise smooth boundary. Let $\Omega \subset \mathbb{C}^n$ be a domain and $Z \subset \Omega$. We denote by $\mathcal{O}(Z)$ the ring of germs of holomorphic functions in a neighborhood of Z. If Z is relatively compact in Ω, we denote by $\|\cdot\|_Z$ the maximum norm on $\mathcal{O}(\bar{Z})$.

Let $A \subset \mathbb{C}^n$ be a ball or a polydisc around a point $p \in \mathbb{C}^n$ and $\delta > 0$. We let A^δ denote the δ^{-1}-rescaling of A around p, i.e., $A^\delta := p + \delta^{-1}(A - p)$.

2.1. *Polynomial interpolation.*

2.1.1. *Univariate interpolation.* Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ a function. Let $\mathbf{a} := \{\mathbf{a}_1, \ldots, \mathbf{a}_k\} \subset \Omega$ be a multiset of points and denote by $\nu(\mathbf{a}_j)$ the number of times $\mathbf{a}_j$ appears in $\mathbf{a}$. We define the *interpolation polynomial* $L[\mathbf{a}; f]$ to be the unique polynomial of degree at most $k - 1$ satisfying

$$L[\mathbf{a}; f]^{(l)}(\mathbf{a}_j) = f(\mathbf{a}_j), \qquad j = 1, \ldots, k, \quad l = 0, \ldots, \nu(\mathbf{a}_j) - 1. \tag{11}$$

Denote $h_{\mathbf{a}}(z) := \prod_{j=1}^{k}(z - \mathbf{a}_j)$. When f is a holomorphic function, the classical proof of the Weierstrass division theorem (see, e.g., [13]) shows that $L[\mathbf{a}; f]$ admits an integral representation as follows.

Proposition 1. *Let $\Omega \subset \mathbb{C}$ be a simply-connected domain and $\mathbf{a} \subset \Omega$. Let $f \in \mathcal{O}(\bar{\Omega})$. Then for $z \in \Omega$,*

$$L[\mathbf{a}; f](z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(\zeta)}{h(\zeta)} \frac{h(\zeta) - h(z)}{\zeta - z} \, \mathrm{d}\zeta, \qquad h := h_{\mathbf{a}}. \tag{12}$$

Proof. The right-hand side of (12) is easily seen to be a polynomial of degree $k - 1$ in z (since this is true for the integrand). Evaluating at $z = \mathbf{a}_j$, we have $h(z) = 0$, and the integral reduces to the Cauchy formula for $f(\mathbf{a}_j)$. □

Next we give norm estimates for $L[\mathbf{a}; f]$ in terms of the norm of f.

Proposition 2. *Let $D \subset \mathbb{C}$ be a disc, $\mathbf{a} \subset D$ a multiset and $f \in \mathcal{O}(\bar{D}^{1/3})$. Then*

$$\|L[\mathbf{a}; f]\|_D \leqslant 3 \|f\|_{D^{1/3}} . \tag{13}$$

Proof. Since the claim is invariant under affine transformations of $\mathbb{C}$, we may assume that D is the unit disc. Then we have for any $z \in D$, an estimate $|h(z)| \leqslant 2^k$ and for any $\zeta \in \partial D^{1/3}$ an estimate $|h(\zeta)| \geqslant (3 - 1)^k = 2^k$. Using the integral representation (12),

$$\begin{aligned} \|L[\mathbf{a}; f]\|_D &= \max_{z \in D} \left| \frac{1}{2\pi i} \oint_{\partial D^{1/3}} \frac{f(\zeta)}{\zeta - z} \left(1 - \frac{h(z)}{h(\zeta)}\right) \mathrm{d}\zeta \right| \\ &\leqslant 3 \frac{\|f\|_{D^{1/3}}}{3 - 1}(1 + 1) \leqslant 3 \|f\|_{D^{1/3}} . \end{aligned} \tag{14}$$

□

2.2. *Weierstrass polydiscs.* We say that $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is a *standard* coordinate system on $\mathbb{C}^n$ if it is obtained from the standard coordinates by an affine unitary transformation.

Let $\Omega \subset \mathbb{C}^n$ be a domain and $X \subset \Omega$ an analytic subset.

Definition 3. We say that a polydisc $\Delta = \Delta_z \times \Delta_w$ in the $\mathbf{x} = \mathbf{z} \times \mathbf{w}$ coordinates is a *pre-Weierstrass polydisc* for X if $\bar{\Delta} \subset \Omega$ and $(\bar{\Delta}_z \times \partial\Delta_w) \cap X = \emptyset$. We call Δ_z the base and Δ_w the fiber of Δ.

If X is pure-dimensional, we say that Δ is a *Weierstrass polydisc* for X if $\dim \mathbf{z} = \dim X$.

When speaking about (pre-)Weierstrass polydiscs we will assume (unless otherwise stated) that the coordinates are given by $\mathbf{x} = \mathbf{z} \times \mathbf{w}$. We will also denote by $\pi_z : \mathbb{C}^n \to \mathbb{C}^{\dim \mathbf{z}}$ the projection to the $\mathbf{z}$-coordinates and by π_z^X its restriction to $\Delta \cap X$.

We recall some standard facts.

FACT 4. *If Δ is a pre-Weierstrass polydisc for X, then π_z^X is proper and finite-to-one.*

Proof. Let $p_i \in \Delta \cap X$ be a sequence of points that escapes to the boundary of Δ. We will show that $\pi_z^X(p_i)$ escapes to the boundary of Δ_z. Assume otherwise. Passing to a subsequence we may assume that $\pi_z^X(p_i)$ converges in Δ_z, and passing to a further subsequence we may also assume that p_i converges in $\bar{\Delta}$. But then it must necessarily converge to a point in $(\Delta_z \times \partial\Delta_w) \cap X$, which is ruled out by the definition of a pre-Weierstrass polydisc. Thus π_z^X is proper, hence its fibers are compact complex submanifolds of Δ_w, and must therefore be finite. $\square$

FACT 5. *If X has pure dimension and Δ is a Weierstrass polydisc for X, then π_z^X is $e(X, \Delta)$-to-1 for some number $e(X, \Delta) \in \mathbb{N}$ (where points in the fiber are counted with multiplicities).*

Proof. In the Weierstrass case, $\dim X = \dim \mathbf{z}$ so $\dim X = \dim \pi_z^X(X)$. Under this condition it is well known that the map π_z^X is a finite unramified cover outside some proper analytic subset $B \subset \Delta_z$ [13, III.B]. Then the map is $e(X, \Delta)$-to-1, where $e(X, \Delta)$ is the cardinality of the fiber over any point in $\Delta_z \setminus B$. $\square$

LEMMA 6. *Let X have pure dimension and Δ be a Weierstrass polydisc for X. For $l = 1, \ldots, \dim \mathbf{w}$, there exists a monic polynomial*

$$P_l(z, \mathbf{w}_l) \in \mathcal{O}(\Delta_z)[\mathbf{w}_l], \qquad \deg P_l = e(X, \Delta) \tag{15}$$

such that for any $z \in \Delta_z$, the roots of $P_l(z, \mathbf{w}_l)$ are precisely the $\mathbf{w}_l$-coordinates of the points of $(\pi_z^X)^{-1}(z)$.

Proof. In the notation of Fact 5 and its proof, set $\nu = e(X, \Delta)$ and let $W_1, \ldots, W_\nu : \Delta_z \setminus B \to X \cap \Delta$ be the (ramified) inverses of π_z^X. Then

$$P_l(z, \mathbf{w}_l) = \prod_{j=1}^{\nu} \left(\mathbf{w}_l - \mathbf{w}_l(W_j(z))\right) \tag{16}$$

has univalued coefficients that are holomorphic outside B, and since $W_1, \dots, W_\nu$ are bounded near B, it follows from the Riemann removable singularity theorem that the coefficients extend to holomorphic functions in Δ_z. □

PROPOSITION 7. *Let X have pure dimension m. Let Δ be a Weierstrass polydisc for X, and set $\nu = e(X, \Delta)$. Let $f \in \mathcal{O}(\bar{\Delta}_z \times \bar{\Delta}_w^{1/3})$. There exists a function*

$$P \in \mathcal{O}(\bar{\Delta}_z)[\mathbf{w}], \qquad \deg_{\mathbf{w}_i} P \leqslant \nu - 1, \quad i = 1, \dots, n-m \tag{17}$$

such that $P|_{X\cap\Delta} = f|_{X\cap\Delta}$, and

$$\|P\|_\Delta \leqslant 3^{n-m} \|f\|_{\Delta_z \times \Delta_w^{1/3}}. \tag{18}$$

Proof. Set $s := \dim \mathbf{w} = n - m$. Let $\Delta_w = \prod_{i=1}^s D_i$. For $l = 1, \dots, s$, write

$$\hat{\mathbf{w}}_l := (\mathbf{w}_1, \dots, \mathbf{w}_{l-1}, \omega_l, \mathbf{w}_{l+1}, \dots, \mathbf{w}_s), \tag{19}$$

$$\Omega_l = \Delta_z \times D_1 \times \cdots \times D_{l-1} \times D_l^{1/3} \times \cdots \times D_s^{1/3}. \tag{20}$$

Consider the operator

$$L_l(g) = \frac{1}{2\pi i} \oint_{\partial D_l^{1/3}} \frac{g(z, \hat{\mathbf{w}}_l)}{P_l(z, \omega_l)} \frac{P_l(z, \omega_l) - P_l(z, \mathbf{w}_l)}{\omega_l - \mathbf{w}_l} \, \mathrm{d}\omega_l, \tag{21}$$

where P_l denotes the polynomial from Lemma 6.

We claim that L_l maps $\mathcal{O}(\bar{\Omega}_l)$ to $\mathcal{O}(\bar{\Omega}_{l+1})$. Indeed, let $g \in \mathcal{O}(\bar{\Omega}_l)$. For every fixed $z \in \Delta_z$, the roots of $P_l(z, \mathbf{w}_l)$ lie in D_l, and it follows that the integrand is holomorphic whenever $(z, w) \in \bar{\Omega}_{l+1}$ and ω_l lies in a neighborhood of $\partial D_l^{1/3}$. By Proposition 2 we have the norm estimate

$$\|L_l(g)\|_{\bar{\Omega}_{l+1}} \leqslant 3 \|g\|_{\bar{\Omega}_l}. \tag{22}$$

It is easy to see that if g is polynomial of degree at most $\nu - 1$ in $\mathbf{w}_j, j \neq l$, then so is $L_l(g)$. Moreover, Proposition 1 shows that $L_l(g)$ is polynomial of degree at most $\nu - 1$ in $\mathbf{w}_l$ and agrees with g for any point $(z, w) \in \bar{\Omega}_{l+1}$ such that $\mathbf{w}_l(w)$ is a root of P_l and, in particular, whenever $w \in (\pi_z^X)^{-1}(z)$.

Finally, setting

$$P = L_s \cdots L_1 f \in \mathcal{O}(\bar{\Omega}_{s+1}) = \mathcal{O}(\bar{\Delta}), \tag{23}$$

we obtain a polynomial of degree $\nu - 1$ in each variable $\mathbf{w}_1, \dots, \mathbf{w}_s$ that agrees with f whenever $w \in (\pi_z^X)^{-1}(z)$. The norm estimate (18) follows by repeated application of (22). □

2.3. *Decomposition data.* We recall the following definition from [3, Def. 4]. Given a standard system of coordinates $\mathbf{x}$, we say that (Δ, Δ') is a pair of polydiscs if $\Delta \subset \Delta'$ are two polydiscs with the same center in the $\mathbf{x}$ coordinates.

We view $\mathbb{N}^n$ as a semigroup with respect to coordinate-wise addition. An ideal is a subset $I \subset \mathbb{N}^n$ satisfying $I + \mathbb{N}^n \subset I$, and a co-ideal is the complement of an ideal. For a co-ideal $\mathcal{M} \subset \mathbb{N}^n$ and $k \in \mathbb{N}$, we denote by

$$\mathcal{M}^{\leqslant k} := \{\alpha \in \mathcal{M} : |\alpha| \leqslant k\} \tag{24}$$

and by $H_{\mathcal{M}}(k) := \#\mathcal{M}^{\leqslant k}$ its Hilbert-Samuel function. The function $H_{\mathcal{M}}(k)$ is eventually a polynomial in k, and we denote its degree by $\dim \mathcal{M}$.

Definition 8. Let $X \subset \mathbb{C}^n$ be a locally analytic subset, $\mathbf{x}$ a standard coordinate system, (Δ, Δ') a pair of polydiscs centered at the $\mathbf{x}$-origin and $\mathcal{M} \subset \mathbb{N}^n$ a co-ideal. We say that X admits *decomposition* with respect to the *decomposition datum*

$$\mathcal{D} := (\mathbf{x}, \Delta, \Delta', \mathcal{M}) \tag{25}$$

if there exists a constant denoted $\|\mathcal{D}\|$ such that for every holomorphic function $F \in \mathcal{O}(\bar{\Delta}')$, there is a decomposition

$$F = \sum_{\alpha \in \mathcal{M}} c_\alpha \mathbf{x}^\alpha + Q, \qquad Q \in \mathcal{O}(\bar{\Delta}), \tag{26}$$

where Q vanishes identically on $X \cap \Delta$ and

$$\|c_\alpha \mathbf{x}^\alpha\|_\Delta \leqslant \|\mathcal{D}\| \cdot \|F\|_{\Delta'} \qquad \forall \alpha \in \mathcal{M}. \tag{27}$$

We define the *dimension* of the decomposition datum, denoted $\dim \mathcal{D}$, to be $\dim \mathcal{M}$.

Since $H_{\mathcal{M}}(k)$ is eventually a polynomial of degree $\dim \mathcal{M}$, the function $H_{\mathcal{M}}(k) - H_{\mathcal{M}}(k-1)$ counting monomials of degree k in $\mathcal{M}$ is eventually a polynomial of degree $\dim \mathcal{M} - 1$. If $\dim \mathcal{D} \geqslant 1$, we denote by $e(\mathcal{D})$ the minimal constant satisfying

$$H_{\mathcal{M}}(k) - H_{\mathcal{M}}(k-1) \leqslant e(\mathcal{D}) \cdot L(\dim \mathcal{M}, k) \qquad \forall k \in \mathbb{N}, \tag{28}$$

where $L(n,k) := \binom{n+k-1}{n-1}$ denotes the dimension of the space of monomials of degree k in n variables. In the case $\dim \mathcal{D} = 0$, the co-ideal $\mathcal{M}$ is finite and we denote by $e(\mathcal{D})$ its size.

The following is the standard Cauchy inequality.

LEMMA 9 ([17, p. 6]). *Let Δ be a polydisc in the $\mathbf{x}$-coordinates and $F \in \mathcal{O}(\bar{\Delta})$. Then the Taylor expansion $F = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ satisfies*

$$\|c_\alpha \mathbf{x}^\alpha\|_\Delta \leqslant \|F\|_\Delta\,. \tag{29}$$

As a direct consequence of Proposition 7 and Lemma 9 we obtain the following theorem.

THEOREM 3. *Let X have pure dimension m. Let Δ be a Weierstrass polydisc for X, and set*

$$\nu = e(X,\Delta), \qquad \mathcal{M} = \mathbb{N}^m \times \{0,\dots,\nu-1\}^{n-m}, \qquad \Delta' = \Delta_z \times \Delta_w^{1/3}. \tag{30}$$

Then $(\mathbf{x},\Delta,\Delta',\mathcal{M})$ is a decomposition datum for X with $\|\mathcal{D}\| \leqslant 3^{n-m}$, $\dim\mathcal{M} = m$ and $e(\mathcal{D}) = \nu^{n-m}$.

3. Interpolation determinants

Let $\Omega \subset \mathbb{C}^n$ be a domain and $X \subset \Omega$ an analytic subset of pure dimension m. Let $\mathbf{x}$ be standard coordinates. Let Δ be a Weierstrass polydisc for X, and set $\Delta' := \Delta_z \times \Delta_w^{1/3}$ and $\nu := e(X,\Delta)$ as in Theorem 3.

3.1. *Interpolation determinants.* Let $\mathbf{f} := (f_1,\dots,f_\mu)$ be a collection of functions and $\mathbf{p} := (p_1,\dots,p_\mu)$ a collection of points. We define the *interpolation determinant*

$$\Delta(\mathbf{f},\mathbf{p}) := \det(f_i(p_j))_{1\leqslant i,j\leqslant\mu}. \tag{31}$$

LEMMA 10. *Assume $m > 0$. Suppose $f_i \in \mathcal{O}(\bar{\Delta}')$ with $\|f_i\|_{\Delta'} \leqslant M$ and $p_i \in \Delta^{1/\delta} \cap X$ for $i = 1,\dots,\mu$ and $0 < \delta \leqslant 1/2$. Then*

$$|\Delta(\mathbf{f},\mathbf{p})| \leqslant (C\mu^3 M)^\mu \cdot \delta^{E\cdot\mu^{1+1/m}}, \tag{32}$$

where

$$C = O_m(\nu^{\frac{n-m}{m}}), \tag{33}$$

$$E = \Omega_m(\nu^{-\frac{n-m}{m}}). \tag{34}$$

Proof. This follows from [3, Lemma 9] and Theorem 3. □

We note that the proof of [3, Lemma 9] is a direct adaptation of the interpolation determinant method of [5], and the reader familiar with this method may recognize that essentially the same arguments go through given the definition of decomposition data.

3.2. *Polynomial interpolation determinants.* Let $d \in \mathbb{N}$, and let μ denote the dimension of the space of polynomials of degree at most d in $m+1$ variables, $\mu = L(m+2,d)$. Let $\mathbf{f} := (f_1,\dots,f_{m+1})$ be a collection of functions and $\mathbf{p} := (p_1,\dots,p_\mu)$ a collection of points. We define the *polynomial interpolation determinant* of degree d to be

$$\Delta^d(\mathbf{f},\mathbf{p}) := \Delta(\mathbf{g},\mathbf{p}), \qquad \mathbf{g} = (\mathbf{f}^\alpha : \alpha \in \mathbb{N}^{m+1}, |\alpha| \leqslant d). \tag{35}$$

Note that $\Delta^d(\mathbf{f}, \mathbf{p}) = 0$ if and only if there exists a polynomial of degree at most d in $m+1$ variables vanishing at the points $\mathbf{f}(p_1), \ldots, \mathbf{f}(p_\mu)$.

In [24], the following height function was introduced. For an algebraic number $\alpha \in \mathbb{Q}^{\mathrm{alg}}$, let $\mathrm{den}(\alpha)$ denotes the *denominator* of α, i.e., the least positive integer K such that $K\alpha$ is an algebraic integer. If $\{\alpha_i\}$ are the conjugates of α, we denote

$$H^{\mathrm{size}}(\alpha) = \max(\mathrm{den}(\alpha), |\alpha_i|). \tag{36}$$

If α has degree t and

$$P \in \mathbb{Z}[X], \qquad P = a_t(X - \alpha_1) \cdots (x - \alpha_t) \tag{37}$$

is its minimal polynomial, then [4, 1.6.5, 1.6.6]

$$H(\alpha)^t = |a_t| \prod_{j=1}^{t} \max(1, |\alpha_j|). \tag{38}$$

In particular, it follows that

$$H(\alpha)^t \geqslant H^{\mathrm{size}}(\alpha). \tag{39}$$

For a set $A \subset \mathbb{C}^m$, we define $A^{\mathrm{size}}(\mathcal{F}, H)$ in analogy with $A(\mathcal{F}, H)$ from (2) by replacing $H(\cdot)$ with $H^{\mathrm{size}}(\cdot)$. The following lemma is essentially contained in the proof of [24, Th. 3.2], and we reproduce the argument for the convenience of the reader.

Lemma 11. *Let $H \in \mathbb{N}$, and suppose that for every $i = 1, \ldots, m+1$ and $j = 1, \ldots, \mu$,*

$$f_i(p_j) \in \mathcal{F}, \qquad H^{\mathrm{size}}(f_i(p_j)) \leqslant H. \tag{40}$$

Then $\Delta^d(\mathbf{f}, \mathbf{p})$ either vanishes or satisfies

$$\left|\Delta^d(\mathbf{f}, \mathbf{p})\right| \geqslant (\mu! H^{(m+2)\mu d})^{-[\mathcal{F}:\mathbb{Q}]}. \tag{41}$$

Proof. Denote the matrix defining $\Delta^d(\mathbf{f}, \mathbf{p})$ by S. Let $Q_{i,j} := \mathrm{den}\, f_i(p_j)$ for $i = 1, \ldots, m+1$ and $j = 1, \ldots, \mu$. By assumption, $Q_{i,j} \leqslant H$. The row corresponding to p_j in $\Delta^d(\mathbf{f}, \mathbf{p})$ consists of algebraic numbers with common denominator dividing $Q_j := \prod_i Q_{i,j}^d$. Setting $K = \prod_{j=1}^{\mu} Q_j$, we see that KS is a matrix of algebraic integers and $|K| \leqslant H^{(m+1)\mu d}$.

Let $G := \mathrm{Gal}(\mathcal{F}/\mathbb{Q})$. If $\det S$ is nonvanishing, then so are its G-conjugates, and then

$$1 \leqslant \Big|\prod_{\sigma \in G} KS^\sigma\Big| = K^{[\mathcal{F}:\mathbb{Q}]} \cdot |\det S| \cdot \prod_{\mathrm{id} \neq \sigma \in G} |\det(S^\sigma)|. \tag{42}$$

We estimate $|\det(S^\sigma)|$ from above. By assumption, each entry of S^σ has absolute value bounded by H^d. Expanding the determinant by the Laplace expansion, we have

$$|\det(S^\sigma)| \leqslant \mu! H^{\mu d} \qquad \forall \sigma \in G. \tag{43}$$

Plugging (43) into (42), we have

$$|\det S| \geqslant K^{-[\mathcal{F}:\mathbb{Q}]}(\mu! H^{\mu d})^{-[\mathcal{F}:\mathbb{Q}]+1} \geqslant (\mu! H^{(m+2)\mu d})^{-[\mathcal{F}:\mathbb{Q}]}. \tag{44}$$ □

Comparing Lemmas 10 and 11 we obtain the following.

Proposition 12. *Let $M, H \geqslant 2$, and suppose that $f_i \in \mathcal{O}(\bar{\Delta}')$ with $\|f_i\|_{\Delta'} \leqslant M$. Let*

$$Y = \mathbf{f}(X \cap \Delta^2) \subset \mathbb{C}^{m+1}. \tag{45}$$

There exists a constant $C_n > 0$ depending only on n such that if

$$d > C_n \nu^{n-m} \left([\mathcal{F} : \mathbb{Q}] \log H + \log M\right)^m, \tag{46}$$

then $Y^{\text{size}}(\mathcal{F}, H)$ is contained in an algebraic hypersurface of degree at most d in $\mathbb{C}^{m+1}$.

Proof. We consider first the case $m = 0$. In this case $\nu = e(X, \Delta)$ is the number of points in $X \cap \Delta$. In particular, this bounds the number of points in Y, all the more in $Y^{\text{size}}(\mathcal{F}, H)$, and the claim holds with any $d \geqslant \nu$.

Now assume $m > 0$ and suppose toward contradiction that $Y^{\text{size}}(\mathcal{F}, H)$ is not contained in an algebraic hypersurface of degree at most d in $\mathbb{C}^{m+1}$. By standard linear algebra it follows that there exist $\mathbf{p} = p_1, \dots, p_\mu \in X \cap \Delta^2$ such that $\{\mathbf{f}(p_j) : j = 1, \dots, \mu\}$ is a subset of Y and does not lie on the zero locus of any nonzero polynomial of degree d. Then $\left|\Delta^d(\mathbf{f}, \mathbf{p})\right| \neq 0$, and from Lemmas 10 and 11, we have

$$(\mu! H^{(m+2)\mu d})^{-[\mathcal{F}:\mathbb{Q}]} \leqslant \left|\Delta^d(\mathbf{f}, \mathbf{p})\right| \leqslant (C\mu^3 M^d)^\mu \cdot (1/2)^{E\cdot\mu^{1+1/m}}. \tag{47}$$

Taking logs and using $\mu \sim_m d^{m+1}$, we have

$$\log 2 \cdot E \cdot d^{1+1/m} \lesssim \log(C\mu^3 M^d) + [\mathcal{F} : \mathbb{Q}]\big[(m+2)d \log H + \log \mu\big]. \tag{48}$$

Therefore

$$d^{1/m} = \nu^{\frac{n-m}{m}} O_n\left(\frac{\log \nu}{d} + \log M + [\mathcal{F} : \mathbb{Q}] \log H\right). \tag{49}$$

Finally note that (46) implies, in the case $m > 0$, that $(\log \nu)/d = O_n(1)$. □

4. Metric entropy, Vitushkin's bound

Let $A \subset \mathbb{R}^n$ be a relatively compact subset. For every $\varepsilon > 0$, we denote by $M(\varepsilon, A)$ the minimal number of closed balls of radius ε needed to cover A. The logarithm of $M(\varepsilon, A)$ is called the *ε-entropy* of A.

For $r > 0$, we denote $Q_r := [0, r] \subset \mathbb{R}$. In our setting it will be more convenient to define $M(\varepsilon, A)$ in terms of covering by ε-cubes, i.e., translates of the cube Q^n_ε. For simplicity, we will also restrict our considerations to the unit cube $Q^n_1 \subset \mathbb{R}^n$.

Vitushkin's bound states that

$$M(\varepsilon, A) \leqslant c_n \sum_{i=0}^{n} \tilde{V}_i(A)/\varepsilon^i, \tag{50}$$

where $\tilde{V}_i(A)$ denotes the i-th variation of A — that is, the average number of connected components of the section $A \cap P$ over all affine $(n-i)$-planes $P \subset \mathbb{R}^n$ with respect to an appropriate measure.

Let $A \subset Q^n_1$, and denote by $V_i(A)$ the maximal number of connected components of the set $A \cap P$, where $P \subset \mathbb{R}^n$ is an affine $(n-i)$-plane (or ∞ if this number is unbounded). We also denote $V(A) := \max_i V_i(A)$. We will use the following result of Friedland and Yomdin [9].

Theorem 4 ([9, Th. 1]). *Let $A \subset Q^n_1$ and $0 < \varepsilon \leqslant 1$. Then*

$$M(\varepsilon, A) \leqslant \mathrm{Vol}(A) + \sum_{i=0}^{n} 2^i \binom{n}{i} V_i(\partial A)/\varepsilon^i. \tag{51}$$

We use the following to slightly improve the asymptotics, but it is otherwise inessential.

Corollary 13. *Let $A \subset Q^n_1$ be subanalytic, and suppose* $\dim A \leqslant m < n$. *Then*

$$M(\varepsilon, A) \leqslant \sum_{i=0}^{m} 2^i \binom{n}{i} V_i(A)/\varepsilon^i. \tag{52}$$

Proof. Note first that in this case $\partial A = A$. In the proof of Theorem 4, for every fixed ε the quantity $V_i(A)$ is in fact only used to estimate the number of connected components of the intersection $A \cap P$ where P varies over a certain finite set of affine $(n-i)$-planes P. It is easy to see that the argument remains valid if one replaces each P by its sufficiently small parallel translate P'. For $i > m$, we can choose these translates so that $A \cap P' = \emptyset$, and the statement follows. $\square$

Corollary 14. *Let $r > 0$ and $A \subset Q^n_r$ with* $\dim A = m < n$. *If*

$$r\varepsilon^{-1} > \sqrt[n-m]{CV(A)}, \qquad C := (m+1)2^{4n}, \tag{53}$$

then there exists an ε-ball disjoint from A.

Proof. Since the claim is invariant under rescaling, we may assume $r = 1$. Let $S \subset Q^n_1$ be a set of at least $(4\varepsilon)^{-n}$ points with pairwise ℓ_∞ distances at least 4ε: for instance, one can choose a grid with $(4\varepsilon)^{-1}$ equally spaced points on each axis. Suppose A touches the ε-ball B_s around each point of $s \in S$. Then every ε-cover of A by cubes must contain a cube that touches each B_s, and since an ε-cube cannot touch two such balls by the triangle inequality, it follows that

$$(4\varepsilon)^{-n} \leqslant M(\varepsilon, A) \leqslant \sum_{i=0}^{m} 2^i \binom{n}{i} V_i(A)/\varepsilon^i \leqslant 2^{2n}(m+1)\varepsilon^{-m}V(A) \tag{54}$$

and the conclusion follows. □

5. **The Pfaffian category, Entropy and Weierstrass polydiscs**

5.1. *Pfaffian functions, semi-Pfaffian and sub-Pfaffian sets.* Let $U \subset \mathbb{R}^n$ be a domain. We denote the coordinates on $\mathbb{R}^n$ by $\mathbf{x}$. The following definition, which plays a key role in our considerations, was introduced by Khovanskii in [15] (see also [11]).

Definition 15. A *Pfaffian chain* of order ℓ and degree α is a sequence of functions $f_1, \ldots, f_\ell : U \to \mathbb{R}$, real analytic in U and satisfying a triangular system of differential equations

$$\mathrm{d}f_j = \sum_{i=1}^{n} P_{i,j}(\mathbf{x}, f_1(\mathbf{x}), \ldots, f_j(\mathbf{x}))\,\mathrm{d}x_i, \qquad j = 1, \ldots, \ell, \tag{55}$$

where P_{ij} are polynomials of degrees not exceeding α. A function $f : U \to \mathbb{R}$ of the form $f(\mathbf{x}) = P(\mathbf{x}, f_1(\mathbf{x}), \ldots, f_\ell(\mathbf{x}))$, where P is a polynomial of degree not exceeding β, is called a *Pfaffian function* of *order* ℓ and *degree* (α, β).

The following Pfaffian analog of the Bezout theorem, due to Khovanskii [15], is the basis for the theory of Pfaffian functions and sets.

THEOREM 5. *Let $f_1, \ldots, f_n : U \to \mathbb{R}$ be Pfaffian functions with a common Pfaffian chain of order ℓ and $\deg f_i = (\alpha, \beta_i)$. Then the number of isolated points in $\{\mathbf{x} \in U : f_1(\mathbf{x}) = \cdots = f_n(\mathbf{x}) = 0\}$ does not exceed*

$$2^{\ell(\ell-1)/2}\beta_1 \cdots \beta_n(\min(n, \ell)\alpha + \beta_1 + \cdots + \beta_n - n + 1)^\ell. \tag{56}$$

We now move to the notion of semi-Pfaffian and sub-Pfaffian sets. For this purpose we restrict our consideration to domains of the form $\prod_{j=1}^n I_j$ where each I_j is an open, possibly unbounded interval in $\mathbb{R}$. By a slight abuse of notation we denote this product by $\mathcal{I}^n$. We will write $X^n := (X_1, \ldots, X_n)$ for a set a variables ranging over $\mathcal{I}^n$.

Definition 16.

- A *basic Pfaffian relation* on $\mathcal{I}^n$ is a relation $f(X^n) * 0$ where $* \in \{=, >\}$ and f is a Pfaffian function on $\mathcal{I}^n$.
- A *semi-Pfaffian formula* $\phi(X^n)$ is a Boolean combination of basic Pfaffian relations. We say that ϕ has complexity $(n, s, \ell, \alpha, \beta)$ if it involves s basic Pfaffian relations, where all the Pfaffian functions have degree at most β in a common Pfaffian chain of order ℓ and degree α.

- A *sub-Pfaffian formula* is a formula of the form $\phi(X^n) := \exists Y^r : \psi(X^n, Y^r)$, where $\psi(X^n, Y^r)$ is a semi-Pfaffian formula on $\mathcal{I}^{n+r}$. The complexity of ϕ is defined to be $(n, r, s, \ell, \alpha, \beta)$ where ψ has complexity $(n + r, s, \ell, \alpha, \beta)$.

If a formula is semi-algebraic, then we omit ℓ, α from the complexity notation (formally $\ell = \alpha = 0$).

We write $\phi(\mathcal{I}^n)$ for the set of points in $\mathcal{I}^n$ satisfying ϕ, and we refer to such sets as semi-Pfaffian (resp. sub-Pfaffian) for ϕ semi-Pfaffian (resp. sub-Pfaffian). The categories of semi-Pfaffian and sub-Pfaffian sets thus defined admit effective estimates for various geometric quantities in terms of the complexity of the formulas. We will require only estimates for the number of connected components, which are provided by the following theorem.

THEOREM 6 ([12, Th. 6.6]). *If ϕ is semi-Pfaffian of complexity $(n, s, \ell, \alpha, \beta)$, then the number of connected components of $\phi(\mathcal{I}^n)$ is bounded by*

$$s^n 2^{\ell(\ell-1)/2} O(n\beta + \min(n, \ell)\alpha)^{n+\ell}. \tag{57}$$

Similarly if ϕ is sub-Pfaffian of complexity $(n, r, s, \ell, \alpha, \beta)$, then the number of connected components of $\phi(\mathcal{I}^n)$ is bounded by

$$s^{n+r} 2^{\ell(\ell-1)/2} O((n + r)\beta + \min(n + r, \ell)\alpha)^{n+r+\ell}. \tag{58}$$

Proof. The first part is [12, Th. 6.6]. The second follows from the first since projection cannot increase the number of connected components. $\square$

COROLLARY 17. *Let ϕ be sub-Pfaffian of complexity $(n, r, s, \ell, \alpha, \beta)$. Then $V(\phi(\mathcal{I}^n))$ is bounded by a polynomial of degree at most $n + r + \ell$ in β.*

Proof. To estimate $V(\phi(\mathcal{I}^n))$ we intersect with additional linear equations and count connected components. The result follows easily from Theorem 6. $\square$

5.2. *Sub-Pfaffian sets and* $\mathbb{R}^{\mathrm{RE}}$. The restricted exponential and sine functions are Pfaffian. As a consequence we have the following proposition.

PROPOSITION 18. *Every $\mathbb{R}^{\mathrm{RE}}$-definable subset of $\mathbb{R}^n$ is sub-Pfaffian.*

Proof. By the main result of [8] every $\mathbb{R}^{\mathrm{RE}}$-definable subset of $\mathbb{R}^n$ is definable by a formula of the type

$$\phi(X^n) := \exists Y^m : \psi(X^n, Y^m), \tag{59}$$

where ψ is a quantifier-free $\mathbb{R}^{\mathrm{RE}}$-formula. (In fact one can replace $\exists$ by "exists a unique," although we shall not use this fact.) By adding additional variables Y one can also assume that the function symbols exp, sin only appear in the form $\exp(Y_j), \sin(Y_j)$: by induction on the construction tree of each term we replace every occurrence of $\exp(T)$ for a term T by $\exp(Y_j)$ for some new variable Y_j, and we add the condition $Y_j = T$ to ψ (and similarly for sin). Then it will follow that ψ is equivalent to a sub-Pfaffian formula once we show that the graph of the restricted exponential and sine functions is sub-Pfaffian.

It is known that the function $\sin(z)$ is Pfaffian in the interval $[0,\pi]$. We claim that the graph of the restricted sine, $X_2 = \sin|_{[0,\pi]}(X_1)$ in $\mathcal{I}^2 := \mathbb{R}^2$ is sub-Pfaffian. Note that in defining this graph we may not use the function $\sin(X_1)$, since sin is not a Pfaffian function in $\mathbb{R}$. To resolve this minor technicality we define the graph by a projection from $\mathcal{I}^3 := \mathbb{R}^2 \times [0,\pi]$ using the sub-Pfaffian formula

$$\begin{aligned}\phi_{\sin}(X_1,X_2) := [(X_1 < 0 \vee X_1 > \pi) \wedge X_2 = 0] \vee \\ [\exists Y \in [0,\pi] : (X_1 = Y \wedge X_2 = \sin(Y))],\end{aligned} \tag{60}$$

where $\sin Y$ is a Pfaffian function over $[0,\pi]$. The restricted exponential function can be treated similarly. (In fact here it is not necessary to add the additional variable over $[0,1]$ because exp is Pfaffian in $\mathbb{R}$ itself.) $\square$

5.3. *Pfaffian functions in the complex domain.* We return now to the complex setting. We fix some standard coordinates $\mathbf{x}$ on $\mathbb{C}^n$ and identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by the map

$$(\mathbf{x}_1,\ldots,\mathbf{x}_n) \to (\operatorname{Re}\mathbf{x}_1, \operatorname{Im}\mathbf{x}_1,\ldots,\operatorname{Re}\mathbf{x}_n,\operatorname{Im}\mathbf{x}_n). \tag{61}$$

Since we work in $\mathbb{C}^n$, it will be convenient to allow unitary changes of variables. To make this consistent with the Pfaffian framework we consider the following setting. We let U denote some fixed ball around the origin, and we will assume that our Pfaffian chain is defined over U. We then let $\mathcal{I}^{2n}$ denote some product of intervals and assume $A \cdot \mathcal{I}^{2n} \subset U$ for any unitary A. Finally we will always work with sub-Pfaffian sets contained in a ball $B \subset \mathcal{I}^n$, and we assume that the formulas explicitly contain the condition $\mathbf{x} \in B$. Under these assumptions we can make a constant unitary change of variable in a Pfaffian formula without affecting the complexity: if $\{f_j(\mathbf{x})\}$ is a Pfaffian chain, then by the chain rule, $\{f_j(A\cdot\mathbf{x})\}$ is a Pfaffian chain of the same order ℓ and degree α. If the coefficients of A are taken to be independent variables, then this transformation increases the degree α by 1.

Our main result in this subsection is a theorem showing that if an analytic set X in a ball $B \subset \mathbb{C}^n$ is sub-Pfaffian, then one can choose a Weierstrass polydisc for X with size depending polynomially on β^{-1}. Since we are mainly concerned with the asymptotic in β, we allow the asymptotic constants to depend on all other parameters. In particular, when dealing with formulas of complexity $(2n, r, s, \ell, \alpha, \beta)$, we view all parameters except β as $O(1)$.

We will require a slight technical extension of the notion of Weierstrass polydiscs.

Definition 19. Suppose $\Delta := \Delta_z \times \Delta_w$ is a Weierstrass polydisc for an analytic set X. If $B \subset \mathbb{C}^n$ is a Euclidean ball around the origin, we will say that Δ has *gap* B if $\Delta_z \times \partial\Delta_w$ is disjoint from the set $B + X$.

We begin with a lemma in codimension one.

LEMMA 20. *Let $B \subset \mathbb{C}^n$ be a Euclidean ball around the origin. Let $X \subset B$ be an analytic subset of pure dimension m. Suppose X is defined by a sub-Pfaffian formula ϕ of complexity $(2n, r, s, \ell, \alpha, \beta)$. Then there exists a pre-Weierstrass polydisc $\Delta := \Delta_z \times \Delta_w$ centered at the origin for X where* $\dim \mathbf{w} = 1$ *and $B^\eta \subset \Delta \subset B$ where*

$$\eta := O(\beta^\nu), \qquad \nu = \nu(n, m, r, \ell) := \frac{2n + r + \ell + 2}{2n - 2m - 1}. \tag{62}$$

Moreover, Δ can be chosen to have gap B^η.

Proof. We may assume without loss of generality that B is the unit ball. The group $S^1 := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ acts on B by multiplication. We consider $Z := S^1 \cdot X$, the S^1-saturation of X. Then Z can be defined as a sub-Pfaffian set using the formula

$$\psi(\mathbf{x}) := \exists(\zeta \in \mathbb{C}) : (|\zeta|^2 = 1) \wedge \phi(\zeta \cdot y) \tag{63}$$

of complexity $(2n, r + 2, O(1), \ell, O(1), \beta)$. By Corollary 17, we have

$$V(Z) = O(\beta^{2n+r+\ell+2}). \tag{64}$$

Note that Z has real dimension at most $2m+1$. Then according to Corollary 14 applied to Z, in the cube $Q := [1/4n, 1/2n]^{2n} \subset B$ there exists a ball $B_v \subset Q$ with center $v \in Q$ and radius $\Omega(\beta^{-\nu})$ such that $B_v \cap Z = \emptyset$. Equivalently, $(S^1 \cdot B_v) \cap X = \emptyset$.

Making a unitary change of coordinates, we may assume that in the $\mathbf{x} = \mathbf{z} \times \mathbf{w}$ coordinates, v is given by $(0, \lambda)$, where $|\lambda| = \Omega(1)$. Let Δ_w denote the disc of radius $|\lambda|$ around the origin in the $\mathbf{w}$ coordinate and Δ_z denote a polydisc of polyradius $\Omega(\beta^{-\nu})$ around the origin in the $\mathbf{z}$ coordinates with $v + \Delta_z \subset B_v$. Since Δ_z is invariant under the S^1 action,

$$\Delta_z \times \partial\Delta_w = (S^1 \cdot v) + \Delta_z = S^1 \cdot (v + \Delta_z) \subset S^1 \cdot B_v \tag{65}$$

is disjoint from X, i.e., $\Delta := \Delta_z \times \Delta_w$ is a Weierstrass polydisc for X. Finally, since each radius of Δ is $\Omega(\beta^{-\nu})$, we have $B^\eta \subset \Delta$ for $\eta = O(\beta^{-\nu})$ as claimed.

To satisfy the gap condition it is enough to choose B_v to be disjoint from $Z + B^{O(\beta^\nu)}$ instead of Z. This is clearly possible for the same reasons: for instance, it is enough to decrease the radius of B_v by a factor of two. □

We now state our main result.

THEOREM 7. *Let $B \subset \mathbb{C}^n$ be a Euclidean ball around the origin. Let $X \subset B$ be an analytic subset of pure dimension m. Suppose X is defined by a sub-Pfaffian formula ϕ of complexity $(2n, r, s, \ell, \alpha, \beta)$. Then there exists a Weierstrass polydisc $\Delta := \Delta_z \times \Delta_w$ centered at the origin for X where $B^\eta \subset \Delta \subset B$ and*

$$\eta := O(\beta^\theta) \qquad \theta = \theta(n, m, r, \ell) \leqslant (2n + r + \ell + 2)\log(2n - 2m + 1). \tag{66}$$

Moreover, Δ can be chosen to have gap B^η.

Proof. We begin with a simple topological remark. Suppose that $\Delta_1 = \Delta_z \times \Delta_w \subset B$ is a pre-Weierstrass polydisc for X. Then π_z^X is proper so $X' := \pi_z^X(X) \subset \Delta_z$ is an analytic subset. Suppose $\Delta_2 := \Delta_{z'} \times \Delta_{w'} \subset \Delta_z$ is a Weierstrass polydisc for X'. Then

$$\Delta := \Delta_2 \times \Delta_w \subset \Delta_1 \tag{67}$$

is a Weierstrass polydisc for X. Indeed,

- X does not meet $\Delta_{z'} \times (\Delta_{w'} \times \partial\Delta_w)$ since it does not meet $\Delta_z \times \partial\Delta_w$;
- X does not meet $\Delta_{z'} \times (\partial\Delta_{w'} \times \Delta_w)$ since its z-projection X' does not meet $\Delta_{z'} \times \partial\Delta_{w'}$.

We now proceed with the proof, by induction on $n-m$. The case $n-m = 1$ is exactly Lemma 20. For $n - m > 1$, consider the pre-Weierstrass polydisc $\Delta_1 = \Delta_z \times \Delta_w \subset B$ provided by Lemma 20. Choose some ball $B' \subset \Delta_z$. Then $X' \cap B' \subset B'$ is a sub-Pfaffian set: after a unitary change from the $\mathbf{x}$ to the $\mathbf{z} \times \mathbf{w}$ coordinates, it is defined by the formula

$$\psi(\mathbf{z}) = \exists \mathbf{w} : (\mathbf{w} \in \Delta_w) \wedge (\mathbf{z} \in B') \wedge \phi(\mathbf{z}, \mathbf{w}) \tag{68}$$

of complexity $(2n - 2, r + 2, O(1), \ell, O(1), \beta)$. Thus, applying the inductive hypothesis we obtain a Weierstrass polydisc $\Delta_2 \subset B' \subset \Delta_z$ for X'. Defining Δ as above we obtain a Weierstrass polydisc for X.

We have $B^{\eta_1} \subset \Delta_1$, where $\eta_1 = O(\beta^\nu)$ for $\nu = \nu(n, m, r, \ell)$. Then B' can be chosen so that $B^{O(\eta_1)} \subset B' \times \Delta_w$. Also $(B')^{\eta_2} \subset \Delta_2$, where $\eta_2 = O(\beta^\theta)$ for $\theta = \theta(n - 1, m, r + 2, \ell)$. Setting

$$\eta = O(\eta_1) \cdot \eta_2 = O(\beta^{\nu+\theta}), \tag{69}$$

we see that

$$B^\eta = (B^{O(\eta_1)})^{\eta_2} \subset (B' \times \Delta_w)^{\eta_2} \subset (B')^{\eta_2} \times \Delta_w \subset \Delta_2 \times \Delta_w = \Delta. \tag{70}$$

Finally, computing θ by induction, we see

$$\begin{aligned} \theta(n, m, r, \ell) &= \sum_{j=0}^{n-m-1} \nu(n - j, m, r + 2j, \ell) = \sum_{j=0}^{n-m-1} \frac{2n + r + \ell + 2}{2n - 2j - 2m - 1} \\ &\leqslant (2n + r + \ell + 2)\log(2n - 2m + 1). \end{aligned} \tag{71}$$

To verify the gap condition, by Lemma 20 we may choose Δ_1 to have gap B^{η_1}, and by induction we may choose Δ_2 to have gap $(B')^{\eta_2}$. Then

$$[\Delta_{z'} \times (\Delta_{w'} \times \partial\Delta_w)] \cap (X + B^{\eta_1}) \subset (\Delta_z \times \partial\Delta_w) \cap (X + B^{\eta_1}) = \emptyset \tag{72}$$

and

$$\begin{aligned}[\Delta_{z'} \times (\partial\Delta_{w'} \times \Delta_w)] \cap (X + (B' \times \Delta_w)^{\eta_2}) \\ \subset \pi_z^{-1}[(\Delta_{z'} \times \partial\Delta_{w'}) \cap (X' + (B')^{\eta_2})] = \emptyset.\end{aligned} \tag{73}$$

Since $B^\eta \subset B^{\eta_1}$ and $B^\eta \subset (B' \times \Delta_w)^{\eta_2}$, we see that Δ indeed has gap B^η. □

Theorem 7 contains the key argument that allows us to cover analytic sets by a polynomial (in β) number of Weierstrass polydiscs. However, in practice the condition of pure-dimensionality of X is somewhat inconvenient. We use a deformation argument to obtain a result valid in the case of mixed dimensions. We begin with a definition.

Definition 21. Let $\Omega \subset \mathbb{C}^n$ be a domain and $\{g_l : \Omega \to \mathbb{C}, l = 1, \ldots, S\}$ a collection of holomorphic functions. Suppose that the graphs of g_α are sub-Pfaffian with complexity bounded by $(2n+2, r, s, \ell, \alpha, \beta)$. Then we say that the analytic set $X \subset \Omega$ of common zeros of $\{g_l\}$ is a *holomorphic-Pfaffian variety.*

If $X \subset \Omega$ is an analytic subset of a domain $\Omega \subset \mathbb{C}^n$ and $k \in \mathbb{N}$, we denote by $X^{\leqslant k}$ the union of the components of X that have dimension k or less. Note that $X^{\leqslant k}$ is also analytic in Ω.

Corollary 22. *Let $X \subset \Omega$ be a holomorphic-Pfaffian variety as in Definition* 21 *and $B \subset \Omega$ a relatively compact Euclidean ball. Let $0 \leqslant m < n$. There exist an analytic set $Z \subset B$ of pure dimension m satisfying $X^{\leqslant m} \subset Z$ and a Weierstrass polydisc Δ for Z such that*

(1) $B^\eta \subset \Delta \subset B$, *where* $\eta = O(\beta^\theta)$ *and*

$$\theta = \theta(n, m, 2S(r+1), \ell) \leqslant (2n + 2S(r+1) + \ell + 2)\log(2n - 2m + 1); \tag{74}$$

(2) $e(Z, \Delta) = O(\beta^{n+2S(r+1)+\ell})$.

Proof. The claim is invariant under translation, and we may assume without loss of generality that B is centered at the origin. Let $\tilde{g}_1$ denote a generic linear combination of the g_l. If not all g_l are identically vanishing, then the zero locus Z_1 of $\tilde{g}_1$ is an analytic subset of $\bar{B}$ of codimension 1. In particular, it has finitely many irreducible components. We write $Z_{1,b}$ for the union of those components of Z_1 that are components of X and $Z_{1,g}$ for the rest.

For every component of $Z_{1,g}$, there is a function g_l that is not identically vanishing on it. Then we may choose a generic linear combination $\tilde{g}_2$ of the g_l that is not identically vanishing on any component of $Z_{1,g}$. We set $Z_2 = Z_{1,g} \cap \{\tilde{g}_2 = 0\}$, which is an analytic subset of $\bar{B}$ of codimension 2. We write

$Z_{2,b}$ for the union of those components of Z_2 that are components of X and $Z_{2,g}$ for the rest.

Proceeding in the same manner, we obtain a set $Z := Z_{n-m}$ with

$$X \subset Z \cup Z_b, \qquad Z_b := Z_{1,b} \cup \cdots \cup Z_{n-m-1,b}. \tag{75}$$

Since the components of the sets $Z_{j,b}$ for $j = 1, \ldots, n-m-1$ have codimension $j < n-m$, we have $X^{\leqslant m} \subset Z$.

Fix a tuple $c_1, \ldots, c_{n-m} \in \mathbb{C}$. Let $\varepsilon > 0$, and set

$$Z_\varepsilon := \{\mathbf{x} \in B : \tilde{g}_1(\mathbf{x}) = c_1\varepsilon, \ldots, \tilde{g}_{n-m}(\mathbf{x}) = c_{n-m}\varepsilon\}. \tag{76}$$

By a Sard-type argument, for generic c_j and sufficiently small $\varepsilon > 0$, we have $\dim Z_\varepsilon = m$. We claim that Z is contained in the Hausdorff limit of Z_{ε_j} along any sequence $0 \neq \varepsilon_j \to 0$. Note that Z has pure dimension m while $Z_b \cap Z$ has dimension strictly smaller than m, so $Z_g := Z \setminus Z_b$ is dense in Z and it will suffice to prove that Z_g is contained in the limit of Z_ε. Let $y \in Z_g$. We will show it is in the limit of Z_{ε_j}.

The equations $\tilde{g}_1 = \cdots = \tilde{g}_{n-m} = 0$ intersect properly, i.e., at an analytic set of dimension m, around y. If we choose m additional generic affine-linear functions $L_1, \ldots, L_m$ vanishing at y, then the intersection

$$\tilde{g}_1 = \cdots = \tilde{g}_{n-m} = L_1 = \cdots = L_m = 0 \tag{77}$$

is a proper isolated intersection. By conservation of proper intersection numbers under deformations, we see that the system

$$\tilde{g}_1 = c_1\varepsilon_j, \ldots, \tilde{g}_{n-m} = c_{n-m}\varepsilon_j, \quad L_1 = \cdots = L_m = 0 \tag{78}$$

must indeed admit at least one solution $y_j \in Z^{\varepsilon_j}$ converging to y as $\varepsilon_j \to 0$.

For $\varepsilon > 0$, the set Z^ε is defined by the sub-Pfaffian formula

$$\phi(\mathbf{x}) = (x \in B) \wedge \exists (y_1, \ldots, y_S) : \bigwedge_{l=1}^{S} (y_l = g_l(\mathbf{x})) \cap \bigwedge_{j=1}^{n-m} (\tilde{g}_j(\mathbf{x}) = c_j\varepsilon), \tag{79}$$

where we write each $\tilde{g}_j$ as an appropriate linear combination of the y_l variables. The complexity of ϕ is bounded by $(2n, 2S(r+1), O(1), \ell, O(1), \beta)$. By Theorem 7, Z^ε admits a Weierstrass polydisc $\Delta(\varepsilon)$ satisfying $B^\eta \subset \Delta(\varepsilon) \subset B$, where

$$\eta := O(\beta^\theta), \qquad \theta = \theta(n, m, 2S(r+1), \ell). \tag{80}$$

Moreover, we may assume that the $\Delta(\varepsilon)$ have a gap bounded from below uniformly over ε.

Since the space of Weierstrass polydiscs satisfying the conditions above is compact, we may choose a sequence $\varepsilon_j \to 0$ such that $\Delta(\varepsilon_j)$ converges (for instance, in the Hausdorff distance) to some polydisc Δ. We claim the Δ is a Weierstrass polydisc for Z. Indeed, suppose Z intersects $\Delta_z \times \partial\Delta_w$. Since Z

is the Hausdorff limit of Z^{ε_j}, we see that points of Z^{ε_j} must come arbitrarily close to $\Delta_z \times \partial\Delta_w$. But this contradicts the fact that $\Delta(\varepsilon_j)$ converges to Δ and Z^{ε_j} stays at a uniformly bounded distance from $\Delta(\varepsilon_j)_z \times \partial\Delta(\varepsilon_j)_w$.

To estimate $e(Z,\Delta)$ recall that $Z \setminus Z_b$ has dimension strictly smaller than m. Since the map π_z^Z is finite, we see that for a generic choice of a point $p \in \Delta_z$, the fiber $(\pi_z^Z)^{-1}(p)$ consists of $\nu = e(Z,\Delta)$ isolated points in $Z \setminus Z_b$. Each such isolated point is an isolated solution of the system

$$\{(\mathbf{z},\mathbf{w}) \in \Delta : \tilde{g}_1(\mathbf{z},\mathbf{w}) = \cdots = \tilde{g}_{n-m}(\mathbf{z},\mathbf{w}) = 0, \mathbf{z} = \mathbf{z}(p)\}. \tag{81}$$

Then the intersection (81) is proper at these isolated points and it follows that for sufficiently small ε, the intersection

$$\{(\mathbf{z},\mathbf{w}) \in \Delta : \tilde{g}_1(\mathbf{z},\mathbf{w}) = c_1\varepsilon, \ldots, \tilde{g}_{n-m}(\mathbf{z},\mathbf{w}) = c_{n-m}\varepsilon, \mathbf{z} = \mathbf{z}(p)\} \tag{82}$$

contains at least ν points. But this intersection is sub-Pfaffian, being the intersection of Z^ε with the equation $\mathbf{z} = \mathbf{z}(p)$. Hence the upper bound for ν follows from Theorem 6. □

Remark 23. In Corollary 22, if some of the functions g_l are in fact *Pfaffian* of degree β rather sub-Pfaffian, then one can take S to be the number of sub-Pfaffian functions. Indeed, in the formula (79) one does not need to add new variables y_l to express the value of the Pfaffian g_l: as Pfaffian functions they can be summed into the linear combinations $\tilde{g}_j$ directly.

6. **Exploring rational points**

We begin with a definition.

Definition 24. Let $X \subset \mathbb{C}^m$ and $W \subset \mathbb{C}^m$ be two sets. We define

$$X(W) := \{w \in W : W_w \subset X\} \tag{83}$$

to be the set of points of W such that X contains the germ of W around w, i.e., such that w has a neighborhood $U_w \subset \mathbb{C}^m$ such that $W \cap U_w \subset X$.

If $A \subset \mathbb{C}^n$, we denote by $A_\mathbb{R} := A \cap \mathbb{R}^n$. We remark that

$$(A(W))_\mathbb{R} \subset (A_\mathbb{R})(W_\mathbb{R}). \tag{84}$$

We will consider Definition 24 in two cases: for $X \subset \mathbb{C}^m$ locally analytic and $W \subset \mathbb{C}^m$ an algebraic variety, and for $X \subset \mathbb{R}^m$ subanalytic and $W \subset \mathbb{R}^m$ a semi-algebraic set.

Our principal motivation for Definition 24 is the following lemma (cf. Theorem 10).

LEMMA 25. *Let $W \subset \mathbb{R}^m$ be a connected positive-dimensional semi-algebraic set and $A \subset \mathbb{R}^m$. Then $A(W) \subset A^{\mathrm{alg}}$.*

We record some simple consequences of Definition 24.

LEMMA 26. *Let* $A, B, W \subset \mathbb{C}^m$. *Then*

$$A(W) \cup B(W) \subset (A \cup B)(W). \tag{85}$$

If $A \subset B$ *is relatively open, then*

$$B(W) \cap A = A(W). \tag{86}$$

6.1. *Projections from admissible graphs.* Let $\Omega_x \subset \mathbb{C}^m$ and $\Omega_y \subset \mathbb{C}^n$ be domains, and set $\Omega := \Omega_x \times \Omega_y \subset \mathbb{C}^{n+m}$. We denote by $\pi : \Omega \to \Omega_x$ the projection map.

Let $U \subset \Omega_x$ be an open subset and $\psi : U \to \Omega_y$ a function, and denote its graph by

$$\Gamma_\psi \subset \Omega, \qquad \Gamma_\psi := \{(x, \psi(x)) : x \in U\}. \tag{87}$$

We denote by $\tilde{\psi} : U \to \Gamma_\psi$ the map $x \to (x, \psi(x))$.

Definition 27. We say that ψ is *admissible* if $\Gamma = \Gamma_\psi$ is relatively compact in Ω, and if there exists an analytic subset X_Γ of Ω that agrees with Γ over U, i.e., $X_\Gamma \cap \pi^{-1}(U) = \Gamma$.

Example 28. Let $\Omega_x \subset \mathbb{C}^2$ be a domain such that the unit ball $B_2 \subset \mathbb{C}^2$ is a relatively compact subset of Ω_x, and let $\Omega_y \subset \mathbb{C}$ be a domain such that the unit ball $B_1 \subset \mathbb{C}$ is a relatively compact subset of Ω_y. Let

$$U := \{(x_1, x_2) \in B_2 : |x_1| < |x_2|\}, \tag{88}$$

and define $\psi : U \to \Omega_y$ by $(x_1, x_2) \to (x_1/x_2)$. Then ψ is an admissible projection, as its graph over U agrees with the analytic subset $X_\Gamma \subset \Omega$ given by $yx_2 = x_1$. This example is essentially the only case that we shall require in the sequel.

In Theorem 8 we prove an analog of the Wilkie conjecture for images of holomorphic Pfaffian varieties under admissible projections. In Section 7 we study $\mathbb{R}^{\mathrm{RE}}$-definable sets and show that (for the purpose of counting rational points) one can reduce any definable set to the image of such an admissible projection (see Proposition 36).

6.2. *Rational points on admissible projections.* We fix an admissible map $\psi : U \to \Omega_y$ and denote $\Gamma := \Gamma_\psi$. In this section we will consider a fixed holomorphic-Pfaffian variety X and compute asymptotics for the number of rational points with respect to the height H. Therefore in our asymptotic notation we allow our constants to depend on X and Γ, as well as on $[\mathcal{F} : \mathbb{Q}]$. We note, however, that the estimates do not depend on $\mathcal{F}$ itself. The *degree* of a pure dimensional algebraic variety $W \subset \mathbb{C}^n$ denoted $\deg W$ is the number of intersections between W and a generic hyperplane of complementary dimension.

The following is our main result in this section.

THEOREM 8. *Let $X \subset \Omega$ be a holomorphic-Pfaffian variety defined by S sub-Pfaffian functions with complexity bounded by $(2n+2m+2, r, s, \ell, \alpha, \beta)$. Suppose $X \subset X_\Gamma$, and set*

$$Y := \pi(X \cap \Gamma) \subset \Omega_x. \tag{89}$$

Then for

$$\kappa = 3^m \left(m+n+2S(r+1)+\ell+1\right)^{3m} \tag{90}$$

and any $H \in \mathbb{N}$, there exist algebraic varieties $V_0, \dots, V_m$ such that V_j has pure dimension j and degree $O((\log H)^\kappa)$, and

$$Y^{\mathrm{size}}(\mathcal{F}, H) \subset Y(V_0) \cup \cdots \cup Y(V_m). \tag{91}$$

We will need the following basic lemma. Below, $\operatorname{Sing} W$ denotes the singular part of the algebraic variety W.

LEMMA 29. *Let $W \subset \mathbb{C}^m$ be a pure dimensional algebraic variety of degree d. Then*

(1) *W is set-theoretically cut out by a set of at most $m+1$ polynomials, each of degree at most d;*
(2) *there exists a polynomial Q of degree at most d vanishing identically on $\operatorname{Sing} W$ but not on W.*

Proof. Assume first that W is a hypersurface. Then $W = \{P = 0\}$, where P is square-free and $\deg P = d$, proving (1). Any derivative of P vanishes on $\operatorname{Sing} W$, and any derivative that is not identically zero has no common factors with P and hence does not vanish identically on W, proving (2). We proceed with the case $k := \dim W < m-1$.

For any sufficiently generic affine-linear projection $L : \mathbb{C}^n \to \mathbb{C}^{k+1}$, the Zariski closure of $L(W) \subset \mathbb{C}^{k+1}$ is a hypersurface of degree d (since the pullback of a generic line in $\mathbb{C}^{k+1}$ by L is a generic $(n-k)$-plane). Let P_L be the (square-free) polynomial of degree d defining this hypersurface, and set $P'_L := P_L \circ L$. Since L is affine linear, we also have $\deg P'_L = d$. We claim that $m+1$ (sufficiently generic) polynomials thus constructed define W set theoretically, proving (1). Indeed, let L_1 be generic as above, and let $W_1 = \{P'_{L_1} = 0\}$. Next, choose L_2 sufficiently generic so that for any component $C \subset W_1$ that is not contained in W, we have $L_2(C) \not\subset L(W)$. Then P'_{L_2} does not vanish identically on any C as above, and setting $W_2 = W_1 \cap \{P'_{L_2} = 0\}$, we see that any component of W_2 not contained in W has codimension at least 2.

Continuing in the same manner we construct $W_3, \dots, W_{m+1}$ such that any component $C \subset W_k$ that is not contained in W has codimension at least k and, in particular, $W_{m+1} = W$.

For the second statement, let $T_p(W) \subset \mathbb{C}^m$ be the common zeros locus of the differentials $\mathrm{d}P_p$ for every polynomial in the ideal of W. By definition we have $\operatorname{Sing} W = \{p : \dim T_p(W) > k\}$. Choose a sufficiently generic $L : \mathbb{C}^n \to \mathbb{C}^{k+1}$ such that $\dim L(W) = k$, and such that at a generic point p of (each component of) $\operatorname{Sing} W$, we have $\mathrm{d}L(T_p(W)) = \mathbb{C}^{k+1}$. We let Q_L denote one of the nonvanishing derivatives of the (square-free) P_L. Then Q_L does not vanish identically on $L(W)$, so $Q'_L := Q_L \circ L$ does not vanish identically on W. Thus (2) will be proved with $Q = Q'_L$ once we show that Q'_L vanishes on (a generic point of) $\operatorname{Sing} W$. Let $p \in \operatorname{Sing} W$ be a generic point such that $\mathrm{d}L(T_p(W)) = \mathbb{C}^{k+1}$. Then

(92)
$$[(\mathrm{d}P_L)_{L(p)}](\mathbb{C}^{k+1}) = [(\mathrm{d}P_L)_{L(p)} \circ (\mathrm{d}L)_p](T_p(W)) = [(\mathrm{d}P'_L)_p](T_p(W)) = \{0\},$$

where the last equality follows from the definition of $T_p(W)$ and the fact that P'_L vanishes on W. In conclusion, we see that Q_L, as a derivative of P_L, vanishes at the point $L(p)$ so that Q'_L vanishes at p as claimed. □

We begin the proof of Theorem 8 with the following proposition.

PROPOSITION 30. *Let $X \subset \Omega$ be a holomorphic-Pfaffian variety defined by S sub-Pfaffian function with complexity bounded by $(2n+2m+2, r, s, \ell, \alpha, \beta)$. Suppose $X \subset X_\Gamma$, and set*

$$Y := \pi(X \cap \Gamma) \subset \Omega_x. \tag{93}$$

Let $W \subset \mathbb{C}^m$ be an algebraic variety of pure dimension k and degree d. Then for

$$\begin{aligned} \lambda(k) = \lambda(n,m,r,S,\ell,k) &:= (n+m+2S(r+1)+\ell)(n+m-k+1) \\ &\quad + (n+m)\theta(n+m, k-1, 2S(r+1), \ell) + 1 \end{aligned} \tag{94}$$

and for any $H \in \mathbb{N}$, there exists an algebraic variety $V \subset W$ of pure dimension $k-1$ and degree $O(d^{\lambda(k)}(\log H)^{k-1})$ such that

$$(Y \cap W)^{\mathrm{size}}(\mathcal{F}, H) \subset Y(W) \cup V. \tag{95}$$

Proof. Set

$$Z := (X \cap (W \times \Omega_y))^{<k}. \tag{96}$$

Let $q \in Y \cap W$, and suppose that $q \notin \operatorname{Sing} W$ and $q \notin Y(W)$. Then the germ W_q of W at q is smooth k-dimensional and not contained in Y. Equivalently, its image $\tilde\psi(W_q)$ is the germ of a smooth k-dimensional analytic set at $\tilde\psi(q)$

that is not contained in X. Since we assume $X \subset \Gamma$ around $\tilde{\psi}(q)$, we deduce that the dimension of

$$X \cap (W_q \times \Omega_y) = X \cap \Gamma \cap (W_q \times \Omega_y) = X \cap \tilde{\psi}(W_q) \tag{97}$$

at $\tilde{\psi}(q)$ is strictly smaller than k, i.e., $\tilde{\psi}(q) \in Z$. In conclusion,

$$Y \cap W \subset Y(W) \cup \operatorname{Sing} W \cup \pi(Z). \tag{98}$$

By Lemma 29 there exists a hypersurface $\mathcal{H}_0 \subset \mathbb{C}^m$ of degree at most d containing $\operatorname{Sing} W$ and not containing W. Also, the holomorphic-Pfaffian variety $X \cap (W \times \Omega_y)$ is cut out by the equations for X and a set of additional polynomials equations (in $\mathbf{x}$) of degrees bounded by d.

Let $p \in \bar{\Gamma}$. Since $\Gamma \subset \Omega$ is relatively compact, there exists a Euclidean ball $B_p \subset \Omega$ around p of radius $\Omega(1)$. Slightly shrinking B_p if necessary, we may also assume $B_p^{1/3} \subset \Omega$. By Corollary 22 there exist an analytic set $Z' \subset B_p$ of pure dimension $k-1$ satisfying $Z \subset Z'$ and a Weierstrass polydisc Δ for Z such that

(1) $B^\eta \subset \Delta \subset B$, where $\eta = O(d^\theta)$ and $\theta = \theta(n+m, k-1, 2S(r+1), \ell)$;
(2) $e(Z', \Delta) = O(d^{n+m+2S(r+1)+\ell})$.

Note that in the estimate above we use S, the number of sub-Pfaffian holomorphic equations for X, and do not count the additional equations used to define W. This is permissible in light of Remark 23 and improves the asymptotics.

Assume first that W is irreducible. Then one can choose a subset of k coordinates on $\mathbb{C}^m$, say $\mathbf{f} = (x_1, \dots, x_k)$, such that $\mathbf{f} : W \to \mathbb{C}^k$ is dominant and, in particular, no nonzero polynomial in $\mathbf{f}$ vanishes identically on W. We apply Proposition 12 to Z' and $\mathbf{f}$. We conclude that

$$[\pi(Z' \cap B_p^{2\eta})]^{\text{size}}(\mathcal{F}, H) \subset \{P_p(\mathbf{f}) = 0\} \tag{99}$$

for some nonzero polynomial $P_p(\mathbf{f})$ of degree $\tilde{d}$, where

$$\begin{aligned} \tilde{d} &= O(e(Z', \Delta)^{n+m-k+1} (\log H)^{k-1}) \\ &= O(d^{(n+m+2S(r+1)+\ell)(n+m-k+1)} (\log H)^{k-1}). \end{aligned} \tag{100}$$

Finally, since the ball $B_p^{2\eta}$ has radius $\Omega(\eta^{-1})$, one can choose a covering of $\bar{\Gamma}$ by $O(\eta^{n+m})$ such balls. We let $\mathcal{H}$ be the union of the corresponding hyperplanes $\{P_p(\mathbf{f}) = 0\}$, and we take $V' = W \cap \mathcal{H}$ and $V = V' \cup (W \cap \mathcal{H}_0)$. Then the degree estimates follow from the Bezout theorem and the statement follows from (98), (99) and the choice of $\mathcal{H}_0$.

If W is reducible with components W_i, then we may repeat the construction above for each W_i separately and take V' to be the union of the resulting V_i' and $V = V' \cup (W \cap \mathcal{H}_0)$ as before. The degree estimates in this case are only improved. $\square$

Proof of Theorem 8. Define $\kappa(m),\dots,\kappa(0)$ by

$$\kappa(m)=0, \qquad \kappa(k)=k-1+\lambda(k)\kappa(k+1). \tag{101}$$

Apply Proposition 30 with $W=V_m:=\mathbb{C}^m$ to obtain an algebraic variety $V_{m-1}\subset\mathbb{C}^m$ of pure dimension $m-1$ and degree $O((\log H)^{\kappa(m-1)})$ such that

$$Y^{\text{size}}(\mathcal{F},H)\subset Y(V_m)\cup V_{m-1}. \tag{102}$$

Apply Proposition 30 again with $W=V_{m-1}$ to obtain an algebraic variety $V_{m-2}\subset\mathbb{C}^m$ of pure dimension $m-2$ and degree $O((\log H)^{\kappa(m-2)})$ such that

$$(Y\cap V_{m-1})^{\text{size}}(\mathcal{F},H)\subset Y(V_{m-1})\cup V_{m-2}. \tag{103}$$

Repeating similarly we obtain an algebraic variety $V_k\subset\mathbb{C}^m$ of pure dimension k and degree $O((\log H)^{\kappa(k)})$ such that

$$(Y\cap V_k)^{\text{size}}(\mathcal{F},H)\subset Y(V_k)\cup V_{k-1}, \tag{104}$$

where $V_{-1}=\emptyset$. Finally, using $Y(V_0)=Y\cap V_0$ and (104) gives

$$Y^{\text{size}}(\mathcal{F},H)\subset Y(V_0)\cup\cdots\cup Y(V_m). \tag{105}$$

An easy estimate on $\kappa(k)$ finishes the proof: as $\kappa(k)+1\leqslant\lambda(k)\left(\kappa(k+1)+1\right)$, we have

$$\kappa(0)\leqslant\prod_{k=0}^{m-1}\lambda(k)\leqslant 3^m\left(m+n+2S(r+1)+\ell+1\right)^{3m}. \qquad \square$$

7. Definable sets in $\mathbb{R}^{\text{RE}}$ and the language L^D_{RE}

Let $I=[-1,1]$. For $m\in\mathbb{N}$, we let $\mathbb{R}^{\text{RE}}\{X_1,\dots,X_m\}$ denote the ring of power series $f\in\mathbb{R}[[X_1,\dots,X_m]]$ such that

(1) f converges in a neighborhood of I^m;
(2) for every point $p\in I^m$, there is a polydisc Δ_p around p such that f converges in Δ_p to a holomorphic function, whose graph (in Δ_p) is definable in $\mathbb{R}^{\text{RE}}$.

We remark that [8] requires *strong definability* in item (2) above, but this is in fact equivalent to definability by the main result of [8]. By Proposition 18 and the compactness of I^m, for every function $f\in\mathbb{R}^{\text{RE}}\{X_1,\dots,X_m\}$, there exists a complex neighborhood Ω_f of I^m such that f is holomorphic and sub-Pfaffian in Ω_f.

We recall the language L^D_{RE} of [8]. There are a countable set of variables $\{X_1,X_2,\dots\}$, a relation symbol $<$ and a binary operation symbol D, and an m-ary operation symbol f for every $f\in\mathbb{R}^{\text{RE}}\{X_1,\dots,X_m\}$ satisfying $f(I^m)\subset I$.

We view I as an L^D_{RE}-structure by interpreting $<$ and f in the obvious way and interpreting D as *restricted division*, namely,

$$D(x,y) = \begin{cases} x/y & |x| \leqslant |y| \text{ and } y \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{106}$$

We denote by L_{RE} the language obtained from L^D_{RE} by omitting D.

For every L^D_{RE}-term $t(X_1,\ldots,X_m)$, we have an associated map $t : I^m \to I$, which we denote $x \to t(x)$.

LEMMA 31. *Let $t(X_1,\ldots,X_m)$ be an L_{RE} term. Then there is a complex neighborhood Ω of I^m such that t corresponds to a holomorphic sub-Pfaffian function in Ω.*

Proof. We prove the claim by induction: if $t = X_i$, then the claim is obvious. Suppose $t = f(T_1,\ldots,T_j)$, where $f \in \mathbb{R}^{\mathrm{RE}}\{Y_1,\ldots,Y_j\}$ and the claim is proved for $T_1,\ldots,T_j$. Then there exist a complex neighborhood Ω_f of I^j such that f is holomorphic and sub-Pfaffian in Ω_f and complex neighborhoods Ω_i of I^m such that T_i corresponds to a holomorphic sub-Pfaffian function in Ω_i. Since $T_i(I^n) \subset I$, we may, shrinking Ω_i if necessary, assume that $(T_1,\ldots,T_j)$ maps $\Omega := \Omega_1 \times \cdots \times \Omega_j$ to Ω_f. Then t corresponds to a holomorphic function in Ω, and since sub-Pfaffian functions are closed under composition, it is also sub-Pfaffian. $\square$

For an L^D_{RE}-formula $\phi(X_1,\ldots,X_m)$, we write $\phi(I^m)$ for the set of points $x \in I^m$ satisfying ϕ. If $A \subset I^m$, we write $\phi(A) := \phi(I^m) \cap A$. We will use the following key result of [8].

THEOREM 9. *I has elimination of quantifiers in L^D_{RE}.*

7.1. *Admissible formulas.* Let $U \subset \mathring{I}^m$ be an open subset. We define the notion of an L^D_{RE}-term being *admissible* in U by recursion as follows: a variable X_j is always admissible in U; a term $f(t_1,\ldots,t_m)$ is admissible in U if and only if the terms $t_1,\ldots,t_m$ are admissible in U; and a term $D(t_1,t_2)$ is admissible in U if t_1,t_2 are admissible in U and if

$$|t_1(x)| \leqslant |t_2(x)| \text{ and } t_2(x) \neq 0 \tag{107}$$

for every $x \in U$. An easy induction gives the following.

LEMMA 32. *If t is admissible in U, then the map $t : U \to I$ is real analytic.*

We will say that an L^D_{RE}-formula ϕ is admissible in U if all terms appearing in ϕ are admissible in U. The following proposition shows that when considering definable subsets of I one can essentially reduce to admissible formulas. The proof is identical to that of [3, Prop. 20].

Proposition 33. *Let $U \subset \mathring{I}^m$ be an open subset and $\phi(X_1, \dots, X_m)$ a quantifier-free L^D_{RE}-formula. There exist open subsets $U_1, \dots, U_k \subset U$ and quantifier-free L^D_{RE}-formulas $\phi_1, \dots, \phi_k$ such that ϕ_j is admissible in U_j and*

$$\phi(U) = \bigcup_{j=1}^{k} \phi_j(U_j). \tag{108}$$

7.2. *Basic formulas and equations.* We say that ϕ is a *basic D-formula* if it has the form

$$\Big(\wedge_{j=1}^{k} t_j(X_1, \dots, X_m) = 0 \Big) \wedge \Big(\wedge_{j=1}^{k'} s_j(X_1, \dots, X_m) > 0 \Big) \tag{109}$$

where t_j, s_j are L^D_{RE}-terms. It is easy to check the following.

Lemma 34. *Every quantifier-free L^D_{RE}-formula ϕ is equivalent in the structure I to a finite disjunction of basic formulas. If ϕ is U-admissible, then so are the basic formulas in the disjunction.*

We say that ϕ is a *basic D-equation* if $k' = 0$, i.e., if it involves only equalities. If ϕ is a basic D-formula, we denote by $\tilde{\phi}$ the basic D-equation obtained by removing all inequalities.

Let ϕ be a U-admissible basic D-formula for some $U \subset I^m$. Then $\tilde{\phi}$ is U-admissible as well. Moreover, since all the terms s_j evaluate to continuous functions in U, the strict inequalities of ϕ are open in U and we have the following.

Lemma 35. *Suppose ϕ is a U-admissible basic D-formula. Then $\phi(U)$ is relatively open in $\tilde{\phi}(U)$.*

The set defined by an admissible D-equation can be described in terms of admissible projections in the sense of Section 6.1.

Proposition 36. *Let $U \subset \mathring{I}^m$, and let ϕ be a U-admissible D-equation,*

$$\phi = (t_1 = 0) \wedge \cdots \wedge (t_k = 0). \tag{110}$$

In the notation of Section 6.1, there exist

(1) *complex domains $\Omega_x \subset \mathbb{C}^m$ and $\Omega_y \subset \mathbb{C}^N$ with $N \in \mathbb{N}$ and $I^m \subset \Omega_x$,*
(2) *an open complex neighborhood $U \subset U_{\mathbb{C}} \subset \Omega_x$,*
(3) *an analytic map $\psi : U_{\mathbb{C}} \to \Omega_y$,*
(4) *an analytic set $X \subset \Omega$,*

such that ψ is admissible, the sets $X, X_\Gamma \subset \Omega$ are sub-Pfaffian, and $Y := \pi(X \cap \Gamma_\psi)$ satisfies $Y_{\mathbb{R}} = \phi(U)$.

Proof. The proof is essentially the same as the proof in [3, Prop. 23] for the language L^D_{an}. We note that in [3] this proposition is proved with an extra set of parameters Λ, and in the current context one can take Λ to be a singleton. To see that the sets X, X_Γ are also sub-Pfaffian use Lemma 31. $\square$

7.3. *Estimate for L^D_{RE}-definable sets.*

PROPOSITION 37. *Let $A \subset \mathring{I}^m$ be an L^D_{RE}-definable set. There exist integers $\kappa = \kappa(A)$ and $N = N(A, [\mathcal{F} : \mathbb{Q}])$ with the following property: for any $H \in \mathbb{N}$, there exists a collection of at most $\beta := N(\log H)^\kappa$ smooth connected semi-algebraic subsets $S_\alpha \subset \mathbb{R}^m$, each of complexity (m, β, β) such that*

$$A(\mathcal{F}, H) \subset \bigcup_\alpha A(S_\alpha). \tag{111}$$

Proof. According to (39), for any $\alpha \in \mathcal{F}$, we have $H^{\mathrm{size}}(\alpha) \leqslant H(\alpha)^t$ where $t := [\mathcal{F} : \mathbb{Q}]$. Thus

$$A(\mathcal{F}, H) \subset A^{\mathrm{size}}(\mathcal{F}, H^t) \tag{112}$$

for every $H \in \mathbb{N}$. Therefore up to minor rescaling it will suffice to prove the claim with $H(\cdot)$ replaced by $H^{\mathrm{size}}(\cdot)$.

By Theorem 9 we may write $A = \phi(\mathring{I}^m)$ for some quantifier-free L^D_{RE}-formula ϕ. By Proposition 33 and Lemma 34 we may write

$$A = \bigcup_{j=1}^{k} \bigcup_{i=1}^{n_j} \phi_{ji}(U_j), \tag{113}$$

where ϕ_{ji} is a U_j-admissible basic D-formula. By the first part of Lemma 26 it is clear that it will suffice to prove the claim with A replaced by each $\phi_{ij}(U_j)$. We thus assume without loss of generality that ϕ is already a U-admissible basic D-formula and prove the claim for $A = \phi(U)$.

Recall that $\tilde{\phi}$ is a U-admissible D-equation. We write $B = \tilde{\phi}(\mathring{I}^m)$. Applying Proposition 36 to $\tilde{\phi}$ and using Theorem 8 we construct a locally analytic set $Y \subset \mathbb{C}^m$ such that $Y_{\mathbb{R}} = B$, and algebraic varieties $V_0, \ldots, V_m$ such that V_j has pure dimension j and degree $O((\log H)^{\kappa'})$ such that

$$Y^{\mathrm{size}}(\mathcal{F}, H) \subset Y(V_0) \cup \cdots \cup Y(V_m). \tag{114}$$

By (84) and $Y_{\mathbb{R}} = B$, we have

$$B(\mathcal{F}, H)^{\mathrm{size}} \subset B(\mathcal{V}_0) \cup \cdots \cup B(\mathcal{V}_m), \qquad \mathcal{V}_j := (V_j)_{\mathbb{R}}. \tag{115}$$

Recall that A is relatively open in B by Lemma 35. Then

$$\begin{aligned} A^{\mathrm{size}}(\mathcal{F}, H) &= A \cap (B^{\mathrm{size}}(\mathcal{F}, H)) \subset A \cap \bigcup_{j=0}^{m} B(\mathcal{V}_j) = \bigcup_{j=0}^{m} (A \cap B(\mathcal{V}_j)) \\ &= \bigcup_{j=0}^{m} A(\mathcal{V}_j), \end{aligned} \tag{116}$$

where the last equality is given by the second part of Lemma 26.

If we write each $\mathcal{V}_j$ as a union of connected smooth strata $\mathcal{V}_j = \cup_l S_{j,l}$, then we have

$$A^{\text{size}}(\mathcal{F}, H) \subset \bigcup_j A(\mathcal{V}_j) \subset \bigcup_{j,l} A(S_{j,l}). \tag{117}$$

It remains to estimate the number and complexity of the strata. Recall from Lemma 29 that each V_j is cut out by $m+1$ complex equations of degree at most $\deg V_j$, which is bounded by $\beta_1 := O((\log H)^{\kappa'})$. The real-part $\mathcal{V}_j$ is cut out by the same equations and additional linear equations for the vanishing of all imaginary parts, and thus has complexity $(m, 3m+2, \beta_1)$. By [10, Th. 2] one can decompose $\mathcal{V}_j$ into a union of $\beta_2 := \beta_1^{(2^{O(m)})}$ smooth (but not necessarily connected) semi-algebraic sets of complexity (m, β_2, β_2). Finally, by [1, Th. 16.13] each such semi-algebraic set can be decomposed into its connected components, with the number of connected components bounded by $\beta_3 = \beta_2^{(m^4)}$ and their complexity bounded by (m, β_3, β_3). (A better estimate for the number of connected components follows from Theorem 6.) □

7.4. *Estimate for $\mathbb{R}^{\text{RE}}$-definable sets.* Consider the map $\tau : \mathbb{R} \to \mathring{I}$ given by $x \to x/\sqrt{1+x^2}$. This map is a bijection between $\mathbb{R}$ and $\mathring{I}$, with the inverse $\tau^{-1} : \mathring{I} \to \mathbb{R}$ given by $y \to y/\sqrt{1-y^2}$. A straightforward verification [8, 4.6] shows that the τ-images of the basic relations of $\mathbb{R}^{\text{RE}}$ are L^D_{RE}-definable in I, and it follows that every $\mathbb{R}^{\text{RE}}$-definable set $A \subset \mathbb{R}^n$ has an L^D_{RE}-definable τ-image $\tau(A) \subset \mathring{I}^m$.

Lemma 38. *Suppose $A \subset \mathring{I}^m$ is $\mathbb{R}^{\text{RE}}$-definable. Then A is L^D_{RE}-definable.*

Proof. Let $I' := \tau I = [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$. By the above, $\tau(A) \subset \mathring{I}'$ is L^D_{RE}-definable. The restriction $\tau^{-1}|_{I'} : I' \to [-1,1]$ is definable in I by the L^D_{RE}-formula

$$\phi(y,x) := \exists z : z \geqslant 0, z^2 = 1 - y^2, x = D(y,z). \tag{118}$$

Then $A = \tau^{-1}(\tau A)$ is L^D_{RE}-definable as well. □

The following theorem is the general form of our main result.

Theorem 10. *Let $A \subset \mathbb{R}^m$ be an $\mathbb{R}^{\text{RE}}$-definable set. There exist integers $\kappa = \kappa(A)$ and $N = N(A, [\mathcal{F} : \mathbb{Q}])$ with the following property: for any $H \in \mathbb{N}$, there exists a collection of at most $\beta := N(\log H)^\kappa$ smooth connected semi-algebraic subsets $S_\alpha \subset \mathbb{R}^m$, each of complexity (m, β, β) such that*

$$A(\mathcal{F}, H) \subset \bigcup_\alpha A(S_\alpha). \tag{119}$$

Proof. For $A \subset \mathring{I}^m$, the claim follows by Lemma 38 and Proposition 37. For the general case, note that the definable transformations $x_i \to 1/x_i$ and

$x_i \to x_i + 1$ do not affect the heights of the points of A by more than a constant factor, their pullbacks preserve the smoothness of S_α and do not affect the degrees of S_α by more than a constant factor. It is easy to construct a finite set $\{T_j\}$ where each $T_j : \mathbb{R}^m \to \mathbb{R}^m$ is a finite composition of the transformations above such that for any $A \subset \mathbb{R}^m$,

$$A = \bigcup_j T_j^{-1}(T_j(A) \cap \mathring{I}^m). \tag{120}$$

The claim now follows from the case $A \subset \mathring{I}^m$ already considered. □

Remark 39. The reader may note that the strata S_α in Theorem 10 play essentially the same role as the "basic blocks" introduced in [23]. They will be used in a similar way for the proof of Theorem 2.

7.5. *Proofs of the main theorems.* In this section we prove our two main results Theorems 1 and 2. We start with Theorem 1, which is essentially an immediate consequence of Theorem 10.

Proof of Theorem 1. Let $H \in \mathbb{N}$. Consider the collection of β smooth connected semi-algebraic sets S_α obtained from Theorem 10, such that

$$A(\mathcal{F}, H) \subset \bigcup_\alpha A(S_\alpha). \tag{121}$$

By Lemma 25 we see that $A^{\text{trans}}(\mathcal{F}, H)$ is contained in the union of the zero-dimensional strata S_α. Then $\#(A^{\text{trans}})(\mathcal{F}, H)$ is bounded by the number of strata, which is of the required form by Theorem 10. □

We now pass to the proof of Theorem 2, which is a direct adaptation of the approach of [23] (with slightly more effort needed to obtain the necessary degree estimates). We introduce some additional notation for this purpose. Let $\mathcal{P}_{\leqslant k} := \mathbb{R}^{k+1} \setminus \{\mathbf{0}\}$ and $\mathcal{P}_k := \{\mathbf{c} \in \mathcal{P}_{\leqslant k} : c_k \neq 0\}$. For $\mathbf{c} \in \mathcal{P}_{\leqslant k}$, let $P_{\mathbf{c}} \in \mathbb{R}[x]$ denote the polynomial

$$P_{\mathbf{c}}(X) := \sum_{j=0}^{k} c_j X^j. \tag{122}$$

We let $D_k \subset \mathcal{P}_k$ denote the *discriminant*, i.e., the set of $\mathbf{c} \in \mathcal{P}_k$ such that $P_{\mathbf{c}}$ has a (possibly complex) double zero. Then D_k is an algebraic subset and its complement $\tilde{\mathcal{P}}_k := \mathcal{P}_k \setminus D_k$ is an open semi-algebraic set. We let $Z_k \subset \tilde{\mathcal{P}}_k \times \mathbb{R}$ be the set

$$Z_k := \{(\mathbf{c}, x) \in \tilde{\mathcal{P}}_k \times \mathbb{R} : P_{\mathbf{c}}(x) = 0\}. \tag{123}$$

LEMMA 40. *Let $\{U_\chi\}$ denote the connected components of $\tilde{\mathcal{P}}_k$. For each χ, there is a tuple of at most k real-analytic algebraic functions $\phi_{\chi,j} : U_\chi \to \mathbb{R}$*

such that

$$Z_k = \bigcup_{\chi,j} \Gamma_{\phi_{\chi,j}}, \tag{124}$$

where Γ_ϕ denotes the graph of ϕ.

Proof. Since $\tilde{\mathcal{P}}_k$ is the complement of the discriminant, the number of zeros of $P_{\mathbf{c}}$ for $\mathbf{c} \in U_\chi$ is some constant $n_\chi \leqslant k$, and since $P_{\mathbf{c}}$ has no double zeros one can uniquely choose the branch $\phi_{\chi,j}$ for $j = 1, \dots, n_\chi$ to be the j-th root of $P_{\mathbf{c}}$ in increasing order. The branches thus constructed are Nash functions, i.e., real analytic and algebraic, on U_χ. Their graphs cover $Z_k \cap (U_\chi \times \mathbb{R})$ by construction. □

Following [23] we introduce the following height function. For an algebraic number $\alpha \in \mathbb{Q}^{\mathrm{alg}}$, we define

$$H_k^{\mathrm{poly}}(\alpha) = \min\{H(\mathbf{c}) : \mathbf{c} \in \mathcal{P}_{\leqslant k}(\mathbb{Q}), \quad P_{\mathbf{c}}(\alpha) = 0\} \tag{125}$$

and $H_k^{\mathrm{poly}}(\alpha) = \infty$ if $[\mathbb{Q}(\alpha) : \mathbb{Q}] > k$. Then whenever $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leqslant k$, we have [23, 5.1]

$$H_k^{\mathrm{poly}}(\alpha) \leqslant 2^k H(\alpha)^k. \tag{126}$$

For a set $A \subset \mathbb{R}^m$ we define $A^{\mathrm{poly}}(k, H)$ in analogy with $A(k, H)$ replacing $H(\cdot)$ by $H_k^{\mathrm{poly}}(\cdot)$.

Proof of Theorem 2, *adapted from* [23]. As in the proof of Theorem 1, from (126) we deduce that up to minor rescaling it will suffice to prove the claim with $A(k, H)$ replaced by $A^{\mathrm{poly}}(k, H)$. Let $\mathbf{k} = (k_1, \dots, k_m) \in \mathbb{N}^m$, and denote

$$A(\mathbf{k}) := \{\mathbf{x} \in A : [\mathbb{Q}(x_1) : \mathbb{Q}] = k_1, \dots, [\mathbb{Q}(x_m) : \mathbb{Q}] = k_m\}, \tag{127}$$

$$A^{\mathrm{poly}}(\mathbf{k}, H) := \{\mathbf{x} \in A(\mathbf{k}) : H_k^{\mathrm{poly}}(\mathbf{x}) \leqslant H\}. \tag{128}$$

Then

$$A^{\mathrm{poly}}(k, H) = \bigcup_{\mathbf{k}: 1 \leqslant k_j \leqslant k} A^{\mathrm{poly}}(\mathbf{k}, H), \tag{129}$$

and it will suffice to prove the claim for fixed $\mathbf{k}$ in place of k.

Denote $\tilde{\mathcal{P}}^{\mathbf{k}} := \prod_{j=1}^m \tilde{\mathcal{P}}^{k_j}$, and let $Z_{\mathbf{k}} \subset \tilde{\mathcal{P}}_{\mathbf{k}} \times \mathbb{R}^m$ be the set

$$Z_{\mathbf{k}} := \{(\mathbf{c}^1, \dots, \mathbf{c}^m, x_1, \dots, x_m) \in \tilde{\mathcal{P}}^k \times \mathbb{R}^m : \\ P_{\mathbf{c}^1}(x_1) = \dots = P_{\mathbf{c}^m}(x_m) = 0\}. \tag{130}$$

Let $\{U_{\boldsymbol{\chi}}\}$ denote the connected components of $\tilde{\mathcal{P}}_{\mathbf{k}}$. For each $\boldsymbol{\chi}$, there is a tuple of at most k^m real-analytic algebraic maps $\phi_{\boldsymbol{\chi},j} : U_{\boldsymbol{\chi}} \to \mathbb{R}^m$ such that

$$Z_{\mathbf{k}} = \bigcup_{\boldsymbol{\chi},j} \Gamma_{\phi_{\boldsymbol{\chi},j}}, \tag{131}$$

where Γ_ϕ denotes the graph of ϕ. Indeed, $U_{\boldsymbol{\chi}}$ are just direct products of the connected components of $\tilde{\mathcal{P}}^{k_j}$, and the claim follows by taking the direct products of the functions constructed in Lemma 40.

For each component $U_{\boldsymbol{\chi}}$ and map $\phi_{\boldsymbol{\chi},j}$, we define the set

$$A_{\boldsymbol{\chi},j} \subset \tilde{\mathcal{P}}^{\mathbf{k}}, \quad A_{\boldsymbol{\chi},j} := \{p \in \tilde{\mathcal{P}}^{\mathbf{k}} : \phi_{\boldsymbol{\chi},j}(p) \in A\}. \tag{132}$$

Since $\phi_{\boldsymbol{\chi},j}$ are semi-algebraic, $A_{\boldsymbol{\chi},j}$ are definable in $\mathbb{R}^{\mathrm{RE}}$. Consider the collection of at most $\beta = N(\log H)^\kappa$ smooth connected semi-algebraic strata $S_{\boldsymbol{\chi},j,\alpha}$ obtained from Theorem 10, such that

$$A_{\boldsymbol{\chi},j}(\mathbb{Q}, H) \subset \bigcup_\alpha A_{\boldsymbol{\chi},j}(S_{\boldsymbol{\chi},j,\alpha}). \tag{133}$$

Let $\mathbf{x} \in A^{\mathrm{poly}}(\mathbf{k}, H)$, and for $l = 1, \ldots, m$, let $\mathbf{c}^l \in \mathcal{P}^{\leqslant k_l}$ be a tuple satisfying $P_{\mathbf{c}^l}(x_l) = 0$ and $H(\mathbf{c}^l) \leqslant H$. Since $[\mathbb{Q}(x_l) : \mathbb{Q}] = k_l$, we see that $P_{\mathbf{c}^l}$ is (up to a scalar) the minimal polynomial of x_l, so it has degree k_l and no multiple roots, i.e., $\mathbf{c}^l \in \tilde{\mathcal{P}}^{k_l}$. Write $p = (\mathbf{c}^1, \ldots, \mathbf{c}^m) \in \tilde{\mathcal{P}}^{\mathbf{k}}$. Then $(p, x) \in Z_{\mathbf{k}}$, and by (131) we have $\mathbf{x} = \phi_{\boldsymbol{\chi},j}(p)$ for some pair $(\boldsymbol{\chi}, j)$. By (132) we have $p \in A_{\boldsymbol{\chi},j}(\mathbb{Q}, H)$. Choose one of the strata $S_{\boldsymbol{\chi},j,\alpha}$ such that $p \in A_{\boldsymbol{\chi},j}(S_{\boldsymbol{\chi},j,\alpha})$, and denote it by $S(p)$ and its germ at p by S_p. Then $S_p \subset A_{\boldsymbol{\chi},j}$, and by (132) we have $Y_p := \phi_{\boldsymbol{\chi},j}(S_p) \subset A$. Note that Y_p is a semi-algebraic set containing $\phi_{\boldsymbol{\chi},j}(p) = \mathbf{x}$.

Suppose $\mathbf{x} \in A^{\mathrm{trans}}$. Then $\phi_{\boldsymbol{\chi},j}$ is constant on S_p, for otherwise Y_p would be a connected positive-dimensional semi-algebraic set containing $\mathbf{x}$. Since $S(p)$ is connected, nonsingular and analytic and $\phi_{\boldsymbol{\chi},j}$ is real analytic, it follows that $\phi_{\boldsymbol{\chi},j}$ is constant (with value $\mathbf{x}$) on the whole of $S(p)$.

From the above it follows that there is an injective correspondence $p \to S(p)$ between the points $\mathbf{x} \in (A^{\mathrm{trans}})^{\mathrm{poly}}(\mathbf{k}, H)$ and the strata $S_{\boldsymbol{\chi},j,\alpha}$. The number of these strata for each $\boldsymbol{\chi}, j$ is at most β, and since the number of pairs $\boldsymbol{\chi}, j$ is independent of H we obtain a bound of the required form. (But note that the exponent κ now depends on the sets $A_{\boldsymbol{\chi},j}$, i.e., on k as well as A.) This finishes the proof. □

8. Concluding remarks

8.1. *Effectivity.* While we do not compute all explicit constants in this paper in the interest of space, all of the estimates presented for holomorphic-Pfaffian varieties and their admissible projections are entirely effective in the complexity of the holomorphic-Pfaffian varieties involved. We do give an effective estimate for the exponent κ in Theorem 8 as we believe this may be of some interest in possible diophantine applications.

As a consequence of the above, our estimates for sets defined by quantifier-free L_{RE}^D-formulas can be made effective in terms of the complexity of the formulas. We do not study the effectivity of the quantifier-elimination result of [8], and we therefore cannot make a direct statement about the effectivity of our main results for general $\mathbb{R}^{\mathrm{RE}}$-definable sets. However, we believe that by a combination of Pfaffian techniques and the proof of [8] it is possible to obtain an effective statement in this context as well.

8.2. *Uniformity with respect to parameters.* Our approach is essentially uniform over definable families, and the results could be developed in this additional generality in the same way as this is done in [3] for the subanalytic context. We avoided this extra generality in this paper in the interest of preserving clarity; and also since in view of Section 8.1, we believe a detailed analysis should give estimates depending only on the complexity of the formulas involved, and hence *a priori* uniform over families.

8.3. *Generalization to other structures.* We have focused in this paper on the structure $\mathbb{R}^{\mathrm{RE}}$ since it contains the restricted form of Wilkie's original conjecture. It is of course natural to make similar conjectures for sets definable in other "tame" geometric structures. We identify two possible categories for such a generalization: structures generated by *elliptic* functions, for instance the Weierstrass $\wp$-function (for a fixed lattice/s) and possibly higher-dimensional abelian functions; and structures generated by *modular* functions, for instance Klein's j-invariant and possibly universal covering maps of more general Shimura curves/varieties. In both cases one must of course restrict the functions to a suitable fundamental domain to avoid periodicity and obtain an O-minimal structure [18]. Both categories are closely related to diophantine problems of unlikely intersections: the former to the circle of problems around the Manin-Mumford conjecture, and the latter to the circle of problems around the André-Oort conjecture [25].

The elliptic case appears to be possibly amenable to our approach. That is, a surprising work of Macintyre [16] shows that the real and imaginary parts of $\wp^{-1}$ (on an appropriate domain) are real-Pfaffian functions, thus placing $\wp$ in the holomorphic-Pfaffian category (in analogy with the function e^z whose real an imaginary parts e^x and $\sin x$, restricted to an appropriate domain, are real-Pfaffian and generate $\mathbb{R}^{\mathrm{RE}}$). From the model-theoretic side, the work of Bianconi [2] establishes a close analog of the results of [8] for the structures generated by real and imaginary parts of elliptic (or more generally abelian) functions. It therefore appears likely that the methods used in this paper could be carried over to the elliptic category (at least for elliptic functions).

The modular category currently appears to be more challenging: we have no reason to believe that the j-function is Pfaffian (or definable from Pfaffian functions). However, we note that the j-function (as well as other modular functions) does satisfy certain natural non-Pfaffian systems of differential equations, and one may hope that some progress in the analysis of such systems could provide a suitable replacement for the Pfaffian category.

References

[1] S. Basu, R. Pollack, and M.-F. Roy, *Algorithms Real Algebraic Geometry*, second ed., *Algorithms in Comput. Math.* no. 10, Springer-Verlag, New York, 2006. MR 2248869. Zbl 1102.14041.

[2] R. Bianconi, Model completeness results for elliptic and abelian functions, *Ann. Pure Appl. Logic* **54** no. 2 (1991), 121–136. MR 1134193. Zbl 0756.03018. https://doi.org/10.1016/0168-0072(91)90028-K.

[3] G. Binyamini and D. Novikov, The Pila-Wilkie theorem for subanalytic families: a complex analytic approach, 2016, preprint, to appear in *Compos. Math.*

[4] E. Bombieri and W. Gubler, *Heights in Diophantine Geometry*, *New Math. Monogr.* **4**, Cambridge Univ. Press, Cambridge, 2006. MR 2216774. Zbl 1115.11034. https://doi.org/10.1017/CBO9780511542879.

[5] E. Bombieri and J. Pila, The number of integral points on arcs and ovals, *Duke Math. J.* **59** no. 2 (1989), 337–357. MR 1016893. Zbl 0718.11048. https://doi.org/10.1215/S0012-7094-89-05915-2.

[6] L. A. Butler, Some cases of Wilkie's conjecture, *Bull. Lond. Math. Soc.* **44** no. 4 (2012), 642–660. MR 2967234. Zbl 1253.03063. https://doi.org/10.1112/blms/bdr126.

[7] J. Denef and L. van den Dries, p-adic and real subanalytic sets, *Ann. of Math.* **128** no. 1 (1988), 79–138. MR 0951508. Zbl 0693.14012. https://doi.org/10.2307/1971463.

[8] L. van den Dries, On the elementary theory of restricted elementary functions, *J. Symbolic Logic* **53** no. 3 (1988), 796–808. MR 0960999. Zbl 0698.03023. https://doi.org/10.2307/2274572.

[9] O. Friedland and Y. Yomdin, Vitushkin-type theorems, in *Geometric Aspects of Functional Analysis*, *Lecture Notes in Math.* **2116**, Springer-Verlag, New York, 2014, pp. 147–157. MR 3364686. Zbl 1327.14240. https://doi.org/10.1007/978-3-319-09477-9_12.

[10] A. Gabrièlov and N. Vorobjov, Complexity of stratifications of semi-Pfaffian sets, *Discrete Comput. Geom.* **14** no. 1 (1995), 71–91. MR 1320587. Zbl 0832.68056. https://doi.org/10.1007/BF02570696.

[11] A. Gabrielov and N. Vorobjov, Complexity of computations with Pfaffian and Noetherian functions, in *Normal Forms, Bifurcations and Finiteness Problems in Differential Equations*, *NATO Sci. Ser. II Math. Phys. Chem.* **137**, Kluwer Acad. Publ., Dordrecht, 2004, pp. 211–250. MR 2083248.

[12] A. Gabrielov and N. Vorobjov, Approximation of definable sets by compact families, and upper bounds on homotopy and homology, *J. Lond. Math. Soc.* **80** no. 1 (2009), 35–54. MR 2520376. Zbl 1177.14097. https://doi.org/10.1112/jlms/jdp006.

[13] R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, AMS Chelsea Publishing, Providence, RI, 2009, reprint of the 1965 original. MR 2568219. Zbl 1204.01045. https://doi.org/10.1090/chel/368.

[14] G. O. Jones and M. E. M. Thomas, The density of algebraic points on certain Pfaffian surfaces, *Q. J. Math.* **63** no. 3 (2012), 637–651. MR 2967167. Zbl 1253.03065. https://doi.org/10.1093/qmath/har011.

[15] A. G. Khovanskiĭ, *Fewnomials*, *Transl. Math. Monogr.* **88**, Amer. Math. Soc., Providence, RI, 1991, translated from the Russian by Smilka Zdravkovska. MR 1108621. Zbl 0728.12002.

[16] A. Macintyre, Some observations about the real and imaginary parts of complex Pfaffian functions, in *Model Theory with Applications to Algebra and Analysis. Vol.* 1, *London Math. Soc. Lecture Note Ser.* **349**, Cambridge Univ. Press, Cambridge, 2008, pp. 215–223. MR 2441381. Zbl 1162.33312. https://doi.org/10.1017/CBO9780511735226.011.

[17] B. Malgrange, *Lectures on the Theory of Functions of Several Complex Variables*, *Tata Inst. Fund. Res. Lect. Math. Phys.* **13**, Springer-Verlag, New York, 1984, reprint of the 1958 edition, notes by Raghavan Narasimhan. MR 0742775. Zbl 0561.32006.

[18] Y. Peterzil and S. Starchenko, Definability of restricted theta functions and families of abelian varieties, *Duke Math. J.* **162** no. 4 (2013), 731–765. MR 3039679. Zbl 1284.03215. https://doi.org/10.1215/00127094-2080018.

[19] J. Pila, Geometric postulation of a smooth function and the number of rational points, *Duke Math. J.* **63** no. 2 (1991), 449–463. MR 1115117. Zbl 0763.11025. https://doi.org/10.1215/S0012-7094-91-06320-9.

[20] J. Pila, Integer points on the dilation of a subanalytic surface, *Q. J. Math.* **55** no. 2 (2004), 207–223. MR 2068319. Zbl 1111.32004. https://doi.org/10.1093/qjmath/55.2.207.

[21] J. Pila, Rational points on a subanalytic surface, *Ann. Inst. Fourier* (*Grenoble*) **55** no. 5 (2005), 1501–1516. MR 2172272. Zbl 1121.11032. Available at http://aif.cedram.org/item?id=AIF_2005__55_5_1501_0.

[22] J. Pila, The density of rational points on a Pfaff curve, *Ann. Fac. Sci. Toulouse Math.* **16** no. 3 (2007), 635–645. MR 2379055. Zbl 1229.11053. Available at http://afst.cedram.org/item?id=AFST_2007_6_16_3_635_0.

[23] J. Pila, On the algebraic points of a definable set, *Selecta Math.* **15** no. 1 (2009), 151–170. MR 2511202. Zbl 1218.11068. https://doi.org/10.1007/s00029-009-0527-8.

[24] J. Pila, Counting rational points on a certain exponential-algebraic surface, *Ann. Inst. Fourier* (*Grenoble*) **60** no. 2 (2010), 489–514. MR 2667784. Zbl 1210.11074. https://doi.org/10.5802/aif.2530.

[25] J. Pila, O-minimality and the André-Oort conjecture for $\mathbb{C}^n$, *Ann. of Math.* **173** no. 3 (2011), 1779–1840. MR 2800724. Zbl 1243.14022. https://doi.org/10.4007/annals.2011.173.3.11.

[26] J. Pila and A. J. Wilkie, The rational points of a definable set, *Duke Math. J.* **133** no. 3 (2006), 591–616. MR 2228464. Zbl 1217.11066. https://doi.org/10.1215/S0012-7094-06-13336-7.

[27] T. Scanlon, Counting special points: logic, Diophantine geometry, and transcendence theory, *Bull. Amer. Math. Soc.* **49** no. 1 (2012), 51–71. MR 2869007. Zbl 1323.11041. https://doi.org/10.1090/S0273-0979-2011-01354-4.

[28] A. J. Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function, *J. Amer. Math. Soc.* **9** no. 4 (1996), 1051–1094. MR 1398816. Zbl 0892.03013. https://doi.org/10.1090/S0894-0347-96-00216-0.

(Received: May 15, 2016)
(Revised: February 13, 2017)

Weizmann Institute of Science, Rehovot, Israel
E-mail: gal.binyamini@weizmann.ac.il

Weizmann Institute of Science, Rehovot, Israel
E-mail: dmitry.novikov@weizmann.ac.il